Structural Analysis

Civil Engineering Course Material from IIT Kharagpur, cover structural analysis, in seven modules (40 lessons)

Version 2 CE

IIT, Kharagpur

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Contents:

Module
1
Energy Methods in Structural Analysis

Lesson 1

General Introduction

Instructional Objectives

After reading this chapter the student will be able to

1. Differentiate between various structural forms such as beams, plane truss, space truss, plane frame, space frame, arches, cables, plates and shells.
2. State and use conditions of static equilibrium.
3. Calculate the degree of static and kinematic indeterminacy of a given structure such as beams, truss and frames.
4. Differentiate between stable and unstable structure.
5. Define flexibility and stiffness coefficients.
6. Write force-displacement relations for simple structure.

1.1 Introduction

Structural analysis and design is a very old art and is known to human beings since early civilizations. The Pyramids constructed by Egyptians around 2000 B.C. stands today as the testimony to the skills of master builders of that civilization. Many early civilizations produced great builders, skilled craftsmen who constructed magnificent buildings such as the Parthenon at Athens (2500 years old), the great Stupa at Sanchi (2000 years old), Taj Mahal (350 years old), Eiffel Tower (120 years old) and many more buildings around the world. These monuments tell us about the great feats accomplished by these craftsmen in analysis, design and construction of large structures. Today we see around us countless houses, bridges, fly-overs, high-rise buildings and spacious shopping malls. Planning, analysis and construction of these buildings is a science by itself. The main purpose of any structure is to support the loads coming on it by properly transferring them to the foundation. Even animals and trees could be treated as structures. Indeed biomechanics is a branch of mechanics, which concerns with the working of skeleton and muscular structures. In the early periods houses were constructed along the riverbanks using the locally available material. They were designed to withstand rain and moderate wind. Today structures are designed to withstand earthquakes, tsunamis, cyclones and blast loadings. Aircraft structures are designed for more complex aerodynamic loadings. These have been made possible with the advances in structural engineering and a revolution in electronic computation in the past 50 years. The construction material industry has also undergone a revolution in the last four decades resulting in new materials having more strength and stiffness than the traditional construction material.

In this book we are mainly concerned with the analysis of framed structures (*beam, plane truss, space truss, plane frame, space frame and grid*), arches, cables and suspension bridges subjected to static loads only. The methods that we would be presenting in this course for analysis of structure were developed based on certain energy principles, which would be discussed in the first module.

1.2 Classification of Structures

All structural forms used for load transfer from one point to another are 3-dimensional in nature. In principle one could model them as 3-dimensional elastic structure and obtain solutions (response of structures to loads) by solving the associated partial differential equations. In due course of time, you will appreciate the difficulty associated with the 3-dimensional analysis. Also, in many of the structures, one or two dimensions are smaller than other dimensions. This geometrical feature can be exploited from the analysis point of view. The dimensional reduction will greatly reduce the complexity of associated governing equations from 3 to 2 or even to one dimension. This is indeed at a cost. This reduction is achieved by making certain assumptions (like Bernoulli-Euler' kinematic assumption in the case of beam theory) based on its observed behaviour under loads. Structures may be classified as 3-, 2- and 1-dimensional (see Fig. 1.1(a) and (b)). This simplification will yield results of reasonable and acceptable accuracy. Most commonly used structural forms for load transfer are: beams, plane truss, space truss, plane frame, space frame, arches, cables, plates and shells. Each one of these structural arrangement supports load in a specific way.

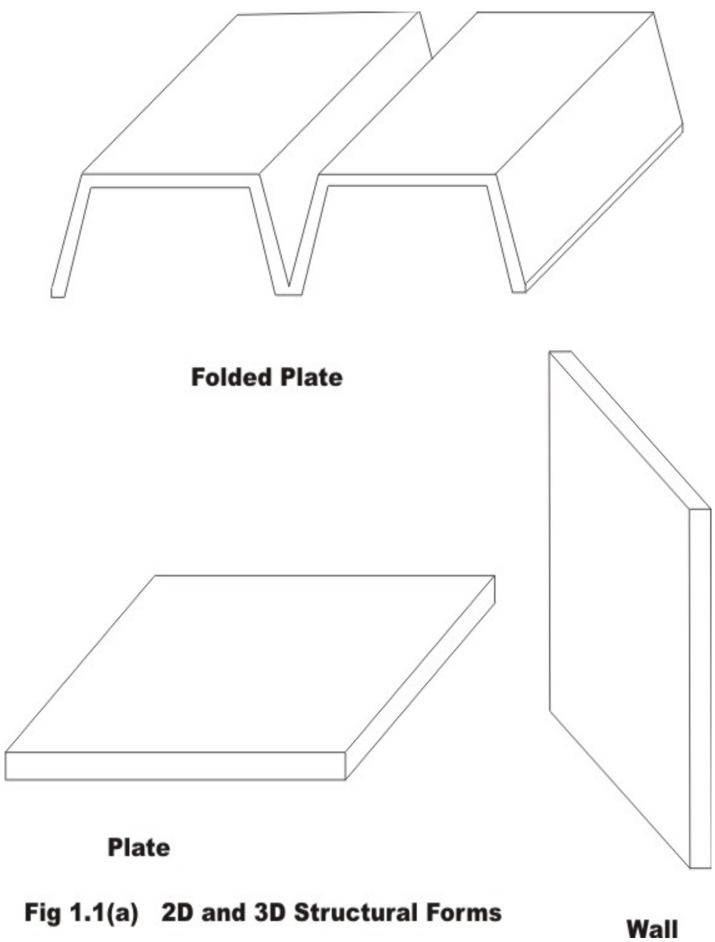


Fig 1.1(a) 2D and 3D Structural Forms

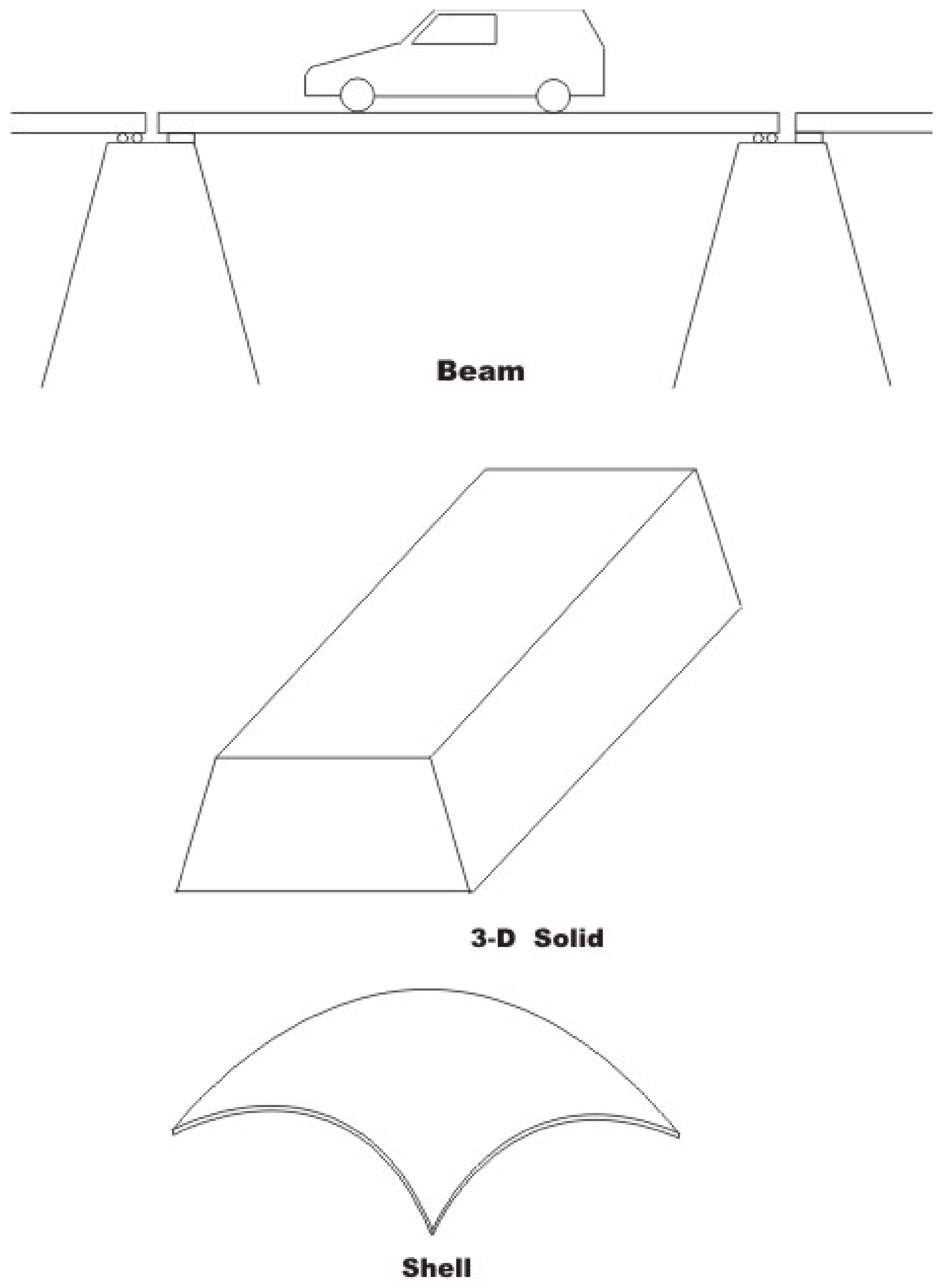
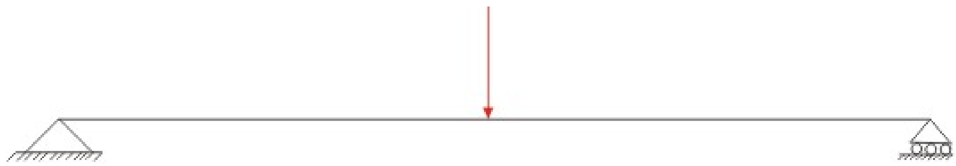
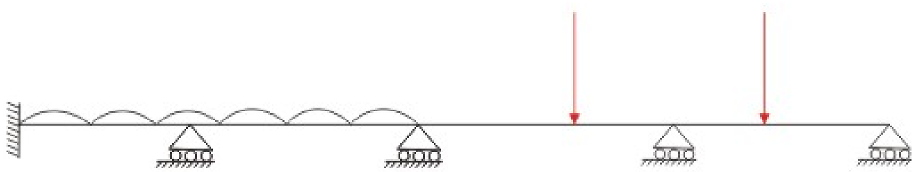


Fig 1.1(b) Commonly Used Structural Forms

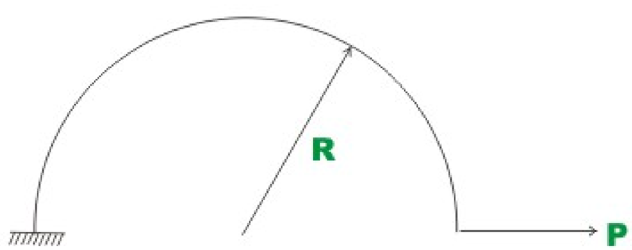
Beams are the simplest structural elements that are used extensively to support loads. They may be straight or curved ones. For example, the one shown in Fig. 1.2 (a) is hinged at the left support and is supported on roller at the right end. Usually, the loads are assumed to act on the beam in a plane containing the axis of symmetry of the cross section and the beam axis. The beams may be supported on two or more supports as shown in Fig. 1.2(b). The beams may be curved in plan as shown in Fig. 1.2(c). Beams carry loads by deflecting in the same plane and it does not twist. It is possible for the beam to have no axis of symmetry. In such cases, one needs to consider unsymmetrical bending of beams. In general, the internal stresses at any cross section of the beam are: bending moment, shear force and axial force.



(a) Simply Supported Beam



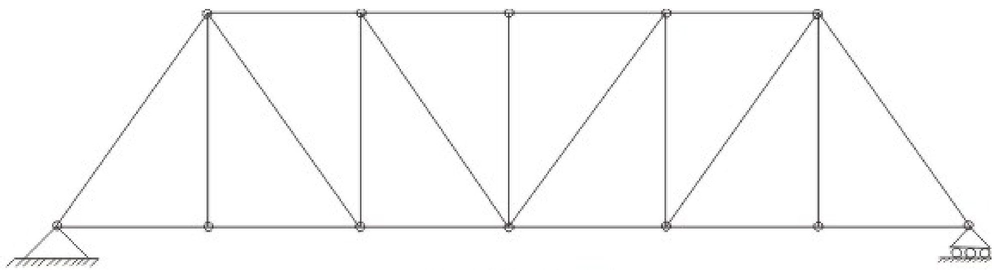
(b) Continuous Beam



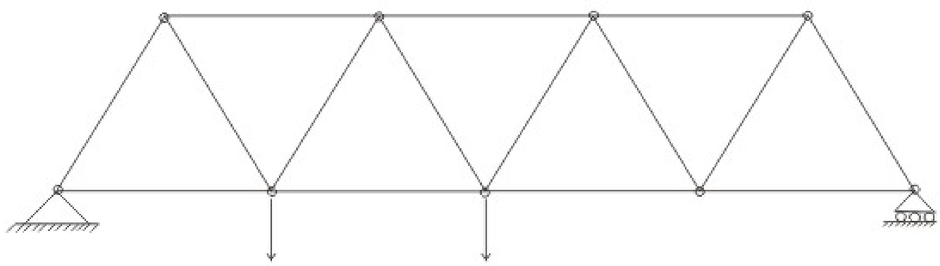
(c) Curved Beam

Fig 1.2 Beams

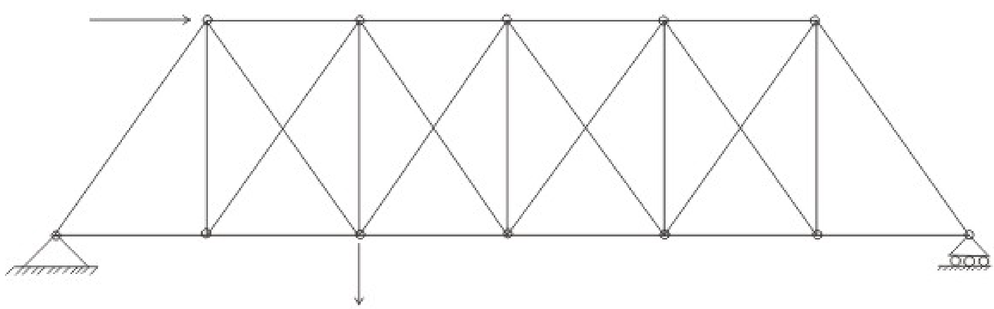
In India, one could see ***plane trusses*** (vide Fig. 1.3 (a),(b),(c)) commonly in Railway bridges, at railway stations, and factories. Plane trusses are made of short thin members interconnected at hinges into triangulated patterns. For the purpose of analysis statically equivalent loads are applied at joints. From the above definition of truss, it is clear that the members are subjected to only axial forces and they are constant along their length. Also, the truss can have only hinged and roller supports. In field, usually joints are constructed as rigid by welding. However, analyses were carried out as though they were pinned. This is justified as the bending moments introduced due to joint rigidity in trusses are negligible. Truss joint could move either horizontally or vertically or combination of them. In ***space truss*** (Fig. 1.3 (d)), members may be oriented in any direction. However, members are subjected to only tensile or compressive stresses. Crane is an example of space truss.



(a) Pratt Truss

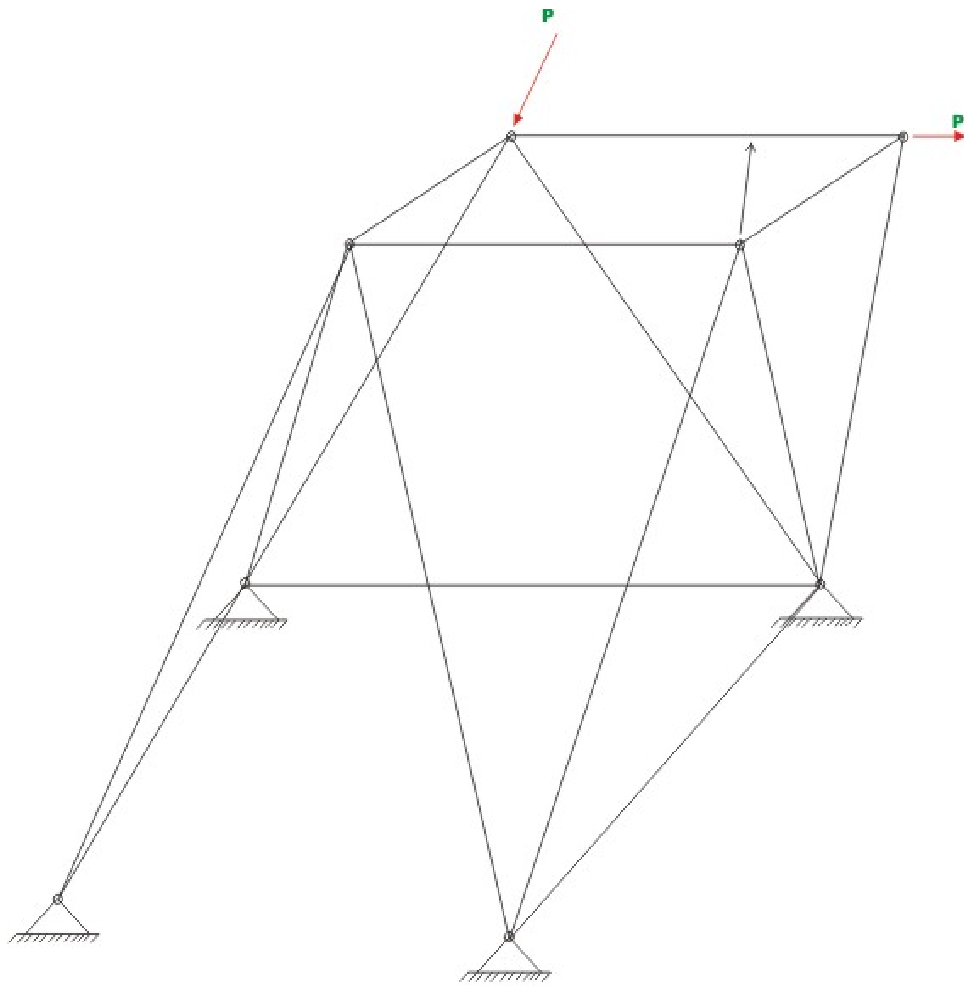


(b) Warren Truss



(c) Double Warren Truss

Fig 1.3 Trusses



(d) Space Truss

Plane frames are also made up of beams and columns, the only difference being they are rigidly connected at the joints as shown in the Fig. 1.4 (a). Major portion of this course is devoted to evaluation of forces in frames for variety of loading conditions. Internal forces at any cross section of the plane frame member are: bending moment, shear force and axial force. As against plane frame, ***space frames*** (vide Fig. 1.4 (b)) members may be oriented in any direction. In this case, there is no restriction of how loads are applied on the space frame.

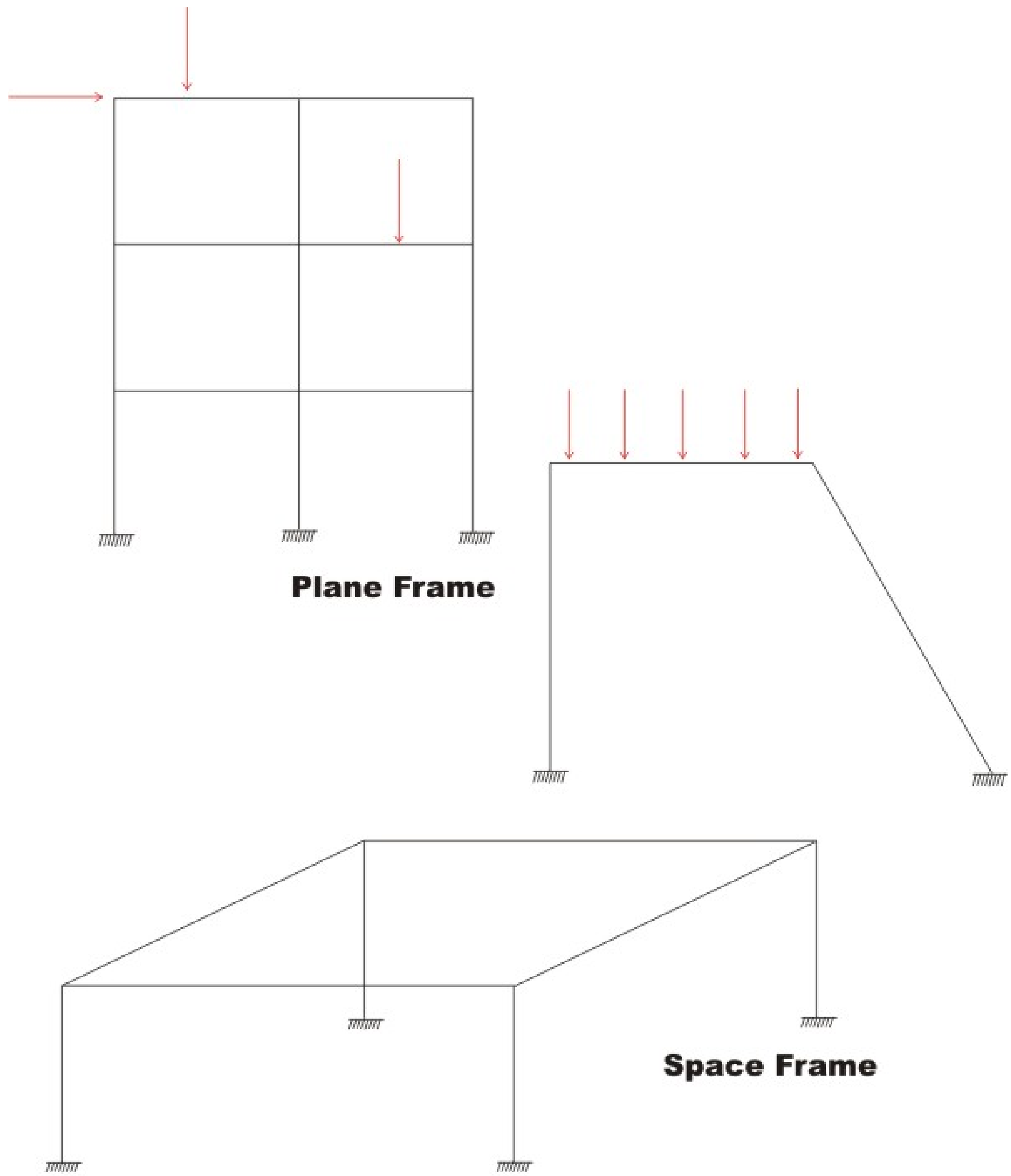


Fig 1.4 Frames

1.3 Equations of Static Equilibrium

Consider a case where a book is lying on a frictionless table surface. Now, if we apply a force F_1 horizontally as shown in the Fig.1.5 (a), then it starts moving in the direction of the force. However, if we apply the force perpendicular to the book as in Fig. 1.5 (b), then book stays in the same position, as in this case the vector sum of all the forces acting on the book is zero. When does an object move and when does it not? This question was answered by Newton when he formulated his famous second law of motion. In a simple vector equation it may be stated as follows:

$$\sum_{i=1}^{n} F_i = ma \tag{1.1}$$

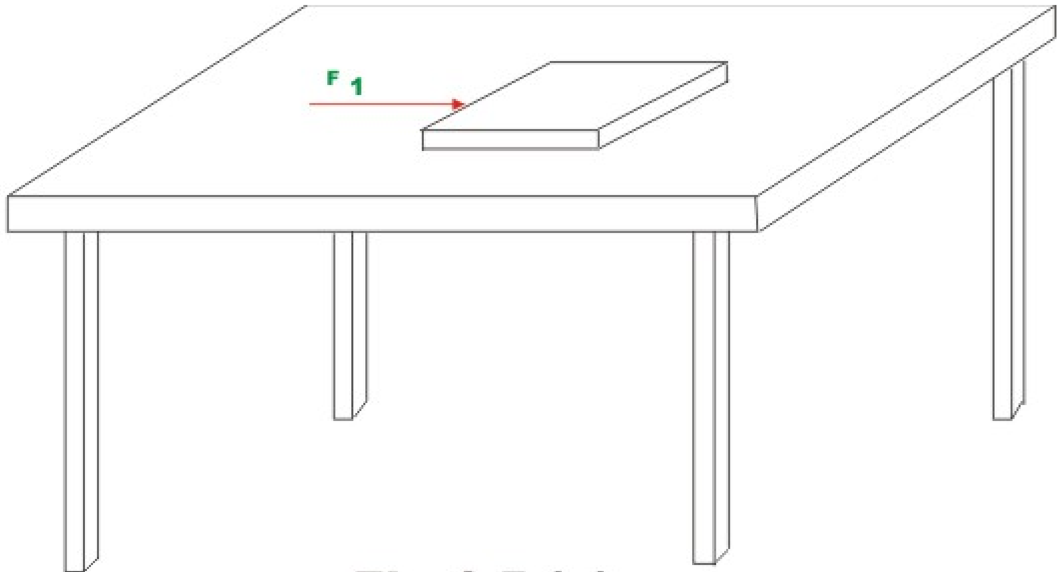


Fig 1.5 (a)

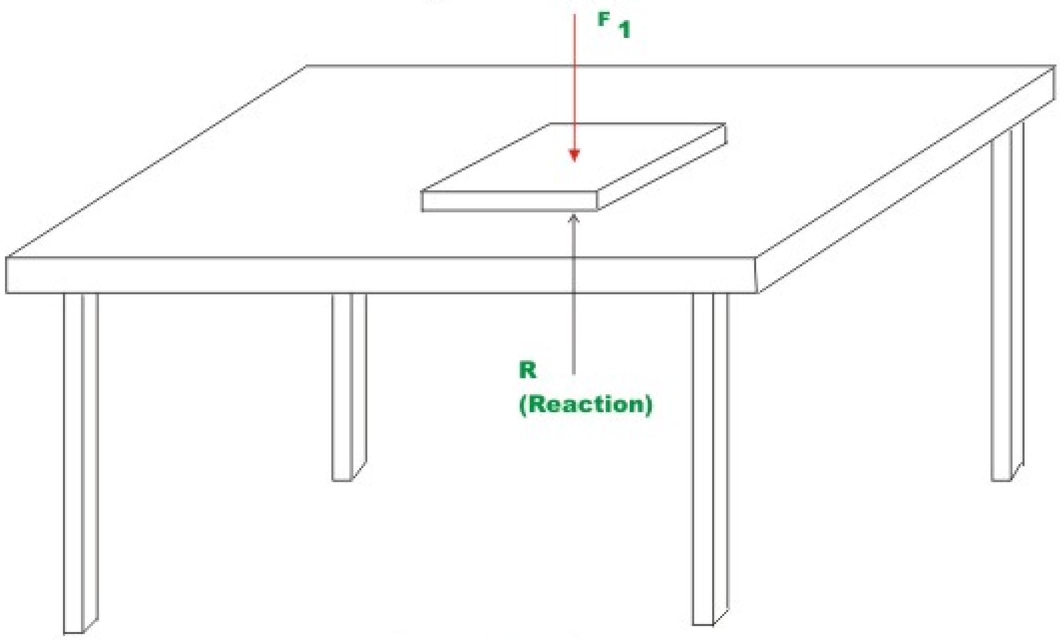


Fig 1.5(b)

where $\sum_{i=1}^{n} F_i$ is the vector sum of all the external forces acting on the body, m is the total mass of the body and a is the acceleration vector. However, if the body is in the state of static equilibrium then the right hand of equation (1.1) must be zero. Also for a body to be in equilibrium, the vector sum of all external moments $(\sum M = 0)$ about an axis through any point within the body must also vanish. Hence, the book lying on the table subjected to external force as shown in Fig. 1.5 (b) is in static equilibrium. The equations of equilibrium are the direct consequences of Newton's second law of motion. A vector in 3-dimensions can be resolved into three orthogonal directions viz., *x, y* and *z* (Cartesian) co-ordinate axes. Also, if the resultant force vector is zero then its components in three mutually perpendicular directions also vanish. Hence, the above two equations may also be written in three co-ordinate axes directions as follows:

$$\sum F_x = 0\,;\ \sum F_y = 0\,;\ \sum F_z = 0 \qquad (1.2a)$$

$$\sum M_x = 0\,;\sum M_y = 0\,;\sum M_z = 0 \qquad (1.2b)$$

Now, consider planar structures lying in $xy-$plane. For such structures we could have forces acting only in x and y directions. Also the only external moment that could act on the structure would be the one about the z-axis. For planar structures, the resultant of all forces may be a force, a couple or both. The static equilibrium condition along x-direction requires that there is no net unbalanced force acting along that direction. For such structures we could express equilibrium equations as follows:

$$\sum F_x = 0\,;\sum F_y = 0\,;\sum M_z = 0 \qquad (1.3)$$

Using the above three equations we could find out the reactions at the supports in the beam shown in Fig. 1.6. After evaluating reactions, one could evaluate internal stress resultants in the beam. Admissible or correct solution for reaction and internal stresses must satisfy the equations of static equilibrium for the entire structure. They must also satisfy equilibrium equations for any part of the structure taken as a free body. If the number of unknown reactions is more than the number of equilibrium equations (as in the case of the beam shown in Fig. 1.7), then we can not evaluate reactions with only equilibrium equations. Such structures are known as the statically indeterminate structures. In such cases we need to obtain extra equations (*compatibility equations*) in addition to equilibrium equations.

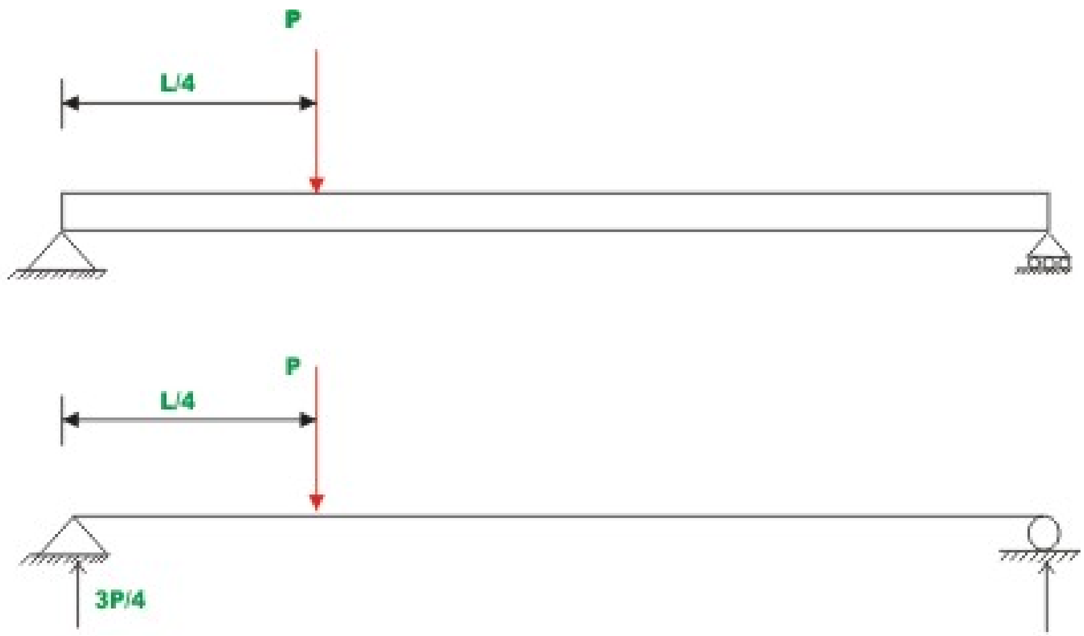


Fig 1.6 Statically Determinate Beam

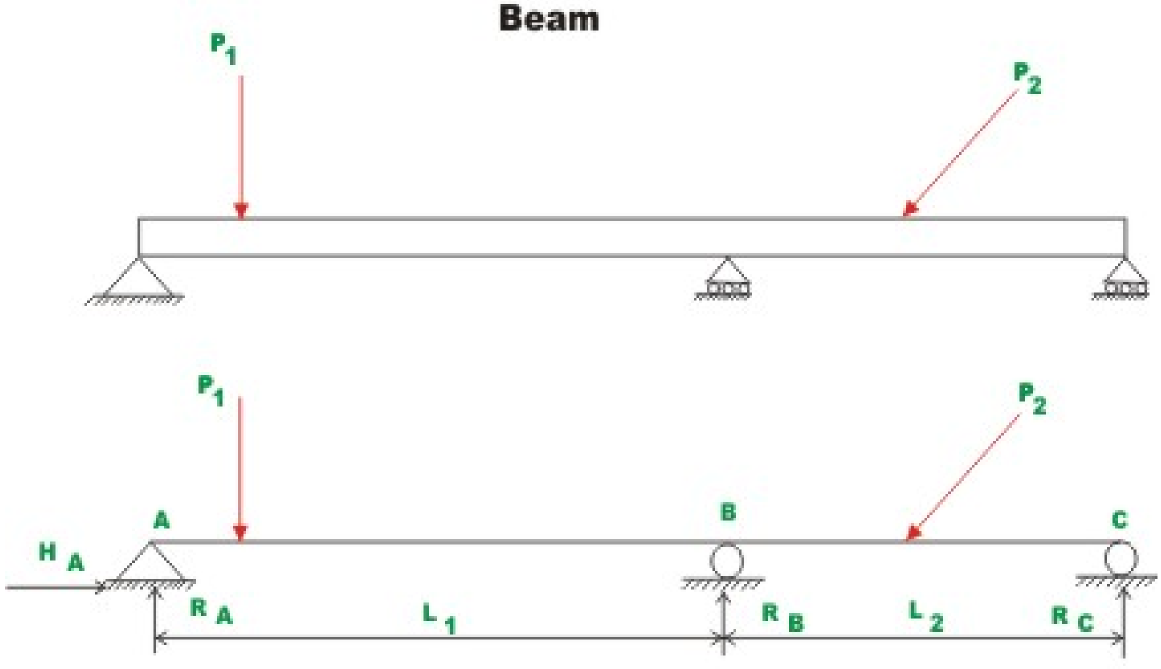


Fig 1.7 Statically Indeterminate Beam

1.4 Static Indeterminacy

The aim of structural analysis is to evaluate the external reactions, the deformed shape and internal stresses in the structure. If this can be accomplished by equations of equilibrium, then such structures are known as determinate structures. However, in many structures it is not possible to determine either reactions or internal stresses or both using equilibrium equations alone. Such structures are known as the statically indeterminate structures. The indeterminacy in a structure may be external, internal or both. A structure is said to be externally indeterminate if the number of reactions exceeds the number of equilibrium equations. Beams shown in Fig.1.8(a) and (b) have four reaction components, whereas we have only 3 equations of equilibrium. Hence the beams in Figs. 1.8(a) and (b) are externally indeterminate to the first degree. Similarly, the beam and frame shown in Figs. 1.8(c) and (d) are externally indeterminate to the 3^{rd} degree.

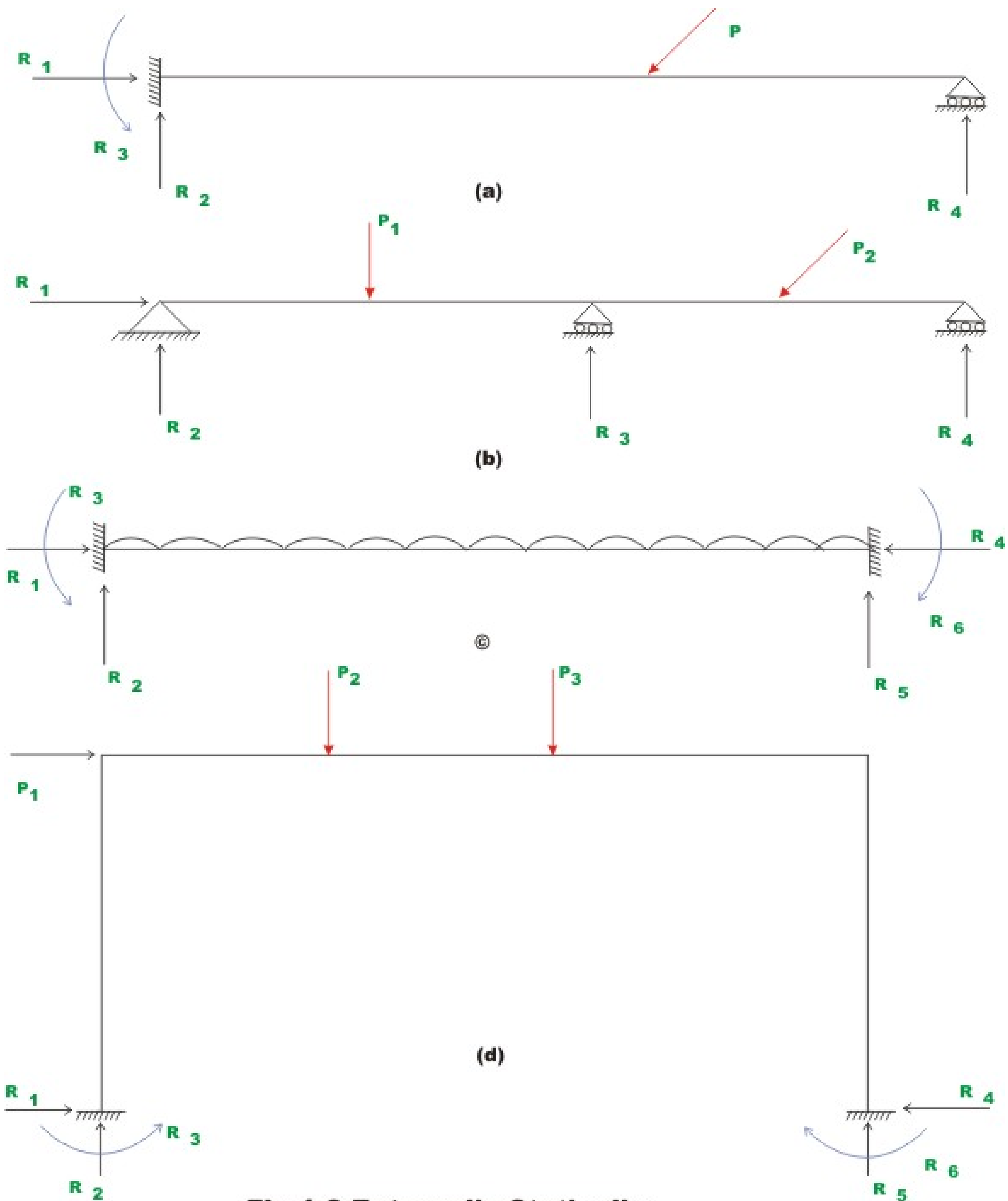
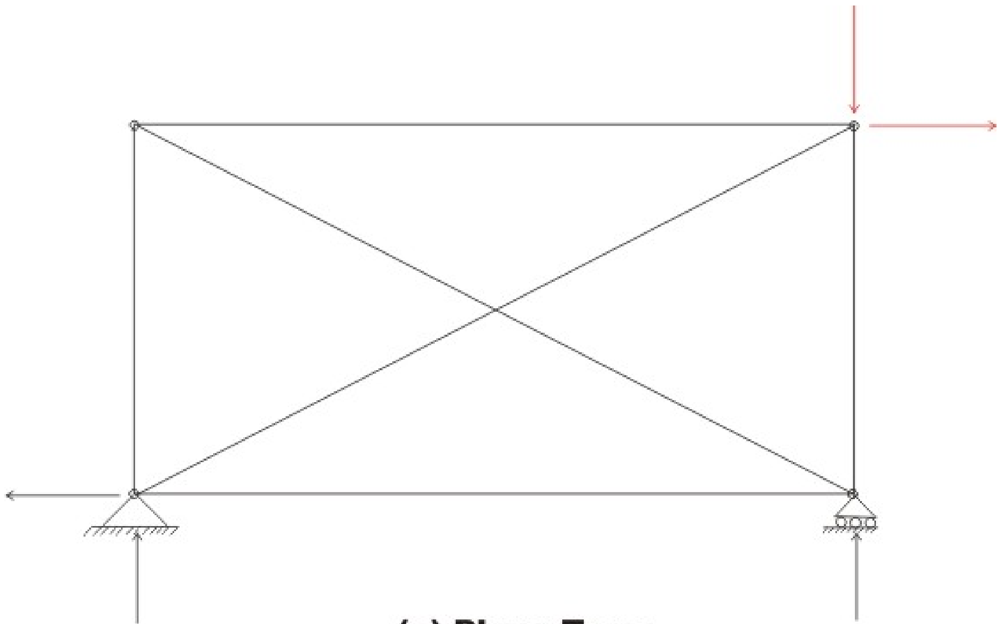
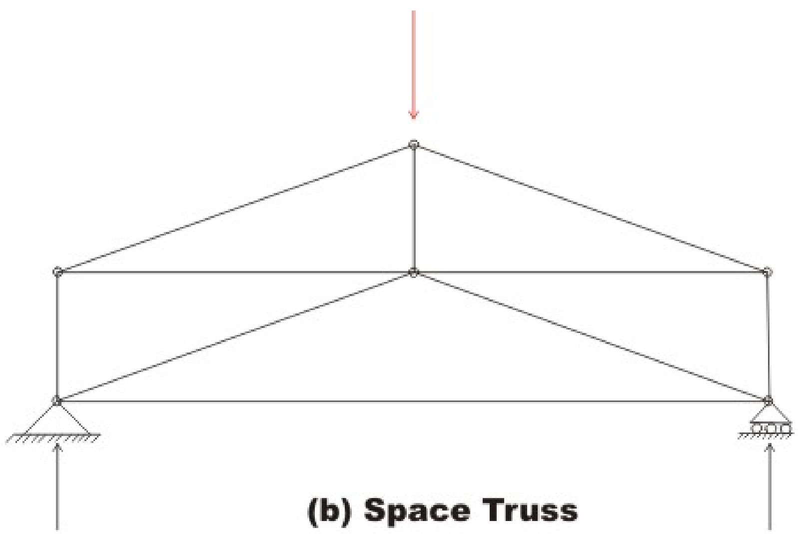


Fig 1.8 Externally Statically Indeterminate Structures

Now, consider trusses shown in Figs. 1.9(a) and (b). In these structures, reactions could be evaluated based on the equations of equilibrium. However, member forces can not be determined based on statics alone. In Fig. 1.9(a), if one of the diagonal members is removed (cut) from the structure then the forces in the members can be calculated based on equations of equilibrium. Thus, structures shown in Figs. 1.9(a) and (b) are internally indeterminate to first degree.The truss and frame shown in Fig. 1.10(a) and (b) are both externally and internally indeterminate.

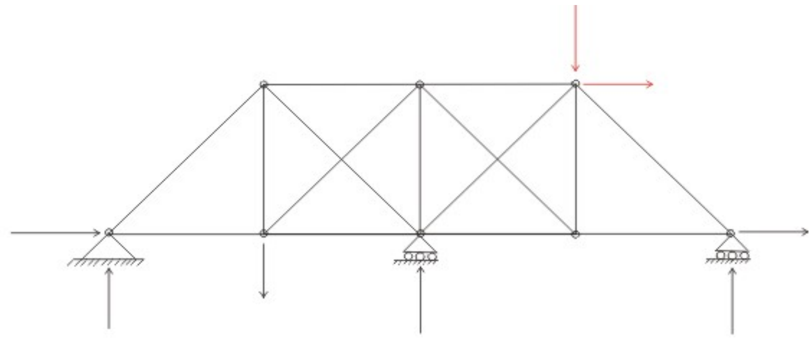


(a) Plane Truss

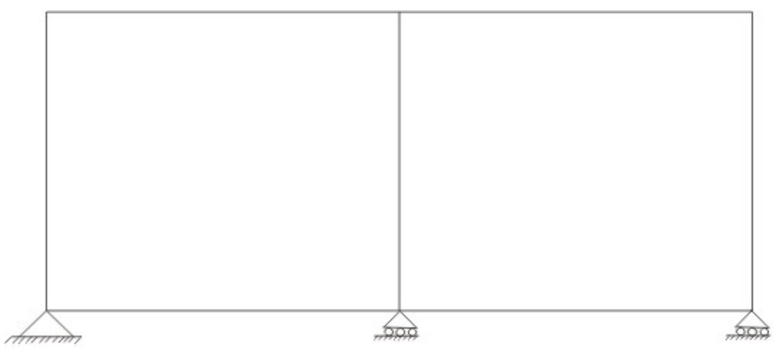


(b) Space Truss

Fig 1.9 Internally Statically Indeterminate Structures



(a) Plane Truss



(b) Plane Frame

Fig 1.10 Externally and Internally Indeterminate Structures

So far, we have determined the degree of indeterminacy by inspection. Such an approach runs into difficulty when the number of members in a structure increases. Hence, let us derive an algebraic expression for calculating degree of static indeterminacy.

Consider a planar stable truss structure having m members and j joints. Let the number of unknown reaction components in the structure be r. Now, the total number of unknowns in the structure is $m+r$. At each joint we could write two equilibrium equations for planar truss structure, viz., $\sum F_x = 0$ and $\sum F_y = 0$. Hence total number of equations that could be written is $2j$.

If $2j = m + r$ then the structure is statically determinate as the number of unknowns are equal to the number of equations available to calculate them.

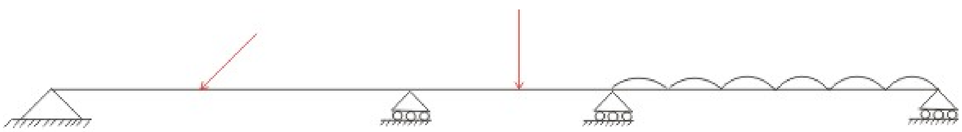
The degree of indeterminacy may be calculated as

$$i = (m+r) - 2j \qquad (1.4)$$

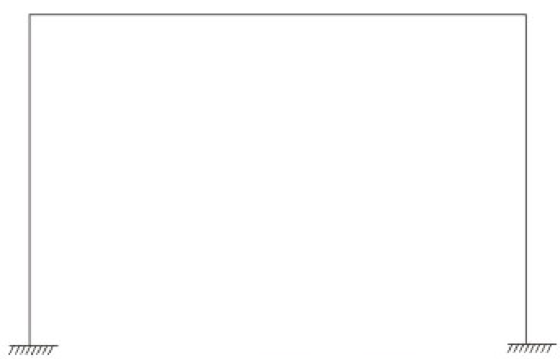
We could write similar expressions for space truss, plane frame, space frame and grillage. For example, the plane frame shown in Fig.1.11 (c) has 15 members, 12 joints and 9 reaction components. Hence, the degree of indeterminacy of the structure is

$$i = (15 \times 3 + 9) - 12 \times 3 = 18$$

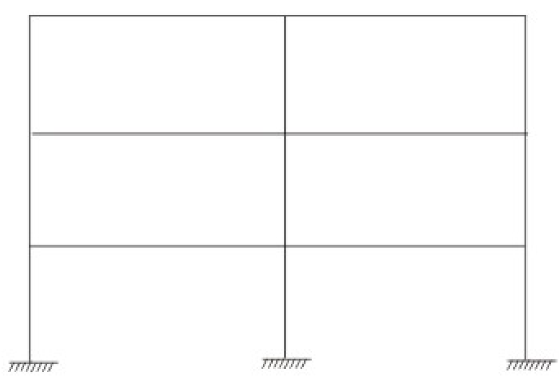
Please note that here, at each joint we could write 3 equations of equilibrium for plane frame.



