Linear Programming

Topics

- Introduction
- The Linear Programming Model
- Examples of Linear Programming Problems
- Developing Linear Programming Models
- Graphical Solution to LP Problems
- The Simplex Method
- Simplex Tableau for Maximization Problem

Introduction

- OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing.
- Mathematical programming is used to find the best or optimal solution to a problem that requires a decision or set of decisions about how best to use a set of limited resources to achieve a state goal of objectives.
- **Steps involved in mathematical programming**
  - Conversion of stated problem into a mathematical model that abstracts all the essential elements of the problem.
  - Exploration of different solutions of the problem.
  - Finding out the most suitable or optimum solution.
- **Linear programming** requires that all the mathematical functions in the model be linear functions.

The Linear Programming Model (1)

Let: $X_1, X_2, X_3, \ldots\ldots\ldots, X_n$ = decision variables

Z = Objective function or linear function

Requirement: Maximization of the linear function Z.

$$Z = c_1X_1 + c_2X_2 + c_3X_3 + \ldots\ldots\ldots + c_nX_n \quad \ldots\ldots Eq\ (1)$$

subject to the following constraints:

.....Eq (2)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_n \\ \text{all } x_j &\geq 0 \end{aligned}$$

where a_{ij}, b_i, and c_j are given constants.

The Linear Programming Model (2)

- The linear programming model can be written in more compact notation as:

Maximize

$$Z = \sum_{j=1}^{n} c_j x_j$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \text{.....Eq (3)}$$

where

$$i = 1, 2, \ldots, m$$

and

$$x_j \geq 0$$

where

$$j = 1, 2, \ldots, n$$

The decision variables, x_1, x_2, ..., x_n, represent levels of n competing activities.

Examples of LP Problems (1)

1. A Product Mix Problem

- A manufacturer has fixed amounts of different resources such as raw material, labor, and equipment.
- These resources can be combined to produce any one of several different products.
- The quantity of the i^{th} resource required to produce one unit of the j^{th} product is known.
- The decision maker wishes to produce the combination of products that will maximize total income.

Examples of LP Problems (2)

2. A Blending Problem

- Blending problems refer to situations in which a number of components (or commodities) are mixed together to yield one or more products.
- Typically, different commodities are to be purchased. Each commodity has known characteristics and costs.
- The problem is to determine how much of each commodity should be purchased and blended with the rest so that the characteristics of the mixture lie within specified bounds and the total cost is minimized.

Examples of LP Problems (3)

3. A Production Scheduling Problem

- A manufacturer knows that he must supply a given number of items of a certain product each month for the next *n* months.
- They can be produced either in regular time, subject to a maximum each month, or in overtime. The cost of producing an item during overtime is greater than during regular time. A storage cost is associated with each item not sold at the end of the month.
- The problem is to determine the production schedule that minimizes the sum of production and storage costs.

Examples of LP Problems (4)

4. A Transportation Problem

- A product is to be shipped in the amounts $a_1, a_2, \ldots, a_m$ from m shipping origins and received in amounts $b_l, b_2, \ldots, b_n$ at each of n shipping destinations.
- The cost of shipping a unit from the ith origin to the jth destination is known for all combinations of origins and destinations.
- The problem is to determine the amount to be shipped from each origin to each destination such that the total cost of transportation is a minimum.

Examples of LP Problems (5)

5. A Flow Capacity Problem

- One or more commodities (e.g., traffic, water, information, cash, etc.) are flowing from one point to another through a network whose branches have various constraints and flow capacities.
- The direction of flow in each branch and the capacity of each branch are known.
- The problem is to determine the maximum flow, or capacity of the network.

Developing LP Model (1)

- **Steps Involved:**
  - Determine the objective of the problem and describe it by a criterion function in terms of the decision variables.
  - Find out the constraints.
  - Do the analysis which should lead to the selection of values for the decision variables that optimize the criterion function while satisfying all the constraints imposed on the problem.

Developing LP Model (2)

Example: Product Mix Problem

N. Dustrious Company produces two products: I and II. The raw material requirements, space needed for storage, production rates, and selling prices for these products are given in Table 1.

