Module
12

Yield Line Analysis for Slabs

Lesson 31

Nodal Forces and Two-way Slabs

Instructional Objectives:

At the end of this lesson, the student should be able to:

- derive the expression for determining the work done by bending and twisting moments when the yield lines are at angles with the directions of reinforcing bars,
- state the need for considering the nodal forces and to estimate their values when one yield line meets another yield line or a free edge,
- to select the possible yield pattern of a two-way slab supported at four sides either by simple supports or fixed supports,
- to finalise the yield patterns and to evaluate the collapse loads of two-way slabs, either simply supported or clamped at four sides,
- apply the theory in solving numerical problems of slabs to finalise the yield pattern and to determine the collapse load employing (i) the method of segmental equilibrium and (ii) the method of virtual work.

12.31.1 Introduction

Lesson 30 introduces the yield line analysis, which is an upper bound method of analysis for slabs. The different rules for predicting the yield lines are stated. The two methods i.e., (i) method of segmental equilibrium and (ii) method of virtual work are explained. Applications of both the methods are illustrated through numerical problems of one-way slabs – either simply supported or continuous.

This lesson presents the derivations of the expressions for determining bending and torsional moments when yield lines are at angles with the directions of reinforcement. The need for the nodal forces and their determinations are explained when yield line meets another yield line or the free edge. Thereafter, different possible yield patterns of two-way slabs are explained. Numerical illustrative problems of two-way slabs with or without nodal forces are illustrated.

12.31.2 Work Done by Yield Line Moments

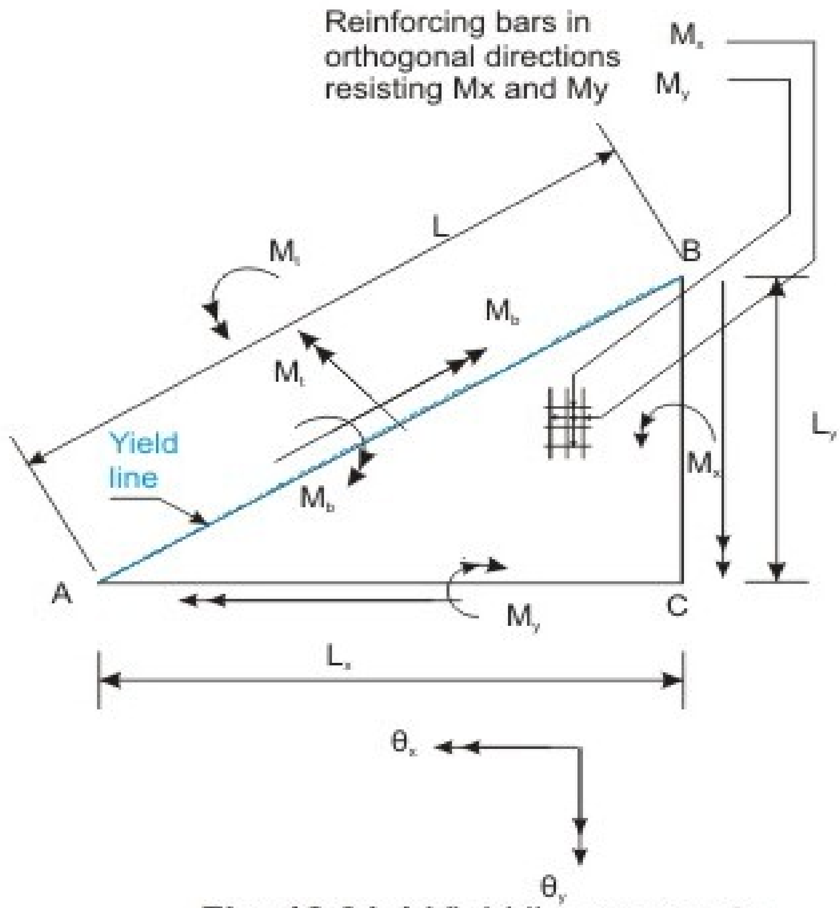

Fig. 12.31.1:Yield line moments

Normally, the reinforcing bars are placed in two mutually perpendicular directions parallel to the sides of rectangular and square slabs. However, the yield lines may be at an angle with the direction of reinforcing bars as shown in Fig. 12.31.1, in which the yield line AB of length L has bending moment M_b and twisting moment M_t per unit length of the yield line. The slab segment is undergoing rigid body rotation whose components are θ_x and θ_y. The horizontal and vertical projections of the yield line are having moment capacities of M_x and M_y per unit length, respectively. All moments and rotations are shown using the right hand thumb rule. The following expression is derived for obtaining the absolute values of the work done by M_b and M_t on the yield line AB.

With reference to Fig. 12.31.1, the total work done by the bending and twisting moments M_b and M_t is,

$$W = M_b\, L\, (\theta_x \cos\theta + \theta_y \sin\theta) + M_t\, L\, (-\,\theta_x \sin\theta + \theta_y \cos\theta)$$

(12.19)

Equating the work done by bending and twisting moments M_b and M_t along the yield line AB with the respective work done by their components along the projections of the yield line, we have the following two equations:

$M_b\, L = M_x\, L \sin\theta \sin\theta + M_y\, L \cos\theta \cos\theta$,   which gives

$M_b = M_x \sin^2\theta + M_y \cos^2\theta$
(12.20)

and

$M_t\, L = M_x \sin\theta \cos\theta - M_y\, L \cos\theta \sin\theta$,   which gives

$M_t = (M_x - M_y) \sin\theta \cos\theta$
(12.21)

Substituting the expressions of M_b and M_t from Eqs.12.20 and 12.21 in Eq.12.19, we have, $W = (M_x \sin^2\theta + M_y \cos^2\theta)\,(\theta_x\, L \cos\theta + \theta_y\, L \sin\theta) + (M_x - M_y) \sin\theta \cos\theta\, (-\theta x\, L \sin\theta + \theta y\, L \cos\theta)$, which gives

$W = M_x\, L \sin\theta\, \theta_y + M_y\, L \cos\theta\, \theta_x$

or

$W = M_x\, L_y\, \theta_y + M_y\, L_x\, \theta_x$
(12.22)

The two terms on the right hand side of Eq. 12.22 give the external work done by the moments M_xL_y and M_yL_x acting on the horizontal and vertical projections of the yield line.

It is thus seen that the expression of Eq.12.19 involving bending and twisting moments M_b and M_t may be replaced by M_x and M_y of Eq.12.22 to get the same work done by Eq. 12.22 as that of Eq. 12.19.

12.31.3 Special Conditions at Edges and Corners

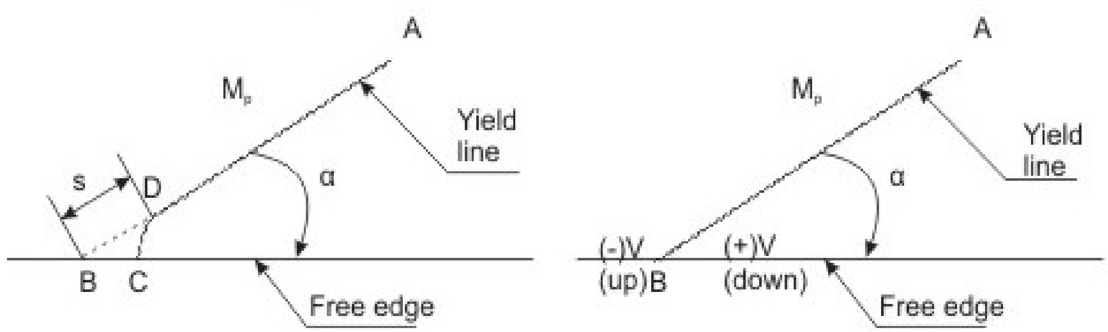

Fig. 12.31.2(a): Actual yield line   Fig. 12.31.2(b): Simplified yield line

Fig. 12.31.2: Actual and simplified yield lines

Figures 12.30.5d, e and f of Lesson 30 present yield patterns of slabs when positive yield lines intersect the free edges at angles. In actual case, however, both bending and twisting moments are zero at the free or simply supported edges. The directions of the principal stresses are, therefore, parallel and perpendicular to the respective edge. Accordingly, the yield lines should enter the edge perpendicular to it, which is confirmed by experimental tests also. In Fig. 12.31.2a, it is shown that the yield line ADC normally turns quite close to the edge, say at D, which is at a distance s. The yield line is approximated by extending it in a straight line AB to the edge introducing a pair of concentrated shear force, (+) V and (-) V, as shown in Fig. 12.31.2b. These shear forces, which are parallel, equal and opposite forces for the equilibrium, are designated as nodal forces acting upward at an obtuse corner, marked by (-)ve sign on the left of the yield line AB and acting downward at an acute corner, marked by (+)ve sign on the right of the yield line AB, as shown in Fig. 12.31.2b.

They are, in fact, the static equivalent of twisting moments and shear forces near the edge. We now establish the required expression for determining the magnitude of these nodal forces V.

