Chapter Two : CRITICAL FLOW

- Criterion for a critical flow
- Calculation of the Critical Depth
- Section factor for critical flow (Z_c)
- First Hydraulic Exponent (M)
- Characteristics of subcritical and supercritical flow
  - Wave Propagation
  - Transition with a change in width
  - Transition with hump
  - Choking

Critical Flow

For a given specific energy and discharge per unit width q, **there are two possible (real) depths of flow**, and that transition from one depth to the other can be accomplished under certain situations.

These two depths represented on the two different limbs of the **E-y curve separated by the crest c, are characteristic of two different kinds of flow;** a rational way to understand the nature of the difference between them is to consider first the flow represented by the point c.

Here the flow is in a critical condition, poised between two alternative flow regimes, and indeed the word “critical " is used to describe this state of flow; it may be defined as the **state at which the specific energy E is a minimum for a given q.**

Criterion for a Critical Flow

- The Froude number is equal to unity.
- The **specific energy** and **specific force** are minimum for the given discharge.
- The **discharge** is maximum at the critical flow for a given specific energy
- The velocity head is equal to half the hydraulic depth in a channel of small slope. Thus $y_c = f(A,D)$ for a given discharge
- The velocity of flow in a channel of small slope with uniform velocity distribution, is **equal to the celerity of small gravity waves** ($C = \sqrt{gh}$) in shallow water caused by local disturbance.

Criterion for a Critical Flow….

- **Flow at the critical state is unstable.**
- Critical flow may occur at a particular section or in the entire channel :
- For a prismatic channel for a given discharge
  - The critical depth is constant at all sections of a channel.

    Critical section

  - The bed slope which sustains a given discharge at a uniform and critical flow

    Critical Slope

Critical Depth

- The condition of minimum specific energy is known as the *critical flow condition* and the corresponding depth is known as ***Critical depth (y_c).***
- At critical depth, the specific energy is minimum, if differentiating the equation of specific energy with respect to y (keeping zero

$$\frac{dE}{dy} = 1 - \frac{Q_1^2}{gA^3} \times \frac{dA}{dy} = 0$$

But,

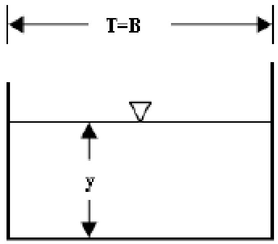
$$\frac{dA}{dy} = \frac{Tdy}{dy} = T = \text{Top width}$$

Designating the critical flow condition by suffix C

$$\frac{Q_1^2 T_c}{gA_c^3} = 1$$

Computation of Critical Depth

A) Rectangular Channel



For a rectangular channel, A = By, and T = B,

$$E = y + \frac{Q^2}{2gB^2y^2}$$

$$q = \frac{Q}{B} = \text{Discharge per unit width}$$

$$E = y + \frac{q^2}{2gy^2}$$

$$\frac{dE}{dy} = 1 - \frac{q^2}{gy^3} = 0$$

$$\frac{q^2}{g} = y_c^3$$

$$y_c = \sqrt[3]{\frac{q^2}{g}}$$

$$\frac{q^2}{g} = y_c^3$$

$$V_c = \frac{Q}{A_c} = \frac{Q}{By_c} = \frac{q}{y_c} = \frac{\sqrt{gy_c^3}}{y_c}$$

$$V_c = \sqrt{gy_c}$$

$$E_{min} = y_c + \frac{q^2}{2gy_c^2}$$

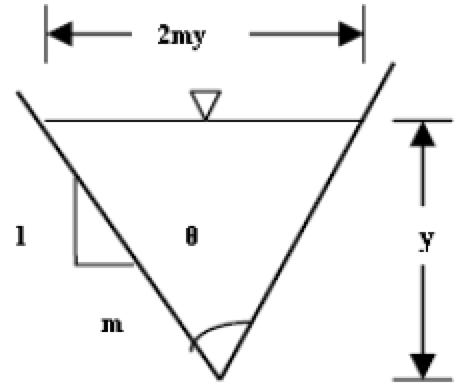
$$E_{min} = y_c + \frac{y^3}{2y_c^2}$$

$$E_{min} = 1.5y_c = Ec$$

The Froude number for a rectangular Channel will be defined as $F_r = \frac{V}{\sqrt{gy}}$