(a) Continuous Beam



(b) Plane Frame

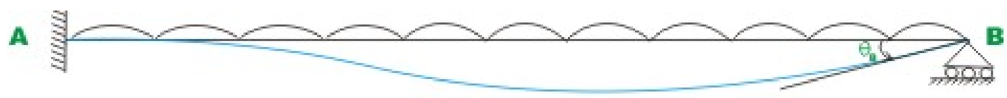


(c) Plane Frame

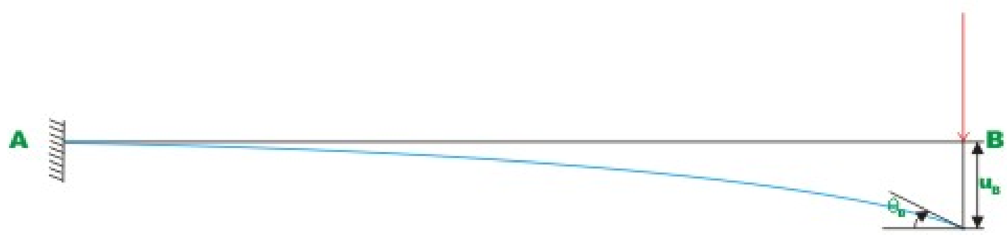
Fig 1.11 Indeterminate Structures

1.5 Kinematic Indeterminacy

When the structure is loaded, the joints undergo displacements in the form of translations and rotations. In the displacement based analysis, these joint displacements are treated as unknown quantities. Consider a propped cantilever beam shown in Fig. 1.12 (a). Usually, the axial rigidity of the beam is so high that the change in its length along axial direction may be neglected. The displacements at a fixed support are zero. Hence, for a propped cantilever beam we have to evaluate only rotation at B and this is known as the kinematic indeterminacy of the structure. A fixed fixed beam is kinematically determinate but statically indeterminate to 3rd degree. A simply supported beam and a cantilever beam are kinematically indeterminate to 2nd degree.



(a) Propped Cantilever Beam



(b) Cantilever Beam

(c) Simply Supported Beam

Fig 1.12 Kinematically Indeterminate Structures

The joint displacements in a structure is treated as independent if each displacement (translation and rotation) can be varied arbitrarily and independently of all other displacements. The number of independent joint displacement in a structure is known as the degree of kinematic indeterminacy or the number of degrees of freedom. In the plane frame shown in Fig. 1.13, the joints B and C have 3 degrees of freedom as shown in the figure. However if axial deformations of the members are neglected then $u_1 = u_4$ and u_2 and u_4 can be neglected. Hence, we have 3 independent joint displacement as shown in Fig. 1.13 i.e. rotations at B and C and one translation.

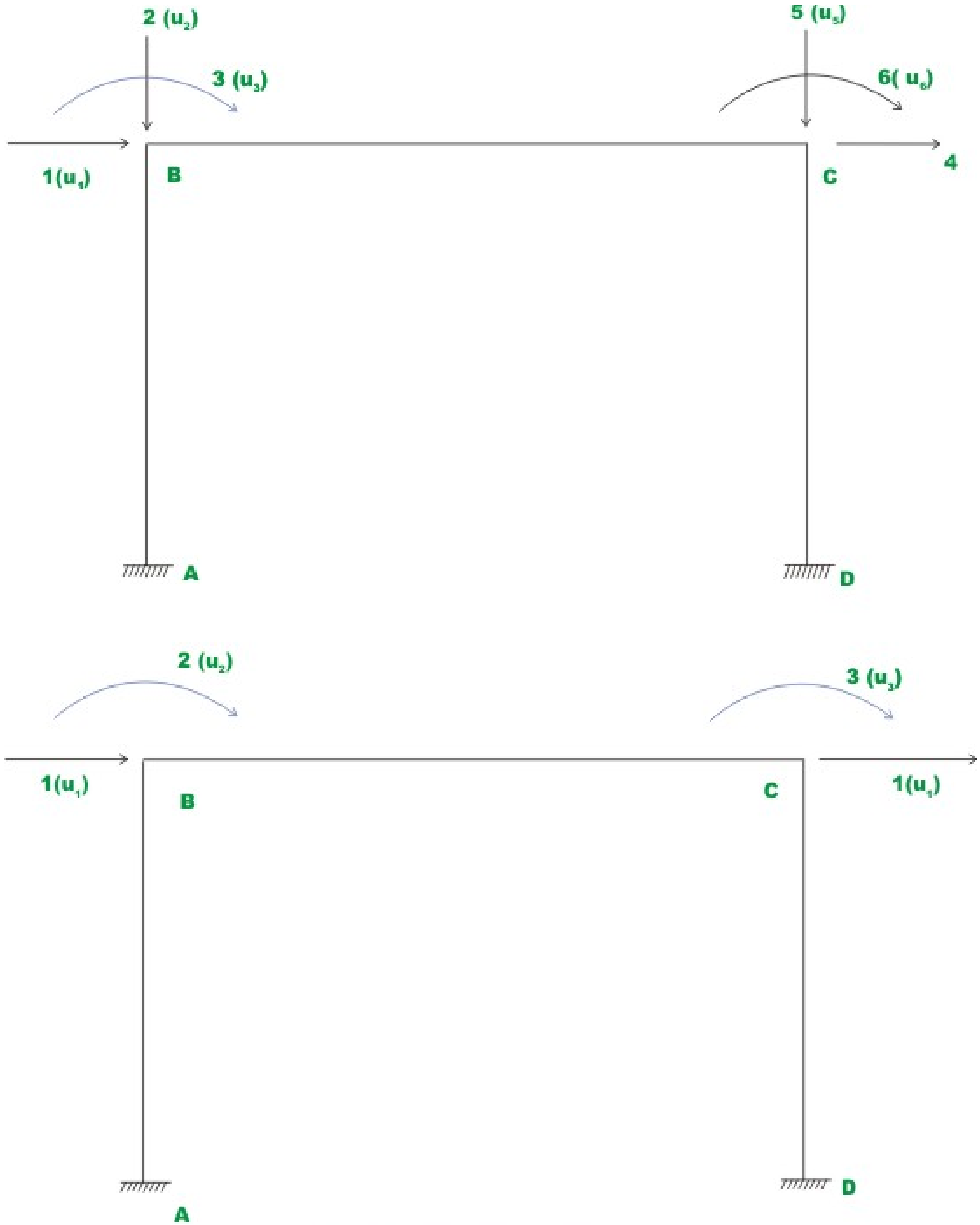


Fig 1.13 Rigid Frame

1.6 Kinematically Unstable Structure

A beam which is supported on roller on both ends (vide. Fig. 1.14) on a horizontal surface can be in the state of static equilibrium only if the resultant of the system of applied loads is a vertical force or a couple. Although this beam is stable under special loading conditions, is unstable under a general type of loading conditions. When a system of forces whose resultant has a component in the horizontal direction is applied on this beam, the structure moves as a rigid body. Such structures are known as kinematically unstable structure. One should avoid such support conditions.

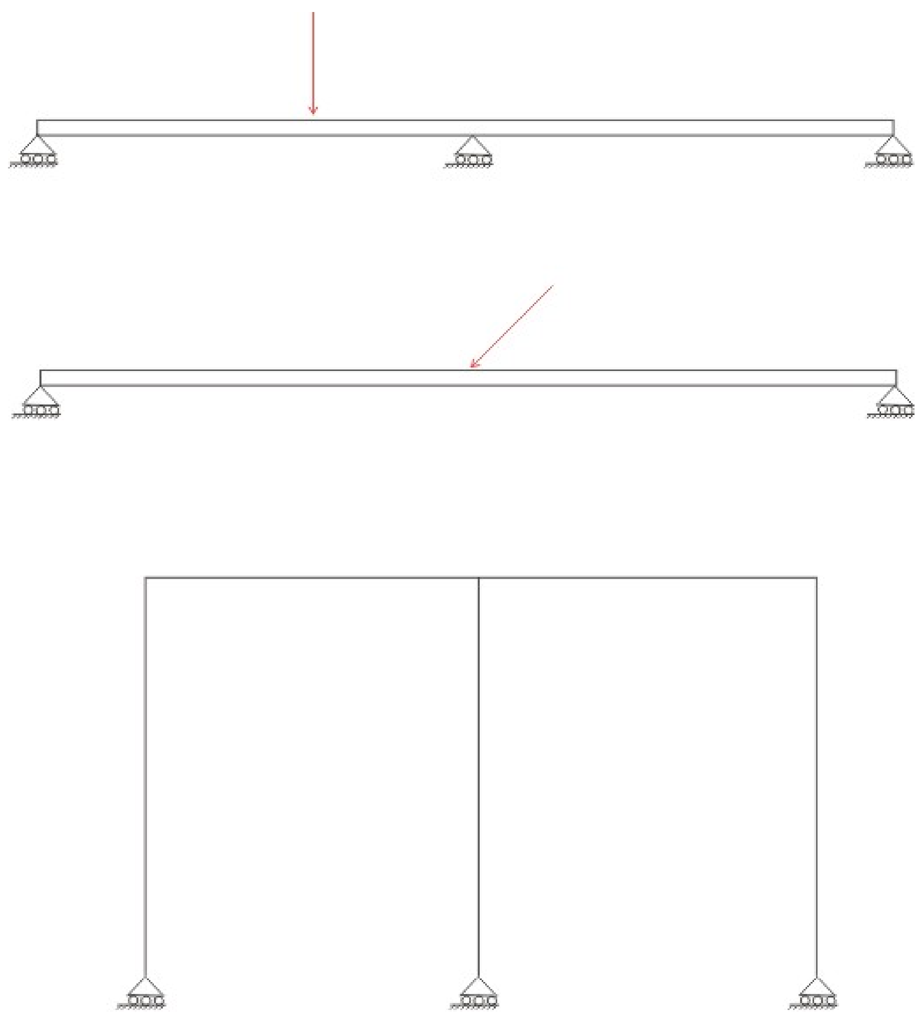


Fig 1.14 Kinematically Unstable Structures

1.7 Compatibility Equations

A structure apart from satisfying equilibrium conditions should also satisfy all the compatibility conditions. These conditions require that the displacements and rotations be continuous throughout the structure and compatible with the nature supports conditions. For example, at a fixed support this requires that displacement and slope should be zero.

1.8 Force-Displacement Relationship

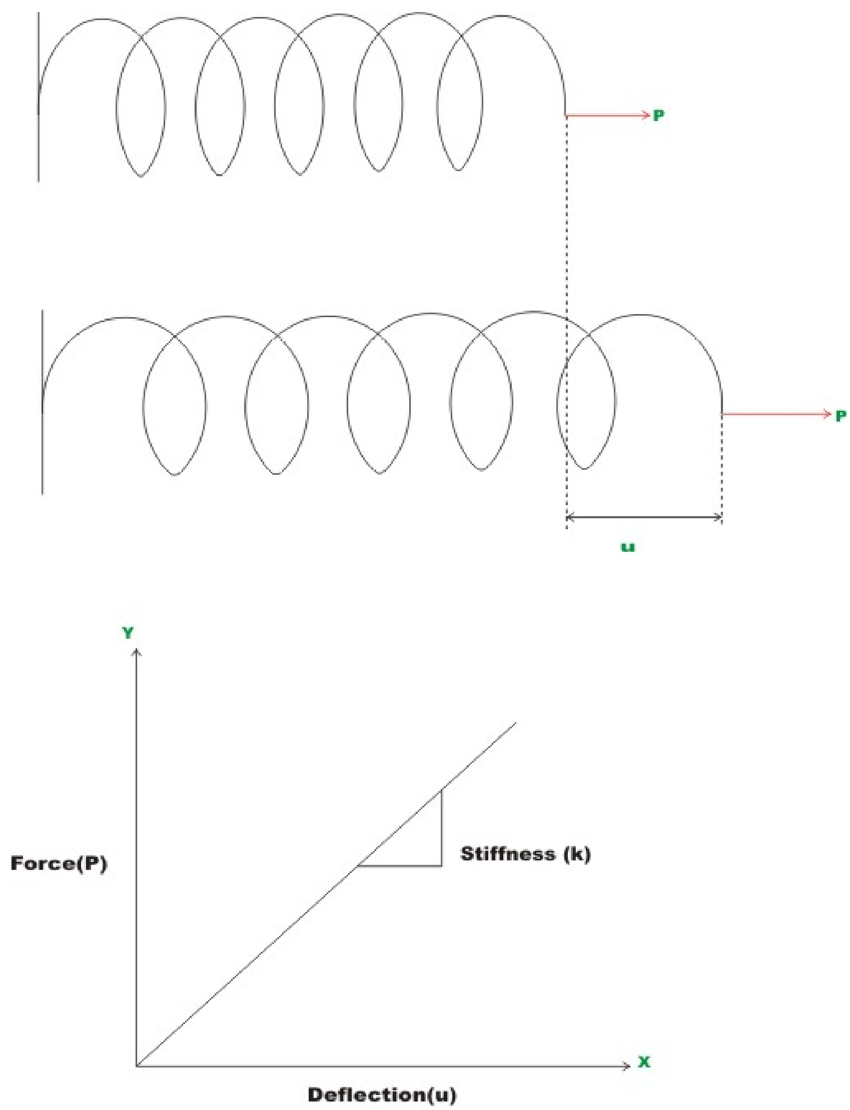


Fig 1.15 Force displacement Relationship

Consider linear elastic spring as shown in Fig.1.15. Let us do a simple experiment. Apply a force P_1 at the end of spring and measure the deformation u_1. Now increase the load to P_2 and measure the deformation u_2. Likewise repeat the experiment for different values of load $P_1, P_2, ..., P_n$. Result may be represented in the form of a graph as shown in the above figure where load is shown on y-axis and deformation on abscissa. The slope of this graph is known as the stiffness of the spring and is represented by k and is given by

$$k = \frac{P_2 - P_1}{u_2 - u_1} = \frac{P}{u} \tag{1.5}$$

$$P = ku \tag{1.6}$$

The spring stiffness may be defined as the force required for the unit deformation of the spring. The stiffness has a unit of force per unit elongation. The inverse of the stiffness is known as flexibility. It is usually denoted by a and it has a unit of displacement per unit force.

$$a = \frac{1}{k} \tag{1.7}$$

the equation (1.6) may be written as

$$P = ku \Rightarrow \qquad u = \frac{1}{k} P = aP \tag{1.8}$$

The above relations discussed for linearly elastic spring will hold good for linearly elastic structures. As an example consider a simply supported beam subjected to a unit concentrated load at the centre. Now the deflection at the centre is given by

$$u = \frac{PL^3}{48EI} \quad \text{or} \quad P = \left(\frac{48EI}{L^3}\right) u \tag{1.9}$$

The stiffness of a structure is defined as the force required for the unit deformation of the structure. Hence, the value of stiffness for the beam is equal to

$$k = \frac{48EI}{L^3}$$

As a second example, consider a cantilever beam subjected to a concentrated load (P) at its tip. Under the action of load, the beam deflects and from first principles the deflection below the load (u) may be calculated as,

$$u = \frac{PL^3}{3EI_{zz}} \tag{1.10}$$

For a given beam of constant cross section, length L, Young's modulus E, and moment of inertia I_{ZZ} the deflection is directly proportional to the applied load. The equation (1.10) may be written as

$$u = a P \tag{1.11}$$

Where a is the flexibility coefficient and is $a = \frac{L^3}{3EI_{zz}}$. Usually it is denoted by a_{ij} the flexibility coefficient at i due to unit force applied at j. Hence, the stiffness of the beam is

$$k_{11} = \frac{1}{a_{11}} = \frac{3EI}{L^3} \tag{1.12}$$

Summary

In this lesson the structures are classified as: beams, plane truss, space truss, plane frame, space frame, arches, cables, plates and shell depending on how they support external load. The way in which the load is supported by each of these structural systems are discussed. Equations of static equilibrium have been stated with respect to planar and space and structures. A brief description of static indeterminacy and kinematic indeterminacy is explained with the help simple structural forms. The kinematically unstable structures are discussed in section 1.6. Compatibility equations and force-displacement relationships are discussed. The term stiffness and flexibility coefficients are defined. In section 1.8, the procedure to calculate stiffness of simple structure is discussed.

Suggested Text Books for Further Reading

- Armenakas, A. E. (1988). *Classical Structural Analysis – A Modern Approach*, McGraw-Hill Book Company, NY, ISBN 0-07-100120-4

- Hibbeler, R. C. (2002). *Structural Analysis*, Pearson Education (Singapore) Pte. Ltd., Delhi, ISBN 81-7808-750-2

- Junarkar, S. B. and Shah, H. J. (1999). *Mechanics of Structures – Vol. II*, Charotar Publishing House, Anand.

- Leet, K. M. and Uang, C-M. (2003). *Fundamentals of Structural Analysis*, Tata McGraw-Hill Publishing Company Limited, New Delhi, ISBN 0-07-058208-4

- Negi, L. S. and Jangid, R.S. (2003). *Structural Analysis*, Tata McGraw-Hill Publishing Company Limited, New Delhi, ISBN 0-07-462304-4

- Norris, C. H., Wilbur, J. B. and Utku, S. (1991). *Elementary Structural Analysis*, Tata McGraw-Hill Publishing Company Limited, New Delhi, ISBN 0-07-058116-9

- MATRIX ANALYSIS of FRAMED STRUCTURES, 3-rd Edition, by Weaver and Gere Publishe, Chapman & Hall, New York, New York, 1990

Module 1

Energy Methods in Structural Analysis

Lesson
2
Principle of Superposition, Strain Energy

Instructional Objectives

After reading this lesson, the student will be able to

1. State and use principle of superposition.
2. Explain strain energy concept.
3. Differentiate between elastic and inelastic strain energy and state units of strain energy.
4. Derive an expression for strain energy stored in one-dimensional structure under axial load.
5. Derive an expression for elastic strain energy stored in a beam in bending.
6. Derive an expression for elastic strain energy stored in a beam in shear.
7. Derive an expression for elastic strain energy stored in a circular shaft under torsion.

2.1 Introduction

In the analysis of statically indeterminate structures, the knowledge of the displacements of a structure is necessary. Knowledge of displacements is also required in the design of members. Several methods are available for the calculation of displacements of structures. However, if displacements at only a few locations in structures are required then energy based methods are most suitable. If displacements are required to solve statically indeterminate structures, then only the relative values of EA, EI and GJ are required. If actual value of displacement is required as in the case of settlement of supports and temperature stress calculations, then it is necessary to know actual values of E and G. In general deflections are small compared with the dimensions of structure but for clarity the displacements are drawn to a much larger scale than the structure itself. Since, displacements are small, it is assumed not to cause gross displacements of the geometry of the structure so that equilibrium equation can be based on the original configuration of the structure. When non-linear behaviour of the structure is considered then such an assumption is not valid as the structure is appreciably distorted. In this lesson two of the very important concepts i.e., principle of superposition and strain energy method will be introduced.

2.2 Principle of Superposition

The principle of superposition is a central concept in the analysis of structures. This is applicable when there exists a linear relationship between external forces and corresponding structural displacements. The principle of superposition may be stated as *the deflection at a given point in a structure produced by several loads acting simultaneously on the structure can be found by superposing deflections at the same point produced by* loads *acting individually*. This is illustrated with the help of a simple beam problem. Now consider a cantilever beam of length L and having constant flexural rigidity EI subjected to two externally applied forces P_1 and P_2 as shown in Fig. 2.1. From moment-area theorem we can evaluate deflection below C, which states that the tangential deviation of point c from the tangent at point A is equal to the first moment of the area of the $\frac{M}{EI}$ diagram between A and C about C. Hence, the deflection u below C due to loads P_1 and P_2 acting simultaneously is (by moment-area theorem),

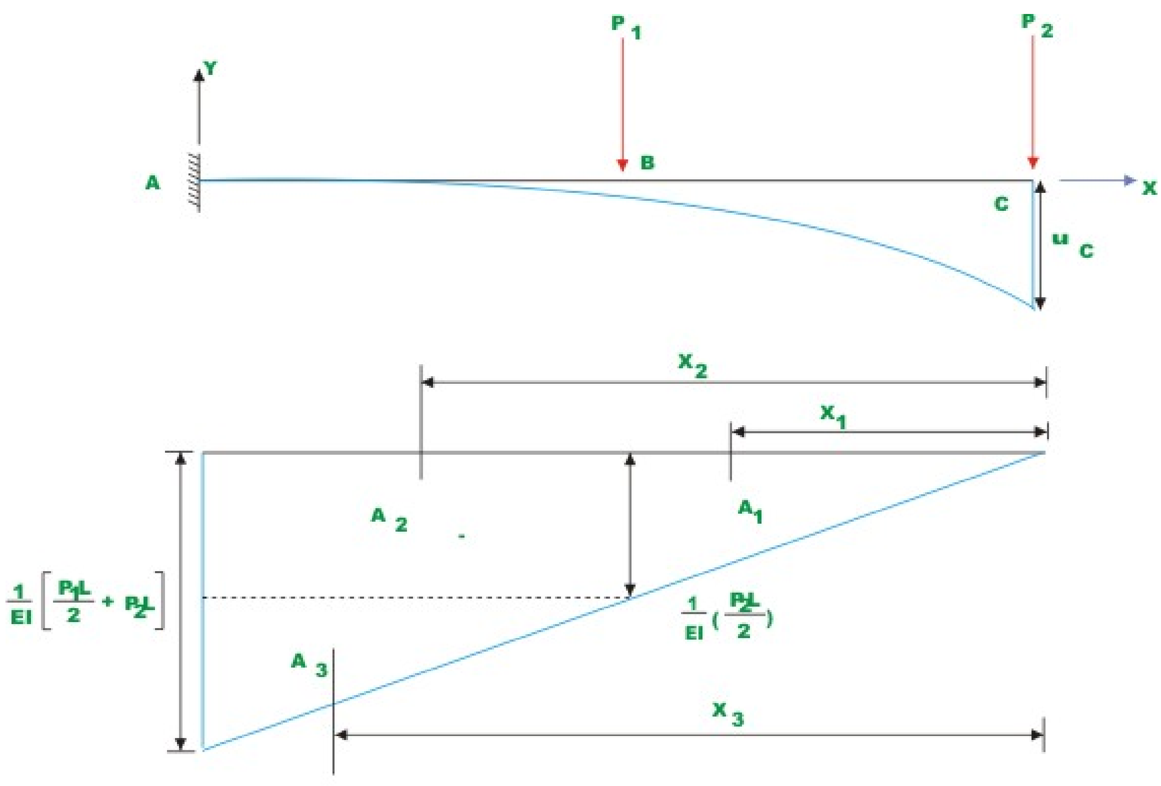


Fig 2.1 Cantilever Beam with Two Concentrated Loads

$$u = A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 \tag{2.1}$$

where u is the tangential deviation of point C with respect to a tangent at A. Since, in this case the tangent at A is horizontal, the tangential deviation of point

C is nothing but the vertical deflection at C. $\bar{x}_1, \bar{x}_2$ and $\bar{x}_3$ are the distances from point C to the centroids of respective areas respectively.

$$\bar{x}_1 = \frac{2}{3}\frac{L}{2} \qquad \bar{x}_2 = \left(\frac{L}{2} + \frac{L}{4}\right) \qquad \bar{x}_3 = \frac{2}{3}\frac{L}{2} + \frac{L}{2}$$

$$A_1 = \frac{P_2 L^2}{8EI} \qquad A_2 = \frac{P_2 L^2}{4EI} \qquad A_3 = \frac{(P_1 L + P_2 L)L}{8EI}$$

Hence,

$$u = \frac{P_2 L^2}{8EI}\frac{2}{3}\frac{L}{2} + \frac{P_2 L^2}{4EI}\left[\frac{L}{2} + \frac{L}{4}\right] + \frac{(P_1 L + P_2 L)L}{8EI}\left[\frac{2}{3}\frac{L}{2} + \frac{L}{2}\right] \tag{2.2}$$

After simplification one can write,

$$u = \frac{P_2 L^3}{3EI} + \frac{5 P_1 L^3}{48EI} \tag{2.3}$$

Now consider the forces being applied separately and evaluate deflection at C in each of the case.

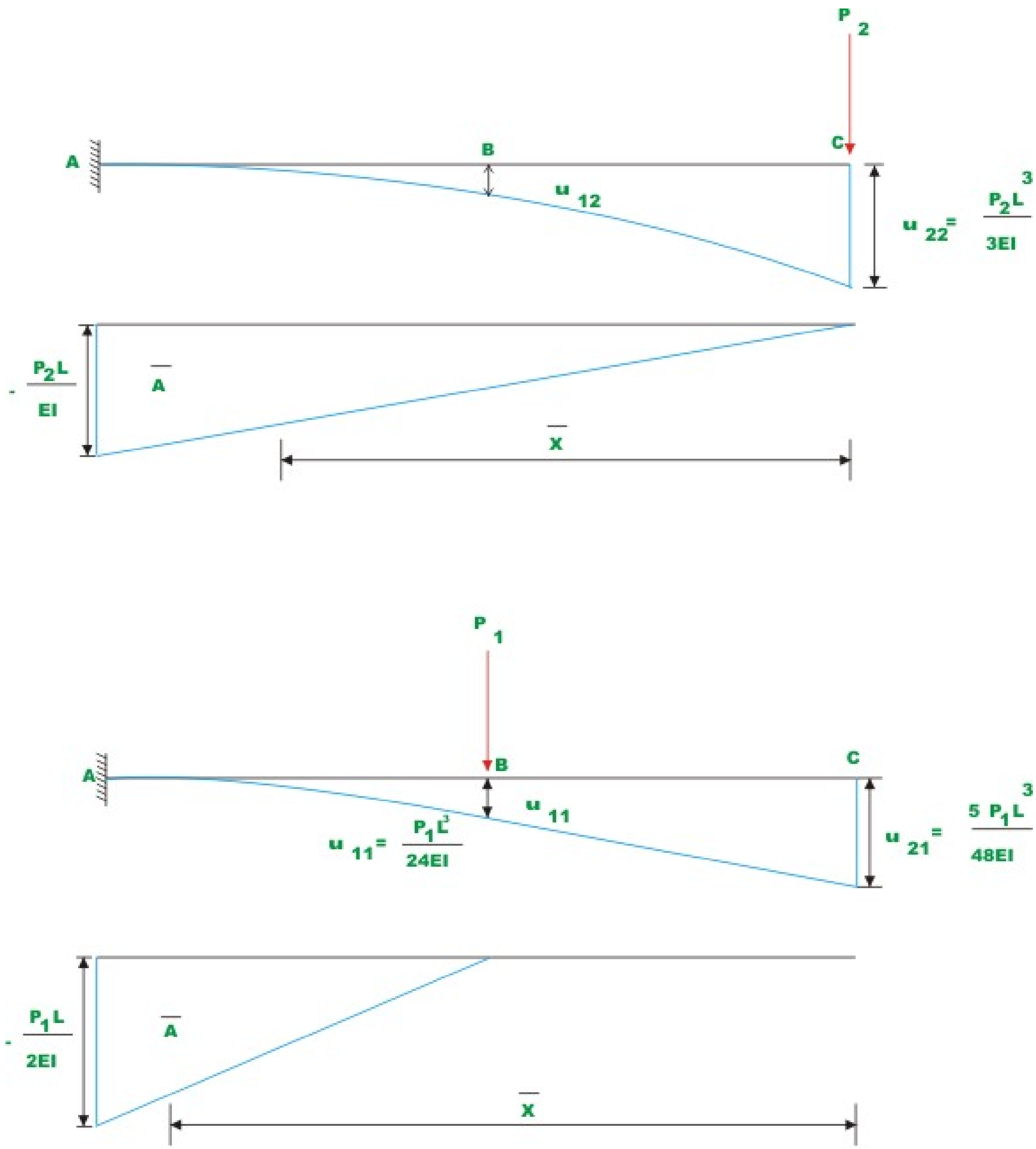


Fig 2.2 Deflection Computation

$$u_{22} = \frac{P_2 L^3}{3EI} \tag{2.4}$$

where u_{22} is deflection at C (2) when load P_1 is applied at C (2) itself. And,

$$u_{21} = \frac{1}{2}\frac{P_1 L}{2EI}\frac{L}{2}\left[\frac{L}{2} + \frac{2}{3}\frac{L}{2}\right] = \frac{5P_1 L^3}{48EI} \qquad (2.5)$$

where u_{21} is the deflection at $C(2)$ when load is applied at $B(1)$. Now the total deflection at C when both the loads are applied simultaneously is obtained by adding u_{22} and u_{21}.

$$u = u_{22} + u_{21} = \frac{P_2 L^3}{3EI} + \frac{5P_1 L^3}{48EI} \qquad (2.6)$$

Hence it is seen from equations (2.3) and (2.6) that when the structure behaves linearly, the total deflection caused by forces $P_1, P_2, ..., P_n$ at any point in the structure is the sum of deflection caused by forces $P_1, P_2, ..., P_n$ acting independently on the structure at the same point. This is known as the Principle of Superposition.

The method of superposition is not valid when the material stress-strain relationship is non-linear. Also, it is not valid in cases where the geometry of structure changes on application of load. For example, consider a hinged-hinged beam-column subjected to only compressive force as shown in Fig. 2.3(a). Let the compressive force P be less than the Euler's buckling load of the structure. Then deflection at an arbitrary point C (say) u_c^1 is zero. Next, the same beam-column be subjected to lateral load Q with no axial load as shown in Fig. 2.3(b). Let the deflection of the beam-column at C be u_c^2. Now consider the case when the same beam-column is subjected to both axial load P and lateral load Q. As per the principle of superposition, the deflection at the centre u_c^3 must be the sum of deflections caused by P and Q when applied individually. However this is not so in the present case. Because of lateral deflection caused by Q, there will be additional bending moment due to P at C. Hence, the net deflection u_c^3 will be more than the sum of deflections u_c^1 and u_c^2.

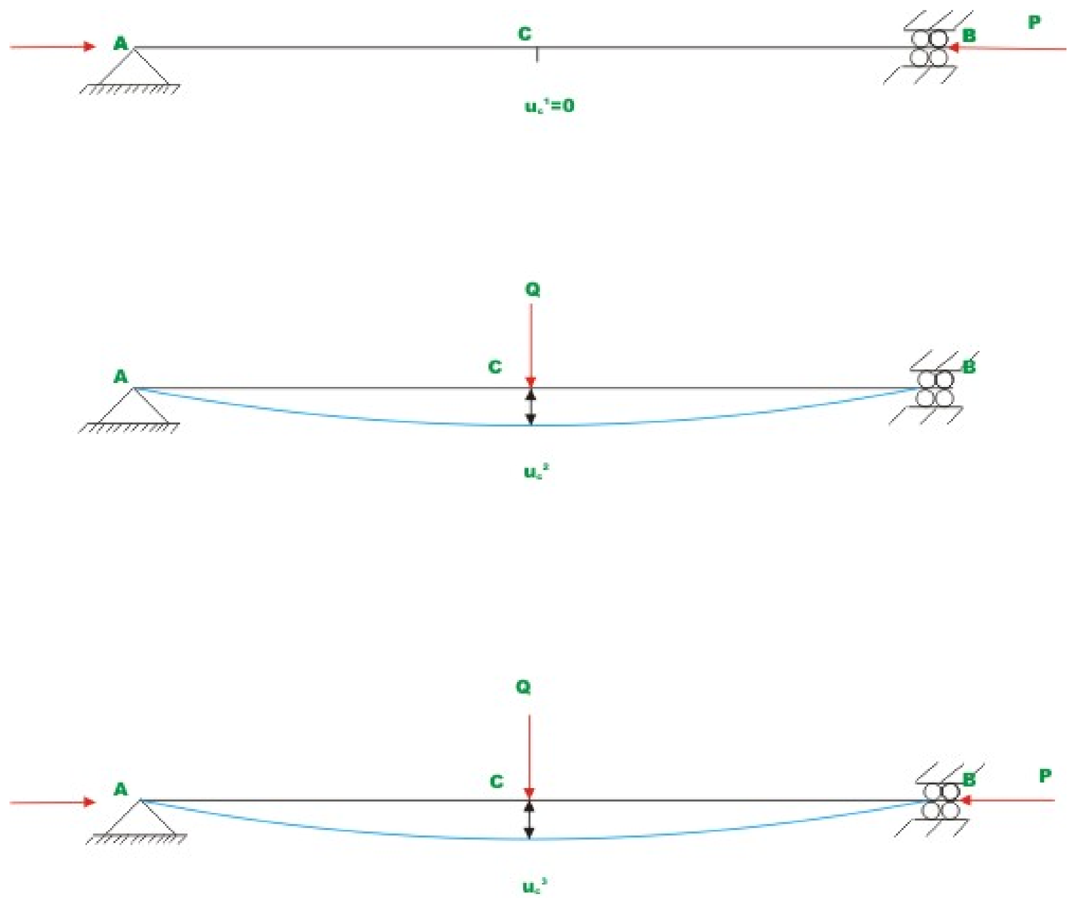


Fig. 2.3

2.3 Strain Energy

Consider an elastic spring as shown in the Fig.2.4. When the spring is slowly pulled, it deflects by a small amount u_1. When the load is removed from the spring, it goes back to the original position. When the spring is pulled by a force, it does some work and this can be calculated once the load-displacement relationship is known. It may be noted that, the spring is a mathematical idealization of the rod being pulled by a force P axially. It is assumed here that the force is applied gradually so that it slowly increases from zero to a maximum value P. Such a load is called static loading, as there are no inertial effects due to motion. Let the load-displacement relationship be as shown in Fig. 2.5. Now, work done by the external force may be calculated as,

$$W_{ext} = \frac{1}{2} P_1 u_1 = \frac{1}{2}(force \times displacement) \tag{2.7}$$

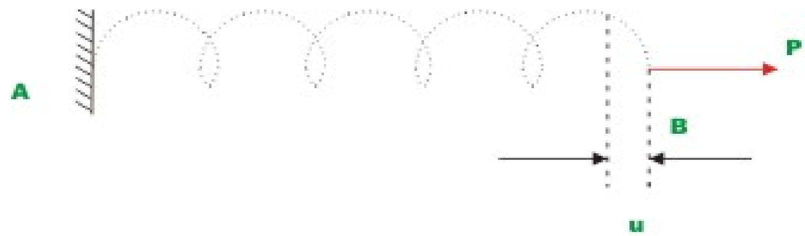


Fig. 2.4 Linear Spring

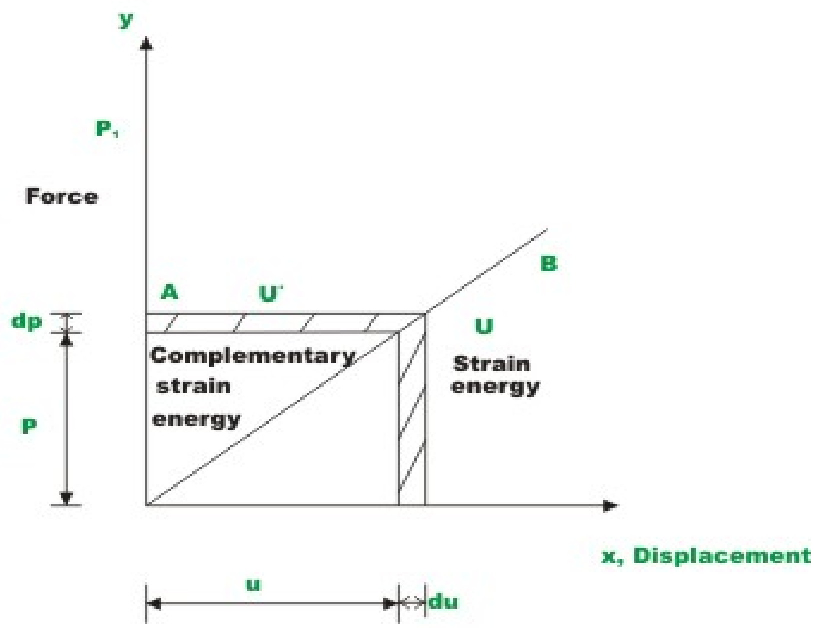


Fig. 2.5 Force-displacement relation

The area enclosed by force-displacement curve gives the total work done by the externally applied load. Here it is assumed that the energy is conserved i.e. the work done by gradually applied loads is equal to energy stored in the structure. This internal energy is known as strain energy. Now strain energy stored in a spring is

$$U = \frac{1}{2} P_1 u_1 \tag{2.8}$$

Work and energy are expressed in the same units. In SI system, the unit of work and energy is the joule (J), which is equal to one Newton metre (N.m). The strain energy may also be defined as the internal work done by the stress resultants in moving through the corresponding deformations. Consider an infinitesimal element within a three dimensional homogeneous and isotropic material. In the most general case, the state of stress acting on such an element may be as shown in Fig. 2.6. There are normal stresses $\left(\sigma_x, \sigma_y \text{ and } \sigma_z\right)$ and shear stresses $\left(\tau_{xy}, \tau_{yz} \text{ and } \tau_{zx}\right)$ acting on the element. Corresponding to normal and shear stresses we have normal and shear strains. Now strain energy may be written as,

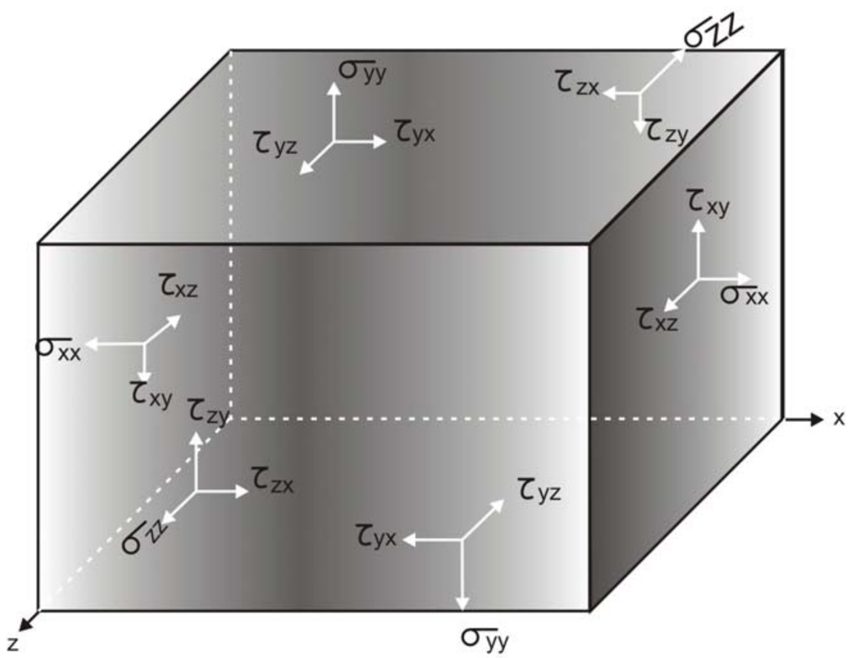


Figure 2.6. Stress on an infinitesimal element .

$$U = \frac{1}{2}\int_v \sigma^T \varepsilon \, dv \tag{2.9}$$

in which σ^T is the transpose of the stress column vector i.e.,

$$\{\sigma\}^T = \left(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}\right) \text{ and } \{\varepsilon\}^T = \left(\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}\right) \tag{2.10}$$

The strain energy may be further classified as elastic strain energy and inelastic strain energy as shown in Fig. 2.7. If the force P is removed then the spring shortens. When the elastic limit of the spring is not exceeded, then on removal of load, the spring regains its original shape. If the elastic limit of the material is exceeded, a permanent set will remain on removal of load. In the present case, load the spring beyond its elastic limit. Then we obtain the load-displacement curve $OABCDO$ as shown in Fig. 2.7. Now if at B, the load is removed, the spring gradually shortens. However, a permanent set of OD is till retained. The shaded area BCD is known as the elastic strain energy. This can be recovered upon removing the load. The area $OABDO$ represents the inelastic portion of strain energy.

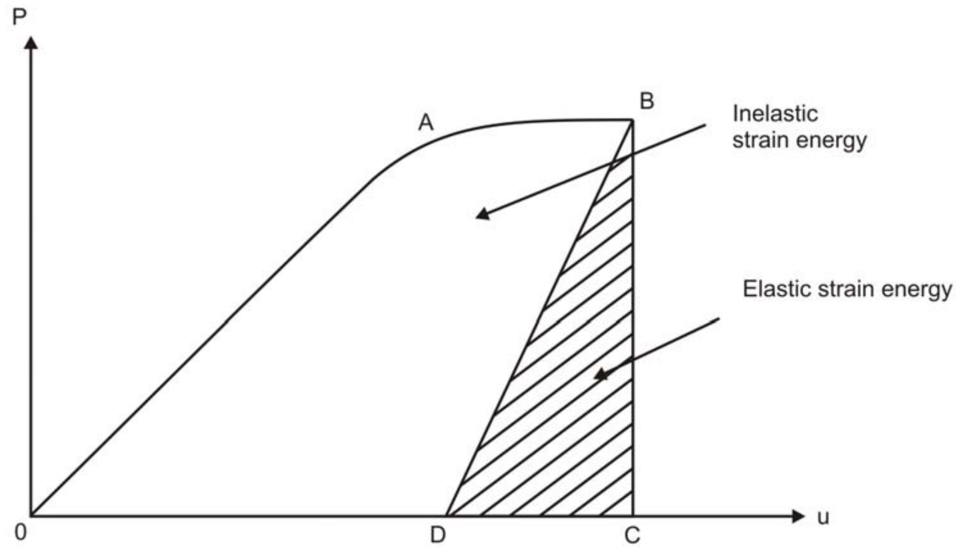


Figure 2.7 Elastic and inelastic strain energy.

The area $OABCDO$ corresponds to strain energy stored in the structure. The area $OABEO$ is defined as the complementary strain energy. For the linearly elastic structure it may be seen that

Area OBC = Area OBE

i.e. Strain energy = Complementary strain energy

This is not the case always as observed from Fig. 2.7. The complementary energy has no physical meaning. The definition is being used for its convenience in structural analysis as will be clear from the subsequent chapters.

Usually structural member is subjected to any one or the combination of bending moment; shear force, axial force and twisting moment. The member resists these external actions by internal stresses. In this section, the internal stresses induced in the structure due to external forces and the associated displacements are calculated for different actions. Knowing internal stresses due to individual forces, one could calculate the resulting stress distribution due to combination of external forces by the method of superposition. After knowing internal stresses and deformations, one could easily evaluate strain energy stored in a simple beam due to axial, bending, shear and torsional deformations.

2.3.1 Strain energy under axial load

Consider a member of constant cross sectional area A, subjected to axial force P as shown in Fig. 2.8. Let E be the Young's modulus of the material. Let the member be under equilibrium under the action of this force, which is applied through the centroid of the cross section. Now, the applied force P is resisted by uniformly distributed internal stresses given by average stress $\sigma = \frac{P}{A}$ as shown by the free body diagram (vide Fig. 2.8). Under the action of axial load P applied at one end gradually, the beam gets elongated by (say) u. This may be calculated as follows. The incremental elongation du of small element of length dx of beam is given by,

$$du = \varepsilon\, dx = \frac{\sigma}{E} dx = \frac{P}{AE} dx \qquad (2.11)$$

Now the total elongation of the member of length *L* may be obtained by integration

$$u = \int_0^L \frac{P}{AE} dx \qquad (2.12)$$

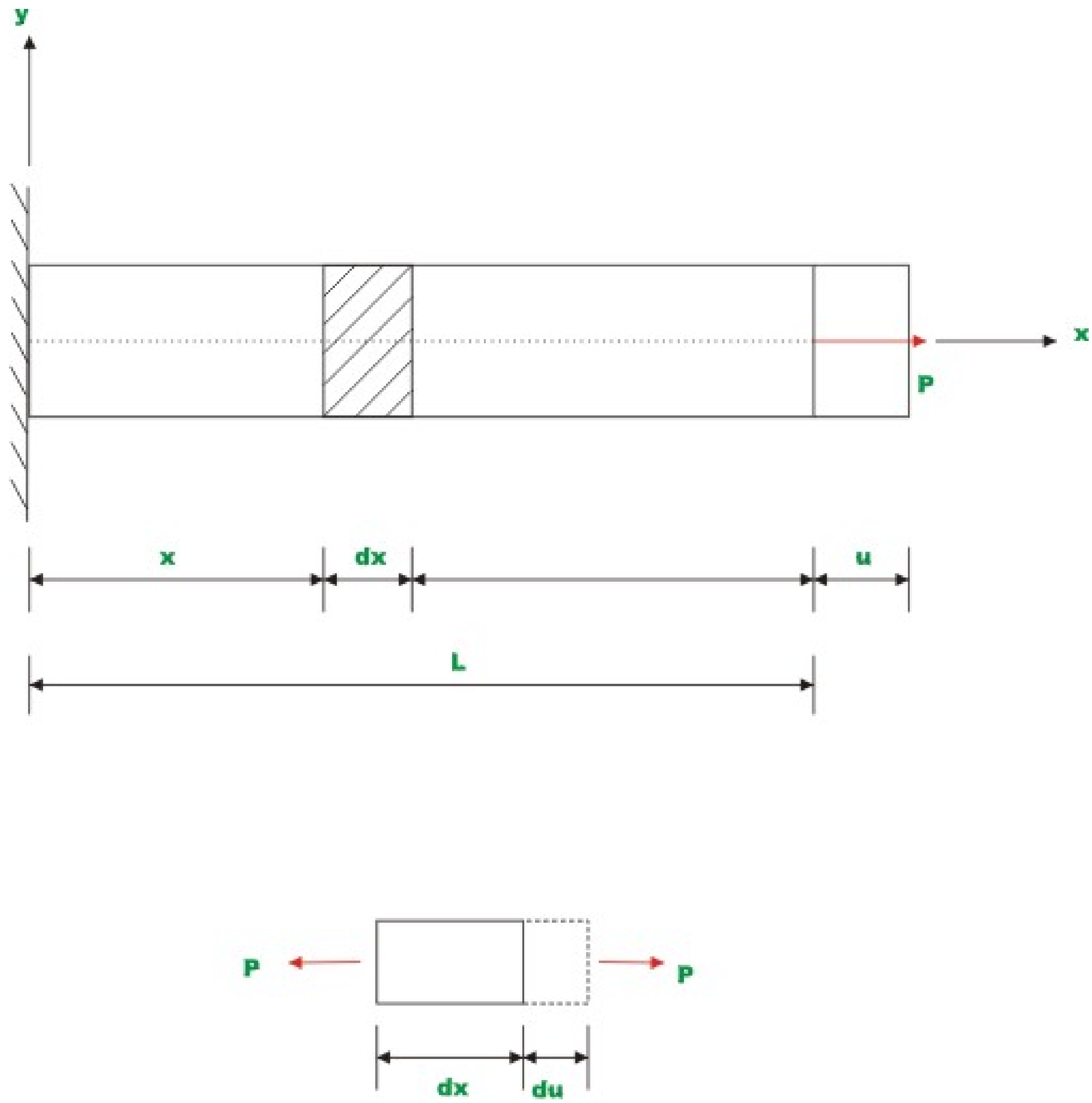


Fig 2.8

Now the work done by external loads $W = \frac{1}{2}Pu$ (2.13)

In a conservative system, the external work is stored as the internal strain energy. Hence, the strain energy stored in the bar in axial deformation is,

$$U = \frac{1}{2}Pu \tag{2.14}$$

Substituting equation (2.12) in (2.14) we get,

$$U = \int_0^L \frac{P^2}{2AE} dx \tag{2.15}$$

2.3.2 Strain energy due to bending

Consider a prismatic beam subjected to loads as shown in the Fig. 2.9. The loads are assumed to act on the beam in a plane containing the axis of symmetry of the cross section and the beam axis. It is assumed that the transverse cross sections (such as AB and CD), which are perpendicular to centroidal axis, remain plane and perpendicular to the centroidal axis of beam (as shown in Fig 2.9).