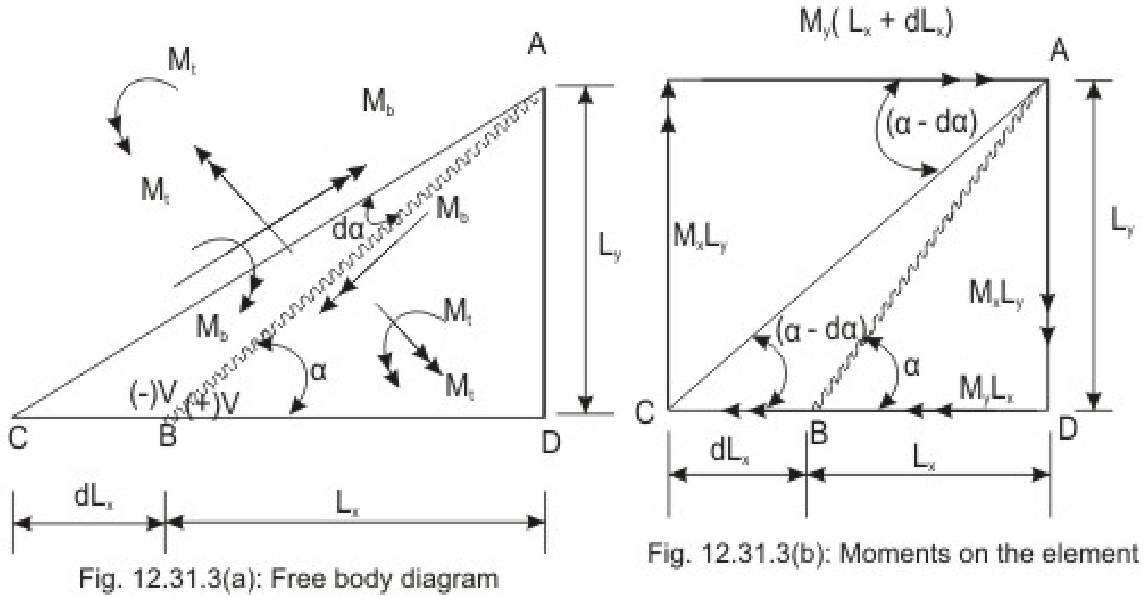

Fig. 12.31.3(a): Free body diagram

Fig. 12.31.3(b): Moments on the element

Fig. 12.31.3: Free body diagram of slab element

Figure 12.31.3a shows an element ACB of a slab where the yield line AB is making an angle α with the free edge CB. The angle between the yield line AB and the element side AC (i.e., angle CAB) is $d\alpha$. The bending and twisting moments M_b and M_t on the yield lines AB and AC are shown in Fig. 12.31.3a, neglecting their differential increments. The free body diagram is shown in Fig. 12.31.3b. The distances BD, AD and BC are L_x, L_y and dL_x, respectively. Taking moments about the line AC and equating it to zero, we get M_y (L_x +dL_x) cos (α -

$d\alpha) – M_y\, L_x \cos(\alpha - d\alpha) + M_x\, L_y \sin(\alpha - d\alpha)\; M_x\, L_y \sin(\alpha - d\alpha) – V\, dL_x \sin(\alpha - d\alpha) = 0$

or $\quad V \sin(\alpha - d\alpha) = M_y \cos(\alpha - d\alpha)$

or $\quad V(\sin\alpha \cos d\alpha - \cos\alpha \sin d\alpha) = M_y(\cos\alpha \cos d\alpha + \sin\alpha \sin d\alpha)$

For small values of $d\alpha$, $\sin d\alpha = 0$ and $\cos d\alpha = 1$. So, we have $V \sin\alpha = M_y \cos\alpha$

or $$V = M_y \cot\alpha \quad (12.23)$$

Equation 12.23 gives the magnitude of the nodal force V where α is measured anticlockwise. When α = 90 degree, V = 0, i.e., the nodal force is zero if the yield line intersects the free edge at an angle of 90 degree.

The nodal force V shall be used in the method of segmental equilibrium as we consider the equilibrium of individual segment. In the method of virtual work, however, the work done is determined for the entire slab involving all the elements, when the total work done by the positive and negative nodal forces is zero. Hence, it is not needed to consider the nodal force in the method of virtual work.

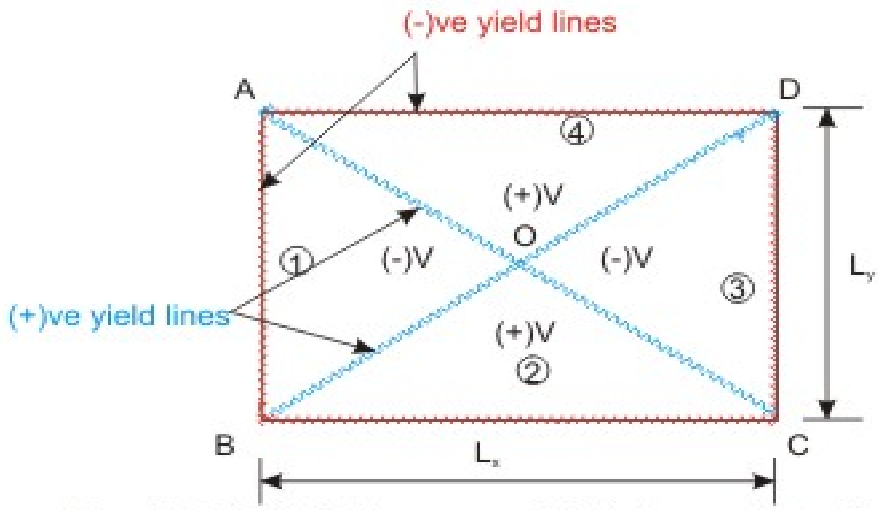

Fig. 12.31.4(a): Two-way slab (clamped at all edges) - yield pattern 1

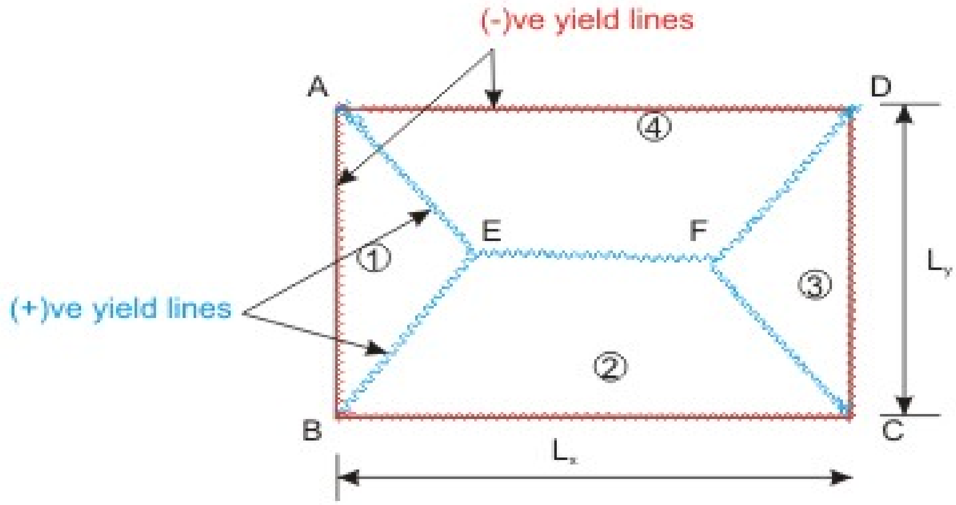

Fig. 12.31.4(b): Two-way slab (clamped at all edges) - yield pattern 2

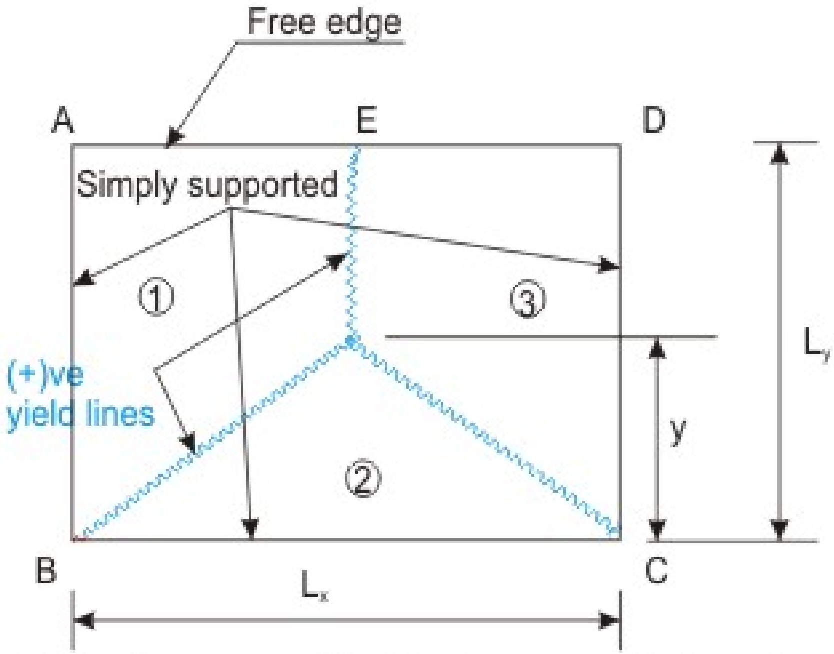

Fig. 12.31.4(c): Two-way slab (simply supported on three sides and free on one side) - yield pattern 1

