Critical flow

Chapter 2

Reminder

- **Depth (y)** the vertical distance from the lowest point of the channel section to the free surface.
- **Stage (z)** - the vertical distance from the free surface to an arbitrary datum
- **Area (A) -** the cross-sectional area of flow, normal to the direction of flow
- **Wetted perimeter (P)** - the length of the wetted surface measured normal to the direction of flow.
- **Surface width (B) -** width of the channel section at the free surface
- **Hydraulic radius (R)** - the ratio of area to wetted perimeter (A/P)
- **Hydraulic mean depth (D)** - the ratio of area to surface width (A/B)

Central principle of open channel flow

- Conservation of energy

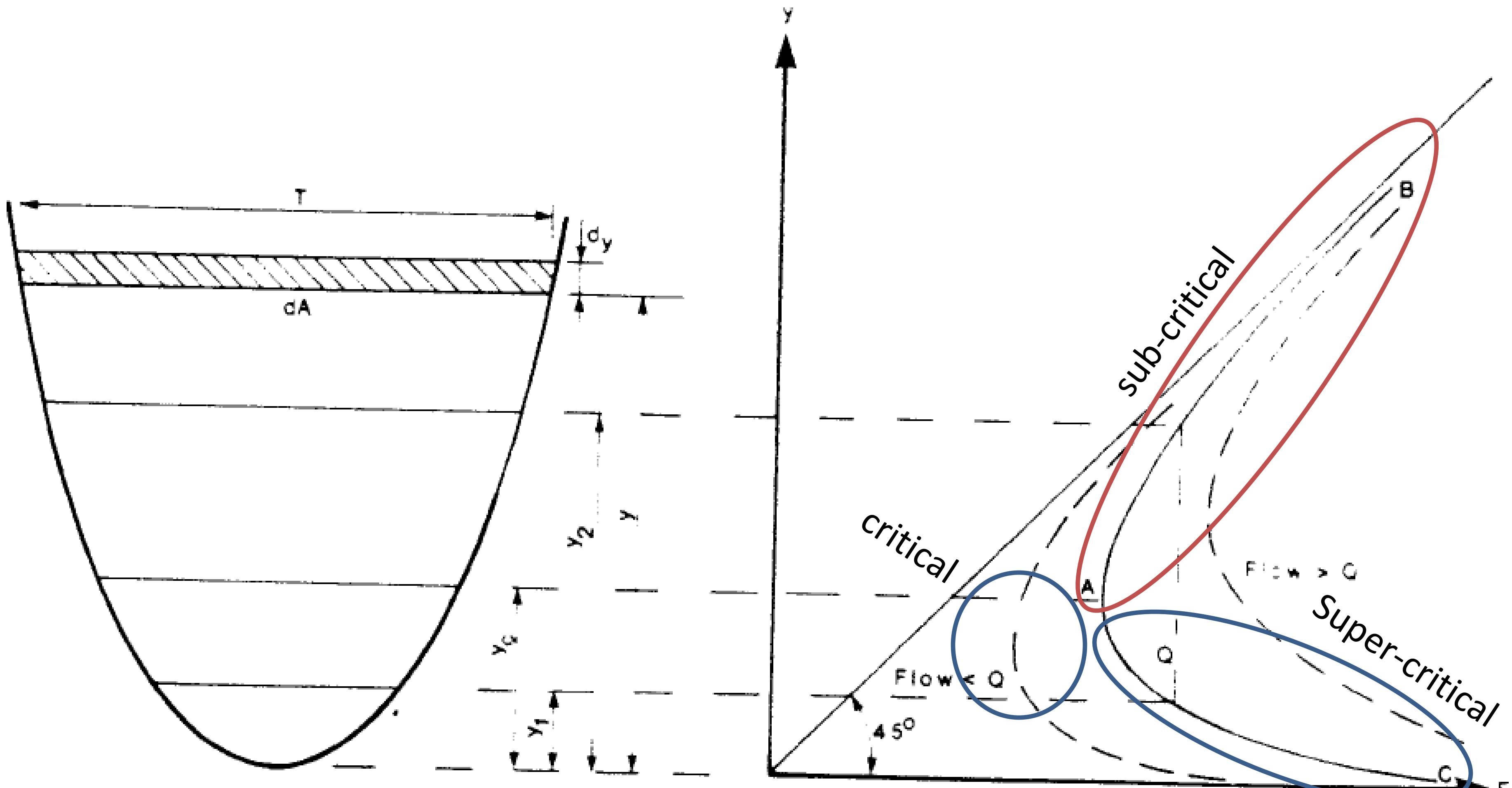
$$H = z + \frac{p}{\gamma} + \frac{u^2}{2g}$$

where H = total energy
z = elevation of streamline above the datum
p = pressure
γ = fluid specific weight
p/γ = pressure head
u = streamline velocity
$u^2/2g$ = velocity head
g = local acceleration of gravity