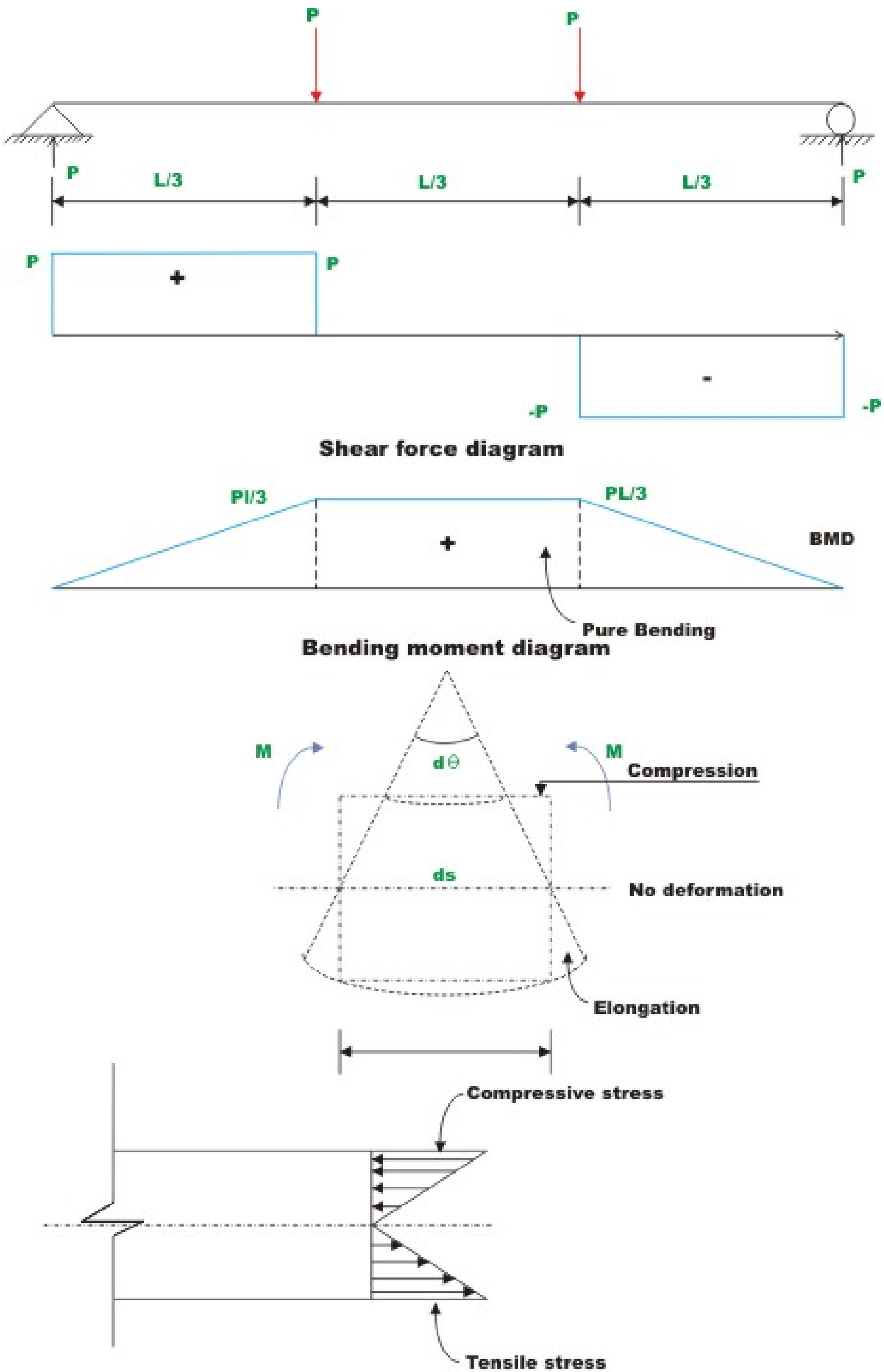


Fig. 2.9 BENDING DEFORMATION

Consider a small segment of beam of length ds subjected to bending moment as shown in the Fig. 2.9. Now one cross section rotates about another cross section by a small amount $d\theta$. From the figure,

$$d\theta = \frac{1}{R} ds = \frac{M}{EI} ds \qquad (2.16)$$

where R is the radius of curvature of the bent beam and EI is the flexural rigidity of the beam. Now the work done by the moment M while rotating through angle $d\theta$ will be stored in the segment of beam as strain energy dU. Hence,

$$dU = \frac{1}{2} M \, d\theta \qquad (2.17)$$

Substituting for $d\theta$ in equation (2.17), we get,

$$dU = \frac{1}{2} \frac{M^2}{EI} ds \qquad (2.18)$$

Now, the energy stored in the complete beam of span L may be obtained by integrating equation (2.18). Thus,

$$U = \int_0^L \frac{M^2}{2EI} ds \qquad (2.19)$$

2.3.3 Strain energy due to transverse shear

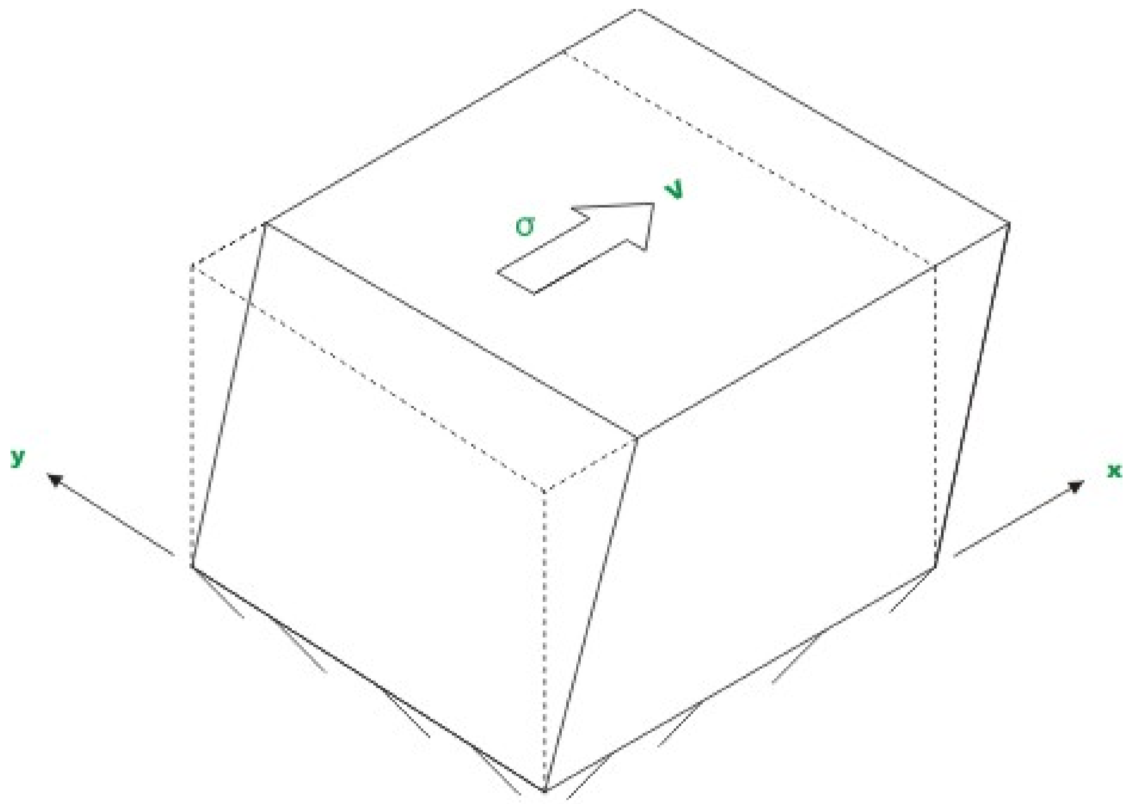


Fig. 2.10 (a) Shear Deformation

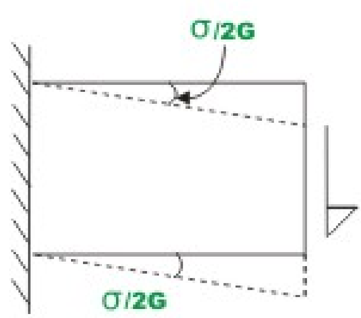


Fig. 2.10 (b)

The shearing stress on a cross section of beam of rectangular cross section may be found out by the relation

$$\tau = \frac{VQ}{bI_{ZZ}} \tag{2.20}$$

where Q is the first moment of the portion of the cross-sectional area above the point where shear stress is required about neutral axis, V is the transverse shear force, b is the width of the rectangular cross-section and I_{zz} is the moment of inertia of the cross-sectional area about the neutral axis. Due to shear stress, the angle between the lines which are originally at right angle will change. The shear stress varies across the height in a parabolic manner in the case of a rectangular cross-section. Also, the shear stress distribution is different for different shape of the cross section. However, to simplify the computation shear stress is assumed to be uniform (which is strictly not correct) across the cross section. Consider a segment of length ds subjected to shear stress τ. The shear stress across the cross section may be taken as

$$\tau = k\frac{V}{A}$$

in which A is area of the cross-section and k is the form factor which is dependent on the shape of the cross section. One could write, the deformation du as

$$du = \Delta\gamma \, ds \tag{2.21}$$

where $\Delta\gamma$ is the shear strain and is given by

$$\Delta\gamma = \frac{\tau}{G} = k\frac{V}{AG} \tag{2.22}$$

Hence, the total deformation of the beam due to the action of shear force is

$$u = \int_0^L k\frac{V}{AG}ds \tag{2.23}$$

Now the strain energy stored in the beam due to the action of transverse shear force is given by,

$$U = \frac{1}{2}Vu = \int_0^L \frac{kV^2}{2AG}ds \tag{2.24}$$

The strain energy due to transverse shear stress is very low compared to strain energy due to bending and hence is usually neglected. Thus the error induced in assuming a uniform shear stress across the cross section is very small.

2.3.4 Strain energy due to torsion

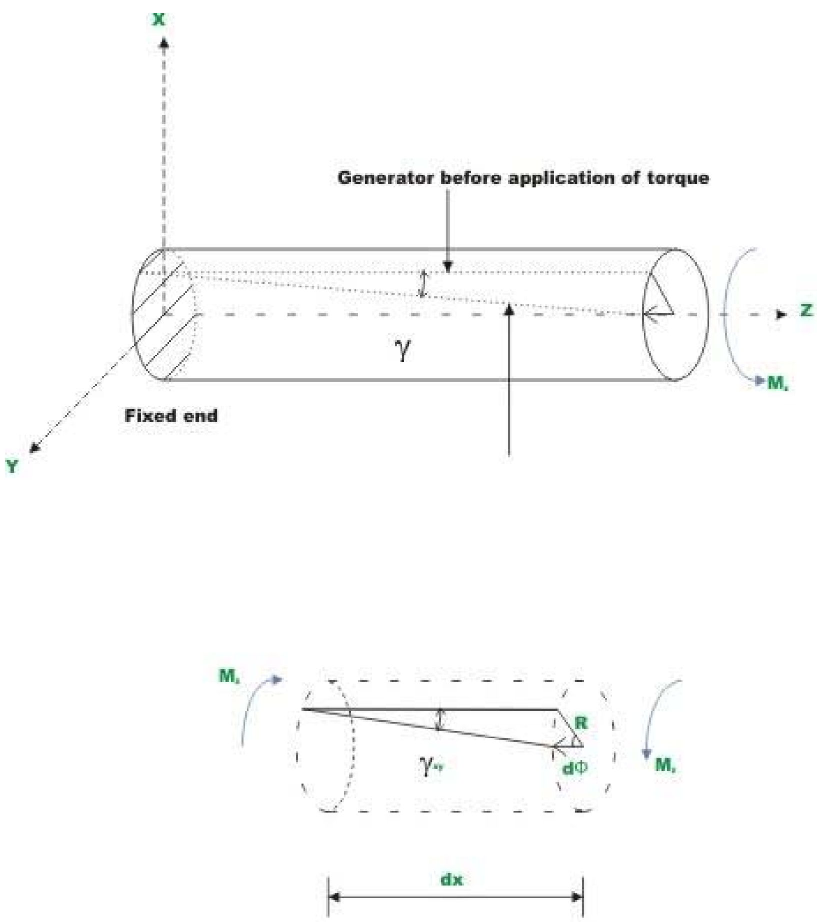


Fig 2.11 Generator after application of torque

Consider a circular shaft of length L radius R, subjected to a torque T at one end (see Fig. 2.11). Under the action of torque one end of the shaft rotates with respect to the fixed end by an angle $d\phi$. Hence the strain energy stored in the shaft is,

$$U = \frac{1}{2}T\phi \qquad (2.25)$$

Consider an elemental length ds of the shaft. Let the one end rotates by a small amount $d\phi$ with respect to another end. Now the strain energy stored in the elemental length is,

$$dU = \frac{1}{2} T d\phi \tag{2.26}$$

We know that

$$d\phi = \frac{Tds}{GJ} \tag{2.27}$$

where, G is the shear modulus of the shaft material and J is the polar moment of area. Substituting for $d\phi$ from (2.27) in equation (2.26), we obtain

$$dU = \frac{T^2}{2GJ} ds \tag{2.28}$$

Now, the total strain energy stored in the beam may be obtained by integrating the above equation.

$$U = \int_0^L \frac{T^2}{2GJ} ds \tag{2.29}$$

Hence the elastic strain energy stored in a member of length s (it may be curved or straight) due to axial force, bending moment, shear force and torsion is summarized below.

1. Due to axial force $\quad U_1 = \int_0^s \frac{P^2}{2AE} ds$

2. Due to bending $\quad U_2 = \int_0^s \frac{M^2}{2EI} ds$

3. Due to shear $\quad U_3 = \int_0^s \frac{V^2}{2AG} ds$

4. Due to torsion $\quad U_4 = \int_0^s \frac{T^2}{2GJ} ds$

Summary

In this lesson, the principle of superposition has been stated and proved. Also, its limitations have been discussed. In section 2.3, it has been shown that the elastic strain energy stored in a structure is equal to the work done by applied loads in deforming the structure. The strain energy expression is also expressed for a 3-dimensional homogeneous and isotropic material in terms of internal stresses and strains in a body. In this lesson, the difference between elastic and inelastic strain energy is explained. Complementary strain energy is discussed. In the end, expressions are derived for calculating strain stored in a simple beam due to axial load, bending moment, transverse shear force and torsion.

Module
1
Energy Methods in Structural Analysis

Lesson 3

Castigliano's Theorems

Instructional Objectives

After reading this lesson, the reader will be able to;

1. State and prove first theorem of Castigliano.
2. Calculate deflections along the direction of applied load of a statically determinate structure at the point of application of load.
3. Calculate deflections of a statically determinate structure in any direction at a point where the load is not acting by fictious (imaginary) load method.
4. State and prove Castigliano's second theorem.

3.1 Introduction

In the previous chapter concepts of strain energy and complementary strain energy were discussed. Castigliano's first theorem is being used in structural analysis for finding deflection of an elastic structure based on strain energy of the structure. The Castigliano's theorem can be applied when the supports of the structure are unyielding and the temperature of the structure is constant.

3.2 Castigliano's First Theorem

For linearly elastic structure, where external forces only cause deformations, the complementary energy is equal to the strain energy. For such structures, the Castigliano's first theorem may be stated as the first partial derivative of the strain energy of the structure with respect to any particular force gives the displacement of the point of application of that force in the direction of its line of action.

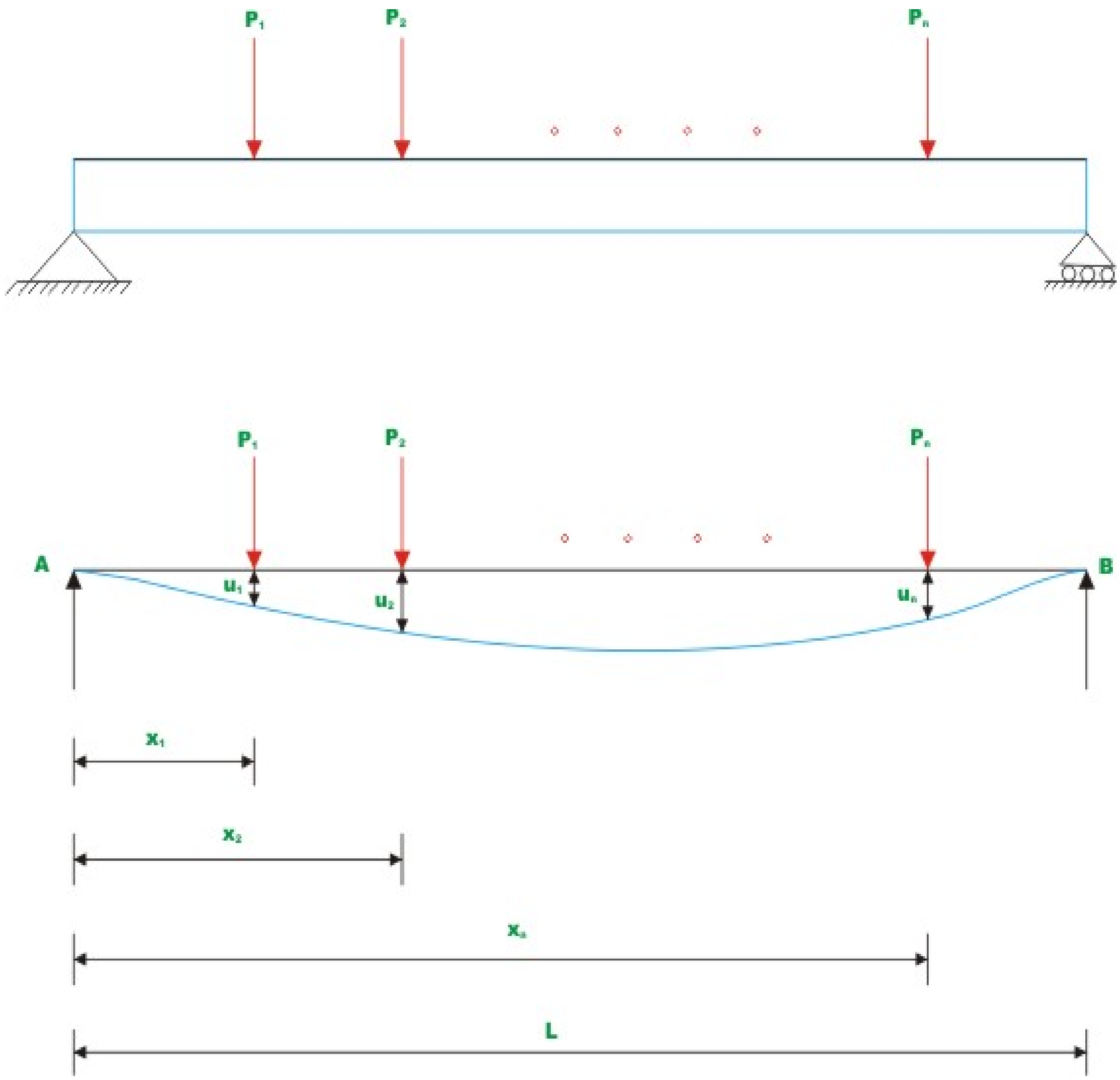


Fig. 3.1 Castigliano's First Theorem

Let $P_1, P_2, ..., P_n$ be the forces acting at $x_1, x_2, ..., x_n$ from the left end on a simply supported beam of span L. Let $u_1, u_2, ..., u_n$ be the displacements at the loading points $P_1, P_2, ..., P_n$ respectively as shown in Fig. 3.1. Now, assume that the material obeys Hooke's law and invoking the principle of superposition, the work done by the external forces is given by (vide eqn. 1.8 of lesson 1)

$$W = \frac{1}{2}P_1 u_1 + \frac{1}{2}P_2 u_2 + ... + \frac{1}{2}P_n u_n \tag{3.1}$$

Work done by the external forces is stored in the structure as strain energy in a conservative system. Hence, the strain energy of the structure is,

$$U = \frac{1}{2}P_1 u_1 + \frac{1}{2}P_2 u_2 + ... + \frac{1}{2}P_n u_n \tag{3.2}$$

Displacement u_1 below point P_1 is due to the action of $P_1, P_2, ..., P_n$ acting at distances $x_1, x_2, ..., x_n$ respectively from left support. Hence, u_1 may be expressed as,

$$u_1 = a_{11}P_1 + a_{12}P_2 + ... + a_{1n}P_n \tag{3.3}$$

In general,

$$u_i = a_{i1}P_1 + a_{i2}P_2 + ... + a_{in}P_n \qquad i = 1,2,...n \tag{3.4}$$

where a_{ij} is the flexibility coefficient at i due to unit force applied at j. Substituting the values of $u_1, u_2, ..., u_n$ in equation (3.2) from equation (3.4), we get,

$$U = \frac{1}{2}P_1[a_{11}P_1 + a_{12}P_2 + ...] + \frac{1}{2}P_2[a_{21}P_1 + a_{22}P_2 + ...] + ... + \frac{1}{2}P_n[a_{n1}P_1 + a_{n2}P_2 + ...] \tag{3.5}$$

We know from Maxwell-Betti's reciprocal theorem $a_{ij} = a_{ji}$. Hence, equation (3.5) may be simplified as,

$$U = \frac{1}{2}\left[a_{11}P_1^2 + a_{22}P_2^2 + ... + a_{nn}P_n^2\right] + \left[a_{12}P_1P_2 + a_{13}P_1P_3 + ... + a_{1n}P_1P_n\right] + ... \tag{3.6}$$

Now, differentiating the strain energy with any force P_1 gives,

$$\frac{\partial U}{\partial P_1} = a_{11}P_1 + a_{12}P_2 + ... + a_{1n}P_n \tag{3.7}$$

It may be observed that equation (3.7) is nothing but displacement u_1 at the loading point.
In general,

$$\frac{\partial U}{\partial P_n} = u_n \tag{3.8}$$

Hence, for determinate structure within linear elastic range the partial derivative of the total strain energy with respect to any external load is equal to the displacement of the point of application of load in the direction of the applied load, provided the supports are unyielding and temperature is maintained constant. This theorem is advantageously used for calculating deflections in elastic structure. The procedure for calculating the deflection is illustrated with few examples.

Example 3.1

Find the displacement and slope at the tip of a cantilever beam loaded as in Fig. 3.2. Assume the flexural rigidity of the beam EI to be constant for the beam.

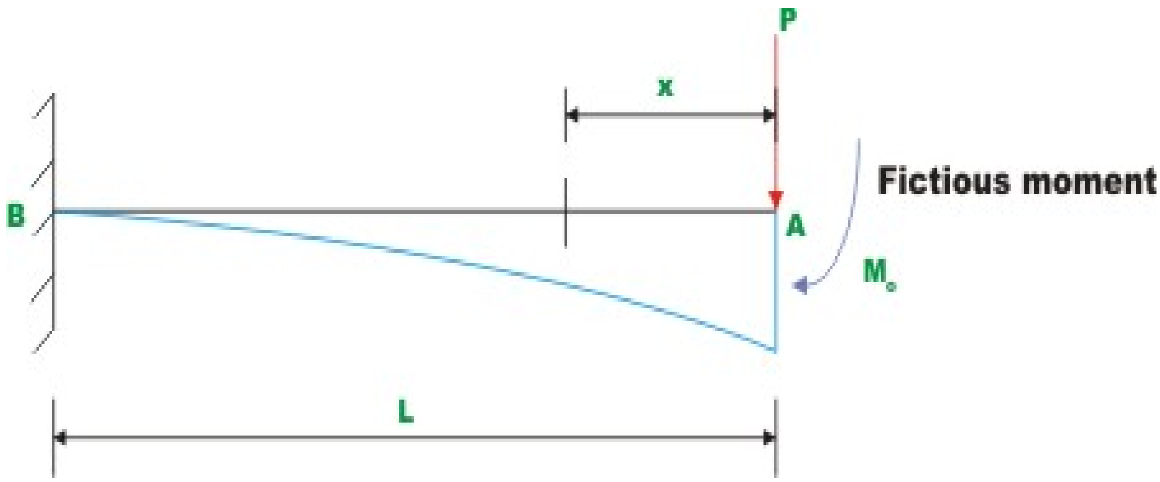


Fig. 3.2 Example 3.1

Moment at any section at a distance x away from the free end is given by

$$M = -Px \quad (1)$$

Strain energy stored in the beam due to bending is $U = \int_0^L \frac{M^2}{2EI} dx$ (2)

Substituting the expression for bending moment M in equation (3.10), we get,

$$U = \int_0^L \frac{(Px)^2}{2EI} dx = \frac{P^2 L^3}{6EI} \quad (3)$$

Now, according to Castigliano's theorem, the first partial derivative of strain energy with respect to external force P gives the deflection u_A at A in the direction of applied force. Thus,

$$\frac{\partial U}{\partial P} = u_A = \frac{PL^3}{3EI} \tag{4}$$

To find the slope at the free end, we need to differentiate strain energy with respect to externally applied moment M at A. As there is no moment at A, apply a fictitious moment M_0 at A. Now moment at any section at a distance x away from the free end is given by

$$M = -Px - M_0$$

Now, strain energy stored in the beam may be calculated as,

$$U = \int_0^L \frac{(Px + M_0)^2}{2EI} dx = \frac{P^2 L^3}{6EI} + \frac{M_0 PL^2}{2EI} + \frac{M_0^{\;2} L}{2EI} \tag{5}$$

Taking partial derivative of strain energy with respect to M_0, we get slope at A.

$$\frac{\partial U}{\partial M_0} = \theta_A = \frac{PL^2}{2EI} + \frac{M_0 L}{EI} \tag{6}$$

But actually there is no moment applied at A. Hence substitute $M_0 = 0$ in equation (3.14) we get the slope at A.

$$\theta_A = \frac{PL^2}{2EI} \tag{7}$$

Example 3.2

A cantilever beam which is curved in the shape of a quadrant of a circle is loaded as shown in Fig. 3.3. The radius of curvature of curved beam is R, Young's modulus of the material is E and second moment of the area is I about an axis perpendicular to the plane of the paper through the centroid of the cross section. Find the vertical displacement of point A on the curved beam.

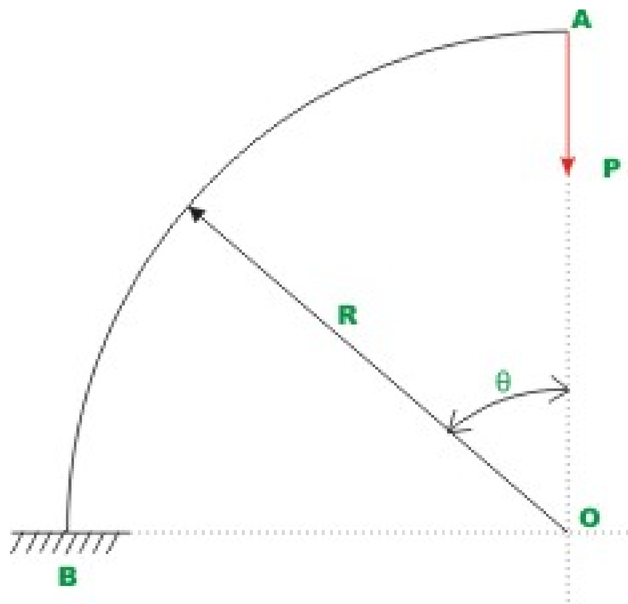


Fig. 3.3 Example 3.2

The bending moment at any section θ of the curved beam (see Fig. 3.3) is given by

$$M = PR\sin\theta \qquad (1)$$

Strain energy U stored in the curved beam due to bending is,

$$U = \int_0^s \frac{M^2}{2EI} ds = \int_0^{\pi/2} \frac{P^2R^2(\sin^2\theta)Rd\theta}{2EI} = \frac{P^2R^3}{2EI}\frac{\pi}{4} = \frac{\pi P^2R^3}{8EI} \qquad (2)$$

Differentiating strain energy with respect to externally applied load, P we get

$$u_A = \frac{\partial U_b}{\partial P} = \frac{\pi PR^3}{4EI} \qquad (3)$$

Example 3.3

Find horizontal displacement at D of the frame shown in Fig. 3.4. Assume the flexural rigidity of the beam EI to be constant through out the member. Neglect strain energy due to axial deformations.

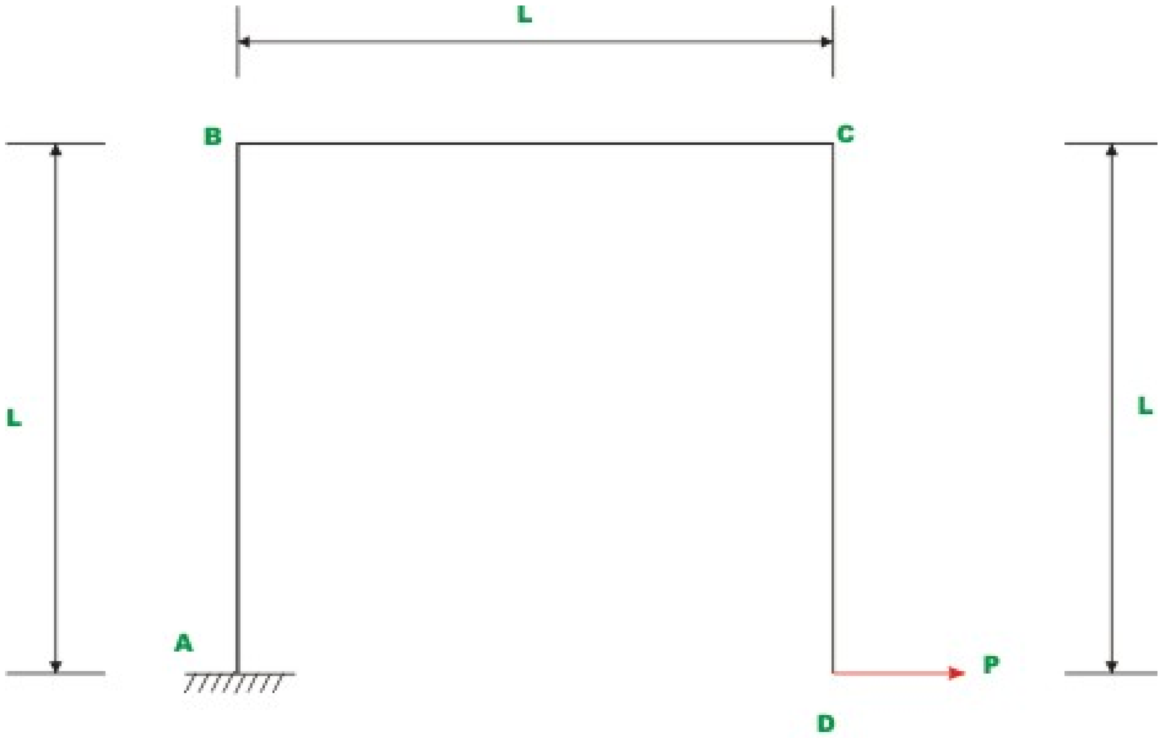


Fig. 3.4 Example 3.3

The deflection D may be obtained via. Castigliano's theorem. The beam segments BA and DC are subjected to bending moment Px $(0 < x < L)$ and the beam element BC is subjected to a constant bending moment of magnitude PL.

Total strain energy stored in the frame due to bending

$$U = 2\int_0^L \frac{(Px)^2}{2EI} dx + \int_0^L \frac{(PL)^2}{2EI} dx \tag{1}$$

After simplifications,

$$U = \frac{P^2 L^3}{3EI} + \frac{P^2 L^3}{2EI} = \frac{5P^2 L^3}{6EI} \tag{2}$$

Differentiating strain energy with respect to P we get,

$$\frac{\partial U}{\partial P} = u_D = 2\frac{5P\ L^3}{6EI} = \frac{5P\ L^3}{3EI}$$

Example 3.4

Find the vertical deflection at A of the structure shown Fig. 3.5. Assume the flexural rigidity EI and torsional rigidity GJ to be constant for the structure.

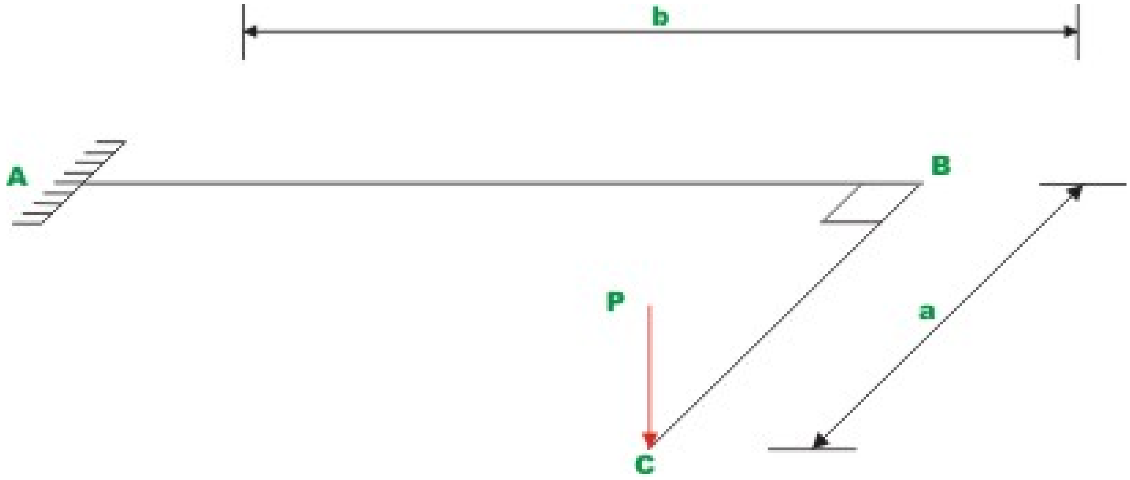


Fig.3.5 Example 3.4

The beam segment BC is subjected to bending moment Px $(0 < x < a$; x is measured from C)and the beam element AB is subjected to torsional moment of magnitude Pa and a bending moment of $Px\,(0 \le x \le b\,;\text{x is measured from B})$. The strain energy stored in the beam ABC is,

$$U = \int_0^a \frac{M^2}{2EI}dx + \int_0^b \frac{(Pa)^2}{2GJ}dx + \int_0^b \frac{(Px)^2}{2EI}dx \tag{1}$$

After simplifications,

$$U = \frac{P^2a^3}{6EI} + \frac{P^2a^2b}{2GJ} + \frac{P^2b^3}{6EI} \tag{2}$$

Vertical deflection u_A at A is,

$$\frac{\partial U}{\partial P} = u_A = \frac{Pa^3}{3EI} + \frac{Pa^2b}{GJ} + \frac{Pb^3}{3EI} \tag{3}$$

Example 3.5

Find vertical deflection at C of the beam shown in Fig. 3.6. Assume the flexural rigidity EI to be constant for the structure.

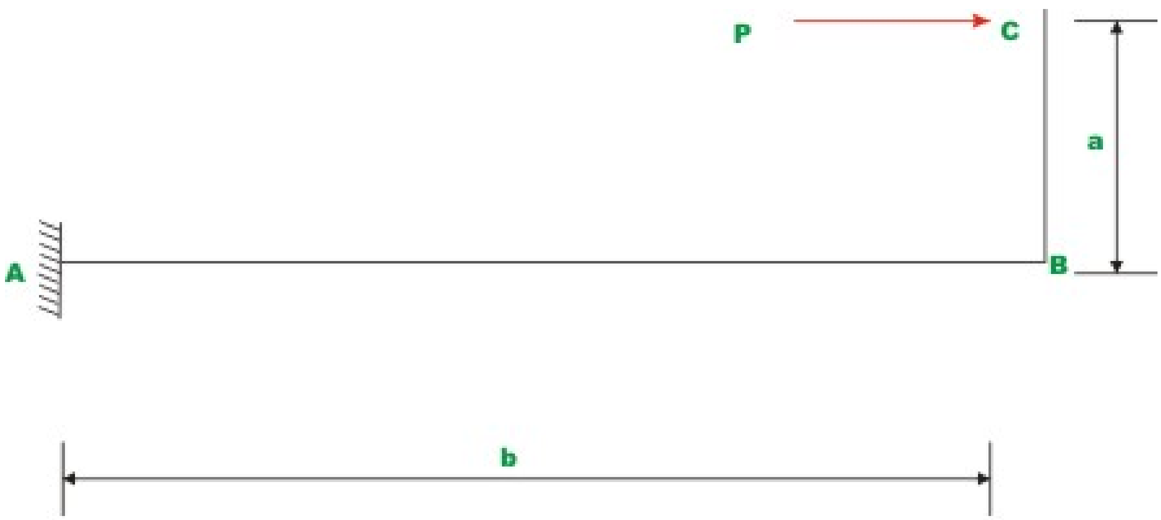


Fig. 3.6 Example 3.5

The beam segment CB is subjected to bending moment Px $(0 < x < a)$ and beam element AB is subjected to moment of magnitude Pa.
To find the vertical deflection at C, introduce a imaginary vertical force Q at C.
Now, the strain energy stored in the structure is,

$$U = \int_0^a \frac{(Px)^2}{2EI} dx + \int_0^b \frac{(Pa + Qy)^2}{2EI} dy \tag{1}$$

Differentiating strain energy with respect to Q, vertical deflection at C is obtained.

$$\frac{\partial U}{\partial Q} = u_C = \int_0^b \frac{2(Pa + Qy)y}{2EI} dy \tag{2}$$

$$u_C = \frac{1}{EI} \int_0^b Pay + Qy^2 dy \tag{3}$$

$$u_C = \frac{1}{EI}\left[\frac{Pab^2}{2} + \frac{Qb^3}{3}\right] \quad (4)$$

But the force Q is fictitious force and hence equal to zero. Hence, vertical deflection is,

$$u_C = \frac{Pab^2}{2EI} \quad (5)$$

3.3 Castigliano's Second Theorem

In any elastic structure having n independent displacements $u_1, u_2, ..., u_n$ corresponding to external forces $P_1, P_2, ..., P_n$ along their lines of action, if strain energy is expressed in terms of displacements then n equilibrium equations may be written as follows.

$$\frac{\partial U}{\partial u_j} = P_j, \quad j = 1, 2, ..., n \quad (3.9)$$

This may be proved as follows. The strain energy of an elastic body may be written as

$$U = \frac{1}{2}P_1 u_1 + \frac{1}{2}P_2 u_2 + ... + \frac{1}{2}P_n u_n \quad (3.10)$$

We know from Lesson 1 (equation 1.5) that

$$P_i = k_{i1}u_1 + k_{i2}u_2 + ... + k_{in}u_n, \quad i = 1, 2, .., n \quad (3.11)$$

where k_{ij} is the stiffness coefficient and is defined as the force at i due to unit displacement applied at j. Hence, strain energy may be written as,

$$U = \frac{1}{2}u_1[k_{11}u_1 + k_{12}u_2 + ...] + \frac{1}{2}u_2[k_{21}u_1 + k_{22}u_2 + ...] + ... + \frac{1}{2}u_n[k_{n1}u_1 + k_{n2}u_2 + ...] \quad (3.12)$$

We know from reciprocal theorem $k_{ij} = k_{ji}$. Hence, equation (3.12) may be simplified as,

$$U = \frac{1}{2}\left[k_{11}u_1^2 + k_{22}u_2^2 + ... + k_{nn}u_n^2\right] + \left[k_{12}u_1u_2 + k_{13}u_1u_3 + ... + k_{1n}u_1u_n\right] + ... \quad (3.13)$$

Now, differentiating the strain energy with respect to any displacement u_1 gives the applied force P_1 at that point, Hence,

$$\frac{\partial U}{\partial u_1} = k_{11}u_1 + k_{12}u_2 + ... + k_{1n}u_n \tag{3.14}$$

Or,

$$\frac{\partial U}{\partial u_j} = P_j, \qquad j = 1, 2, ..., n \tag{3.15}$$

Summary

In this lesson, Castigliano's first theorem has been stated and proved for linearly elastic structure with unyielding supports. The procedure to calculate deflections of a statically determinate structure at the point of application of load is illustrated with examples. Also, the procedure to calculate deflections in a statically determinate structure at a point where load is applied is illustrated with examples. The Castigliano's second theorem is stated for elastic structure and proved in section 3.4.

Module
1
Energy Methods in Structural Analysis

Lesson
4
Theorem of Least Work

Instructional Objectives

After reading this lesson, the reader will be able to:

1. State and prove theorem of Least Work.
2. Analyse statically indeterminate structure.
3. State and prove Maxwell-Betti's Reciprocal theorem.

4.1 Introduction

In the last chapter the Castigliano's theorems were discussed. In this chapter theorem of least work and reciprocal theorems are presented along with few selected problems. We know that for the statically determinate structure, the partial derivative of strain energy with respect to external force is equal to the displacement in the direction of that load at the point of application of load. This theorem when applied to the statically indeterminate structure results in the theorem of least work.

4.2 Theorem of Least Work

According to this theorem, the partial derivative of strain energy of a statically indeterminate structure with respect to statically indeterminate action should vanish as it is the function of such redundant forces to prevent any displacement at its point of application. The forces developed in a redundant framework are such that the total internal strain energy is a minimum. This can be proved as follows. Consider a beam that is fixed at left end and roller supported at right end as shown in Fig. 4.1a. Let $P_1, P_2, ..., P_n$ be the forces acting at distances $x_1, x_2, ..., x_n$ from the left end of the beam of span L. Let $u_1, u_2, ..., u_n$ be the displacements at the loading points $P_1, P_2, ..., P_n$ respectively as shown in Fig. 4.1a. This is a statically indeterminate structure and choosing R_a as the redundant reaction, we obtain a simple cantilever beam as shown in Fig. 4.1b. Invoking the principle of superposition, this may be treated as the superposition of two cases, viz, a cantilever beam with loads $P_1, P_2, ..., P_n$ and a cantilever beam with redundant force R_a (see Fig. 4.2a and Fig. 4.2b)

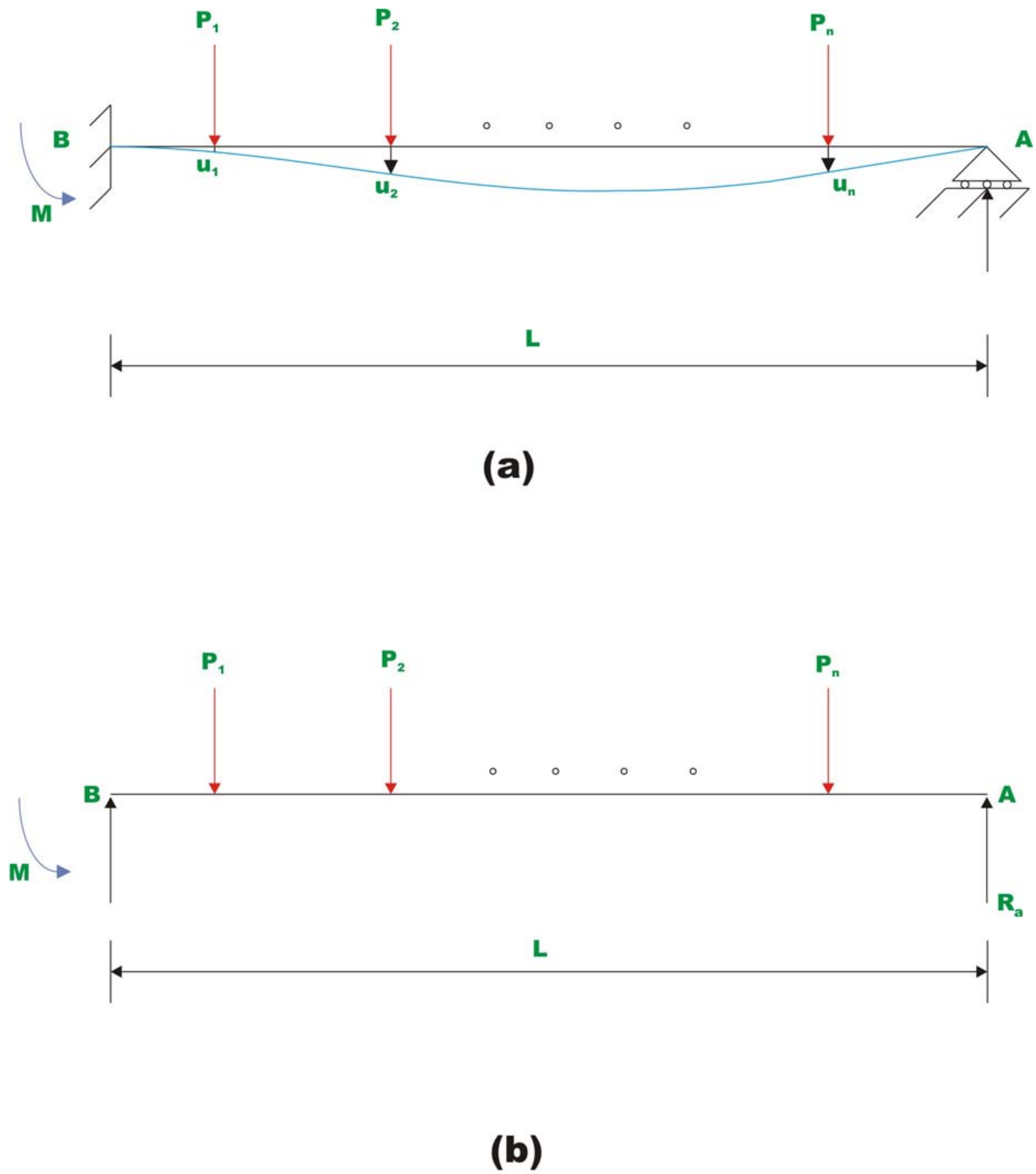


Fig.4.1 Theorem of Least work

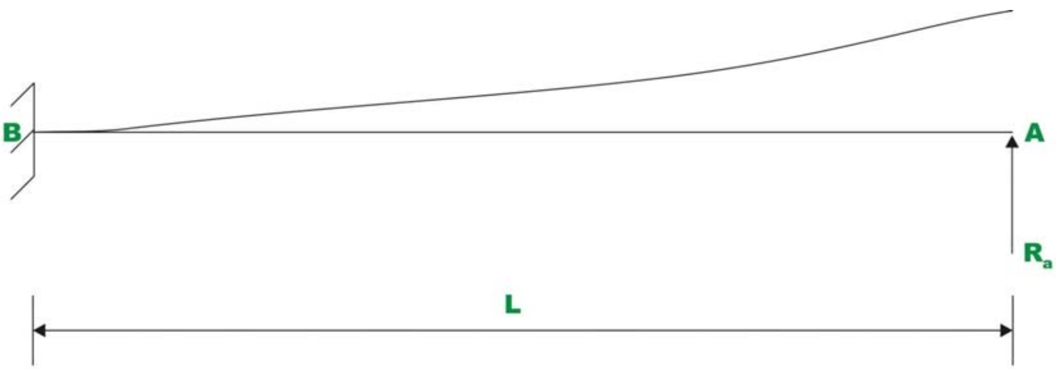


Fig. 4.2 (a)Cantilever beam with redundant

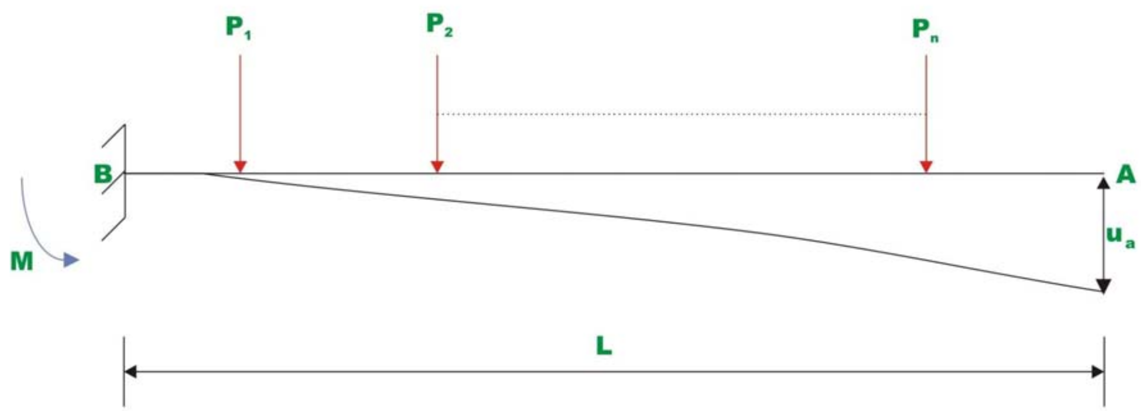


Fig. 4.2 (b)Cantilever beam with externally applied loads and a fictious load

In the first case (4.2a), obtain deflection below A due to applied loads $P_1, P_2, ..., P_n$. This can be easily accomplished through Castigliano's first theorem as discussed in Lesson 3. Since there is no load applied at A, apply a fictitious load Q at A as in Fig. 4.2. Let u_a be the deflection below A.

