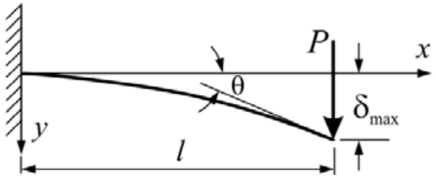
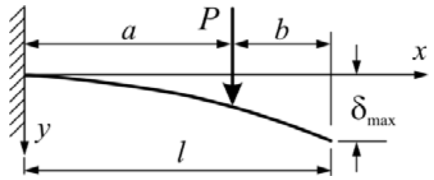
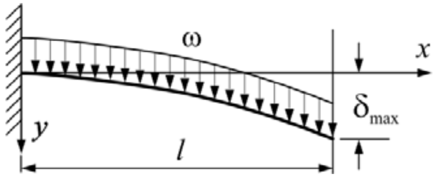
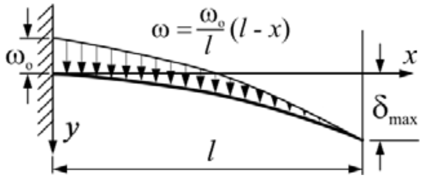
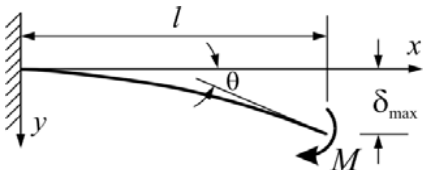
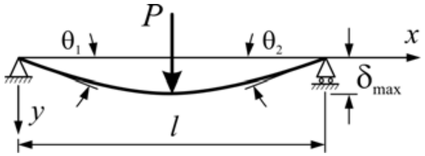
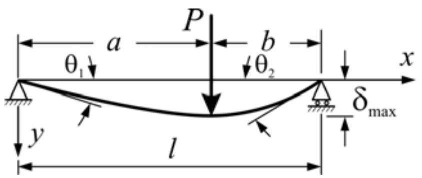
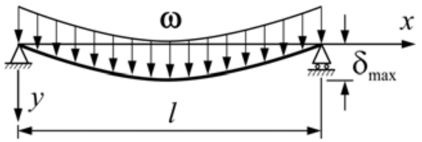
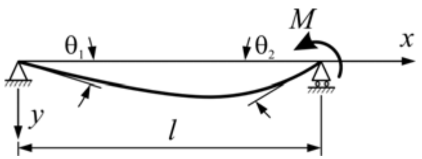
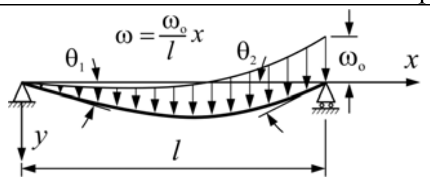
BEAM DEFLECTION FORMULAE

BEAM TYPE	SLOPE AT FREE END	DEFLECTION AT ANY SECTION IN TERMS OF *x*	MAXIMUM DEFLECTION
1. Cantilever Beam – Concentrated load *P* at the free end			
	$\theta = \frac{Pl^2}{2EI}$	$y = \frac{Px^2}{6EI}(3l - x)$	$\delta_{max} = \frac{Pl^3}{3EI}$
2. Cantilever Beam – Concentrated load *P* at any point			
	$\theta = \frac{Pa^2}{2EI}$	$y = \frac{Px^2}{6EI}(3a - x)$ for $0 < x < a$ $y = \frac{Pa^2}{6EI}(3x - a)$ for $a < x < l$	$\delta_{max} = \frac{Pa^2}{6EI}(3l - a)$
3. Cantilever Beam – Uniformly distributed load ω (N/m)			
	$\theta = \frac{\omega l^3}{6EI}$	$y = \frac{\omega x^2}{24EI}(x^2 + 6l^2 - 4lx)$	$\delta_{max} = \frac{\omega l^4}{8EI}$
4. Cantilever Beam – Uniformly varying load: Maximum intensity ω_o (N/m)			
	$\theta = \frac{\omega_o l^3}{24EI}$	$y = \frac{\omega_o x^2}{120lEI}(10l^3 - 10l^2 x + 5lx^2 - x^3)$	$\delta_{max} = \frac{\omega_o l^4}{30EI}$
5. Cantilever Beam – Couple moment *M* at the free end			
	$\theta = \frac{Ml}{EI}$	$y = \frac{Mx^2}{2EI}$	$\delta_{max} = \frac{Ml^2}{2EI}$