Computation of Critical Depth

B) Triangular Channel



For triangular channel having side slope of m or (H: V= m: 1)

$$A = my^2 \quad \text{and} \quad T = 2my$$

$$\frac{Q^2 T_c}{g A_c^3} = 1$$

$$\frac{Q^2}{g} = \frac{A_c^3}{T_c} = \frac{m^3 y_c^6}{2my_c} = \frac{m^2 y_c^5}{2}$$

$$y_c = \left(\frac{2Q^2}{gm^2}\right)^{1/5}$$

The specific energy at critical water depth,

$$E_c = y_c + \frac{V_c^2}{2g} = y_c + \frac{Q^2}{2gA_c^2}$$

$$E_c = y_c + \frac{gm^2 y_c^5}{2 \times 2 \times g \times m^2 \times y_c^4}$$

$$E_c = y_c + \frac{y_c}{4} = 1.25 y_c$$

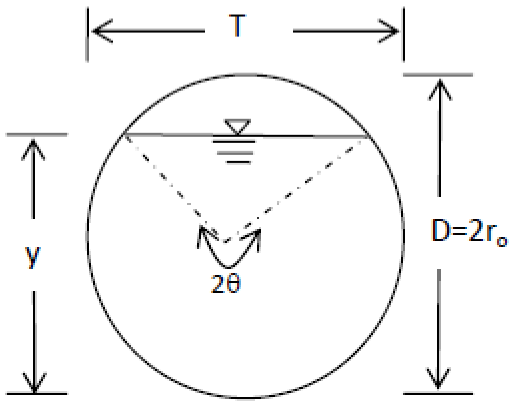
$$F_r = \frac{V}{\sqrt{g\frac{A}{T}}} = \frac{V}{\sqrt{g\frac{my^2}{2my}}}$$

$$F_r = \frac{V\sqrt{2}}{\sqrt{gy}}$$

Computation of Critical Depth

C) Circular Channel

Let D be the diameter of a circular channel and 2θ be the angel in radians subtended by the water surface at the center.



A = area of the flow section

= area of the sector + area of triangular portion

$$A = \frac{1}{2} r_o^2 2\theta + \frac{1}{2} 2r_o \sin(\pi - \theta) r_o \cos(\pi - \theta)$$

$$A = \frac{1}{2} r_o^2 2\theta + \frac{1}{2} 2r_o^{\,2} (2\sin(\pi - \theta)\cos(\pi - \theta))$$

$$A = \frac{1}{2} r_o^2 2\theta - \frac{1}{2} 2r_o^{\,2} (\sin 2\theta)$$

$$A = \frac{1}{2} r_o^2 (2\theta - \sin 2\theta)$$

$$\boxed{A = \frac{1}{8} D^2 (2\theta - \sin 2\theta)} \qquad \boxed{T = D\sin\theta}$$

For critical flow state

$$\frac{Q^2}{g} = \frac{A_c^3}{T_c} = \frac{[\frac{D^2}{8}(2\theta_c - \sin 2\theta_c)]^3}{D\sin\theta_c}$$

Example 2-3

- Calculate the critical depth and corresponding specific energy for a discharge of 5.0m3/sec in the following channels
  a) Rectangular channel B=2.0m
  b) Triangular channel m=0.5
  c) Circular channel D=2.0m and θ =600

The section Factor for critical flow

- The section factor (Z) is the product of the water area and the square root of the hydraulic depth.

$$Z = A\sqrt{\frac{A}{T}} = A\sqrt{D} \Rightarrow Z^2 = A^2 D \Rightarrow D = \frac{Z^2}{A^2}$$

- For critical flow $\frac{V^2}{2g} = \frac{D}{2}$ by substituting

$$\frac{V^2}{2g} = \frac{D}{2} = \frac{z^2}{2A^2} \Rightarrow Z^2 = \frac{V^2 A^2}{g} \Rightarrow Z = \frac{VA}{\sqrt{g}} \Rightarrow Z = \frac{Q_c}{\sqrt{g}}$$

- Therefore

$$Q_c = Z_c\sqrt{g} \quad \text{------------------------------} \quad (3.1)$$

- When the energy coefficient is not assumed to be unity

$$Q_c = Z_c\sqrt{\frac{g}{\alpha}} \quad \text{------------------------------} \quad (3.2)$$