Now the strain energy U_s stored in the determinate structure (i.e. the support A removed) is given by,

$$U_S = \frac{1}{2}P_1 u_1 + \frac{1}{2}P_2 u_2 + ... + \frac{1}{2}P_n u_n + \frac{1}{2}Q u_a \tag{4.1}$$

It is known that the displacement u_1 below point P_1 is due to action of $P_1, P_2, ..., P_n$ acting at $x_1, x_2, ..., x_n$ respectively and due to Q at A. Hence, u_1 may be expressed as,

$$u_1 = a_{11}P_1 + a_{12}P_2 + ...+ a_{1n}P_n + a_{1a}Q \tag{4.2}$$

where, a_{ij} is the flexibility coefficient at i due to unit force applied at j. Similar equations may be written for $u_2, u_3, ..., u_n$ and u_a. Substituting for $u_2, u_3, ..., u_n$ and u_a in equation (4.1) from equation (4.2), we get,

$$\begin{aligned} U_S = & \frac{1}{2}P_1[a_{11}P_1 + a_{12}P_2 + ...+ a_{1n}P_n + a_{1a}Q] + \frac{1}{2}P_2[a_{21}P_1 + a_{22}P_2 + ...a_{2n}P_n + a_{2a}Q] + ... \\ & + \frac{1}{2}P_n[a_{n1}P_1 + a_{n2}P_2 + ...a_{nn}P_n + a_{na}Q] + \frac{1}{2}Q[a_{a1}P_1 + a_{a2}P_2 + ...+ a_{an}P_n + a_{aa}Q] \end{aligned} \tag{4.3}$$

Taking partial derivative of strain energy U_s with respect to Q, we get deflection at A.

$$\frac{\partial U_s}{\partial Q} = a_{a1}P_1 + a_{a2}P_2 + ...+ a_{an}P_n + a_{aa}Q \tag{4.4}$$

Substitute $Q = 0$ as it is fictitious in the above equation,

$$\frac{\partial U_s}{\partial Q} = u_a = a_{a1}P_1 + a_{a2}P_2 + ...+ a_{an}P_n \tag{4.5}$$

Now the strain energy stored in the beam due to redundant reaction R_A is,

$$U_r = \frac{R_a^2 L^3}{6EI} \tag{4.6}$$

Now deflection at A due to R_a is

$$\frac{\partial U_r}{\partial R_a} = -u_a = \frac{R_a L^3}{3EI} \tag{4.7}$$

The deflection due to R_a should be in the opposite direction to one caused by superposed loads $P_1, P_2, ..., P_n$, so that the net deflection at A is zero. From equation (4.5) and (4.7) one could write,

$$\frac{\partial Us}{\partial Q} = u_a = -\frac{\partial U_r}{\partial R_a} \tag{4.8}$$

Since Q is fictitious, one could as well replace it by R_a. Hence,

$$\frac{\partial}{\partial R_a}(U_s + U_r) = 0 \qquad (4.9)$$

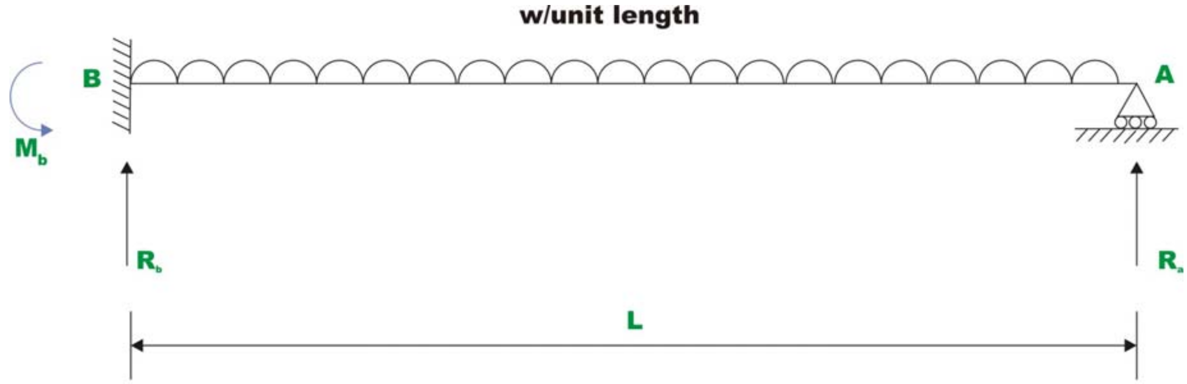
or,

$$\frac{\partial U}{\partial R_a} = 0 \qquad (4.10)$$

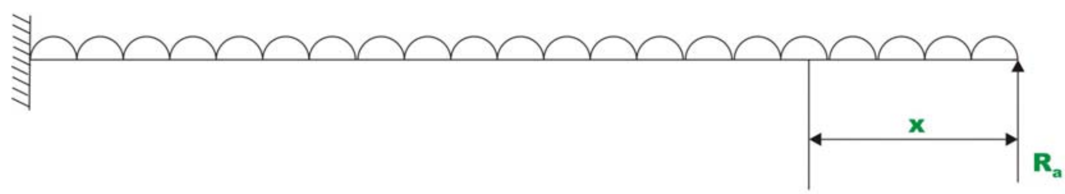
This is the statement of theorem of least work. Where U is the total strain energy of the beam due to superimposed loads $P_1, P_2, ..., P_n$ and redundant reaction R_a.

Example 4.1

Find the reactions of a propped cantilever beam uniformly loaded as shown in Fig. 4.3a. Assume the flexural rigidity of the beam EI to be constant throughout its length.



(a)



(b)

Fig.4.3 Example 4.1

There three reactions R_a, R_b and M_b as shown in the figure. We have only two equation of equilibrium viz., $\sum F_y = 0$ and $\sum M = 0$. This is a statically indeterminate structure and choosing R_b as the redundant reaction, we obtain a simple cantilever beam as shown in Fig. 4.3b.
Now, the internal strain energy of the beam due to applied loads and redundant reaction, considering only bending deformations is,

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad (1)$$

According to theorem of least work we have,

$$\frac{\partial U}{\partial R_b} = 0 = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial R_b} \quad (2)$$

Bending moment at a distance x from B, $M = R_b x - \frac{wx^2}{2}$ (3)

$$\frac{\partial M}{\partial R_b} = x \quad (4)$$

Hence,

$$\frac{\partial U}{\partial R_b} = \int_0^L \frac{(R_b x - wx^2/2)x}{EI} dx \quad (5)$$

$$\frac{\partial U}{\partial R_b} = \left[\frac{R_B L^3}{3} - \frac{wL^4}{8}\right]\frac{1}{EI} = 0 \quad (6)$$

Solving for R_b, we get,

$$R_B = \frac{3}{8}wL$$

$$R_a = wL - R_b = \frac{5}{8}wL \text{ and } M_a = -\frac{wL^2}{8} \quad (7)$$

Example 4.2

A ring of radius R is loaded as shown in figure. Determine increase in the diameter AB of the ring. Young's modulus of the material is E and second moment of the area is I about an axis perpendicular to the page through the centroid of the cross section.

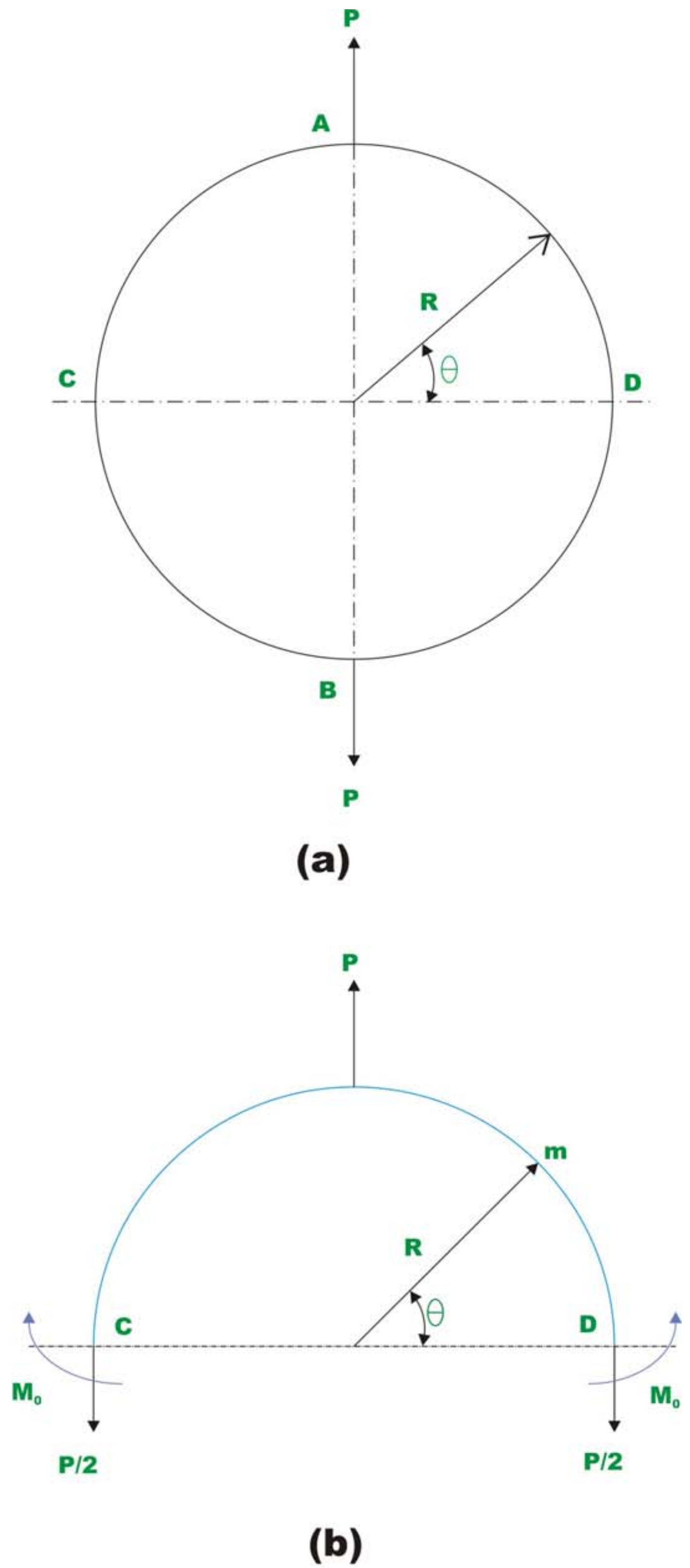


Fig.4.4 Example 4.2

The free body diagram of the ring is as shown in Fig. 4.4. Due to symmetry, the slopes at C and D is zero. The value of redundant moment M_0 is such as to make slopes at C and D zero. The bending moment at any section θ of the beam is,

$$M = M_0 - \frac{PR}{2}(1-\cos\theta) \tag{1}$$

Now strain energy stored in the ring due to bending deformations is,

$$U = \int_0^{2\pi} \frac{M^2 R}{2EI} d\theta \tag{2}$$

Due to symmetry, one could consider one quarter of the ring. According to theorem of least work,

$$\frac{\partial U}{\partial M_0} = 0 = \int_0^{2\pi} \frac{M}{EI} \frac{\partial M}{\partial M_0} R d\theta \tag{3}$$

$$\frac{\partial M}{\partial M_0} = 1$$

$$\frac{\partial U}{\partial M_0} = \int_0^{2\pi} \frac{M}{EI} R d\theta \tag{4}$$

$$0 = \frac{4R}{EI} \int_0^{\frac{\pi}{2}} [M_0 - \frac{PR}{2}(1-\cos\theta)]\, d\theta \tag{5}$$

Integrating and solving for M_0,

$$M_0 = PR\left(\frac{1}{2} - \frac{1}{\pi}\right) \tag{6}$$

$$M_0 = 0.182PR$$

Now, increase in diameter Δ, may be obtained by taking the first partial derivative of strain energy with respect to P. Thus,

$$\Delta = \frac{\partial U}{\partial P}$$

Now strain energy stored in the ring is given by equation (2). Substituting the value of M_0 and equation (1) in (2), we get,

$$U = \frac{2R}{EI}\int_0^{\pi/2}\{\frac{PR}{2}(\frac{2}{\pi}-1)-\frac{PR}{2}(1-\cos\theta)\}^2\,d\theta \qquad (7)$$

Now the increase in length of the diameter is,

$$\frac{\partial U}{\partial P} = \frac{2R}{EI}\int_0^{\pi/2} 2\{\frac{PR}{2}(\frac{2}{\pi}-1)-\frac{PR}{2}(1-\cos\theta)\}\{\frac{R}{2}(\frac{2}{\pi}-1)-\frac{R}{2}(1-\cos\theta)\}d\theta \qquad (8)$$

After integrating,

$$\Delta = \frac{PR^3}{EI}\{\frac{\pi}{4}-\frac{2}{\pi}) = 0.149\frac{PR^3}{EI} \qquad (9)$$

4.3 Maxwell–Betti Reciprocal theorem

Consider a simply supported beam of span L as shown in Fig. 4.5. Let this beam be loaded by two systems of forces P_1 and P_2 separately as shown in the figure. Let u_{21} be the deflection below the load point P_2 when only load P_1 is acting. Similarly let u_{12} be the deflection below load P_1, when only load P_2 is acting on the beam.

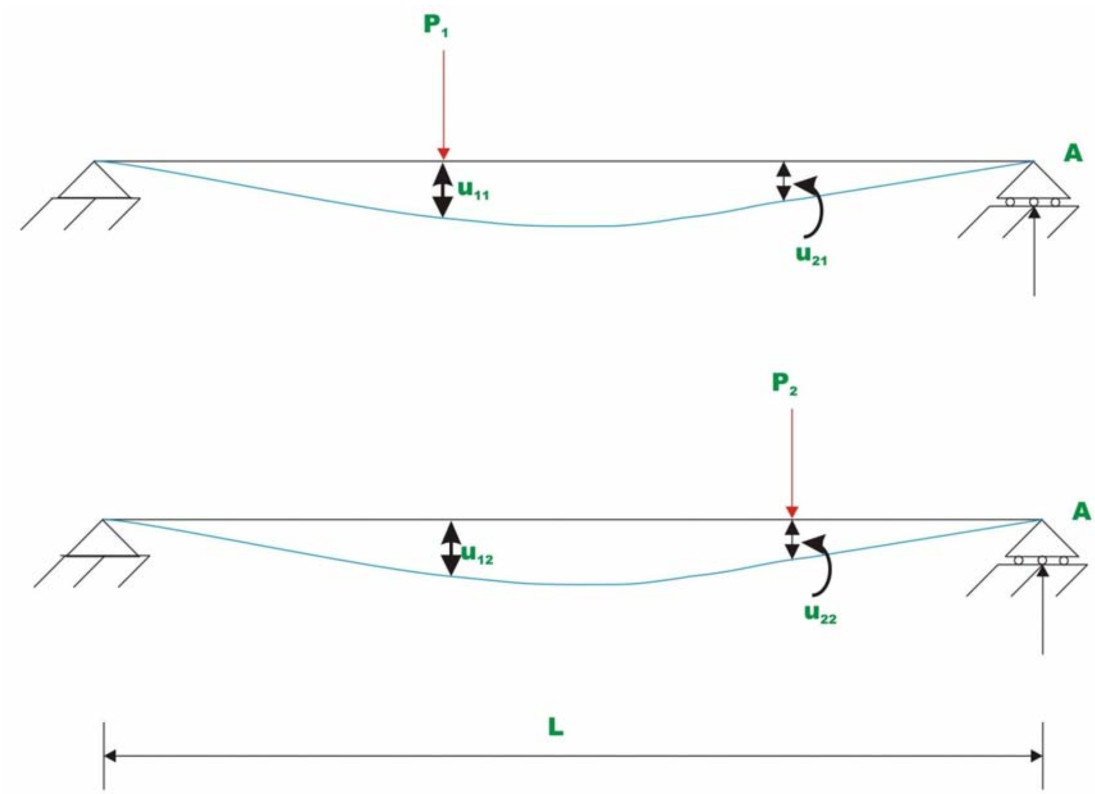


Fig. 4.5 Reciprocal theorem

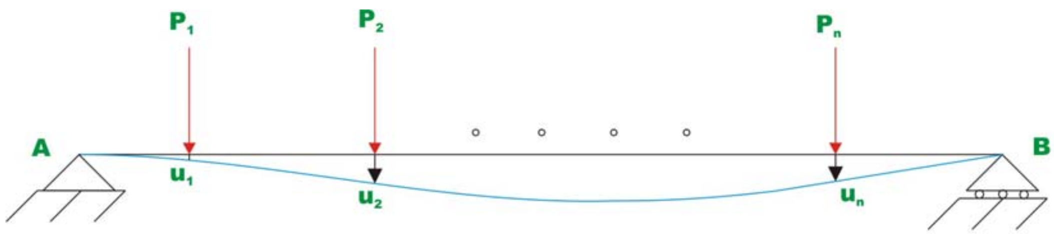
The reciprocal theorem states that the work done by forces acting through displacement of the second system is the same as the work done by the second system of forces acting through the displacements of the first system. Hence, according to reciprocal theorem,

$$P_1 \times u_{12} = P_2 \times u_{21} \tag{4.11}$$

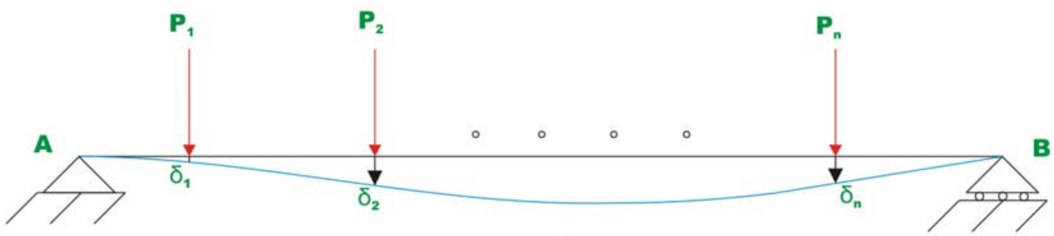
Now, u_{12} and u_{21} can be calculated using Castiglinao's first theorem. Substituting the values of u_{12} and u_{21} in equation (4.27) we get,

$$P_1 \times \frac{5P_2L^3}{48EI} = P_2 \times \frac{5P_1L^3}{48EI} \tag{4.12}$$

Hence it is proved. This is also valid even when the first system of forces is $P_1, P_2, ..., P_n$ and the second system of forces is given by $Q_1, Q_2, ..., Q_n$. Let $u_1, u_2, ..., u_n$ be the displacements caused by the forces $P_1, P_2, ..., P_n$ only and $\delta_1, \delta_2, ..., \delta_n$ be the displacements due to system of forces $Q_1, Q_2, ..., Q_n$ only acting on the beam as shown in Fig. 4.6.



(a)



(b)

Fig. 4.6 Generalized statement of Reciprocal Theorem

Now the reciprocal theorem may be stated as,

$$P_i \, \delta_i = Q_i \, u_i \qquad i = 1,2,....,n \tag{4.13}$$

Summary

In lesson 3, the Castigliano's first theorem has been stated and proved. For statically determinate structure, the partial derivative of strain energy with respect to external force is equal to the displacement in the direction of that load at the point of application of the load. This theorem when applied to the statically indeterminate structure results in the theorem of Least work. In this chapter the theorem of Least Work has been stated and proved. Couple of problems is solved to illustrate the procedure of analysing statically indeterminate structures. In the end, the celebrated theorem of Maxwell-Betti’s reciprocal theorem has been sated and proved.

Module 1

Energy Methods in Structural Analysis

Lesson
5
Virtual Work

Instructional Objectives

After studying this lesson, the student will be able to:

1. Define Virtual Work.
2. Differentiate between external and internal virtual work.
3. Sate principle of virtual displacement and principle of virtual forces.
4. Drive an expression of calculating deflections of structure using unit load method.
5. Calculate deflections of a statically determinate structure using unit load method.
6. State unit displacement method.
7. Calculate stiffness coefficients using unit-displacement method.

5.1 Introduction

In the previous chapters the concept of strain energy and Castigliano's theorems were discussed. From Castigliano's theorem it follows that for the statically determinate structure; the partial derivative of strain energy with respect to external force is equal to the displacement in the direction of that load. In this lesson, the principle of virtual work is discussed. As compared to other methods, virtual work methods are the most direct methods for calculating deflections in statically determinate and indeterminate structures. This principle can be applied to both linear and nonlinear structures. The principle of virtual work as applied to deformable structure is an extension of the virtual work for rigid bodies. This may be stated as: if a rigid body is in equilibrium under the action of a $F-$system of forces and if it continues to remain in equilibrium if the body is given a small (virtual) displacement, then the virtual work done by the $F-$system of forces as 'it rides' along these virtual displacements is zero.

5.2 Principle of Virtual Work

Many problems in structural analysis can be solved by the principle of virtual work. Consider a simply supported beam as shown in Fig.5.1a, which is in equilibrium under the action of real forces $F_1, F_2, ..., F_n$ at co-ordinates $1,2,.....,n$ respectively. Let $u_1, u_2, ..., u_n$ be the corresponding displacements due to the action of forces $F_1, F_2, ..., F_n$. Also, it produces real internal stresses σ_{ij} and real internal strains ε_{ij} inside the beam. Now, let the beam be subjected to second system of forces (which are virtual not real) $\delta F_1, \delta F_2, ..., \delta F_n$ in equilibrium as shown in Fig.5.1b. The second system of forces is called virtual as they are imaginary and they are not part of the real loading. This produces a displacement configuration $\delta u_1, \delta u_2, ..., \delta u_n$. The virtual loading system produces virtual internal stresses $\delta\sigma_{ij}$ and virtual internal strains $\delta\varepsilon_{ij}$ inside the beam. Now, apply the second system of forces on the beam which has been deformed by first system of forces. Then, the external loads F_i and internal stresses σ_{ij} do virtual work by moving along δu_i and $\delta\varepsilon_{ij}$. The product $\sum F_i \delta u_i$ is known as the external virtual work. It may be noted that the above product does not represent the conventional work since each component is caused due to different source i.e. δu_i is not due to F_i. Similarly the product $\sum \sigma_{ij} \delta\varepsilon_{ij}$ is the internal virtual work. In the case of deformable body, both external and internal forces do work. Since, the beam is in equilibrium, the external virtual work must be equal to the internal virtual work. Hence, one needs to consider both internal and external virtual work to establish equations of equilibrium.

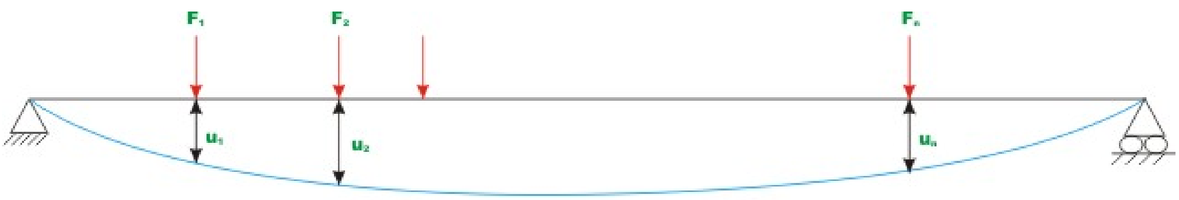


Fig. 5.1a : Actual system of forces.

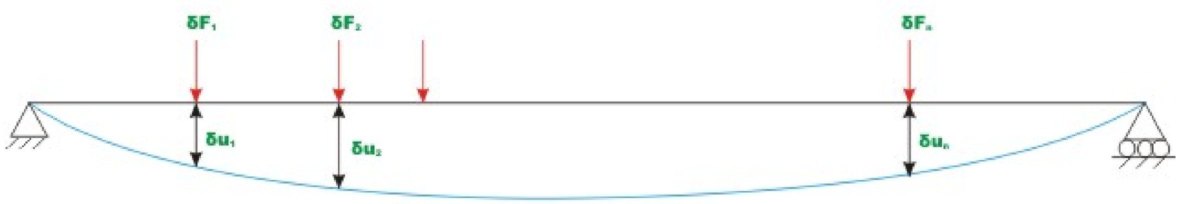


Fig. 5.1b : virtual system of forces.

5.3 Principle of Virtual Displacement

A deformable body is in equilibrium if the total external virtual work done by the system of true forces moving through the corresponding virtual displacements of the system i.e. $\sum F_i \delta u_i$ is equal to the total internal virtual work for every kinematically admissible (consistent with the constraints) virtual displacements.

That is virtual displacements should be continuous within the structure and also it must satisfy boundary conditions.

$$\sum F_i \, \delta u_i = \int \sigma_{ij} \, \delta \varepsilon_{ij} \, dv \tag{5.1}$$

where σ_{ij} are the true stresses due to true forces F_i and $\delta\varepsilon_{ij}$ are the virtual strains due to virtual displacements δu_i.

5.4 Principle of Virtual Forces

For a deformable body, the total external complementary work is equal to the total internal complementary work for every system of virtual forces and stresses that satisfy the equations of equilibrium.

$$\sum \delta F_i \, u_i = \int \delta\sigma_{ij} \, \varepsilon_{ij} \, dv \tag{5.2}$$

where $\delta\sigma_{ij}$ are the virtual stresses due to virtual forces δF_i and ε_{ij} are the true strains due to the true displacements u_i.

As stated earlier, the principle of virtual work may be advantageously used to calculate displacements of structures. In the next section let us see how this can be used to calculate displacements in a beams and frames. In the next lesson, the truss deflections are calculated by the method of virtual work.

5.5 Unit Load Method

The principle of virtual force leads to unit load method. It is assumed throughout our discussion that the method of superposition holds good. For the derivation of unit load method, we consider two systems of loads. In this section, the principle of virtual forces and unit load method are discussed in the context of framed structures. Consider a cantilever beam, which is in equilibrium under the action of a first system of forces $F_1, F_2, \ldots\ldots, F_n$ causing displacements $u_1, u_2, \ldots\ldots, u_n$ as shown in Fig. 5.2a. The first system of forces refers to the actual forces acting on the structure. Let the stress resultants at any section of the beam due to first system of forces be axial force (P), bending moment (M) and shearing force (V). Also the corresponding incremental deformations are axial deformation ($d\Delta$), flexural deformation ($d\theta$) and shearing deformation ($d\lambda$) respectively.

For a conservative system the external work done by the applied forces is equal to the internal strain energy stored. Hence,

$$\frac{1}{2}\sum_{i=1}^{n} F_i u_i = \frac{1}{2}\int P \ \mathrm{d}\Delta + \frac{1}{2}\int M \ \ \mathrm{d}\theta + \frac{1}{2}\int V \ \ \mathrm{d}\lambda$$

$$= \int_0^L \frac{P^2 ds}{2EA} + \int_0^L \frac{M^2 ds}{2EI} + \int_0^L \frac{V^2 ds}{2AG} \tag{5.3}$$

Now, consider a second system of forces $\delta F_1, \delta F_2, ..., \delta F_n$, which are virtual and causing virtual displacements $\delta u_1, \delta u_2, ..., \delta u_n$ respectively (see Fig. 5.2b). Let the virtual stress resultants caused by virtual forces be $\delta P_v, \delta M_v$ and δV_v at any cross section of the beam. For this system of forces, we could write

$$\frac{1}{2}\sum_{i=1}^{n} \delta F_i \delta u_i = \int_0^L \frac{\delta P_v{}^2 ds}{2EA} + \int_0^L \frac{\delta M_v{}^2 ds}{2EI} + \int_0^L \frac{\delta V_v{}^2 ds}{2AG} \tag{5.4}$$

where $\delta P_v, \delta M_v$ and δV_v are the virtual axial force, bending moment and shear force respectively. In the third case, apply the first system of forces on the beam, which has been deformed, by second system of forces $\delta F_1, \delta F_2, ..., \delta F_n$ as shown in Fig 5.2c. From the principle of superposition, now the deflections will be $(u_1 + \delta u_1), (u_2 + \delta u_2), ..., (u_n + \delta u_n)$ respectively

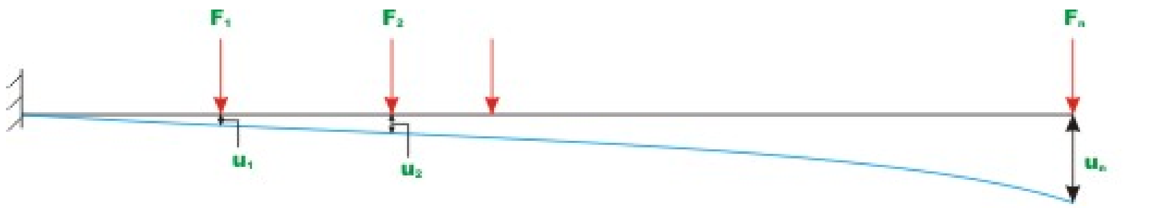


Fig. 5.2a : Actual system.

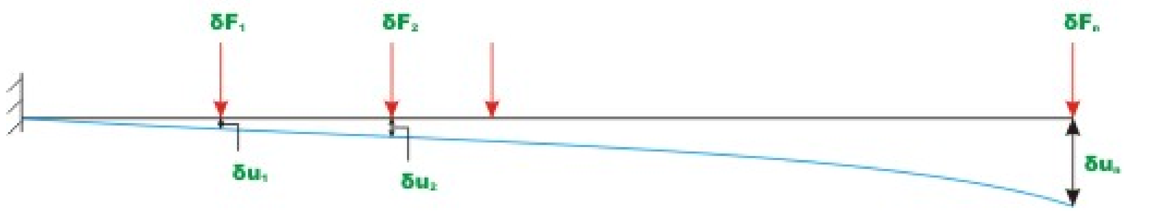


Fig. 5.2b : Virtual system of forces.

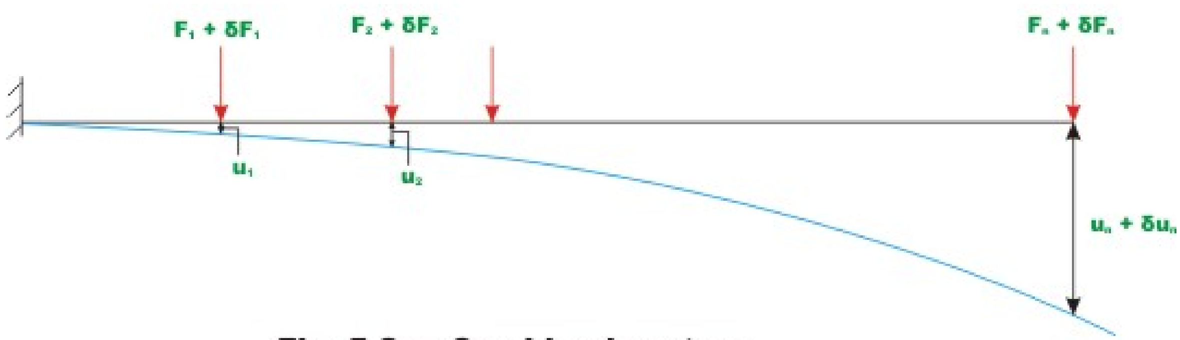


Fig. 5.2c : Combined system.

Since the energy is conserved we could write,

$$\frac{1}{2}\sum_{j=1}^{n} F_j u_j + \frac{1}{2}\sum_{j=1}^{n} \delta F_j \delta u_j + \sum_{j=1}^{n} \delta F_j u_j = \int_0^L \frac{\delta P_v^2 ds}{2EA} + \int_0^L \frac{\delta M_v{}^2 ds}{2EI} + \int_0^L \frac{\delta V_v{}^2 ds}{2AG} + \int_0^L \frac{P^2 ds}{2EA} + \int_0^L \frac{M^2 ds}{2EI} + \int_0^L \frac{V^2 ds}{2AG} + \int_0^L \delta P_v d\Delta + \int_0^L \delta M_v d\theta + \int_0^L \delta V_v d\lambda \tag{5.5}$$

In equation (5.5), the term on the left hand side $\left(\sum \delta F_j u_j\right)$, represents the work done by virtual forces moving through real displacements. Since virtual forces act at its full value, $\left(\frac{1}{2}\right)$ does not appear in the equation. Subtracting equation (5.3) and (5.4) from equation (5.5) we get,

$$\sum_{j=1}^{n} \delta F_j u_j = \int_0^L \delta P_v d\Delta + \int_0^L \delta M_v d\theta + \int_0^L \delta V_v d\lambda \qquad (5.6)$$

From Module 1, lesson 3, we know that

$d\Delta = \frac{Pds}{EA}, d\theta = \frac{Mds}{EI}$ and $d\lambda = \frac{Vds}{AG}$. Hence,

$$\sum_{j=1}^{n} \delta F_j u_j = \int_0^L \frac{\delta P_v Pds}{EA} + \int_0^L \frac{\delta M_v Mds}{EI} + \int_0^L \frac{\delta V_v Vds}{AG} \qquad (5.7)$$

Note that $\left(\frac{1}{2}\right)$ does not appear on right side of equation (5.7) as the virtual system resultants act at constant values during the real displacements. In the present case $\delta P_v = 0$ and if we neglect shear forces then we could write equation (5.7) as

$$\sum_{j=1}^{n} \delta F_j u_j = \int_0^L \frac{\delta M_v Mds}{EI} \qquad (5.8)$$

If the value of a particular displacement is required, then choose the corresponding force $\delta F_i = 1$ and all other forces $\delta F_j = 0$ $(j = 1,2,....,i-1,i+1,....,n)$. Then the above expression may be written as,

$$(1)u_i = \int_0^L \frac{\delta M_v Mds}{EI} \qquad (5.9)$$

where δM_v are the internal virtual moment resultants corresponding to virtual force at *i*-th co-ordinate, $\delta F_i = 1$. The above equation may be stated as,

$$\begin{aligned}&(unit \quad virtual \quad load\ )\ unknown \quad true\ displacement \\ &= \int (virtual \quad stress \quad resultants)(real \quad deformations)\ ds.\end{aligned} \qquad (5.10)$$

The equation (5.9) is known as the unit load method. Here the unit virtual load is applied at a point where the displacement is required to be evaluated. The unit load method is extensively used in the calculation of deflection of beams, frames and trusses. Theoretically this method can be used to calculate deflections in statically determinate and indeterminate structures. However it is extensively used in evaluation of deflections of statically determinate structures only as the method requires a priori knowledge of internal stress resultants.

Example 5.1

A cantilever beam of span L is subjected to a tip moment M_0 as shown in Fig 5.3a. Evaluate slope and deflection at a point $\left(\frac{3L}{4}\right)$ from left support. Assume EI of the given beam to be constant.

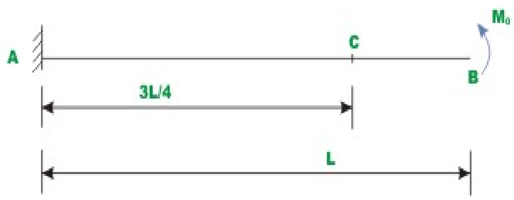


Fig. 5.3a Example 5.1

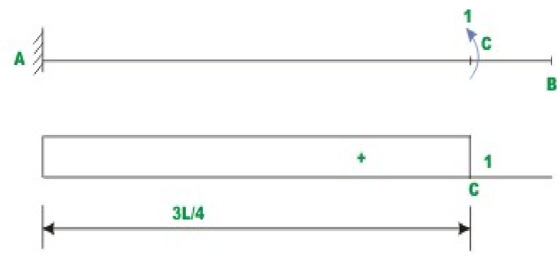
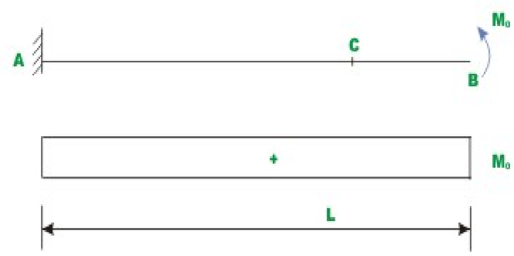


Fig. 5.3c. B. M. diagram of the beam due to unit moment at C.



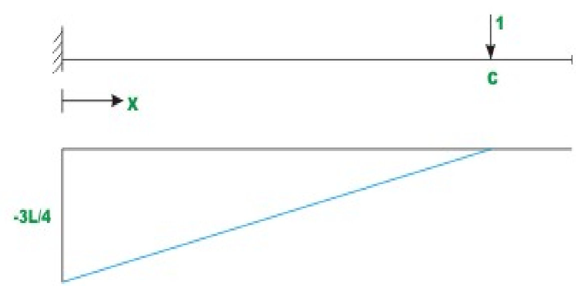


Fig. 5.3d B.M.D due to unit load at C

Fig. 5.3b : B. M. diagram of the beam due to moment M_0.

Slope at C

To evaluate slope at C, a virtual unit moment is applied at C as shown in Fig 5.3c. The bending moment diagrams are drawn for tip moment M_0 and unit moment applied at C and is shown in fig 5.3b and 5.3c respectively. Let θ_c be the rotation at C due to moment M_0 applied at tip. According to unit load method, the rotation at C, θ_c is calculated as,

$$(1)\theta_c = \int_0^L \frac{\delta M_v(x) M(x) dx}{EI} \quad (1)$$

where $\delta M_v(x)$ and $M(x)$ are the virtual moment resultant and real moment resultant at any section x. Substituting the value of $\delta M_v(x)$ and $M(x)$ in the above expression, we get

$$(1)\theta_c = \int_0^{3L/4} \frac{(1)Mdx}{EI} + \int_{3L/4}^{L} \frac{(0)Mdx}{EI}$$

$$\theta_c = \frac{3ML}{4EI} \quad (2)$$

Vertical deflection at C

To evaluate vertical deflection at C, a unit virtual vertical force is applied ac C as shown in Fig 5.3d and the bending moment is also shown in the diagram. According to unit load method,

$$(1)u_A = \int_0^L \frac{\delta M_v(x) M(x) dx}{EI} \quad (3)$$

In the present case, $\delta M_v(x) = -\left(\frac{3L}{4} - x\right)$

and $M(x) = +M_0$

$$u_A = \int_0^{\frac{3L}{4}} \frac{-\left(\frac{3L}{4} - x\right)M}{EI} dx$$

$$= -\frac{M}{EI}\int_0^{\frac{3L}{4}} \left(\frac{3L}{4} - x\right) dx$$

$$= -\frac{M}{EI}\left[\frac{3L}{4}x - \frac{x^2}{2}\right]_0^{\frac{3L}{4}}$$

$$= -\frac{9ML^2}{32EI} (\uparrow) \quad (4)$$

Example 5.2

Find the horizontal displacement at joint B of the frame ABCD as shown in Fig. 5.4a by unit load method. Assume EI to be constant for all members.

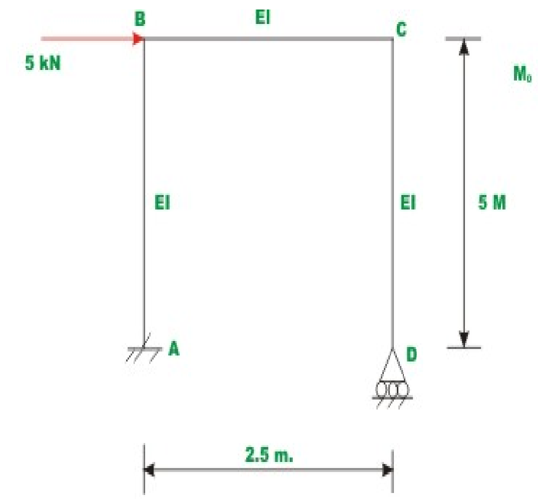


Fig. 5.4 a Example 5.2

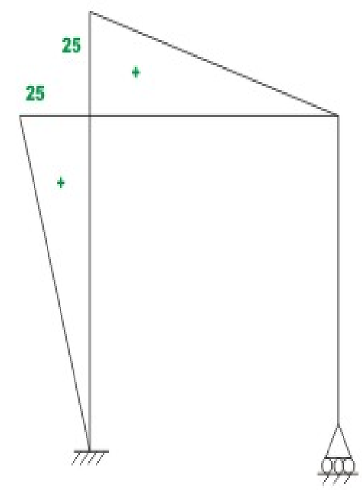


Fig. 5.4 c. Bending moment diagram of the frame for external loading.

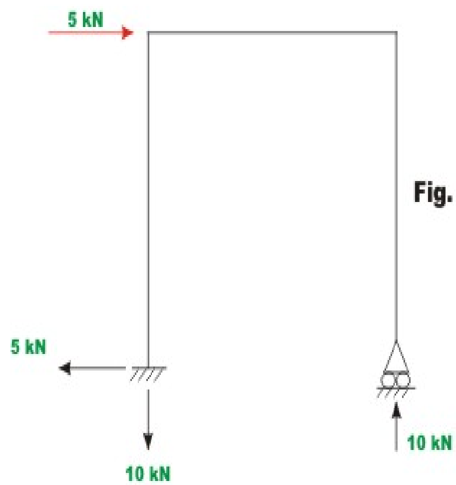


Fig. 5.4 b. Reactions.

The reactions and bending moment diagram of the frame due to applied external loading are shown in Fig 5.4b and Fig 5.4c respectively. Since, it is required to calculate horizontal deflection at B, apply a unit virtual load at B as shown in Fig. 5.4d. The resulting reactions and bending moment diagrams of the frame are shown in Fig 5.4d.

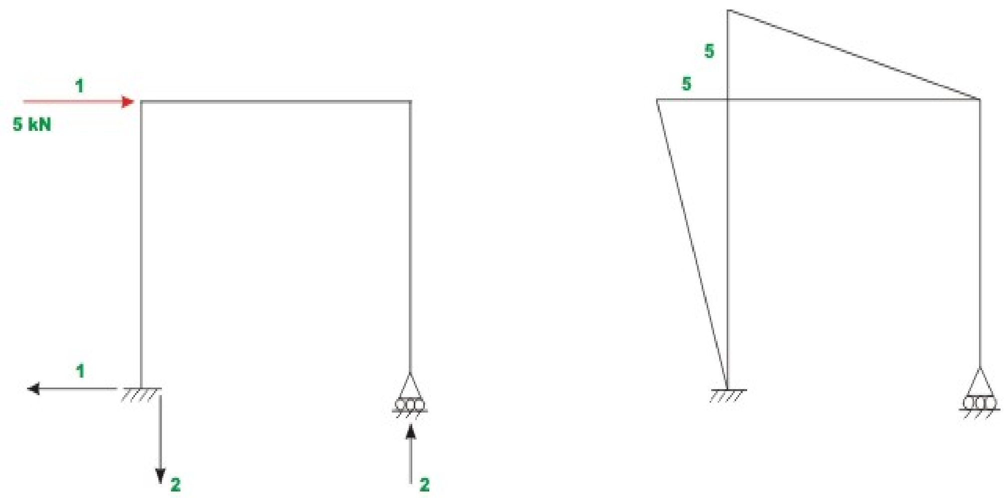


Fig. 5.4 d. Reactions and bending moment diagram of the frame for unit vertical load applied at B.

Now horizontal deflection at B, u_B may be calculated as

$$(1)\times u_H^B = \int_A^D \frac{\delta M_v(x) M(x) dx}{EI} \qquad (1)$$

$$= \int_A^B \frac{\delta M_v(x) M(x) dx}{EI} + \int_B^C \frac{\delta M_v(x) M(x) dx}{EI} + \int_C^D \frac{\delta M_v(x) M(x) dx}{EI}$$

$$= \int_0^5 \frac{(x)(5x)dx}{EI} + \int_0^{2.5} \frac{2(2.5-x)10(2.5-x)dx}{EI} + 0$$

$$= \int_0^5 \frac{(5x^2)dx}{EI} + \int_0^{2.5} \frac{20(2.5-x)^2 dx}{EI}$$

$$= \frac{625}{3EI} + \frac{312.5}{3EI} = \frac{937.5}{3EI}$$

Hence, $\quad u_A = \dfrac{937.5}{3EI} (\rightarrow) \qquad (2)$

Example 5.3

Find the rotations of joint B and C of the frame shown in Fig. 5.4a. Assume EI to be constant for all members.