Production Data for N. Dustrious Company

	Product	
	I	II
Storage space (ft^2/unit)	4	5
Raw material (lb/unit)	5	3
Production rate (units/hr)	60	30
Selling price ($/unit)	13	11

The total amount of raw material available per day for both products is 1575 lb. The total storage space for all products is 1500 ft^2, and a maximum of 7 hours per day can be used for production.

Developing LP Model (3)

Example Problem

All products manufactured are shipped out of the storage area at the end of the day. Therefore, the two products must share the total raw material, storage space, and production time.

The company wants to determine how many units of each product to produce per day to maximize its total income.

Solution

- The company has decided that it wants to maximize its sale income, which depends on the number of units of product I and II that it produces.
- Therefore, the decision variables, x_1 and x_2 can be the number of units of products I and II, respectively, produced per day.

Developing LP Model (4)

- The object is to maximize the equation:

$$Z = 13x_1 + 11x_2$$

subject to the constraints on storage space, raw materials, and production time.

- Each unit of product I requires 4 ft^2 of storage space and each unit of product II requires 5 ft^2. Thus a total of $4x_1 + 5x_2$ ft^2 of storage space is needed each day. This space must be less than or equal to the available storage space, which is 1500 ft^2. Therefore,

$$4X_1 + 5X_2 \leq 1500$$

- Similarly, each unit of product I and II produced requires 5 and 3 lbs, respectively, of raw material. Hence a total of $5x_l + 3x_2$ Ib of raw material is used.

Developing LP Model (5)

- This must be less than or equal to the total amount of raw material available, which is 1575 lb. Therefore,

$$5x_1 + 3x_2 \leq 1575$$

- Product I can be produced at the rate of 60 units per hour. Therefore, it must take I minute or 1/60 of an hour to produce I unit. Similarly, it requires 1/30 of an hour to produce 1 unit of product II. Hence a total of $x_1/60 + x_2/30$ hours is required for the daily production. This quantity must be less than or equal to the total production time available each day. Therefore,

$$x_1 / 60 + x_2 / 30 \leq 7$$

$$\text{or} \quad x_1 + 2x_2 \leq 420$$

- Finally, the company cannot produce a negative quantity of any product, therefore x_1 and x_2 must each be greater than or equal to zero.

Developing LP Model (6)

- The linear programming model for this example can be summarized as:

