Chapter 3

HYDROSTATICS

INTRODUCTION:-

Hydrostatics deals with the study of fluids at rest or moving with uniform velocity as a solid body, so that there is no relative motion between fluid elements (or layers). There is no shear stress in a fluid at rest. Hence, only normal pressure forces are present in hydrostatics.

Engineering applications of hydrostatics principles include the study of forces acting on submerged bodies such as dam faces, gates & others and the analysis of stability of floating bodies.

OR in other word

Fluid static's is a branch of hydraulics that deals with fluids (water) at rest. The particles of fluid are at rest, there is no tangential or shear stress between the fluid particles.

In hydrostatics, all forces act normally to the boundary surface and are independent of viscosity. The analysis made on hydrostatics is based on straightforward application of the mechanical principles of forces and moment and exact solution can be obtained without experimental evidence.

As in solid mechanics we shall build our knowledge by first considering static's followed by the more difficult problem of dynamics. Considering Newton's second law, that is, d(mv)/dt = 0. This can be achieved either when the fluid velocity is constant or the very special case where the acceleration is constant everywhere in the flow. The first case is the case of fluid static's (the branch of fluid mechanics, which is concerned with fluids at rest), while the latter is the special case of solid body acceleration. The overriding assumption necessary to achieve these two conditions is that there is no relative motion of adjacent fluid layers, and consequently the shear stresses are zero. Therefore, only normal or pressure forces are considered to be acting on the fluid surfaces.

Fluid Pressure

The pressure intensity or more simply the pressure on a surface is the pressure force per unit area expressed by the relation $P = \dfrac{dF}{dA}$ *but the force should be applied normal to the surface.*

Pressure at a point

Consider a finite but small element (the small triangular prism) of liquid at rest, acted upon by the fluid around it. The values of average unit pressures on the three surfaces are P_1, P_2 and P_3. In the Z direction the forces are equal and opposite and cancel each other.

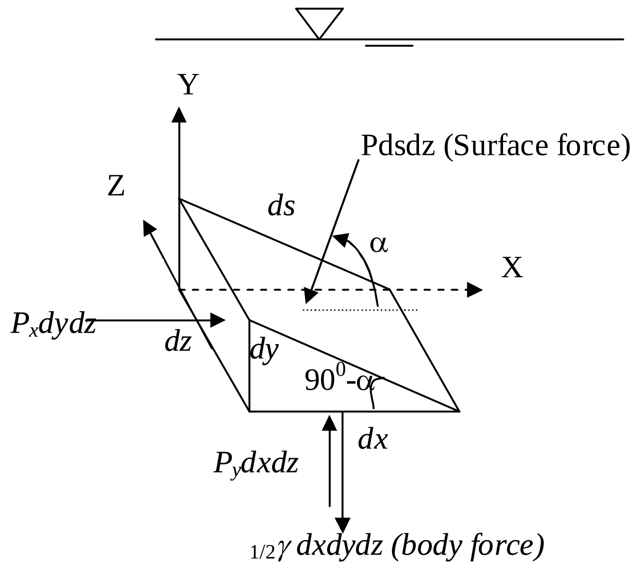

Fig.3.1 Definition sketch for normal stress at a point.

P_x and P_y are the average pressure in the horizontal and vertical directions.

For equilibrium condition,

$\Sigma F_x = 0$,

$$P_x dy d_Z - P d_S d_Z \cos\alpha = 0$$

but $ds \cos\alpha = dy$

$$P_x dyd Z - P d Z dy = 0$$

$$\Rightarrow P_x = P$$

$\Sigma Fy = 0$

$$P_y dxdz - Pdsdz \sin\alpha - 1/2\gamma dxdydz = 0$$

$ds \sin\alpha = dx$

$$P_y dxdz - Pdxdz - 1/2\gamma dxdydz = 0$$

$P_y - P - 1/2\gamma dy = 0$ as compared to others dy is small so, $1/2\gamma dy$ is ignored.

$$\Rightarrow P_y = P$$

The pressure force can also be considered and it will be the same with others.

$$\therefore P = Pz = P_X = Py$$

As the triangular prism approaches a point, dy approaches zero as a limit and the average pressures become uniform or even "point pressures". Then putting dy = 0 in equation, we obtain $p_1 = p_3$ and hence $p_1 = p_2 = p_3$. Therefore, the pressure is independent of its orientation.

3.1 Pressure Distribution PASCAL's Law

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The pressure variation throughout a fluid at rest can be obtained by again applying Newton's second law to a differential element such as shown in Fig.3.2. Note that the pressures shown are all compressive. This, by convention, is defined as positive pressure, since tensile stresses in fluids are relatively rare. The pressure on the left hand face is taken as P. If the rate of change of pressure (or pressure gradient) in the x direction is ∂_p/∂_x, then the total change in pressure between the left face and the right face is the rate of change of pressure times the distance between the two faces, or $(\partial_p/\partial_x)\,dx$.

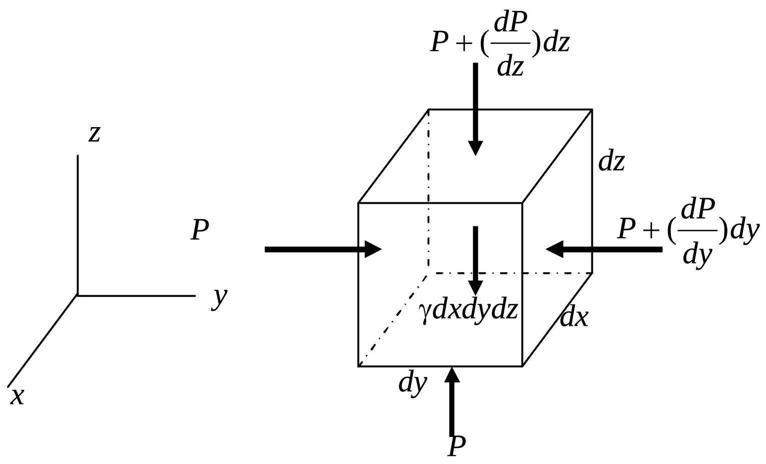


Fig.3.2 Definition sketch for pressure variation

For fluid element at rest $\Sigma F_X=0$, $\Sigma Fy=0$, $\Sigma Fz=0$, the pressure force in the opposite vertical faces must be equal.

$$\Sigma F_x = 0 \Rightarrow \; p\,dy\,dz - \left(p + \frac{\partial p.dx}{\partial x} \right) dy\,dz = 0$$

$$\Rightarrow \frac{\partial p}{\partial x} = 0$$

$$\Sigma Fy = 0 \Rightarrow \; p\,dx\,dz - \left(p + \frac{\partial p.dy}{\partial y} \right) dx\,dz = 0$$

$$\Rightarrow \frac{\partial p}{\partial y} = 0$$

The preceding two equations show, respectively, that the pressure does not change in the x and y directions. Thus, the pressure is constant throughout a horizontal plane.

With reference to Fig.3.2 the vertical direction will now be examined.

Similar to the foregoing procedure, if the pressure on the bottom face is taken as P, the pressure on the top face becomes p + (∂P/∂z) dz.

$$\sum Fz \Rightarrow p\,dx\,dy - \left(p + \frac{\partial p}{\partial z}dz\right)dx\,dy - \gamma\ dx\,dy\,dz = 0$$

$$\Rightarrow \frac{\partial p}{\partial z} = -\gamma$$

It has been shown that p is not a function of x or y. If it is further assumed that the pressure does not change with time, the relationship may be replaced by the total differential equation.

$$\frac{dp}{dz} = -\gamma$$

From the above equation the pressure variation is not a function of x and y.
This equation can now be integrated to give the actual pressure variation in the vertical direction. The negative sign indicates that as z gets higher up ward, the pressure gets smaller. For incompressible fluids, (where γ = constant) the above equation can be directly used.

If the fluid can be assumed incompressible so that γ = constant, this can be integrated to give

$$P + \gamma z = \text{constant}$$

This expression defines what is often referred to as the hydrostatic pressure variation, in which the pressure increases linearly with decreasing elevation. The constant of integration can be absorbed by integrating between two elevations z_1 and z_2 with corresponding pressure P_1 and P_2,

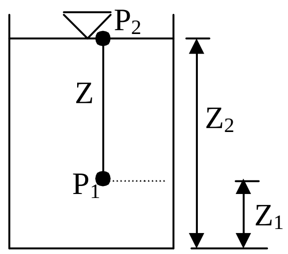


Fig.3.3. Pressure relative to the surface of a liquid

$$\int_{P_1}^{P_2} \partial P = -\gamma \int_{Z_1}^{Z_2} \partial Z$$

$P_2 - P_1 = -\gamma(z_2 - z_1)$ Showing pressure decreases linearly with an increase in elevation.

Since the pressure at the surface is atmospheric it can be taken to be zero gage pressure. So, the above expression will be $P_1 = \gamma(z_2 - z_1)$ But $z_2 - z_1 = z$ and substituting,

$$P_1 = \gamma z$$

And the pressure is proportional to the depth below the free surface. In other words, the pressure at a point in a stationary liquid is the product of the depth of the point and the specific weight of the fluid. If a free surface does not exist, for example in a closed container completely filled with liquid, The above equation can be applied in reverse to determine the position of a line of zero pressure, provided that the actual pressure is known at some point in the container.
When water fills a containing vessel, it automatically seeks a horizontal surface upon which the pressure is constant everywhere. In practice, the free surface of water in vessel is the surface that is not in contact with the cover of the vessel. Such a surface may be subjected to the atmospheric pressure (open vessel) or any other pressure that is exerted in the vessel (closed vessel).

N.B: The pressure in a homogeneous, incompressible fluid at rest depends on the depth of the fluid relative to some reference plane, and it is not influenced by the size or shape of the container in which the fluid is held.

3.2 Pressure measurement

Absolute and gage pressures

The pressure at a point within a fluid mass can be designated as either an absolute pressure or a gage pressure.
In a region such as outer space, which is virtually void of gases, the pressure is essentially zero. Such a condition can be approached very nearly in a laboratory when a vacuum pump is used to evacuate a bottle. The pressure in a vacuum is called absolute zero, and all pressures referenced with respect to this zero pressure are termed **absolute pressures**.

Many pressure-measuring devices measure not absolute pressure but only difference in pressure. For example, a Bourdon-tube gage indicates only the difference between the pressure in the fluid to which it is tapped and the pressure in the atmosphere. In this case, then, the reference pressure is actually the atmospheric pressure. This type of pressure reading is called **gage pressure**. For example, if a pressure of 50 kPa is measured with a gage referenced to the atmosphere and the atmospheric pressure is 100 kPa, then the pressure can be expressed as either p = 50 kPa gage or p = 150 kPa absolute.

Whenever atmospheric pressure is used as a reference, the possibility exists that the pressure thus measured can be either positive or negative. Negative gage pressures are also termed as vacuum or suction pressures. Hence, if a gage tapped into a tank indicates a vacuum pressure of 31 kPa, this can also be stated as 70 kPa absolute, or -31 kPa gage, assuming that the atmospheric pressure is 101 kPa absolute.

