Ambo university woliso campus
April, 2020

Chapter 4

Graph Theory

4.1 Definition and Representations of Graph

4.2 Types of Graph

4.3 Isomorphic graphs

4.4 Path and Connectivity of Graph

4.5 Eulerian and Hamiltonian Graph

4.6 Trees and Forests

4.7 Planar Graph

4.8 Graph coloring and Chromatic polynomials

4.1 Definition and Representations of graph

1. **Definition**: A Graph G = (V, E,) is an order triple consists of a non-empty set of vertices, set of edges together with an incidence function which associates each elements of E with a pair of vertices in V of G.

Note:

- {u, v} and {v, u} is the same in undirected graph but if both appears on E they each represent an edge.

The number of vertices of G is the called Order(cardinality) of G denoted by |V|=|G|

The number of edges of G is the called Size of G denoted by |E| = ||G||

- Let G be a graph for any edge e = {u, v} then
- u and v are the end point of e
- e is a loop if u = v
- e is a link if u ‡ v
- u and v are adjacent vertices (neighbor)
- e is Incident with u and v and vice versa

Definition: Two or more edges are said to be parallel or Multiple edges if they are incident with the same pair of vertices.

Example: Let G = (V, E,φG) where V = {a, b, c, d, e, f }

E = {e1, e2, e3,e4,e5}

φG (e1) = {a, b}; φG (e2) = {b, a}; φG (e3) = {c, d};

φG (e4) = {c, c}; φG (e5) = {e, d}; φG (e6) = {b, c}, then find

1 Find the size and order of G

2 The multiple edge are .... and ....

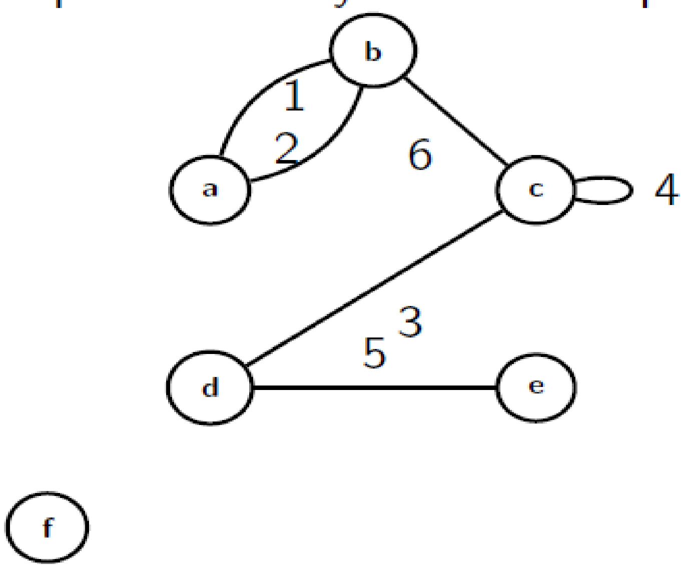
3 .... is a loop

4 The links are ....

5 In e3 = {c; d}, then c and d are ....vertices

6 In e3 = {c; d}, e3 is .... with c and d

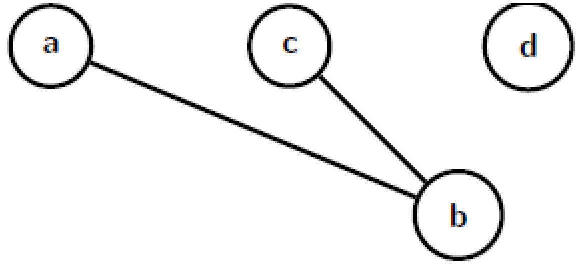
2.Pictorial representation of a graph Let $G = (V, E, \varphi_G)$ be a graph
vertices-represented by **Nodes or Points**
Edges- represented by **Line segments**
Note: The position of the vertex or the shape and length of edges doesn't matter at all.
Example: Find The pictorial representation of the Graph represented by ordered triple in the above example?



Definition: The degree(valancy) of a vertex v ε VG is the number of edges incident with it plus twice the Number of loops at v if any denoted by (dG (v), degG (v))

Note:

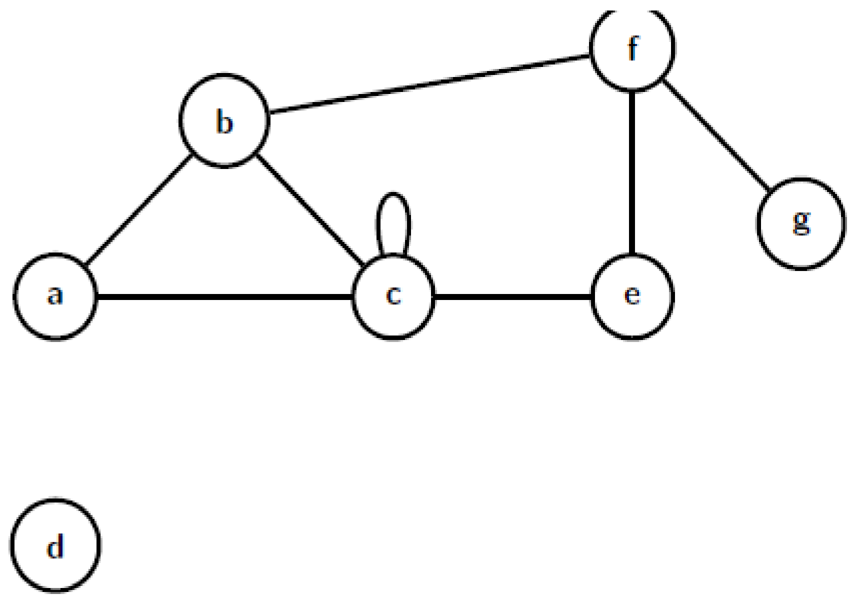
- if d(v) = 0, then v is isolated vertex
- if d(v) = 1, then v is Pendant vertex
- if d(v) = 2k(even number), then v is even vertex
- if d(v) = 2k $\pm$ 1(odd number), then v is odd vertex
- δ(G) : Smallest degree of a vertex in G.
- Δ(G) : Largest degree of a vertex in G.
- Example: Find the even and odd vertices and δ(G) and Δ(G)



The handshaking lemma: let G = (V, E) be any graph, then , the sum of deg(v) = 2|E|= 2||G||

3.**Degree sequence Representations of a graph**: for a graph of order n, the degree sequence of G is d(v1), d(v2), d(v3),…., d(vn).

Example: Find the degree sequence and the number of edges of G?



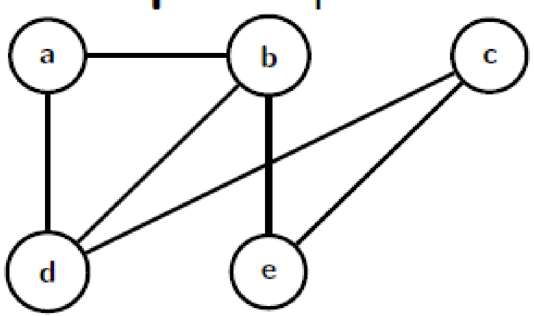
Note: if summation of degree of vertices is odd there is no graph representation (so always it should be even)

- Corollary :The number of odd vertices in any graph is even

4.Matrix Representation of a graph Suppose that $G(V, E)$ is a simple graph where $|V| = n$ suppose that $V = \{v_1, v_2, ..., v_n\}$
The **Adjacency matrix** $A_G = [a_{i,j}]_{n\times n}$ is a zero-one matrix

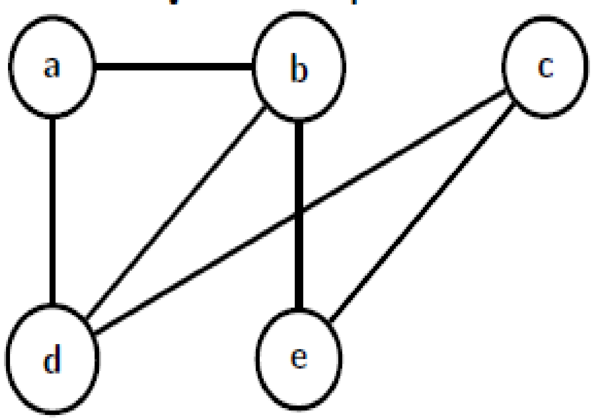
$$a_{i,j} = \begin{cases} 1, & \text{if } v_i v_j \in E_G \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Example: represent the graph below with adjacency matrix

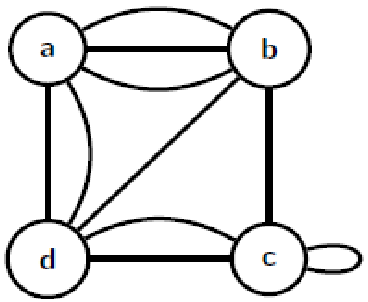


$$\begin{array}{c} \\ a \\ b \\ c \\ c \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Example: represent the graph below with adjacency matrix



$$\begin{array}{c} \\ a \\ b \\ c \\ c \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