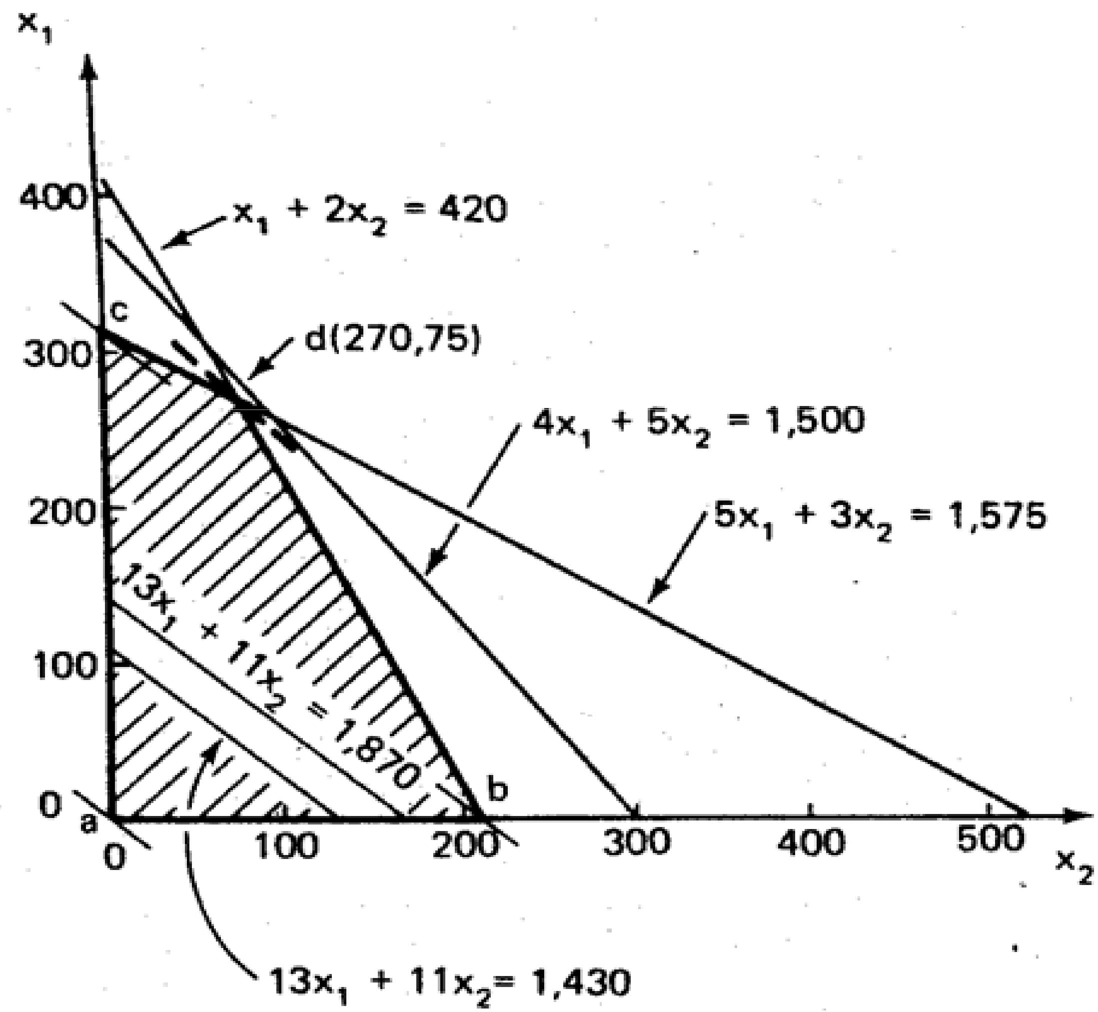
Maximize .....Eq (4)

$$Z = 13x_1 + 11x_2$$

subject to:

$$\begin{aligned} 4x_1 + 5x_2 &\leq 1500 \\ 5x_1 + 3x_2 &\leq 1575 \\ x_1 + 2x_2 &\leq 420 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Graphical Solution to LP Problems (1)



Graphical solution for linear programming problem.

Graphical Solution to LP Problems (2)

- An equation of the form $4x_1 + 5x_2 = 1500$ defines a straight line in the x_1-x_2 plane. An inequality defines an area bounded by a straight line. Therefore, the region below and including the line $4x_1 + 5x_2 = 1500$ in the Figure represents the region defined by $4x_1 + 5x_2 \leq 1500$.
- Same thing applies to other equations as well.
- The <u>shaded area</u> of the figure comprises the area common to all the regions defined by the constraints and contains all pairs of x_l and x_2 that are feasible solutions to the problem.
- This area is known as the feasible region or feasible solution space. <u>The optimal solution must lie within this region</u>.
- There are various pairs of x_1 and x_2 that satisfy the constraints such as:

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Graphical Solution to LP Problems (3)

- Trying different solutions, the optimal solution will be:

$$X_1 = 270$$

$$X_2 = 75$$

- This indicates that maximum income of $4335 is obtained by producing 270 units of product I and 75 units of product II.
- In this solution, all the raw material and available time are used, because the optimal point lies on the two constraint lines for these resources.
- However, 1500- [4(270) + 5(75)], or 45 ft^2 of storage space, is not used. Thus the storage space is not a constraint on the optimal solution; that is, more products could be produced before the company ran out of storage space. Thus this constraint is said to be slack.

Graphical Solution to LP Problems (4)

- If the objective function happens to be parallel to one of the edges of the feasible region, any point along this edge between the two extreme points may be an optimal solution that maximizes the objective function. When this occurs, there is no unique solution, but there is an infinite number of optimal solutions.

- The graphical method of solution may be extended to a case in which there are three variables. In this case, each constraint is represented by a plane in three dimensions, and the feasible region bounded by these planes is a polyhedron.