Critical flow

- **state at which the specific energy E is a minimum for a given q**
- the corresponding depth is known as ***Critical depth (y_c).***

$$\frac{\bar{u}}{\sqrt{gD}} = \mathbf{F} = 1$$



T
d_y
dA
y_2
y
y_c
y_1
y
E
B
A
Q
C
Flow > Q
Flow < Q
45°
sub-critical
critical
Super-critical

Fr value also determines the regime of flow

- Fr < 1 sub-critical ($y > y_c$)
    - upstream levels affected by downstream controls
- Fr = 1 critical
- Fr > 1 super-critical ($y < y_c$)
    - upstream levels not affected by downstream controls

Methods of estimating Y_c

- Algebraic solution
- Semi empirical equations
- Design charts

1. Algebraic solution

$$E = y + \alpha \frac{\overline{u}^2}{2g} = y + \alpha \frac{Q^2}{2gA^2}$$

$$\frac{dE}{dy} = 1 - \alpha \frac{Q^2}{gA^3} \frac{dA}{dy} = 0$$

$$D = \frac{A}{T}$$

Substituting $\alpha = 1$

$$1 - \frac{Q^2}{gA^3} \frac{dA}{dy} = 1 - \frac{Q^2}{gA^2} \frac{T}{A} = 1 - \frac{\overline{u}^2}{g} D = 0$$

$$\frac{Q_1^2 T_c}{g A_c^3} = 1$$

$$\frac{\overline{u}^2}{2g} = \frac{D}{2}$$

$$\frac{\overline{u}}{\sqrt{gD}} = \mathbf{F} = 1$$

Rectangular channels

$$q = \frac{Q}{b} \quad \Rightarrow \quad \overline{u} = \frac{q}{y}$$

and for a rectangular channel, $y = D$. With these definitions, can be rearranged to yield

$$y_c = \left(\frac{q^2}{g}\right)^{1/3}$$

Substitution of the above definitions

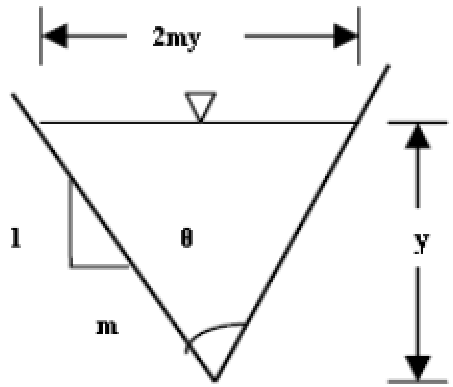
$$\frac{\overline{u}_c^2}{2g} = \frac{1}{2} y_c$$

And using the definition of specific energy,

$$y_c = \tfrac{2}{3} E_c$$

where E_c = specific energy at critical depth and velocity.

B) Triangular Channel



For triangular channel having side slope of m or (H: V= m: 1)

$$A = my^2 \quad \text{and} \quad T = 2my$$

$$\frac{Q^2 T_c}{g A_c^3} = 1$$

$$\frac{Q^2}{g} = \frac{A_c^3}{T_c} = \frac{m^3 y_c^6}{2my_c} = \frac{m^2 y_c^5}{2}$$

