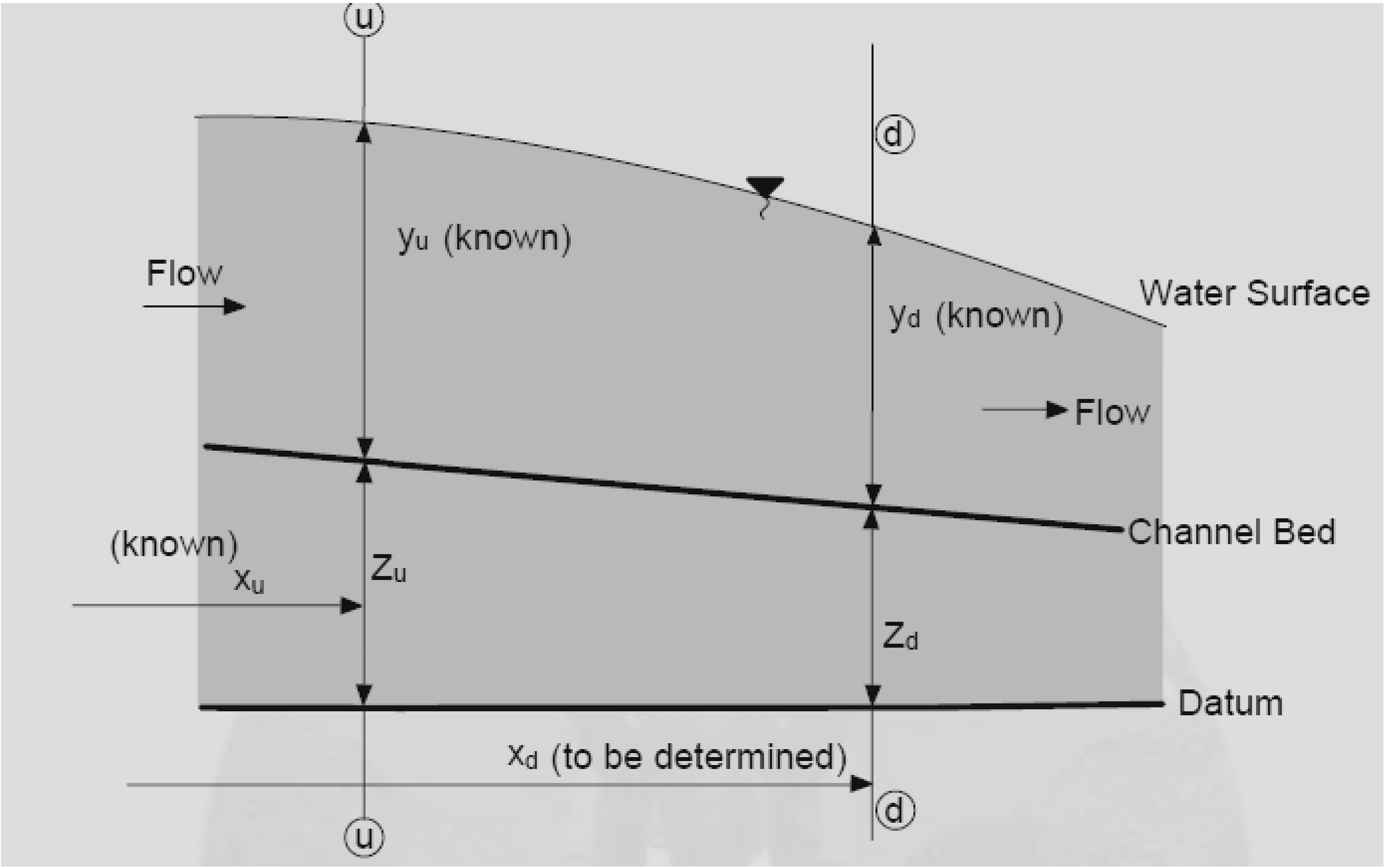
Simple numerical Solution of the GVF equation

Fitsume T.