BEAM DEFLECTION FORMULAS

BEAM TYPE	SLOPE AT ENDS	DEFLECTION AT ANY SECTION IN TERMS OF *x*	MAXIMUM AND CENTER DEFLECTION
6. Beam Simply Supported at Ends – Concentrated load *P* at the center			
	$\theta_1 = \theta_2 = \frac{Pl^2}{16EI}$	$y = \frac{Px}{12EI}\left(\frac{3l^2}{4} - x^2\right)$ for $0 < x < \frac{l}{2}$	$\delta_{max} = \frac{Pl^3}{48EI}$
7. Beam Simply Supported at Ends – Concentrated load *P* at any point			
	$\theta_1 = \frac{Pb(l^2 - b^2)}{6lEI}$ $\theta_2 = \frac{Pab(2l - b)}{6lEI}$	$y = \frac{Pbx}{6lEI}\left(l^2 - x^2 - b^2\right)$ for $0 < x < a$ $y = \frac{Pb}{6lEI}\left[\frac{l}{b}(x-a)^3 + \left(l^2 - b^2\right)x - x^3\right]$ for $a < x < l$	$\delta_{max} = \frac{Pb\left(l^2 - b^2\right)^{3/2}}{9\sqrt{3}\, lEI}$ at $x = \sqrt{\left(l^2 - b^2\right)/3}$ $\delta = \frac{Pb}{48EI}\left(3l^2 - 4b^2\right)$ at the center, if $a > b$
8. Beam Simply Supported at Ends – Uniformly distributed load ω (N/m)			
	$\theta_1 = \theta_2 = \frac{\omega l^3}{24EI}$	$y = \frac{\omega x}{24EI}\left(l^3 - 2lx^2 + x^3\right)$	$\delta_{max} = \frac{5\omega l^4}{384EI}$
9. Beam Simply Supported at Ends – Couple moment *M* at the right end			
	$\theta_1 = \frac{Ml}{6EI}$ $\theta_2 = \frac{Ml}{3EI}$	$y = \frac{Mlx}{6EI}\left(1 - \frac{x^2}{l^2}\right)$	$\delta_{max} = \frac{Ml^2}{9\sqrt{3}\, EI}$ at $x = \frac{l}{\sqrt{3}}$ $\delta = \frac{Ml^2}{16EI}$ at the center
10. Beam Simply Supported at Ends – Uniformly varying load: Maximum intensity ω_o (N/m)			
	$\theta_1 = \frac{7\omega_o l^3}{360EI}$ $\theta_2 = \frac{\omega_o l^3}{45EI}$	$y = \frac{\omega_o x}{360lEI}\left(7l^4 - 10l^2x^2 + 3x^4\right)$	$\delta_{max} = 0.00652\frac{\omega_o l^4}{EI}$ at $x = 0.519l$ $\delta = 0.00651\frac{\omega_o l^4}{EI}$ at the center

http://www.advancepipeliner.com/Resources/Others/Beams/Beam_Deflection_Formulae.pdf