The Simplex Method (1)

- When decision variables are more than 2, it is always advisable to use Simplex Method to avoid lengthy graphical procedure.

- The simplex method is not used to examine all the feasible solutions.

- It deals only with a small and unique set of feasible solutions, the set of vertex points (i.e., extreme points) of the convex feasible space that contains the optimal solution.

The Simplex Method (2)

- **Steps involved:**

1. Locate an extreme point of the feasible region.
2. Examine each boundary edge intersecting at this point to see whether movement along any edge increases the value of the objective function.
3. If the value of the objective function increases along any edge, move along this edge to the adjacent extreme point. If several edges indicate improvement, the edge providing the greatest rate of increase is selected.
4. Repeat steps 2 and 3 until movement along any edge no longer increases the value of the objective function.

The Simplex Method (3)

Example: Product Mix Problem

The N. Dustrious Company produces two products: I and II. The raw material requirements, space needed for storage, production rates, and selling prices for these products are given below:

	Product	
	I	II
Storage space (ft^2/unit)	4	5
Raw material (lb/unit)	5	3
Production rate (units/hr)	60	30
Selling price ($/unit)	13	11

The total amount of raw material available per day for both products is 1575lb. The total storage space for all products is 1500 ft^2, and a maximum of 7 hours per day can be used for production. **The company wants to determine how many units of each product to produce per day to maximize its total income.**

The Simplex Method (4)

Solution

- **Step 1:** Convert all the inequality constraints into equalities by the use of slack variables. Let:

S_1 = unused storage space

S_2 = unused raw materials

S_3 = unused production time

As already developed, the LP model is:

Maximize

$$Z = 13x_1 + 11x_2$$

subject to:

$$4x_1 + 5x_2 \leq 1500$$
$$5x_1 + 3x_2 \leq 1575$$
$$x_1 + 2x_2 \leq 420$$

.....Eq (4)

$$x_1 \geq 0$$
$$x_2 \geq 0$$

The Simplex Method (5)

- Introducing these slack variables into the inequality constraints and rewriting the objective function such that all variables are on the left-hand side of the equation. Equation 4 can be expressed as:

$$
\begin{aligned}
Z - 13x_1 - 11x_2 &= 0 && \text{(A1)} \\
4x_1 + 5x_2 + S_1 &= 1500 && \text{(B1)} \\
5x_1 + 3x_2 + S_2 &= 1575 && \text{(C1)} \\
x_1 + 2x_2 + S_3 &= 420 && \text{(D1)} \\
x_i \geq 0, \quad i = 1, 2
\end{aligned}
\qquad \textbf{Eq. (i)}
$$

From the equations above, it is obvious that one feasible solution that satisfies all the constraints is: $x_1 = 0$, $x_2 = 0$, $S_1 = 1500$, $S_2 = 1575$, $S_3 = 420$, and $Z = 0$.

The Simplex Method (6)

- Since the coefficients of x_1 and x_2 in Eq. (A1) are both negative, the value of Z can be increased by giving either x_1 or x_2 some positive value in the solution.
- In Eq. (B1), if $x_2 = S_1$, then $x_1 = 1500/4 = 375$. That is, there is only sufficient storage space to produce 375 units at product I.
- From Eq. (C1), there is only sufficient raw materials to produce 1575/5 = 315 units of product I.
- From Eq. (D1), there is only sufficient time to produce 420/1 = 420 units of product I.
- Therefore, considering all three constraints, there is sufficient resource to produce only 315 units of x_1. Thus the maximum value of x_1 is limited by Eq. (C1).

The Simplex Method (7)

❖ **Step 2:** From Equation C1, which limits the maximum value of x_1.