Water surface in contact with the earth's atmosphere is subjected to the atmospheric pressure, which is approximately equal to a 10.33-m- high column at sea level. In still water, any element located below the water surface is subjected to a pressure greater than the atmospheric pressure.

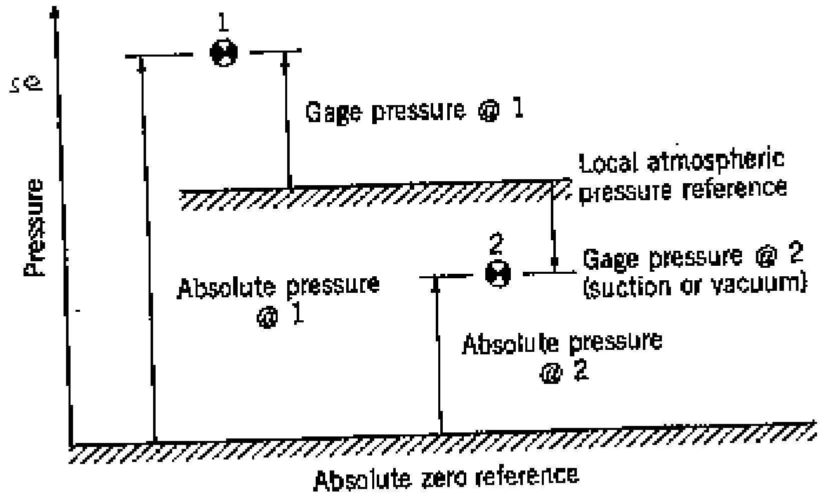


Fig.3.4. Graphical representation of gage and absolute pressure.

Measurement of pressure

Since pressure is a very important characteristic of a fluid field, it is not surprising that numerous devices and techniques are used in its measurement

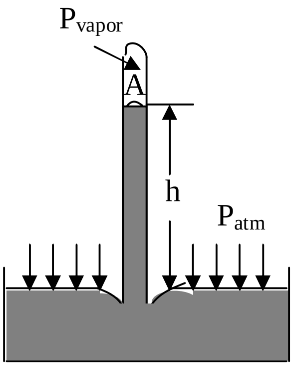
All the devices designed for measurement of the intensity of hydraulic pressure are based on either of the two fundamental principles of measurement of pressure: firstly by balancing the column of liquid (whose pressure is to be found) by the same or another column of liquid and secondly by balancing the column of liquid by spring or dead weight.

1. Mercury Barometer

The measurement of atmospheric pressure is usually accomplished with a mercury barometer, which in its simplest form, consists of a glass tube closed at one end with the open end immersed in a container of mercury as shown in Fig. The tube is initially filled with mercury (inverted with its open end up) and then turned upside down (open end down) with the open end in the container of mercury. The column of mercury will come to an equilibrium position where its weight plus the force due to the vapor pressure (which develops in the space above the column) balances the force due to the atmospheric pressure. Thus,

$$P_{atm} = \gamma h + P_{vapor}$$

Where: γ is the specific weight of mercury. For most practical purposes the contribution of the vapor pressure can be neglected since it is extremely small at room temperatures (e.g. 0.173 Pa at 20oC).



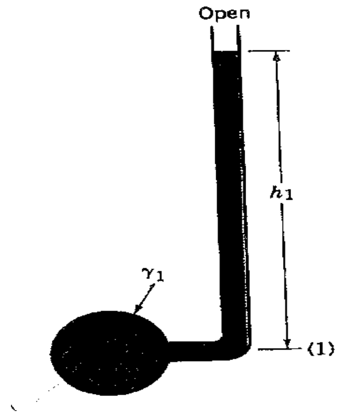


Fig. 3.5 a) Mercury barometer b) Piezometer tube

2. Manometry

A standard technique for measuring pressure involves the use of liquid columns in vertical or inclined tubes containing one or more liquid of different specific gravities. Pressure measuring devices based on this technique are called **manometers**. In using a manometer, generally a known pressure (which may be atmospheric) is applied to one end of the manometer tube and the unknown pressure to be determined is applied to the other end. In some cases, however, the difference between pressures at ends of the manometer tube is desired rather than the actual pressure at the either end. A manometer to determine this differential pressure is known as differential pressure manometer.
The mercury barometer is an example of one type of manometer, but there are many other configurations possible, depending on the particular application. The common types of manometers include the piezometer tube, the U-tube manometer, micro- manometer and the inclined - tube manometer.

i. Piezometer Tube

The simplest type of manometer consists of a vertical tube, open at the top, and attached to the container in which the pressure is desired, as illustrated in Fig.3.5. Since manometers involve columns of fluids at rest, the fundamental equation describing their use is the Eq.

$$P = \gamma h + P_0$$

Which gives the pressure at any elevation within a homogeneous fluid in terms of a reference pressure p_0 and the vertical distance h between p and p_0? Remember that in fluid at rest pressure will increase as we move downward, and will decrease as we move upward. Application of this equation to the piezometer tube Fig.3.5 indicates that the pressure P_A can be determined by a measurement of h_1 through the relationship.

$$P_A = \gamma_1 h_1$$

Where, γ_1 is the specific weight of the liquid in the container. Note that since the tube is open at the top, the pressure P_o can be set equal to zero (we are now using gage pressure), with the height h_1 measured from the meniscus at the upper surface to point (1). Since point (1) and point A within the container are at the same elevation, $P_A = P_1$.

Although the piezometer tube is a very simple and accurate pressure-measuring device, it has several disadvantages. It is only suitable if the pressure in the container is greater than atmospheric pressure (otherwise air would be sucked into the system), and the pressure to be measured must be relatively small so that required height of the column is reasonable. Also, the fluid in the container in which the pressure is to be measured must be a liquid rather than a gas.

ii. U- Tube Manometer

To overcome the difficulties noted previously, another type of manometer, which is widely used, consists of a tube formed into the shape of U as is shown in Fig.3.5. The fluid in the manometer is called the **gage fluid**. To measure larger pressure differences we can choose a manometer with higher density, and to measure smaller pressure differences with accuracy we can choose a manometer fluid which is having a density closer to the fluid density.

To find the pressure p_a in terms of the various column heights, we can use one of the two ways of manometer reading techniques:

I) Surface of equal pressure(*SEP*)

II) Step by step procedure(*SS*)

a) Start at one end and write the pressure there
b) Add the change in pressure there
   + If next meniscus is lower.
   - If next meniscus is higher
c) Continue until the other end of the gage and equate the pressure at that point

Thus, for the U- tube manometer shown in Fig.3.5, using SS method we will start at point A and work around to the open end. The pressure at points A and (1) are the same, and as we move from point (1) to (2) the pressure will increase by $\gamma_1 h_1$. The pressure at point (2) is equal to the pressure at point (3), since the pressures at equal elevation in a continuous mass of fluid at rest must be the same. Note that we could not simply "jump across" from point (1) to a point at the same elevation in the right – hand tube since these would not be points within the same continuous mass of fluid. With the pressure at point (3) specified we now move to the open end where the pressure is zero. As we move vertically upward the pressure decreases by an amount $\gamma_2 h_2$. In equation form these various steps can be expressed as

$$P_A + \gamma_1 h_1 - \gamma_2 h_2 = 0$$

And therefore, the pressure P_A can be written in terms of the column heights as

$$P_A = \gamma_2 h_2 - \gamma_1 h_1$$

A major advantage of the U- tube manometer lies in the fact that the gage fluid can be different from the fluid in the container in which the pressure is to be determined. For example, the fluid in A in Fig. 3.5b can be either a liquid or a gas. If A does contain a gas, the contribution of the gas column, $\gamma_1 h_1$, is almost always negligible so that $P_A \approx p_2$ and in this instance the above Eq. becomes.

$$P_A = \gamma_2 h_2$$

Thus, for a given pressure the height, h_2 is governed by the specific weight, γ_2, of the gage fluid used in the manometer. If the pressure P_A is large, then a heavy gage fluid, such as mercury, can be used and a reasonable column height (not too long) can still be maintained. Alternatively, if the pressure P_A is small, a lighter gage fluid, such as water, can be used so that a relatively large column height (which is easily read) can be achieved.

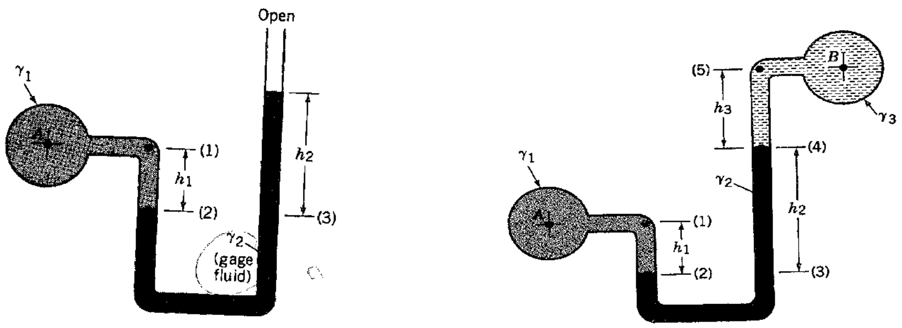


Fig.3.5 Simple U-tube and Differential U-tube manometer

The U- tube manometer is also widely used to measure the **difference** in pressure between two containers or two points in a given system. Consider a manometer connected between container A and B as is shown in Fig.3.5. The difference in pressure between A and B can be found by again starting at one end of the system and working around to the other end. For example, at A the pressure is P_A, which is equal to p_1, and as we move to point (2) pressure increases by $\gamma_1 h_1$. The pressure at p_2 is equal to p_3, and as we move upward to from point (4) to (5) the pressure decreases by $\gamma_3 h_3$. Finally, $P_5 = P_B$, since they are at equal elevation. Thus,

$$P_A + \gamma_1 h_1 - \gamma_3 h_3 = P_B$$

And the pressure difference is

$$P_A - P_B = \gamma_2 h_2 + \gamma_3 h_3 - \gamma_1 h_1$$

When substituting in numbers, be sure to use a consistent system of units!

iii) Differential U-tube

Inverted U-tube manometer is used for measuring pressure differences in liquids. The space above the liquid in the manometer is filled with air which can be admitted or expelled through the tap on the top, in order to adjust the level of the liquid in the manometer.