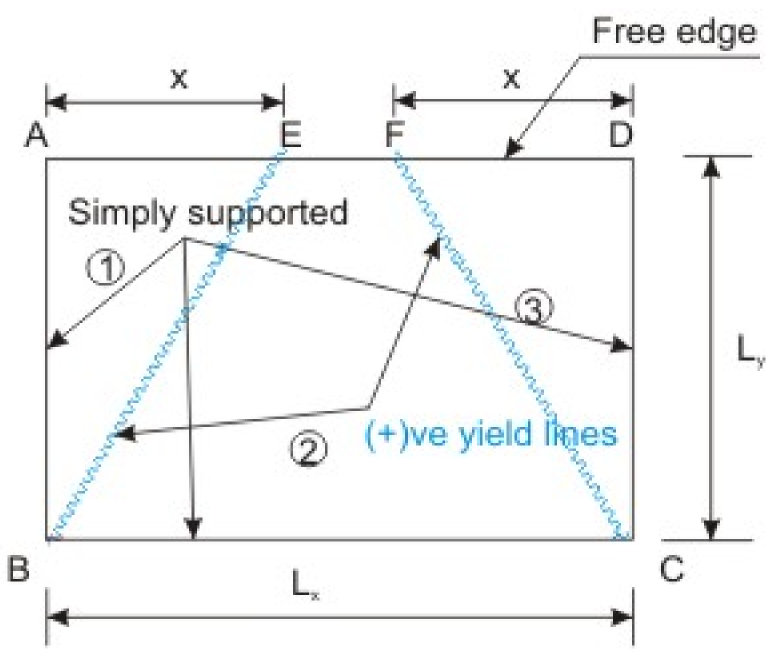

Fig. 12.31.4(d): Two-way slab (simply supported on three sides and free on one side) - yield pattern 2

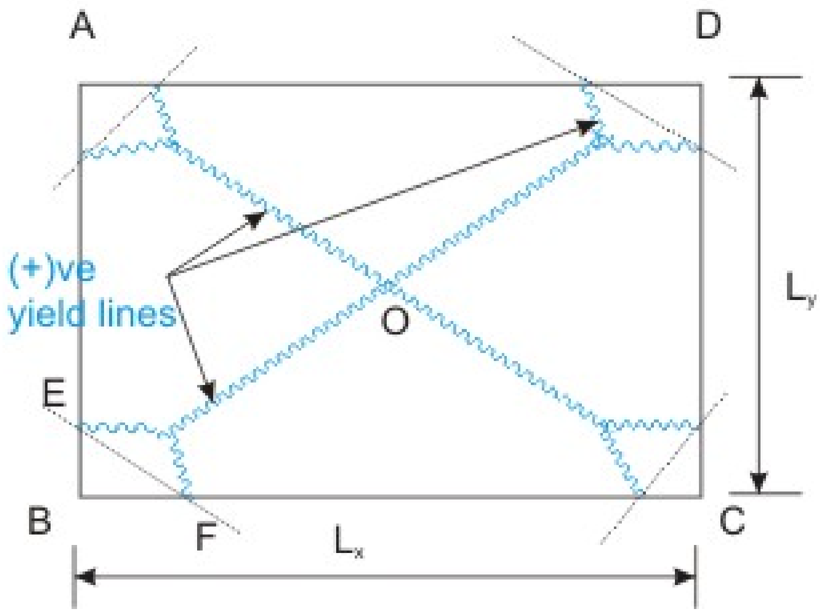

Fig. 12.31.4(e): Two-way slab (simply supported at all edge) - alternative yield pattern

Figures 12.31.4a and b present two typical yield patterns of two-way slabs clamped at four sides and Fig. 12.31.4c and d show two typical yield patterns of a slab simply supported on three sides and free at one side. In all the yield patterns, it is assumed that the yield lines enter the corners between the two intersecting sides. It may not be the case always. Sometimes, yield lines fork before they reach the corner as shown in Fig. 12.31.4e and thus form corner lever. The triangular element EFB in Fig. 12.31.4e will pivot about the axis EF and lift off the supports if the corner is not held down. It has been observed that such yield patterns with corner levers are more critical than those without them. However, these patterns are generally neglected. It should be mentioned that the introduction of corner levers makes the analysis more complicated and does not produce much error by neglecting them. This is illustrated in Lesson 32.

12.31.4 Two-way Slabs of Yield Pattern 1 of Figure 12.31.4a

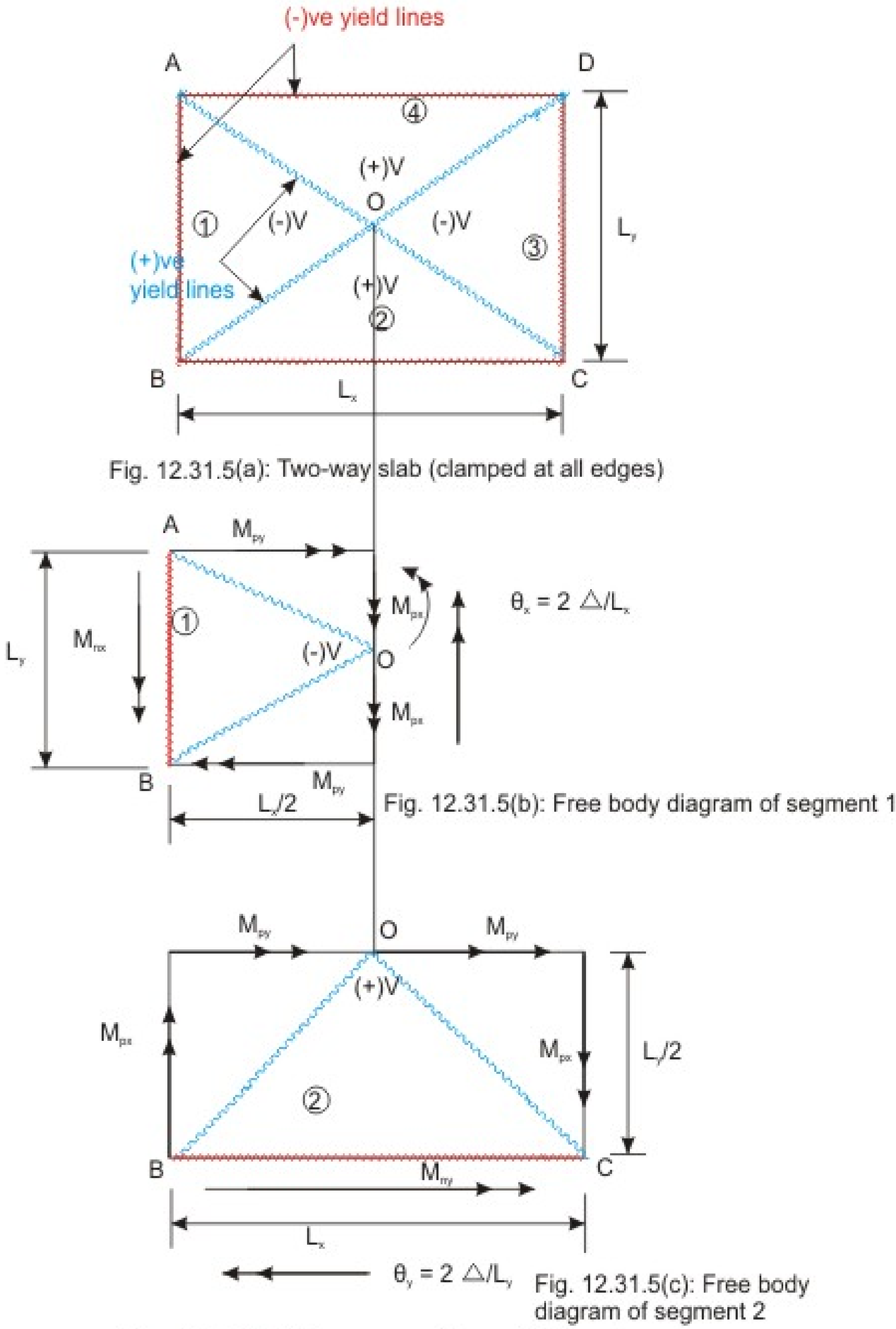

Fig. 12.31.5(a): Two-way slab (clamped at all edges)

Fig. 12.31.5(b): Free body diagram of segment 1

Fig. 12.31.5(c): Free body diagram of segment 2

Fig. 12.31.5: Two-way slab - yield pattern 1

We consider the two-way slab of Fig. 12.31.4a subjected to uniformly distributed collapse load of intensity w kN/m^2. Yield pattern 1 divides the slab into four segments marked by 1, 2, 3 and 4 in Fig. 12.31.5a, where the positive and negative yield lines are shown along with the positive and negative nodal forces. The slab undergoes a displacement Δ at the center point O. The free body diagrams of segments 1 and 2 are shown in Figs. 12.31.5b and c, respectively. The positive and negative moments along x and y directions are designated by M_{px}, M_{py}, M_{nx} and M_{ny} in these two figures. We are employing both (A) the method of segmental equilibrium and (B) the method of virtual work to determine the magnitude of the collapse load w of the slab.