.....Eq (6)

$$x_1 = -\frac{3}{5}x_2 - \frac{1}{5}S_2 + 315$$

Substituting this equation into Eq. (i) above yields the following new formulation of the model.

$$
\begin{aligned}
Z - \frac{16}{5}x_2 \qquad\qquad + \frac{13}{5}S_2 \qquad &= 4095 \quad \text{(A2)} \\
+ \frac{13}{5}x_2 + S_1 - \frac{4}{5}S_2 \qquad &= 240 \quad \text{(B2)} \\
x_1 + \frac{3}{5}x_2 \qquad\qquad + \frac{1}{5}S_2 \qquad &= 315 \quad \text{(C2)} \\
\frac{7}{5}x_2 \qquad\qquad - \frac{1}{5}S_2 + S_3 &= 105 \quad \text{(D2)}
\end{aligned}
$$

.....Eq (7)

Eq. (ii)

The Simplex Method (8)

- It is now obvious from these equations that the new feasible solution is:

  $x_1 = 315$, $x_2 = 0$, $S_1 = 240$, $S_2 = 0$, $S_3 = 105$, and $Z = 4095$

- It is also obvious from Eq.(A2) that it is also not the <u>optimum solution</u>. The coefficient of x_1 in the objective function represented by A2 is negative (-16/5), which means that the value of Z can be further increased by giving x_2 some positive value.

The Simplex Method (9)

- Following the same analysis procedure used in step 1, it is clear that:
- In Eq. (B2), if $S_1 = S_2 = 0$, then $x_2 = (5/13)(240) = 92.3$.
- From Eq. (C2), x_2 can take on the value $(5/3)(315) = 525$ if $x_1 = S_2 = 0$
- From Eq. (D2), x_2 can take on the value $(5/7)(105) = 75$ if $S_2 = S_3 = 0$
- Therefore, constraint D_2 limits the maximum value of x_2 to 75. Thus a new feasible solution includes $x_2 = 75$, $S_2 = S_3 = 0$.

The Simplex Method (10)

.....Eq (8)

- **Step 3:** From Equation D2:

$$x_2 = \frac{1}{7}S_2 - \frac{5}{7}S_3 + 75$$

Substituting this equation into Eq. (ii) above yields:

$$
\begin{aligned}
Z \quad & + \frac{15}{7}S_2 + \frac{16}{7}S_3 = 4335 \quad (\text{A}\,3) \\
S_1 & - \frac{3}{7}S_2 - \frac{13}{7}S_3 = 45 \quad (\text{B}\,3) \\
x_1 \quad & + \frac{2}{7}S_2 - \frac{3}{7}S_3 = 270 \quad (\text{C}\,3) \\
x_2 & - \frac{1}{7}S_2 + \frac{5}{7}S_3 = 75 \quad (\text{D}\,3)
\end{aligned}
$$

.....Eq (9)

Eq. (iii)

From these equations, the new feasible solution is readily found to be: $x_1 = 270$, $x_2 = 75$, $S_1 = 45$, $S_2 = 0$, $S_3 = 0$, $Z = 4335$.

Simplex Tableau for Maximization (1)

❖ **Step I:** Set up the initial tableau using Eq. (i).

.....Eq (5)

$$Z - 13x_1 - 11x_2 = 0 \quad (A1)$$
$$4x_1 + 5x_2 + S_1 = 1500 \quad (B1)$$
$$5x_1 + 3x_2 + S_2 = 1575 \quad (C1)$$
$$x_1 + 2x_2 + S_3 = 420 \quad (D1)$$
$$x_i \geq 0, \quad i = 1, 2$$

In any iteration, a variable that has a nonzero value in the solution is called a **basic variable**.

Row Number	Basic Variable	Coefficients of: Z	x_1	x_2	S_1	S_2	S_3	Right-Hand Side	Upper Bound on Entering Variable
Initial tableau									
A1	Z	1	−13	−11	0	0	0	0	
B1	S_1	0	4	5	1	0	0	1500	375
C1	S_2	0	5	3	0	1	0	1575	315
D1	S_3	0	1	2	0	0	1	420	420

Simplex Tableau for Maximization (2)

- **Step II:** . Identify the variable that will be assigned a nonzero value in the next iteration so as to increase the value of the objective function. This variable is called the entering variable.
  - It is that non-basic variable which is associated with the smallest negative coefficient in the objective function.
  - If two or more non-basic variables are tied with the smallest coefficients, select one of these arbitrarily and continue.