Capillarity due to surface tension at the various fluid interfaces in the manometer is usually not considered, since for a simple U –tube with a meniscus in each leg, the capillary effects cancel (assuming the surface tension and tube diameters are the same at each meniscus), or we can make the capillary rise negligible by using relatively large bore tubes (with diameters of about 0.5 in, or larger). Two common gage fluids are water and mercury. Both give a well –defined meniscus, a very important characteristic for a gage fluid, and their properties are well known. Of course, the gage fluid must be immiscible with respect to the other fluids in contact with it. For highly accurate measurements, special attention should be given to temperature since the various specific weights of the fluids in the manometer well vary with temperature.

iv) Inclined – tube Manometer

To measure small pressure changes, a manometer of the type shown in Fig. 3.6 is frequently used. One leg of the manometer is inclined at an angleθ, and the differential reading ℓ_2 is measured along the inclined tube. The difference in pressure $P_A - P_B$ can be expressed as

$$P_A + \gamma_1 h_{12} - \gamma_2 \ell_2 \sin\theta - \gamma_3 h_3 = P_B$$

Or

$$p_A - p_B = \gamma_2 \ell_2 \sin\theta + \gamma_3 h_3 - \gamma_1 h_1$$

Where it is to be noted that the pressure difference between points (1) and (2) is due to the vertical distance between the points, which can be expressed as $\ell_2 \sin\theta$. Thus, for relatively small angles the differential reading along the inclined tube can be made large even for small pressure differences. The inclined- tube manometer is often used to measure small differences in gas pressures so that if pipes A and B contain a gas then

$$p_A - p_B = \gamma_2 \ell_2 \sin\theta$$

Or

$$\ell_2 = \frac{p_A - p_B}{\gamma_2 \sin\theta}$$

Where the contributions of the gas columns h_1 and h_3 have been neglected. The above Equation shows that the differential reading ℓ_2 (for a given pressure difference) of the inclined –tube manometer can be increased over that obtained with a conventional U-tube manometer by the factor $1/\sin\theta$. Recall that $\sin\theta \to 0$ as $\theta \to 0$.

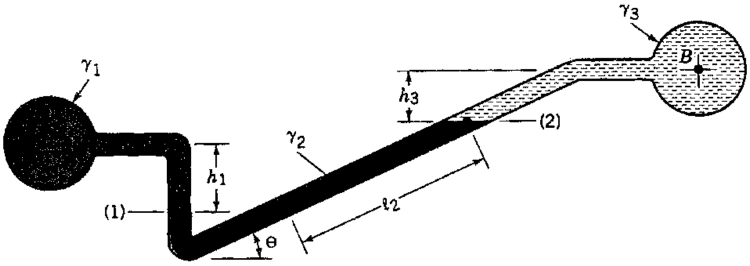
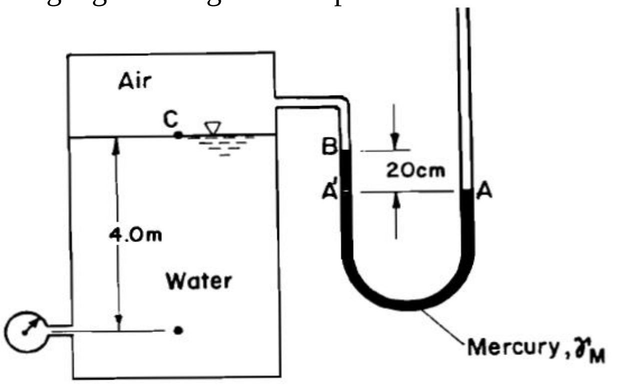


Fig.3.6 Inclined Tube manometer

Example 1

A closed tank is partly filled with water and connected to the manometer containing mercury (S = 13.6) as shown in the figure below. A gauge is connected to the tank at a depth of 4 m below the water surface.

If the manometer reading is 20 cm, determine the gauge reading in N/ m^2. What will be the gauge reading when expressed as head of water in m?



Solution

Using the letter designation in the Figure, $p_A = p'_A$

$$P_B = P'_A - 0.20\ \gamma_M$$

$$P_C = P_B \text{ and } P_D = P_{gauge} = P_C + 4\ \gamma_w$$

$$\therefore \qquad P_D = P'_A - 0.20\gamma_m + 4\gamma_w$$

$$= 0 - 0.20x\gamma_w.S_m + 4\gamma_w = \gamma_w(-0.2S_m + 4)$$

$$= 9810\ \frac{N}{m^3}\ (-0.2 \times 13.6 + 4)m = 9810(-2.72 + 4)N/m^2$$

$$P_D = 9810 \times 1.28 = 12556.8\ n/M^2$$

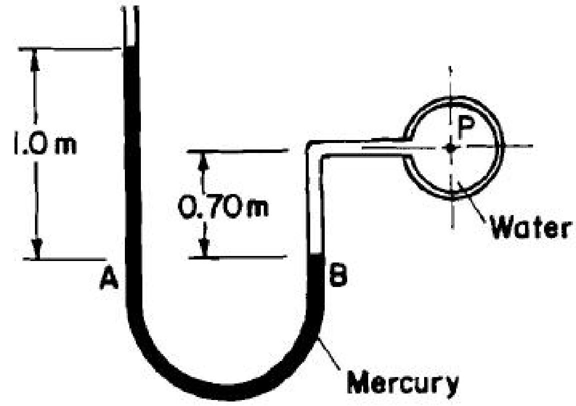
Therefore, the gauge reading is 12556.8 N/m^2

When expressed as head of water, the gauge reading will be

$$h = \frac{P}{\gamma_w} = \frac{12556.8\ N/m^2}{9810\ N/m^2} = 1.28\ m$$

Example 2

A manometer is mounted in a city water supply main pipe to monitor the water pressure in the pipe as shown below. Determine the water pressure in the pipe.



Solution

$P_A = P_B$

$\gamma_{Hg} \cdot 1 = P_p + 0.70\ \gamma_w$

$P_p = \gamma_{Hg} \times 1 - 0.7\gamma_w = 13.6\gamma_w - 0.7\gamma_w$

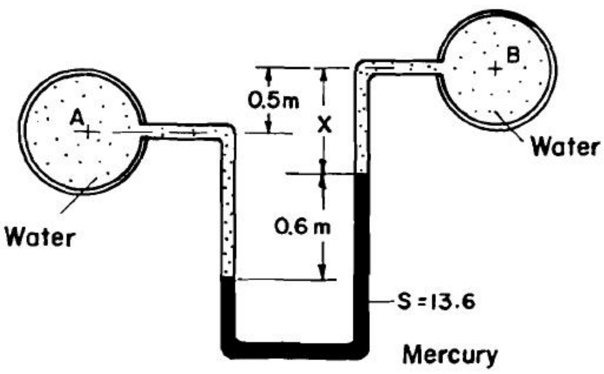
$= 12.9\ \gamma_w = (12.9 \times 9810)\ N/m^2$

$= 1,2655 \times 10^5\ N/m = 1.249 atmospheres$

$(Note:\ 1\ standard\ atmosphere = 1.01325 \times 10^5\ N/m^2)$

Example 3

Calculate the pressure difference between points A and B in the differential manometer shown in Figure below.



Solution

Starting from A,

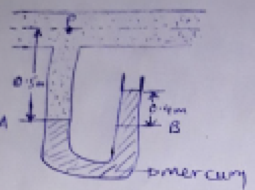
$$P_A + (x - 0.5)\gamma_w + 0.6\gamma_w - 0.6 \times 13.6\gamma_w - x\gamma_w = P_B$$

$$PA + x\gamma_w - 0.5\gamma_w + 0.6\gamma_w - 8.16\gamma_w - x\gamma_w = P_B$$

$$\therefore \quad P_A - PB = (8.16 - 0.1)\gamma_w = 8.06\gamma_w$$

$$= 8.06 \times 9.81 = 79.07\ kN/m^2$$

1. The lower part of the U-tube manometer below contains mercury [$\rho_m = 13600\ kg/m^3$]. The pipe contains water [$\rho = 1000\ kg/m^3$]. Determine the gauge pressure P, at the center of the pipe.



Sol^n

$P_A = P_B$ → pressure at the same level

$\Rightarrow P + \gamma_w * 0.5m = P_{atm} + \gamma_m * 0.4m$ (~0)

$\Rightarrow P = \gamma_m * 0.4 - \gamma_w * 0.5$

$= \cancel{13600 kg/m^3 * 0.4m - 1000 kg/m^3 * 0.5m}$ * 9.81 m/s²

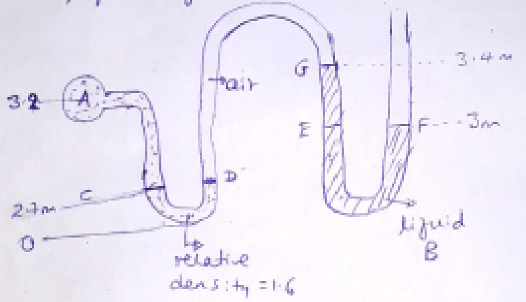
$= \cancel{48.5 * 10^3\ kg\,m/s}$

$= 13600\ kg/m^3 * 9.81 m/s^2 * 0.4m - 1000 kg/m^3 * 9.81 * 0.5$

$= 48461.4\ (kg\,m/s^2)\ N$

$\simeq 48.5 * 10^3\ N/m^2$

② For a gage pressure at A of $-11,000$ Pa, find the relative density of the gage liquid B.



Sol^n

$P_C = P_D$

$\Rightarrow P_D = P_A + \gamma_A * [3.2 - 2.7]$

but, to get γ_A

$S_A = 1.6$

$\Rightarrow S_A = \frac{\rho_A}{\rho_w} \Rightarrow \rho_A = S_A * \rho_w$

$\Rightarrow \rho_A = 1.6 * 1000 = 1600 kg/m^3$

$\therefore \gamma_A = 1600\ kg/m^3 * 9.81\ m/s^2 = 15696\ Pa$

$kg/m/s^2 = N/m^2 = Pa$

$\Rightarrow P_D = -11,000 Pa + [15696 * 0.5] = -3152 Pa$

$P_D = P_G = -3152 Pa$ → air | but

$P_E = P_F = 0$

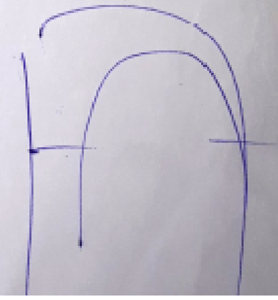
$\Rightarrow P_G + [\gamma_B * (3.4 - 3)] = 0 = P_E$

$\Rightarrow \gamma_B = \frac{3152}{0.4} = 7880$

but $\gamma_B = \rho_B g$

$\Rightarrow \rho_B = \frac{\gamma_B}{g} = \frac{7880}{9.81} = 803.3$

$\Rightarrow S_B = \frac{\rho_B}{\rho_w} = \frac{803}{1000} = 0.8$



3. Mechanical and Electronic pressure measuring devices

Although manometers are widely used, they are not well suited for measuring very high pressures, or pressures that are changing rapidly with time. In addition, they require the measurement of one or more column heights, which although not particularly difficult, can be time consuming. To overcome some of these problems numerous other types of pressure –measuring instruments have been developed. Most of these make use of the idea that *when a pressure acts on an elastic structure the structure will deform*, and *this deformation can be related to the magnitude of the pressure*. Probably the most familiar device of this kind is the *Bourdon pressure gage*, which is shown in Fig.3.7.

