# Process Dynamics and Control

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## Chapter Four

## Dynamic Behavior of First and Second Order Processes

- Standard Process Inputs
- Response of First-Order Processes
- Response of Integrating Processes
- Response of Second-Order Processes

## Dynamic Behavior of First-Order Processes

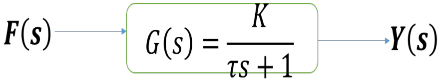
- In analyzing process dynamic and process control systems, it is important to know how the process responds to changes in the process inputs.
- Process inputs falls into two categories:
  1) Inputs that can be manipulated to control the process.
  2) Inputs that are not manipulated, such as disturbance variables.
- A number of standard types of input changes are widely used for two reasons:
  1) They are representative of the types of changes that occur in plants.
  2) They are easy to analyze mathematically.

## First Order Process System

- It is one whose output y(t) is modeled by the first order of differential equation.
- Consider the differential equation:

$a_1 y' + a_o y = bf(t)$ take the Laplace

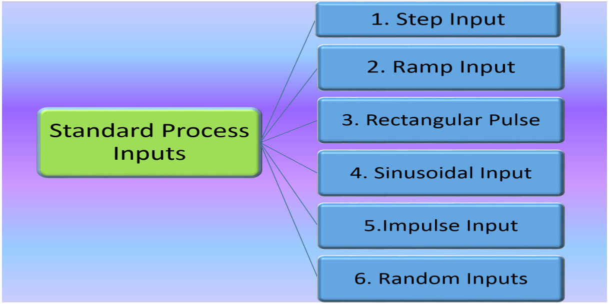
$$\Rightarrow a_1 sY(s) + a_o Y(s) = bF(s)$$
$$\Rightarrow \frac{a_1}{a_o} sY(s) + Y(s) = \frac{b}{a_o} F(s)$$
$$\Rightarrow \tau sY(s) + Y(s) = KF(s) \Rightarrow (\tau s + 1)Y(s) = KF(s)$$
$$\Rightarrow \frac{Y(s)}{F(s)} = \frac{K}{\tau s + 1} = G(s) \Rightarrow G(s) = \frac{K}{\tau s + 1}$$



- This equation is called the transfer function of first order system.
  - $\tau$- time constant of the process
  - $K$- static gain of the process (or simply gain)
- The parameters the first-order system are:
  - $\tau$ (time constant)
  - $K$ (steady state gain)
- Time constant of a process:
  - is a measure of the time necessary for the process to adjust to a change in the input.
- Steady state gain:
  - is the steady state change in output divided by the sustained change in the input.
  - characterizes the sensitivity of the output to the change in input.

- The steady state of a transfer function can be used to calculate the steady-state change in an output due to a steady-state change in the input.

Standard Process Inputs for first order



## 1) Unit Step Response

- A sudden change in a process variable can be approximated by a step change of magnitude: $f(t) = M$
- The step change occurs at an arbitrary time denoted as t = 0.
- Example of a step change: A reactor feedstock is suddenly switched from one supply to another, causing sudden changes in feed concentration, flow, etc.

- If the input u(t) is a step function with amplitude M:

$$f(t) = M \Rightarrow F(S) = \frac{M}{s} \Rightarrow Y(S) = \frac{K}{\tau s + 1} F(s)$$

$$\Rightarrow Y(s) = \frac{K}{\tau s + 1} . \frac{M}{s}$$

$$\Rightarrow Y(s) = \frac{MK}{s(\tau s + 1)}$$

$$\Rightarrow Y(S) = \frac{c_1}{s} + \frac{c_2}{\tau s + 1}$$

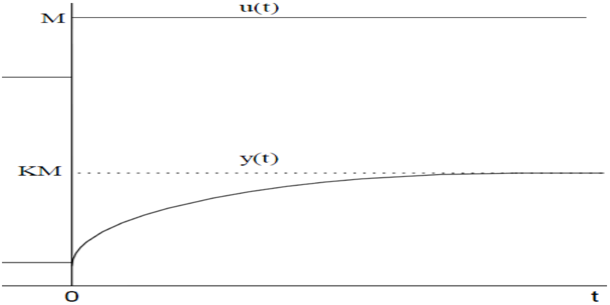
$$\Rightarrow Y(S) = \frac{MK}{s} - \frac{MK\tau}{\tau s + 1}$$

$$\Rightarrow y(t) = MK - MKe^{\frac{-t}{\tau}}$$

$$\Rightarrow y(t) = MK(1 - e^{\frac{-t}{\tau}})$$

- This is the response of first order of unit step change.