Section factor (Z) for different channel section

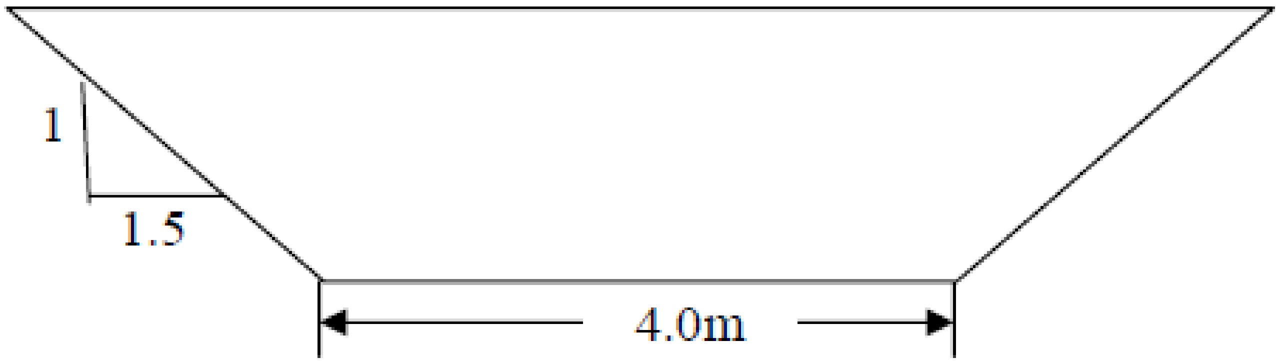
- **Rectangular** $Z = by^{1.5}$

- **Triangular** $Z = \frac{\sqrt{2}}{2} my^{2.5}$

- **Trapezoidal** $Z = \frac{[y * (b + my)]^{1.5}}{\sqrt{b + 2my}}$

Example 3.1.

Compute the critical depth the channel with its cross section presented in the figure below and carrying a discharge of 45m³/sec



First Hydraulic Exponent (M)

- In many computations involving a wide range of depths in channel, such as in the GVF computations, it is convenient to express the variation of Z with y in an exponential form.
- The (Z-y) relationship

$$Z^2 = C_1 y^M$$

- In this equation
  - C_1= a coefficient and
  - M= an exponent called first hydraulic exponent.
- It is found that generally M is a slowly –varying function of the aspect ratio for most of the channel shape

$$M = \frac{y}{A}\left[3T - \frac{A}{T}\frac{dT}{dy}\right]$$

Example 3.2:

Obtain the value of the first hydraulic exponent (M) for

a) Rectangular channel

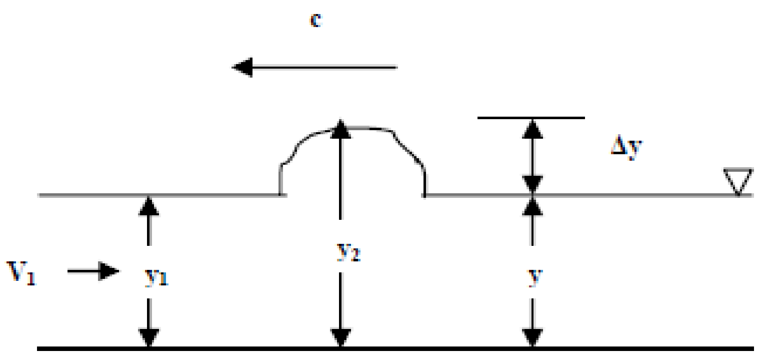
b) Exponential channel where the area $A=K^1y^a$

Characteristics of Flow Subcritical and Supercritical

- **Wave Propagation**
- **Transition with a change in width**
- **Transition with hump**
- **Choking**

Wave Propagation

- C is the wave pro r with velocity V_1.



- If we take the celerity **C** equal but opposite to the flow velocity V_1, then the wave stays still and the steady state conditions may be applied.

- Writing the energy equation between cross-sections 1 and 2 and neglecting the energy loss

$$y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g} \qquad \text{(a)}$$

Wave Propagation ....

$$q = V_1 y_1 = V_2 y_2$$

$$V_2 = V_1 \frac{y_1}{y_2}$$

- For rectangular channels

- Substituting this relation to Equ. (a),

$$y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_1^2}{2g} \times \left(\frac{y_1}{y_2} \right)^2$$

$$\frac{V_1^2}{2g} \left[1 - \left(\frac{y_1}{y_2} \right)^2 \right] = y_2 - y_1$$

$$\frac{V_1^2}{2g} = \frac{y_2 - y_1}{1 - \left(\frac{y_1}{y_2} \right)^2}$$

- If $y_1 = y$ then $y_2 = y + \Delta y$ and $V_1 = -c$, in which Δy = Wave height, the above equation may be written as,