The essential mechanical element in this gage is the hollow, elastic curved tube (Bourdon tube) which is connected to the pressure source as shown in Fig. As the pressure within the tube increases the tube tends to straighten, and although the deformation is small, it can be translated into the motion of a pointer on a dial as illustrated. Since it is the difference in pressure between the outside of the tube (atmospheric pressure) and the inside of the tube that causes the movement of the tube, the indicated pressure is gage pressure. The Bourdon gage must be calibrated so that the dial reading can directly indicate the pressure in suitable units. A zero reading on the gage indicates that the measured pressure is equal to the local atmospheric pressure. This type of gage can be used to measure a negative gage pressure (vacuum) as well positive pressure.

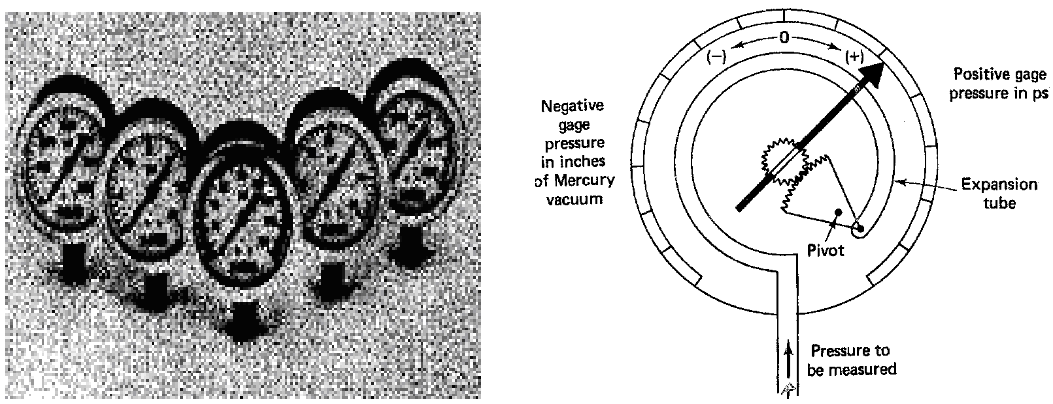


Figure 3.7 Bourdon Gauge

Manometers-advantages and limitations

The manometer in its various forms is an extremely useful type of pressure measuring instrument, but suffers from a number of limitations.

While it can be adapted to measure very small pressure differences, it cannot be used conveniently for large pressure differences - although it is possible to connect a number of manometers in series and to use mercury as the manometric fluid to improve the range.

(Limitation)
A manometer does not have to be calibrated against any standard; the pressure difference can be calculated from first principles. (Advantage)
Some liquids are unsuitable for use because they do not form well-defined menisci. Surface tension can also cause errors due to capillary rise; this can be avoided if the diameters of the tubes are sufficiently large - preferably not less than 15 mm diameter. (Limitation)
A major disadvantage of the manometer is its slow response, which makes it unsuitable for measuring fluctuating pressures. (Limitation)
It is essential that the pipes connecting the manometer to the pipe or vessel containing the liquid under pressure should be filled with this liquid and there should be no air bubbles in the liquid. (Important point to be kept in mind)

3.3 Hydrostatic pressure on plane and curved surfaces

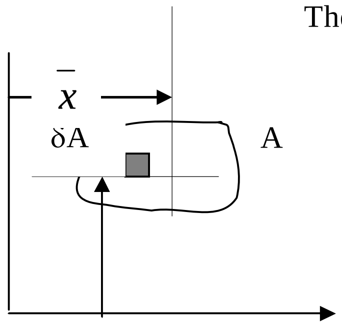
When a surface is submerged in a fluid, forces develop on the surface due to the fluid. The determination of these forces is important in the design of storage tanks, ships, dams, and other hydraulic structures. For fluid at rest we know that the force must be perpendicular to the surface since there are no shearing stresses present. We also know that the pressure will vary linearly with depth if the fluid is incompressible.

1. Forces on plane surface

The distributed forces resulting from the action of fluid on a finite area can be conveniently replaced by resultant force. The magnitude of resultant force and its line of action (pressure center) are determined by integration, by formula and by using the concept of the pressure prism.

i. Horizontal surfaces

A plane surface in a horizontal position in a fluid at rest is subjected to a constant pressure.



The magnitude of the force acting on one side of the surface is

$$\int P dA = P\int dA = PA$$

The elemental forces PdA acting on A are all parallel. The summation of all elements yields the magnitude of the resultant force. Its direction is normal to the surface.

- To find line of action of the resultant, the moment of resultant is equated to the moment of the distributed system about any axis (y-axis).

i.e. $PAx^1=\int_A x\, PdA$

x^1 is the distance from the y axis to the resultant.

$$x^1=\frac{1}{A}\int_A x\, dA=\bar{x} \qquad \text{p-is constant.}$$

$\bar{x}$ is the distance to the centroid of the area.

Hence, for a horizontal area subjected to static fluid pressure, the resultant passes through the centroid of the area.

ii. Inclined surfaces

A plane surface which is inclined to the water surface may be subjected to hydrostatic pressure. For a plane inclined θ^0 from the horizontal, the intersection of the plane of area and the free surface is taken as the x-axis. The y-axis is taken in the plane of the area with origin 0 at the free surface. Thus, the x-y plane portrays the arbitrary inclined area. We wish to determine the magnitude, direction and line of the action of the resultant force acting on one side of this area due to the liquid in contact with the area.

For an element area δA at y distance from the origin, the magnitude of the force δF acting on it is

$$\delta F = P\delta A = \Upsilon h\delta A = \Upsilon y \sin\theta\, \delta A$$

Since all such elemental forces are parallel, the integral over the area yields the magnitude of force, F, acting on one side of the area.

$$F=\int PdA=\gamma\sin\theta\int ydA=\gamma\sin\theta\ \bar{Y}A=\gamma\bar{h}A=P_G.A$$
$$=\Upsilon h_c.A$$

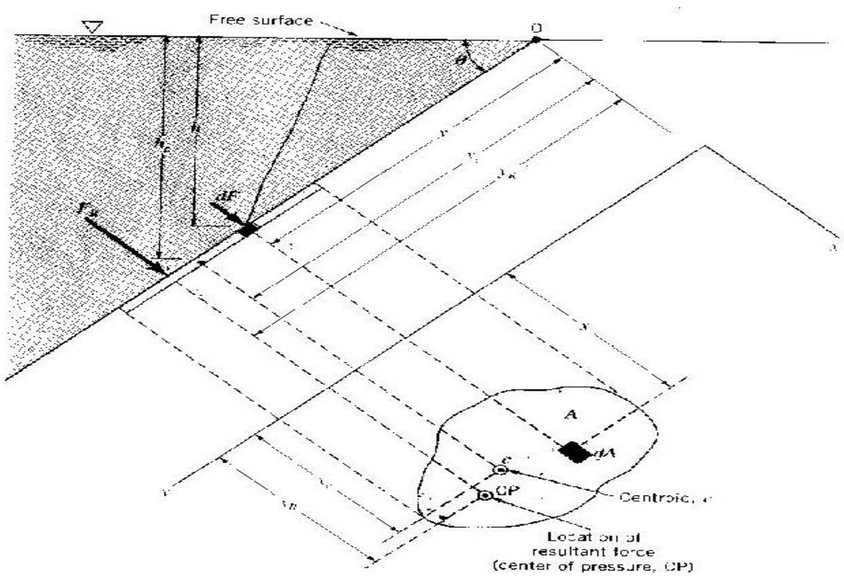


Fig.

$h_c=y_c.\sin\theta$

$\bar{Y}\sin\theta=\bar{h}$ *and* $p_G=\gamma\bar{h}$; The pressure at the centroid of the area.

Hence, the force exerted on one side of a plane area submerged in a liquid is the product of the area and the pressure at its centroid.

The point on the plane surface where this resultant force acts is known as the *center of pressure*. Considering the plane surface as free body we see that the distributed forces can be replaced by a single resultant force at the pressure center with out altering any reactions or moments in the system.

$$F = \int dF = \int_A P dA$$

Let x_p and y_p be distances measured from the y-axis and x-axis to the pressure center respectively, then

$$F.y_p = \int y.\ dF, \qquad y_P = \frac{1}{F}\int ydF$$

But F= ϒsinθ A$\overline{Y}$ $and \;\; dF = \gamma y \sin\theta \; dA$

$$y_p = \frac{1}{\gamma \sin\theta A\overline{Y}}\int \gamma y^2 \sin\theta \; dA = \frac{1}{A\overline{Y}}\int y^2 \; dA = \frac{I_o}{A\overline{Y}}$$

But, $I_o = A\overline{Y}^2 + I_g$

$$y_p = \overline{Y} + \frac{I_g}{A\overline{Y}}$$

$y_p - \overline{Y} \geq 0 \,, b/c \; I_g \;\; is \;\; positive.$

This shows that center of pressure is below the center of gravity (or centroid).

Where I_g-is the moment of inertia of the plane with respect to its own centroid.

$$x_p.F \;\; = \int x.dF = \int x.\gamma y \sin\theta.dA$$

$$X_p = \frac{1}{\gamma \overline{Y} A \sin\theta}\int_A x\gamma y \sin\theta.dA = \frac{1}{A\overline{Y}}\int_A x.y.dA = \frac{I_{xy}}{A\overline{Y}}$$

$$I_{xy} = \overline{X}.\overline{Y}.A + I_{xyg}$$

$$X_p = \frac{I_{xyg}}{A\overline{Y}} + \overline{X}$$ (Product of inertia at (x, y)).

The pressure prism

The pressure prism is an approach, which is developed for determining the resultant hydrostatic force and line of action of the force on a plane surface. It is a prismatic volume with its base the given surface area and with altitude at any point of the base given by p=γh. Where h is the vertical distance to the free surface.

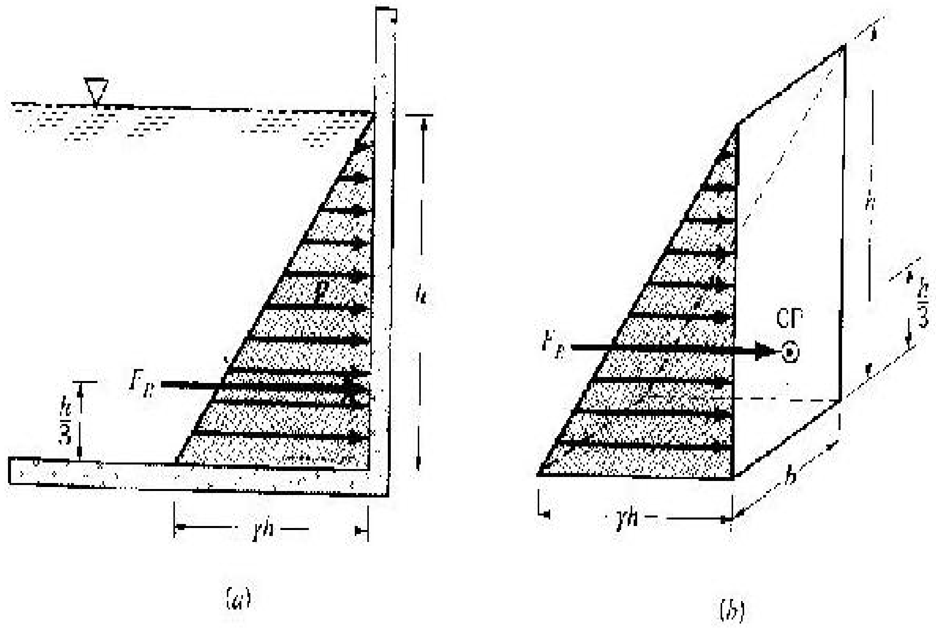


Fig. Pressure prism

Force acting on the element area δA is:

$\boxed{\delta \mathbf{F} = \gamma \mathbf{h}\, \delta \mathbf{A} = \delta \forall}$, which is an element of volume of the pressure prism.