- **Step III:** Identify the variable, called the leaving variable, which will be changed from a nonzero to a zero value in the next solution.

Simplex Tableau for Maximization (3)

- **Step IV:** . Enter the basic variables for the second tableau. The row sequence of the previous tableau should be maintained, with the leaving variable being replaced by the entering variable.

Row Number	Basic Variable	Coefficients of: Z	x_1	x_2	S_1	S_2	S_3	Right-Hand Side	Upper Bound on Entering Variable
Initial tableau									
A1	Z	1	−13	−11	0	0	0	0	
B1	S_1	0	4	5	1	0	0	1500	375
C1	S_2	0	5	3	0	1	0	1575	315
D1	S_3	0	1	2	0	0	1	420	420
Second tableau at end of first iteration									
A2	Z	1	0	$-\frac{16}{5}$	0	$+\frac{13}{5}$	0	4095	
B2	S_1	0	0	$\frac{13}{5}$	1	$-\frac{4}{5}$	0	240	92.3
C2	x_1	0	1	$\frac{3}{5}$	0	$\frac{1}{5}$	0	315	525
D2	S_3	0	0	$\frac{7}{5}$	0	$-\frac{1}{5}$	1	105	75

Simplex Tableau for Maximization (4)

- **Step V:** Compute the coefficients for the second tableau. A sequence of operations will be performed so that at the end the x_1 column in the second tableau will have the following coefficients:

	x_1
Z	0
S_1	0
x_1	1
S_3	0

The second tableau yields the following feasible solution:

x_1 = 315, x_2 = 0, SI = 240, S2 = 0, S3 = 105, and Z = 4095

Simplex Tableau for Maximization (5)

- The row operations proceed as follows:
  - The coefficients in row C2 are obtained by dividing the corresponding coefficients in row C1 by 5.
  - The coefficients in row A2 are obtained by multiplying the coefficients of row C2 by 13 and adding the products to the corresponding coefficients in row A1.
  - The coefficients in row B2 are obtained by multiplying the coefficients of row C2 by -4 and adding the products to the corresponding coefficients in row B1.
  - The coefficients in row D2 are obtained by multiplying the coefficients of row C2 by -1 and adding the products to the corresponding coefficients in row D1.

Simplex Tableau for Maximization (6)

- **Step VI:** Check for optimality. The second feasible solution is also not optimal, because the objective function (row A2) contains a negative coefficient. Another iteration beginning with step 2 if necessary.

- In the third tableau (next slide), all the coefficients in the objective function (row A3) are positive. Thus an optimal solution has been reached and it is as follows:

$x_1 = 270$, $x_2 = 75$, SI = 45, S2 = 0, S3 = 0, and Z = 4335

Row Number	Basic Variable	Z	x_1	x_2	S_1	S_2	S_3	Right-Hand Side	Upper Bound on Entering Variable
		Coefficients of:							
Initial tableau									
A1	Z	1	−13	−11	0	0	0	0	
B1	S_1	0	4	5	1	0	0	1500	375
C1	S_2	0	5	3	0	1	0	1575	315
D1	S_3	0	1	2	0	0	1	420	420
Second tableau at end of first iteration									
A2	Z	1	0	$-\frac{16}{5}$	0	$+\frac{13}{5}$	0	4095	
B2	S_1	0	0	$\frac{13}{5}$	1	$-\frac{4}{5}$	0	240	92.3
C2	x_1	0	1	$\frac{3}{5}$	0	$\frac{1}{5}$	0	315	525
D2	S_3	0	0	$\frac{7}{5}$	0	$-\frac{1}{5}$	1	105	75
Third tableau at end of second and final iteration									
A3	Z	1	0	0	0	$+\frac{15}{7}$	$+\frac{16}{7}$	4335	
B3	S_1	0	0	0	1	$-\frac{3}{7}$	$-\frac{13}{7}$	45	
C3	x_1	0	1	0	0	$\frac{2}{7}$	$-\frac{3}{7}$	270	
D3	x_2	0	0	1	0	$-\frac{1}{7}$	$\frac{5}{7}$	75	