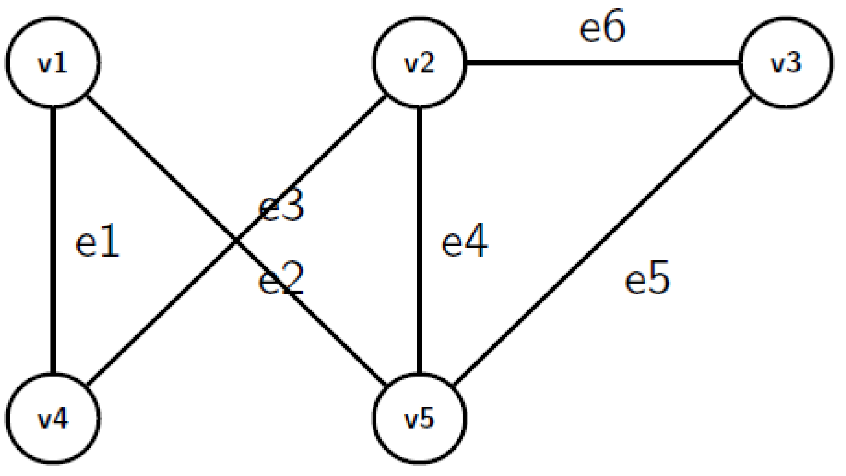


$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{pmatrix} \end{array}$$

Suppose that $G(V, E)$ is a simple unordered graph where $|V| = n$
suppose that $V = \{v_1, v_2, ..., v_n\}$
and $E = \{e_1, e_2, ..., e_m\}$
Then the **Incident matrix** with respect to this ordering of V and E is $n \times m$ matrix $M_G = [m_{i,j}]_{n\times m}$ is a zero-one matrix

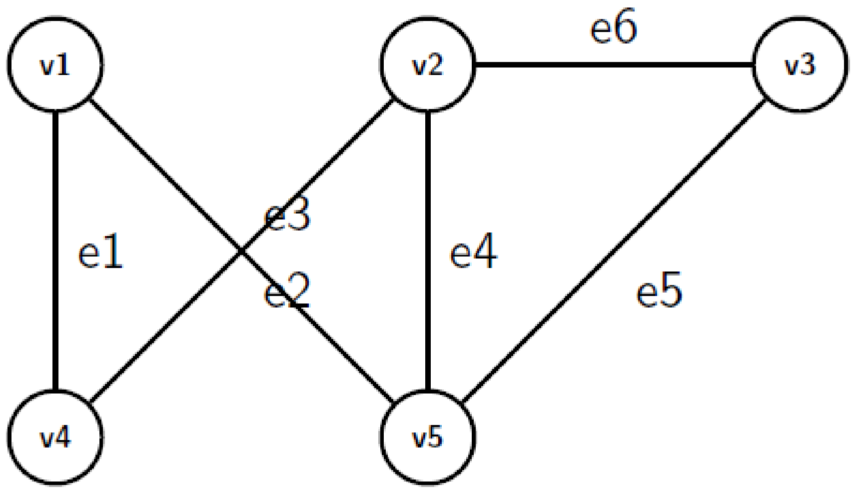
$$m_{i,j} = \begin{cases} 1, & \text{if edge } e_j \text{is incident with vertex} v_i \in E_G \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Example: represent the graph below with incidence matrix



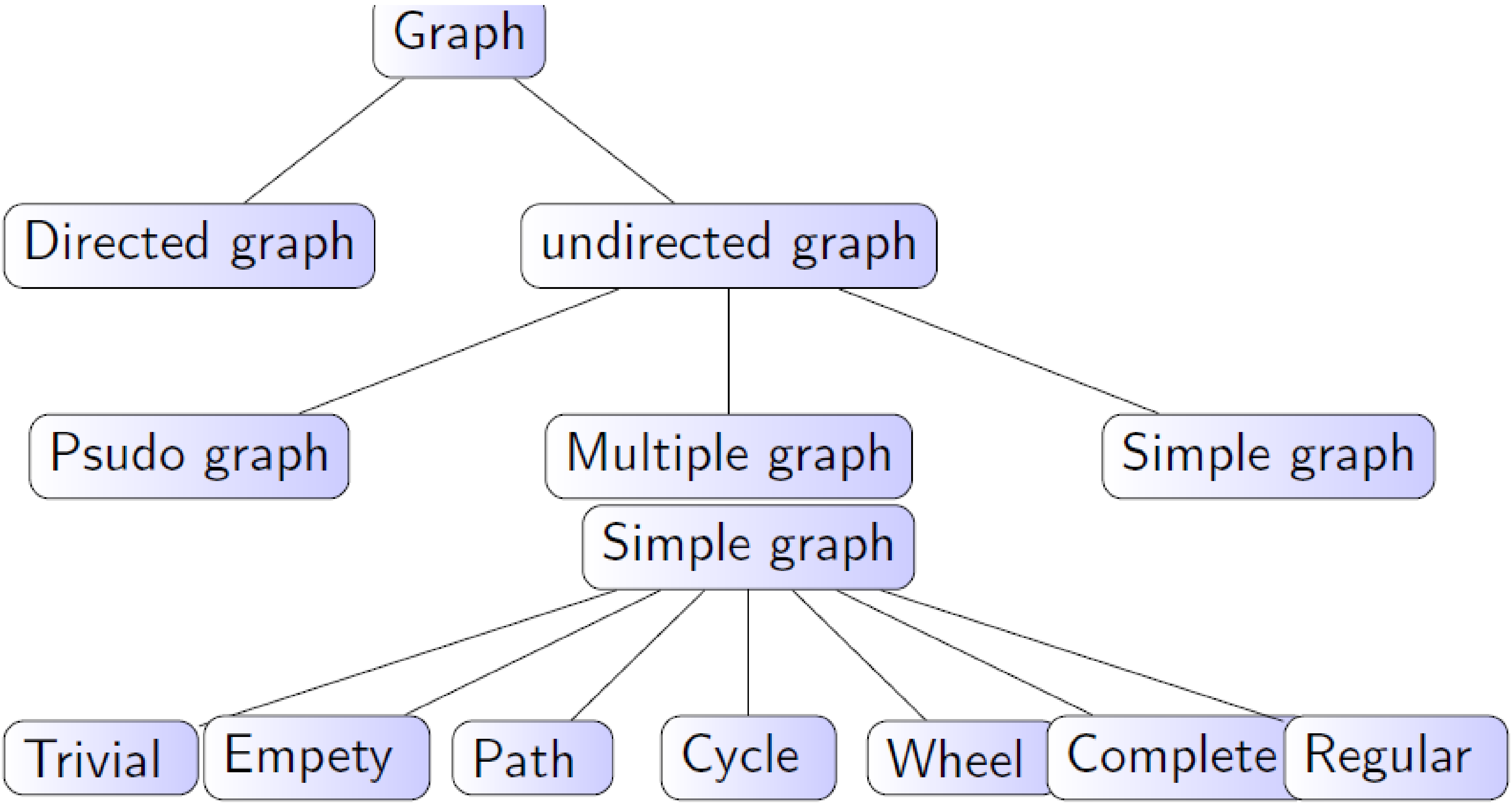
	e1	e2	e3	e4	e5	e6
v1	1	1	0	0	0	0
v2	0	0	1	1	0	1
v3	0	0	0	0	1	1
v4	1	0	1	0	0	0
v5	0	1	0	1	1	0

Example: represent the graph below with incidence matrix



	e1	e2	e3	e4	e5	e6
v1	1	1	0	0	0	0
v2	0	0	1	1	0	1
v3	0	0	0	0	1	1
v4	1	0	1	0	0	0
v5	0	1	0	1	1	0

4.2 Types of graph



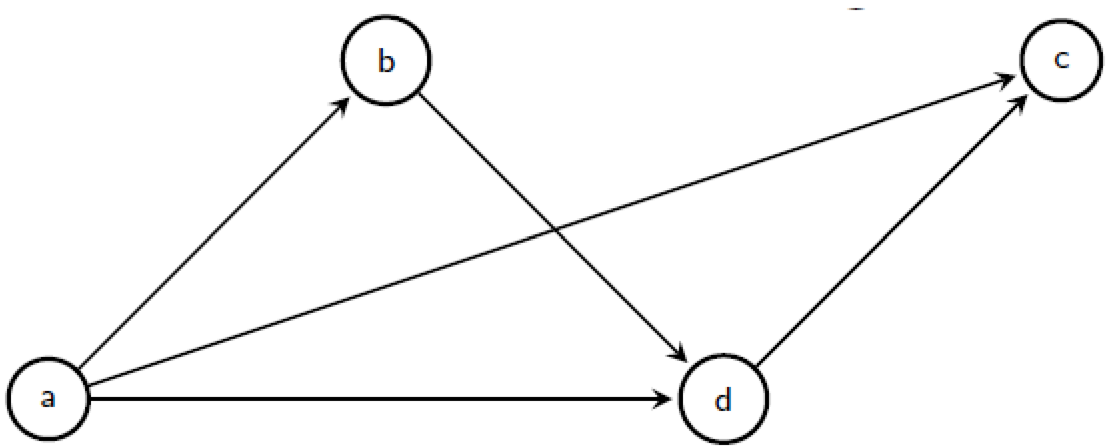
Directed graph: Is an order triple D = (V, E, φG) which contains a non-empty set of vertex set of edge and an incidence function which associate each directed edge with ordered pair of vertices.