Wave Propagation ....

$$\frac{c^2}{2g} = \frac{y+\Delta y - y}{1-\left(\frac{y}{y+\Delta y}\right)^2}$$

$$\frac{c^2}{2g} = \frac{\Delta y(y+\Delta y)^2}{(y+\Delta y)^2 - y^2}$$

$$\frac{c^2}{2g} = \frac{\Delta y\left(y^2+2y\Delta y+\Delta y^2\right)}{y^2+2y\Delta y+\Delta y^2 - y^2}$$

Neglecting Δy^2 values,

$$\frac{c^2}{2g} \cong \frac{\Delta y\left(y^2+2y\Delta y\right)}{2y\Delta y}$$

$$\frac{c^2}{2g} \cong \frac{y^2\left(1+2\frac{\Delta y}{y}\right)}{2y}$$

$$c \cong \sqrt{gy}\left(1+2\frac{\Delta y}{y}\right)^{1/2}$$

This Equ. is valid for shallow waters. **Generally Δy/y** may be taken as zero. The celerity equation is then

$$c = \sqrt{gy}$$

Wave Propagation

- Froude number for rectangular or wide channels is,

$$F_r = \frac{V}{\sqrt{gy}}$$

- Since celerity, $c = \sqrt{gy}$

$$F_r = \frac{V}{c} = \frac{Flow velocity}{Celerity}$$

- **Subcritical**

$$F_r = \frac{V}{c} = \frac{Flow velocity}{Celerity} < 1 \Rightarrow \text{Flow velocity} < \text{Celerity}$$

- The generated wave will be seen in the entire flow surface. That is why subcritical flows is also called ***downstream controlled flows***.

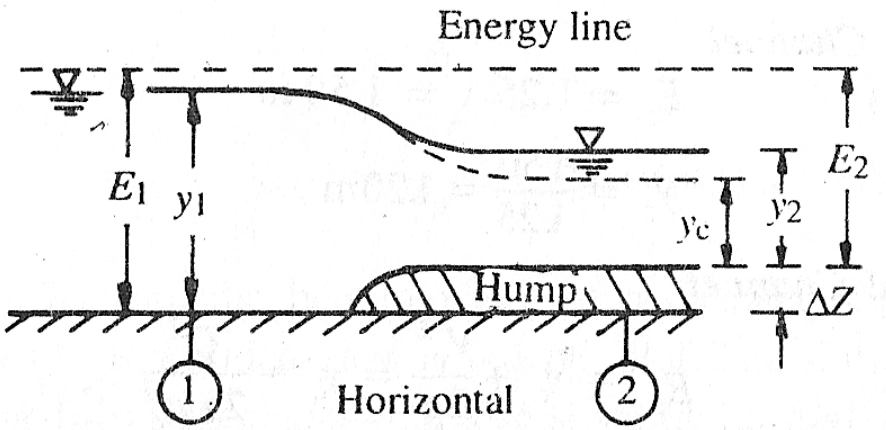
- **Supercritical Flows**

$$F_r = \frac{V}{c} = \frac{Flow velocity}{Celerity} > 1 \Rightarrow \text{Flow velocity} > \text{Celerity}$$

- Since flow velocity is greater than the wave celerity, a generated wave will propagate only in the downstream direction. That is why supercritical flows are called ***upstream controlled flows*.**