(A) Method of segmental equilibrium

At the equilibrium, the moment of all the forces and moments of segment 1 about the left edge AB is zero. This gives (Fig. 12.31.5b): $(1/2)\ (L_x/2)\ (L_y)\ w(L_x/6) - V(L_x/2) - M_{nx} L_y - M_{px} L_y = 0$

or $\quad wL_xL_y - 12\,V = 24\,(M_{px} + M_{nx})\,(L_y / L_x)$ (12.24)

Similarly, at the equilibrium, the moment of all forces and moments of segment 2 about the bottom edge BC is zero. This gives (Fig. 12.31.5c): $(1/2)\ L_x\ (L_y/2)\ w\ (L_y/6) + V\ (L_y/2) - M_{ny} L_x - M_{py} L_x = 0$

or, $\quad wL_xL_y + 12\,V = 24\,(M_{py} + M_{ny})\,(L_x / L_y)$ (12.25)

Eliminating V from Eqs. 12.24 and 12.25 by adding the two equations, we have

$$2w\,L_x\,L_y = 24\,\{(M_{px} + M_{nx})(L_y / L_x) + (M_{py} + M_{ny})\,(L_x / L_y)\}$$

or $\quad w = 12\,\{(M_{px} + M_{nx}) / L_x^2 + (M_{py} + M_{ny}) / L_y^2\}$ (12.26)

The collapse load w is determined from Eq. 12.26 from known values of M_{px}, M_{nx} M_{py} and M_{ny}.

(B) Method of virtual work

The total external work done by the load in causing the segments to undergo deflection and the total internal work done by the moments in rotating all the four segments are computed. As mentioned in sec. 12.31.3, the effect of all the nodal forces is zero. Accordingly, the total external work done (TEW) is,

$$TEW = 4\,(1/2)\,L_x\,(L_y/2)\,w\,(\Delta/3) = wL_x\,L_y\,(\Delta/3) \quad (12.27)$$

The total internal work done (TIW) is,

$$TIW = 2\,(M_{px} + M_{nx})\,L_y\,\theta_x + 2\,(M_{py} + M_{ny})\,L_x\,\theta_y$$
$$= 4\Delta\,\{(M_{px} + M_{nx})\,(L_y/L_x) + (M_{py} + M_{ny})\,(L_x/L_y)\}$$
(12.28)

Equating the two works from Eqs. 12.27 and 12.28

$$w\,L_x\,L_y\,(\Delta/3) = 4\Delta\{(M_{px} + M_{nx})\,(L_y/L_x) + (M_{py} + M_{ny})\,(L_x/L_y)\}$$

or $$w = 12\,\{(M_{px} + M_{nx})/L_x^2 + (M_{py}+M_{ny})/L_y^2\}$$
(12.26)

The above equation is the same as obtained by the method of segmental equilibrium to get the value of the collapse load w.

We now consider the two cases below, first the square and simply supported at all edges and secondly the square and clamped at all edges.

Case (i) Square and simply supported slab

For square and simply supported slab, $L_x = L_y = L$, $M_{nx} = M_{ny} = 0$ and let us assume $M_{px} = M_{py} = M_p$. Using the conditions mentioned above in Eq. 12.26, we get

$$w = 24\,M_p/L^2$$
(12.29)

or, $$M_p = w\,L^2/24$$
(12.30)

Case (ii) Square and clamped slab

For square and clamped slab, let us assume $M_{px} = M_{py} = M_{nx} = M_{ny} = M_p$. Using these conditions in Eq. 12.26, we get

$$w = 48\,M_p/L^2$$
(12.31)

or $$M_p = wL^2/48$$
(12.32)

The values of w and M_p in the two cases above reveal that the factored load intensity of a clamped slab is twice of that of simply supported slab having the same moment carrying capacity M_p. Further, we observe that for the same factored load intensity w, a clamped slab would have half the factored moment of simply supported slab and, therefore, would be economic.

12.31.5 Two-way Slabs of Yield Pattern 2 of Figure 12.31.4b

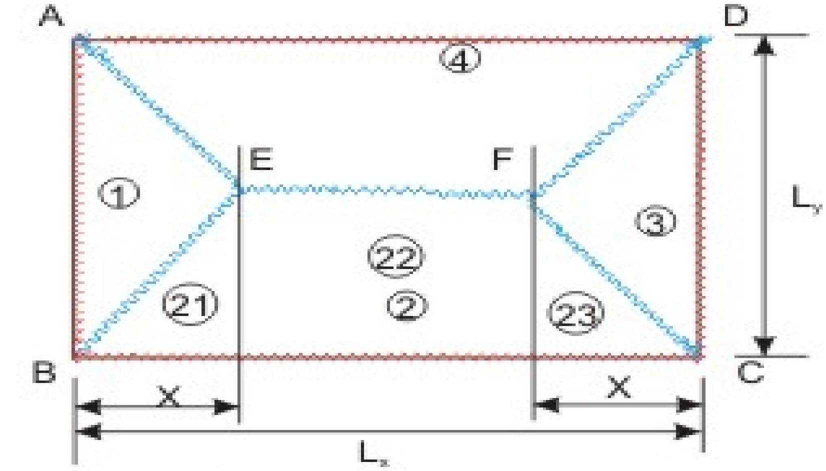

Fig. 12.31.6(a): Two-way slab (clamped at all edges)

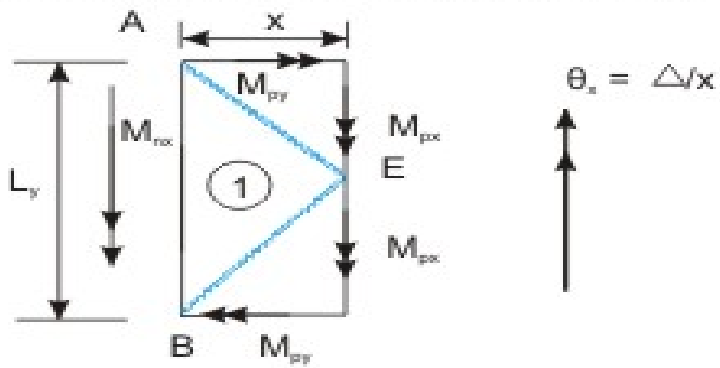

Fig. 12.31.6(b): Free body diagram of segment 1

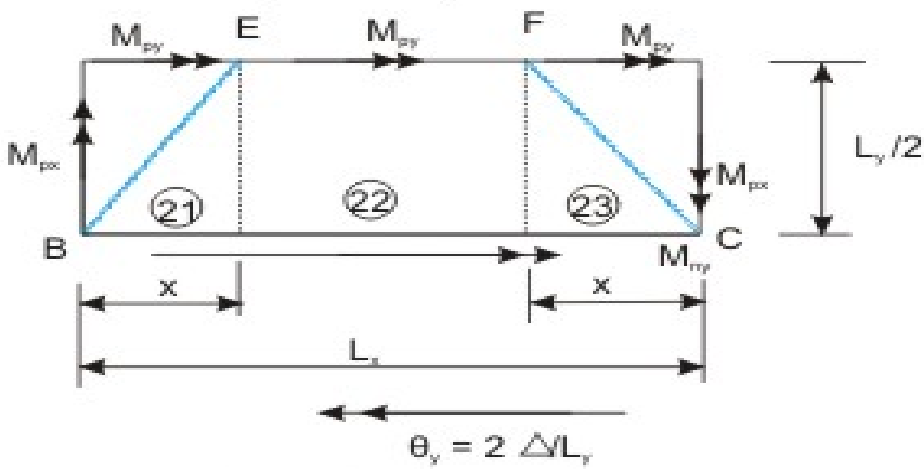

Fig. 12.31.6(c): Freebody diagram of segment 2

Fig. 12.31.6: Two-way slab - yield pattern 2

We now consider the two-way slab having the yield pattern 2 as shown in Fig. 12.31.4b and subjected to uniformly distributed collapse load of intensity *w* kN/m^2. Yield pattern 2 divides the slab into four segments marked by 1, 2, 3 and 4 in Fig. 12.31.6a, where the positive and negative yield lines are shown. The slab undergoes a displacement of Δ along the yield line EF. The free body diagrams of segments 1 and 2 are shown in Figs. 12.31.6b and c, respectively. The positive and negative moments along *x* and *y* directions are designated by M_{px}, M_{py}, M_{nx} and M_{ny} in these two figures. We are employing both (A) method of segmental equilibrium and (B) method of virtual work to determine the distance *x* and the magnitude of the collapse load *w* of the slab.

(A) Method of segmental equilibrium

At the equilibrium, the moment of all the forces and moments of segment 1 about the left edge AB is zero. This gives (Fig. 12.31.6b): $(L_y/2)\, x\, w\, (x/3) - (M_{px} + M_{nx}) = 0$

or $\quad w = 6\,(M_{px} + M_{nx}) / x^2 \qquad (12.33)$

Similarly, at the equilibrium, the moment of all forces and moments of segment 2 about the bottom edge BC is zero. This gives (Fig. 12.31.6c): $(1/2)\, x\, (L_y/2)\, (L_y/6)\, 2w + w(L_x - 2x)\, (L_y/2)\, (L_y/4) - (M_{py} + M_{ny})L_x = 0$

or, $\quad w = 24\,(M_{py} + M_{ny})\, L_x /\{2x\, L_y^2 + 3L_y^2\, (L_y - 2x)\} \qquad (12.34)$

Equating the two expressions of *w* from Eqs. 12.33 and 12.34, we have

$$\frac{6(M_{px} + M_{nx})}{x^2} = \frac{24(M_{py} + M_{ny})L_x}{2xL_y^{\,2} + 3L_y^{\,2}(Lx - 2x)}$$

or, $\quad 4\,(M_{py} + M_{ny})\, L_x x^2 + 4\,(M_{px} + M_{nx})\, L_y^2\, x - 3\,(M_{px} + M_{nx})\, L_x\, L_y^2 = 0 \qquad (12.35)$

Equation 12.35 is used to determine the values of *x* when M_{px}, M_{py}, M_{nx} and M_{ny} are known. Thereafter, Eq. 12.34 is used to determine the magnitude of the collapse load *w*.