Undirected graph: Is an order triple G = (V, E, φG) which contains a non-empty set of vertex set of edge and an incidence function which associate each undirected edge with unordered pair of vertices.

Undirected graph is divided in to three

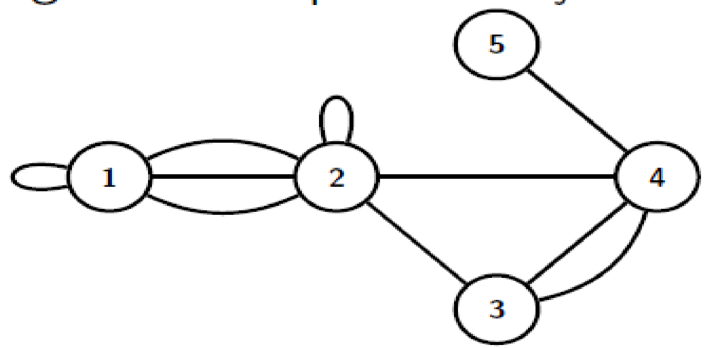
i.e pseudo,multiple and simple;

Example of Directed Graph



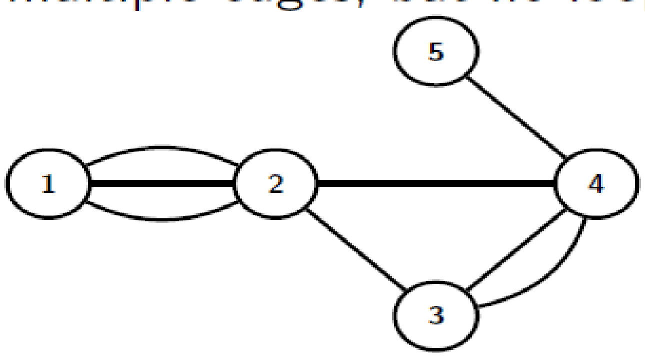
[D]

Definition: A Pseudo graph $G = (V_G, E_G)$ is a graph which has both multiple edges and loops i.e every undirected graph is a

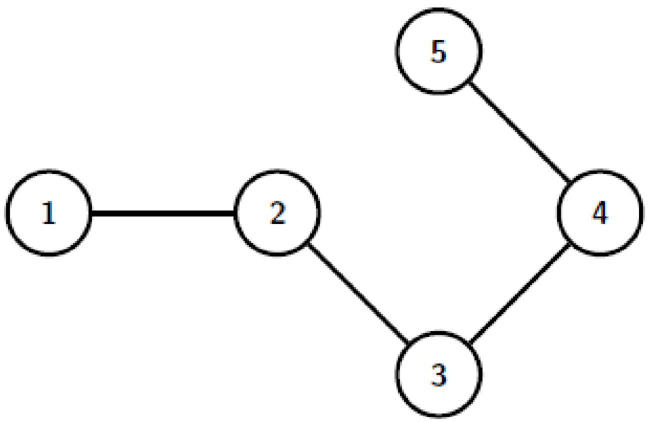


pseudo graph

Definition: A Multiple graph $G = (V_G, E_G)$ is a graph which has multiple edges, but no loops



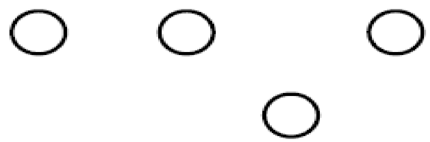
Definition: A Simple graph $G = (V_G, E_G)$ is a graph with no loop and multiple edges.
i.e E has a set of unordered pair of distinct elements of V.



Simple graph is divided in to three i.e Trivial ,Empty ,Path ,Cycle ,Wheel ,Complete ,Hyperbolic ,Regular ,Bipartite ,complete bipartite

1.Trivial Graph: A Graph with only one vertex and no edge.

2.Empty Graph(O_n): Is a graph with n-vertices and no edge.



O_4

Note:$|O_n| = 0$ and $||O_n|| = 0$

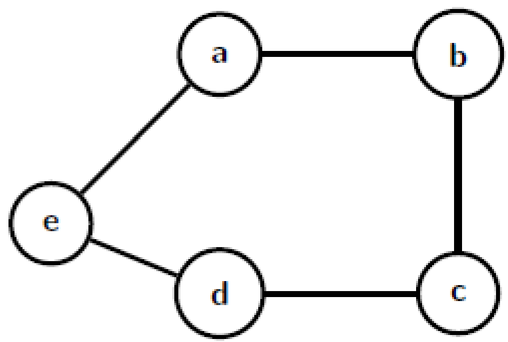
3.Path Graph(P_n): for Is a graph whose vertices could be ordered in such a way that consecuative vertices are adjacent and every vertex is adjacent to exactly two other vertices in the graph.

i.e $E = \{v_i v_{i+1:\forall i}\}$

P_5

Note:$|P_n| = n$ and $||P_n|| = n - 1$

4.Cycle Graph(C_n): for $n > 2$ is a closed path where the initial and the terminal vertices are the same.

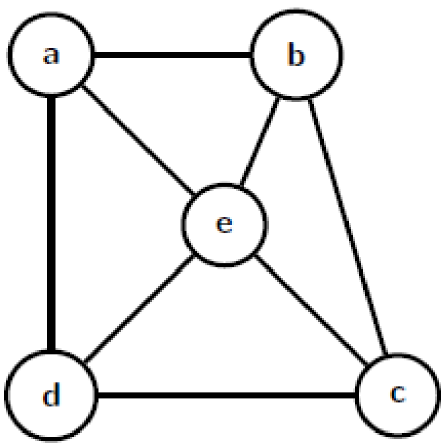


C_5

Note:$|C_n| = n$ and $||C_n|| = n$

5.Wheel graph of order (n) **(W_n):** to obtain W_n by adding additional vertex to C_{n-1} for $n \geq 3$ and connect this new vertex to each of $n-1$ vertices in C_{n-1} by new $n-1$ edges.

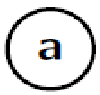
Note: $|W_n| = n$ and $||W_n|| = 2n - 2$



W_5

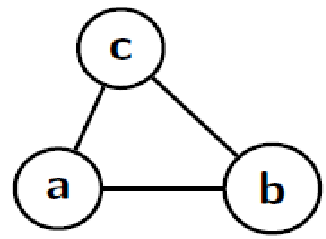
7.Complete graph of order n (K_n):- is a smple graph that contains exactly one edge between each pair of distinct vertices.

Example:

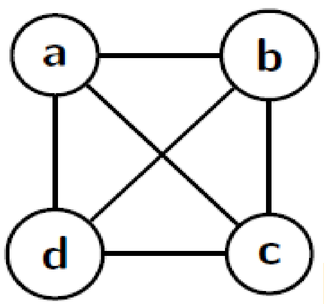


$[k_1]$

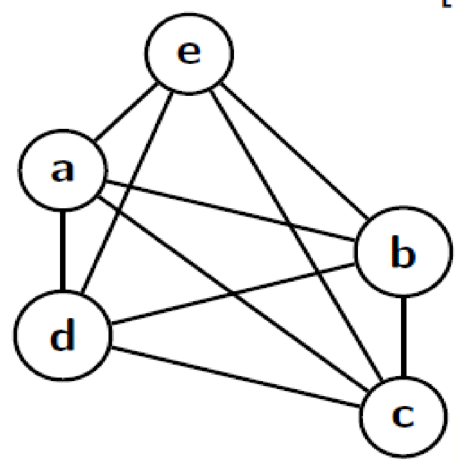
$[K_2]$



$[K_3]$



$[K_4]$



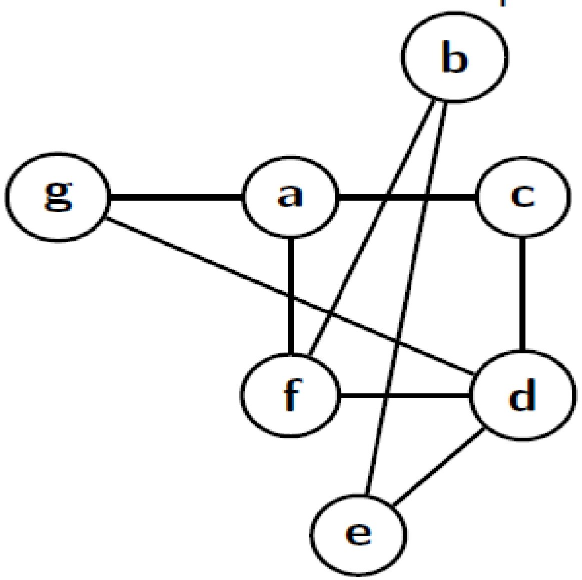
$[K_5]$

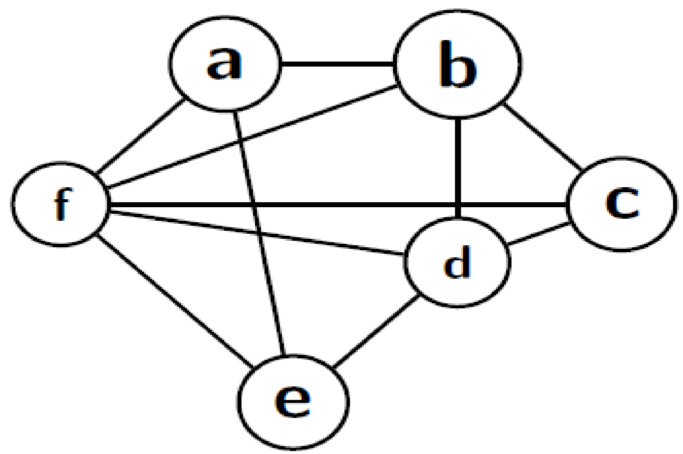
Note:$|K_n| = n$ and $\|K_n\| = \dfrac{n(n-1)}{2}$

8.Bipartite graph A simple graph G is called **Bipartite** if its vertex set V can be partitioned in to two disjoint non-empty sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or V_2)

Example: show that C_6 is bipartite and K_3 is not?