Direct–step Method

- In the Direct Step method, the location where the specified depth, y_d occurs is determined, given the location for the occurrence of depth, y_u. Consider the channel shown in figure below.
- In this channel, say depth y_u occurs at a distance x_u from the reference point. Discharge, Q, channel bottom slope, S_0, the roughness coefficient, n and cross - sectional shape parameters (which relate A, P and R to y) are also known.
- The problem now is to determine the location x_d

Direct–step Method



Direct–step Method

$$z_u + y_u + \frac{\alpha_u V_u^2}{2g} = z_d + y_d + \frac{\alpha_d V_d^2}{2g} + \bar{S}_f (x_d - x_u)$$

$$S_f = \frac{n^2 Q^2}{A^2 R^{4/3}}$$

- S_f varies between sections u and d since the flow depth, and consequently A and R vary between these two sections.
- S_f may also due to variation in the roughness between the two sections. Following equations may be used to determine S_f .

Direct–step Method

Arithmetic mean

$$\overline{S_f} = \frac{1}{2}\left(S_{f_u} + S_{f_d}\right)$$

Geometric mean

$$\overline{S_f} = \sqrt{\left(S_{f_u} * S_{f_d}\right)}$$

Harmonic mean

$$\overline{S_f} = \frac{2S_{f_u}S_{f_d}}{S_{f_u} + S_{f_d}}$$

Direct–step Method

- Experience has indicated that the arithmetic mean gives the lowest maximum error, although it is not always the smallest error. Also, it is the simplest of the three approximations. Therefore, its use is generally recommended.
- Noting that the bed elevations Z_u and Z_d are related through the bed slope, S_0 and the distance between the sections, $(x_d - x_u)$, can be written as

Direct–step Method

$$-\left(y_d + \alpha_d \frac{V_d^2}{2g}\right) + \left(y_u + \alpha_u \frac{V_u^2}{2g}\right) = \bar{S}_f\left(x_d - x_u\right) - S_0\left(x_d - x_u\right)$$

However,

$$y_u + \alpha_u \frac{V_u^2}{2g} = y_u + \frac{\alpha_u Q^2}{2gA_u^2} = E_u$$

$$y_d + \alpha_d \frac{V_d^2}{2g} = y_d + \frac{\alpha_d Q^2}{2gA_d^2} = E_d$$

- where E_u and E_d are specific energies at section u and d, respectively.

$$x_d = x_u + \frac{E_d - E_u}{S_0 - \frac{1}{2}\left(S_{f_u} + S_{f_d}\right)}$$

$$x_d = \frac{x_u + \left(y_d - y_u\right) + \frac{q^2}{2g}\left[\frac{1}{y_d^2} - \frac{1}{y_u^2}\right]}{S_0 - \frac{q^2 n^2}{2}\left[\frac{1}{y_d^{10/3}} + \frac{1}{y_u^{10/3}}\right]}$$