After integrating, $\boldsymbol{F} = \forall$, the volume of the pressure prism equals the magnitude of the resultant force acting on one side of the surface. The center of pressure is given by

$$x_p = \frac{1}{\forall}\int_{\forall} x.d\forall \quad \text{and} \quad y_p = \frac{\mathbf{1}}{\forall}\int_{\forall} y.d\forall .$$

This shows that the resultant force passes through the centroid of the pressure prism.

Therefore; the pressure force is the volume of the prism in magnitude acting at the centroid of the prism normal to the surface.

2. Forces on curved surfaces

When the elemental forces pδA vary in direction, as in the case of a curved surface, they must be added as vector quantities that is, their components in three mutually perpendicular directions are added as scalars and then the three components are added vector ally. With two horizontal components at right angle and with vertical component- all easily computed for a curved surface the resultant can be determined. The lines of action of the components also are readily determined.

Horizontal component of Forces on a curved surface

The horizontal component of pressure force on a curved surface is equal to the pressure force exerted on a vertical projection of the cured surface. The vertical plane of the projection is normal to the direction of the component.

Thus, the magnitude and the line of action of the horizontal component of force on a curved surface can be determined by using the relations developed for plane surface.

Vertical component of force on a curved surface

The vertical component of pressure force on a curved surface is equal to the weight of liquid vertically above the curved surface and extending up to the free surface and acts through the center of gravity of the fluid mass within the volume.

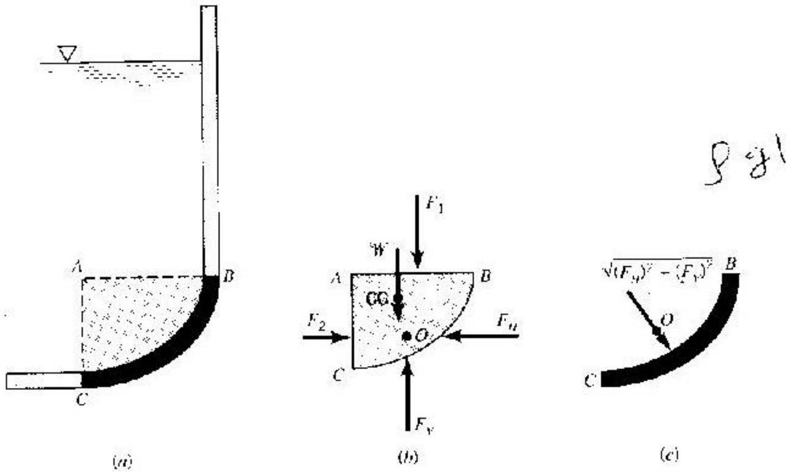


Fig. Forces on curved surface

3.4 Tensile stress in a pipe and spherical shell

A circular pipe under the action of an internal pressure is in tension around its periphery. Assuming that no longitudinal stress occurs, the walls are in tension, as shown in Fig. below.

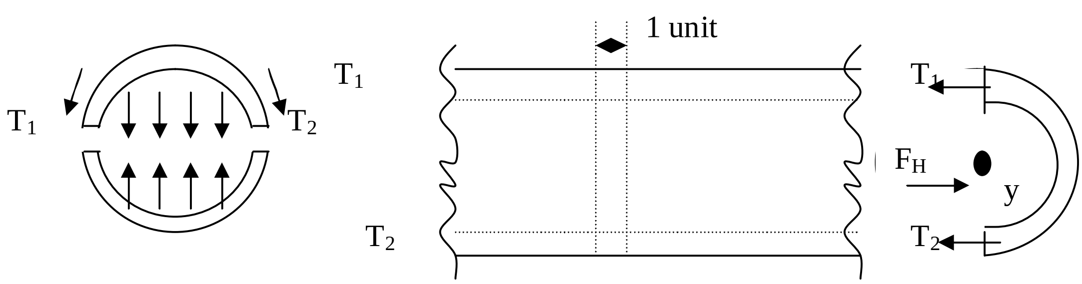


Fig. Internal forces on walls of a pipe. A section of pipe of unit length is considered

The bursting of a pipe can be thought of as a tendency for the top half to separate from the bottom half. The only force acting against this tendency is the hoop tension (T) of the pipe walls then the bursting pressure force must exactly equal the hoop tension.

Total bursting pressure = P* 2r * 1

P = pressure at the centre line

r = the internal pipe radius

For high pressures the pressure centre can be taken at the pipe centre; then $T_1 = T_2$

$\Rightarrow$ T = Pr

For wall thickness t, the tensile stress in the pipe wall, $\sigma = \frac{T}{t} = \frac{pr}{t}$

For larger variations in pressure b/n top and bottom of pipe, the location of pressure centre y is computed.

[$\sum$FH = 0} $\Rightarrow$ $T_1 + T_2 = F_H = 2pr$

[$\sum$M @T_2] $2rT_1 - 2pry = 0$ (Neglecting the vertical component)

$\Rightarrow$ $T_1 = py$ $T_2 = p(2r-y)$

Thin spherical shell subjected to an internal pressure

Fluid force $F_H = P\pi r^2$ (considering half of the sphere)

Resisting force = stress in the wall * cut wall area = $\sigma * 2\pi r * t$

Neglecting the weight

$\Rightarrow$ $\sigma = Pr/2t$

3.5 Relative Equilibrium

Translation and Rotation of fluid masses

A general class of problems involving fluid motion in which there are no shearing stresses occur when a mass of fluid undergoes rigid body motion. For example, if a container of fluid accelerates along ***a straight path***, the fluid will move as a rigid mass (after the initial sloshing motion has died out) with each particle having the same acceleration. Since there is no deformation, there will be no shearing stresses and similarly if a fluid is contained in a tank that ***rotates*** about a fixed axis, the fluid will simply rotate with the tank as a rigid body. In both cases there is no relative motion between particles; hence no shear stress occurs in the fluid. This condition of fluid is called ***relative equilibrium***. Generally there is no motion between the fluid and the containing vessel, however, there is an additional force acting to cause the acceleration.

Specific results for these two cases (rigid body uniform motion and rigid body rotation) are developed in the following two sections. Although problems relating to fluids having rigid body motion are not strictly speaking, “fluid static” problems, they are induced in this chapter because as we will see the analysis and resulting pressure relationships are similar to those for fluid at rest. (Laws of fluid static's can still be applied by modifying to allow for effects of acceleration.)

i. Uniform linear acceleration

Consider a small rectangular element of fluid of size δx, δy and δz as shown in the figure below, δz being measured perpendicular to the paper.

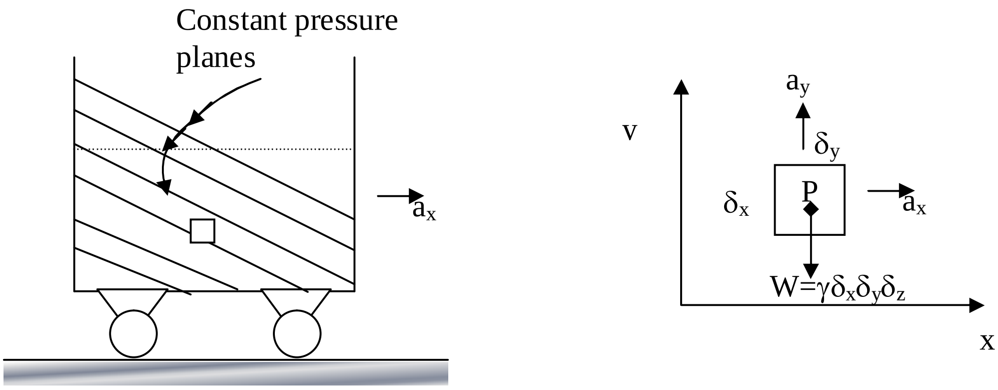


Fig. Linear acceleration of a liquid with a free surface

The pressure on the left face of the elemental fluid

$$P-\left(\frac{\partial p}{\partial x}\right)\frac{1}{2}\delta x$$

and on the right face $P+\left(\frac{\partial p}{\partial x}\right)1/2\,\delta x$

For equilibrium in the x –direction

$$\left[P-\left(\frac{\partial p}{\partial x}\right)1/2\,\delta x-P+\left(\frac{\partial p}{\partial x}\right)1/2\,\delta x\right]\delta y\delta z \;=\; \rho\,\delta x\,\delta y\,\delta z\; a_x$$

$$\left(\frac{-\partial p}{\partial x}\right)\delta x\,\delta y\,\delta z \;=\; \rho\,\delta x\,\delta y\,\delta z\; a_x$$

$$\frac{\partial p}{\partial x} = -\,\rho a_x \;---\; (*)$$

Similarly in the y direction

$$\left(\frac{-\partial p}{\partial y}\right) \delta x\,\delta y\,\delta z - \gamma \delta x \delta y \delta z \;=\; \rho\;\delta_x\,\delta_y\,\delta_z\;a_y$$

$$\frac{\partial p}{\partial y} = -\left(p a_y + \gamma\right)$$

$$\frac{\partial p}{\partial y} = -\,\gamma\left(\frac{a_y}{g}+1\right) ----(^{**})$$

$$dp \;=\; \frac{\partial p}{\partial x}.dx + \frac{\partial p}{\partial y} d_y + \frac{\partial p}{\partial z} dz \qquad (az \;=0)$$

$$dp = -\rho\, a_x\; d_x \;-\gamma\left(\frac{a_y}{g}+1\right)dy$$

$$P \;\;= -\;\rho\, a_x x - \gamma\left(\frac{ay}{g}+1\right) Y \;\; + C$$

If x = y = 0

P = C = P_0

P = - ρ a_x x - $\gamma\left(\frac{a_y}{g}+1\right)Y + P_0$

If the origin is taken at a point on the free surface, P_0 = 0 (= Patm)

Thus, p = -ρa_x x – $\gamma\left(\frac{a_y}{g}+1\right)Y$

$$\gamma\left(\frac{ay}{g}+1\right) Y = -\,P - \rho\;\; a_x x$$

$$Y = \frac{-P}{\gamma\left(\frac{a_y}{g}+1\right)} - \frac{\rho\, a_x x}{\gamma\left(\frac{a_y}{g}+1\right)}$$

$$Y \;\; = -\;\frac{ax}{a_y + g} x \;-\; \frac{p}{\gamma\left(\frac{a_y}{g}+1\right)} \;\; ----(^{***})$$

Y = m x + b

∴ The slope of the free surface will be m = - $\frac{a_x}{a_y \;\; + g}$

and the y – intercept $b \;= \frac{-P}{\gamma\left(\frac{a_y}{g}+1\right)}$

Along a free surface the pressure is constant, so that for the accelerating mass shown in the figure the free surface will be inclined if $a_x \neq 0$. In addition, all lines of constant pressure will be parallel to the free surface.
For the special circumstance in which $a_x = 0$, $a_y \neq 0$, which corresponds to the mass of fluid accelerating in the vertical direction, equation (***) indicates that the fluid surface will be horizontal. However, from Eq (**) we see that the pressure distribution is not hydrostatics, but is given by the equation $\frac{\partial p}{\partial y} = -\gamma\left(\frac{a_y}{g}+1\right) = \rho\,(a_y + g)$

For fluids of constant density this equation shows that the pressure will vary linearly with depth but the variation is due to the combined effects of gravity and the externally induced acceleration, $\rho\,(g + a_y)$ rather than simply the specific weight ρg.