(B) Method of virtual work

The total external work done by the load in causing deflection of the four segments of Fig. 12.31.6a, *TEW* is:

$TEW = 2W_1 + 2\,(W_{21} + W_{22} + W_{23})$, where W_1 is the work done for the segment 1 and the work done of segment 2 is subdivided into three parts as W_{21}, W_{22} and W_{23}. Noting that $W_{21} = W_{23}$, we get the TEW as,

$TEW = 2w\{(1/2)x\,L_y\,(\Delta/3)\} + 2w\,\{(1/2)x\,(L_y/2)\,(\Delta/3) + 2w\,\{(L_x\text{-}2x)\,(L_y/2)\,(\Delta/2)\}$

$$= (w\Delta/6)\,(3\,L_x\,L_y - 2x\,L_y) \tag{12.36}$$

The total internal work done by the yield moments (TIW) is,

$$TIW = 2\,(M_{px} + M_{nx})\,\theta x\,L_y + 2\,(M_{py} + M_{ny})\,\theta y\,L_x$$
$$= 2\,(M_{px} + M_{nx})\,(\Delta/x)\,L_y + 2\,(M_{py} + M_{ny})\,(2\Delta/L_y)\,L_x \tag{12.37}$$

Equating the two works from Eqs. 12.36 and 12.37, we have,

$$(w\,\Delta/6)\,(3L_x\,L_y - 2x\,L_y) = 2\,(M_{px} + M_{nx})\,(\Delta/x) + 2\,(M_{py} + M_{ny})\,(2\Delta/L_y)\,L_x$$

or $$w = \frac{12(M_{px} + M_{nx})L_y^{\,2} + 24\,(M_{py} + M_{ny})xL_x}{L_y^{\,2}(3xL_x - 2x^2)} \tag{12.38}$$

To get the minimum collapse load, we put $dw/dx = 0$, which gives

$L_y^2\,(3x\,L_x - 2x^2)\,\{24\,L_x\,(M_{py} + M_{ny})\} - 12\,[(M_{px} + M_{nx})\,L_y^2 + 24\,x\,L_x\,(M_{py} + M_{ny})]\,(3L_x - 4x)\,L_y^2 = 0$

or $$4\,(M_{py} + M_{ny})\,L_x\,x^2 + 4\,(M_{px} + M_{nx})\,L_y^2\,x - 3\,(M_{px} + M_{nx})\,L_xL_y^2 = 0 \tag{12.35}$$

The above equation is the same as obtained by the method of segmental equilibrium to determine the values of x. Thereafter, Eq. 12.38 is used to get the value of w, the collapse load.

We observe two points from secs. 12.31.4 and 12.31.5. They are as follows:

(i) Only one equation (Eq. 12.26) is needed for determining the value of the collapse load w for the yield pattern 1 of Fig. 12.31.5a. On the other hand, two equations are need for the yield pattern 2. This is because the yield pattern 1 is already determined but the yield pattern 2 is determined only after finding the values of x. Therefore, two equations are needed for determining the two unknowns x and w in the case of yield pattern 2.

(ii) The one equation needed for the yield pattern 1 is the same equation (Eq. 12.26) by the two methods, viz. method of segmental equilibrium and method of virtual work. Out of the two equations needed for the yield pattern 2, only one equation (Eq. 12.35) is the same by both the methods. After getting the values of *x* from Eq. 12.35, the values of collapse load *w* is determined from Eq. 12.33 by the method of segmental equilibrium and Eq. 12.38 by the method of virtual work.

Let us now take up Eq. 12.35, which is used to determine the values of *x* in the case of yield pattern 2. Substituting the value of $x = L_x/2$ in Eq. 12.35, we get

$$4\,(M_{py} + M_{ny})\,(L_x^3)(1/4) + 4\,(M_{px} + M_{nx})\,L_y^2\,(L_x/2) - 3\,(M_{px} + M_{nx})\,L_x\,L_y^2 = 0$$

or, $$\frac{M_{px} + M_{nx}}{M_{py} + M_{ny}} = \frac{Lx^2}{Ly^2} \qquad (12.39)$$

Noting that, $x = L/2$ gives the yield pattern 1 of Fig.12.31.5a, Eq.12.39 is used to check if the slab has the yield pattern 1 from the known values of M_{px}, M_{nx}, M_{py}, M_{ny}, L_x and L_y of the slab.

Similarly, it can be shown that for the yield pattern 2, i.e., when $x < L_x/2$, the required condition is:

$$\frac{M_{px} + M_{nx}}{M_{py} + M_{ny}} < \left(\frac{L_x}{L_y}\right)^2 \qquad (12.40)$$

We now take up numerical problems in the next section for the purpose of illustration.

12.31.6 Numerical Problems

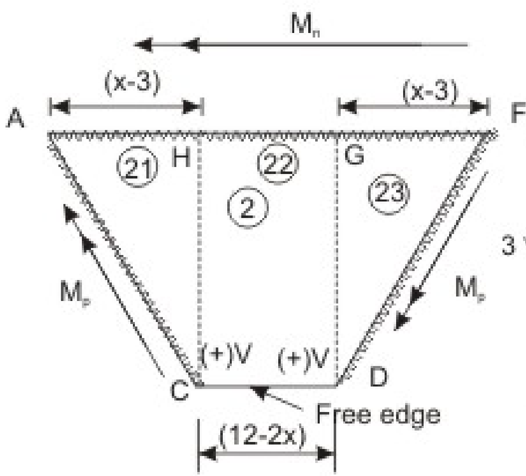

Fig. 12.31.7(c): Free body diagram of segment 2

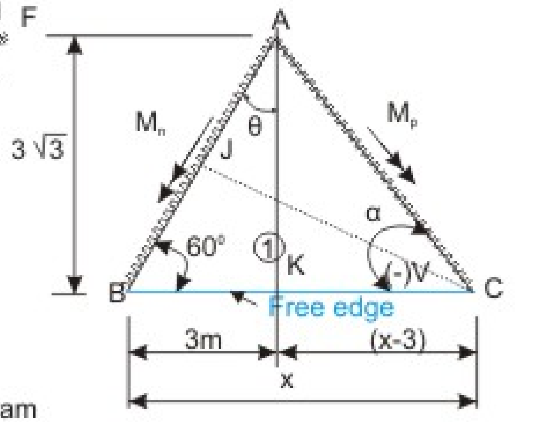

Fig. 12.31.7(b): Free body diagram of segment 1

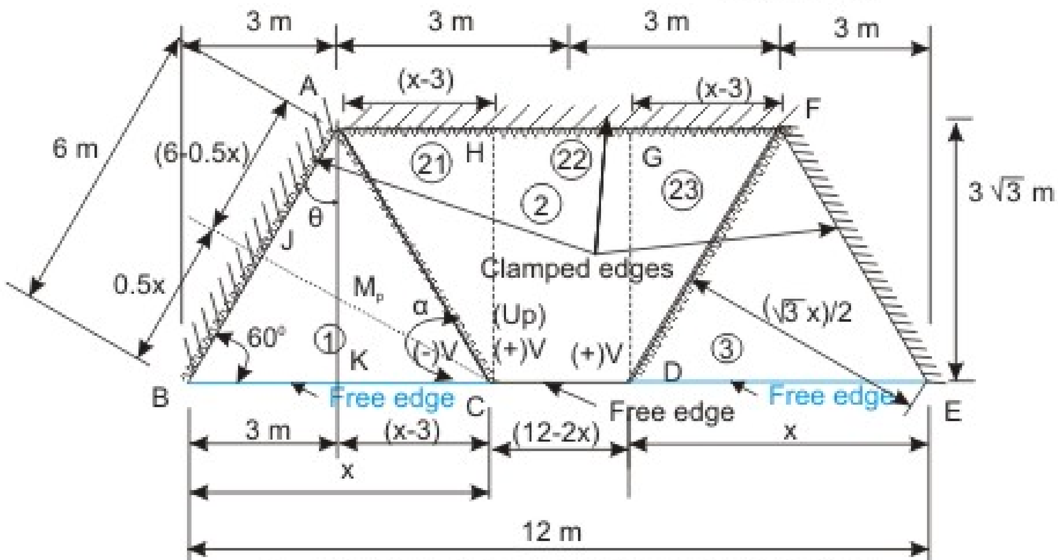

Fig. 12.31.7(a): Problem 1 - Geometric dimensions and yield pattern

Fig. 12.31.7: Problem 1 (Trapezoidal slab)

Problem 1. Determine the yield pattern and collapse load of the trapezoidal slab of Fig.12.31.7a having clamped edges along BA, AF and FE while the edge BE is free. Given that $M_n = M_p = 70$ kNm/m. Employ the method of segmental equilibrium.