Transition with a Hump

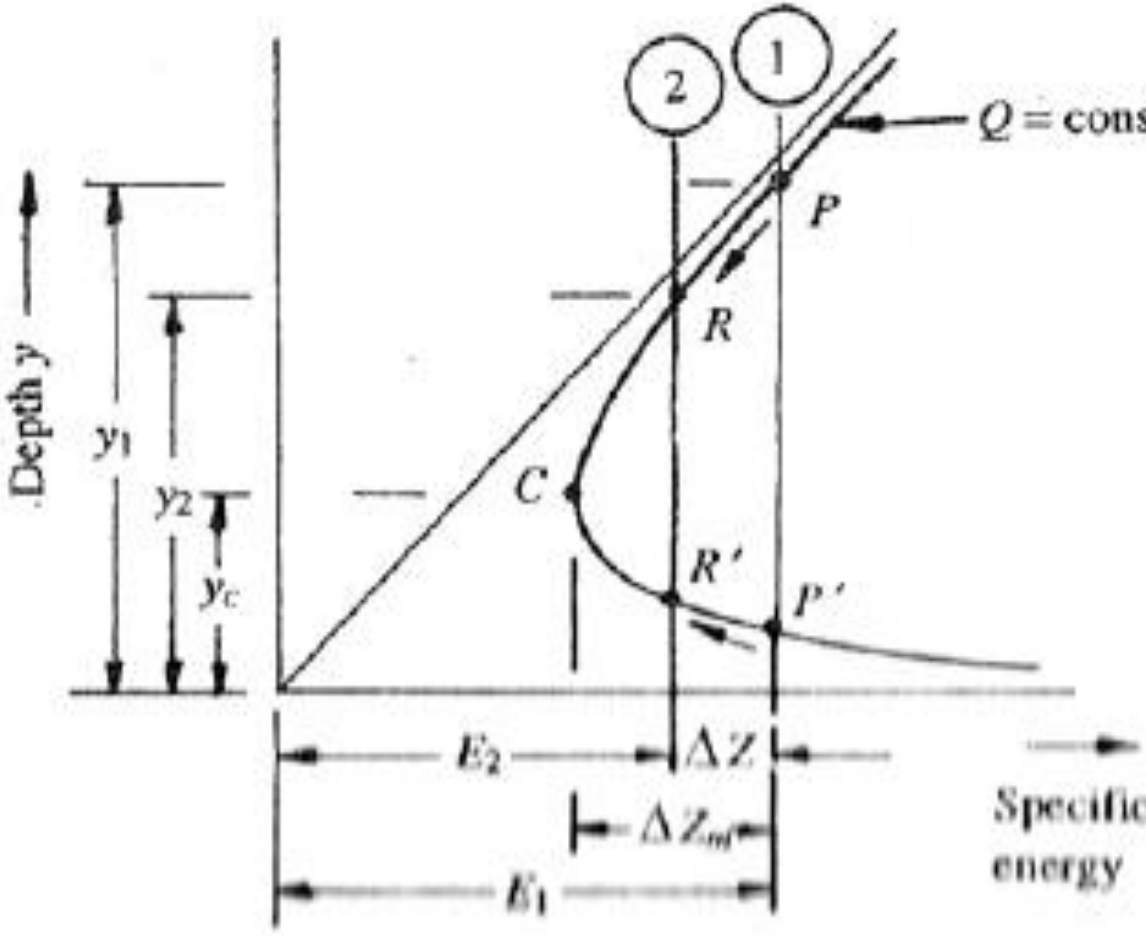
- Consider a horizontal, frictionless rectangular channel of width B carrying discharge Q at depth y_1
- At a section 2 a smooth hump of height ΔZ is built on the floor.



- Since there are no energy losses between sections 1 and 2, construction of a hump causes the specific energy at section to decrease by ΔZ. Thus the specific energies at sections 1 and 2 are,

$$E_1 = y_1 + \frac{V_1^2}{2g} \qquad\qquad E_2 = E_1 - \Delta Z$$

…….with a Hump



…….with a Hump

Subcritical

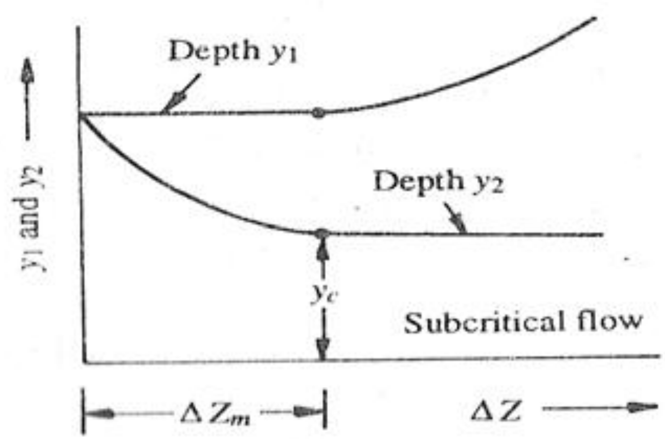
- Increase $\Delta Z \Rightarrow$ decrease in y_2
- ΔZ_{max} become when $y_2 = y_c$
- If $\Delta Z > \Delta Zmax$ no flow is possible in the given conation so that adjustment is expected
  - **At upper stream (section 1)**
    - Y_1 should increase to y_1^A
    - E_1 also increase to E_1^A
  - **At downstream (section 2)**
    - The flow will continue at the minimum specific energy level (critical condition)