$$y_c = \left(\frac{2Q^2}{gm^2} \right)^{1/5}$$

The specific energy at critical water depth,

$$E_c = y_c + \frac{V_c^2}{2g} = y_c + \frac{Q^2}{2gA_c^2}$$

$$E_c = y_c + \frac{gm^2 y_c^5}{2 \times 2 \times g \times m^2 \times y_c^4}$$

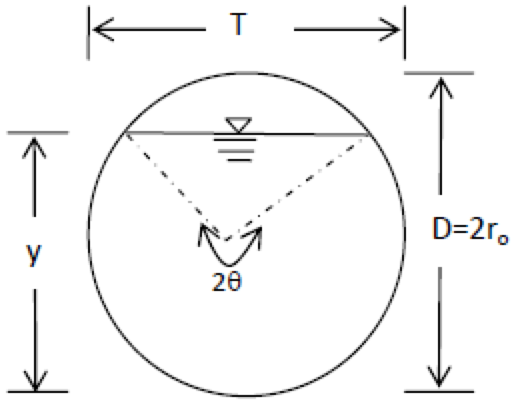
$$E_c = y_c + \frac{y_c}{4} = 1.25 y_c$$

$$F_r = \frac{V}{\sqrt{g \frac{A}{T}}} = \frac{V}{\sqrt{g \frac{my^2}{2my}}}$$

$$F_r = \frac{V\sqrt{2}}{\sqrt{gy}}$$

C) Circular Channel

Let D be the diameter of a circular channel and 2θ be the angel in radians subtended by the water surface at the center.



A = area of the flow section

= area of the sector + area of triangular portion

$$A=\frac{1}{2}r_o^2 2\theta+\frac{1}{2}2r_o\sin(\pi-\theta)r_o\cos(\pi-\theta)$$

$$A=\frac{1}{2}r_o^2 2\theta+\frac{1}{2}2r_o{}^2(2\sin(\pi-\theta)\cos(\pi-\theta))$$

$$A=\frac{1}{2}r_o^2 2\theta-\frac{1}{2}2r_o{}^2(\sin 2\theta)$$

$$A=\frac{1}{2}r_o^2(2\theta-\sin 2\theta)$$

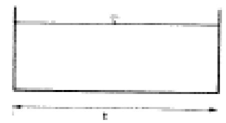
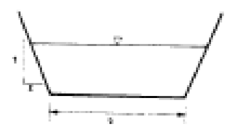
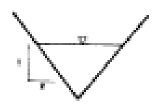
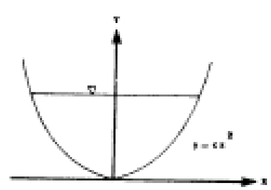
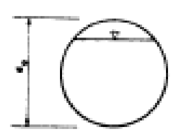
$$\boxed{A=\frac{1}{8}D^2(2\theta-\sin 2\theta)}$$

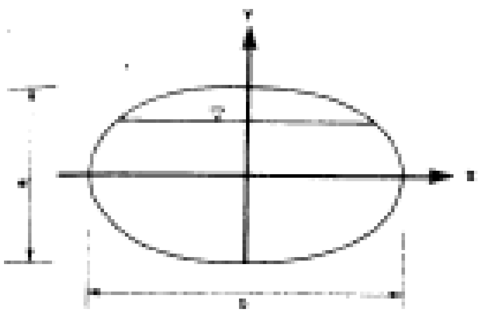
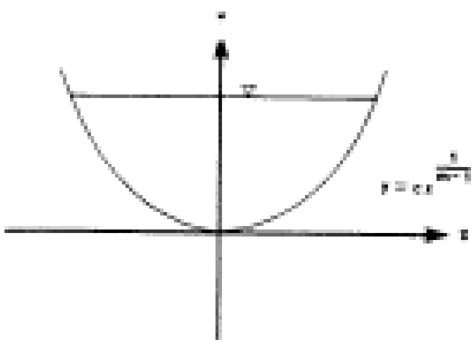
$$\boxed{T=D\sin\theta}$$