## First order step change characteristics

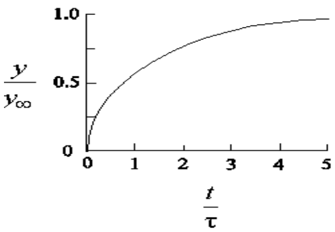



- A first-order process is self-regulating.
- The process reaches a new steady state.
- The ultimate value of the output is K for a unit step change in the input, or KM for a step of size M.

$y \to KM$ as $t \to \infty$ ; $K = \dfrac{\triangle(Output)}{\triangle(Input)}$

- This equation tells us by how much we should change the value of the input in order to achieve a desired change in the output, for a process with given K.
- A small change in the input if K is large (very sensitive systems)
- A large change in the input if K is small.

Let $y_\infty$ = steady-state value of $y(t)$. $y_\infty = KM$.




| t | $\frac{y}{y_\infty}$ |
|---|---|
| 0 | 0 |
| $\tau$ | 0.632 |
| $2\tau$ | 0.865 |
| $3\tau$ | 0.950 |
| $4\tau$ | 0.982 |
| $5\tau$ | 0.993 |

*Note:* Large $\tau$ means a slow response.

- After a time interval equal to the process time constant ($t = \tau$), the process response is still only 63.2 percent complete.
- When the time elapsed is $2\tau, 3\tau, 4\tau, 5\tau$ the response is 86, 95, 98 and 99 percent respectively.
- If the initial rate of change of y(t) were to be maintained, the response would reach its final value in one time constant.
- The smaller the value of time constant, the steeper the initial response of the System.
- Theoretically, the process output never reaches the new steady-state value except as $t \to \infty$
- It does approximate the final steady-state value when $t = 5\tau$.

**Example 1**: Let a transfer function $G(s) = \dfrac{15}{2s+4}$ then calculate:

a) The static gain, K

b) The time constant, $\tau$

c) The response of the system for step change if input is 5.

d) Ultimate value of the response

Solution: $G(s) = \dfrac{15}{2s+4}$

a) $K = \dfrac{15}{2} = 3.75$

b) $\tau = \dfrac{2}{4} = 0.5$

c) The response of the system for step change if input is 5. $f(t) = 5$

$$\Rightarrow F(S) = \frac{5}{s} \Rightarrow y(t) = MK(1 - e^{\frac{-t}{\tau}})$$

$$\Rightarrow y(t) = 18.75(1 - e^{-2t})$$

d) Ultimate value of the response

$$y_\infty = lim_{t \to \infty} y(t)$$

$$y_\infty = lim_{t \to \infty}[18.75(1 - e^{-2t})]$$

$$y_\infty = 18.75$$

**Example 2**: A stirred-tank heating system is used to preheat a reactant containing a suspended solid catalyst at a constant flow rate of 1000kg/h. The volume in the tank is $2m^3$, and the density and specific heat of the suspended mixture are, respectively, 900kg/$m^3$ and 1cal/g.C. The process initially is operating with inlet and outlet temperatures of 100 and 130C respectively.

a) What is the heater input at the initial steady state and the values of K and $\tau$ ?

b) If the heater input is increased by $+30$ percent, how long will it takes for the tank temperature to achieve 99 percent of the final temperature change?

c) Assume the tank is at its initial steady state. If the inlet temperature is increased suddenly from 100 to 120C, how long will it take before the outlet temperature changes from 130 to 135C?

## 2) Ramp Input

- Industrial processes often experience drifting disturbances, that is, relatively slow changes up or down for some period of time with a roughly constant slope.
- The rate of change is approximately constant.
- We can approximate a drifting disturbance by a ramp input: $f(t) = at$
- Examples of ramp changes:
  - Ramp a set point to a new value rather than making a step change.
  - Feed composition, heat exchanger fouling, catalyst activity, ambient temperature.