Supercritical

- Increase $\Delta Z \Rightarrow$ increase in y'_2
- ΔZ_{max} become when $y'_2 = y_c$
- If $\Delta Z > \Delta Zmax$ no flow is possible in the given conation so that adjustment is expected
  - **At upper stream (section 1)**
    - Y_1 should decrease to y'^A_1
    - E_1 also decrease to E'^A_1
  - **At downstream (section 2)**
    - The flow will continue at the minimum specific energy level (critical condition)

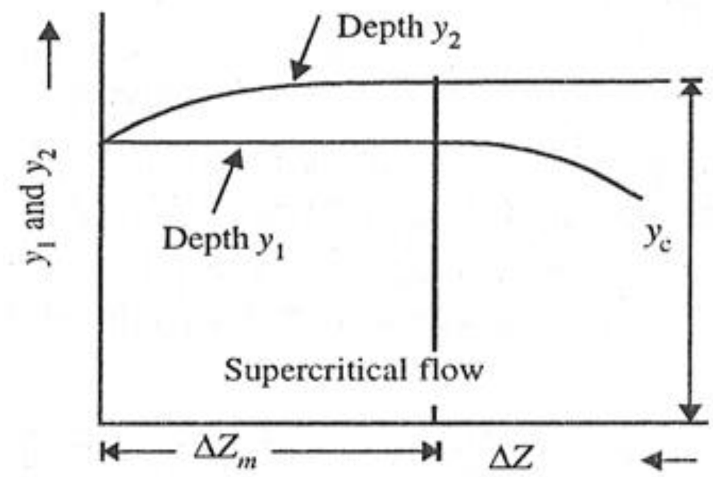
…….with a Hump

Subcritical



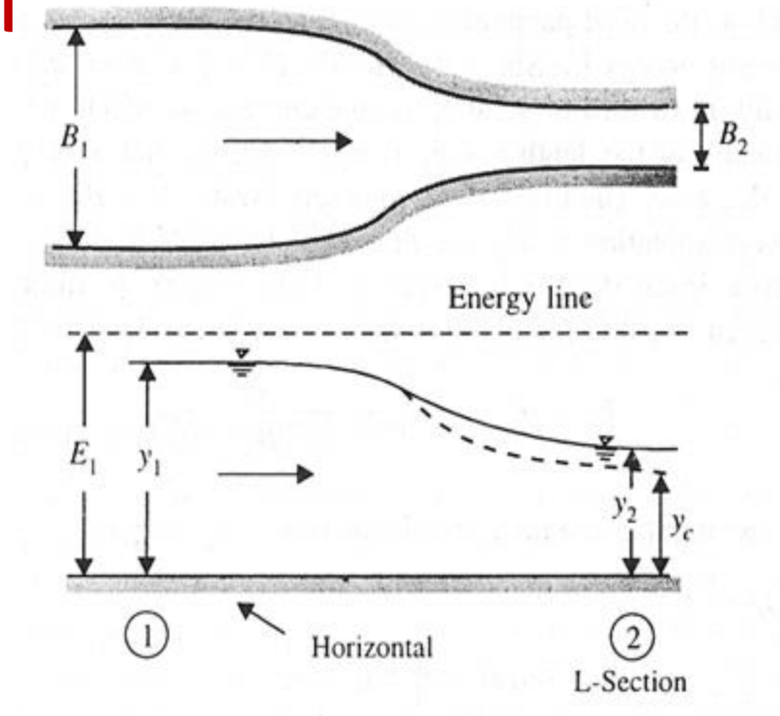
Recollecting the various sequences, when 0 < ΔZ < ΔZmax the upstream water level remains stationary at y1 while the depth of flow at section 2 decreases with ΔZ reaching a minimum value of yc at ΔZ = ΔZmax. With further increase in the value of ΔZ, i.e. for ΔZ > ΔZmax , y1 will change to y1` while y2 will continue to remain yc.

Supercritical



For ΔZ > ΔZmax , the depth over the hump y2 = yc will remain constant and the upstream depth y1 will change. It will decrease to have a higher specific energy E1`by increasing velocity V1.