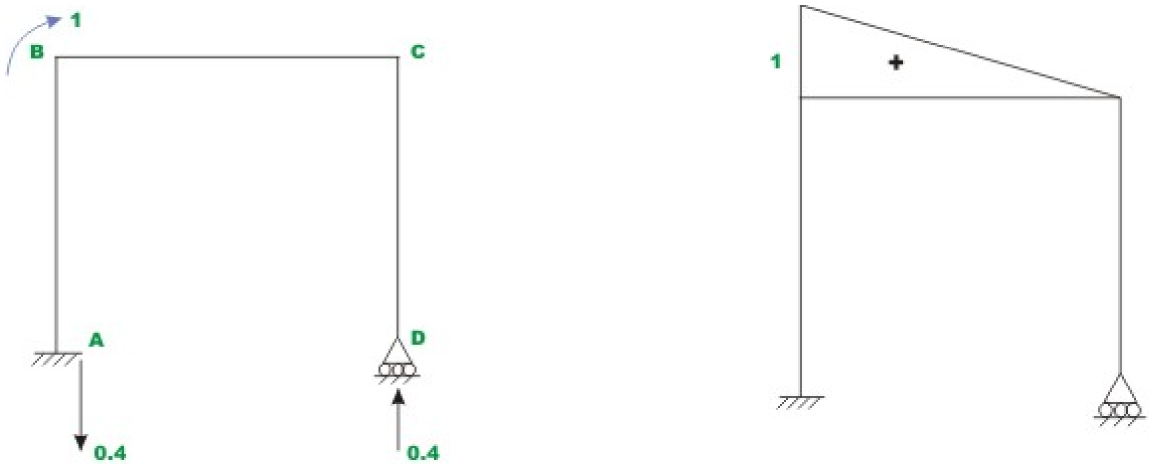


Fig. 5.5a. Reaction and B. M. diagram for the unit moment applied at B.

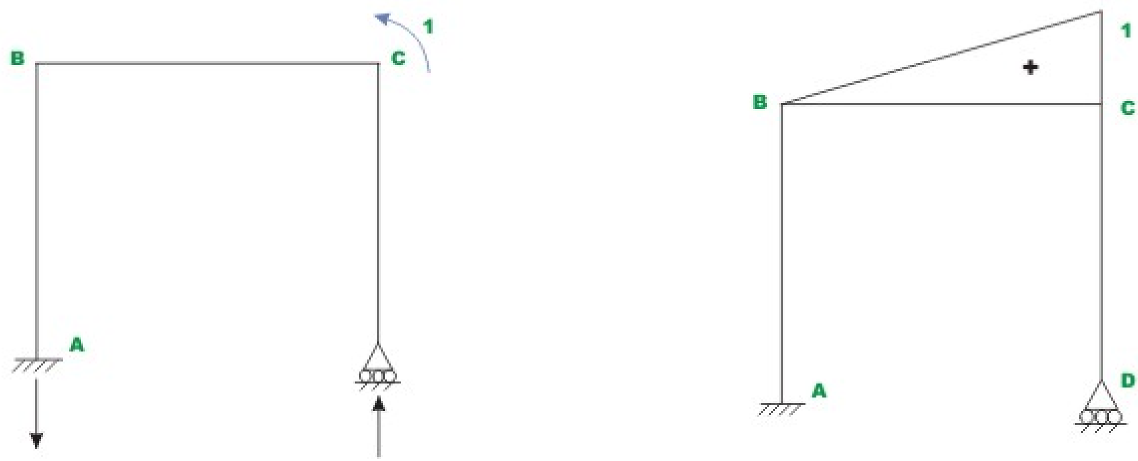


Fig. 5.5b. Reaction and B. M. diagram for the unit moment applied at C.

Rotation at B

Apply unit virtual moment at B as shown in Fig 5.5a. The resulting bending moment diagram is also shown in the same diagram. For the unit load method, the relevant equation is,

$$(1) \times \theta_B = \int_A^D \frac{\delta M_v(x) M(x) dx}{EI} \tag{1}$$

wherein, θ_B is the actual rotation at B, $\delta M_v(x)$ is the virtual stress resultant in the frame due to the virtual load and $\int_A^D \frac{M(x)}{EI} dx$ is the actual deformation of the frame due to real forces.

Now, $M(x)=10(2.5-x)$ and $\delta M_v(x)=0.4(2.5-x)$

Substituting the values of $M(x)$ and $\delta M_v(x)$ in the equation (1),

$$\theta_B = \frac{4}{EI}\int_0^{2.5}(2.5-x)^2\,dx$$

$$= \frac{4}{EI}\left[6.25x - \frac{5x^2}{2} + \frac{x^3}{3}\right]_0^{2.5} = \frac{62.5}{3EI} \tag{2}$$

Rotation at C

For evaluating rotation at C by unit load method, apply unit virtual moment at C as shown in Fig 5.5b. Hence,

$$(1)\times\theta_C = \int_A^D \frac{\delta M_v(x)M(x)dx}{EI} \tag{3}$$

$$\theta_C = \int_0^{2.5}\frac{10(2.5-x)(0.4x)}{EI}dx$$

$$= \frac{4}{EI}\left[\frac{2.5x^2}{2} - \frac{x^3}{3}\right]_0^{2.5} = \frac{31.25}{3EI} \tag{4}$$

5.6 Unit Displacement Method

Consider a cantilever beam, which is in equilibrium under the action of a system of forces $F_1, F_2, ..., F_n$. Let $u_1, u_2, ..., u_n$ be the corresponding displacements and P, M and V be the stress resultants at section of the beam. Consider a second system of forces (virtual) $\delta F_1, \delta F_2, ..., \delta F_n$ causing virtual displacements $\delta u_1, \delta u_2, ..., \delta u_n$. Let $\delta P_v, \delta M_v$ and δV_v be the virtual axial force, bending moment and shear force respectively at any section of the beam.

Apply the first system of forces $F_1, F_2, ..., F_n$ on the beam, which has been previously bent by virtual forces $\delta F_1, \delta F_2, ..., \delta F_n$. From the principle of virtual displacements we have,

$$\sum_{j=1}^{n} F_j \delta u_j = \int \frac{M(x)\delta M_v(x)ds}{EI}$$

$$= \int_V \sigma^T \delta\varepsilon\, \delta v \tag{5.11}$$

The left hand side of equation (5.11) refers to the external virtual work done by the system of true/real forces moving through the corresponding virtual displacements of the system. The right hand side of equation (5.8) refers to internal virtual work done. The principle of virtual displacement states that the external virtual work of the real forces multiplied by virtual displacement is equal to the real stresses multiplied by virtual strains integrated over volume. If the value of a particular force element is required then choose corresponding virtual displacement as unity. Let us say, it is required to evaluate F_1, then choose $\delta u_1 = 1$ and $\delta u_i = 0 \quad i = 2,3,.....,n$. From equation (5.11), one could write,

$$(1)\ F_1 = \int \frac{M\,(\delta M_v)_1\,ds}{EI} \tag{5.12}$$

where, $(\delta M_v)_1$ is the internal virtual stress resultant for $\delta u_1 = 1$. Transposing the above equation, we get

$$F_1 = \int \frac{(\delta M_v)_1 M ds}{EI} \tag{5.13}$$

The above equation is the statement of unit displacement method. The above equation is more commonly used in the evaluation of stiffness co-efficient k_{ij} . Apply real displacements $u_1,.....,u_n$ in the structure. In that set $u_2 = 1$ and the other all displacements $u_i = 0 \quad (i = 1,3,......,n)$. For such a case the quantity F_j in equation (5.11) becomes k_{ij} *i.e.* force at 1 due to displacement at 2. Apply virtual displacement $\delta u_1 = 1$. Now according to unit displacement method,

$$(1)\ k_{12} = \int \frac{(\delta M_v)_1 M_2 ds}{EI} \tag{5.14}$$

Summary

In this chapter the concept of virtual work is introduced and the principle of virtual work is discussed. The terms internal virtual work and external virtual work has been explained and relevant expressions are also derived. Principle of virtual forces has been stated. It has been shown how the principle of virtual load leads to unit load method. An expression for calculating deflections at any point of a structure (both statically determinate and indeterminate structure) is derived. Few problems have been solved to show the application of unit load method for calculating deflections in a structure.

Module 1

Energy Methods in Structural Analysis

Lesson 6

Engesser's Theorem and Truss Deflections by Virtual Work Principles

Instructional Objectives

After reading this lesson, the reader will be able to:

1. State and prove Crotti-Engesser theorem.
2. Derive simple expressions for calculating deflections in trusses subjected to mechanical loading using unit-load method.
3. Derive equations for calculating deflections in trusses subjected to temperature loads.
4. Compute deflections in trusses using unit-load method due to fabrication errors.

6.1 Introduction

In the previous lesson, we discussed the principle of virtual work and principle of virtual displacement. Also, we derived unit – load method from the principle of virtual work and unit displacement method from the principle of virtual displacement. In this lesson, the unit load method is employed to calculate displacements of trusses due to external loading. Initially the Engesser's theorem, which is more general than the Castigliano's theorem, is discussed. In the end, few examples are solved to demonstrate the power of virtual work.

6.2 Crotti-Engesser Theorem

The Crotti-Engesser theorem states that the first partial derivative of the complementary strain energy (U^*) expressed in terms of applied forces F_j is equal to the corresponding displacement.

$$\frac{\partial U^*}{\partial F_j} = \sum_{k=1}^{n} a_{jk} F_k = u_j \tag{6.1}$$

For the case of indeterminate structures this may be stated as,

$$\frac{\partial U^*}{\partial F_j} = 0 \tag{6.2}$$

Note that Engesser's theorem is valid for both linear and non-linear structures. When the complementary strain energy is equal to the strain energy (i.e. in case of linear structures) the equation (6.1) is nothing but the statement of Castigliano's first theorem in terms of complementary strain energy.

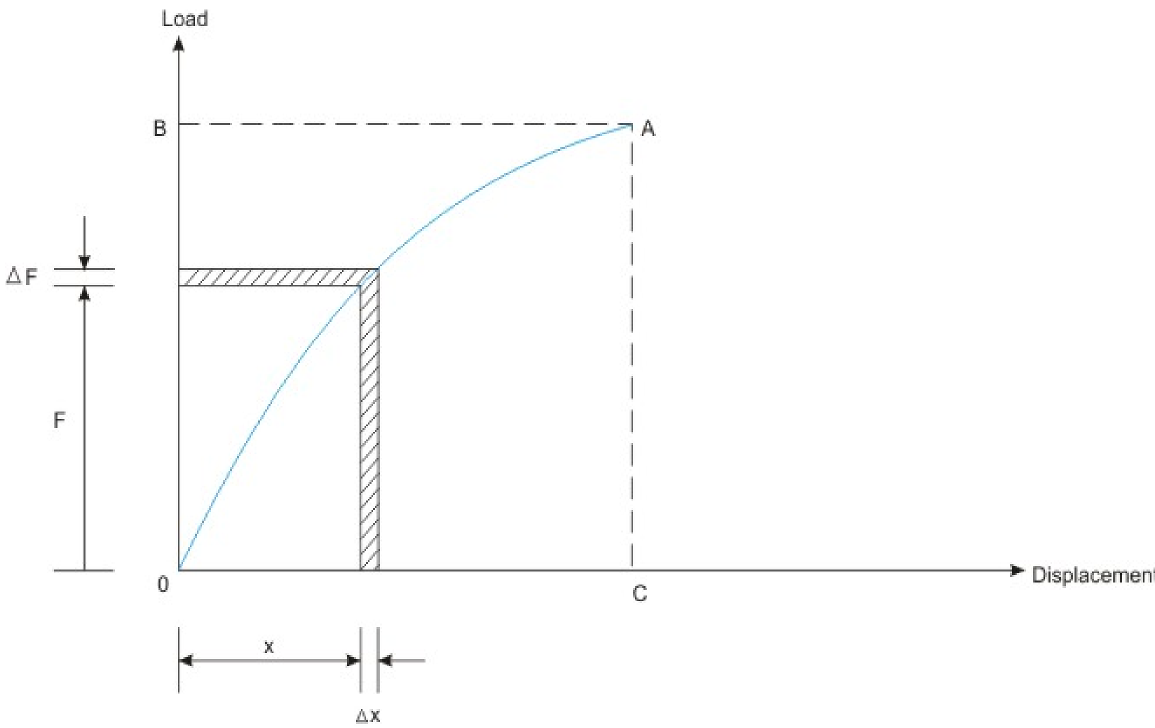


Fig. 6.1 Non-linear Load-displacement curve.

In the above figure the strain energy (area OACO) is not equal to complementary strain energy (area OABO)

$$Area\ \ OACO\ = U = \int_0^u F\,du \tag{6.3}$$

Differentiating strain energy with respect to displacement,

$$\frac{dU}{du} = F \tag{6.4}$$

This is the statement of Castigliano's second theorem. Now the complementary energy is equal to the area enclosed by OABO.

$$U^* = \int_0^F u\ dF \tag{6.5}$$

Differentiating complementary strain energy with respect to force F,

$$\frac{dU^*}{dF} = u \tag{6.6}$$

This gives deflection in the direction of load. When the load displacement relationship is linear, the above equation coincides with the Castigliano's first theorem given in equation (3.8).

6.3 Unit Load Method as applied to Trusses

6.3.1 External Loading

In case of a plane or a space truss, the only internal forces present are axial as the external loads are applied at joints. Hence, equation (5.7) may be written as,

$$\sum_{j=1}^{n} \delta F_j u_j = \int_0^L \frac{\delta P_v P ds}{EA} \tag{6.7}$$

wherein, δF_j is the external virtual load, u_j are the actual deflections of the truss, δP_v is the virtual stress resultant in the frame due to the virtual load and $\int_0^L \frac{P}{EA} ds$ is the actual internal deformation of the frame due to real forces. In the above equation L, E, A respectively represent length of the member, cross-sectional area of a member and modulus of elasticity of a member. In the unit load method, $\delta F_j = 1$ and all other components of virtual forces $\delta F_i \ (i = 1,2,..., j-1, j+1,...,n)$ are zero. Also, if the cross sectional area A of truss remains constant throughout, then integration may be replaced by summation and hence equation (6.7) may be written as,

$$u_j = \sum_{i=1}^{m} \frac{(\delta P_v)_{ij} P_i L_i}{E_i A_i} \tag{6.8}$$

where m is the number of members, $(\delta P_v)_{ij}$ is the internal virtual axial force in member i due to unit virtual load at j and $(\frac{P_i}{E_i A_i}) L_i$ is the total deformation of member i due to real loads. If we represent total deformation by Δ_i, then

$$u_j = \sum_{i=1}^{m} (\delta P_v)_{ij} \Delta_i \tag{6.9}$$

where, Δ_i is the true change in length of member i due to real loads.

6.3.2 Temperature Loading

Due to change in the environmental temperature, the truss members either expand or shrink. This in turn produces joint deflections in the truss. This may be calculated by equation (6.9). In this case, the change in length of member Δ_i is calculated from the relation,

$$\Delta_i = \alpha T L_i \tag{6.10}$$

where α is the co-efficient of thermal expansion member, L_i is the length of member and T is the temperature change.

6.3.3 Fabrication Errors and Camber

Sometimes, there will be errors in fabricating truss members. In some cases, the truss members are fabricated slightly longer or shorter in order to provide camber to the truss. Usually camber is provided in bridge truss so that its bottom chord is curved upward by an equal to its downward deflection of the chord when subjected to dead. In such instances, also, the truss joint deflection is calculated by equation (6.9). Here,

$$\Delta_i = e_i \tag{6.11}$$

where, e_i is the fabrication error in the length of the member. e_i is taken as positive when the member lengths are fabricated slightly more than the actual length otherwise it is taken as negative.

6.4 Procedure for calculating truss deflection

1. First, calculate the real forces in the member of the truss either by method of joints or by method of sections due to the externally applied forces. From this determine the actual deformation (Δ_i) in each member from the equation $\frac{P_i L_i}{E_i A_i}$. Assume tensile forces as positive and compressive forces as negative.

2. Now, consider the virtual load system such that only a unit load is considered at the joint either in the horizontal or in the vertical direction, where the deflection is sought. Calculate virtual forces $(\delta P_v)_{ij}$ in each member due to the applied unit load at the *j*-th joint.

3. Now, using equation (6.9), evaluate the *j*-th joint deflection u_j.

4. If deflection of a joint needs to be calculated due to temperature change, then determine the actual deformation (Δ_i) in each member from the equation $\Delta_i = \alpha T L_i$.

The application of equation (6.8) is shown with the help of few problems.

Example 6.1

Find horizontal and vertical deflection of joint C of truss ABCD loaded as shown in Fig. 6.2a. Assume that, all members have the same axial rigidity.

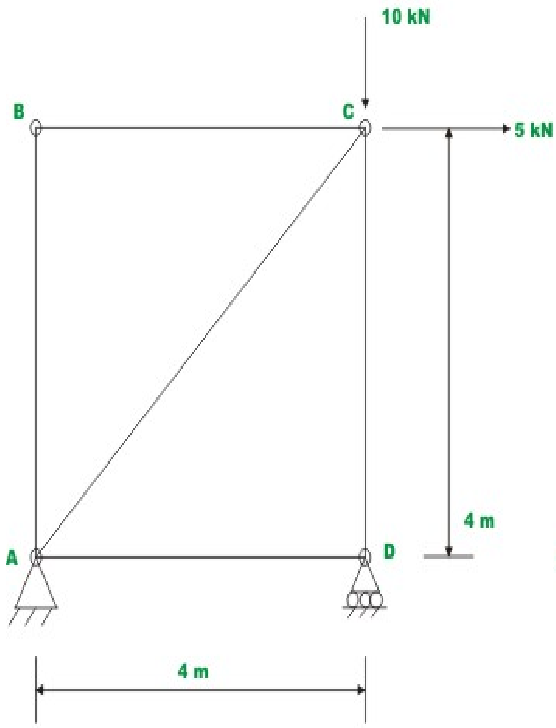


Fig. 6.2a Example 6.1

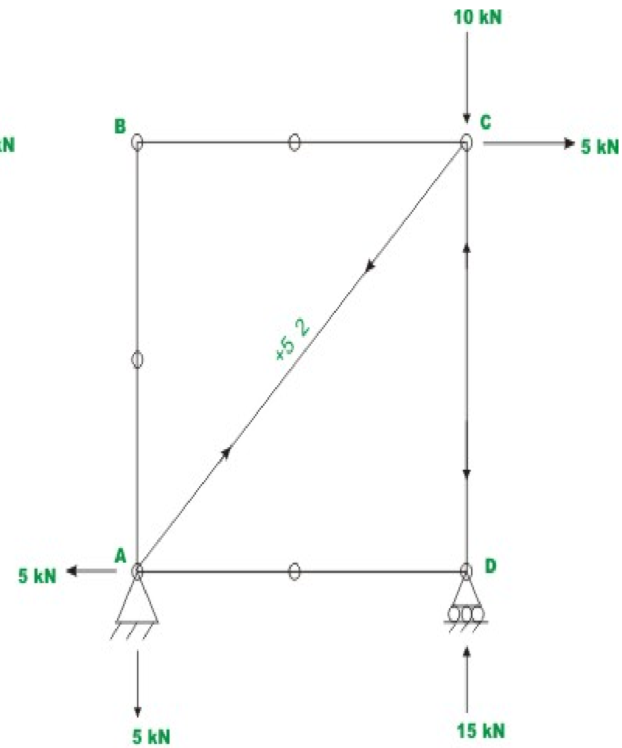
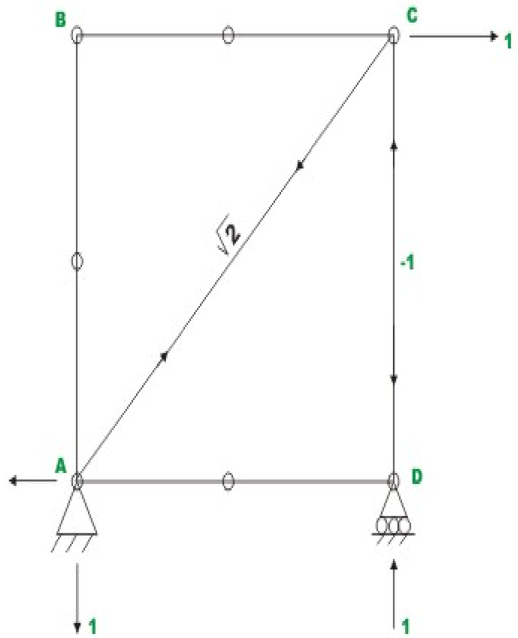


Fig. 6.2b Reaction and forces in members.

The given truss is statically determinate one. The reactions are as shown in Fig 6.2b along with member forces which are determined by equations of static equilibrium. To evaluate horizontal deflection at 'C', apply a unit load as shown in Fig 6.2c and evaluate the virtual forces δP_v in each member. The magnitudes of internal forces are also shown in the respective figures. The tensile forces are shown as +ve and compressive forces are shown as –ve. At each end of the bar, arrows have been drawn indicating the direction in which the force in the member acts on the joint.



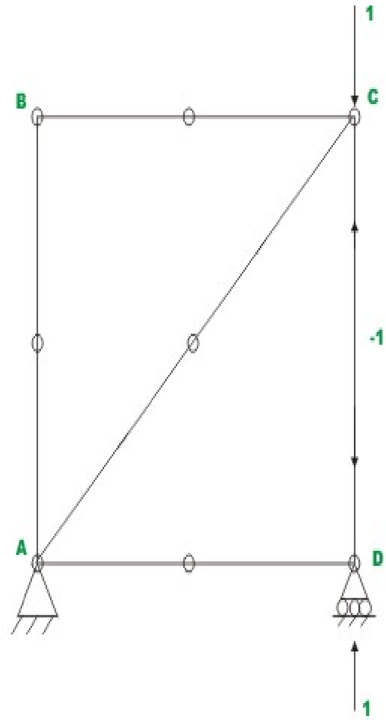


Fig. 6.2c Reaction and member forces due to vertical horizontal force at C.

Fig. 6.2d Reaction and member forces due to vertical horizontal force at C.

Horizontal deflection at joint C is calculated with the help of unit load method. This may be stated as,

$$1 \times u_c^H = \sum \frac{(\delta P_v)_{ic} P_i L_i}{E_i A_i} \tag{1}$$

For calculating horizontal deflection at C, u_c, apply a unit load at the joint C as shown in Fig.6.2c. The whole calculations are shown in table 6.1. The calculations are self explanatory.

Table 6.1 Computational details for horizontal deflection at C

Member	Length	$L_i / A_i E_i$	P_i	$(\delta P_v)_i$	$\frac{(\delta P_v)_i P_i L_i}{E_i A_i}$
units	m	m/kN	kN	kN	kN.m
AB	4	4/AE	0	0	0
BC	4	4/AE	0	0	0
CD	4	4/AE	-15	-1	60/AE
DA	4	4/AE	0	0	0
AC	$4\sqrt{2}$	$4\sqrt{2}$ /AE	$5\sqrt{2}$	$\sqrt{2}$	$40\sqrt{2}$ /AE
				$\sum$	$\frac{60+40\sqrt{2}}{AE}$

$$(1)(u_C^H) \rightarrow = \frac{60+40\sqrt{2}}{AE} = \frac{116.569}{AE} \qquad \text{(Towards right)} \qquad (2)$$

Vertical deflection at joint C

$$1 \times u_c^v = \sum \frac{(\delta P_v^v)_{ic} P_i L_i}{E_i A_i} \qquad (3)$$

In this case, a unit vertical load is applied at joint C of the truss as shown in Fig. 6.2d.

Table 6.2 Computational details for vertical deflection at C

Member	Length	$L_i / A_i E_i$	P_i	$(\delta P_v^v)_i$	$\frac{(\delta P_v)_i P_i L_i}{E_i A_i}$
units	m	m/kN	kN	kN	kN.m
AB	4	4/AE	0	0	0
BC	4	4/AE	0	0	0
CD	4	4/AE	-15	-1	60/AE
DA	4	4/AE	0	0	0
AC	$4\sqrt{2}$	$4\sqrt{2}$ /AE	$5\sqrt{2}$	0	0
				$\sum$	$\frac{60}{AE}$

$$(1)(u_C^v) \downarrow = \frac{60}{AE} = \frac{60}{AE} \qquad \text{(Downwards)} \qquad (4)$$

Example 6.2

Compute the vertical deflection of joint b and horizontal displacement of joint D of the truss shown in Fig. 6.3a due to

a) Applied loading as shown in figure.

b) Increase in temperature of 25^0 C in the top chord BD. Assume $\alpha = \frac{1}{75000}$ $per°\text{C}$, $E = 2.00 \times 10^5\ N/mm^2$. The cross sectional areas of the members in square centimeters are shown in parentheses.

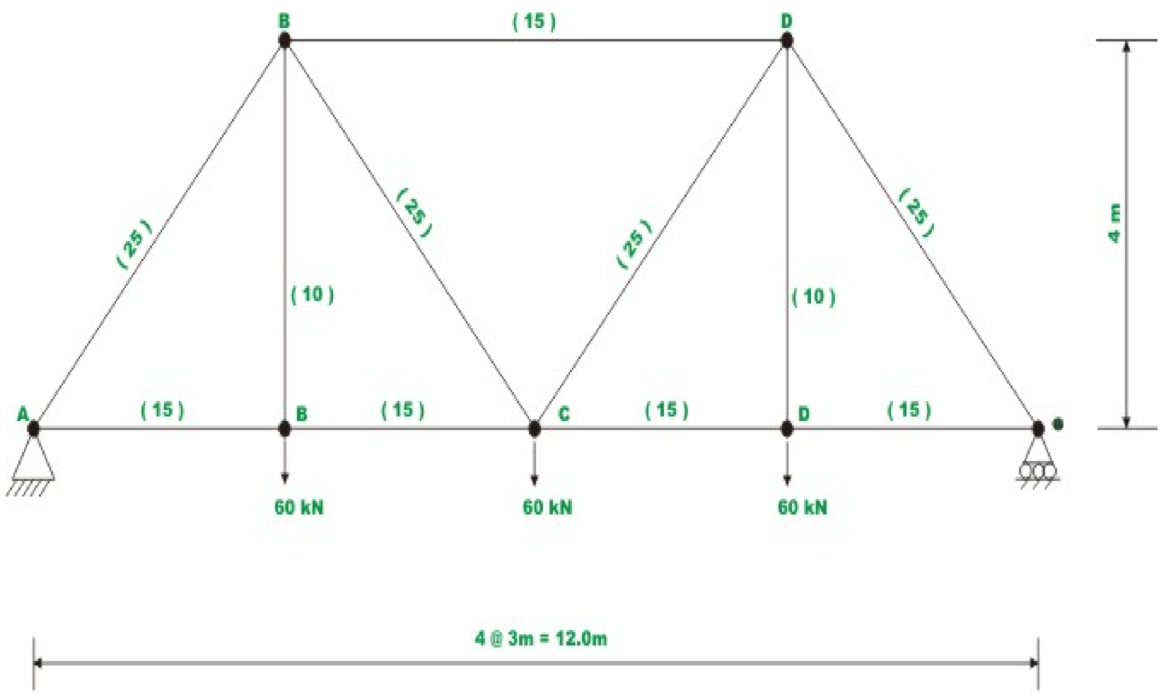


Fig. 6.3a. Example 6.2

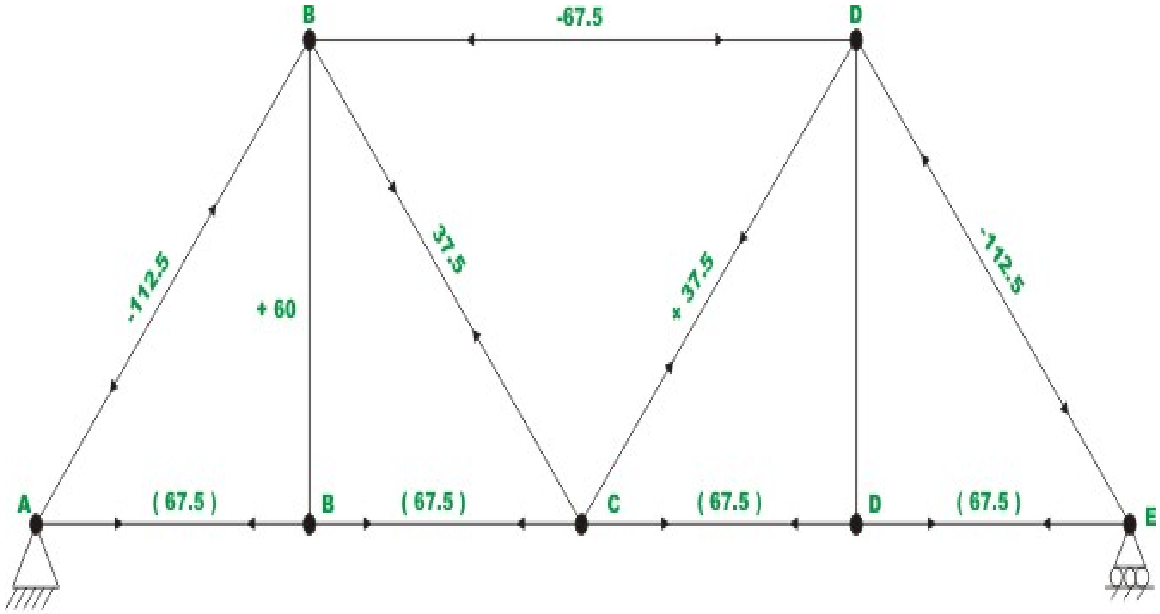


Fig. 6.3b Reaction and member forces due to applied load

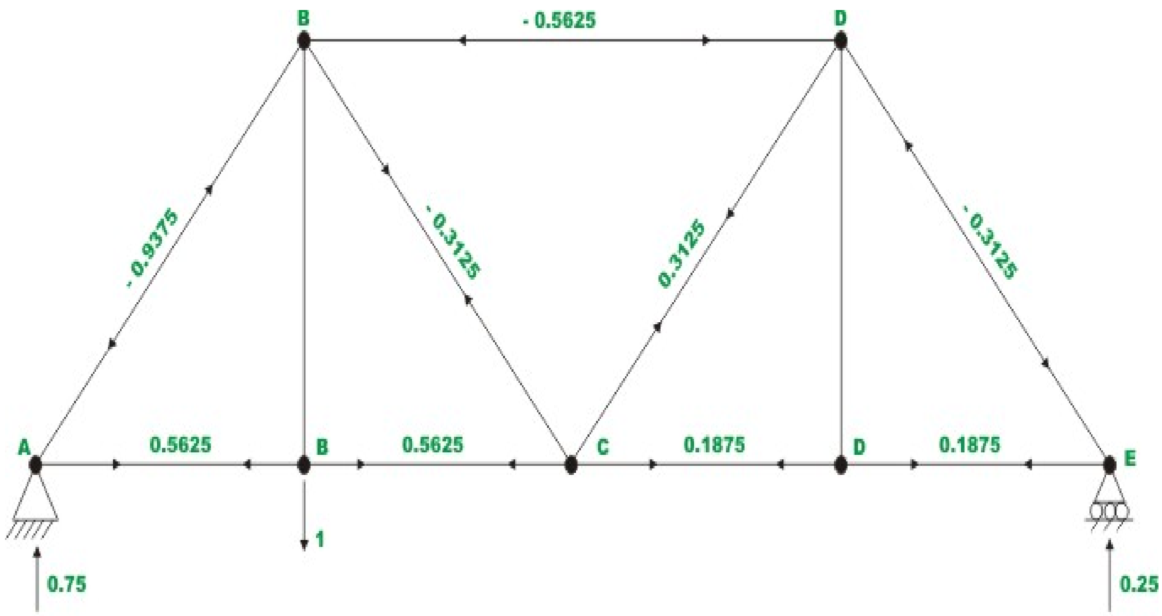


Fig. 6.3c Forces in members due to unit virtual vertical force at b.

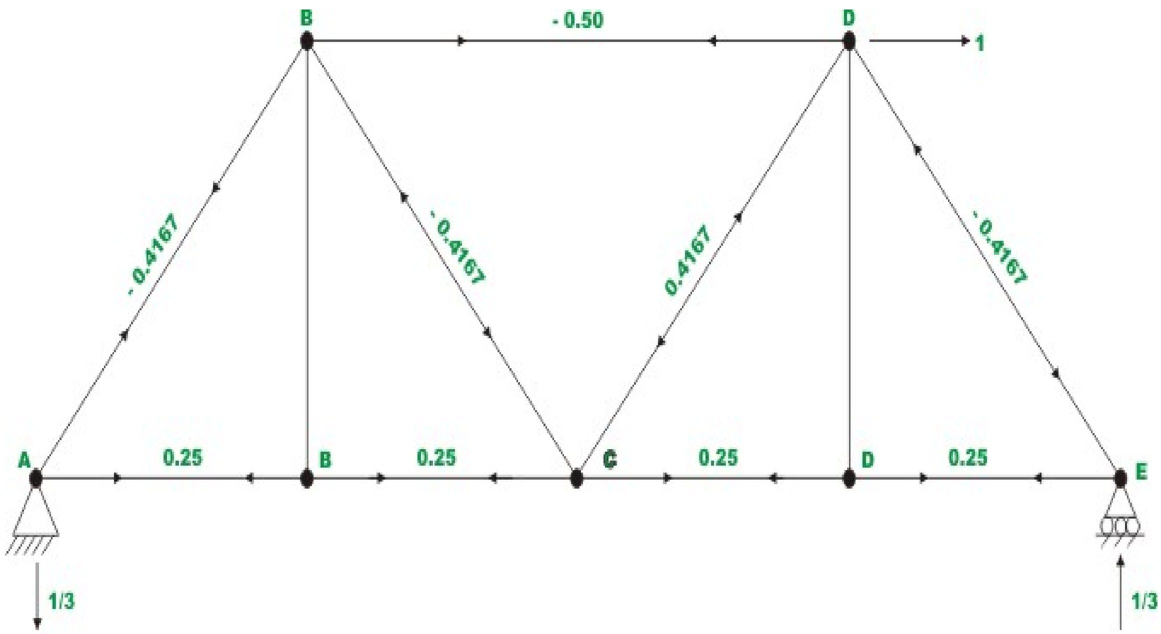


Fig. 6.3d Forces in members due to unit horizontal force at D.

The complete calculations are shown in the following table.

Table 6.3 Computational details for example 6.2

Mem	L_i	$L_i / A_i E_i$	P_i	$(\delta P_v^v)_i$	$(\delta P_v^H)_i$	$\Delta_{ti} = \alpha t L_i$	$\frac{(\delta P_v^v)_i P_i L_i}{E_i A_i}$	$\frac{(\delta P_v^H)_i P_i L_i}{E_i A_i}$	$(\delta P_v^v)_i \Delta_{ti}$	$(\delta P_v^H)_i \Delta_{ti}$
units	m	(10^{-5}) m/kN	kN	kN	kN	m	(10^{-3}) kN.m	(10^{-3}) kN.m	(10^{-3}) kN.m	(10^{-3}) kN.m
aB	5	1.0	-112.5	-0.937	+0.416	0	1.05	-0.47	0	0
ab	3	1.0	+67.5	+0.562	+0.750	0	0.38	0.51	0	0
bc	3	1.0	+67.5	+0.562	+0.750	0	0.38	0.51	0	0
Bc	5	1.0	+37.5	-0.312	-0.416	0	-0.12	-0.16	0	0
BD	6	2.0	-67.5	-0.562	+0.500	0.002	0.76	-0.68	-1.13	1
cD	5	1.0	+37.5	+0.312	+0.416	0	0.12	0.16	0	0
cd	3	1.0	+67.5	+0.187	+0.250	0	0.13	0.17	0	0
de	3	1.0	+67.5	+0.187	+0.250	0	0.13	0.17	0	0
De	5	1.0	-112.5	-0.312	-0.416	0	0.35	0.47	0	0
Bb	4	2.0	+60.0	1	0	0	1.2	0	0	0
Dd	4	2.0	+60.0	0	0	0	0	0	0	0
						$\sum$	4.38	0.68	-1.13	1

a) Vertical deflection of joint b

Applying principle of virtual work as applied to an ideal pin jointed truss,

$$\sum_{j=1}^{1} \delta F_j u_j = \sum_{i=1}^{m} \frac{(\delta P_v)_{ij} P_i L_i}{E_i A_i} \tag{1}$$

For calculating vertical deflection at b, apply a unit virtual load $\delta F_b = 1$. Then the above equation may be written as,

$$1 \times u_b^v = \sum \frac{(\delta P_v^v)_i P_i L_i}{E_i A_i} \tag{2}$$

1) Due to external loads

$$u_b \downarrow = \frac{+0.00438 \;\; KNm}{1 \; KN} = 0.00438 \;\; m$$

$$= 4.38 \;\; mm \downarrow$$

2) Due to change in temperature

$$(1)(u_b^t \downarrow) = \sum (\delta P_v^v)_i \Delta_{ti}$$

$$u_b^t \downarrow = \frac{-0.001125 \;\; KN.m}{1 \;\; KN} = -0.00113 m$$

$$u_b^t = 1.13 \;\; mm \uparrow$$

b) Horizontal displacement of joint ‘D’

1) Due to externally applied loads

$$1 \times u_b^H = \sum \frac{(\delta P_v^H)_i P_i L_i}{E_i A_i}$$

$$u_D^H \rightarrow = \frac{+0.00068 \;\; KNm}{1 \; KN} = 0.00068 \;\; m$$

$$= 0.68 \;\; mm \rightarrow$$

2) Due to change in temperature

$$(1)(u_D^{Ht} \rightarrow) = \sum (\delta P_v^H)_i \Delta_{ti}$$

$$u_D^{Ht} \rightarrow = \frac{0.001 \;\; KN.m}{1 \;\; KN} = 0.001m$$

$$u_D^{Ht} = 1.00 \;\; mm \rightarrow$$

Summary

In this chapter the Crotti-Engessor's theorem which is more general than the Castigliano's theorem has been introduced. The unit load method is applied statically determinate structure for calculating deflections when the truss is subjected to various types of loadings such as: mechanical loading, temperature loading and fabrication errors.

Module 2

Analysis of Statically Indeterminate Structures by the Matrix Force Method

Lesson
7
The Force Method of Analysis: An Introduction

Since twentieth century, indeterminate structures are being widely used for its obvious merits. It may be recalled that, in the case of indeterminate structures either the reactions or the internal forces cannot be determined from equations of statics alone. In such structures, the number of reactions or the number of internal forces exceeds the number of static equilibrium equations. In addition to equilibrium equations, compatibility equations are used to evaluate the unknown reactions and internal forces in statically indeterminate structure. In the analysis of indeterminate structure it is necessary to satisfy the equilibrium equations (implying that the structure is in equilibrium) compatibility equations (requirement if for assuring the continuity of the structure without any breaks) and force displacement equations (the way in which displacement are related to forces). We have two distinct method of analysis for statically indeterminate structure depending upon how the above equations are satisfied:

1. Force method of analysis (also known as flexibility method of analysis, method of consistent deformation, flexibility matrix method)

2. Displacement method of analysis (also known as stiffness matrix method).

In the force method of analysis, primary unknown are forces. In this method compatibility equations are written for displacement and rotations (which are calculated by force displacement equations). Solving these equations, redundant forces are calculated. Once the redundant forces are calculated, the remaining reactions are evaluated by equations of equilibrium.

In the displacement method of analysis, the primary unknowns are the displacements. In this method, first force -displacement relations are computed and subsequently equations are written satisfying the equilibrium conditions of the structure. After determining the unknown displacements, the other forces are calculated satisfying the compatibility conditions and force displacement relations. The displacement-based method is amenable to computer programming and hence the method is being widely used in the modern day structural analysis.

In general, the maximum deflection and the maximum stresses are small as compared to statically determinate structure. For example, consider two beams of identical cross section and span carrying uniformly distributed load as shown in Fig. 7.1a and Fig. 7.1b.

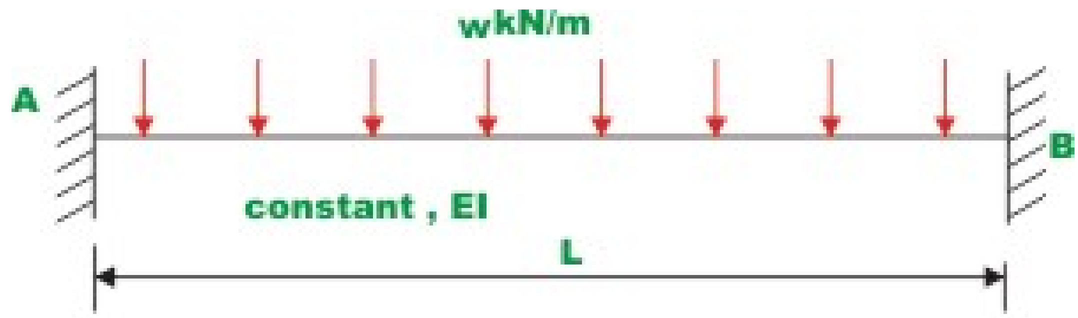


Fig. 7.1a Fixed - Fixed beam

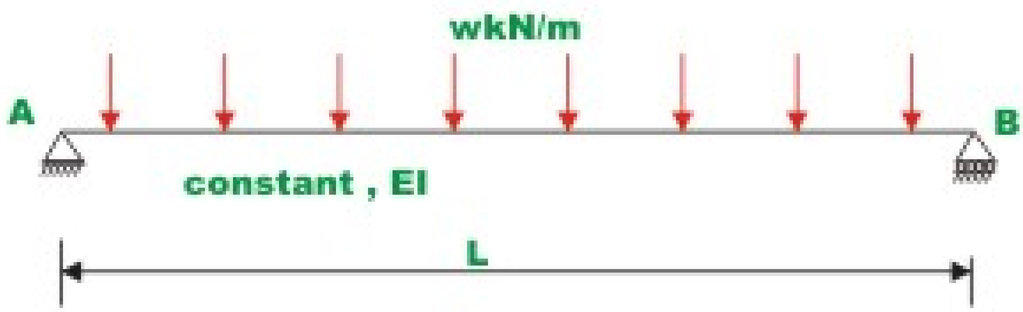


Fig. 7.1b Simply supported beam

The loads are also the same in both cases. In the first case, the beam is fixed at both ends and thus is statically indeterminate. The simply supported beam in Fig. 7.1b is a statically determinate structure. The maximum bending moment in case of fixed- fixed beam is $\frac{wL^2}{12}$ (which occurs at the supports) as compared to $\frac{wL^2}{8}$ (at the centre) in case of simply supported beam. Also in the present case, the deflection in the case of fixed- fixed beam $\left(\frac{wL^4}{384EI}\right)$ is five times smaller than that of simply supported beam $\left(\frac{5wL^4}{384EI}\right)$. Also, there is redistribution of stresses in the case of redundant structure. Hence if one member fails, structure does not collapse suddenly. The remaining members carry the load. The determinate structural system collapses if one member fails. However, there are disadvantages in using indeterminate structures. Due to support settlement,

there will be additional stresses in the case of redundant structures where as determinate structures are not affected by support settlement.
The analysis of indeterminate structure differs mainly in two aspects as compared to determinate structure.

a) To evaluate stresses in indeterminate structures, apart from sectional properties (area of cross section and moment of inertia), elastic properties are also required.
b) Stresses are developed in indeterminate structure due to support settlements, temperature change and fabrication errors etc.

Instructional Objectives

After reading this chapter the student will be
1. Able to analyse statically indeterminate structure of degree one.
2. Able to solve the problem by either treating reaction or moment as redundant.
3. Able to draw shear force and bending moment diagram for statically indeterminate beams.
4. Able to state advantages and limitations of force method of analysis.

7.1 Introduction.

In this lesson, a general introduction is given to the force method of analysis of indeterminate structure is given. In the next lesson, this method would be applied to statically indeterminate beams. Initially the method is introduced with the help of a simple problem and subsequently it is discussed in detail. The flexibility method of analysis or force method of analysis (or method of consistent deformation) was originally developed by J. Maxwell in 1864 and O. C. Mohr in 1874. Since flexibility method requires deflection of statically determinate structure, a table of formulas for deflections for various load cases and boundary conditions is also given in this lesson for ready use. The force method of analysis is not convenient for computer programming as the choice of redundant is not unique. Further, the bandwidth of the flexibility matrix in the force method is much larger than the stiffness method. However it is very useful for hand computation.

7.2 Simple Example

Consider a propped cantilever beam (of constant flexural rigidity EI, and span L), which is carrying uniformly distributed load of w kN/m., as shown in Fig. 7.2a. The beam is statically indeterminate i.e. its reaction cannot be evaluated from equations of statics alone. To solve the above problem by force method proceeds as follows.

1) Determine the degree of statical indeterminacy. In the present case it is one. Identify the reaction, which can be treated as redundant in the analysis. In the present case R_B or M_A can be treated as redundant. Selecting R_B as the redundant, the procedure is illustrated. Subsequently, it will be shown how to attack the problem by treating M_A as redundant.