## Ramp Response

- If the input f(t) is a ramp function with amplitude A

$$f(t) = At \Rightarrow F(S) = \frac{A}{s^2} \Rightarrow Y(S) = \frac{K}{\tau s + 1} F(s)$$

$$\Rightarrow Y(s) = \frac{K}{\tau s + 1} . \frac{A}{s^2} = \frac{AK}{s^2(\tau s + 1)}$$

$$\Rightarrow Y(S) = \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{\tau s + 1}$$

$$\Rightarrow Y(s) = \frac{-AKt}{s} + \frac{AK}{s^2} + \frac{AK\tau}{s + 1/\tau}$$

$$\Rightarrow y(t) = -AK\tau + AKt + AK\tau e^{\frac{-t}{\tau}}$$

$$\Rightarrow y(t) = AK(t - \tau + \tau e^{\frac{-t}{\tau}})$$

$$\Rightarrow y(t) = AK(t - \tau)$$

- This is the response of first order of ramp change.

## Dynamic Behavior of Second-Order Processes

- A second-order system is one whose output, y(t), is described by a second-order differential equation.
- The following equation describes a second-order linear system:

$$a_2 y'' + a_1 y' + a_o y = bf(t)$$

$$\Rightarrow \frac{Y(s)}{F(s)} = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1} = G(s)$$

$$\Rightarrow G(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1} \text{ where}$$

- K = Process steady-state gain
- $\tau$ = Process time constant
- $\xi$ = Damping coefficient

- The very large majority of the second- or higher-order systems encountered in a chemical plant come from multi-capacity processes, i.e. processes that consist of two or more first-order systems in series, or the effect of process control systems.

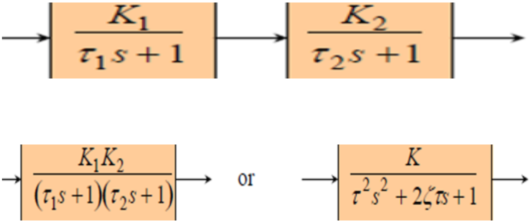
**Standard Form**

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

K = Gain
τ = Natural Period of Oscillation
ζ = Damping Factor (zeta)

**Note: this has to be 1.0!!!**

## Multiple Capacity Systems in Series



## Dynamic response

- For a step change of magnitude M, F(s) = M/s

$$Y(s) = \frac{KM}{s(\tau^2 s^2 + 2\xi\tau s + 1)}$$

- The two poles of the second-order transfer function are given by the roots of the characteristic polynomial.

$$\tau^2 s^2 + 2\xi\tau s + 1 = 0$$

- The form of the response of y(t) will depend on the location of the two poles in the complex plane.

- Characteristic Behavior of Second-Order Transfer Functions

| Case | Damping factor | Pole location | Characteristic behavior |
|---|---|---|---|
| 1 | $\varepsilon > 1$ | Two real, distinct roots | Over damped |
| 2 | $\varepsilon = 1$ | Two real, equal roots | Critically damped |
| 3 | $\varepsilon < 1$ | Two complex conjugate roots | Underdamped |

- The value of $\xi$ completely determines the degree of oscillation in a process response after a perturbation.

  a) $\xi > 1$ (Over-damped)

     - Two distinct real roots
     - Sluggish response
     - No oscillation

  - The response of the over damped is:

$$y(t) = KM[1 - e^{-\xi t/\tau} cosh(\frac{\sqrt{1-\xi^2}}{\tau} t) + \frac{\xi}{\sqrt{1-\xi^2}} sinh(\frac{\sqrt{1-\xi^2}}{\tau} t)]$$

- b) $\xi = 1$ (Critically-damped)
- The transition between over damped and under damped.
  - Two equal real roots
  - Faster than over damped
  - No oscillation