Transition with a Change in Width



Consider a frictionless horizontal channel of width B_1 carrying a discharge Q at a depth y_1 and at a section 2 channel width has been constricted to B_2 by a smooth transition

$$E_1 = y_1 + \frac{V_1^2}{2g} = y_1 + \frac{Q^2}{2gB_1^2 y_1^2}$$

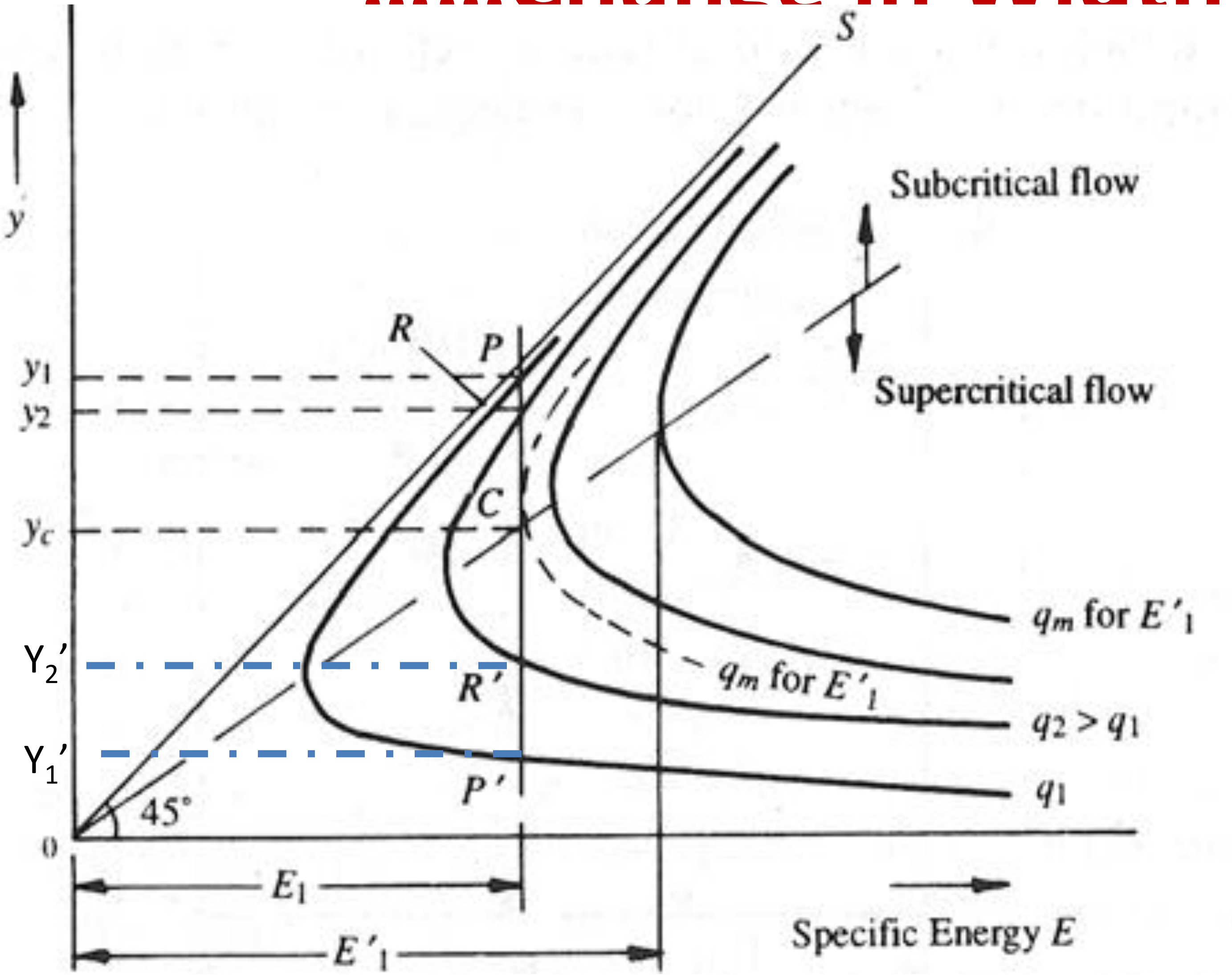
$$E_2 = y_2 + \frac{V_2^2}{2g} = y_2 + \frac{Q^2}{2gB_2^2 y_2^2}$$

It is convenient to analyze the flow in terms of the discharge intensity q = Q/B.