Solution 1. Here, in this problem the yield pattern drawn in Fig.12.31.7a shows that there is one additional unknown *x* to finalise the yield pattern. Thus, we have two unknowns *x* and the collapse load *w*.

The yield pattern divides the slab into three segments marked by 1, 2 and 3. Yield lines AC and FD meet the free edge BE. So, we have to consider the nodal forces (+)V and (-)V, as shown on the left and right of C. Since the problem is symmetrical, segments 1 and 3 are identical. We first consider the equilibrium of segment 1 having positive moment M_p along yield line AC and negative moment M_n along yield line AB (Fig. 12.31.7b). The nodal force V at C of segment 1 is negative, i.e., acting downward.

Equation 12.23 gives the magnitude of the nodal force $V = M_n \cot \alpha$, where α is the angle ACB.

Geometric properties:

In triangle ABK, the side AB = $(BK^2 + AK^2)^{1/2}$ = 6 m. The angle ABC = 60^0. Assuming the angle BAK = θ, we have $\sin\theta$ = BK/BA = 0.5 giving $\theta = 30^0$. The line CJ is perpendicular to AB. The angle JCB = 30^0 gives BJ = BC $\cos$ 60 = 0.5x. Therefore, AJ = 6 - 0.5x. The distance CJ = BC $\sin$ 60 =$3^{1/2}$ (x/2). From triangle ACK, we have $\cot\alpha$ = CK/AK = (x-3)/3$(3)^{1/2}$. Area of the triangle ABC = (1/2) (BC) (AK) = (x/2) 3$(3)^{1/2}$. The load of segment 1 is acting at a distance of CJ/3 from the side AB, which is equal to 05x/$(3)^{1/2}$. Taking moments of load of the segment 1, nodal force V and considering M_n and M_p about the edge AB, we have:

M_p (AJ) + M_n (AB) – V (CJ) – w (area of segment ABC) (CJ/3) = 0

Substituting the values of M_p, M_n, V, AJ, AB, CJ and the area of triangle ABC, we have

$$70\,(6 - 0.5x) + 70\,(6) - \{70\,(x - 3) / 3\,(3)^{1/2}\}\,(3)^{1/2}\,(x/2) - (x/2)\,3(3)^{1/2}\,w\,(3)^{1/2}\,(x/2) / 3 = 0$$

$$\text{or} \quad w = 4(2520 - 35x^2) / 9x^2 \qquad \text{(i)}$$

Now, we take up segment 2 for writing the equilibrium equation. The segment is subdivided into three parts marked by 21, 22 and 23. The sub-parts 21 and 23 are identical. The area of the triangle ACH = (1/2) (x - 3) 3 $(3)^{1/2}$ and the load is considered at a distance of CH/3 = $(3)^{1/2}$ m from the edge AF. The area of rectangle CDGH = (12 - 2x) 3$(3)^{1/2}$ and the load is considered at a distance of (3/2) $(3)^{1/2}$ from the edge AF. The equilibrium equation of segment 2 is obtained by taking moments of loads on this segment, two nodal forces of (+) V at C and D about the edge AF and considering M_p along AC and FD and M_n along AF (Fig.12.31.7c). This gives 2M_p (AH) + M_n(AF) + 2V (CH) – 2w(area of sub segment AHC) (HC/3) – w (area of rectangle CDGH) (HC/2) = 0. Substituting the values of M_p, M_n, V, AH, AF, CH and the areas of triangle AHC and rectangle CDGH, we have,

$2\,(70)\,(x - 3) + 70\,(6) + 2\{(70)\,(x\text{-}3) / 3\,(3)^{1/2}\}\,3(3)^{1/2} - 2\,w\,(1/2)\,(x - 3)\,(3)\,(3)^{1/2}\,(3)^{1/2} + w\,[\{6 - 2(x - 3)\}\,3\,(3)^{1/2}\,(3/2)\,(3)^{1/2}\,] = 0$

This gives

$$w = (280\,x - 420) / (135 - 18\,x) \qquad \text{(ii)}$$

Equating the two expressions of w from Eqs. (i) and (ii), we have:

$$4\,(2520 - 35x^2) / 9x^2 = (280x - 420) / (135 - 18x)$$

This gives

$$x^2 + 12x - 90 = 0 \qquad \text{(iii)}$$

Solving we get $x = 5.2249$ m

From Eq. (i) $w = 4\,(2520 - 35x^2) / 9x^2 = 25.465$ kN/m^2
From Eq. (ii) $w = (280x - 420) / (135 - 18x) = 25.4695$ kN/m^2

So, we get the same value of w from Eqs. (i) and (ii). Thus, the yield pattern is finalised when $x = 5.2249$ m and the collapse load of the slab is 25.4695 kN/m^2.

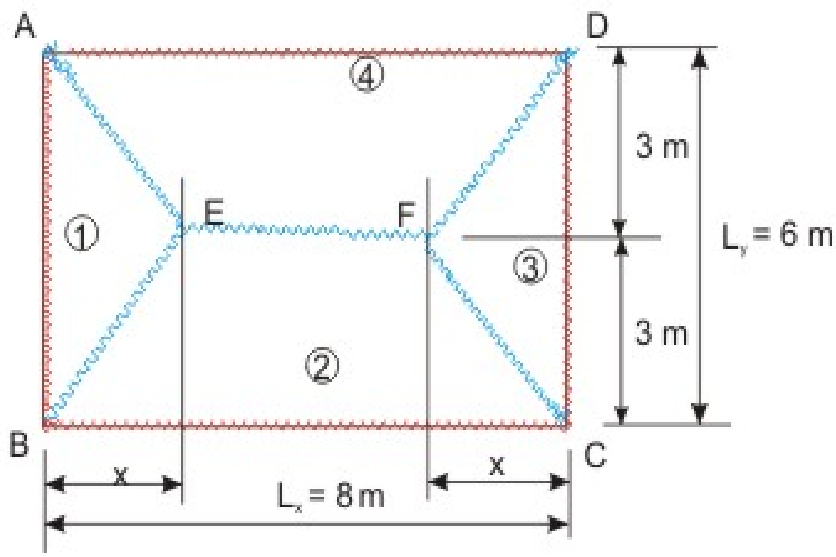

Fig. 12.31.8: Problem 2 (Two-way slab - clamped at all edges)

Problem 2. Determine the yield pattern and the collapse load of the two-way slab of Fig.12.31.8 which is having clamped edges along four sides. Assume that $M_{px} = M_{nx} = 50$ kNm/m and $M_{py} = M_{ny} = 70$ kNm/m.

Solution 2. In this problem, we have to find out whether yield pattern 1 or 2 will be the governing from Eqs.12.39 and 12.40 of sec.12.31.5.

From the given data, we have:

$(M_{px} + M_{nx}) / (M_{py} + M_{ny})$ = 100/140 = 0.714 and L_x^2 / L_y^2 = 64/36 = 1.77. Therefore, yield pattern 2 will govern (see Eq. 12.40).

Thus, we have two unknowns: (i) the value of x for finalising the yield pattern and (ii) the value of the collapse load w. Since, the governing equations of yield pattern 2 are derived in sec.12.31.5, the problem is solved by direct application of the equations.

We have Eq.12.35 to determine the value of x. Using the values of M_{px}, M_{nx}, M_{py}, M_{ny}, L_x and L_y in Eq. 12.35, we have:

$$4\ (M_{py} + M_{ny})\ L_x\ x^2 + 4\ (M_{px} + M_{nx})\ L_y^2\ x - 3\ (M_{px} + M_{nx})\ L_x\ L_y^2 = 0 \quad (12.35)$$

or $\quad 4\ (140)\ (8)\ x^2 + 4\ (100)\ (36)\ x - 3\ (100)\ (8)\ (36) = 0$

or $\quad 14\ x^2 + 45\ x - 270 = 0,$ which gives x = 3.069 m

Now, we use the three equations (Eqs.12.33 and 12.34 by the method of segmental equilibrium and Eq. 12.38 by the method of virtual work). Using the values of M_{px}, M_{py}, M_{nx}, M_{ny}, L_x, L_y and x in those three equations, we determine the value of w for comparing them.

(i) Eq.12.33 (i.e., $w = 6\ (M_{px} + M_{nx})\ x^2$) gives w = 63.69 kN/m^2

(ii) Eq. 12.34 (i.e., $w = \dfrac{24(M_{py} + M_{ny})L_x}{2xL_y^2 + 3L_y^2(L_x - 2x)}$) gives w = 63.69 kN/m^2

(iii) Eq.12.38 (i.e., $w = \dfrac{12(M_{px} + M_{nx})L_y^2 + 24(M_{py} + M_{ny})xL_x}{L_y^2(3xL_x - 2x^2)}$) gives w = 63.69 kN/m^2.

Thus, we observe that the collapse load is the same from the three equations. Usually, they may differ marginally depending on the truncation of the value of x.

12.31.7 Practice Questions and Problems with Answers.

Q.1: Establish the work done by the yield line moments M_b and M_t when the yield line is at an angle with the two orthogonal directions of the reinforcement.

A.1: See sec. 12.31.2

Q.2: Explain the nodal force and derive the expression to determine its value.

A.2: See sec. 12.31.3

Q.3: Draw the possible two yield patterns of a two-way slab clamped at four sides. Derive the equation to find out which one will govern in a particular case given the values of M_{px}, M_{nx}, M_{py}, M_{ny}, L_x and L_y.