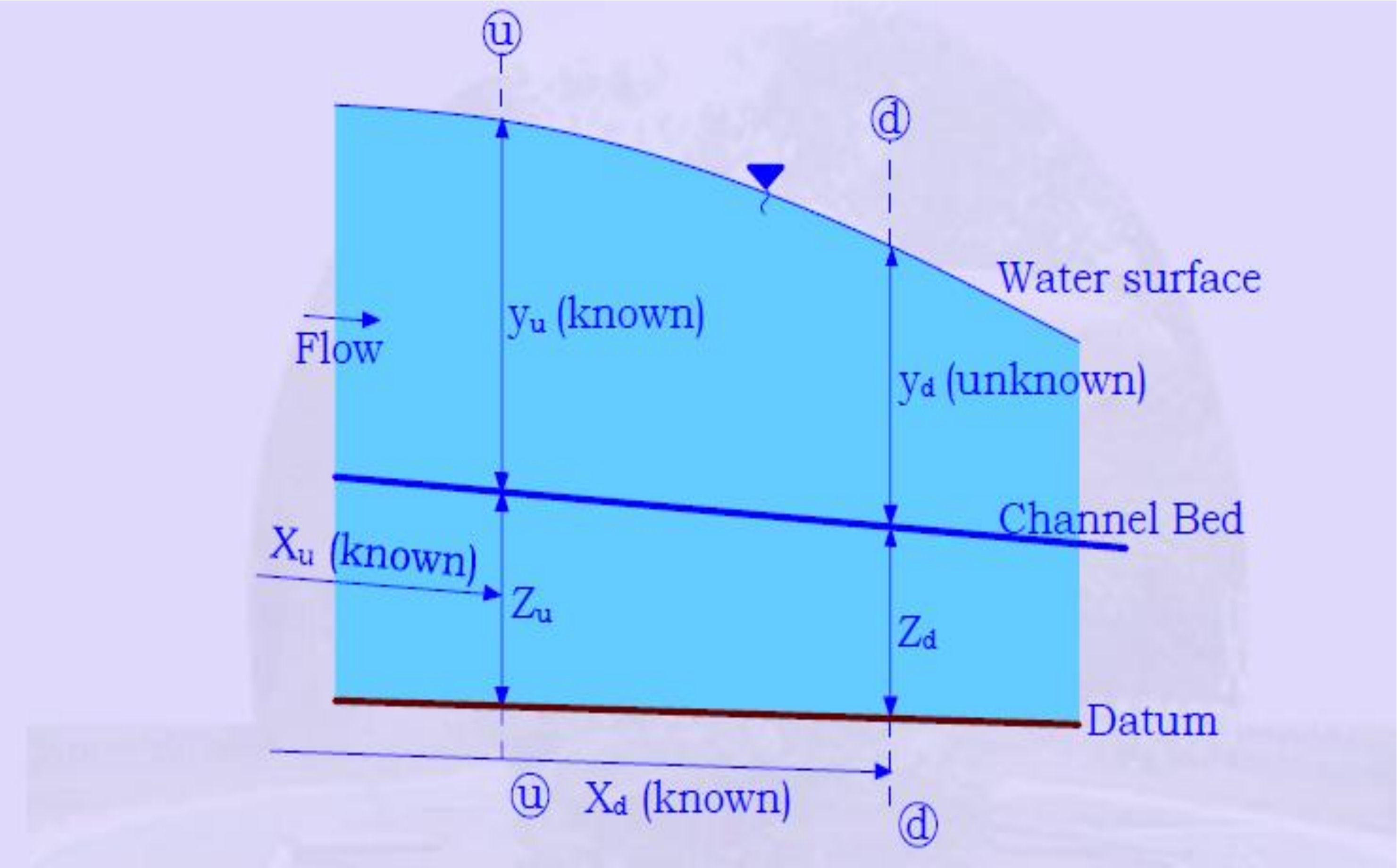
Disadvantages of Direct step method

- Interpolations become necessary if the flow depths are required at specified locations.
- It is inconvenient to apply this method to non prismatic channels because the cross-sectional shape at the unknown location should be known a priori.

Standard Step Method

- In the standard step method, flow depth at a specified location, y_d is determined, given the flow depth, Y_u at another specified location. Consider the channel shown in Figure below. In this channel, say Y_u occurs at a distance X_u from the reference point.
- Discharge, Q, Channel bottom slope, S_0, the roughness coefficient, n and cross-sectional shape parameters (which relate A, P and R to y) are also known.
- The problem now is to determine the flow depth, Y_d at the specified location X_d

Standard Step Method



Standard Step Method

$$-\left(y_d+\alpha_d\frac{V_d^2}{2g}\right)+\left(y_u+\alpha_u\frac{V_u^2}{2g}\right)=\bar{S}_f\left(x_d-x_u\right)-S_0\left(x_d-x_u\right)$$