At section 1, q1 = Q/B1 and

At section 2, q2 = Q/B2

......Change in Width



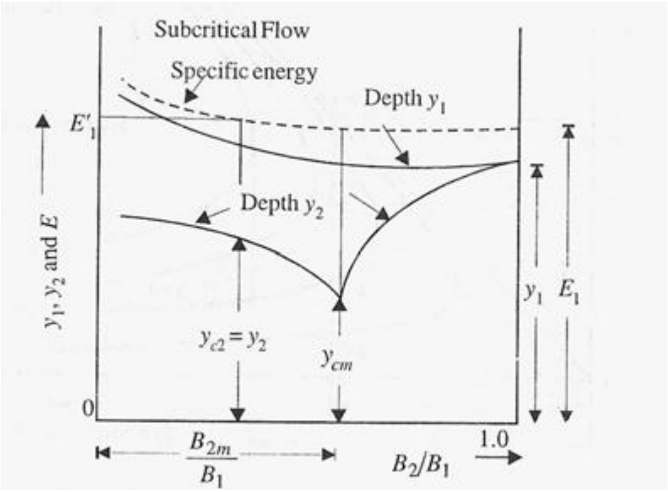
….Change in Width

Subcritical

since $B_2 < B_1$,

$\Rightarrow q_2 > q_1$ and $y_1 > y_2$

- If B_2 is made smaller, then q_2 will increase and y_2 will decrease.
- The limit of the contracted width $B_2 = B_{2min}$ is reached when corresponding to E_1, the discharge intensity $q_2 = q_{2max}$



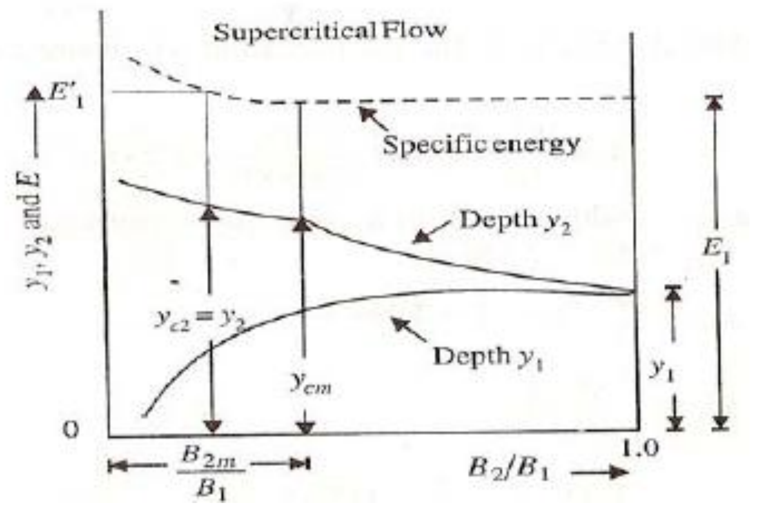
Supercritical

since $B_2 < B_1$,

$\Rightarrow q_2 >_{q1}$ and $y'_1 < y'_2$

As the width decrease R' moves up till B2 = B_{2min}

Further reduction in B_2 causes the upstream depth to decrease to Y'_1



…….Change in Width

- At the minimum width, $y_2 = y_{cm}$ = critical depth.

$$E_1 = E_{C\,\min} = y_{cm} + \frac{Q^2}{2g(B_{2\min})^2 y^2}$$

- For a rectangular channel, at critical flow $y_c = \frac{2}{3}E_C$

Since $E_1 = E_{C\min}$,

$$y_2 = y_{Cm} = \frac{2}{3}E_{C\,\min} = \frac{2}{3}E_1$$

$$y_c = \left(\frac{Q^2}{B_{2\min}^2 g}\right)^{1/3} \rightarrow B_{2\min} = \sqrt{\frac{Q^2}{g y_{cm}^3}}$$

$$B_{2\min} = \sqrt{\frac{Q^2}{g} \times \left(\frac{3}{2E_1}\right)^3}$$

$$B_{2\min} = \sqrt{\frac{27Q^2}{8gE_1^3}}$$

Choking

- In the case of a hump for all $\Delta Z \leq \Delta Z_{max}$, the upstream water depth is constant and for all $\Delta Z > \Delta Z_{max}$ the upstream depth is different from y_1. Similarly, in the case of the width constriction, for $B_2 \geq B_{2min}$, the upstream depth y_1 is constant; while for all $B_2 < B_{2min}$, the upstream depth undergoes a change.
- Thus all cases with $\Delta Z > \Delta Z_{max}$ or $B_2 < B_{2min}$ are known as *choked conditions* .
- Obviously, choked conditions are undesirable and need to be watched in the design of culverts and other surface drainage features involving channel transitions.

Examples