Exercise:which Graph is bipartite and which is not.

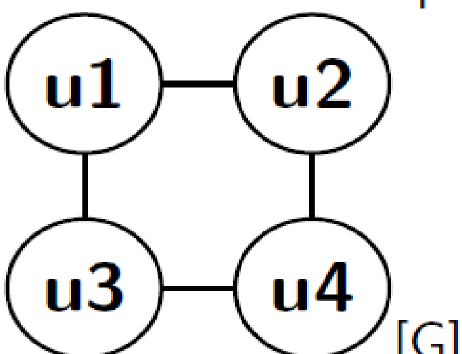




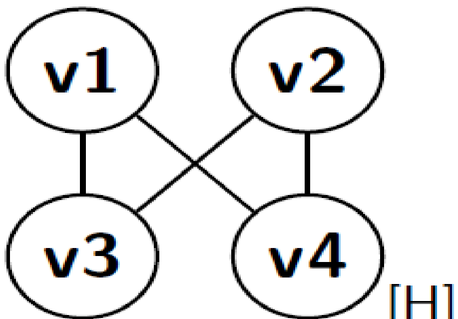
4.3 Isomorphic Graphs

Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic ($G_1 \cong G_2$) if there is a one-to-one correspondence function $f : v_1 \to v_2$ such that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2. such a function is called isomorphic (equal form).

Example: Show that the graph $G = (V, E)$ and $H = (W, F)$ below are isomorphic?



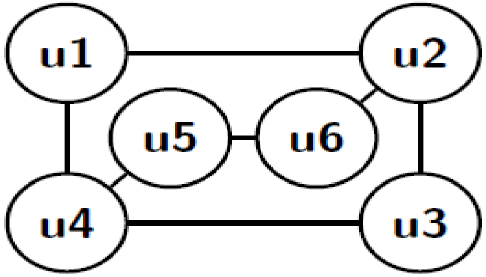
	u1	u2	u3	u4
u1	0	1	1	0
u2	1	0	0	1
u3	1	0	0	1
u4	0	1	1	0



$$\begin{array}{c} \\ v1 \\ v2 \\ v3 \\ v4 \end{array} \begin{array}{c} \begin{array}{cccc} v1 & v2 & v3 & v4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

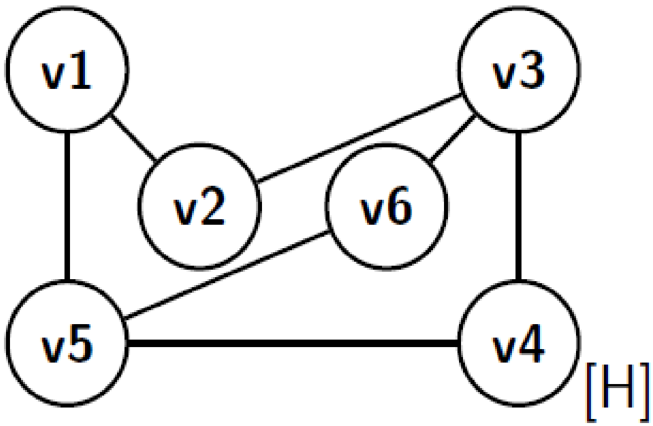
Solution: Let f be a function with
$f(u1) = v1, f(u2) = v4, f(u3) = v3$ and $f(u4) = v2$ is a $1-1$ correspondence $f : V \to W$
and every adjacent vertex in G are adjacent vertices in H.
$u1u2 \in E_G$ implies $f(u1)f(u2) = v1v4 \in E_H$
$u1u3 \in E_G$ implies $f(u1)f(u3) = v1v3 \in E_H$
$u2u4 \in E_G$ implies $f(u2)f(u4) = v4v2 \in E_H$
$u3u4 \in E_G$ implies $f(u3)f(u4) = v3v2 \in E_H$
Therefore G and H are isomorphic $G \cong H$

Note: If the adjacency matrix of G and H are the same when rows and columns are labeled to correspond to the images under f of vertices in G that are the labels of the rows and columns in the adjacency matrix of G, then f is isomorphic from V_G to V_H
from the above example if we interchange the second and forth row and columns in the second matrix ($v2$ implies $v4$) then we get a row equivalent matrix to the first.
Example: determine whether or not G and H are isomorphic or not?



[G]

	u1	u2	u3	u4	u5	u6
u1	0	1	0	1	0	0
u2	1	0	1	0	0	1
u3	0	1	0	1	0	0
u4	1	0	1	0	1	0
u5	0	0	0	1	0	1
u6	0	1	0	0	1	0

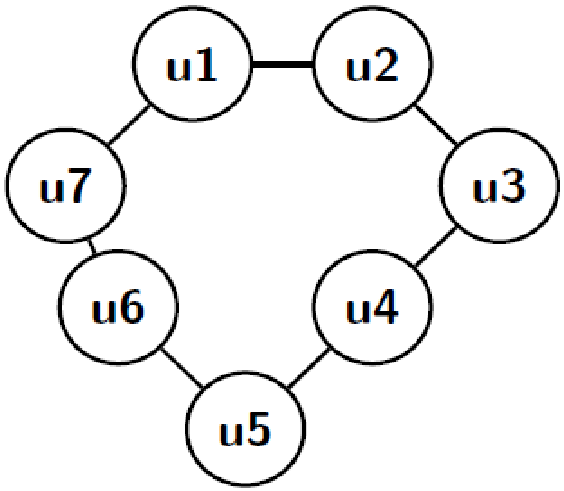


	v6	v3	v4	v5	v1	v2
v6	0	1	0	1	0	0
v3	1	0	1	0	0	1
v4	0	1	0	1	0	0
v5	1	0	1	0	1	0
v1	0	0	0	1	0	1
v2	0	1	0	0	1	0

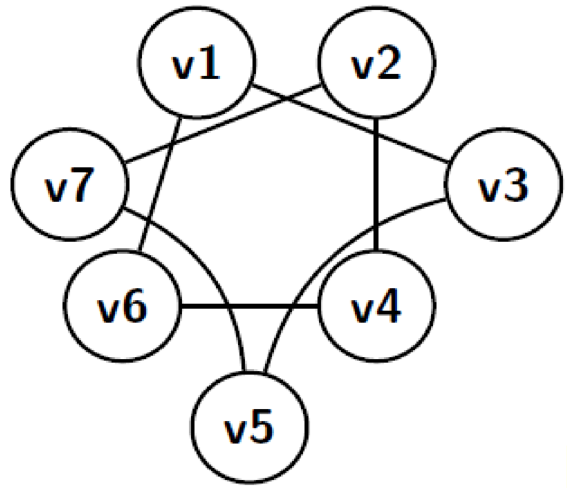
$A_G = A_H$ that is f is an isomorphic if $f(u1) = v6$, $f(u2) = v3$, $f(u3) = v4$, $f(u4) = v5$, $f(u5) = v1$, $f(u6) = v2$

Therefore $G \cong H$

Exercise: show that $G \cong H$ using adjacency matrix.



[G]



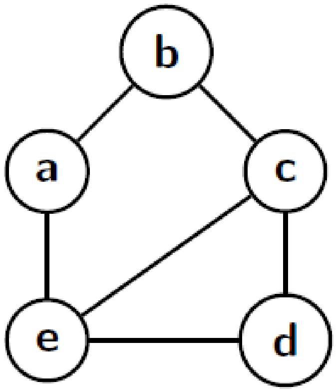
[H]

Invariant: is a property which helps us to check two graphs are not isomorphic depending on the property of isomorphic graph.
i.e if two simple graphs are isomorphic then they have

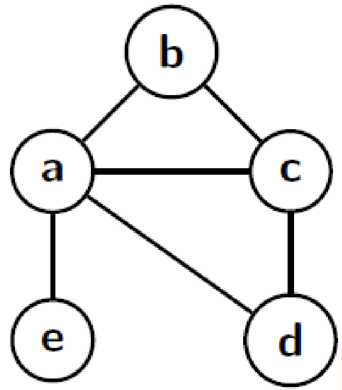
1 the same number of vertices and edges.

2 the degree of each associated vertex must be the same.

Example: Show that the graph below are not isomorphic.