For critical flow state

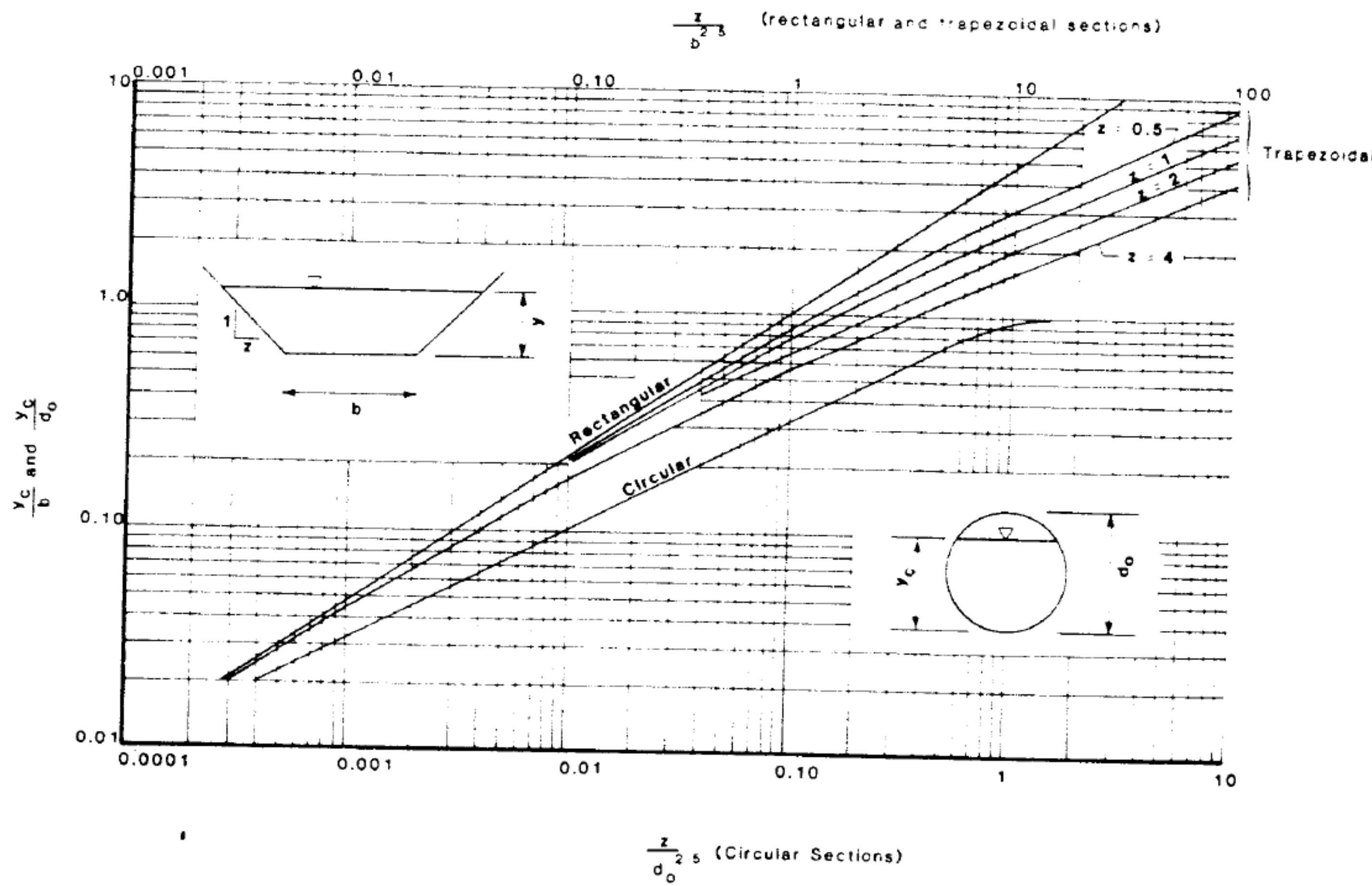
$$\frac{Q^2}{g}=\frac{A_c^3}{T_c}=\frac{[\frac{D^2}{8}(2\theta_c-\sin 2\theta_c)]^3}{D\sin\theta_c}$$

2.Semi empirical Equations of estimating Y_c (straub 1982)

Channel type	Equation for y_c in terms of $\psi = \alpha Q^2/g$	
Rectangular 	$\left(\frac{\psi}{b^2}\right)^{1/3}$	
Trapezoidal 	$0.81\left(\frac{\psi}{z^{0.75}b^{1.25}}\right)^{0.27} - \frac{b}{30z}$	Range of applicability $0.1 < \frac{Q}{b^{2.5}} < 0.4$ For $\frac{Q}{b^{2.5}} < 0.1$ use equation for rectangular channel
Triangular 	$\left(\frac{2\psi}{z^2}\right)^{0.20}$	
Parabolic 	$(0.84c\psi)^{0.25}$	Perimeter equation $y = cx^2$
Circular 	$\left(\frac{1.01}{d_0^{0.26}}\right)\psi^{0.25}$	Range of applicability $0.02 \le \frac{y_c}{d_0} \le 0.85$