Note:

Pressure along the bottom of a liquid filled tank which is resting i.e accelerating upward will be increased over that which exists when the tank is at rest (or moving with a constant velocity).

For a freely falling fluid mass ($a_y = -g$) the pressure gradients in all the three coordinate directions are zero which means that if the pressure surrounding the mass is zero the pressure throught will be zero.

ii. Vortex flow

The flow of a fluid along a curved path or the flow of a rotating mass of fluid is known as vortex flow. The vortex flow is of two types namely: forced vortex flow and free vortex flow.

Forced vortex flow:

It is defined as that type of vortex flow, in which some external torque is required to rotate the fluid mass. The fluid mass in this type of flow rotates at constant angular velocity, ω. The tangential velocity of any fluid particle is given by

$v = \omega r$

Where r = radius of fluid particle from the axis of rotation.

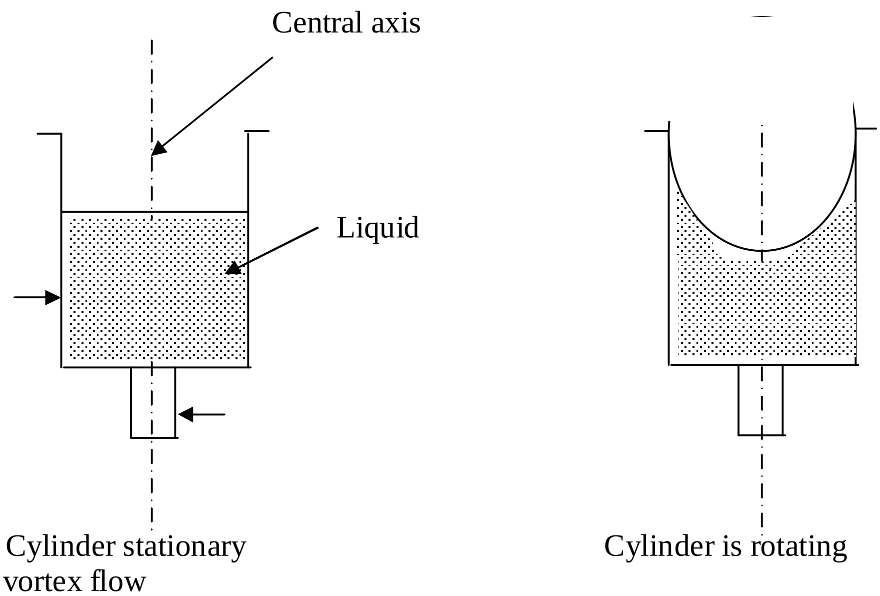


Fig. Forced vortex flow

Hence angular velocity ω is given by

$$\omega = \frac{v}{r} = constant$$

Examples of forced vortex are:

1. A vertical cylinder containing liquid which is rotated about its central axis with a constant angular velocity ω, as shown in figure above.
2. Flow of liquid inside the impeller of a centrifugal pump.
3. Flow of water through the runner of a turbine.

Free vortex flow

Type of flow when no external torque is required to rotate the fluid mass is called free vortex flow. Thus the liquid incase of free vortex is rotating due to the rotation which is imparted to the fluid previously. Examples of the free vortex flow are:

1. Flow of a liquid through a hole provided at the bottom of a container.
2. Flow of liquid around a circular bend in a pipe.
3. Flow of fluid in a centrifugal pump casing.

Free vortex flow will be treated in the coming chapter.

Equation of forced vortex flow
Uniform rotation about a vertical axis

Consider a small element of fluid to move in a circular path about an axis with radius r, and angular velocity ω.

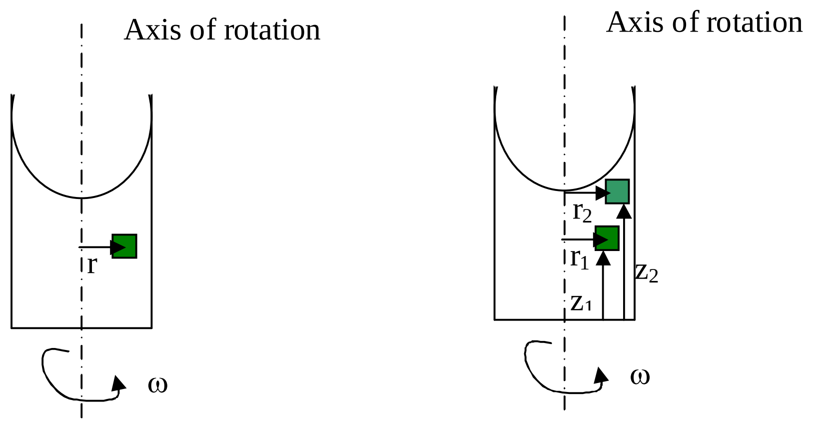


Fig. Rigid body rotation

For constant angular velocity ω, the particle will have an acceleration of $\omega^2 r$ directed radically inward.

$$\frac{\partial p}{\partial r} = \rho\omega^2 r \;=\; \frac{\gamma}{g} wr \quad \left(\sum F_r = ma_r\right)$$

$$\frac{\partial P}{\partial \theta} = 0$$

$$\frac{\partial p}{\partial y} = -\gamma \quad \left(\sum F_y = \right)$$

These results show that for this type of rigid body rotation, the pressure is a function of two variables r and y, and therefore the differential pressure is

$$dp = \frac{dp}{dy}dy + \frac{dp}{dr}dr$$

$$dp = -\gamma dy \;+\; \frac{\gamma}{g}^{2} dr$$

This equation gives the variation of pressure of a rotating fluid.

$$p = -\gamma y + \frac{\gamma}{g}\omega^2 \frac{r^2}{2} \; C$$

For r=0, y=0, $P=C=P_0$

$P=P_0-\gamma y + \gamma \dfrac{\omega^2 r^2}{2g}$

Consider two points 1 and 2in the fluid as shown above. Integrating the above equation for point 1 and 2 we get,

$$\int_1^2 dp = \int_1^2 \frac{\gamma\omega^2}{g} r dr - \int_1^2 \gamma dy$$

$$p_2 - p_1 = \left[\frac{\gamma\omega^2}{2g} r^2\right]_1^2 - \gamma[y]_1^{21}$$

$$p_2 - p_1 = \frac{\gamma\omega^2}{2g}\left[r_2^2 - r_1^2\right] - \gamma\left[y_2 - y_1\right]$$

$$= \frac{\gamma}{2g}\left[\omega r_2^{\,2} - \omega r_1^2\right] - \gamma\left[y_2 - y_1\right]$$

If the points 1 and 2 lie on the free surface of the liquid, then $p_1=p_2$ and hence the above equation becomes

$$\frac{\gamma}{2g}\left[\omega r_2^{\,2} - \omega r_1^2\right] - \left[y_2 - y_1\right] = 0$$

$$\Rightarrow y_2 - y_1 = \frac{1}{2g}\left[\omega r_2^{\,2} - \omega r_1^2\right]$$

If the point 1 lies on the axis of rotation, then the above equation becomes

$$y_2 - y_1 = \frac{\omega^2 r_2^2}{2g}$$

Let y_2-y_1 = y, then

$$y = \frac{\omega^2 r_2^2}{2g}$$

Thus y varies with the square of r. Hence the equation is an equation of parabola. This means the free surface of the liquid is a paraboloid.
Note: Volume of paraboloid of revolution is half of the volume of the circumscribing cylinder.

3.5 Buoyancy and stability of floating bodies

1. Buoyant force (Resultant fluid force in a body)

The buoyant force on a submerged body is the difference between the vertical components of pressure force on its underside and the vertical component of pressure force on its upper side. The buoyant force always acts vertically upward. There can be no horizontal component of the resultant because the projection of the submerged body or submerged portion of the floating body on a vertical plane is always zero.

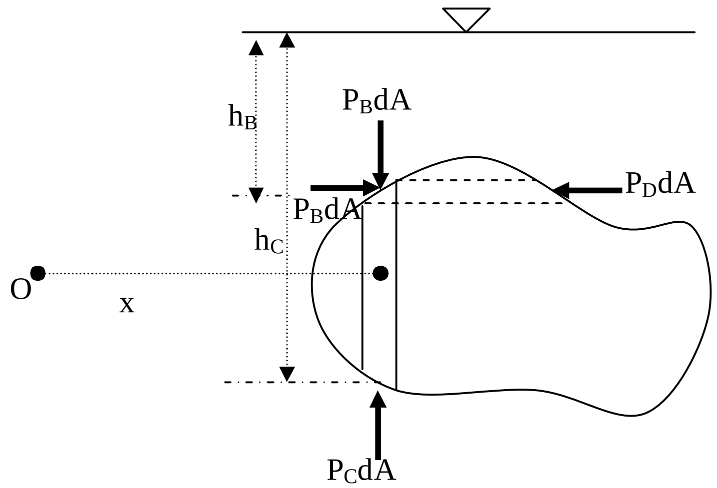


Fig 3.13 Buoyant force on a submerged body

Assume a vertical cylindrical element of cross- sectional area dA. As dA is small, the pressure on the exposed ends of the cylinder may be taken as p_1 and p_2.
Since $p_2 > p_1$, there will be an upward force $(p_2 - p_1)$ dA acting on the cylindrical element.

$$\therefore\ dF_B = (p_2 - p_1)\, dA = \gamma(h_2 - h_1)\, dA = \gamma dv$$

Where dv = volume of the prism

The entire body may be considered to be made up of small cylindrical elements, then integrating over the complete body gives

$$F_B = \int dF_B = \int^{v} \gamma dv = \gamma \int^{v} dv = \gamma V$$

γ is assumed constant through out the volume.
V= Volume of the body

The basic principle of buoyancy and flotation was fist discovered and stated by Archimedes over 2200 years ago. Archimedes principle states that the up thrust or the buoyancy on a body immersed in a fluid is equal to the weight, of the fluid displaced. The up thrust will act through the center of gravity of the displaced fluid, which is called the ***center of buoyancy.***

By applying Archimedes's principle, volumes of irregular solids can be found by determining the apparent loss of weight when a body is wholly immerse in a liquid of known specific gravity. Specific gravities of liquids can be determined by observing the depth of flotation of a hydrometer. Further applications include problems of general flotation and of naval architectural design.