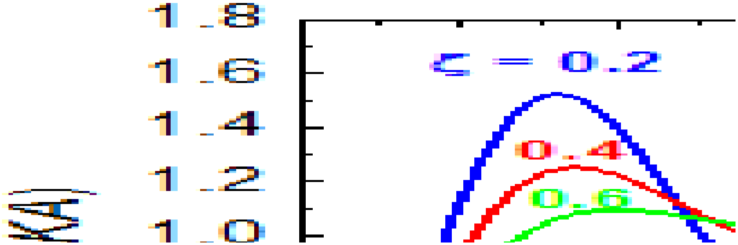
  $$y(t) = KM[1 - (1 + t/\tau)e^{-t/\tau}]$$

- Responses exhibit a higher degree of oscillation and overshoot $\frac{y}{KM} > 1$ as $\xi \rightarrow 0$.
- Large values of $\xi$ yield a sluggish (slow) response.
- The fastest response without overshoot is obtained for the critically damped case ($\xi = 1$ ).

- c) $0 < \xi < 1$ (Under-damped)
  - Two complex conjugate roots
  - Fastest response
  - Oscillating response (the damping is attenuated as $\xi$ decreases)

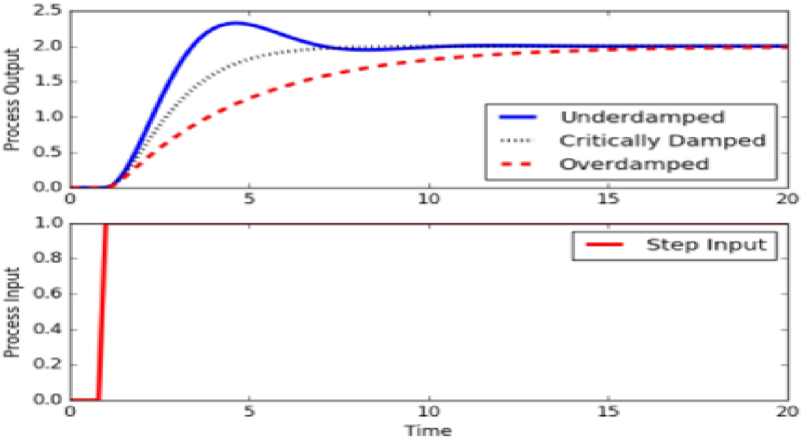
$$y(t) = KM[1 - e^{-\xi t/\tau} cos(\frac{\sqrt{1-\xi^2}}{\tau} t) + \frac{\xi}{\sqrt{1-\xi^2}} sin(\frac{\sqrt{1-\xi^2}}{\tau} t)]$$

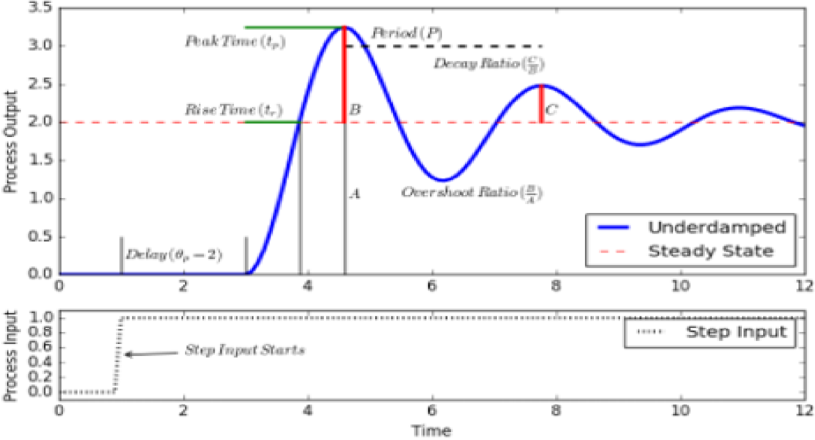
- d) $\xi \leq 0$ (Not-damped)
- Unstable system (the oscillation amplitude grows indefinitely)




Step response of under damped second order processes

- Ways to describe under-damped responses:
  - **Rise time, tr**: amount of time to first cross the steady state level (after accounting for dead time).
  - **Peak time, tp**: amount of time to reach the first peak (after accounting for dead time).
  - Settling time
  - **Overshoot ratio, OS**: amount that first oscillation surpasses the steady state level relative to the steady state change.
  - **Decay ratio, DR**: fractional size of successive peaks. DR is a measure of how rapidly the oscillations decreasing.
  - **Period of Oscillation(P)**: is the time between two successive peaks.