- Can be written as

$$y_d+\frac{\alpha_d Q^2}{2gA_d^2}+\frac{S_{f_d}\left(x_d-x_u\right)}{2.0}=y_u+\frac{\alpha_u Q^2}{2gA_u^2}-\frac{S_{f_u}}{2}\left(x_d-x_u\right)+S_0\left(x_d-x_u\right)$$

- the flow rate (Q), the roughness coefficient (n), distances X_d and X_u, the channel slope (S_0), the flow conditions at section u (y_u , α_u and A_u) are known.
- Therefore the right hand side of Eq. Above can be determined. On the left hand side, the area, A_d and the friction slope, $S_{f\ d}$ are functions of the flow depth Y_d. Thus we have one equation in one unknown Y_d. Therefore, Y_d can be determined by solving the a non-linear equation.

Standard Step Method

- Either trial and error or numerical techniques such as bisection, Newton –Raphson techniques etc. can be used for solving

$$y_d + \frac{\alpha_d Q^2}{2gA_d^2} + \frac{S_{f_d}(x_d - x_u)}{2.0} = y_u + \frac{\alpha_u Q^2}{2gA_u^2} - \frac{S_{f_u}}{2}(x_d - x_u) + S_0(x_d - x_u)$$

- For example, for a wide rectangular channel (assuming αu =αd =1.0)

$$y_d + \frac{q^2}{2gy_d^2} + \frac{n^2q^2(x_d - x_u)}{2y_d^{10/3}} = y_u + \frac{q^2}{2gy_u^2} - \frac{n^2q^2}{2y_u^{10/3}}(x_d - x_u) + S_0(x_d - x_u)$$

Example

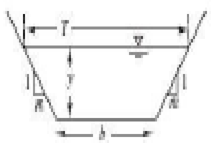
- A grouted-riprap, trapezoidal channel (n= 0.0025) with a bottom width of 4 meters and side slopes of m = 1 carries a discharge 12.5 m^3/sec on a 0.001 slope. Compute the backwater curve (upstream water surface profile) created by a low dam that backs water up to a depth of 2 m immediately behind the dam. Specifically, water depths are required at critical diversion points that are located at distances of 188 m, 423 m, 748 m, and 1,675 m upstream of the dam.

Solution:

1. Calculate normal depth,

$S_o = S_e$ for uniform flow conditions

$$\frac{Q}{A} = \frac{1}{n} \, . \, R_h^{\,2/3} . \, S_o^{\,1/2}$$

Section Type	Area (A)	Wetted perimeter (P)	Hydraulic Radius (R_h)	Top Width (T)	Hydraulic Depth (D)
Trapezoidal 	$(b + my)y$	$b + 2y\sqrt{1 + m^2}$	$\frac{(b + my)y}{b + 2y\sqrt{1 + m^2}}$	$b + 2my$	$\frac{(b + my)y}{b + 2my}$