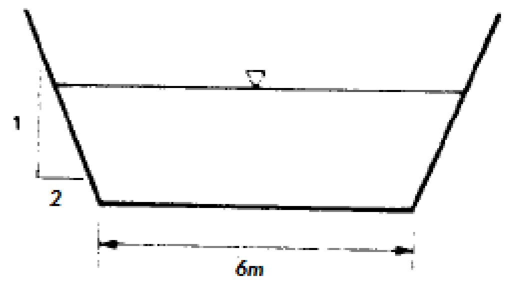
Channel type	Equation for y_c in terms of $\psi = \alpha Q^2/g$	
Elliptical 	$0.84 b^{0.22} \left(\frac{\psi}{a^2}\right)^{0.25}$	Range of applicability $0.05 \leq \frac{y_c}{2b} \leq 0.85$ a = major axis b = minor axis
Exponential 	$\left[\frac{m^3 \psi c^{2m-2}}{4}\right]^{1/(2m+1)}$	Perimeter equation $y = cx^{1/(m-1)}$

3.Curves for estimating critical depth



Example 1

For a trapezoidal channel with base width $b = 6.0$ m and side slope $z = 2$, calculate the critical depth of flow if $Q = 17$ m^3/s



Solution

$$A = (b + zy)y = (6.0 + 2y)y$$

$$T = b + 2zy = 6 + 4y$$

$$D = \frac{A}{T} = \frac{(3 + y)y}{3 + 2y}$$

and

$$\bar{u} = \frac{Q}{A} = \frac{17}{2\,(3 + y)y}$$

Substitution of the above .

$$\frac{[17/(6 + 2y)]^2}{g} = \frac{(3 + y)y}{3 + 2y}$$

Simplifying,

$$7.4(3 + 2y) = [(3 + y)y]^3$$

By trial and error, the critical depth is approximately

$$y_c = 0.84 \text{ m}$$

and the corresponding critical velocity is

$$u_c = \frac{17}{[6 + 2(0.84)]0.84} = 2.6 \text{ m/s}$$

Section factor for critical flow

$$Z = A\sqrt{\frac{A}{T}} = A\sqrt{D} \Rightarrow Z^2 = A^2 D \Rightarrow D = \frac{Z^2}{A^2}$$

For critical flow $\frac{V^2}{2g} = \frac{D}{2}$

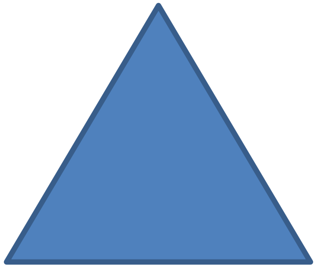
$$\frac{V^2}{2g} = \frac{D}{2} = \frac{z^2}{2A^2} \Rightarrow Z^2 = \frac{V^2 A^2}{g} \Rightarrow Z = \frac{VA}{\sqrt{g}} \Rightarrow Z = \frac{Q_c}{\sqrt{g}}$$

$$Q_c = Z_c\sqrt{g}$$

$$Q_c = Z_c\sqrt{\frac{g}{\alpha}}$$

Section factor ….

$$Z = by^{1.5}$$



$$Z = \frac{\sqrt{2}}{2} my^{2.5}$$

$$Z = \frac{[y * (b + my)]^{1.5}}{\sqrt{b + 2my}}$$

First Hydraulic Exponent (M)

- In many computations involving a wide range of depths in channel, such as in the GVF computations, it is convenient to express the variation of Z with y in an exponential form.
- The (Z-y) relationship

$$Z^2 = C_1 y^M$$

- In this equation
  - C_1= a coefficient and
  - M= an exponent called first hydraulic exponent.
- It is found that generally M is a slowly –varying function of the aspect ratio for most of the channel shape

$$M = \frac{y}{A}\left[3T - \frac{A}{T}\frac{dT}{dy}\right]$$

Example 3

Obtain the value of the first hydraulic exponent

(M) for

a) Rectangular channel

b) Exponential channel where the area

$A=K^{1}y^{a}$

The introduction of the concepts of specific energy and critical flow makes it possible to discuss the reaction of the flow in a channel to changes in the shape of the channel and hydraulic structures for different steady-flow regimes.

At any cross section, the total energy is

$$H = \frac{\overline{u}^2}{2g} + y + z$$