A.3: Figures 12.31.4a and b are the two possible yield patterns. For finding the governing yield pattern, the equations (Eqs. 12.39 and 12.40) are established in sec. 12.31.5.

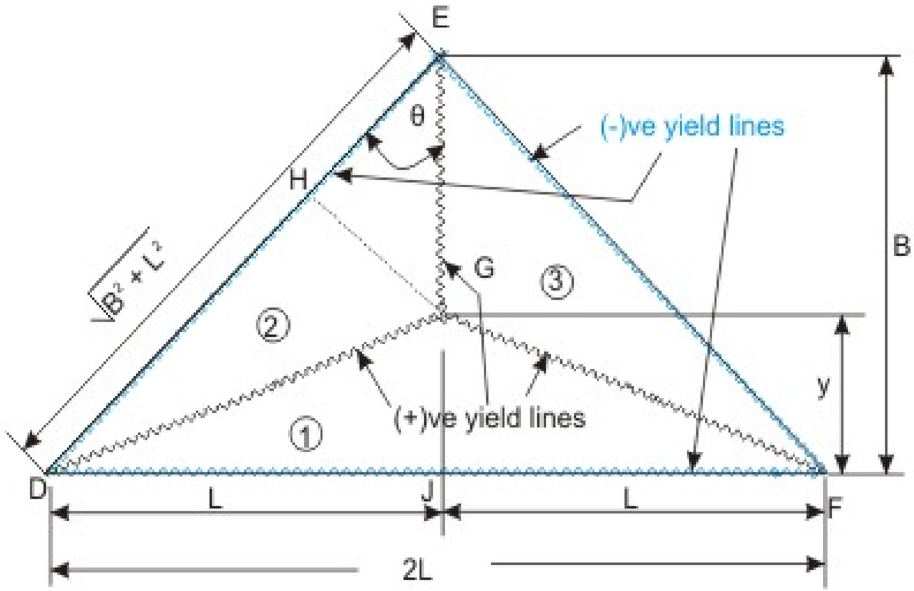

Fig. 12.31.9: Isosceles triangular slab problem of Q.4(a)

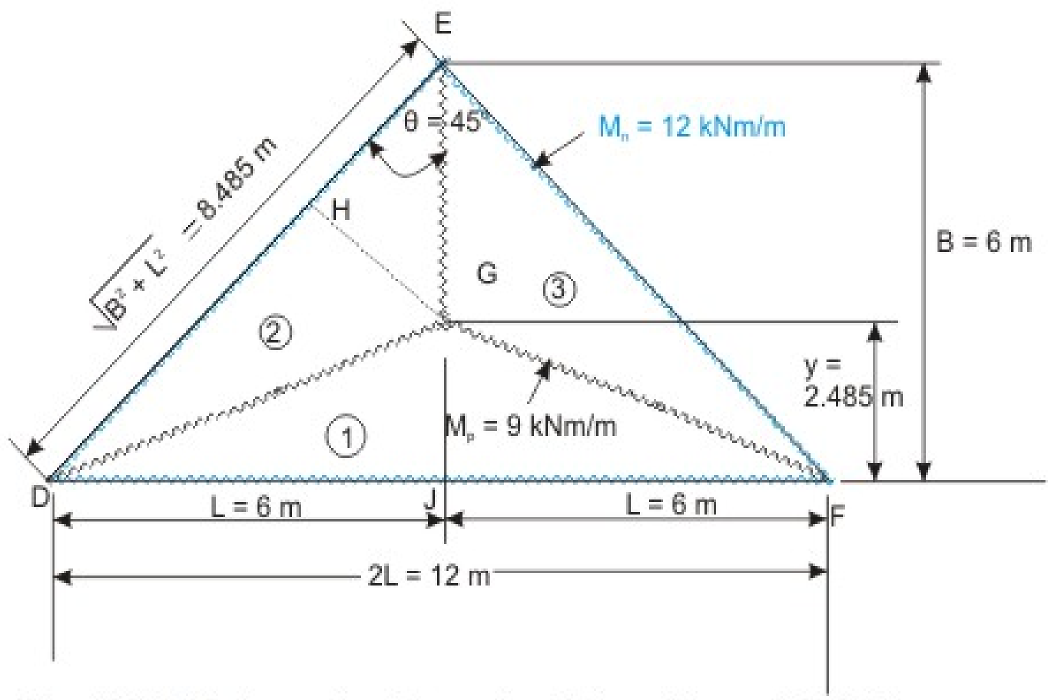

Fig. 12.31.10: Isosceles triangular slab problem of Q.4(b)

Q.4: **(a)** Establish the general equations for determining the yield pattern and uniformly distributed collapse load of an isosceles triangular slab, shown in Fig. 12.31.9 having the positive and negative moment capacity of M_p and M_n, respectively. Use the method of virtual work.

(b) Determine the specific yield pattern and uniformly distributed collapse load of such a slab when B = 6 m, 2L = 12 m, M_p = 9 kNm/m and M_n = 12 kNm/m, as shown in Fig.12.31.10. Use method of virtual work.

A.4: **(a):** The possible yield pattern is drawn in Fig.12.31.9 having unknown y. Thus, we have two unknowns y and the collapse load w of the slab. The yield pattern divides the slab into three segments of which segments 2 and 3 are symmetrical.

Geometric properties.

The angle DEJ = $\theta = tan^{-1} (L/B)$. The perpendicular distance from G to ED is GH = EG $sin\theta = (B - y) \{sin\ tan^{-1} (L/B)\}$. The length of side DE = $(B^2 + L^2)^{1/2}$.

Let us assume the displacement of the slab at point G = Δ. The rotation of the segment 1 = $\theta_1 = \Delta/y$ and the rotation of the segments 2 and 3 = θ_2 = Δ/GH = Δ / $[(B-y)\{sin\ tan^{-1} (L/B)\}]$.

The total external work (TEW) done by the loads of three segments is obtained considering the displacement of the centroid of all three segments as $\Delta/3$.

Thus,

$$TEW = (1/2)\ (2L)\ (B)\ w\ (\Delta/3) = B\ L\ w\ \Delta/3 \quad (1)$$

The internal work done by the negative yield lines DF, DE and EF is:

$$M_n \{(DF)\ (\theta_1) + 2(DE)\ (\theta_2)\} \quad (2A)$$

For the positive yield moment along DG, let us project DG along DE and DF. The projected lengths are DH and DJ, respectively. Similarly, projecting the moments of positive yield lines of EG and FG along the sides of the triangle DEF, we have the total internal works done by the three positive yield lines = $M_p \{(DF)\ (\theta_1) + 2(DE)(\theta_2)\}$ (2B)

Therefore, the total internal work done (TEW) is obtained adding the two expressions of 2A and 2B as:

$$TIW = (M_p + M_n) \{(DF)\ (\theta_1) + 2\ (DE)\ (\theta_2)\} \quad (2)$$

Equating TEW and TIW from Eqs. 1 and 2, we have

$(B)(L)\ w\ \Delta/3 = (M_p + M_n)\ \{(\mathrm{DF})\ (\theta_1) + 2\ (\mathrm{DE})\ (\theta_2)\}$ (3)

Substituting the values of DF, DE, θ_1 and θ_2, we have from Eq. 3

$(B)(L)\ w\ \Delta/3 = (M_p + M_n)\ [(2L)\ (\Delta/y) + 2\ (B^2 + L^2)^{1/2}\ \Delta\ /\ [(B-y)\ \{\sin \tan^{-1} (L/B)\}]]$

$$\text{or, } w = \frac{3(M_p + M_n)}{(B)(L)}\left[\frac{2L}{y} + \frac{2(B^2+L^2)^{1/2}}{(B-y)\{\sin\tan^{-1}(L/B)\}}\right] \quad (4)$$

Equation 4 is the only one equation from the method of virtual work to determine *y* and *w*. Therefore, we differentiate *w* with respect to *y* to get the lowest value of the load and determine *y*. Thereafter, Eq. 4 shall be used to find the value of *w*.

$$\frac{dw}{dy} = 0 \text{ gives: } -\frac{2L}{y^2} = \frac{-2(B^2+L^2)^{1/2}}{(B-y)\{\sin\tan^{-1}(L/B)\}}$$

or $[L\{\sin \tan^{-1} (L/B)\} - (B^2 + L^2)^{1/2}]\ y^2 - [2\ (L)\ (B)\ \{\sin \tan^{-1} (L/B)\}]\ y + LB^2 \{\sin \tan^{-1} (L/B)\} = 0$ (5)

Thus, Eqs. 4 and 5 are the general equations to determine y and *w* of the slab.

A 4. **(b):** For the specific case when $2L = 12$ m, $B = 6$ m, $M_p = 9$ kNm/m and $M_n = 12$ kNm/m, we have angle DEJ = $\theta = 45^0$. The distance GH = EG $\sin\theta = (6-y)\ /\ (2)^{1/2}$. The length of side DE = EF = $(36 + 36)^{1/2} = 6(2)^{1/2}$ m. The rotations $\theta_1 = \Delta/y$ and $\theta_2 = \Delta/[(B-y)\ \{\sin \tan^{-1} (L/B)\}] = \Delta(2)^{1/2}\ /\ (6-y)$.