[G]

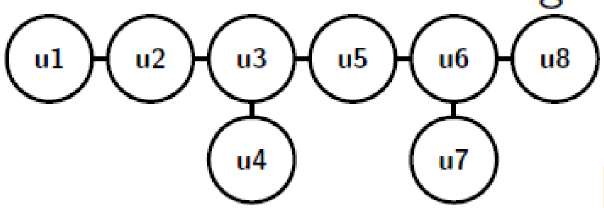


[H]

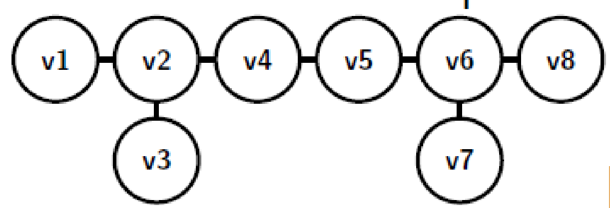
Implies G and H have the same number of vertices and edges, but degree of $G = 2, 2, 3, , 2, 3$ and $H = 4, 2, 3, 2, 1$

Therefor $G \not\cong H$

Exercise: Show that the graph below are not isomorphic.



[G]



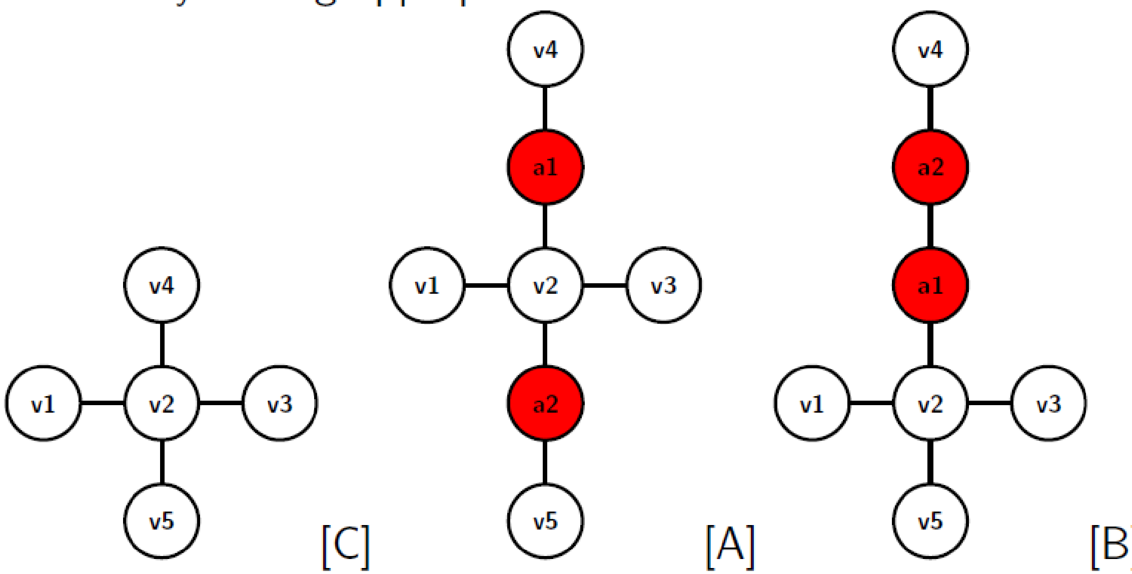
[H]

Homeomorphic graphs

Given any graph G, we can obtain a new graph by dividing an edge of G with additional vertices

G and G^* are Homeomorphic if they can be obtained from the same graph or isomorphic graphs by the above method.

Example: A is Homeomorphic to B since they can be obtained from C by adding approprate vertices.



4.4 Path and connectivity of Graph

Definition: A walk in G is a finite non-empty sequence $W = v_0 e_1 v_1 e_2 \ldots v_{k-1} e_k v_k$ such that for $1 \leq i \leq k$ the end of e_i are v_{i-1} and v_i

Note:

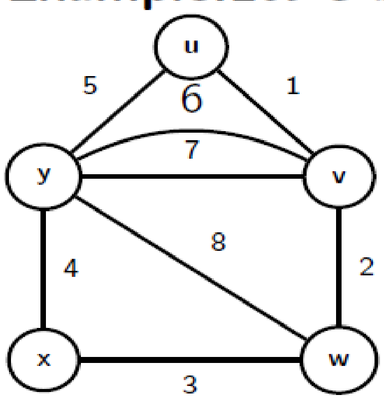
- W-is a walk from v_0 to v_k or (v_0, v_k)-walk.
- v_0 is called the origin and v_k-terminus of W.
- $v_2, v_1, v_3, \ldots, v_{k-1}$ are internal vertices of W.
- $k \in Z^+$ the length of the walk W.

If walk $W = v_0 e_1 v_1 e_2 \ldots v_{k-1} e_k v_k$ and $W' = v_k e_{k+1} v_{k+1} \ldots e_1 v_1$ are walks.

- the walk $W^{-1} = v_k e_k v_{k-1} \ldots e_1 v_0$ is a walk obtained by reversing W.
- the walk $WW' = v_0 e_1 v_1 e_2 \ldots v_{k-1} e_k v_k e_{k+1} v_{k+1} \ldots e_1 v_1$ is a walk obtained by concatenating W and W' at v_k.
- (v_i, v_j) section of W (or sub sequence) is $v_i e_{i+1} v_{i+1} \ldots e_j v_j$ of consecutive terms of W.

- in simple graph $W = v_0 v_1 ... v_{k-1} v_k$ the walk is described by only set of vertices sequence.

Definition: if the edge of a walk W are distinct W is called a **trail**.in this case the length of W is number of edges.
Definition: if the vertices of a trail are distinct, then W is called **a Path**.denoted by (v_0, v_k)-path.
Example: Let G be

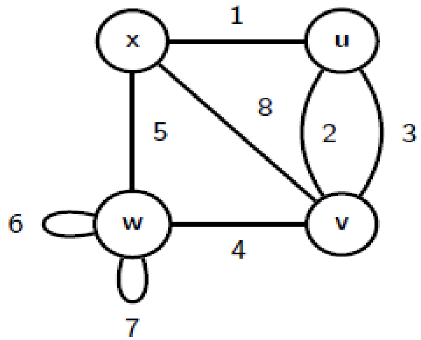


[G] then

- Walk: $W = u1v6y6v7y8w2v$ length of $W = 6$
- Trail: $W_1 = w3x4y8w2v7y$ length of $W_1 = 5$
- Path: $W_2 = x3w8y5u1v$ length of $W_2 = 4$

Definition: A (u, v)-path which is closed $(u = v)$ is known as a cycle or circuit (the only vertex revisited is u).

Example: Let G be



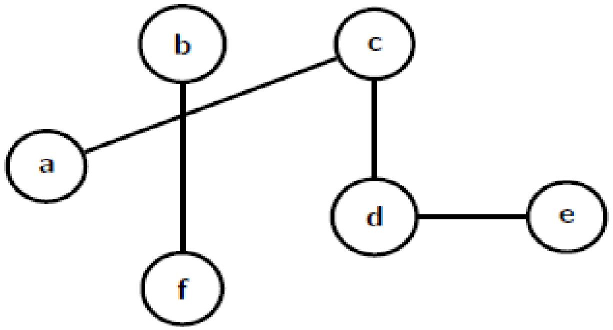
[G] find the cycle?$x1u2v8x$ length 3

Definition: A graph (sub-graph) which is a path (cycle) is known as a path graph or cycle graph and if the order is n denoted by P_n or C_n respectively.

Example:

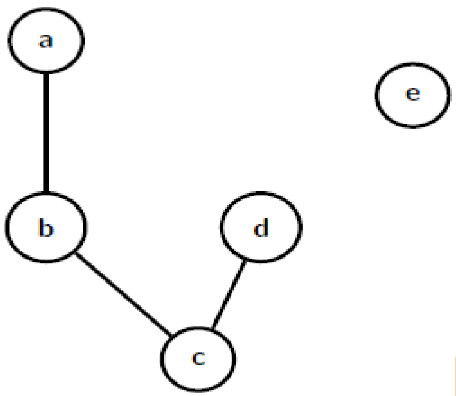
- a loop is considered to be a cycle of order 1.
- two parallel edges also form a cycle of order 2.
- if n is even P_n and C_n are called even path and even cycle of order n.

Connectedness of a graph
Definition: In a graph G if there is a (u, v)-path, then u and v are connected in G. So a graph G is connected if any two distinct vertices are connected.
Example:

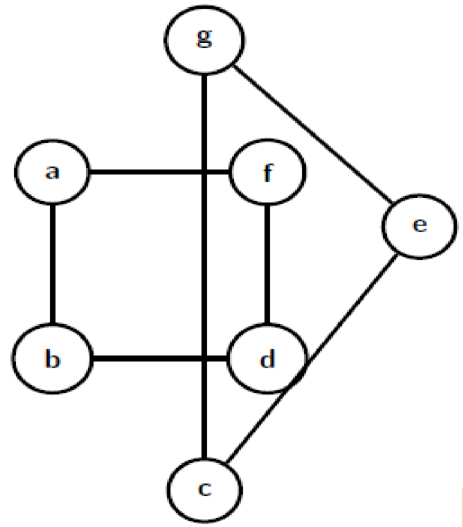


[G]

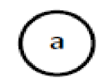
a, b are not connected, a, c and b, f are connected
Therefor G is not connected (disconnected).
Definition: A graph G is connected if it has a (u, v)-path between any two vertices $u, v \in V_G$ otherwise G is disconnected.