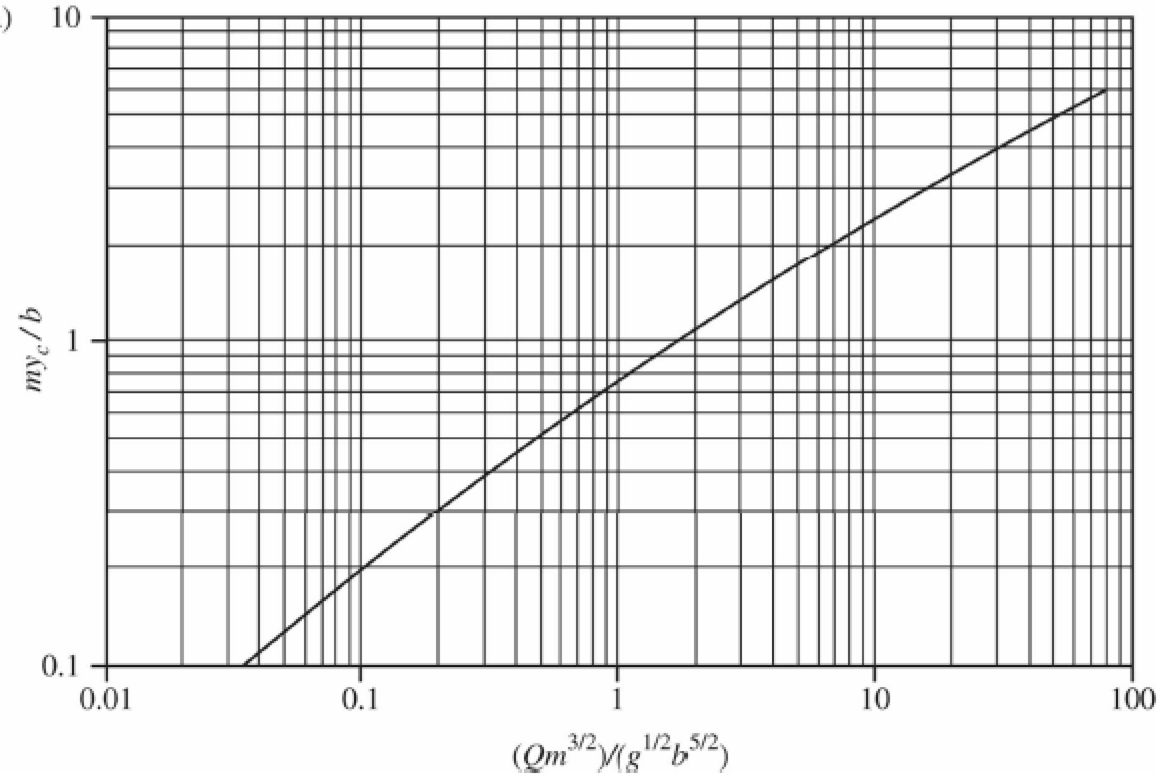
$$\frac{12.5}{(b+m.y).y} = \frac{1}{0.025} \, . \left(\frac{(b+m.y).y}{b+2.y.\sqrt{1+m^2}}\right)^{2/3} . \, (0.001)^{1/2}$$

$$\frac{12.5}{(1.y+4.m).y} = \frac{1}{0.025} \, . \left(\frac{(4+y).y}{4+2.y.\sqrt{1+1}}\right)^{2/3} . \, (0.001)^{1/2}$$

By trial and error ; $y_n = 1.66$ m

2. Calculate critical depth

(a)



$$\frac{Q.m^{3/2}}{g^{1/2}.b^{5/2}} = \frac{(12.5).(1)^{3/2}}{(9.81)^{1/2}.(4)^{5/2}} = 0.125$$

$$\frac{m.y_c}{b} = 0.230$$

$$y_c = \frac{(0.23).(4\ m)}{1}$$

$$y_c = 0.92\ m$$

Known water depth ; $y = 2$ m

Normal depth; $y_n = 1.66$ m

Critical depth; $y_c = 0.92$ m

$y > y_n$ & $y > y_c$ ➔ y>yn>yc ➔ MILD SLOPE ➔ M-1 CURVE

The depth (y= 2 m) just upstream from the dam is the control section designated as section 1. Energy balance computations begin here and progress upstream (backwater) because the flow is subcritical (yc < yn).

Since the profile has an M-1 classification, the flow depth will approach normal depth asymptotically as the computations progress upstream.

Since y/yc >1 and y/yn >1; the value dy/dx is positive, indicating that water depth increases in the direction of flow.