To find the line of action of the buoyant force, moments are taken about a convenient axis 0.

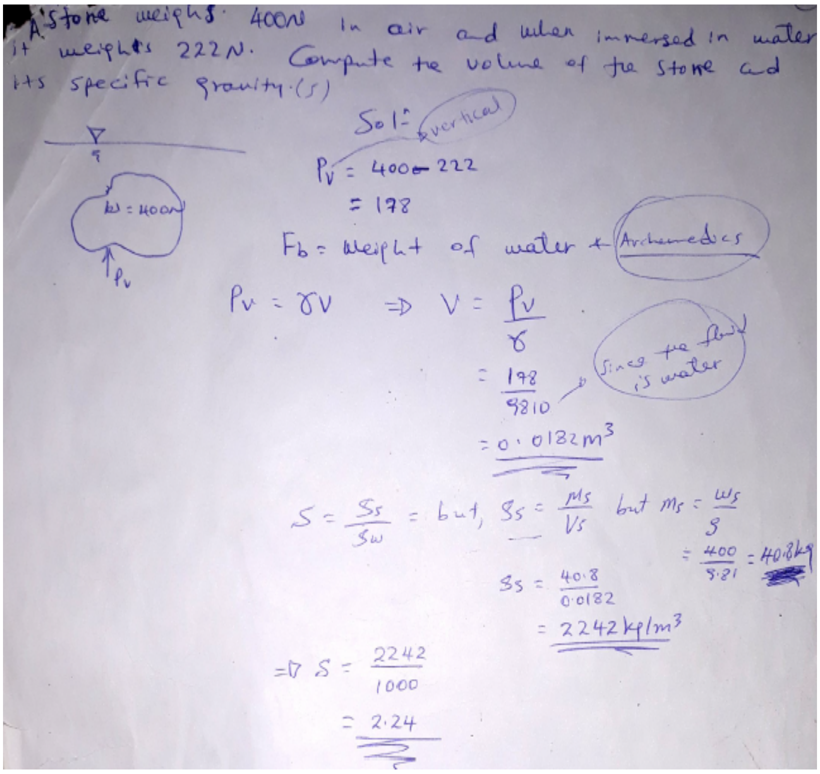
$\gamma V \quad \bar{x} = \gamma \int x.\, dv$ $\qquad \bar{x}$ = The distance from the axis to the line of action.

$\bar{x} = \dfrac{1}{v} \int^{v} x.dv$ (Centroid of the displaced volume of fluid) i.e. B.

A body immersed in two different fluids

Up thrust on body = weight of fluid displaced by the body (Archimedes principle.)

If the body is immersed so that part of its volume V_1 is immersed in a fluid of density ρ_1 and the rest of its volume V_2 in another immiscible fluid of mass density ρ_2,
Up thrust on upper part, $R_1 = \rho_1 g V_1$
acting through G_1, the centroid of V1,
Up thrust on lower part,$R_2 = \rho_2 g V_2$
acting through G_2, the centroid of V_2,
Total up thrust = $\rho_1 g V_1 + \rho_2 g V_2$.
The positions of G_1 and G_2 are not necessarily on the same vertical line, and the centre of buoyancy of the whole body is, therefore, not bound to pass through the centroid of the whole body.



A Stone weighs 400N in air and when immersed in water it weights 222N. Compute the volume of the stone and its specific gravity (S)

Sol:- vertical

$P_v = 400 - 222$

$= 178$

F_b = weight of water ← Archemedics

$P_v = \gamma V \Rightarrow V = \frac{P_v}{\gamma}$

$= \frac{178}{9810}$ → Since the fluid is water

$= 0.0182 m^3$

$S = \frac{\rho_s}{\rho_w}$ = but, $\rho_s = \frac{m_s}{V_s}$ but $m_s = \frac{w_s}{g}$

$= \frac{400}{9.81} = 40.8kg$

$\rho_s = \frac{40.8}{0.0182}$

$= 2242 kg/m^3$

$\Rightarrow S = \frac{2242}{1000}$

$= 2.24$

Hydrometers

Precise measurement of the specific weight of a liquid is done by utilising the principle of buoyancy. The device used for this, the hydrometer, is a glass bulb that is weighted on one end to make the hydrometer float in a vertical position and has a stem of constant diameter extending from the other end. The hydrometer is so designed that only the stem end extends above the liquid surface. Therefore, appreciable vertical movement of the hydrometer is required to change the buoyant force or displaced volume of the device. Because the buoyant force (equal to the weight of the hydrometer) must be constant, the hydrometer will float deeper or shallower depending on the specific weight of the liquid. Consequently graduation on the stem, corresponding to different depths of submergence of the hydrometer, can be made to indicate directly the specific weight or specific gravity of the liquid being measured. Consider the following figure

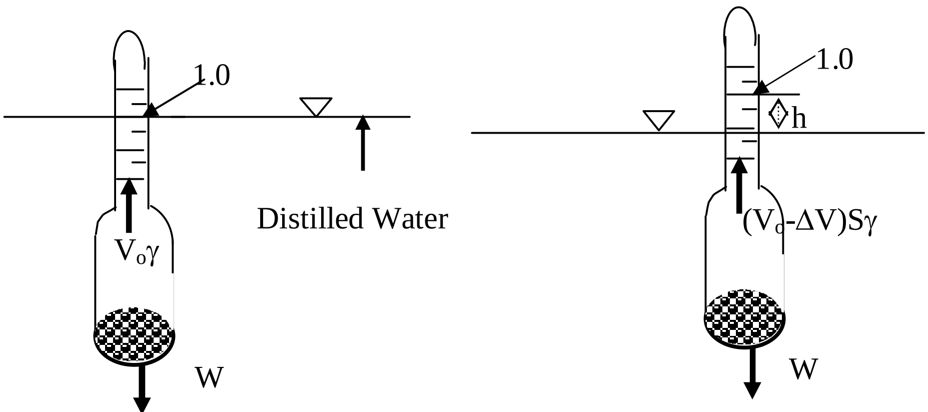


Fig.3.14 Hydrometer in water and in liquid of specific gravity

In the distilled water, the hydrometer floats in equilibrium when $V_0\gamma = W$

In which V_0 is the volume submerged, γ is the specific weight of water, and W is the weight of the hydrometer. The position of the liquid surface is marked as 1.0 on the stem to indicate unit specific gravity S. When the hydrometer is floated in another liquid, the equation of equilibrium becomes

$(V_o - \Delta V)S\gamma = W$ in which $\Delta V = a\Delta h$. Solving for Δh with the above equations gives

$$\Delta h = \frac{V_o}{a}\frac{S-1}{S}$$

One common use of hydrometers is in checking the state of charge of a car battery. When a battery is fully charged the specific gravity of the acid in it is about 1.28, and during discharge this specific gravity falls. The instrument used to check the state of charge is called a battery tester and it consists of a small hydrometer inside a glass container.

Floating in salt water and in fresh water

Exercise: People find that it is easier to float in salt water than in fresh water. Explain

If an egg is placed in a tall vessel and water is added, the egg remains on the bottom, but if salt is added and the water is stirred, the egg rises and floats. Why?

3.5.2. Stability of submerged and floating bodies.

3 – Possible conditions of equilibrium of solid body.

1. Stable equilibrium – A small displacement from the equilibrium produces a righting moment tending to restore the body to the equilibrium position.
2. Unstable equilibrium – A small displacement produces an over turning moment tending to displace the body further from its equilibrium position
3. Neutral equilibrium - The body remains at rest in any position to which it may be displaced. No couple.

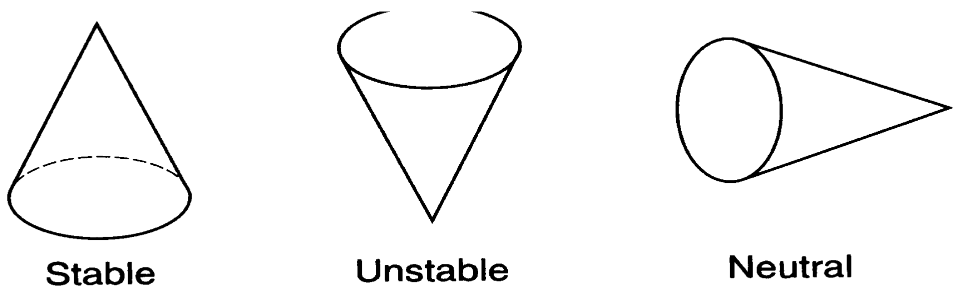
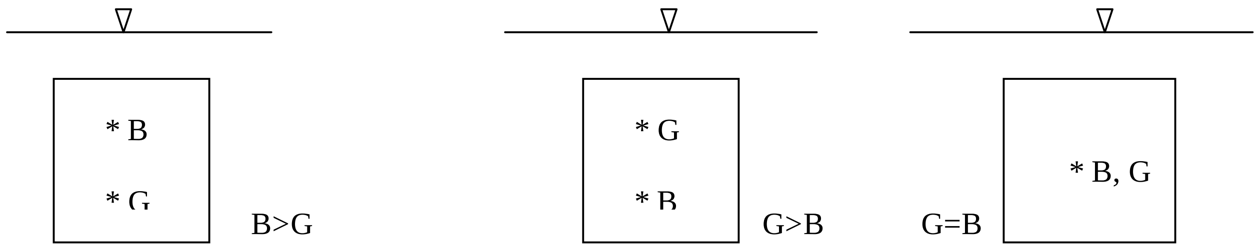


Fig. 3.14 Conditions of equilibrium

1. ***Submerged body***



Stable equilibrium (+ve stability) Unstable equilibrium (-ve stability) Neutral equilibrium (0 stability)

For a submerged body, ***the centre of buoyancy remains constant***. If an object is fully submerged, whether it is a balloon in air or a submarine in water, it must be designed that the centre of buoyancy lies some distance above the centre of gravity.

Exercise

Explain with example why the centre of buoyancy and the centre of gravity are located at different points for a fully submerged object.

2. *Floating body*

The following figure shows a solid body floating in equilibrium (weight acts through G & the buoyancy through B). Both act in the same straight line. When the body is displaced from its equilibrium, weight continues to act at G. The volume of liquid displaced remains constant but the shape of this volume will change and the position of its G and B will move relative to the body.

The point at which the line of action of the buoyant force for the displaced position cuts the original vertical through the center of gravity of the body G is called metacenter, designated M stable equilibrium. Metacentric height is the distance GM.