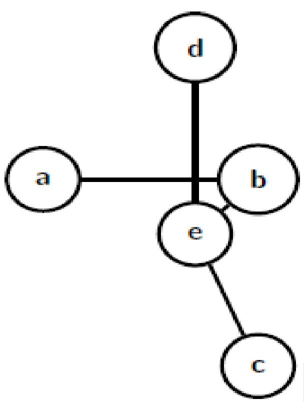
[G]



[G']



[G''']



[G'']

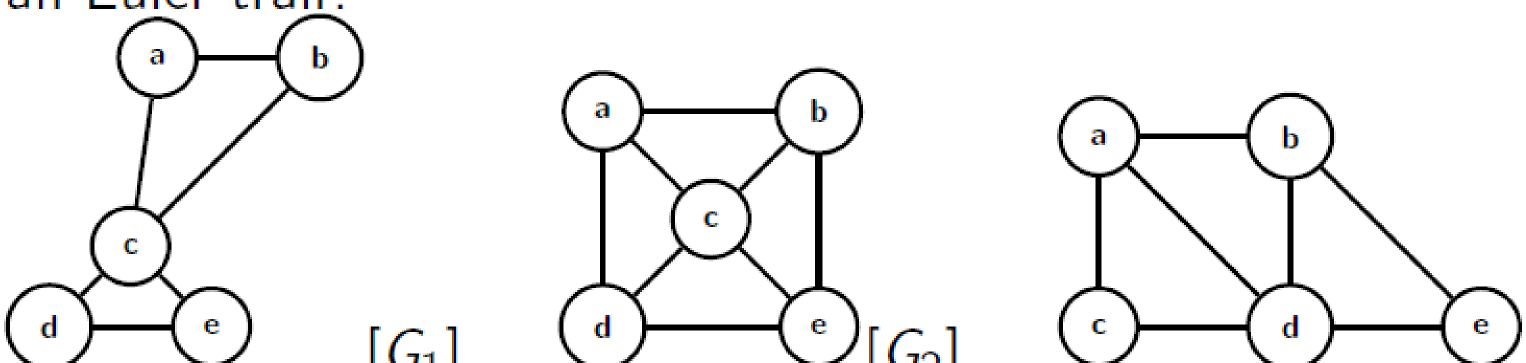
4.5 Euler and Hamilton Graph

Definition: Atrail that visits every edge in a graph G is called **Euler trail**

Definition: A closed Euler trail is called **Euler Circuit or Euler tour**

Definition: A connected graph G is **Eulerian** if it has an Euler tour.

Example: Which on of the graph has Euler tour? and which has an Euler trail?

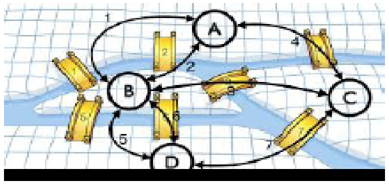


$[G_1]$ $[G_2]$ $[G_3]$

Theorem

A connected graph G is Eulerian if and only if each of its vertex has an even degree.

Corollary: A connected graph has non-closed Eulerian trail if and only if it has exactly two odd vertices.
Konigsberg bridge problem(Leonhard Euler 1736:Swiss mathematician): A famous problem which questions if it is possible to visit all the four cities and the seven bridges separated by the pregal river and connected by seven bridges with out crossing the bridge twice?



Exercise: If possible draw an Euler graph G with $|G|$ even and $||G||$ is odd; otherwise explane why there is no such graph?

Hamilton graph

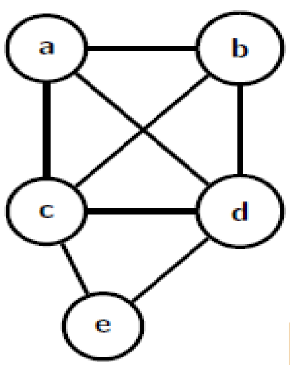
Definition: A path which contains every vertex of a graph G is known as **Hamilton path**.

A closed hamilton path is called **Hamilton cycle or circuit**.

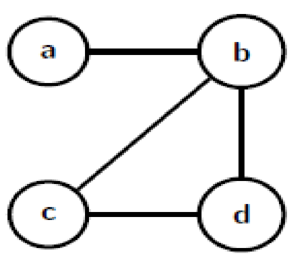
Note: A hamilton cycle is a spanning cycle of a graph.

Definition: A graph is **Hamiltonian** if it contains a hamilton cycle.

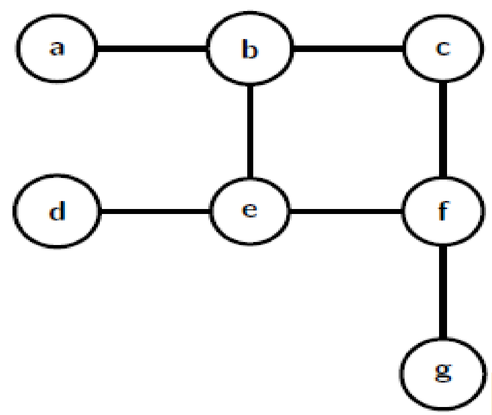
Example: Is the graph below Hamiltonian? find a hamilton path if any?



$[G_1]$



$[G_2]$



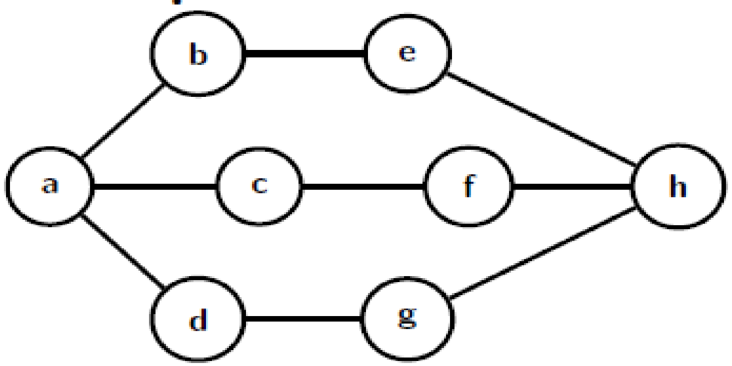
$[G_3]$

Example: When is $K_n, K_{m,n}, C_n$ and Q_n are hamiltonian

Theorem: If G is hamiltonian,then for any $S \subset V_G, S \neq$

$C(G - S) \leq |S|$

Example: Check whether the graph is hamiltonian or nc



$[G]$

Note: If G is hamiltonian , then $C(G - S) \leq |S|$, but if $C(G - S) \leq |S|$ it doesn't implies G is hamiltonian.

A Peterson graph is not hamiltonian, but satisfies the co $C(G - S) \leq |S|$.

Theorem

Ore's Theorem: *Let G be a simple graph of order $V_G \geq 3$ and let $u, v \in V_G$ such that $d_G(u) + d_G(v) \geq |V_G|$, then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Theorem: *Let G be a graph of order ≥ 3 suppose that for all non-adjacent vertices u and v $d_G(u) + d_G(v) \geq |V_G|$, then G is hamiltonian. In particular if $\delta(G) \geq \frac{1}{2}|G|$, then G is hamiltonian.*

Example: Show that K_n is always hamiltonian.

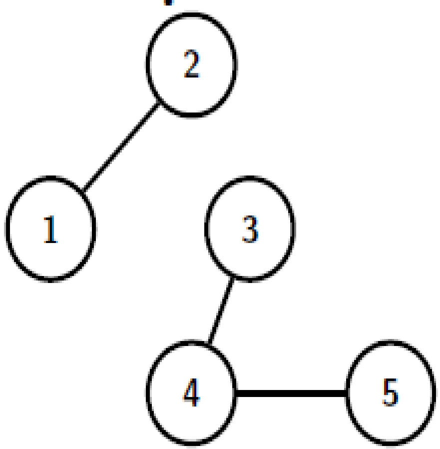
4.6 Trees and Forests

Definition: A graph with no cycle is said to be **acyclic**.

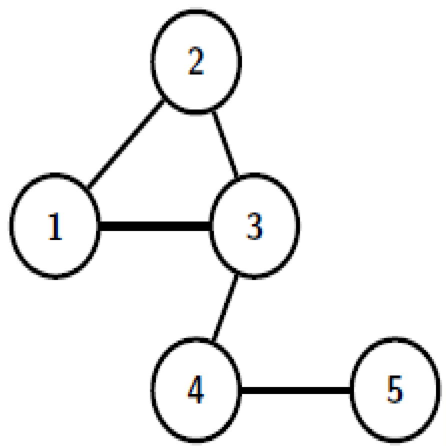
Definition: An acyclic graph is known as a **Forest**.

Definition: A connected graph with no cycle is called a **Tree**.

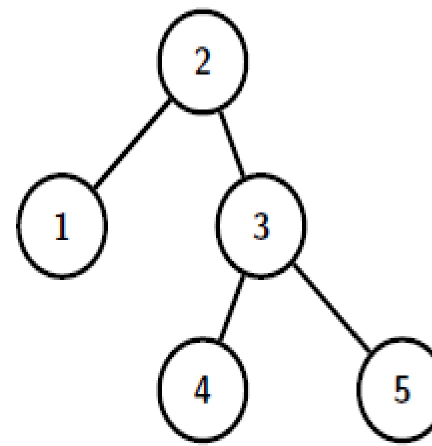
Example which of this is a tree?