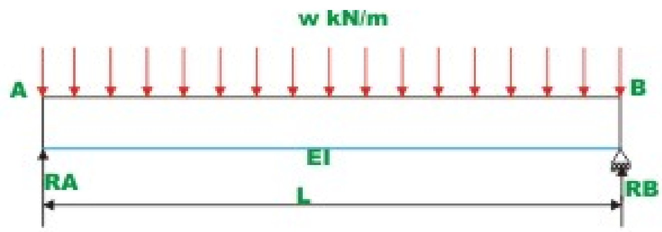


Fig. 7.2(a) Fixed - simply supported beam

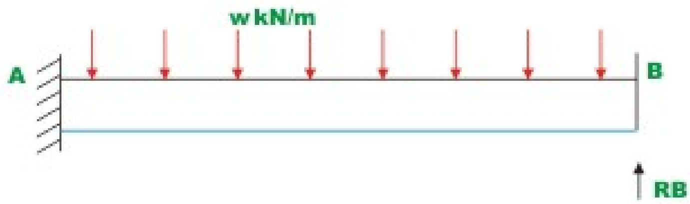


Fig. 7.2(b) Treating reaction R as redundant

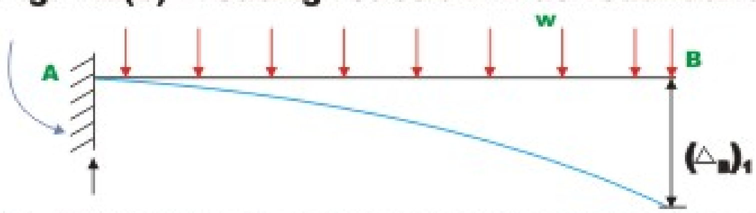


Fig. 7.2(c) Cantilever beam with external loading

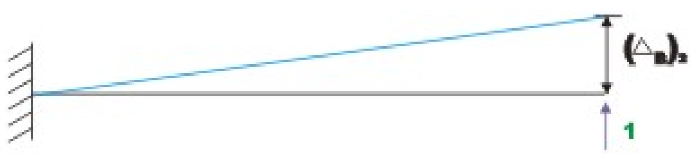
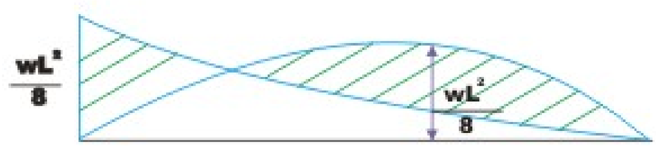


Fig. 7.2(d) Cantilever beam with a unit value of load along redundant R_B



Bending moment diagram

Shear force diagram

Fig. 7.2(e)

Solution with R_B as the redundant

2) After selecting R_B as redundant, express all other reactions in terms of the redundant R_B. This can be accomplished with the help of equilibrium equations. Thus,

$$R_A = wL - R_B \tag{7.1a}$$

and

$$M_A = \frac{wL^2}{2} - R_B L \tag{7.1b}$$

3) Now release the restraint corresponding to redundant reaction R_B. Releasing restraint in the present case amounts to removing the support at B. Now on the resulting cantilever beam (please note that the released structure is statically determinate structure), apply uniformly distributed load w and the redundant reaction R_B as shown in Fig. 7.2b. The released structure with the external loads is also sometimes referred as the primary structure.

4) The deflection at B of the released structure (cantilever beam, in the present case) due to uniformly distributed load and due to redundant reaction R_B could be easily computed from any one of the known methods (moment area method or unit load method). However it is easier to compute deflection at B due to uniformly distributed load and R_B in two steps. First, consider only uniformly distributed load and evaluate deflection at B, which is denoted by $(\Delta_B)_1$ as shown in Fig. 7.2c. Since R_B is redundant, calculate the deflection at B due to unit load at B acting in the direction of R_B and is denoted by $(\Delta_B)_2$ as shown in

In the present case the positive direction of redundant and deflections are assumed to act upwards. For the present case, $(\Delta_B)_1$ and $(\Delta_B)_2$ are given by,

$$(\Delta_B)_1 = -\frac{wL^4}{8EI} \tag{7.2a}$$

and

$$(\Delta_B)_2 = -\frac{L^3}{3EI} \tag{7.2b}$$

From the principle of superposition, the deflection at B, (Δ_B), is the sum of deflection due to uniformly distributed load $(\Delta_B)_1$ and deflection $R_B(\Delta_B)_2$ due to redundant R_B. Hence,

$$\Delta_B = (\Delta_B)_1 + R_B(\Delta_B)_2 \tag{7.2c}$$

5) It is observed that, in the original structure, the deflection at B is zero. Hence the compatibility equation can be written as,

$$\Delta_B = (\Delta_B)_1 + R_B(\Delta_B)_2 = 0 \tag{7.3a}$$

Solving the above equation, the redundant R_B can be evaluated as,

$$R_B = -\frac{(\Delta_B)_1}{(\Delta_B)_2} \tag{7.3b}$$

Substituting values of $(\Delta_B)_1$ and $(\Delta_B)_2$, the value of R_B is obtained as,

$$R_B = \frac{3wL}{8} \tag{7.3d}$$

The displacement at B due to unit load acting at B in the direction of R_B is known as the flexibility coefficient and is denoted in this course by a_{BB}.

6) Once R_B is evaluated, other reaction components can be easily determined from equations of statics. Thus,

$$M_A = \frac{wL^2}{8} \tag{7.4a}$$

$$R_A = wL - \frac{3wL}{8} = \frac{5wL}{8} \tag{7.4b}$$

7) Once the reaction components are determined, the bending moment and shear force at any cross section of the beam can be easily evaluated from equations of static equilibrium. For the present case, the bending moment and shear force diagram are shown in Fig. 7.2e.

Solution with M_A as redundant

1) As stated earlier, in the force method the choice of redundant is arbitrary. Hence, in the above problem instead of R_B one could choose M_A as the redundant reaction. In this section the above problem is solved by taking M_A as redundant reaction.

2) Now release (remove) the restraint corresponding to redundant reaction M_A. This can be done by replacing the fixed support at A by a pin. While releasing the structure, care must be taken to see that the released structure is stable and statically determinate.

Fig.7.3(a) Actual structure

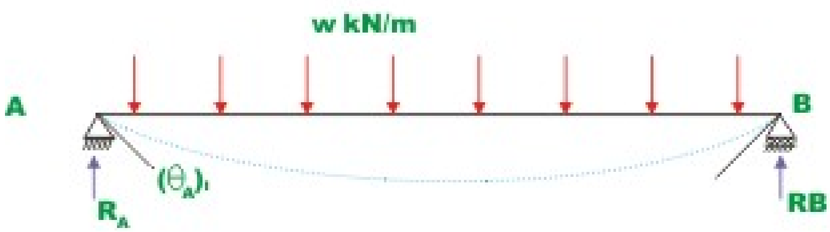


Fig. 7.3(b) Primary structure with external load load applied

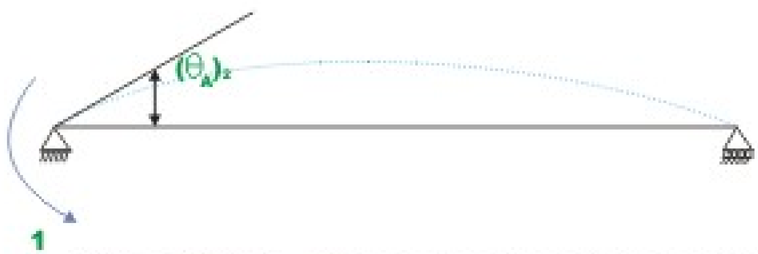


Fig. 7.3(c) Primary structure with unit moment applied in the direction of M_A

3)

Calculate the slope at A due to external loading and redundant moment M_A. This is done in two steps as shown in Fig. 7.3b and Fig.7.3c. First consider only uniformly distributed load (see Fig. 7.3b) and compute slope at A, i.e. $(\theta_A)_1$ from force displacement relations. Since M_A is redundant, calculate the slope at A due to unit moment acting at A in the direction of M_A which is denoted by $(\theta_A)_2$ as in Fig. 7.3c. Taking anticlockwise moment and anticlockwise rotations as positive, the slope at A, due to two different cases may be written as,

$$(\theta_A)_1 = -\frac{wL^3}{24EI} \tag{7.5a}$$

$$(\theta_A)_2 = \frac{L}{3EI} \tag{7.5b}$$

From the principle of superposition, the slope at A, θ_A is the sum of slopes $(\theta_A)_1$ due to external load and $M_A(\theta_A)_2$ due to redundant moment M_A. Hence

$$M_A = (\theta_A)_1 + M_A(\theta_A)_2 \tag{7.5c}$$

4) From the geometry of the original structure, it is seen that the slope at A is zero. Hence the required compatibility equation or geometric condition may be written as,

$$(\theta_A) = (\theta_A)_1 + M_A(\theta_A)_2 = 0 \tag{7.5d}$$

Solving for M_A,

$$M_A = -\frac{(\theta_A)_1}{(\theta_A)_2} \tag{7.5e}$$

Substituting the values of $(\theta_A)_1$, and $(\theta_A)_2$ in equation (7.5e), the value of M_A is calculated as

$$M_A = -\frac{-wL^3/24EI}{L/3EI}$$

$$M_A = \frac{wL^2}{8} \tag{7.5f}$$

5) Now other reaction components can be evaluated using equilibrium equations. Thus,

$$R_A = \frac{5wL}{8} \tag{7.6a}$$

$$R_B = \frac{3wL}{8} \tag{7.6b}$$

7.3 Summary

The force method of analysis may be summarized as follows.

Step 1. Determine the degree of statical indeterminacy of the structure. Identify the redundants that would be treated as unknowns in the analysis. Now, release the redundants one by one so that a statically determinate structure is obtained. Releasing the redundant reactions means removing constraint corresponding to that redundant reaction. As in the above propped cantilever beam, either reactions R_B or M_A can be treated as unknown redundant. By choosing R_B as the redundant, the propped cantilever beam can be converted into a cantilever beam (statically determinate) by releasing the roller support. Similarly by choosing moment as the redundant reaction, the indeterminate structure can be released into a determinate structure (i.e. a simply supported beam) by turning the fixed support into a hinged one. If the redundant force is an internal one, then releasing the structure amounts to introducing discontinuity in the corresponding member. The compatibility conditions for the redundant internal forces are the continuity conditions. That would be discussed further in subsequent lessons.

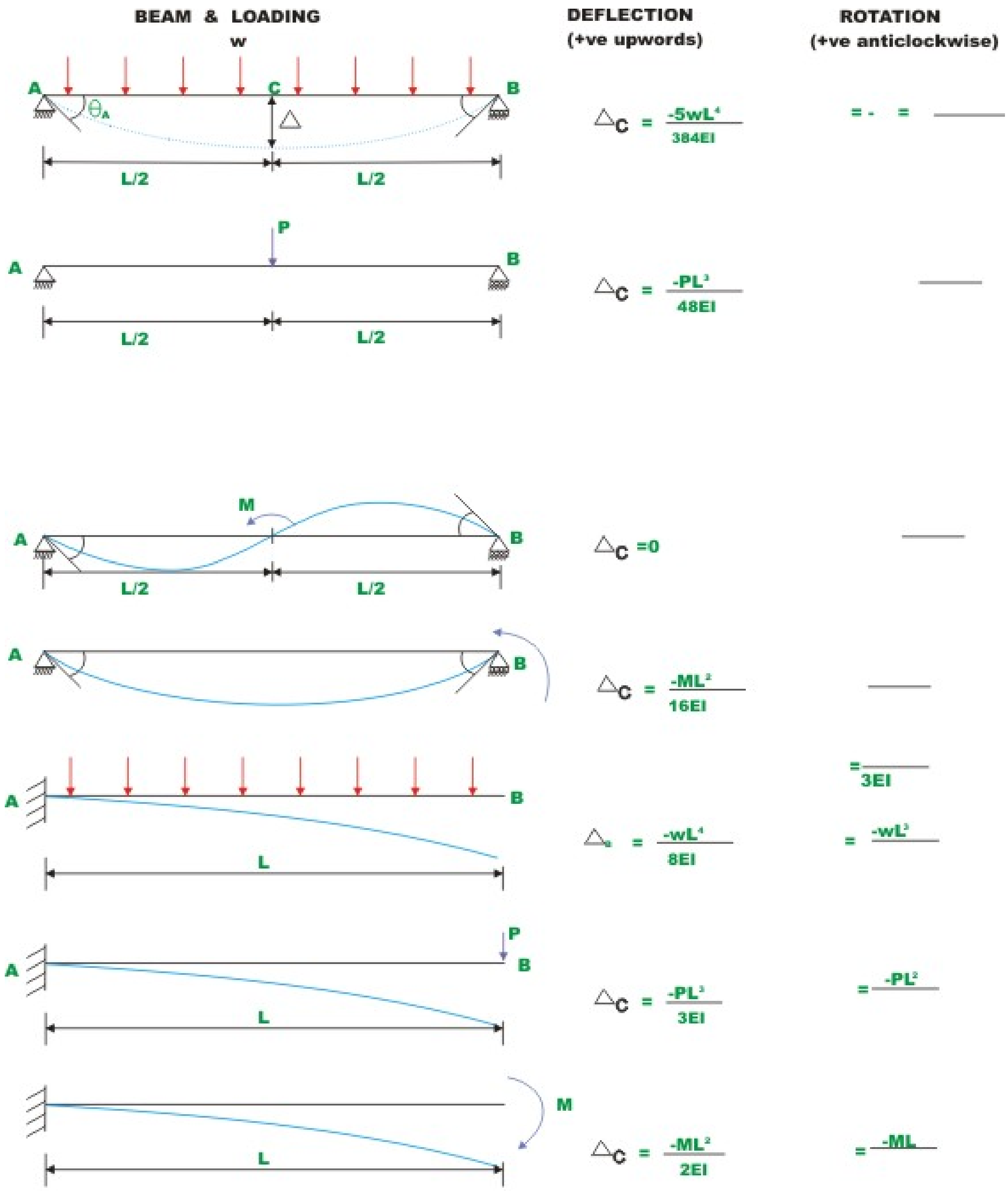
Step 2. In this step, calculate deflection corresponding to redundant action, separately due to applied loading and redundant forces from force displacement relations. Deflection due to redundant force cannot be evaluated without knowing the magnitude of the redundant force. Hence, apply a unit load in the direction of redundant force and determine the corresponding deflection. Since the method of superposition is valid, the deflections due to redundant force can be obtained by simply multiplying the unknown redundant with the deflection obtained from applying unit value of force.

Step 3. Now, calculate the total deflection due to applied loading and the redundant force by applying the principle of superposition. This computed total deflection along the redundant action must be compatible with the actual boundary conditions of the original structure. For example, if in the original structure, the deflection corresponding to the redundant reaction is zero then the total deflection must be equal to zero. If there is more than one redundant force then one could construct a set of equations with redundant forces as unknowns and flexibility coefficients as coefficients of the equations. The total number of equations equals the number of unknown redundants.

Step 4. In the last step, evaluate all other reactions and internal forces from the equilibrium equations.

The method of superposition or the force method as discussed above is applied to any type of structures, i.e. beams, truss and frames or combination of these structures. It is applicable for all general type of loadings.

The deflection of statically determinate structure can be obtained by unit-load method or by moment-area theorem or by any method known to the reader. However, the deflections of few prismatic beams with different boundary conditions and subjected to simple loadings are given in Fig. 7.4. These values will be of help in solving the problems of the present and subsequent lessons. However the students are strongly advised to practice deriving them instead of simply memorizing them.



Example 7.1

A continuous beam ABC is carrying a uniformly distributed load of 1 kN/m in addition to a concentrated load of 10 kN as shown in Fig.7.5a, Draw bending moment and shear force diagram. Assume EI to be constant for all members.

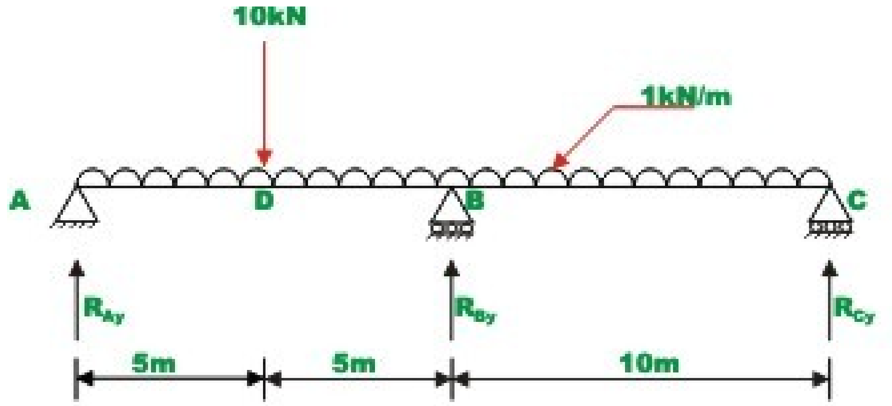


Fig. 7.5a Continuous beam

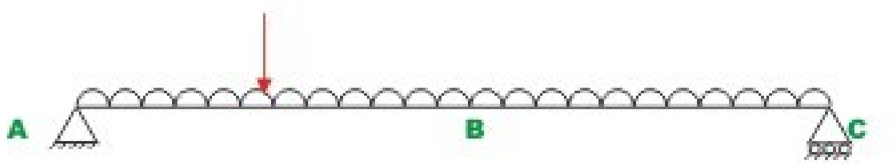


Fig. 7.5b Primary structure

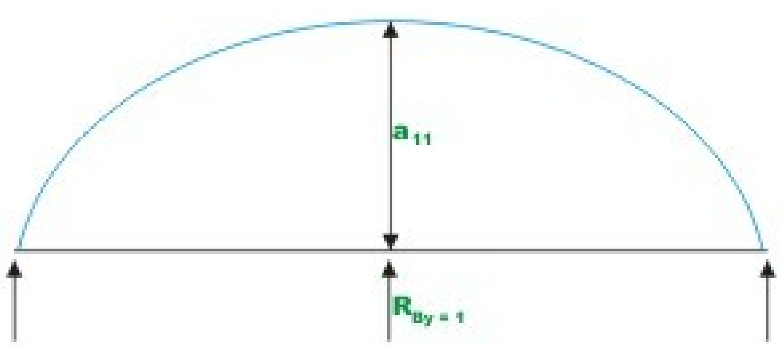


Fig. 7.5c Flexibility co-efficients

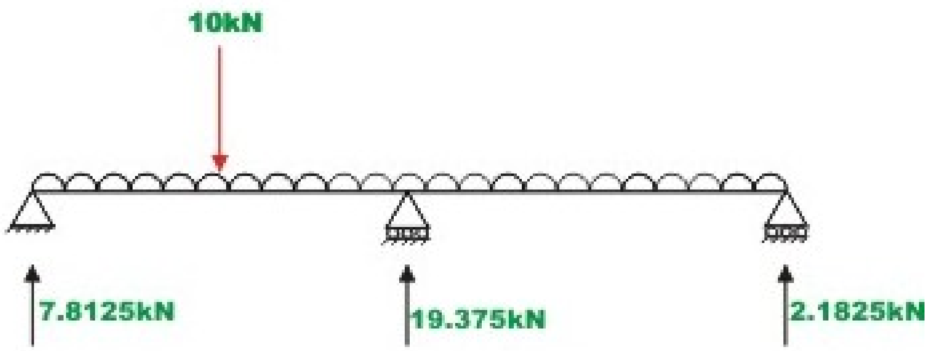


Fig. 7.5d Reactions

It is observed that the continuous beam is statically indeterminate to first degree. Choose the reaction at B, R_{By} as the redundant. The primary structure is a simply supported beam as shown in Fig.7.5b.
Now, compute the deflection at B, in the released structure due to uniformly distributed load and concentrated load. This is accomplished by unit load method. Thus,

$$\Delta_L = \frac{-2083.33}{EI} - \frac{1145.84}{EI}$$

$$\Delta_L = \frac{-3229.17}{EI} \qquad (1)$$

In the next step, apply a unit load at B in the direction of R_{By} (upwards) and calculate the deflection at B of the following structure. Thus (see Fig. 7.5c),

$$a_{11} = \frac{L^3}{48EI} = \frac{166.67}{EI} \qquad (2)$$

Now, deflection at B in the primary structure due to redundant R_B is,

$$\Delta_B = \frac{166.67}{EI} \times R_B \qquad (3)$$

In the actual structure, the deflection at B is zero. Hence, the compatibility equation may be written as

$$\Delta_L + \Delta_B = 0 \tag{4}$$

Substituting for Δ_L and Δ_B in equation (4),

$$\frac{-3229.17}{EI} + \frac{166.67}{EI} R_B = 0 \tag{5}$$

Thus,

$$R_B = 19.375 \text{ kN}$$

The other two reactions are calculated by static equilibrium equations (vide Fig. 7.5d)

$$R_A = 7.8125 \text{ kN}$$
$$R_B = 2.8125 \text{ kN}$$

The shear force and bending moment diagrams are shown in Fig. 7.5e and Fig. 7.5f respectively.

Example 7.2

A propped cantilever beam AB is subjected to a concentrated load of 60 kN at 3m from end A as shown in Fig. 7.6a. Draw the bending moment and shear force diagrams by the force method. Assume that the flexural rigidity of the beam, EI to be constant throughout.

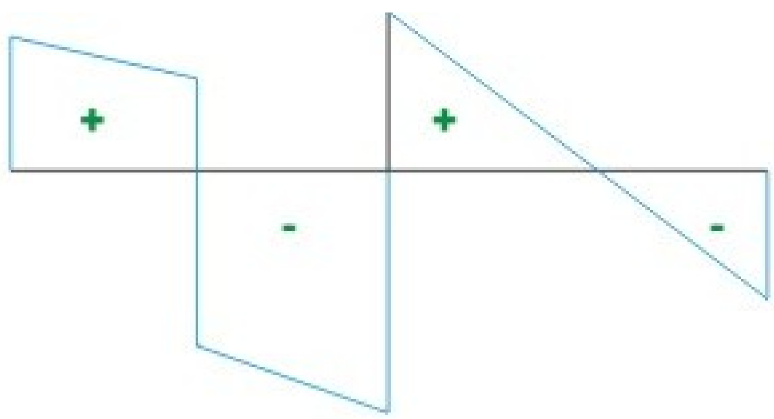


Fig. 7.5e Shear force diagram

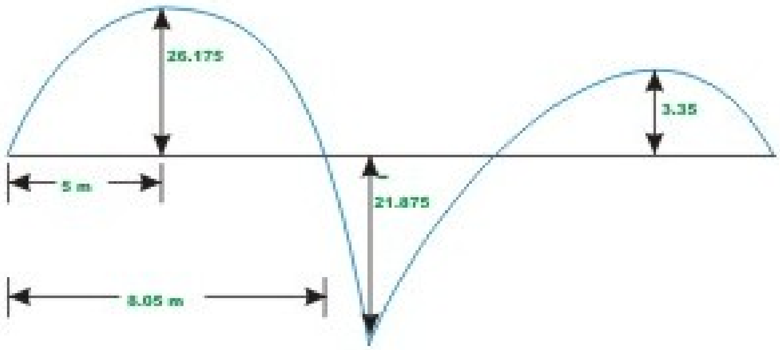


Fig. 7.5f BENDING MOMENT DIAGRAM.

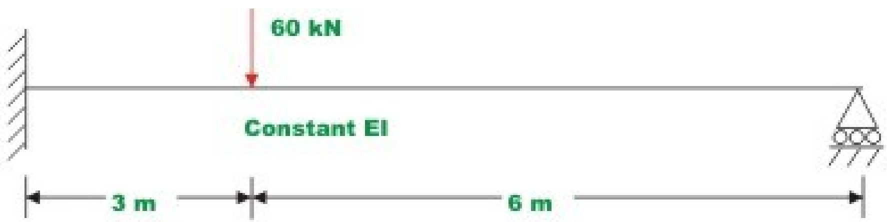


Fig. 7.6a Example 7.2

Fig. 7.6b Primary structure with external loading

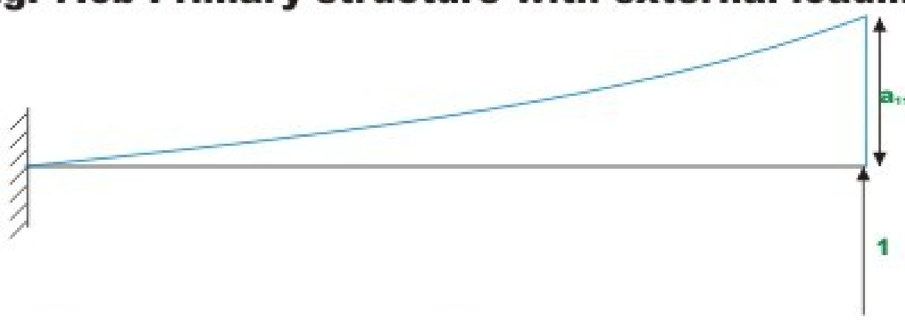


Fig. 7.6c Primary structure with unit load applied along R_B

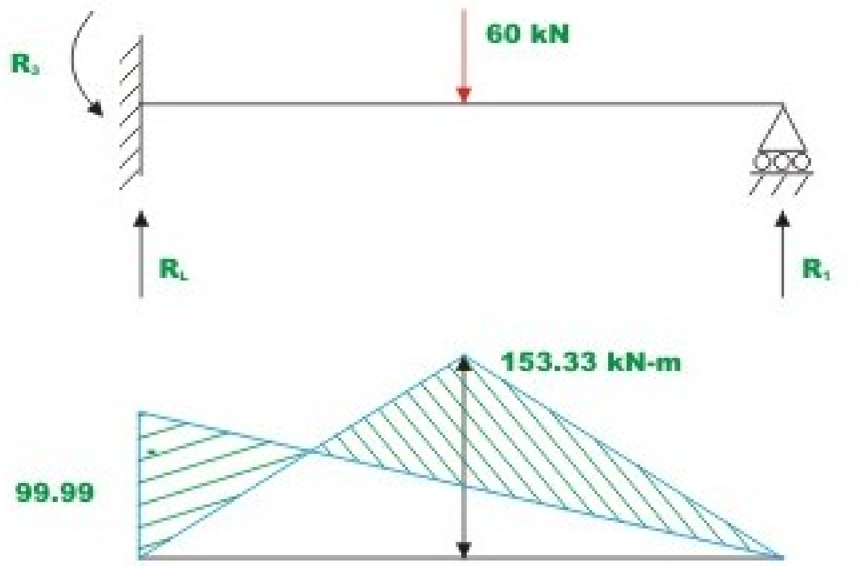


Fig.7.6d Bending moment diagram

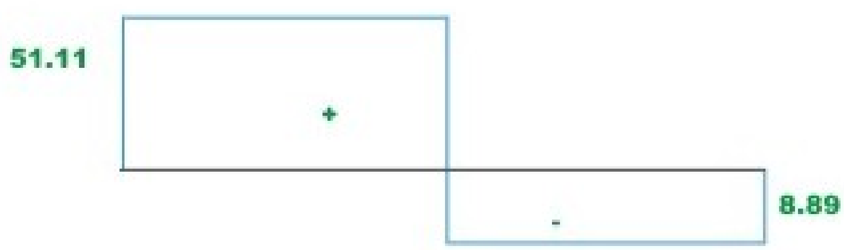


Fig.7.6e Shear force diagram

The given problem is statically indeterminate to first degree. Choose the reaction at B, R_1 as the redundant. After releasing the redundant, the determinate structure, a cantilever beam in this case is obtained. The cantilever beam with the applied loading is chosen in Fig 7.6b.

The deflection of the released structure is,

$$(\Delta_L)_1 = -\frac{60 \times 3^3}{3EI} - \frac{60 \times 3^2 \times 6}{2EI}$$

$$(\Delta_L)_1 = \frac{-2160}{EI} \qquad (1)$$

The deflection at point B due to unit load applied in the direction of redundant R_1 is (vide Fig 7.6c)

$$a_{11} = \frac{9^3}{3EI} = \frac{243}{EI} \tag{2}$$

Now the deflection at B due to redundant R_1 is

$$(\Delta)_1 = \frac{243R_1}{EI} \tag{3}$$

From the original structure it is seen that the deflection at B is zero. Hence, the compatibility condition for the problem may be written as,

$$-\frac{2160}{EI} + \frac{243R_1}{EI} = 0 \tag{4}$$

Solving equation (4), the redundant R_1 is obtained.

$$R_1 = \frac{2160}{243} \\ = 8.89 \text{ kN} \tag{5}$$

The vertical reaction and fixed end moment at A can be determined from equations of statics. Thus,

$$R_2 = 51.11 \text{ kN}$$

$$R_3 = 99.99 \text{ kN.m} \tag{6}$$

Shear force and bending moment diagrams are shown in Fig. 7.6d and Fig. 7.6e respectively.

Summary

In this lesson flexibility matrix method or the method of consistent deformation or the force method of analysing statically indeterminate structures has been introduced with the help of simple problems. The advantages and limitations of flexibility matrix method have been discussed. Only simple indeterminate beam problem has been solved to illustrate the procedure. The principle of superposition has been used to solve statically indeterminate problems.

Module 2

Analysis of Statically Indeterminate Structures by the Matrix Force Method

Lesson
8
The Force Method of Analysis: Beams

Instructional Objectives

After reading this chapter the student will be able to

1. Solve statically indeterminate beams of degree more than one.
2. To solve the problem in matrix notation.
3. To compute reactions at all the supports.
4. To compute internal resisting bending moment at any section of the continuous beam.

8.1 Introduction

In the last lesson, a general introduction to the force method of analysis is given. Only, beams, which are statically indeterminate to first degree, were considered. If the structure is statically indeterminate to a degree more than one, then the approach presented in the previous example needs to be organized properly. In the present lesson, a general procedure for analyzing statically indeterminate beams is discussed.

8.2 Formalization of Procedure

Towards this end, consider a two-span continuous beam as shown in Fig. 8.1a. The flexural rigidity of this continuous beam is assumed to be constant and is taken as EI. Since, the beam is statically indeterminate to second degree, it is required to identify two redundant reaction components, which need be released to make the beam statically determinate.

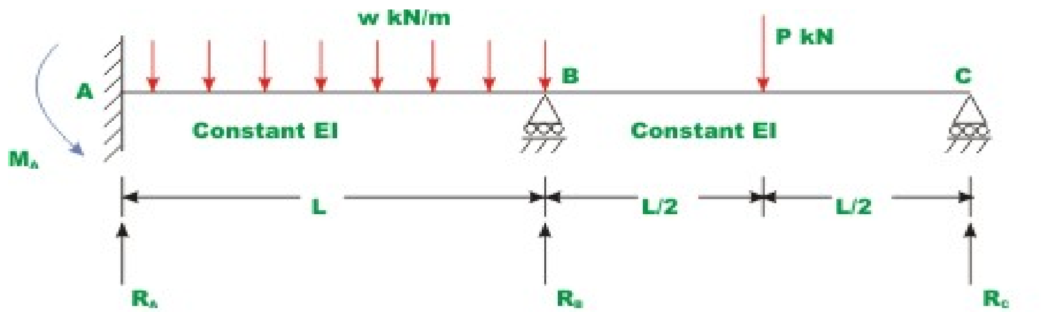


Fig. 8.1(a). Continuous beam.

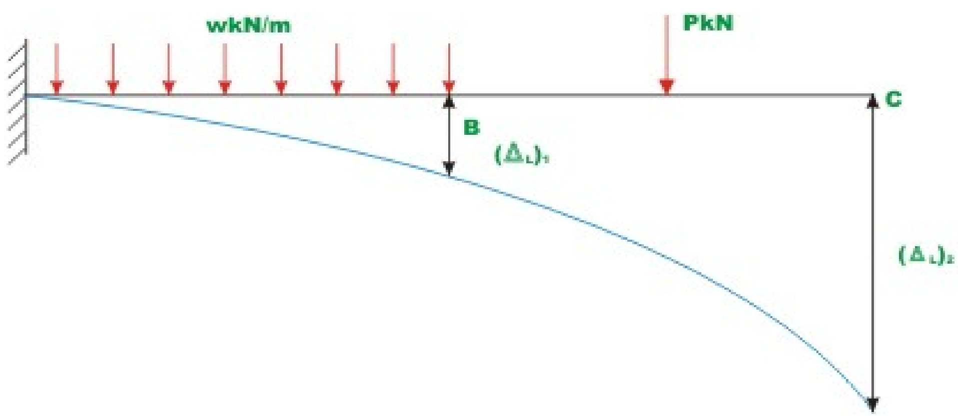


Fig. 8.1(b). Primary structure with applied loading.

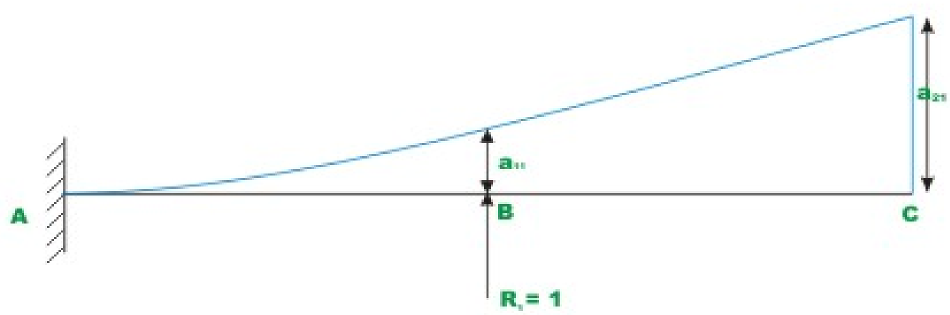


Fig. 8.1(c) Primary structure with unit load applied along R_1

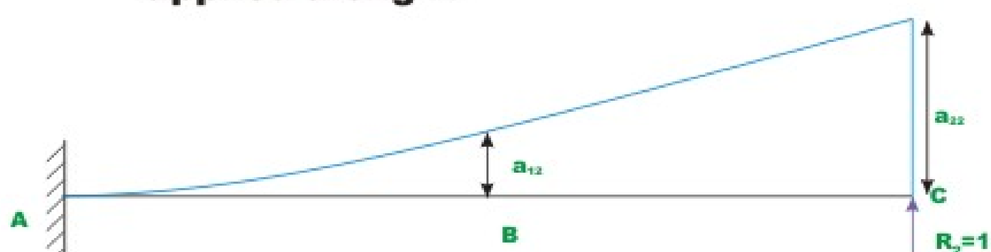


Fig. 8.1(d) Primary structure with unit load applied along R_2

The redundant reactions at A and B are denoted by R_1 and R_2 respectively. The released structure (statically determinate structure) with applied loading is shown in Fig. 8.1b. The deflection of primary structure at B and C due to applied loading is denoted by $(\Delta_L)_1$ and $(\Delta_L)_2$ respectively. Throughout this module $(\Delta_L)_i$ notation is used to denote deflection at i^{th} redundant due to applied loads on the determinate structure.

$$(\Delta_L)_1 = -\frac{wL^4}{8EI} - \frac{7PL^3}{12EI} \tag{8.1a}$$

$$(\Delta_L)_2 = -\frac{7wL^4}{24EI} - \frac{27PL^3}{16EI} \tag{8.1b}$$

In fact, the subscript 1 and 2 represent, locations of redundant reactions released. In the present case $R_A(=R_1)$ and $R_B(=R_2)$ respectively. In the present and subsequent lessons of this module, the deflections and the reactions are taken to be positive in the upward direction. However, it should be kept in mind that the positive sense of the redundant can be chosen arbitrarily. The deflection of the point of application of the redundant should likewise be considered positive when acting in the same sense.
For writing compatibility equations at B and C, it is required to know deflection of the released structure at B and C due to external loading and due to redundants. The deflection at B and C due to external loading can be computed easily. Since redundants R_1 and R_2 are not known, in the first step apply a unit load in the direction of R_1 and compute deflection, a_{11} at B, and deflection, a_{21} at C, as shown in Fig.8.1c. Now deflections at B and C of the given released structure due to redundant R_1 are,

$$(\Delta_R)_{11} = a_{11}\ R_1 \tag{8.2a}$$

$$(\Delta_R)_{21} = a_{21}\ R_1 \tag{8.2b}$$

In the second step, apply unit load in the direction of redundant R_2 and compute deflection at B (point 1), a_{12} and deflection at C, a_{22} as shown in Fig 8.1d. It may be recalled that the flexibility coefficient a_{ij} is the deflection at i due to unit value of force applied at j. Now deflections of the primary structure (released structure) at B and C due to redundant R_2 is

$$(\Delta_R)_{12} = a_{12}\ R_2 \tag{8.3a}$$

$$(\Delta_R)_{22} = a_{22}\ R_2 \tag{8.3b}$$

It is observed that, in the actual structure, the deflections at joints B and C is zero. Now the total deflections at B and C of the primary structure due to applied external loading and redundants R_1 and R_2 is,

$$\Delta_1 = (\Delta_L)_1 + a_{11}R_1 + a_{12}R_2 \tag{8.4a}$$

$$\Delta_2 = (\Delta_L)_2 + a_{21}R_1 + a_{22}R_2 \tag{8.4b}$$

The equation (8.4a) represents the total displacement at B and is obtained by superposition of three terms:

1) Deflection at B due to actual load acting on the statically determinate structure,
2) Displacement at B due to the redundant reaction R_1 acting in the positive direction at B (point 1) and
3) Displacement at B due to the redundant reaction R_2 acting in the positive direction at C.

The second equation (8.4b) similarly represents the total deflection at C. From the physics of the problem, the compatibility condition can be written as,

$$\Delta_1 = (\Delta_L)_1 + a_{11}R_1 + a_{12}R_2 = 0 \tag{8.5a}$$

$$\Delta_2 = (\Delta_L)_2 + a_{21}R_1 + a_{22}R_2 = 0 \tag{8.5b}$$

The equation (8.5a) and (8.5b) may be written in matrix notation as follows,

$$\begin{Bmatrix} (\Delta_L)_1 \\ (\Delta_L)_2 \end{Bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{8.6a}$$

$$\{(\Delta_L)_1\} + [A]\{R\} = \{0\} \tag{8.6b}$$

In which,

$$\{(\Delta_L)_1\} = \begin{Bmatrix} (\Delta_L)_1 \\ (\Delta_L)_2 \end{Bmatrix}; \; [A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } \{R\} = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

Solving the above set of algebraic equations, one could obtain the values of redundants, R_1 and R_2.

$$\{R\} = -[A]^{-1}\{\Delta_L\} \tag{8.7}$$

In the above equation the vectors $\{\Delta_L\}$ contains the displacement values of the primary structure at point 1 and 2, $[A]$ is the flexibility matrix and $\{R\}$ is column vector of redundants required to be evaluated. In equation (8.7) the inverse of the flexibility matrix is denoted by $[A]^{-1}$. In the above example, the structure is indeterminate to second degree and the size of flexibility matrix is 2×2. In general, if the structure is redundant to a degree n, then the flexibility matrix is of the order $n \times n$. To demonstrate the procedure to evaluate deflection, consider the problem given in Fig. 8.1a, with loading as given below

$$w = w; \qquad P = wL \tag{8.8a}$$

Now, the deflection $(\Delta_L)_1$ and $(\Delta_L)_2$ of the released structure can be evaluated from the equations (8.1a) and (8.1b) respectively. Then,

$$(\Delta_L)_1 = -\frac{wL^4}{8EI} - \frac{7wL^4}{12EI} = -\frac{17wL^4}{24EI} \tag{8.8b}$$

$$(\Delta_L)_2 = -\frac{7wL^4}{24EI} - \frac{27wL^4}{16EI} = -\frac{95wL^4}{48EI} \tag{8.8c}$$

The negative sign indicates that both deflections are downwards. Hence the vector $\{\Delta_L\}$ is given by

$$\{\Delta_L\} = -\frac{wL^4}{48EI}\begin{Bmatrix} 34 \\ 95 \end{Bmatrix} \tag{8.8d}$$

The flexibility matrix is determined from referring to figures 8.1c and 8.1d. Thus, when the unit load corresponding to R_1 is acting at B, the deflections are,

$$a_{11} = \frac{L^3}{3EI}, \qquad a_{21} = \frac{5L^3}{6EI} \tag{8.8e}$$

Similarly when the unit load is acting at C,

$$a_{12} = \frac{5L^3}{6EI}, \qquad a_{22} = \frac{8L^3}{3EI} \tag{8.8f}$$

The flexibility matrix can be written as,

$$[A] = \frac{L^3}{6EI}\begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix} \tag{8.8g}$$

The inverse of the flexibility matrix can be evaluated by any of the standard method. Thus,

$$[A]^{-1} = \frac{6EI}{7L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix} \tag{8.8h}$$

Now using equation (8.7) the redundants are evaluated. Thus,

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{6EI}{7L^3} \times \frac{wL^4}{48EI}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}\begin{Bmatrix} 34 \\ 95 \end{Bmatrix}$$

Hence, $R_1 = \frac{69}{56}wL$ and $R_2 = \frac{20}{56}wL$ (8.8i)

Once the redundants are evaluated, the other reaction components can be evaluated by static equations of equilibrium.

Example 8.1

Calculate the support reactions in the continuous beam ABC due to loading as shown in Fig. 8.2a. Assume EI to be constant throughout.

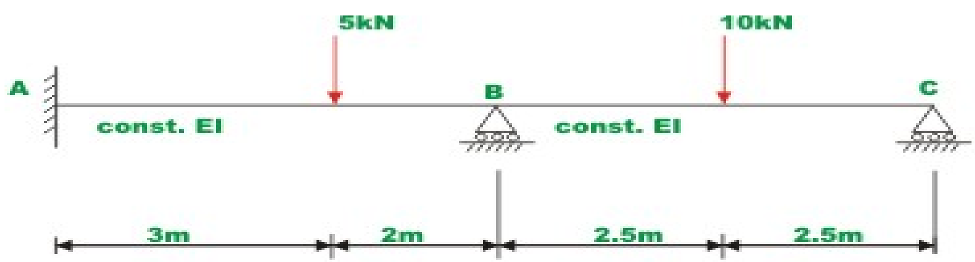


Fig. 8.2 (a) Example 8.2

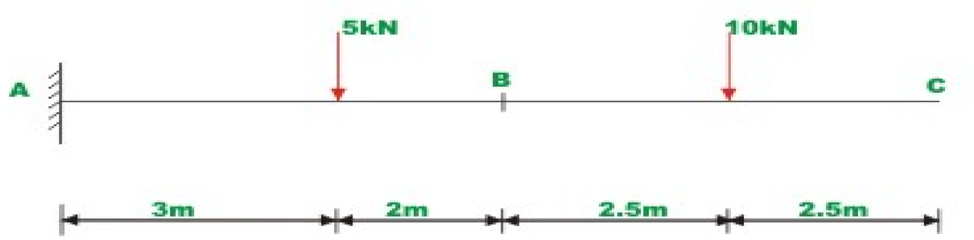


Fig. 8.2 (b) Primary structure with external load

Select two reactions viz, at $B(R_1)$ and $C(R_2)$ as redundants, since the given beam is statically indeterminate to second degree. In this case the primary structure is a cantilever beam AC. The primary structure with a given loading is shown in Fig. 8.2b.

In the present case, the deflections $(\Delta_L)_1$, and $(\Delta_L)_2$ of the released structure at B and C can be readily calculated by moment-area method. Thus,

$$(\Delta_L)_1 = -\frac{819.16}{EI}$$

and
$$(\Delta_L)_2 = -\frac{2311.875}{EI} \tag{1}$$

For the present problem the flexibility matrix is,

$$a_{11} = \frac{125}{3EI} \qquad a_{21} = \frac{625}{6EI}$$

$$a_{12} = \frac{625}{6EI} \qquad a_{22} = \frac{1000}{3EI} \tag{2}$$

In the actual problem the displacements at B and C are zero. Thus the compatibility conditions for the problem may be written as,

$$\begin{aligned} a_{11}R_1 + a_{12}R_2 + (\Delta_L)_1 &= 0 \\ a_{21}R_1 + a_{22}R_2 + (\Delta_L)_2 &= 0 \end{aligned} \tag{3}$$

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{3EI}{27343.75}\begin{bmatrix} 1000 & -312.5 \\ -312.5 & 125 \end{bmatrix} \times \frac{1}{EI}\begin{Bmatrix} 819.16 \\ 2311.875 \end{Bmatrix} \tag{5}$$

Substituting the value of E and I in the above equation,

$$R_1 = 10.609\text{kN} \quad \text{and} \quad R_2 = 3.620 \ \text{ kN}$$

Using equations of static equilibrium,

$$R_3 = 0.771 \ \text{ kN} \qquad \text{and} \quad R_4 = -0.755 \ \text{ kN.m}$$

Example 8.2

A clamped beam AB of constant flexural rigidity is shown in Fig. 8.3a. The beam is subjected to a uniform distributed load of w kN/m and a central concentrated moment $M = wL^2$ kN.m. Draw shear force and bending moment diagrams by force method.

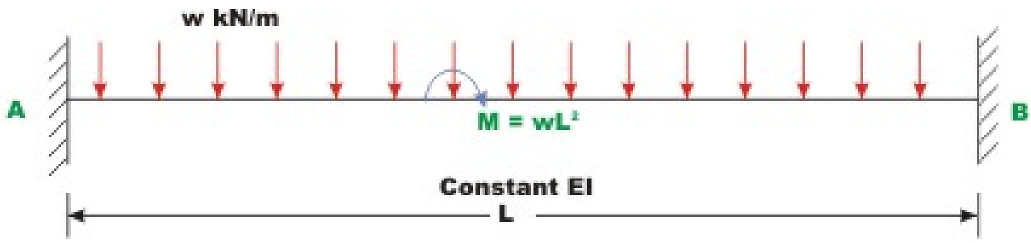


Fig. 8.3(a) Clamped beam (Example 8.1)

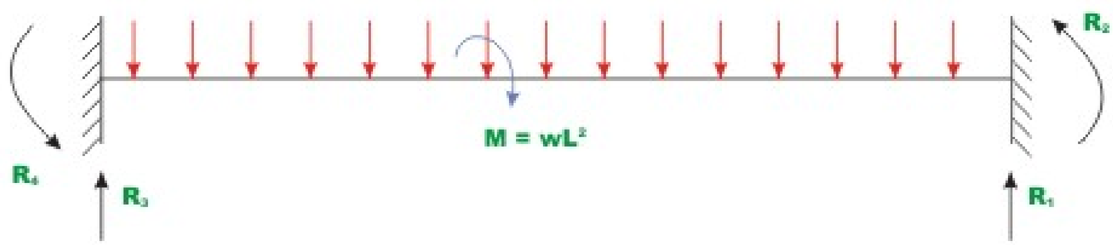


Fig. 8.3(b) Clamped beam with R_1 and R_2 as redundants

Select vertical reaction (R_1) and the support moment (R_2) at B as the redundants. The primary structure in this case is a cantilever beam which could be obtained by releasing the redundants R_1 and R_2. The R_1 is assumed to be positive in the upward direction and R_2 is assumed to be positive in the counterclockwise direction. Now, calculate deflection at B due to only applied loading. Let $(\Delta_L)_1$ be the transverse deflection at B and $(\Delta_L)_2$ be the slope at B due to external loading. The positive directions of the selected redundants are shown in Fig. 8.3b.

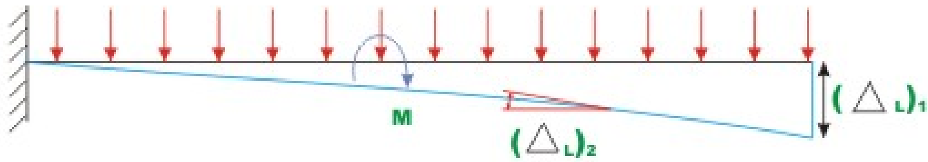


Fig. 8.3(c) Primary structure with external loading

Fig. 8.3(d)Primary structure with unit load along R_1

Fig. 8.3 (e) Primary structure with unit moment along R_2

Fig.8.3f Reaction

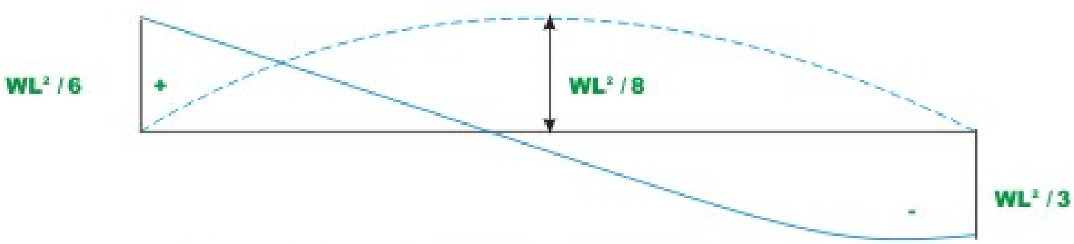


Fig 8.3(g) Bending moment diagram

Fig.8.3(h) Shear force diagram.