From "Handbook of Eng'g Mechanics", W. Flugge (editor), McGraw-Hill, 1962

Table 61.1. Frequencies and Eigenfunctions for Uniform Beams

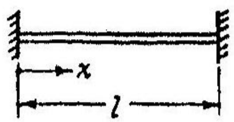
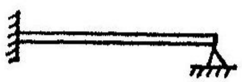
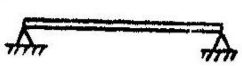
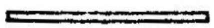
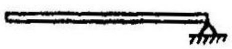
Type	Boundary conditions	Frequency equation	Eigenfunction $\phi_n(x)$	Roots of frequency equation λ_n
Clamped-clamped 	$\phi(0) = \phi'(0) = 0$ $\phi(l) = \phi'(l) = 0$	$\cos\lambda \operatorname{Cosh}\lambda = 1$	$J\left(\frac{\lambda_n x}{l}\right) - \frac{J(\lambda_n)}{H(\lambda_n)} H\left(\frac{\lambda_n x}{l}\right)$	$\lambda_1 = 4.7300$ $\lambda_2 = 7.8532$ $\lambda_3 = 10.9956$ $\lambda_4 = 14.1372$ For n large, $\lambda_n \approx (2n+1)\pi/2$
Clamped-hinged 	$\phi(0) = \phi'(0) = 0$ $\phi(l) = \phi''(l) = 0$	$\tan\lambda = \operatorname{Tanh}\lambda$	$J\left(\frac{\lambda_n x}{l}\right) - \frac{J(\lambda_n)}{H(\lambda_n)} H\left(\frac{\lambda_n x}{l}\right)$	$\lambda_1 = 3.9266$ $\lambda_2 = 7.0686$ $\lambda_3 = 10.2102$ $\lambda_4 = 13.3518$ For n large, $\lambda_n \approx (4n+1)\pi/4$
Clamped-free 	$\phi(0) = \phi'(0) = 0$ $\phi''(l) = \phi'''(l) = 0$	$\cos\lambda \operatorname{Cosh}\lambda = -1$	$J\left(\frac{\lambda_n x}{l}\right) - \frac{G(\lambda_n)}{F(\lambda_n)} H\left(\frac{\lambda_n x}{l}\right)$	$\lambda_1 = 1.8751$ $\lambda_2 = 4.6941$ $\lambda_3 = 7.8548$ $\lambda_4 = 10.9955$ For n large, $\lambda_n \approx (2n-1)\pi/2$
Clamped-guided 	$\phi(0) = \phi'(0) = 0$ $\phi'(l) = \phi'''(l) = 0$	$\tan\lambda = -\operatorname{Tanh}\lambda$	$J\left(\frac{\lambda_n x}{l}\right) - \frac{F(\lambda_n)}{J(\lambda_n)} H\left(\frac{\lambda_n x}{l}\right)$	$\lambda_1 = 2.3650$ $\lambda_2 = 5.4978$ $\lambda_3 = 8.6394$ $\lambda_4 = 11.7810$ For n large, $\lambda_n \approx (4n-1)\pi/4$
Hinged-hinged 	$\phi(0) = \phi''(0) = 0$ $\phi(l) = \phi''(l) = 0$	$\sin\lambda = 0$	$\sin\frac{n\pi x}{l}$	$\lambda_n = n\pi$

Table 61.1. Frequencies and Eigenfunctions for Uniform Beams (*Continued*)

Type	Boundary conditions	Frequency equation	Eigenfunction $\phi_n(x)$	Roots of frequency equation λ_n
Hinged-guided 	$\phi(0) = \phi''(0) = 0$ $\phi'(l) = \phi'''(l) = 0$	$\cos \lambda = 0$	$\sin \frac{(2n-1)\pi x}{2l}$	$\lambda_n = (2n-1)\pi/2$
Guided-guided 	$\phi'(0) = \phi'''(0) = 0$ $\phi'(l) = \phi'''(l) = 0$	$\sin \lambda = 0$	$\cos \frac{n\pi x}{l}$	$\lambda_n = n\pi$
Free-free 	$\phi''(0) = \phi'''(0) = 0$ $\phi''(l) = \phi'''(l) = 0$	$\cos \lambda \operatorname{Cosh} \lambda = 1$	$G\left(\frac{\lambda_n x}{l}\right) - \frac{J(\lambda_n)}{H(\lambda_n)} F\left(\frac{\lambda_n x}{l}\right)$	Same as for clamped-clamped beam
Free-hinged 	$\phi''(0) = \phi'''(0) = 0$ $\phi(l) = \phi''(l) = 0$	$\tan \lambda = \operatorname{Tanh} \lambda$	$G\left(\frac{\lambda_n x}{l}\right) - \frac{G(\lambda_n)}{F(\lambda_n)} F\left(\frac{\lambda_n x}{l}\right)$	Same as for clamped-hinged beam
Free-guided 	$\phi''(0) = \phi'''(0) = 0$ $\phi'(l) = \phi'''(l) = 0$	$\tan \lambda = -\operatorname{Tanh} \lambda$	$G\left(\frac{\lambda_n x}{l}\right) - \frac{H(\lambda_n)}{F(\lambda_n)} F\left(\frac{\lambda_n x}{l}\right)$	Same as for clamped-guided beam

1. The circular frequency is

$$\omega_n = \frac{\lambda_n^2}{l^2}\sqrt{\frac{EI}{\mu}}$$

where

EI = bending stiffness
μ = mass per unit length
l = length of the beam

2. Notation used in expressions for the eigenfunctions:

$$F(u) = \operatorname{Sinh} u + \sin u$$
$$G(u) = \operatorname{Cosh} u + \cos u$$
$$H(u) = \operatorname{Sinh} u - \sin u$$
$$J(u) = \operatorname{Cosh} u - \cos u$$