[G]



[G]



[G]

Remark: A connected forest is a tree

Theorem

The following statements are equivalent about a graph T of order n (T_n)

- i *T is a tree with n vertices.*
- ii *T contains no cycle and has n-1 edges.*
- iii *T is connected and has n-1 edges.*
- iv *T is connected and every edge is a bridge.*
- v *Any two vertices of T are connected by exactly one path.*
- vi *T contains no cycle, but the addition of any new edge creates exactly one cycle.*

Definition: A **Leaf** is a vertex of degree one in a tree.
Corollary: Every tree other than the trivial graph K_1 has at least two leaves.
Remark: A tree is a maximal acyclic graph (i.e $T + e$ is cyclic) and a minimal connected graph (i.e$T - e$ disconnected)

Rooted and Binary trees

Definition:A rooted tree is atree with a designated vertex called the root. Each edge is implicitly directed away from the root

Note: There are two common ways of drawing a rooted tree that is horizontally or vertically

Rooted tree terminology

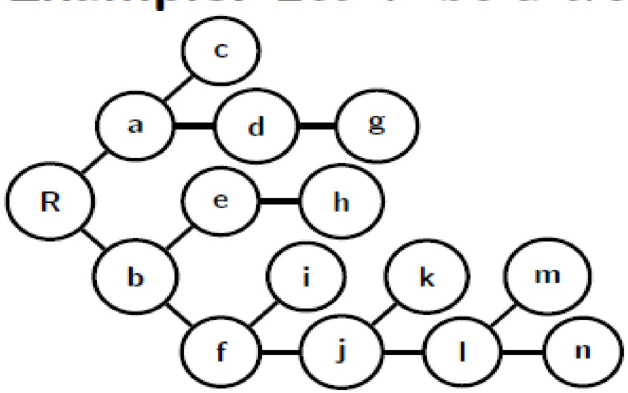
Definition: In a rooted tree the **Depth or level** of a vertex v is its distance from the root i.e $d(R, v)$ where depth of the root is zero $d(R, R) = 0$.

Definition: The**hight** of a rooted tree is the length of the longest path from the root (the greatest depth in the tree).

Definition: If vertex v immededeiately precedes vertex w on the path from the root to w, then v is **Parent** of w and w is **Child** of v.

Definition: vertices having the same parent are called **Siblings**

Definition: A vertex w is called a **Descendent** of v (and v is called **Ancestor** of w) if v is in the unique path from the root to w, if $w \neq v$ then w is a **Proper descendent** of v (and v is called **A proper Ancestor** of w).
Definition: A **Leaf** in a rooted tree is any vertex having no children.
Definition: An **Internal vertex** in a rooted tree is any vertex that has at least one child.
Note: The root is internal vertex, unless the rooted tree is trivial (single vertex)
Example: Let T be a tree given below



[T]

a the depth of j is ....

b The hight of T is ....

c The internal vertices are....

d The leaves are ....

e The siblings are........

f ..... is an ancestor of g and ..... is the descendent of j.

Definition: An **m-ary tree** $(m \geq 2)$ is a rooted tree in which every vertex has at most m-child.

Definition: A**complete m-ary tree** is an m-ary tree in which every internal vertex has exactly m-chidren and all leaves have the same depth.

Binary Tree

Definition: A **Binary tree** is an ordered 2-ary tree in which each child is designated either a left child or a right child.

Note: The complete binary tree of hight h has $2^{h+1} - 1$ vertices. Every binary tree of hight h has at most $2^{h+1} - 1$ vertices. a binary tree of hight 3 has at most 15 vertices

Theorem: The number b_n (the n^{th} catalan number) of different binary trees on n vertices

is given by $b_n = \dfrac{1}{n+1} C(2n, n)$

Example Applying Catalan number to find different binary trees when the vertices are 2,3 and 4?

4.7 Planar Graph

Definition: A graph is said to be embeddable in the plane or **Planar** if it can be drown in the plane so that its edges intersect only at their end (drawing with out crossing)

Definition: A drawing without crossing is a planar embedding of a graph G.

Definition: A particular planar embedding is a plane graph.

Example: which of the graph is planar? $K_4, K_5, K_{3,3}$.

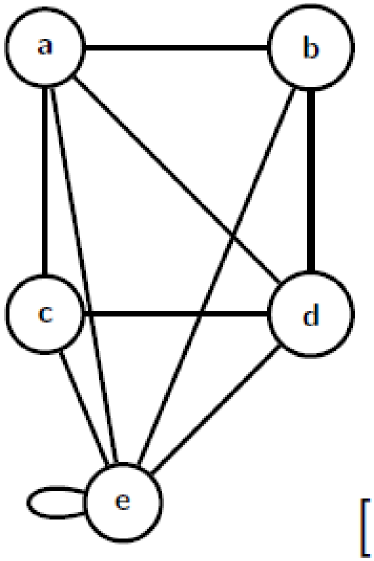
Definition: A plane graph G partitions the rest of the plane in to a number of connected regions; The closures of these regions are called the **Faces of G($F(G)$)** and the number of faces of G denoted by $\phi(G)$

Example: find the faces and number of faces of the planar graph K_4

Note: A finite plane graph has one unbounded face called **Outer face or exterior face**

:- $p, q \in \mathbb{R}^2$ which doesn't lie on any edge are in the same face if and only if $\exists ap, q-$ polygonal curve which doesn't cross any edge.

Example for the graph G below check if it is planar? if planar find the plane embeding of G? and also find $F(G)$ and $\phi(G)$?

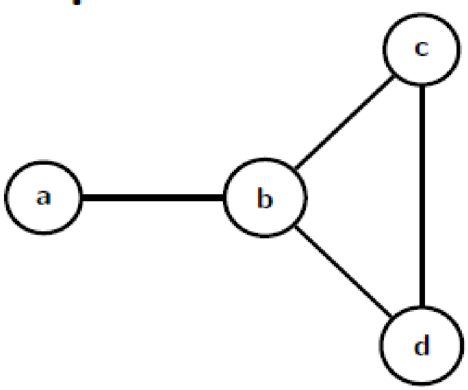


$[G]$

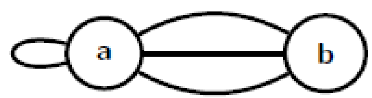
Definition: The **Dual graph (G^*)** of a plane graph G is a plane graph whose vertices corresponds to the face of G i.e $V(G^*) = F(G)$ and if $e \in E_G$ is on the boundary of two faces X and Y then $e^* = xy \in E_{G^*}$ where x and y represent the faces X and Y.

Example: find the duality of A) C_4, C_3 and K_4

B)



$[G_1]$



$[G_2]$

Note: A cut edge in a graph G is in the boundary of the same face leads to a loop in G^*

Theorem

(Euler's Formula) Let G be a connected planar simple graph with $|E| = e$ *and* $|V| = v$. *Let r be the number of regions in a planar representation of G, then* $r = e - v + 2$

Example: suppose that a connected planar simple graph has 8 vertices each of degree 3. In to how many regions is the plane divided by a planar representation of this graph?

Corollary: If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$ then $e \leq 3v - 6$.

Corollary: If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$ and no circuit of length 3 then $e \leq 2v - 4$.

Example: Show that the Kuratowiski's graphs K_5 and $K_{3,3}$ are not planar.

Theorem

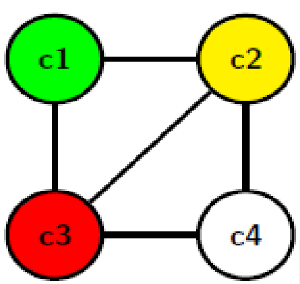
A graph is non-planar if and only if it contains a sub-graph homomorphic to K_5 and $K_{3,3}$.

Note:

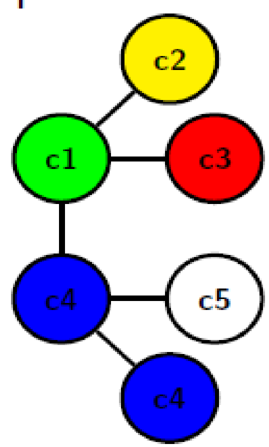
- Any graph isomorphic to any of K_5 and $K_{3,3}$ is non-planar.
- Both K_5 and $K_{3,3}$ are regular graphs.
- K_5 is non-planar with the smallest number of vertices.
- $K_{3,3}$ is non-planar with the smallest number of edges.
- A cut-edge in G counted as 2 in boundary or degree of the region.