The deflection $(\Delta_L)_1$ and $(\Delta_L)_2$ of the released structure can be evaluated from unit load method. Thus,

$$(\Delta_L)_1 = -\frac{wL^4}{8EI} - \frac{3wL^4}{8EI} = -\frac{wL^4}{2EI} \qquad (1)$$

and

$$(\Delta_L)_2 = -\frac{wL^3}{6EI} - \frac{wL^3}{2EI} = -\frac{2wL^3}{3EI} \qquad (2)$$

The negative sign indicates that $(\Delta_L)_1$ is downwards and rotation $(\Delta_L)_2$ is clockwise. Hence the vector $\{\Delta_L\}$ is given by

$$\{\Delta_L\} = -\frac{wL^3}{6EI}\begin{Bmatrix} 3L \\ 4 \end{Bmatrix} \qquad (3)$$

The flexibility matrix is evaluated by first applying unit load along redundant R_1 and determining the deflections a_{11} and a_{21} corresponding to redundants R_1 and R_2 respectively (see Fig. 8.3d). Thus,

$$a_{11} = \frac{L^3}{3EI} \qquad \text{and } a_{21} = \frac{L^2}{2EI} \qquad (4)$$

Similarly, applying unit load in the direction of redundant R_2, one could evaluate flexibility coefficients a_{12} and a_{22} as shown in Fig. 8.3c.

$$a_{12} = \frac{L^2}{2EI} \qquad \text{and } a_{22} = \frac{L}{EI} \qquad (5)$$

Now the flexibility matrix is formulated as,

$$[A] = \frac{L}{6EI}\begin{bmatrix} 2L^2 & 3L \\ 3L & 6 \end{bmatrix} \qquad (6)$$

The inverse of flexibility matrix is formulated as,

$$[A]^{-1} = \frac{6EI}{3L^3}\begin{bmatrix} 6 & -3L \\ -3L & 2L^2 \end{bmatrix}$$

The redundants are evaluated from equation (8.7). Hence,

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = -\frac{6EI}{3L^3}\begin{bmatrix} 6 & -3L \\ -3L & 2L^2 \end{bmatrix} \times \left(-\frac{wL^3}{6EI}\right)\begin{Bmatrix} 3L \\ 4 \end{Bmatrix}$$

$$= \frac{w}{3}\begin{Bmatrix} 6L \\ -L^2 \end{Bmatrix}$$

$$R_1 = 2wL \qquad and \quad R_2 = -\frac{wL^2}{3} \qquad (7)$$

The other two reactions (R_3 and R_4) can be evaluated by equations of statics. Thus,

$$R_4 = M_A = -\frac{wL^2}{6} \qquad \text{and} \quad R_1 = R_A = -wL \qquad (8)$$

The bending moment and shear force diagrams are shown in Fig. 8.3g and Fig.8.3h respectively.

Summary

In this lesson, statically indeterminate beams of degree more than one is solved systematically using flexibility matrix method. Towards this end matrix notation is adopted. Few illustrative examples are solved to illustrate the procedure. After analyzing the continuous beam, reactions are calculated and bending moment diagrams are drawn.

Module 2

Analysis of Statically Indeterminate Structures by the Matrix Force Method

Lesson 9

The Force Method of Analysis: Beams (Continued)

Instructional Objectives

After reading this chapter the student will be able to

1. Calculate additional stresses developed in statically indeterminate structures due to support settlements.
2. Analyse continuous beams which are supported on yielding supports.
3. Sketch the deflected shape of the member.
4. Draw banding moment and shear force diagrams for indeterminate beams undergoing support settlements.

9.1 Introduction

In the last lesson, the force method of analysis of statically indeterminate beams subjected to external loads was discussed. It is however, assumed in the analysis that the supports are unyielding and the temperature remains constant. In the design of indeterminate structure, it is required to make necessary provision for future unequal vertical settlement of supports or probable rotation of supports. It may be observed here that, in case of determinate structures no stresses are developed due to settlement of supports. The whole structure displaces as a rigid body (see Fig. 9.1). Hence, construction of determinate structures is easier than indeterminate structures.

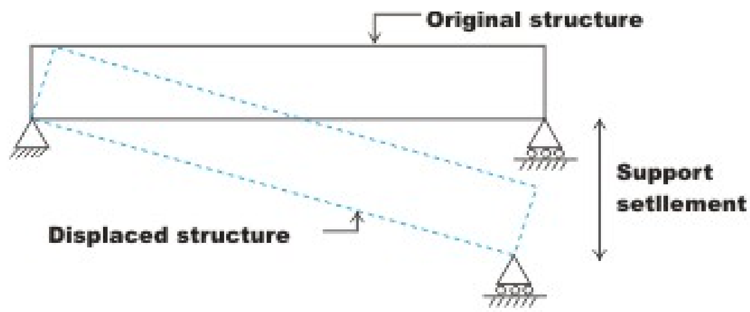


Fig. 9.1 Effect of support settlement on determinate structure

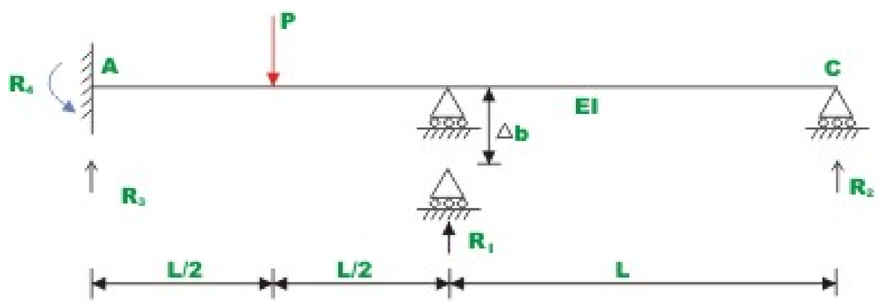


Fig. 9.2 Continuous beam with yielding of support

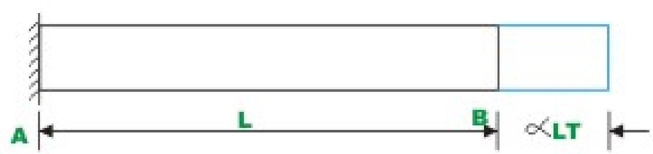


Fig. 9.3 Effect of temperature change in beam

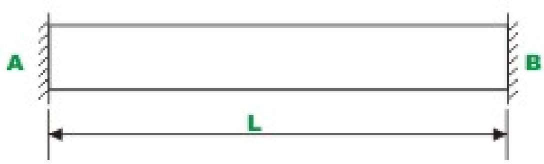
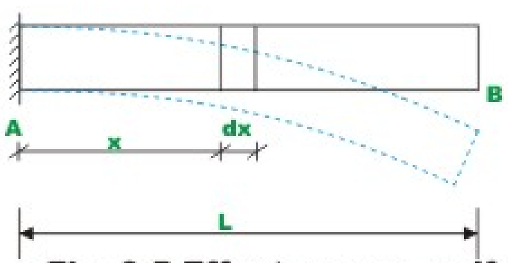


Fig. 9.4 Fixed - fixed beam



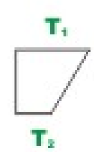


Fig. 9.5 (b)

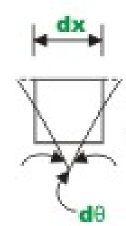


Fig. 9.5 (c)

Fig. 9.5 Effect on non- uniform temperature change across the depth of a beam

The statically determinate structure changes their shape due to support settlement and this would in turn induce reactions and stresses in the system. Since, there is no external force system acting on the structures, these forces form a balanced force system by themselves and the structure would be in equilibrium. The effect of temperature changes, support settlement can also be easily included in the force method of analysis. In this lesson few problems, concerning the effect of support settlement are solved to illustrate the procedure.

9.2 Support Displacements

Consider a two span continuous beam, which is statically indeterminate to second degree, as shown in Fig. 9.2. Assume the flexural rigidity of this beam to be constant throughout. In this example, the support B is assumed to have settled by an amount Δ_b as shown in the figure.

This problem was solved in the last lesson, when there was no support settlement (vide section 8.2). In section 8.2, choosing reaction at B and C as the redundant, the total deflection of the primary structure due to applied external loading and redundant R_1 and R_2 is written as,

$$\Delta_1 = (\Delta_L)_1 + a_{11}R_1 + a_{12}R_2 \tag{9.1a}$$

$$\Delta_2 = (\Delta_L)_2 + a_{21}R_1 + a_{22}R_2 \tag{9.1b}$$

wherein, R_1 and R_2 are the redundants at B and C respectively, and $(\Delta_L)_1$, and $(\Delta_L)_2$ are the deflections of the primary structure at B and C due to applied loading. In the present case, the support B settles by an amount Δ_b in the direction of the redundant R_1. This support movement can be readily incorporated in the force method of analysis. From the physics of the problem the total deflection at the support may be equal to the given amount of support movement. Hence, the compatibility condition may be written as,

$$\Delta_1 = -\Delta_b \tag{9.2a}$$

$$\Delta_2 = 0 \tag{9.2b}$$

It must be noted that, the support settlement Δ_b must be negative as it is displaces downwards. It is assumed that deflections and reactions are positive in the upward direction. The equation (9.1a) and (9.1b) may be written in compact form as,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} - \begin{Bmatrix} (\Delta_L)_1 \\ (\Delta_L)_2 \end{Bmatrix} \quad (9.3a)$$

$$[A]\{R\} = \{\Delta\} - \{(\Delta_L)\} \quad (9.3b)$$

Solving the above algebraic equations, one could evaluate redundants R_1 and R_2 due to external loading and support settlement.

9.3 Temperature Stresses

Internal stresses are also developed in the statically indeterminate structure if the free movement of the joint is prevented.

For example, consider a cantilever beam AB as shown in Fig. 9.3. Now, if the temperature of the member is increased uniformly throughout its length, then the length of the member is increased by an amount

$$\Delta_T = \alpha \; L \; T \quad (9.4)$$

In which, Δ_T is the change in the length of the member due to temperature change, α is the coefficient of thermal expansion of the material and T is the change in temperature. The elongation (the change in the length of the member) and increase in temperature are taken as positive. However if the end B is restrained to move as shown in Fig 9.4, then the beam deformation is prevented. This would develop an internal axial force and reactions in the indeterminate structure.

Next consider a cantilever beam AB, subjected to a different temperature, T_1 at the top and T_2 at the bottom as shown in Fig. 9.5(a) and (b). If the top temperature T_1 is higher than the bottom beam surface temperature T_2, then the beam will deform as shown by dotted lines. Consider a small element dx at a distance x from A. The deformation of this small element is shown in Fig. 9.5c. Due to rise in temperature $T_1 °C$ on the top surface, the top surface elongates by

$$\Delta_{T_1} = \alpha \;\; T_1 \, dx \quad (9.5a)$$

Similarly due to rise in temperature T_2, the bottom fibers elongate by

$$\Delta_{T_2} = \alpha \;\; T_2 \, dx \quad (9.5b)$$

As the cross section of the member remains plane, the relative angle of rotation $d\theta$ between two cross sections at a distance dx is given by

$$d\theta = \frac{\alpha(T_1 - T_2)dx}{d} \tag{9.6}$$

where, d is the depth of beam. If the end B is fixed as in Fig. 9.4, then the differential change in temperature would develop support bending moment and reactions.

The effect of temperature can also be included in the force method of analysis quite easily. This is done as follows. Calculate the deflection corresponding to redundant actions separately due to applied loading, due to rise in temperature (either uniform or differential change in temperature) and redundant forces. The deflection in the primary structure due to temperature changes is denoted by $(\Delta_T)_i$ which denotes the deflection corresponding to i^{th} redundant due to temperature change in the determinate structure. Now the compatibility equation for statically indeterminate structure of order two can be written as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} - \begin{Bmatrix} (\Delta_L)_1 \\ (\Delta_L)_2 \end{Bmatrix} - \begin{Bmatrix} (\Delta_T)_1 \\ (\Delta_T)_2 \end{Bmatrix}$$

$$[A]\{R\} = \{\Delta\} - \{(\Delta_L)\} - \{(\Delta_T)\} \tag{9.7}$$

wherein, $\{\Delta_L\}$ is the vector of displacements in the primary structure corresponding to redundant reactions due to external loads; $\{\Delta_T\}$ is the displacements in the primary structure corresponding to redundant reactions and due to temperature changes and $\{\Delta\}$ is the matrix of support displacements corresponding to redundant actions. Equation (9.7) can be solved to obtain the unknown redundants.

Example 9.1

Calculate the support reactions in the continuous beam ABC (see Fig. 9.6a) having constant flexural rigidity EI throughout, due to vertical settlement of the support B by 5 mm as shown in the figure. $E = 200 \text{ GPa}$ and $I = 4 \times 10^{-4} \text{m}^4$.

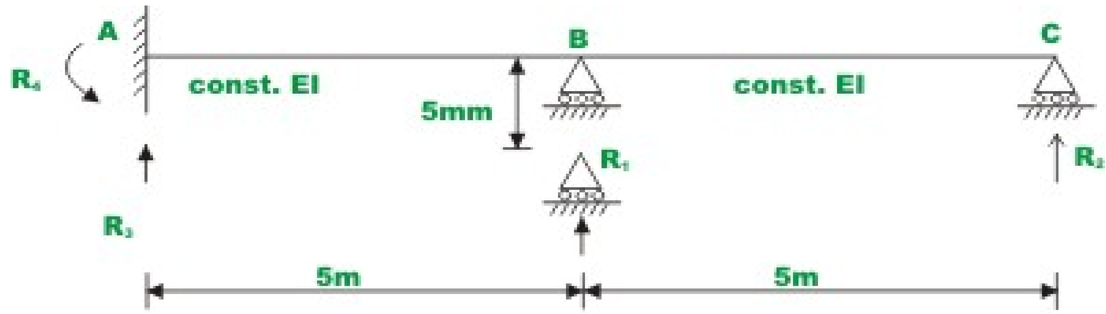


Fig. 9.6 (a) Continuous beam

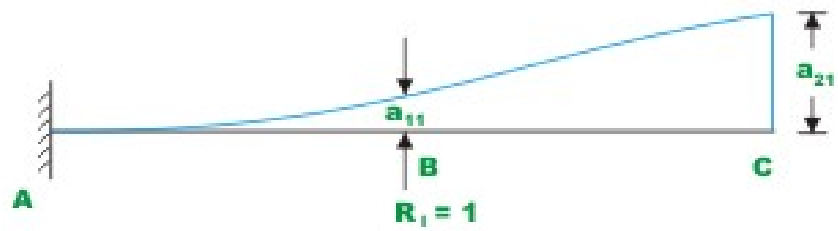


Fig. 9.6 (b) primary structure with unit along R_1

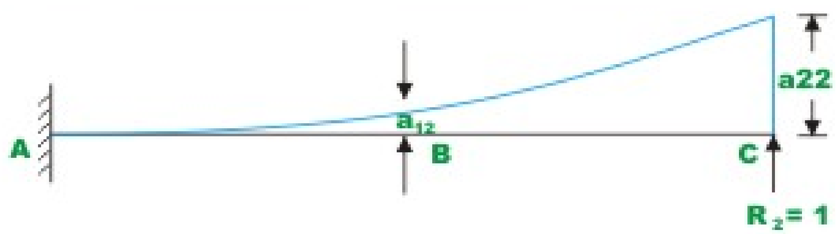


Fig. 9.6 (c) primary structure with unit load along R_2

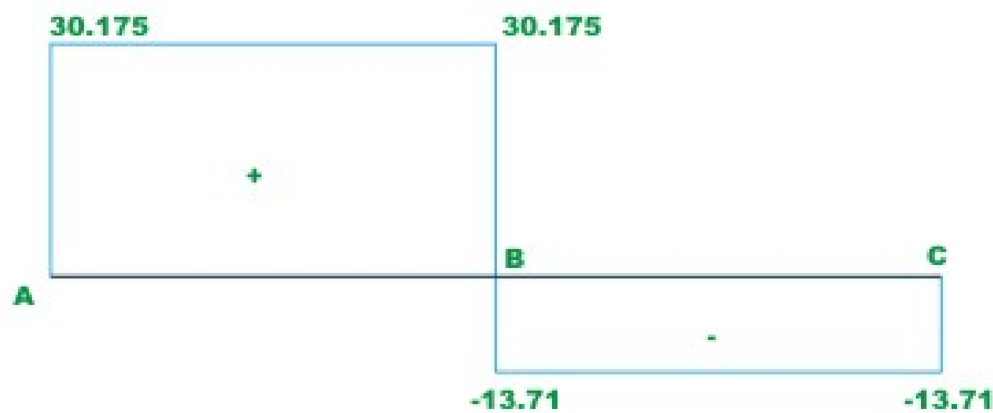


Fig. 9.6(d) Shear force diagram

As the given beam is statically indeterminate to second degree, choose reaction at B (R_1) and C (R_2) as the redundants. In this case the cantilever beam AC is the basic determinate beam (primary structure). On the determinate beam only redundant reactions are acting. The first column of flexibility matrix is evaluated by first applying unit load along the redundant R_1 and determining deflections a_{11} and a_{21} respectively as shown in Fig. 9.6b.

$$a_{11} = \frac{5^3}{3EI} = \frac{125}{3EI}$$

$$a_{21} = \frac{125}{3EI} + \frac{25}{2EI} \times 5 = \frac{625}{6EI} \qquad (1)$$

Simply by applying the unit load in the direction of redundant R_2, one could evaluate flexibility coefficients a_{12} and a_{22} (see Fig. 9.6c).

$$a_{12} = \frac{625}{6EI} \quad \text{and} \quad a_{22} = \frac{1000}{3EI} \qquad (2)$$

The compatibility condition for the problem may be written as,

$$\begin{aligned} a_{11}R_1 + a_{12}R_2 &= -5\times10^{-3} \\ a_{21}R_1 + a_{22}R_2 &= 0 \end{aligned} \qquad (3)$$

The redundant reactions are,

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} -5\times10^{-3} \\ 0 \end{Bmatrix} \qquad (4)$$

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{3EI}{27343.75} \begin{bmatrix} 1000 & -312.5 \\ -312.5 & 125 \end{bmatrix} \times \begin{Bmatrix} -5\times10^{-3} \\ 0 \end{Bmatrix} \qquad (5)$$

Substituting the values of E and I in the above equation, the redundant reactions are evaluated.

$$R_1 = -43.885 \text{ kN} \quad \text{and} \quad R_2 = 13.71 \text{ kN}$$

R_1 acts downwards and R_2 acts in the positive direction of the reaction $i.e.$ upwards. The remaining two reactions R_3 and R_4 are evaluated by the equations of equilibrium.

$$\sum F_y = 0 \Rightarrow R_1 + R_2 + R_3 = 0$$

Hence $\quad R_3 = 30.175 \text{ kN}$

$$\sum M_A = 0 \Rightarrow \quad R_4 + 5 \times R_1 + 10 \times R_2 = 0$$

Solving for R_4,

$R_4 = 82.325 \text{ kN.m} \quad (\text{counter} \quad \text{clockwise})$

The shear force and bending moment diagrams are shown in Figs. 9.6d and 9.6e respectively.

Example 9.2

Compute reactions and draw bending moment diagram for the continuous beam $ABCD$ loaded as shown in Fig. 9.7a, due to following support movements.
Support B, $0.005m$ vertically downwards.
Support C, $0.01m$ vertically downwards.

Assume, $E = 200GPa$; $I = 1.35 \times 10^{-3}\, m^4$.

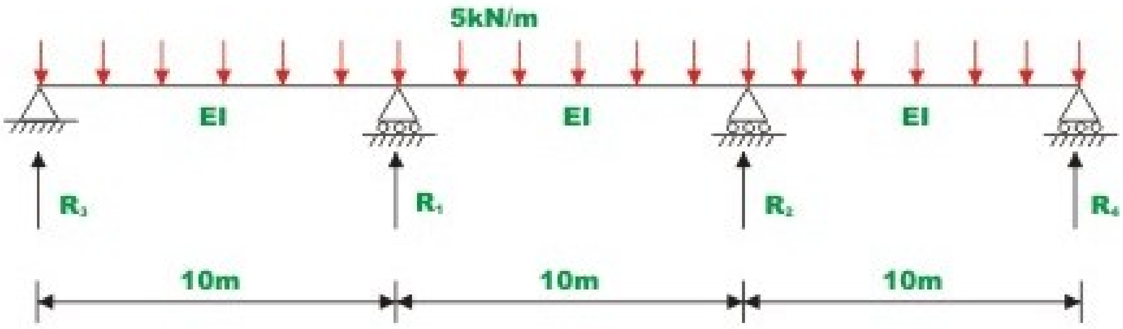


Fig. 9.7(a) Continuous beam

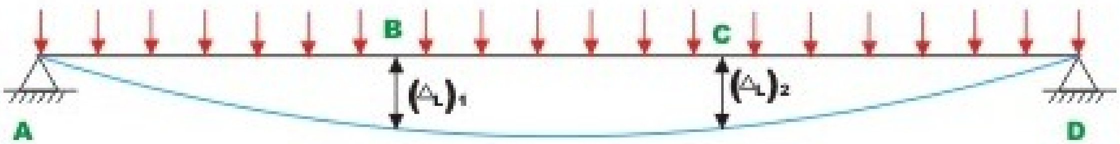


Fig. 9.7(b) Primary structure with external load

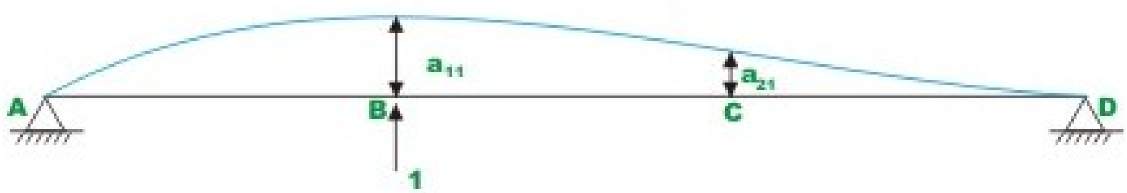


Fig. 9.7 (c) Primary structure with unit load along R

Fig. 9. 7 (d) Primary structure with unit load along R_2

The given beam is statically indeterminate to second degree. Select vertical reactions at $B(R_1)$ and $C(R_2)$ as redundants. The primary structure in this case is a simply supported beam AD as shown in Fig. 9.7b.

The deflection $(\Delta_L)_1$ and $(\Delta_L)_2$ of the released structure are evaluated from unit load method. Thus,

$$(\Delta_L)_1 = \frac{-45833.33\times 10^3}{EI} = \frac{-45833.33\times 10^3}{200\times 10^9 \times 1.35\times 10^{-3}} = -0.169\,m$$

$$(\Delta_L)_2 = \frac{-45833.33\times 10^3}{EI} = -0.169\,m \qquad (1)$$

The flexibility matrix is evaluated as explained in the previous example, *i.e.* by first applying unit load corresponding to the redundant R_1 and determining deflections a_{11} and a_{21} respectively as shown in Fig. 9.7c. Thus,

$$a_{11} = \frac{444.44}{EI}$$

$$a_{21} = \frac{388.89}{EI} \qquad (2)$$

$$a_{22} = \frac{444.44}{EI}$$

$$a_{12} = \frac{388.89}{EI}$$

In this case the compatibility equations may be written as,

$$\begin{aligned} -0.169 + a_{11}R_1 + a_{12}R_2 &= -0.005 \\ -0.169 + a_{21}R_1 + a_{22}R_2 &= -0.01 \end{aligned} \qquad (3)$$

Solving for redundant reactions,

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{EI}{46291.48}\begin{bmatrix} 444.44 & -388.89 \\ -388.89 & 444.44 \end{bmatrix} \times \begin{Bmatrix} 0.164 \\ 0.159 \end{Bmatrix} \qquad (4)$$

Substituting the value of E and I in the above equation,

$R_1 = 64.48 kN$ and $R_2 = 40.174 kN$

Both R_1 and R_2 acts in the upward direction. The remaining two reactions R_3 and R_4 are evaluated by the equations of static equilibrium.

$$\sum M_A = 0 \qquad 10 \times R_1 + 20 \times R_2 + 30 \times R_4 - 5 \times 30 \times 15 = 0$$

Hence $\quad R_4 = 26.724 \;\; \text{kN}$

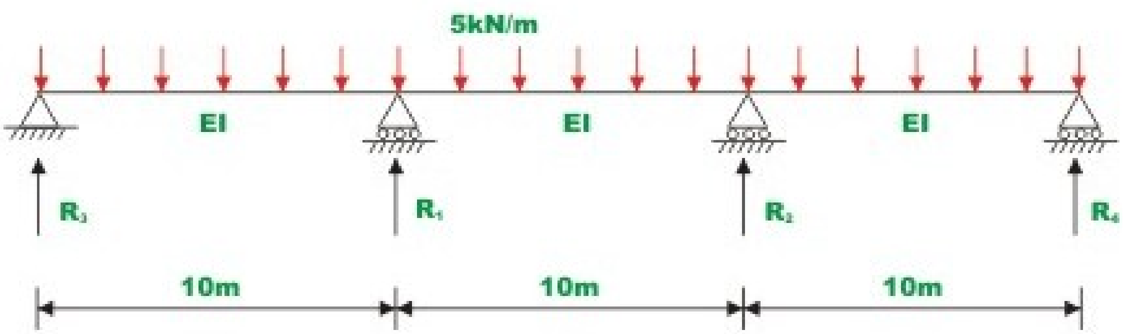
$$\sum F_y = 0 \qquad R_3 + R_1 + R_2 + R_4 - 5 \times 30 = 0$$

Hence $\quad R_3 = 18.622 \;\; \text{kN}$ (5)

The shear force and bending moment diagrams are now constructed and are shown in Figs. 9.7e and 9.7f respectively.

83.325

Fig. 9.6 (e) Bending moment diagram



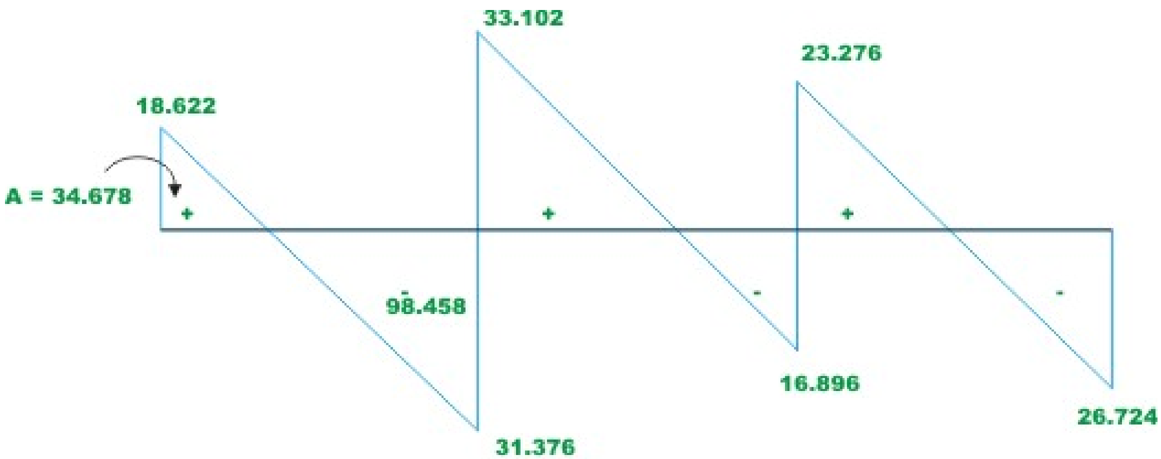


Fig. 9.7 (e) Shear force diagram

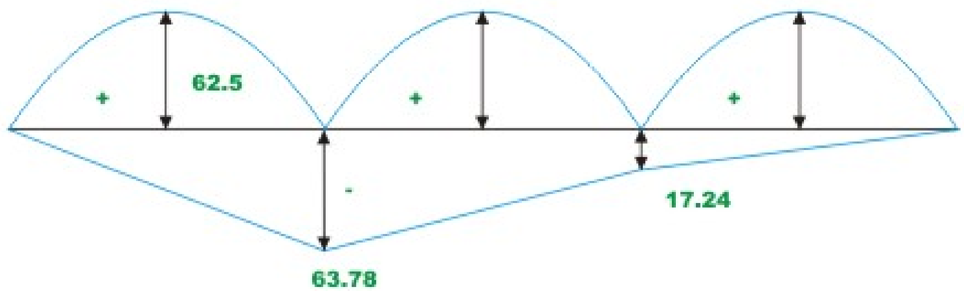


Fig. 9.7 (f) Bending moment diagram

Summary

In this lesson, the effect of support settlements on the reactions and stresses in the case of indeterminate structures is discussed. The procedure to calculate additional stresses caused due to yielding of supports is explained with the help of an example. A formula is derived for calculating stresses due to temperature changes in the case of statically indeterminate beams.

Module 2

Analysis of Statically Indeterminate Structures by the Matrix Force Method

Lesson

10

The Force Method of Analysis: Trusses

Instructional Objectives

After reading this chapter the student will be able to

1. Calculate degree of statical indeterminacy of a planar truss
2. Analyse the indeterminate planar truss for external loads
3. Analyse the planar truss for temperature loads
4. Analyse the planar truss for camber and lack of fit of a member.

10.1 Introduction

The truss is said to be statically indeterminate when the total number of reactions and member axial forces exceed the total number of static equilibrium equations. In the simple planar truss structures, the degree of indeterminacy can be determined from inspection. Whenever, this becomes tedious, one could use the following formula to evaluate the static indeterminacy of static planar truss (see also section 1.3).

$$i = (m + r) - 2j \qquad (10.1)$$

where m, j and r are number of members, joints and unknown reaction components respectively. The indeterminacy in the truss may be external, internal or both. A planar truss is said to be externally indeterminate if the number of reactions exceeds the number of static equilibrium equations available (three in the present case) and has exactly $(2j-3)$ members. A truss is said to be internally indeterminate if it has exactly three reaction components and more than $(2j-3)$ members. Finally a truss is both internally and externally indeterminate if it has more than three reaction components and also has more than $(2j-3)$ members.

The basic method for the analysis of indeterminate truss by force method is similar to the indeterminate beam analysis discussed in the previous lessons. Determine the degree of static indeterminacy of the structure. Identify the number of redundant reactions equal to the degree of indeterminacy. The redundants must be so selected that when the restraint corresponding to the redundants are removed, the resulting truss is statically determinate and stable. Select redundant as the reaction component in excess of three and the rest from the member forces. However, one could choose redundant actions completely from member forces. Following examples illustrate the analysis procedure.

Example 10.1

Determine the forces in the truss shown in Fig.10.1a by force method. All the members have same axial rigidity.

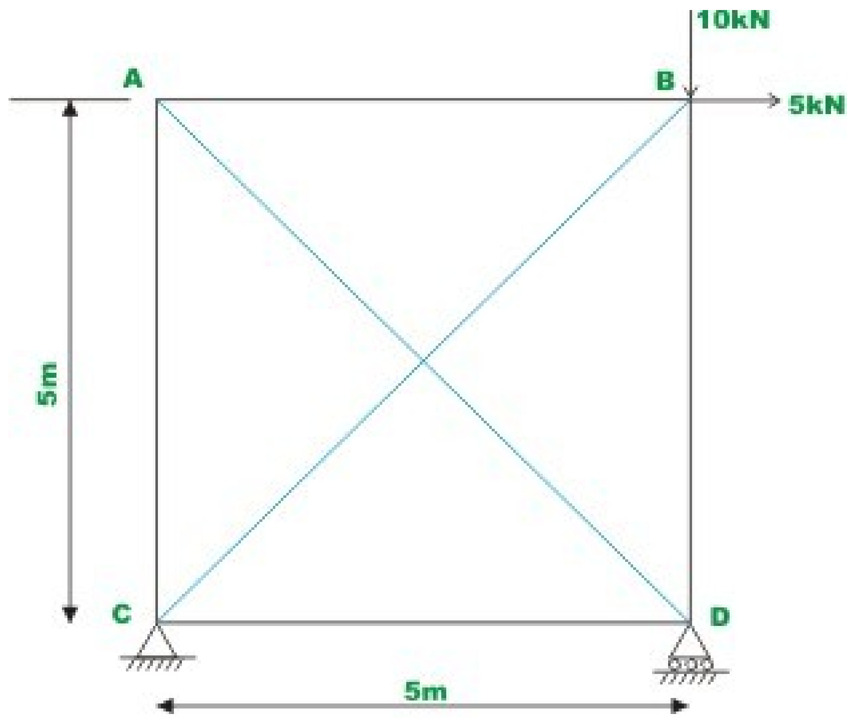


Fig.10.1 (a) Example 10.1

The plane truss shown in Fig.10.1a is statically indeterminate to first degree. The truss is externally determinate $i.e.$ the reactions can be evaluated from the equations of statics alone. Select the bar force F_{AD} in member AD as the redundant. Now cut the member AD to obtain the released structure as shown in Fig. 10.1b. The cut redundant member AD remains in the truss as its deformations need to be included in the calculation of displacements in the released structure. The redundant (F_{AD}) consists of the pair of forces acting on the released structure.

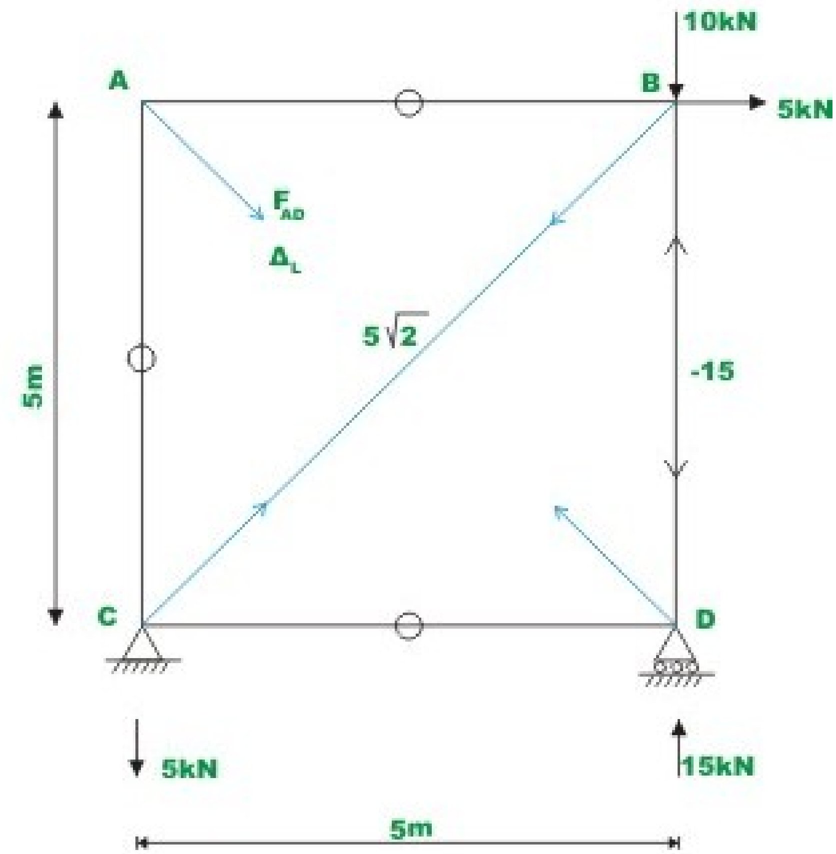


Fig.10.1 (b)

Evaluate reactions of the truss by static equations of equilibrium.

$$
\begin{aligned}
R_{Cy} &= -5 \quad \text{kN} \quad \text{(downwards)} \\
R_{Cx} &= -5 \quad \text{kN} \quad \text{(downwards)} \\
R_{Dy} &= 15 \quad \text{kN(upwards)}
\end{aligned} \qquad (1)
$$

Please note that the member tensile axial force is taken as positive and horizontal reaction is taken as positive to the right and vertical reaction is taken as positive when acting upwards. When the member cut ends are displaced towards one another then it is taken as positive.

The first step in the force method is to calculate displacement (Δ_L) corresponding to redundant bar force F_{AD} in the released structure due to applied external loading. This can be readily done by unit-load method.

To calculate displacement (Δ_L), apply external load and calculate member forces (P_i) as shown in Fig. 10.1b and apply unit virtual load along F_{AD} and calculate member forces $(P_v)_i$ (see Fig. 10.1c). Thus,

$$\Delta_L = \sum P_i (P_v)_i \frac{L_i}{AE} \\ = \frac{103.03}{AE} \tag{2}$$

In the next step, apply a real unit load along the redundant F_{AD} and calculate displacement a_{11} by unit load method. Thus,

$$a_{11} = \sum (P_v)_i^2 \frac{L_i}{A_i E_i} \\ = \frac{24.142}{AE} \tag{3}$$

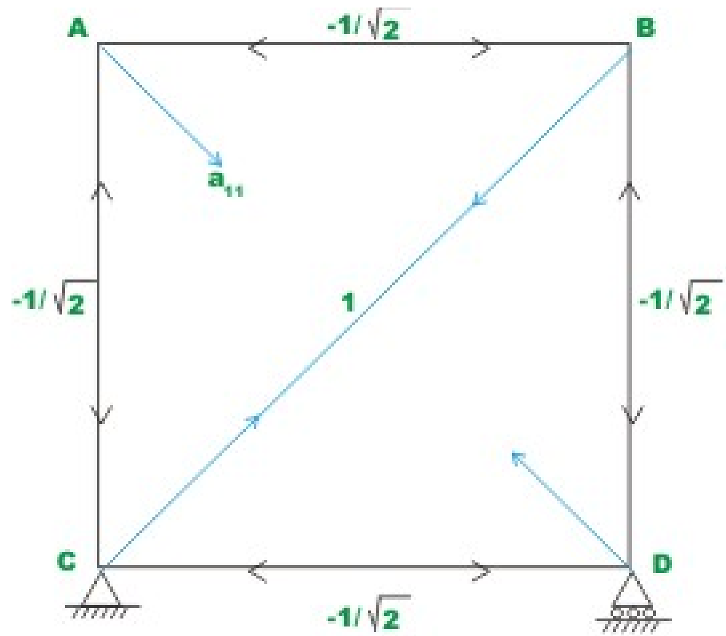


Fig. 10.1 (c) Plane truss of Example 10.1

The compatibility condition of the problem is that the relative displacement Δ_L of the cut member AD due to external loading plus the relative displacement of the member AD caused by the redundant axial forces must be equal to zero *i.e.*

$$\Delta_L + a_{11} F_{AD} = 0 \tag{4}$$

$$F_{AD} = \frac{-103.03}{24.142}$$

$$= -4.268 \quad \text{kN(compressive)}$$

Now the member forces in the members can be calculated by method of superposition. Thus,

$$F_i = P_i + F_{AD} (P_v)_i \tag{5}$$

The complete calculations can be done conveniently in a tabular form as shown in the following table.

Table 10.1 Computation for example 10.1

Member	Length L_i	Forces in the released truss due to applied loading P_i	Forces in the released truss due to unit load $(P_v)_i$	$P_i (P_v)_i \frac{L_i}{AE}$	$(P_v)_i^2 \frac{L_i}{A_i E_i}$	$F_i = P_i + F_{AD} (P_v)_i$
	m	kN	kN	m	m/kN	kN
AB	5	0	$-1/\sqrt{2}$	0	$5/2AE$	3.017
BD	5	-15	$-1/\sqrt{2}$	$75/\sqrt{2}AE$	$5/2AE$	-11.983
DC	5	0	$-1/\sqrt{2}$	0	$5/2AE$	3.017
CA	5	0	$-1/\sqrt{2}$	0	$5/2AE$	3.017
CB	$5\sqrt{2}$	$5\sqrt{2}$	1	$50/AE$	$5\sqrt{2}/AE$	2.803
AD	$5\sqrt{2}$	0	1	0	$5\sqrt{2}/AE$	-4.268
			Total	$\frac{103.03}{AE}$	$\frac{24.142}{AE}$	

Example 10.2

Calculate reactions and member forces of the truss shown in Fig. 10.2a by force method. The cross sectional areas of the members in square centimeters are shown in parenthesis. Assume $E = 2.0 \times 10^5 \text{ N/mm}^2$.

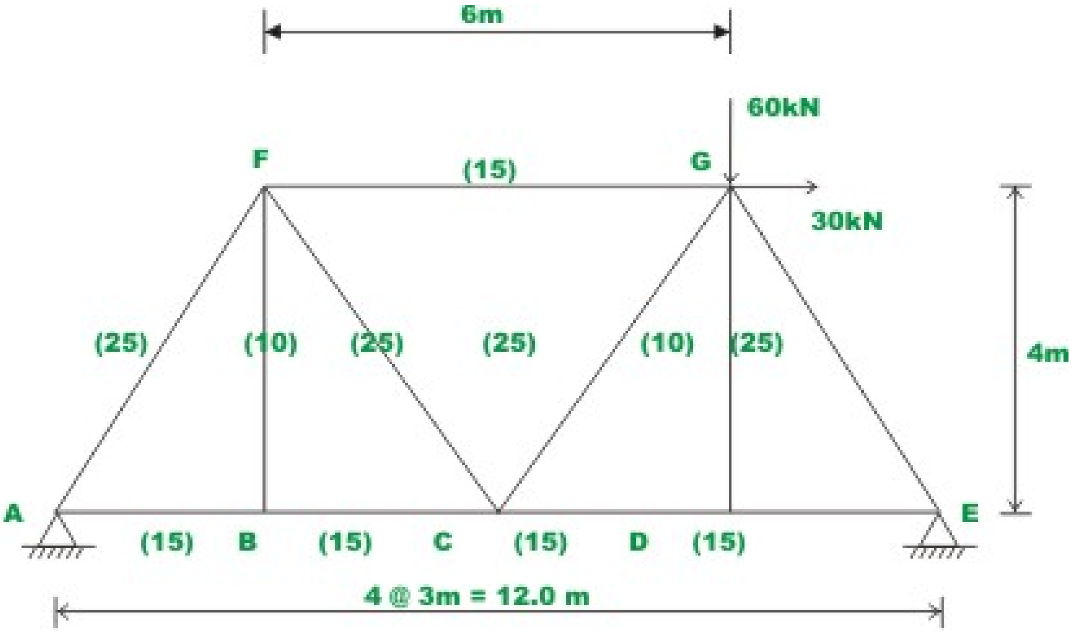


Fig.10.2 (a)

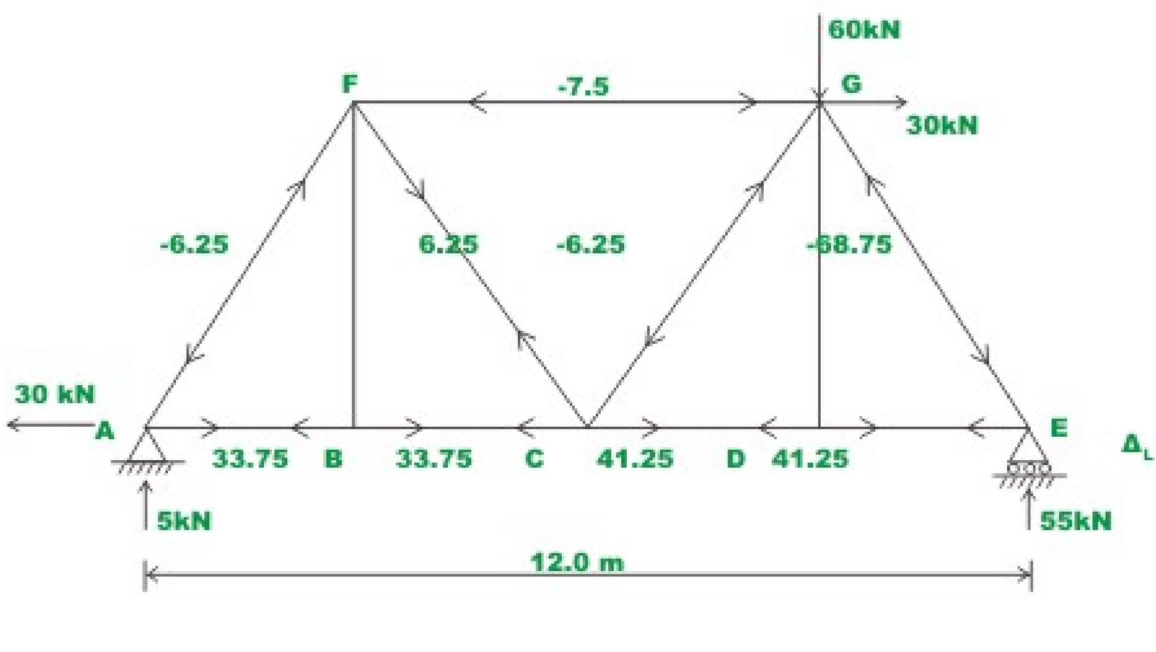


Fig.10.2 (b)

The plane truss shown in Fg.10.2a is externally indeterminate to degree one. Truss is internally determinate. Select the horizontal reaction at E, R_{Ex} as the redundant. Releasing the redundant (replacing the hinge at E by a roller support) a stable determinate truss is obtained as shown in Fig. 10.2b. The member axial forces and reactions of the released truss are shown in Fig. 10.2b.