Equating the *TEW* and *TIW* from Eqs. 1 and 2 of A.4a, we have:

$$w = \frac{(M_p + M_n)}{12}\left\{\frac{12}{y} + \frac{24}{6-y}\right\} \quad (6)$$

$$\frac{dw}{dy} = 0 \text{ gives: } y^2 + 12y - 36 = 0 \quad (7)$$

Solution of Eq. 7 gives y = 2.485 m. Using the value of y in Eq. 6, we get w = 20.399 kN/m^2.

12.31.8 References

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12.31.9 Test 31 with Solutions

Maximum Marks = 50 Maximum Time = 30 minutes

Answer all questions.

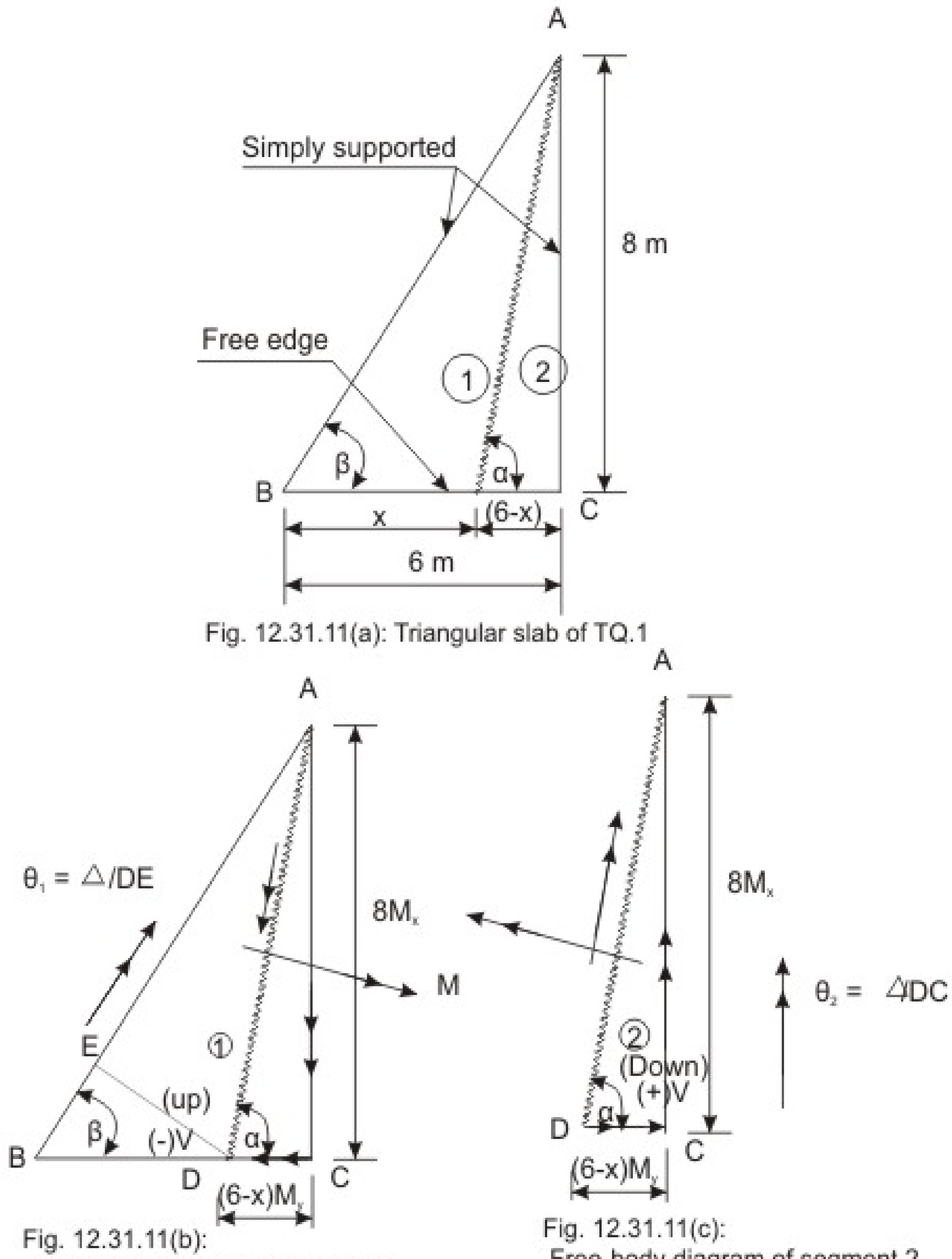

Fig. 12.31.11(a): Triangular slab of TQ.1

Fig. 12.31.11(b): Free-body diagram of segment 1

Fig. 12.31.11(c): Free-body diagram of segment 2

Fig. 12.31.11: Triangular slab of TQ. 1

TQ.1: Determine the yield pattern and the uniformly distributed collapse load *w* kN/m^2 of the triangular slab shown in Fig. 12.31.11a having simple supports along AB and AC and the edge BC is free. The reinforcing bars along x and y directions have the moment capacities M_x = 50 kNm/m and M_y = 60 kNm/m, respectively. Use both the methods i.e., (i) method of segmental equilibrium and (ii) method of virtual work.

[25×2 = 50]

A.TQ.1: (i) Method of segmental equilibrium. The yield pattern of the slab is drawn in Fig.12.31.11a involving x as unknown. Thus, we have two unknowns x and w here.

The yield pattern divides the slab into two segments 1 and 2, whose free body diagrams are shown in Figs. 12.31.11b and c, respectively. The nodal force V has the magnitude (90 – 15x)/ 2 obtained from Eq.12.23 ($V = M_y \cot\alpha$).

Taking moment of all forces and moments of segment 1 about AB and equating it to zero gives

$(1/2)\, x\, (8)\, w\, (DE)/3 – V\, (DE) – 8\, M_x \sin\beta - M_y\, (6-x) \cos\beta = 0,$ which gives

$$w = (-18\, x^2 + 1608) / (3.2)\, x^2 \qquad (1)$$

Similarly, taking moment of all forces and moments about AC of segment 2 and equating it to zero gives: $(1/2)\, (6-x)\, w\, (8)\, (6-x) / 3 + V\, (CD) – 8\, M_x = 0$, which gives,

$$w = (-22.5\, x^2 + 270\, x + 390) / (144 + 4x^2 – 48x) \qquad (2)$$

Equating Eqs. 1 and 2

$$\frac{-18x^2 + 1608}{3.2x^2} = \frac{-22.5x^2 + 270x + 390}{144 + 4x^2 - 48x}$$

or, $$27\, x^2 – 804\, x + 2412 = 0 \qquad (3)$$

which gives x = 3.3847 m.

From Eq. 1: $w = (1608 – 18x^2) / 3.2\, x^2$ = 38.2369 kN/m^2 and
From Eq.2: $w = (-22.5\, x^2 + 270\, x + 390) / (144 + 4\, x^2 – 48\, x)$ = 38.2369 kN/m^2.

(ii) Method of virtual work

Total external work $TEW = \{w\,(\Delta) / 3\,(2)\}\,\{8x + (6 - x)\,8\} = 8\,w\Delta$ (4)

Total internal work $TIW = 8M_x\,(\theta_1)\,\sin\beta + (6\text{-}x)\,M_y\,\theta_1 \cos\beta + (6\text{-}x)\,M_y\,\theta_2 \cos 90 + 8\,M_x\,\theta_2$

$= 8\,(50)\,(\Delta / DE)\,(0.8) + (6 - x)\,(60)\,(\Delta/DE)\,(0.6) + 0 + 8\,(50)\,\Delta/(6\text{-}x)\}$

$$= \Delta\left[\frac{400}{x} + \frac{45(6-x)}{x} + \frac{400}{(6-x)}\right] \tag{5}$$

Equating Eqs. 4 and 5

$$8w\Delta = \Delta\left[\frac{400}{x} + \frac{45(6-x)}{x} + \frac{400}{6-x}\right]$$

or, $$w = \frac{1}{8}\left[\frac{670}{x} - 45 + \frac{400}{6-x}\right] \tag{6}$$

Equation 6 is the only equation to determine x and w. Differentiating w with respect to x and equating it to zero will give the lowest value of w. Thus, $dw/dx = 0$ gives:

$$(-)\,670/(x^2) + 400 / (6 - x)^2 = 0$$

or, $$27\,x^2 - 804\,x + 2412 = 0 \tag{7}$$

Equation 7 is the same as Eq. 3, obtained by the method of segmental equilibrium. The solution of Eq.7 is the same as that of Eq.3 and so, we get x = 3.3847 m.

Using the value of x in Eq.6, we get

$$w = \frac{1}{8}\left[\frac{670}{x} - 45 + \frac{400}{6-x}\right] = 38.2369 \text{ kN/m}^2$$

Thus, we get the same values of x and w by both methods.

12.31.10 Summary of this Lesson

This lesson presents the derivations of the expressions for determining the work done by bending and twisting moments when yield lines are at angles with the two orthogonal directions of the reinforcing bars. The need for the nodal forces and their determination are explained when one yield line meets another yield line or the free edge. Different possible yield patterns of two-way slabs are explained. Numerical problems are solved for the purpose of illustration taking examples with or without nodal forces employing both (i) method of segmental equilibrium and (ii) method of virtual work. Illustrative examples, practice problems and problem of test will give a clear understanding of analysing the slabs by the two methods.