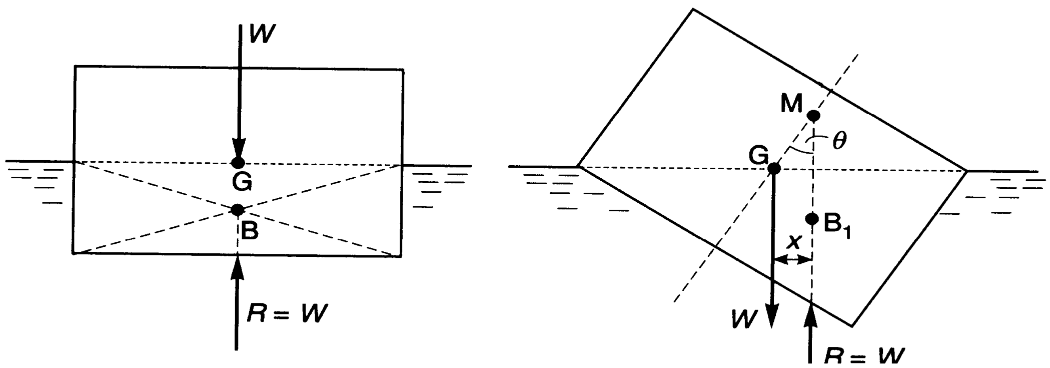


Fig.3.16 Stable equilibrium a) b)

The displaced fluid is rectangular in section (fig. a) but it is triangular in fig.b and the center of buoyancy moves to B_1. As a result F_8 and W are not in the same straight line producing a turning moment WX that is a righting moment.

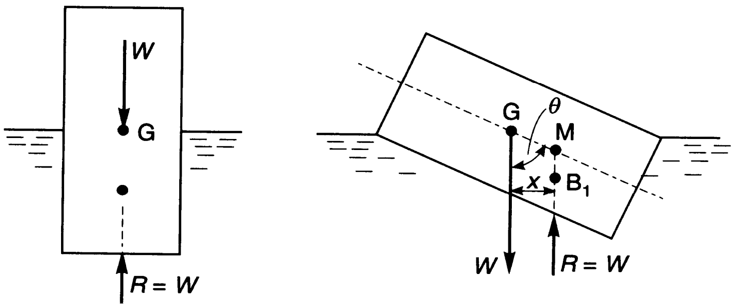


Fig.3.17 Unstable equilibrium a) b)

Comparing the above figures, it can be seen that:

1. If M lies above G a righting moment is produced, GM is regarded as positive, and equilibrium is stable.
2. If M lies below G an overturning moment is produced, GM is regarded as negative, and equilibrium is unstable.
3. If M and G coincide the body is in neutral equilibrium.

Evaluation of Metacentric height

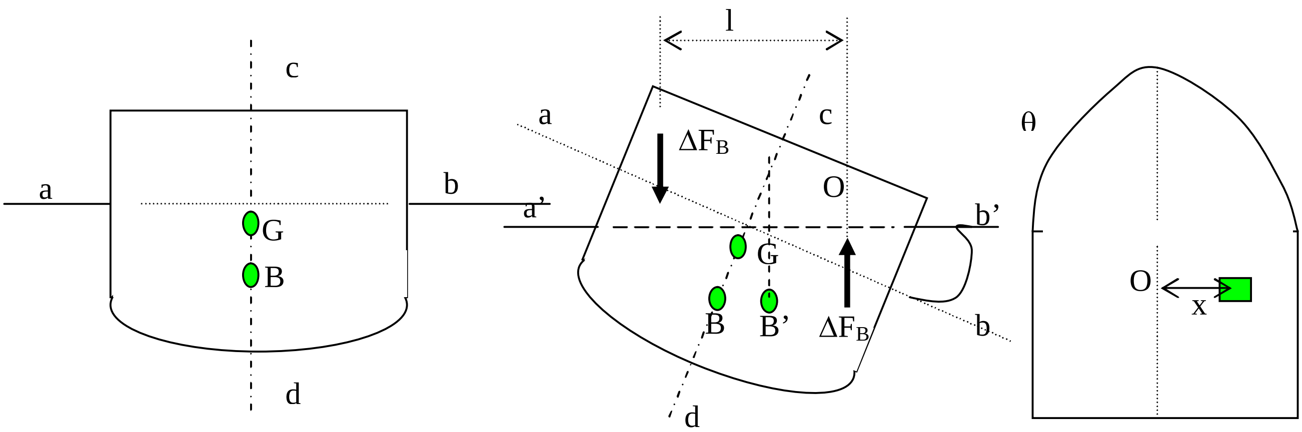


Fig.3.18 x –section and plan in upright position

Consider a non –prismatic floating object, such as a ship. Assume an outside force is applied causing the body to tilt through a small angle θ. The relative position of the G remains unchanged but B shifts from B to B'. The volume of fluid displaced is of course unchanged and in effect a wedge shaped volume of water represented by aoa' has shifted across the central axis to bob'. These wedges represent a gain in the buoyant force on the right side and a corresponding loss in buoyancy on the left side of c-d.

The buoyant force F_B acting through B' may be considered as the resultant of the original buoyant force through B and the gain & loss of buoyant force.

Taking moment about B, we have

$F_B * \overline{BM} \sin\theta = \Delta F_B * \ell$ (Moment of resultant = $\sum$ moment of components.)

Consider an element of area dA in plan at a distance x, from O. The buoyant force acting on this element is γ x θ dA. θ << Small tan $\theta \approx \sin\theta \approx \theta$

$$F_B = \gamma * \text{volume} = \gamma x \theta dA$$

Then $\Delta F_B = \int \Upsilon x \theta dA$ (integrated half of the water line)

Moment of this force about O

$$\Delta F_B * \ell / 2 = \int \gamma x^2 \theta dA$$
$$\Delta F_B * \ell = 2 \int \gamma x^2 \theta dA$$

If the integration is performed over the entire area

$$\Delta F_B * \ell = \int^A \gamma\theta x^2 dA = \gamma\theta \int^A x^2 dA$$

$\int^A x^2 dA =$ Moment of inertia I of a horizontal section of the body taken at the surface of the fluid

$$\Delta F_B * \ell = \gamma\theta I$$
$$F_B \overline{BM} \sin\theta = \Delta F_B * \ell$$
$$W \overline{BM} \theta = \gamma\theta I$$
$$\overline{BM} = \frac{\gamma I}{W} = \frac{\gamma I}{\gamma v} = \frac{I}{V}$$

V = volume of water displaced by the vessel.

Metacentric height
$$\overline{GM} = \overline{BM} - \overline{BG}$$
$$= \frac{I}{V} - \overline{BG}$$

If the G is below B, then $G\overline{M} = \frac{I}{V} + B\overline{G}$

$$GM = \frac{I}{V} \pm B\overline{G}$$

Attend the laboratory session for experimental determination of the metacentric height.

Time of oscillation

Consider a floating body, which is tilted through an angle by an overturning couple as shown below. Let the overturning couple is suddenly removed. The body will start oscillating. Thus, the body will be in a state of oscillation as if suspended at the meta-center M. This is similar to a case of a pendulum. The only force acting on the body is due to the restoring couple due to the weight w of the body force of buoyancy F_B.

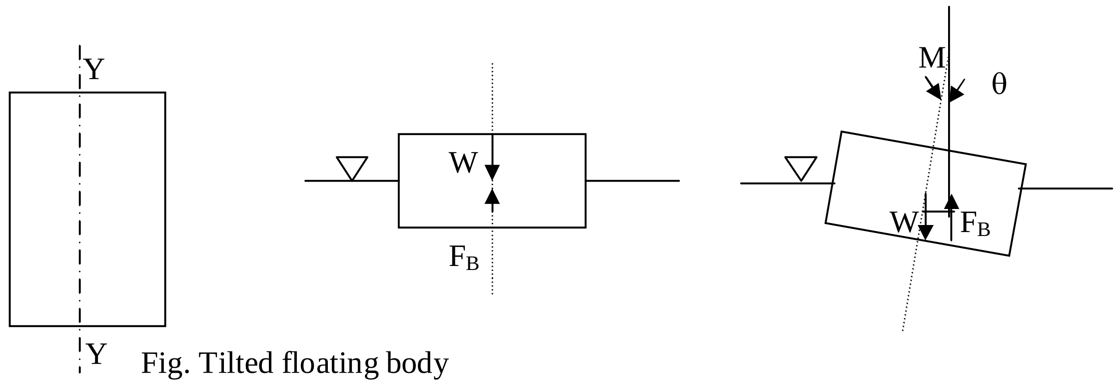


Fig. Tilted floating body

Restoring couple = W GM sinθ

Angular acceleration of the body, $\alpha = \frac{d^2\theta}{dt^2}$

Negative sign has been introduced as the restoring couple tries to decrease the angle θ.

Torque due to inertia = $I_{Y\text{-}Y}$ $\left(\frac{d^2\theta}{dt^2}\right)$

But $I_{Y\text{-}Y}$ = (W/g) K2

Where W=weight of body, K=radius of gyration about Y-Y

Inertia torque = - (W/g) K^2 $\left(\frac{d^2\theta}{dt^2}\right)$

Equating the above equations

W GM sinθ = - (W/g) K2 $\left(\frac{d^2\theta}{dt^2}\right)$ or GM sinθ = - (K^2/) $\left(\frac{d^2\theta}{dt^2}\right)$

For small angle θ, sin θ =θ

GM θ = - (K^2/g) $\left(\frac{d^2\theta}{dt^2}\right)$ or (K^2/g) $\left(\frac{d^2\theta}{dt^2}\right)$ + GM θ = 0

$$\Rightarrow \frac{d^2\theta}{dt^2} + \frac{\mathrm{Gg}\,\theta}{K^2} = 0$$

This is second-degree differential equation, the solution is

$$\theta = C_1 \sin\sqrt{\frac{\mathrm{GM\,g}}{K^2}} * t \;+\; C_2\, Cos\sqrt{\frac{\mathrm{GM\,g}}{K^2}} * t$$

Where C_1 and C_2 are constants of integration.
The values of C_1 and C_2 are obtained from boundary conditions, which are

i) at $t=0$, $\theta=0$
ii) at $t=(T/2)$, $\theta=0$

Where T=time of one complete oscillation
Substituting the first boundary condition, $C_2=0$
Substituting the second boundary condition, we get $0=C_1 \sin\sqrt{\frac{\mathrm{M\,g}}{K^2}} * \frac{T}{2}$
But C_1 cannot be equal to zero and so the other alternative is

$$\sin\sqrt{\frac{\mathrm{GM\,g}}{K^2}} * \frac{T}{2} = 0 = \sin\pi$$

$$\sqrt{\frac{\mathrm{GM\,g}}{K^2}} * \frac{T}{2} = \pi \quad or \quad T = 2\pi\sqrt{\frac{K^2}{\mathrm{GM\,g}}}$$

This gives the time period of oscillation or rolling of a floating body.