Standard Step Method

(1) Section	(2) U/D	(3) y (m)	(4) z (m)	(5) A (m^2)	(6) V (m/sec)	(7) $V^2/2g$ (m)	(8) P (m)	(9) R_h (m)	(10) S_e	(11) $S_{e(avg)}$	(12) h_L (m)	(13) Total Energy (m)
1	*D*	2.00	0.000	12.00	1.042	0.0553	9.657	1.243	0.000508	0.000538	0.1011	2.156
2	*U*	1.94	0.188	11.52	1.085	0.0600	9.487	1.215	0.000567	(ΔL = 188 m)		2.188
Note: The trial depth of 1.94 m is too high; the energy does not balance. Try a lower upstream depth.												
1	*D*	2.00	0.000	12.00	1.042	0.0553	9.657	1.243	0.000508	0.000554	0.1042	2.159
2	*U*	1.91	0.188	11.29	1.107	0.0625	9.402	1.201	0.000600	(ΔL = 188 m)		2.160
Note: The trial depth of 1.91 m is correct. Now balance energy between sections 2 and 3.												
2	*D*	1.91	0.188	11.29	1.107	0.0625	9.402	1.201	0.000601	0.000673	0.1582	2.319
3	*U*	1.80	0.423	10.44	1.197	0.0731	9.091	1.148	0.000745	(ΔL = 235 m)		2.296
Note: The trial depth of 1.80 m is too low; the energy does not balance. Try a higher upstream depth.												
2	*D*	1.91	0.188	11.29	1.107	0.0625	9.402	1.201	0.000601	0.000659	0.1549	2.315
3	*U*	1.82	0.423	10.59	1.180	0.0710	9.148	1.158	0.000716	(ΔL = 235 m)		2.314
Note: The trial depth of 1.82 m is correct. Now balance energy between sections 3 and 4.												

Column (1) Section numbers are arbitrarily designated from downstream to upstream.

Column (2) Sections are designated as either downstream (*D*) or upstream (*U*) to assist in the energy balance.

Column (3) Depth of flow (meters) is known at section 1 and assumed at section 2. Once the energies balance, the depth is now known at section 2, and the depth at section 3 is assumed until the energies at sections 2 and 3 balance.

Column (4) The channel bottom elevation (meters) above some datum (e.g., mean sea level) is given. In this case, the datum is taken as the channel bottom at section 1. The bottom slope and distance interval are used to determine subsequent bottom elevations.

Column (5) Water cross-sectional area (square meters) corresponds to the depth in the trapezoidal cross section.

Column (6) Mean velocity (meters per second) is obtained by dividing the discharge by the area in column 5.

Standard Step Method

Column (7)	Velocity head (meters).
Column (8)	Wetted perimeter (meters) of the trapezoidal cross section based on the depth of flow.
Column (9)	Hydraulic radius (meters) equal to the area in column 5 divided by the wetted perimeter in column 8.
Column (10)	Energy slope obtained from Manning equation (Equation 6.27a).
Column (11)	Average energy grade line slope of the two sections being balanced.
Column (12)	Energy loss (meters) from friction between the two sections found using $h_L = S_{e(avg)}(\Delta L)$ from Equation 6.26b.
Column (13)	Total energy (meters) must balance in adjacent sections (Equation 6.26b). Energy losses are always added to the downstream section. Also, the energy balance must be very close before proceeding to the next pair of adjacent sections or errors will accumulate in succeeding computations. Thus, even though depths were only required to the nearest 0.01 m, energy heads were calculated to the nearest 0.001 m.

Direct step Method

Section	U/D	y (m)	A (m^2)	P (m)	R_h (m)	V (m/sec)	$V^2/2g$ (m)	E (m)	S_e	ΔL(m)	Distance to Dam (m)
1	*D*	2.00	12.00	9.657	1.243	1.042	0.0553	2.0553	0.000508		0
2	*U*	1.91	11.29	9.402	1.201	1.107	0.0625	1.9725	0.000601	186	186
A distance of 186 m separates the two flow depths (2.00 m and 1.91 m).											
2	*D*	1.91	11.29	9.402	1.201	1.107	0.0625	1.9725	0.000601		186
3	*U*	1.82	10.59	9.148	1.158	1.180	0.0710	1.8910	0.000716	239	425
A distance of 239 m separates the two flow depths (1.91 m and 1.82 m).											