4.8 Graph coloring and chromatic polynomial

Definition: A **K-vertex coloring** of a graph G is an assignment of k-colors to the vertex of G.
i.e $f : V_G \to C = \{c_1, c_2, ..., c_k\}$
Definition: A vertex coloring is **Proper** if no two adjacent vertices are colored the same.
Note: A proper k-vertex coloring will be refered as k-coloring.
Coloring loop doesn't make sence so we assume loopless graphs.
K-vertex coloring partitions the vertex set in to k subsets
Example: Which of them is proper coloring

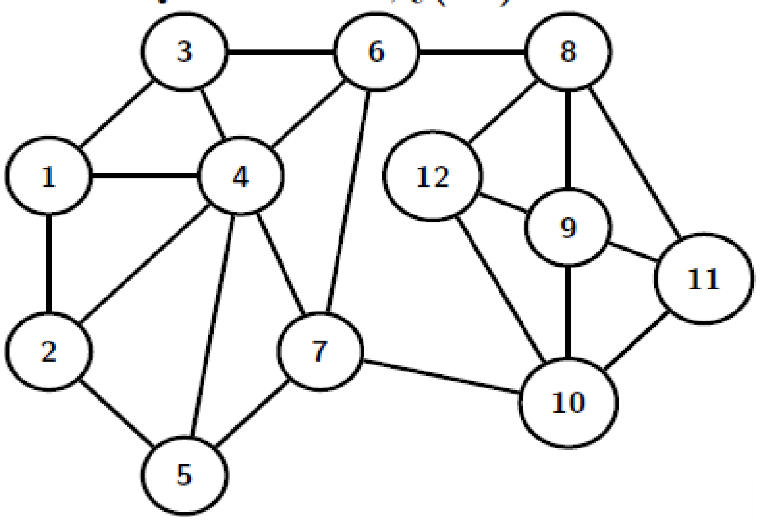


$[G_1]$

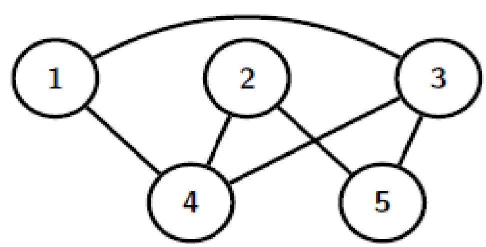


$[G_2]$

Definition: The **Chromatic number of a graph G ($\chi(G)$)** is the minimum number of different colors required for a proper vertex coloring of G.
i.e $\chi(G) = k$ if G is k-colorable, but not (k-1)-colorable.
Definition: if $\chi(G) = k$, then G is called **k-Chromatic**
Example: find $\chi(G)$ if



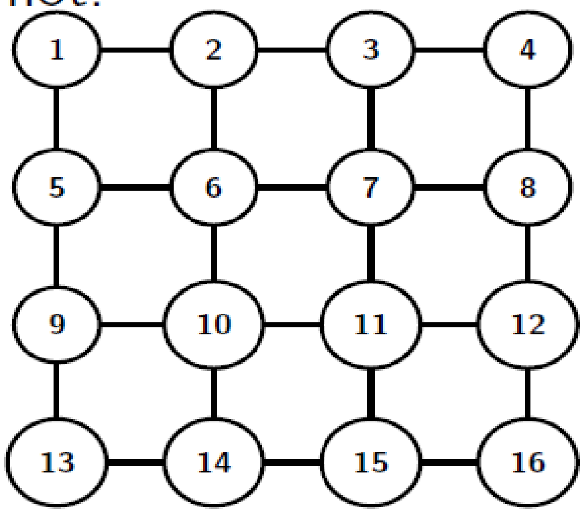
$[G_1]$



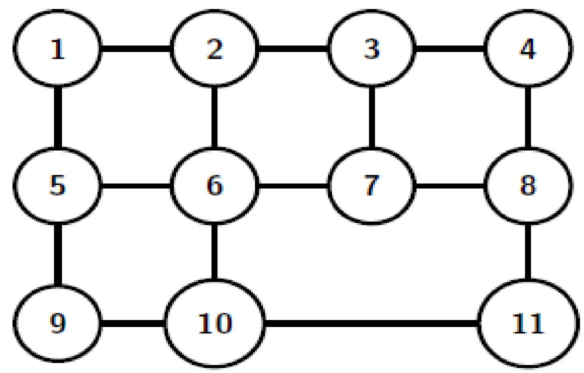
$[G_2]$

Lemma: A simple graph G has $\chi(G) = 1$ if and only if G has no edge or $G = O_n$ and $\chi(G) = 2$ if and only if G is bipartite.

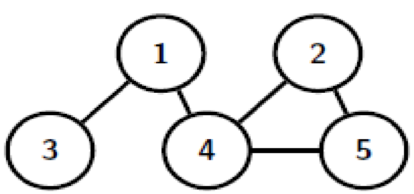
Example: Check Whether the following Graphs are bipartite or not.



$[G_1]$



$[G_2]$



$[G_3]$

Edge -Coloring

Definition: a **k-edge coloring** of a graph G is an assignment of k-colors to the edges of G.

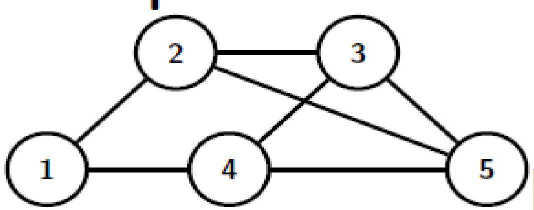
Definition: An edge coloring is **proper** if no to adjacent edges are colored the same.

Definition: G is **k-edge colorable** if it has a proper k-edge coloring.

Definition: The value of k for which a loopless graph G has a k-edge coloring is the**edge chromatic number of G($\chi^{'}(G)$)**.

Note: Edge coloring partitions the edge sets in to matchings.

Example: Find the matchings in the graph below?



[G]

Exercise: Find the Chromatic number($\chi(G)$) and edge chromatic number of G($\chi^{'}(G)$)

A) K_n B) P_n C) C_n D) W_n

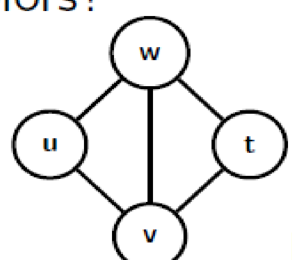
Chromatic polynomials

Definition: For a graph G and $\lambda \in \mathbb{Z}^+$, the number of proper λ-coloring of G is denoted by $P(G, \lambda)$ and is called the **chromatic polynomial** of G.

- Two λ-colorings c and c' of G from the same set $\{1, 2, 3, ..., \lambda\}$ of λ-colors are considered different if $c(v) \neq c_{\prime}(v)$ for some vertex $v \in V_G$
- If $\lambda < \chi(G)$,then $P(G, \lambda) = 0$(which is a convention)

proposition: Let G be a Graph, then $\chi(G) = k$ if and only if k is the smallest positive integer for which $P(G, \lambda) > 0$

Example: For the graph G given below find the number of ways that the vertex of G can be colored from the set which contains five colors?



$[G]$

Theorem

For every $\lambda \in \mathbb{Z}^+$
$P(K_n, \lambda) = \lambda(\lambda - 1)(\lambda - 2)...(\lambda - n + 1) = \lambda^{(n)} = \frac{\lambda!}{(\lambda - n)!}$

Theorem

Let G be a graph containing non-adjacent vertices u and v. and let H be the graph obtained from G by identifying(contracting) u and v, then $P(G, \lambda) = P(G + uv, \lambda) + P(H, \lambda)$

Theorem

Let G be a graph containing adjacent vertices u and v. and let F be the graph obtained from G by identifying(contracting) u and v, then $P(G, \lambda) = P(G - uv, \lambda) - P(F, \lambda)$

Theorem

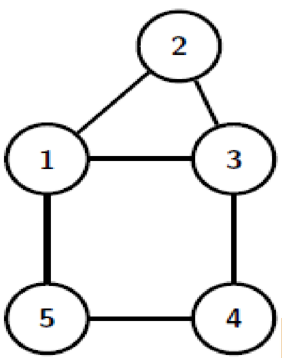
If T is a tree of order $n \geq 1$, then $P(T, \lambda) = \lambda(\lambda - 1)^{n-1}$

Theorem

The chromatic polynomial $P(G, \lambda)$ of a graph G is a polynomial in λ. $P(T, \lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + ... + c_n$

Example: Find the chromatic polynomial of

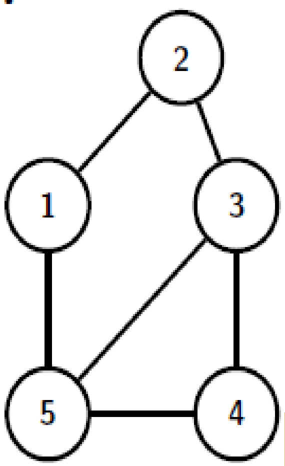
A)



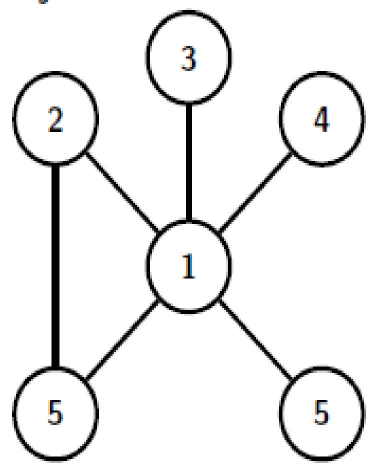
$[G_1]$

B) $G_2 = C_4$

Example: Find the chromatic polynomial for G as a polynomial



[G_1]



[G_2]

$G_3 = C_6$