Process Output
2.5
2.0
1.5
1.0
0.5
0.0
0
5
10
15
20
Underdamped
Critically Damped
Overdamped
Process Input
1.0
0.8
0.6
0.4
0.2
0.0
Step Input
Time


Peak Time ($t_p$)
Period ($P$)
Decay Ratio ($\frac{C}{B}$)
Rise Time ($t_r$)
B
C
A
Overshoot Ratio ($\frac{B}{A}$)
Delay ($\theta_p = 2$)
Underdamped
Steady State
Process Output
Step Input
Step Input Starts
Process Input
Time

Summary for under-damped response formula:

- $OS = exp(\frac{-\Pi\xi}{\sqrt{1-\xi^2}})$
- $DR = (OS)^2 = exp(\frac{-2\Pi\xi}{\sqrt{1-\xi^2}})$
- $\omega = \frac{2\Pi}{T}$
- $T = \frac{2\Pi}{\omega} = \frac{2\Pi\tau}{\sqrt{1-\xi^2}}$
- $t_p = \frac{-\Pi\xi}{\sqrt{1-\xi^2}}$

**Example 3**: Let a transfer function: $G(s) = \dfrac{9}{s^2 + 2s + 9}$; determine:

a) Natural period of oscillation

b) Static gain, Damping coefficient

c) Overshoot, Decay ratio

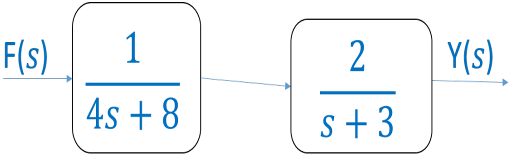
d) Period of oscillation

e) Natural and angular frequency

f) Nature of response

g) The peak time

**Example 4**: For the system




- Determine the:
  a) Transfer function of standard form of second order system
  b) Natural period of oscillation
  c) Static gain
  d) Nature of the response

**Example 5**: A step change from 15 to 31$N/mm^2$ in actual pressure results in the measured response from a pressure indicating element shown in the following figure. Assuming second-order dynamics, calculate:

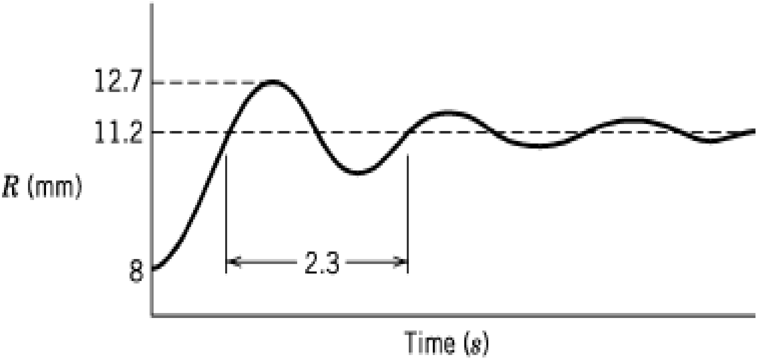
a) Steady state gain of the process and time constant of the system

b) Overshoot percent, Decay ratio and Damping factor

c) Natural frequency and Angular frequency

d) Write an approximate transfer function in the form:
$\frac{R'(s)}{P'(s)} = \frac{K}{\tau^2 s^2 + 2\tau\xi + 1}$ ; where R' is the instrument output deviation (mm), P' is the actual pressure deviation (pa).

e) Write an equivalent differential equation model in terms of actual (not deviation) variables.


12.7
11.2
R (mm)
8
2.3
Time (s)

**Example 6**: An electrically heated process is known to exhibit second-order dynamics with the following parameter values: $K = 3C/kW$, $\tau = 3min, \xi = 0.7$. If the process initially is at steady state at 70C with heater input of 20 kW and the heater input is suddenly changed to 26 kW and held there:

a) Write the expression of process temperature as a function of time?

b) What will be the ultimate temperature?

c) What will be the maximum temperature?

d) When will maximum temperature occur?