Now calculate the displacement Δ_L corresponding to redundant reaction R_{Ex} in the released structure. This can be conveniently done in a table (see Figs. 10.2b, 10.2c and the table). Hence from the table,

$$\Delta_L = \sum P_i (P_v)_i \frac{L_i}{A_i E_i} \tag{1}$$

$$= 15 \times 10^{-4} \quad \text{m}$$

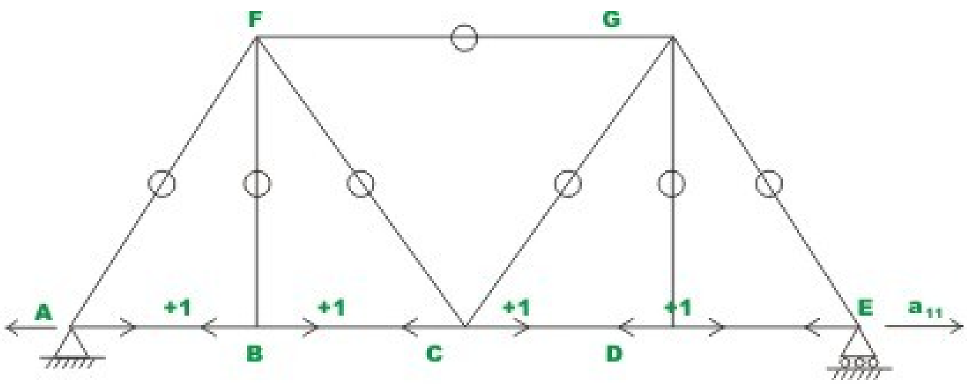


Fig. 10. 2 (c)

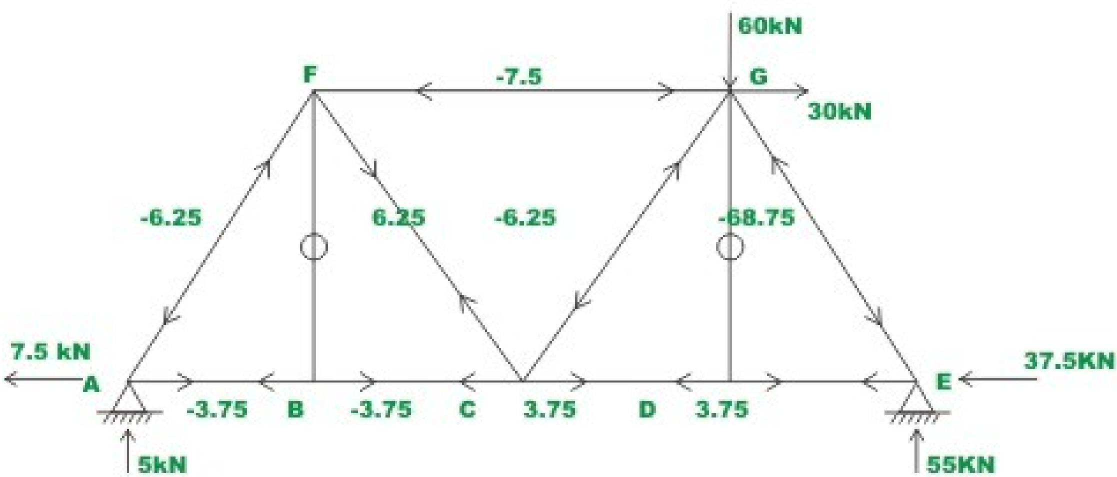


Fig. 10. 2 (d) Plane truss (Example 10.2)

In the next step apply a unit load, along the redundant reaction R_{Ex} and calculate the displacement a_{11} using unit load method.

$$a_{11} = \sum (P_v)_i^2 \frac{L_i}{A_i E_i} \tag{2}$$
$$= 4 \times 10^{-5} \quad \text{m}$$

The support at E is hinged. Hence the total displacement at E must vanish. Thus,

$$\Delta_L + a_{11} F_{AD} = 0 \tag{3}$$

$$15 \times 10^{-4} + 4 \times 10^{-5} R_{Ex} = 0$$

$$R_{Ex} = -\frac{15 \times 10^{-4}}{4 \times 10^{-5}}$$

$$= -37.5 \quad \text{kN(towards left)}$$

The actual member forces and reactions are shown in Fig. 10.2d.

Table 10.2 Numerical computation for example 10.2

Member	L_i	$A_i E_i$	Forces in the released truss due to applied loading P_i	Forces in the released truss due to unit load $(P_v)_i$	$P_i (P_v)_i \frac{L_i}{AE}$	$(P_v)_i^2 \frac{L_i}{A_i E_i}$	$F_i = P_i + F_{AD} (P_v)_i$
	m	(10^5)kN	kN	kN	(10^{-4})m	(10^{-5})m/Kn	kN
AB	3	3	33.75	+1	3.375	1	-3.75
BC	3	3	33.75	+1	3.375	1	-3.75
CD	3	3	41.25	+1	4.125	1	3.75
DE	3	3	41.25	+1	4.125	1	3.75
FG	6	3	-7.50	0	0	0	-7.5
FB	4	2	0.00	0	0	0	0
GD	4	2	0.00	0	0	0	0
AF	5	5	-6.25	0	0	0	-6.25
FC	5	5	6.25	0	0	0	6.25
CG	5	5	-6.25	0	0	0	-6.25
GE	5	5	-68.75	0	0	0	-68.75
				Total	15	4	

Example 10.3

Determine the reactions and the member axial forces of the truss shown in Fig.10.3a by force method due to external load and rise in temperature of member FB by $40°C$. The cross sectional areas of the members in square centimeters are shown in parenthesis. Assume $E = 2.0 \times 10^5 \text{ N/mm}^2$ and $\alpha = \frac{1}{75000}$ per °C.

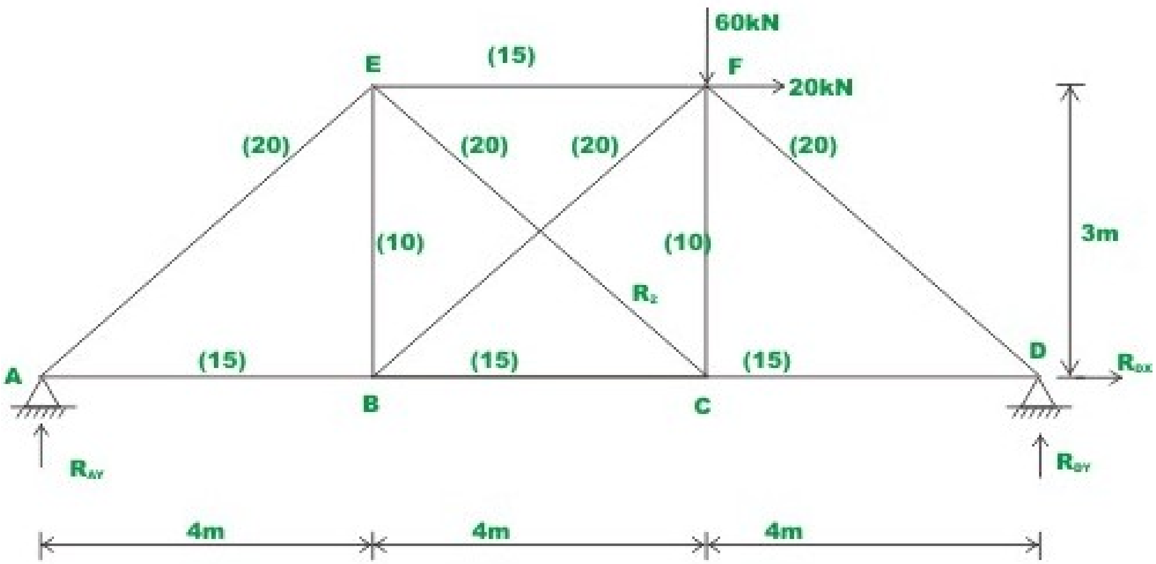


Fig. 10. 3 (a) Plane truss of Example 10.3

The given truss is indeterminate to second degree. The truss has both internal and external indeterminacy. Choose horizontal reaction at $D\,(R_1)$ and the axial force in member $EC\,(R_2)$ as redundant actions. Releasing the restraint against redundant actions, a stable determinate truss is obtained as shown in Fig. 10.3b.

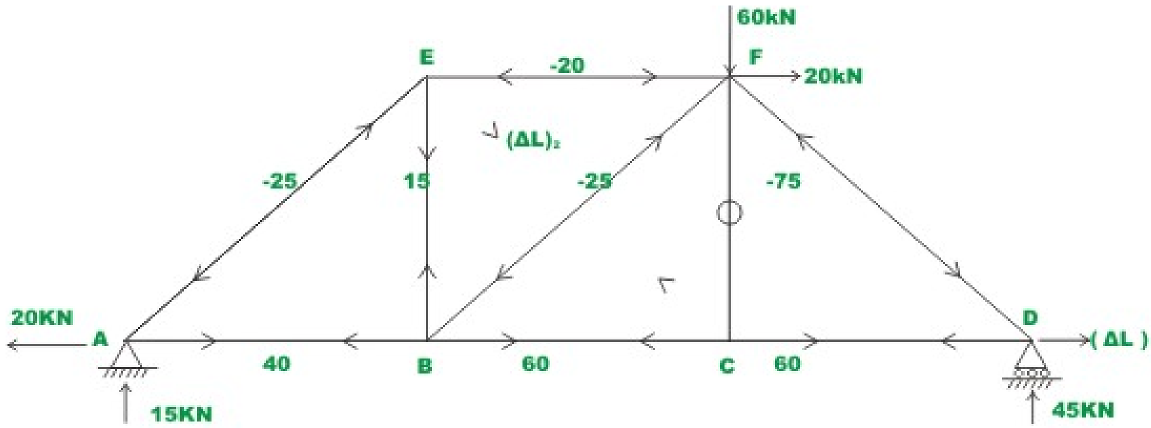


Fig. 10.3 (b)

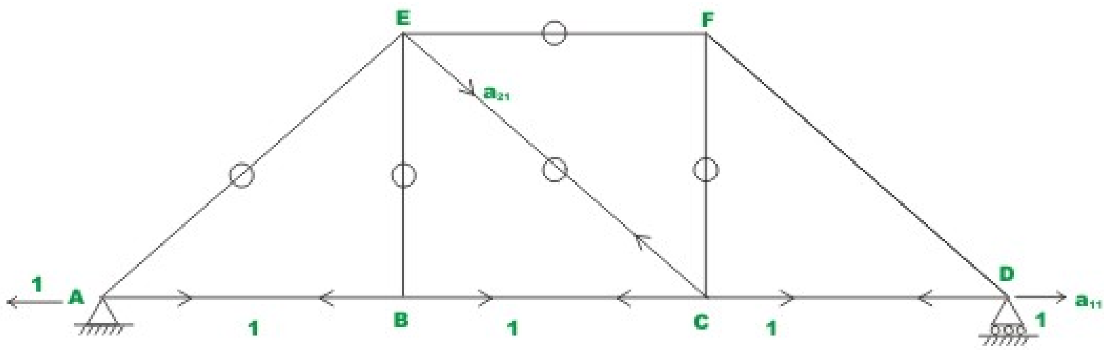


Fig. 10.3 (c)

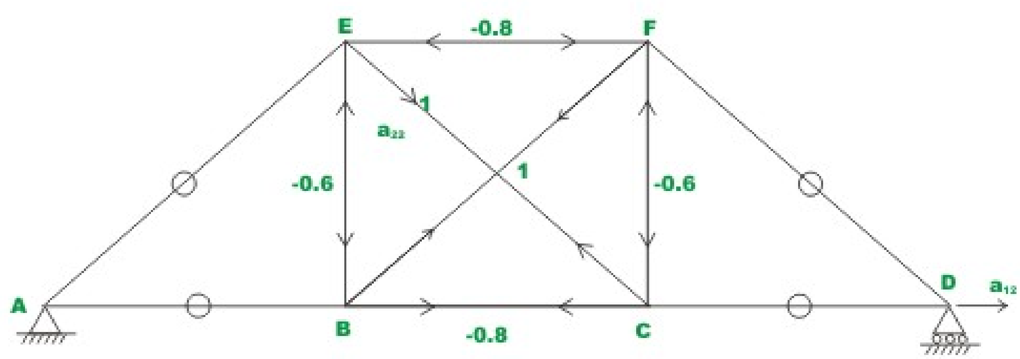


Fig. 10.3 (d)

Table 10.3a Deflection due to external loading

Member	L_i	$A_i E_i$	Forces in the released truss due to applied loading P_i	Forces in the released truss due to unit load $(P_v)_i$	Forces in the released truss due to unit load $(Q_v)_i$	$P_i (P_v)_i \frac{L_i}{AE}$	$P_i (Q_v)_i \frac{L_i}{AE}$
	m	(10^5)kN	kN	kN	kN	(10^{-4})m	(10^{-4})m
AB	4	3	40	+1	0	5.333	0.000
BC	4	3	60	+1	-0.8	8.000	-6.400
CD	4	3	60	+1	0	8.000	0.000
EF	4	3	-20	0	-0.8	0.000	2.133
EB	3	2	15	0	-0.6	0.000	-1.350
FC	3	2	0	0	-0.6	0.000	0.000
AE	5	4	-25	0	0	0.000	0.000
BF	5	4	-25	0	+1	0.000	-3.125
FD	5	4	-75	0	0	0.000	0.000
EC	5	4	0	0	+1	0.000	0.000
				Total		21.333	-8.742

Deflection of the released structure along redundant R_1 and R_2 respectively are,

$$(\Delta_L)_1 = 21.33 \times 10^{-4} \quad \text{m} \quad \text{(towards right)}$$

$$(\Delta_L)_2 = -8.742 \times 10^{-4} \quad \text{m} \quad \text{(shortening)} \qquad (1)$$

In the next step, compute the flexibility coefficients (ref. Fig. 10.3c and Fig. 10.3d and the accompanying table)

Table 10.3b Computation of flexibility coefficients

Member	L_i	A_iE_i	$(P_v)_i$	$(P_v)_i^2 \frac{L_i}{A_iE_i}$	$(Q_v)_i$	$(Q_v)_i^2 \frac{L_i}{A_iE_i}$	$(P_v)_i(Q_v)_i \frac{L_i}{AE}$
	m	(10^5)kN	kN	(10^{-5})m/kN	kN	(10^{-5})m/kN	(10^{-5})m/kN
AB	4	3	+1.00	1.333	0.000	0.000	0.000
BC	4	3	+1.00	1.333	-0.800	0.853	-1.067
CD	4	3	+1.00	1.333	0.000	0.000	0.000
EF	4	3	0	0.000	-0.800	0.853	0.000
EB	3	2	0	0.000	-0.600	0.540	0.000
FC	3	2	0	0.000	-0.600	0.540	0.000
AE	5	4	0	0.000	0.000	0.000	0.000
BF	5	4	0	0.000	1.000	1.250	0.000
FD	5	4	0	0.000	0.000	0.000	0.000
EC	5	4	0	0.000	1.000	1.250	0.000
			Total	4.000		5.286	-1.064

Thus,

$$a_{11} = 4 \times 10^{-5}$$

$$a_{12} = a_{21} = -1.064 \times 10^{-5} \qquad (2)$$

$$a_{22} = 5.286 \times 10^{-5}$$

Analysis of truss when only external loads are acting

The compatibility conditions of the problem may be written as,

$$(\Delta_L)_1 + a_{11}R_1 + a_{12}R_2 = 0$$

$$(\Delta_L)_2 + a_{21}R_1 + a_{22}R_2 = 0 \qquad (3)$$

Solving $R_1 = -51.73 \text{ kN (towards left)}$ and $R_2 = 6.136 \text{ kN (tensile)}$

The actual member forces and reactions in the truss are shown in Fig 10.3c. Now, compute deflections corresponding to redundants due to rise in temperature in the member FB. Due to rise in temperature, the change in length of member FB is,

$$\Delta_T = \alpha T L \tag{4}$$

$$= \frac{1}{75000} \times 40 \times 5 = 2.67 \times 10^{-3}\, m$$

Due to change in temperature, the deflections corresponding to redundants R_1 and R_2 are

$$(\Delta_T)_1 = \sum (P_v)_i (\Delta_T)_i = 0$$

$$(\Delta_T)_2 = \sum (Q_v)_i (\Delta_T)_i = 2.67 \times 10^{-3}\, m \tag{5}$$

When both external loading and temperature loading are acting

When both temperature loading and the external loading are considered, the compatibility equations can be written as,

$$(\Delta_L)_1 + (\Delta_T)_1 + a_{11} R_1 + a_{12} R_2 = 0$$

$$(\Delta_L)_2 + (\Delta_T)_2 + a_{21} R_1 + a_{22} R_2 = 0 \tag{6}$$

Solving $R_1 = -65.92 \text{ kN(towards left)}$ and $R_2 = -47.26 \text{ kN (compressive)}$

The actual member forces and reactions are shown in Fig. 10.3f

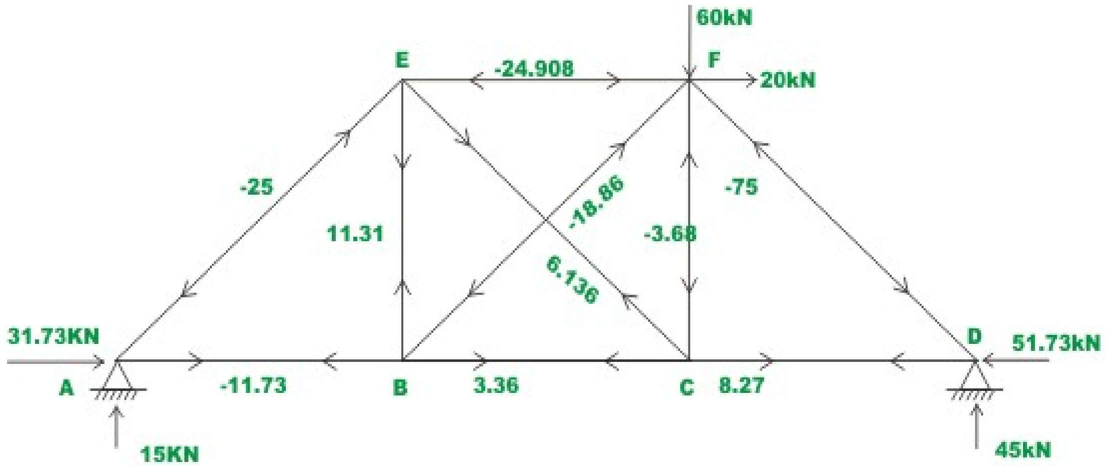


Fig. 10.3 (e)

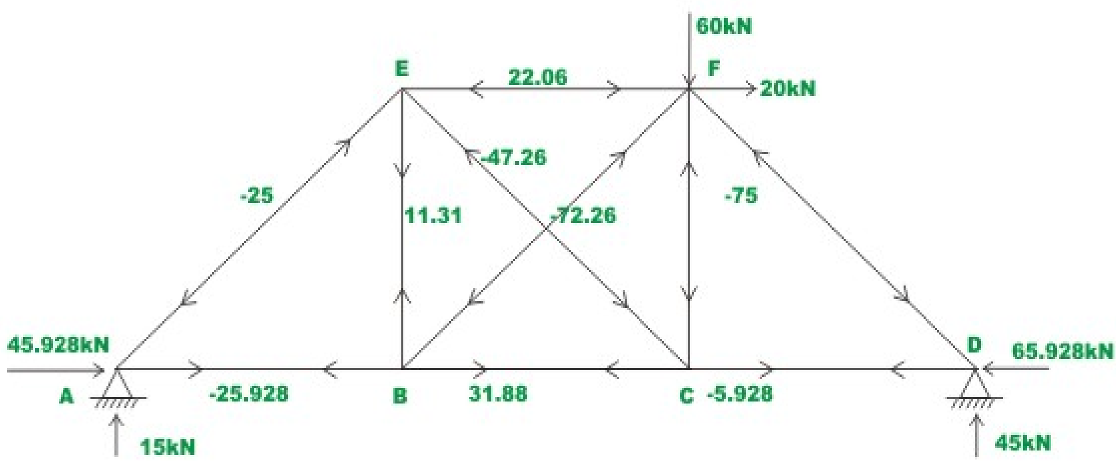


Fig. 10.3 (f)

Fig. 10.3 Plane truss of example 10.3

Summary

In this lesson, the flexibility matrix method is used to analyse statically indeterminate planar trusses. The equation to calculate the degree of statical indeterminacy of a planar truss is derived. The forces induced in the members due to temperature loading and member lack of fit is also discussed in this lesson. Few examples are solved to illustrate the force method of analysis as applied to statically indeterminate planar trusses.

Module
2
Analysis of Statically Indeterminate Structures by the Matrix Force Method

Lesson 11

The Force Method of Analysis: Frames

Instructional Objectives

After reading this chapter the student will be able to

1. Analyse the statically indeterminate plane frame by force method.
2. Analyse the statically indeterminate plane frames undergoing support settlements.
3. Calculate the static deflections of a primary structure (released frame) under external loads.
4. Write compatibility equations of displacements for the plane deformations.
5. Compute reaction components of the indeterminate frame.
6. Draw shear force and bending moment diagrams for the frame.
7. Draw qualitative elastic curve of the frame.

11.1 Introduction

The force method of analysis can readily be employed to analyze the indeterminate frames. The basic steps in the analysis of indeterminate frame by force method are the same as that discussed in the analysis of indeterminate beams in the previous lessons. Under the action of external loads, the frames undergo axial and bending deformations. Since the axial rigidity of the members is much higher than the bending rigidity, the axial deformations are much smaller than the bending deformations and are normally not considered in the analysis. The compatibility equations for the frame are written with respect to bending deformations only. The following examples illustrate the force method of analysis as applied to indeterminate frames.

Example 11.1

Analyse the rigid frame shown in Fig.11.1a and draw the bending moment diagram. Young's modulus E and moment of inertia I are constant for the plane frame. Neglect axial deformations.

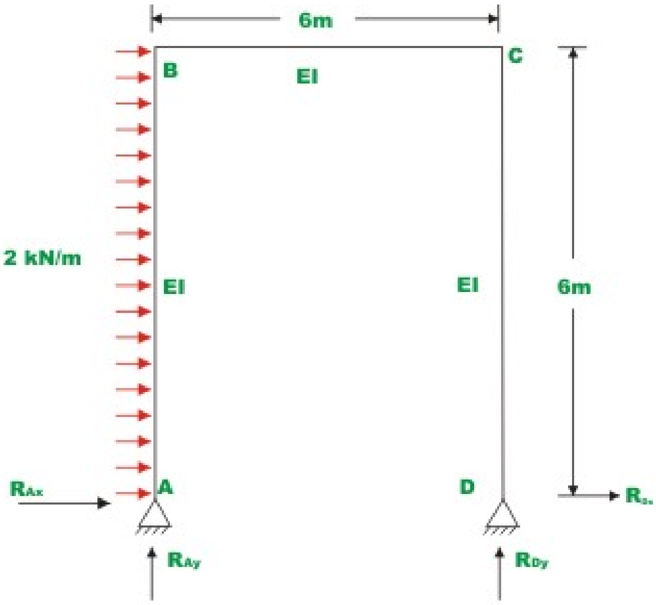


Fig 11.1 (a) Example 11.1

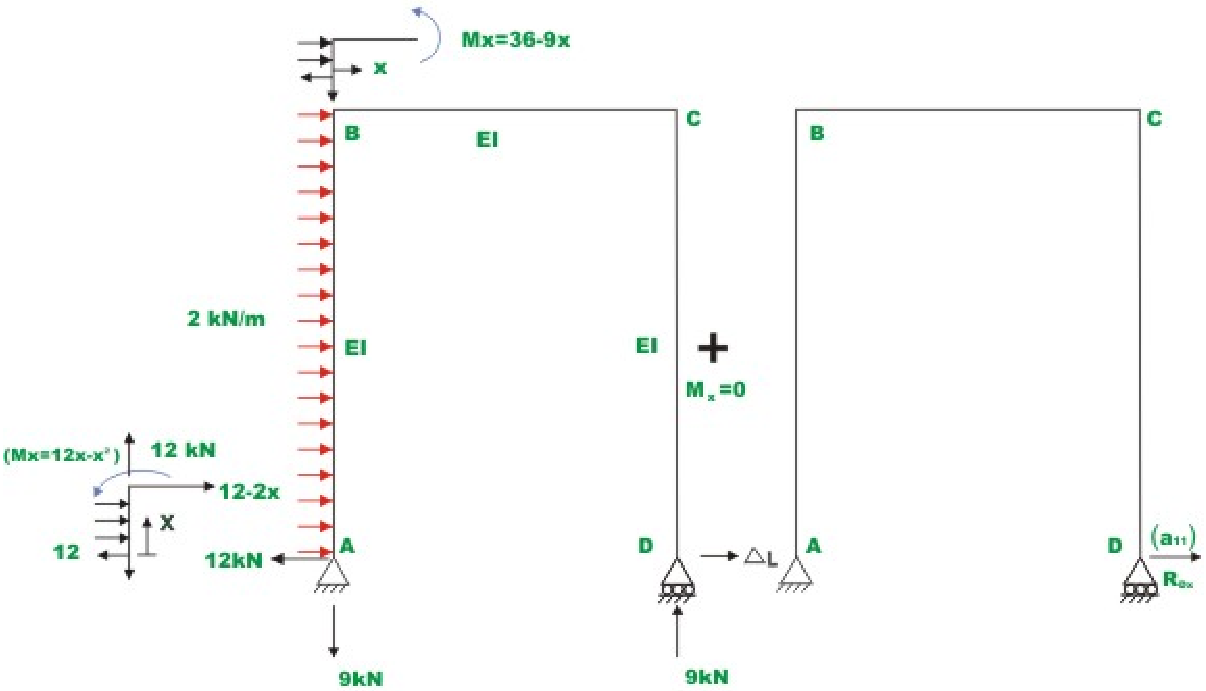


Fig 11.1 (b) Primary Structure

Fig 11.1 (c)Primary Structure with redundant R_{dx}

The given one- storey frame is statically indeterminate to degree one. In the present case, the primary structure is one that is hinged at A and roller supported at D. Treat horizontal reaction at D, R_{Dx} as the redundant. The primary structure (which is stable and determinate) is shown in Fig.11.1.b.The compatibility condition of the problem is that the horizontal deformation of the primary structure (Fig.11.1.b) due to external loads plus the horizontal deformation of the support D, due to redundant

R_{Dx} (vide Fig.11.1.b) must vanish. Calculate deformation a_{11} due to unit load at D in the direction of R_{Dx}. Multiplying this deformation a_{11} with R_{Dx}, the deformation due to redundant reaction can be obtained.

$$\Delta = a_{11} R_{Dx} \tag{1}$$

Now compute the horizontal deflection Δ_L at D due to externally applied load. This can be readily determined by unit load method. Apply a unit load along R_{Dx} as shown in Fig.10.1d.

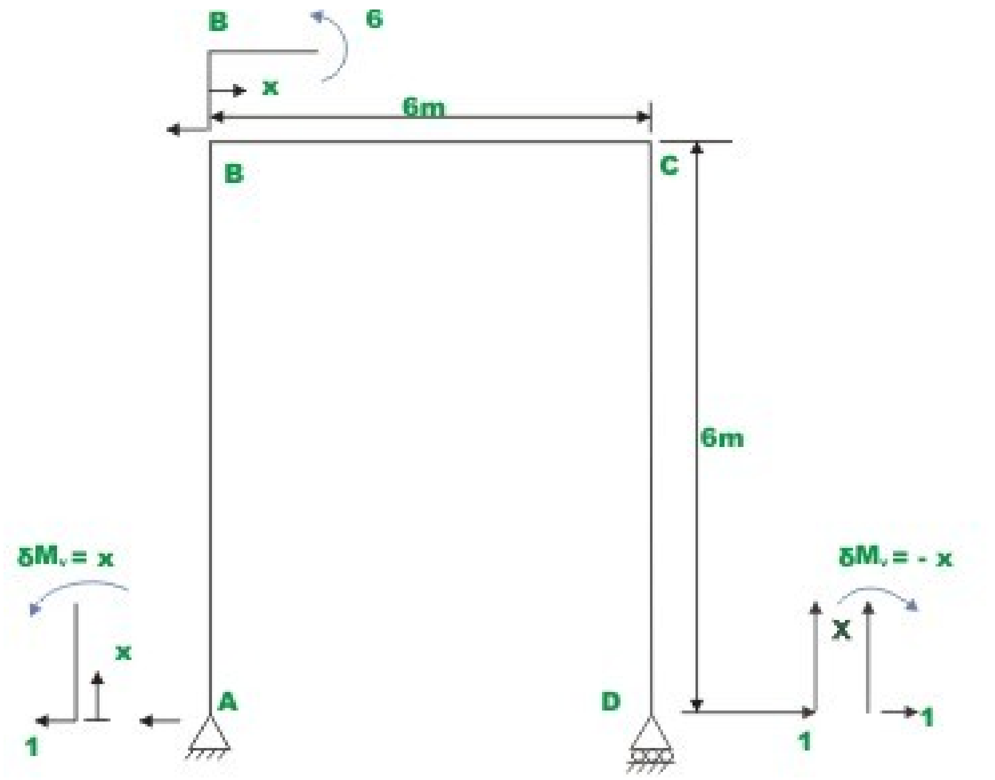


Fig 11.1 (d) Primary Structure with unit load along R_{dx}

The horizontal deflection Δ_L at D in the primary structure due to external loading is given by

$$\Delta_L = \int_A^D \frac{\delta M_v M}{EI} dx \tag{2}$$

where δM_v is the internal virtual moment resultant in the frame due to virtual load applied at D along the resultant R_{Dx} and M is the internal bending moment of the frame due to external loading (for details refer to Module 1,Lesson 5).Thus,

$$\Delta_L = \int_0^6 \frac{\left(12x - x^2\right)x}{EI} dx \quad + \int_0^6 \frac{(36 - 9x)6}{EI} dx + \quad \int_0^6 \frac{0(x)}{EI} dx$$

(*span AB, origin at A*) (*span BC, origin at B*) (*span DC, origin at* D)

$$\Delta_L = \frac{864}{EI} \tag{3}$$

In the next step, calculate the displacement a_{11} at D when a real unit load is applied at D in the direction of R_{Dx} (refer to Fig.11.1 d). Please note that the same Fig. 11.1d is used to represent unit virtual load applied at D and real unit load applied at D. Thus,

$$a_{11} = \int_A^D \frac{\delta m_v m}{EI} dx$$

$$= \int_0^6 \frac{x^2 dx}{EI} + \int_0^6 \frac{36 dx}{EI} + \int_0^6 \frac{x^2}{EI} dx$$

$$= \frac{360}{EI} \tag{4}$$

Now, the compatibility condition of the problem may be written as

$$\Delta_L + a_{11} R_{Dx} = 0 \tag{5}$$

Solving,

$$R_{Dx} = -2.40 \text{ kN} \tag{6}$$

The minus sign indicates that the redundant reaction R_{Dx} acts towards left. Remaining reactions are calculated from equations of static equilibrium.

$$\sum F_x = 0 \Rightarrow R_{Ax} = -12 + 2.40 = -9.60 \quad \text{kN} \quad \text{(towards left)}$$

$$\sum M_D = 0 \Rightarrow R_{Ay} = -9 \text{ kN} \quad \text{(dowwards)}$$

$$R_{Dy} = +9 \text{ kN} \qquad \text{(upwards)}$$

The bending moment diagram for the frame is shown in Fig. 11.1e

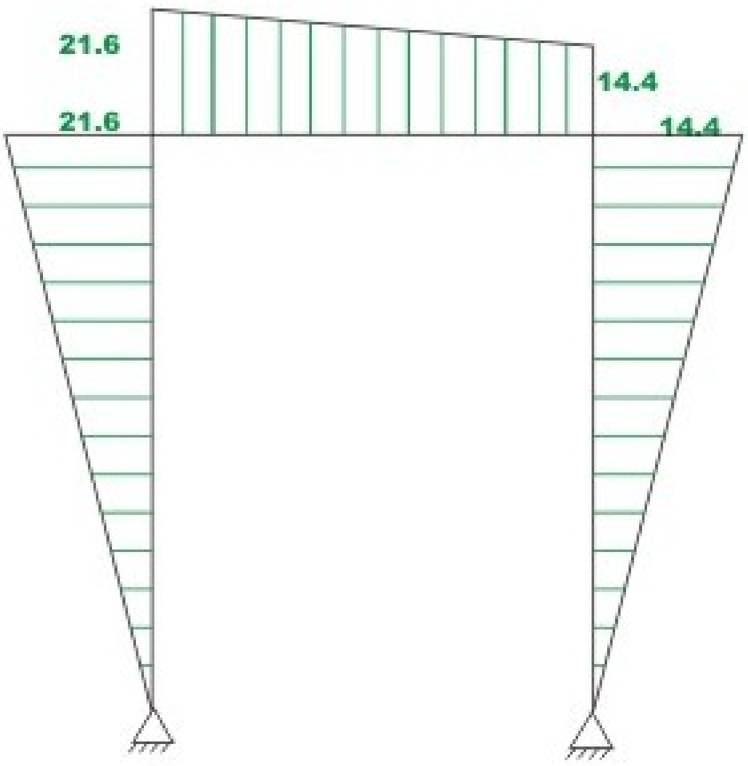


Fig 11.1 (e) Bending moment diagram

Example 11.2

Analyze the rigid frame shown in Fig.11.2a and draw the bending moment and shear force diagram. The flexural rigidity for all members is the same. Neglect axial deformations.

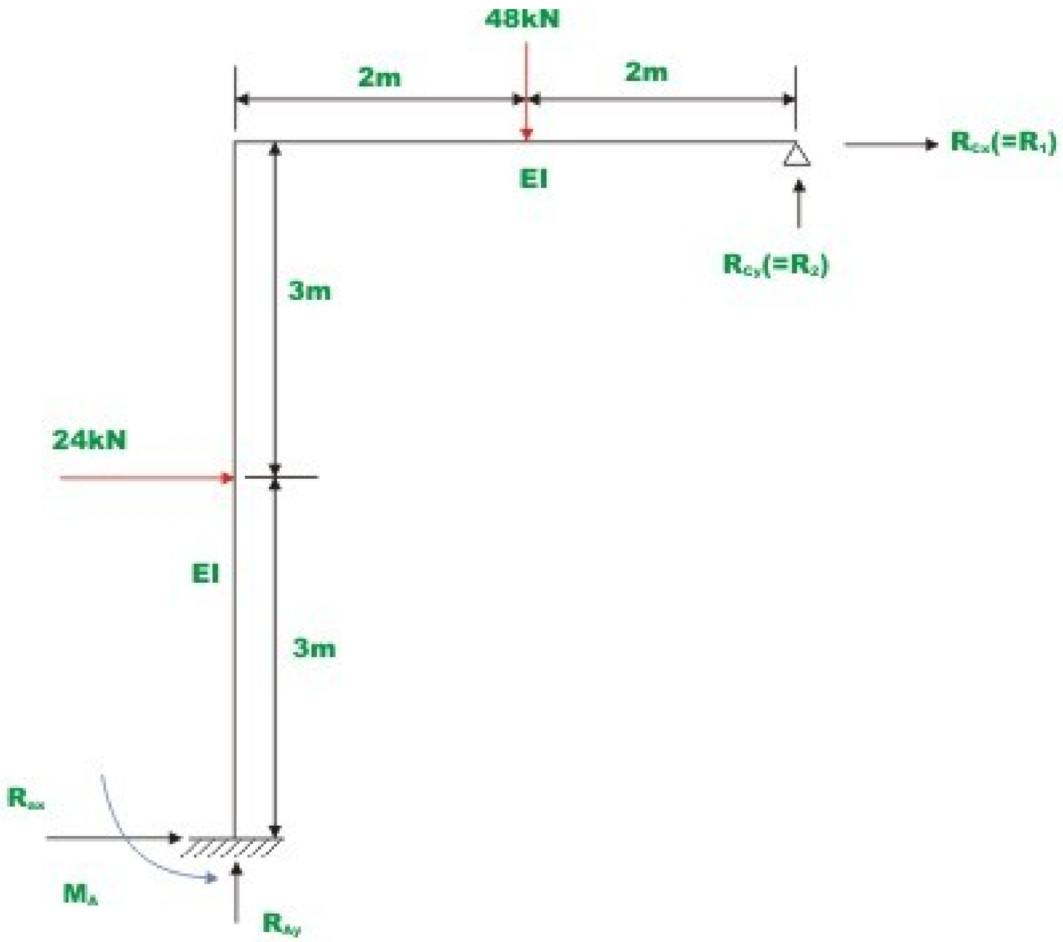


Fig 11.2 (a) Example 11.2

Five reactions components need to be evaluated in this rigid frame; hence it is indeterminate to second degree. Select $R_{cx}(=R_1)$ and $R_{cy}(=R_2)$ as the redundant reactions. Hence, the primary structure is one in which support A is fixed and the support C is free as shown in Fig.11.2b. Also, equations for moments in various spans of the frame are also given in the figure.

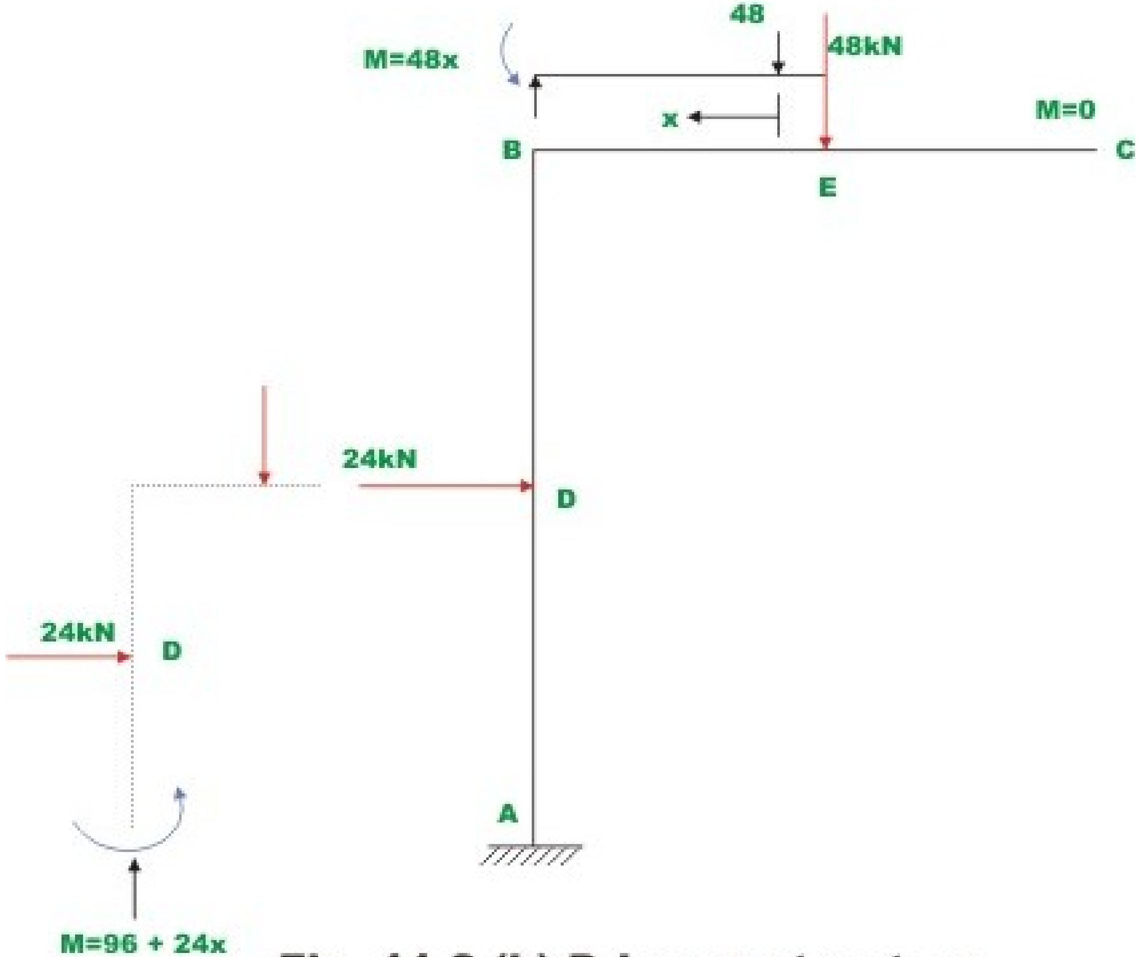


Fig 11.2 (b) Primary structure with external load

Calculate horizontal $(\Delta_L)_1$ and vertical $(\Delta_L)_2$ deflections at C in the primary structure due to external loading. This can be done by unit load method. Thus,

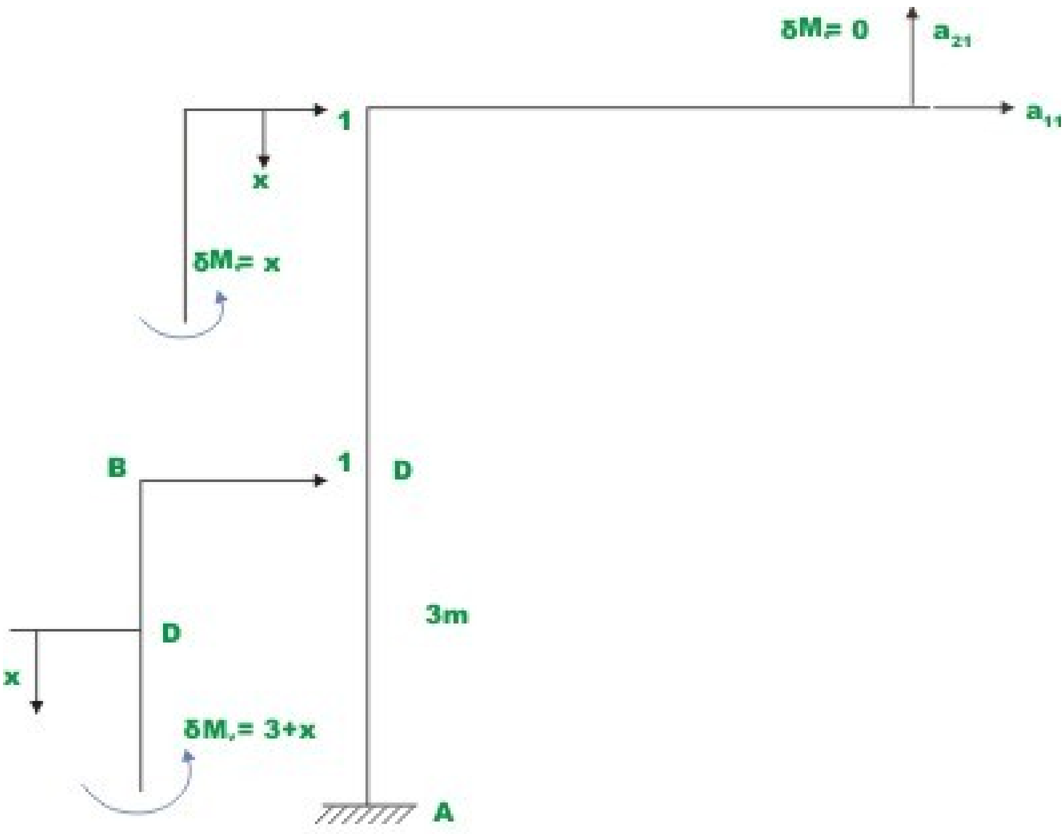


Fig 11.2(c) Primary structure with unit load along R_1

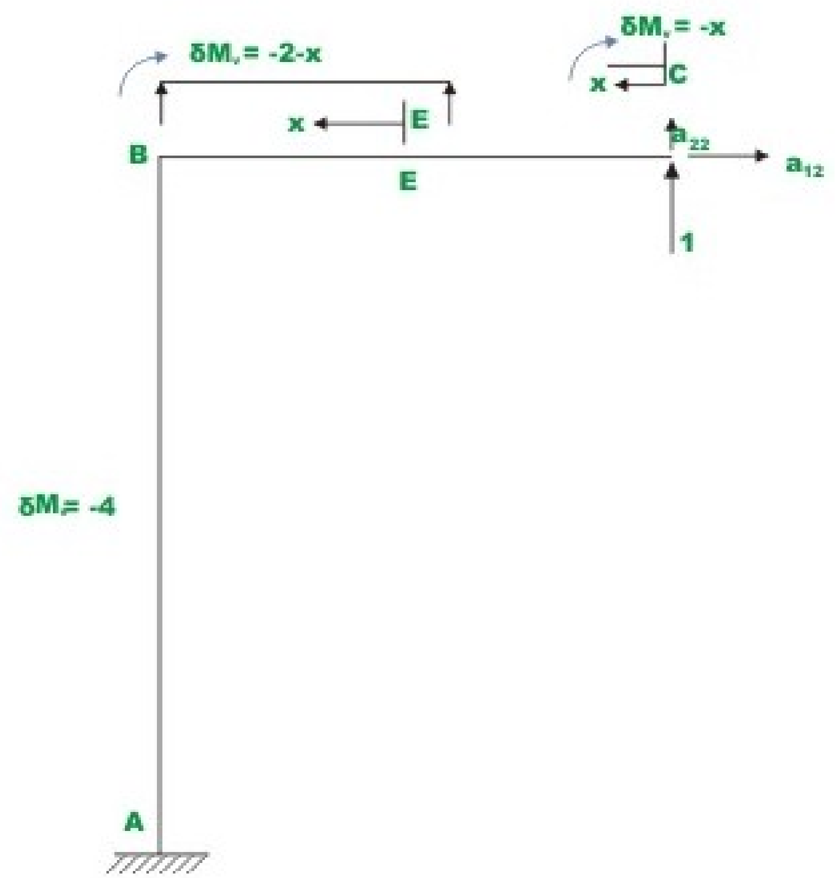


Fig 11.2 (d) Primary structure with unit load along R_2

$$(\Delta_L)_1 = \int_0^3 \frac{(96+24x)(3+x)}{EI}dx \qquad + \int_0^3 \frac{96x}{EI}dx \qquad + 0$$

(*Span DA, origin at D*) (*Span BD, origin at B*) (*span BC, Origin B*)

$$= \frac{2268}{EI} \qquad (1)$$

$$(\Delta_L)_2 = \int_0^3 \frac{(96+24x)(-4)dx}{EI} \qquad + \int_0^3 \frac{96(-4)dx}{EI} \qquad + \int_0^2 \frac{48x(-2-x)dx}{EI} \qquad + 0$$

(Span DA, origin at D) *(Span BD, origin at B)* *(Span BE, origin at E)* *(Span EC, Origin C)*

$$(\Delta_L)_2 = -\frac{3056}{EI} \qquad (2)$$

In the next step, evaluate flexibility coefficients, this is done by applying a unit load along, R_1 and determining deflections a_{11} and a_{21} corresponding to R_1 and R_2 respectively (vide, Fig .11.2 c). Again apply unit load along R_2 and evaluate deflections a_{22} and a_{12} corresponding to R_2 and R_1 and respectively (ref. Fig.11.2d).

$$a_{11} = \int_0^6 \frac{x^2}{EI} dx = \frac{72}{EI} \qquad (3)$$

$$a_{12} = a_{21} = \int_0^6 \frac{x(-4)}{EI} dx + 0$$

$$= \frac{72}{EI} \qquad (4)$$

and

$$a_{22} = \int_0^6 \frac{16}{EI} dx + \int_0^4 \frac{(x)^2}{EI} dx$$

$$= \frac{117.33}{EI} \qquad (5)$$

In the actual structure at *C*, the horizontal and vertical displacements are zero .Hence, the compatibility condition may be written as,

$$(\Delta_L)_1 = a_{11}R_1 + a_{12}R_2 = 0$$

$$(\Delta_L)_2 = a_{12}R_1 + a_{22}R_2 = 0 \qquad (6)$$

Substituting the values of $(\Delta_L)_1, (\Delta_l)_2$, a_{11}, a_{12} and a_{22} in the above equations and solving for and R_1 , R_2 we get

$$R_1 = -1.056 \text{ kN } (\textit{towards left})$$
$$R_2 = 27.44 \text{ kN } (\textit{upwards})$$

The remaining reactions are calculated from equations of statics and they are shown in Fig 11.2e. The bending moment and shear force diagrams are shown in Fig. 11.2f.
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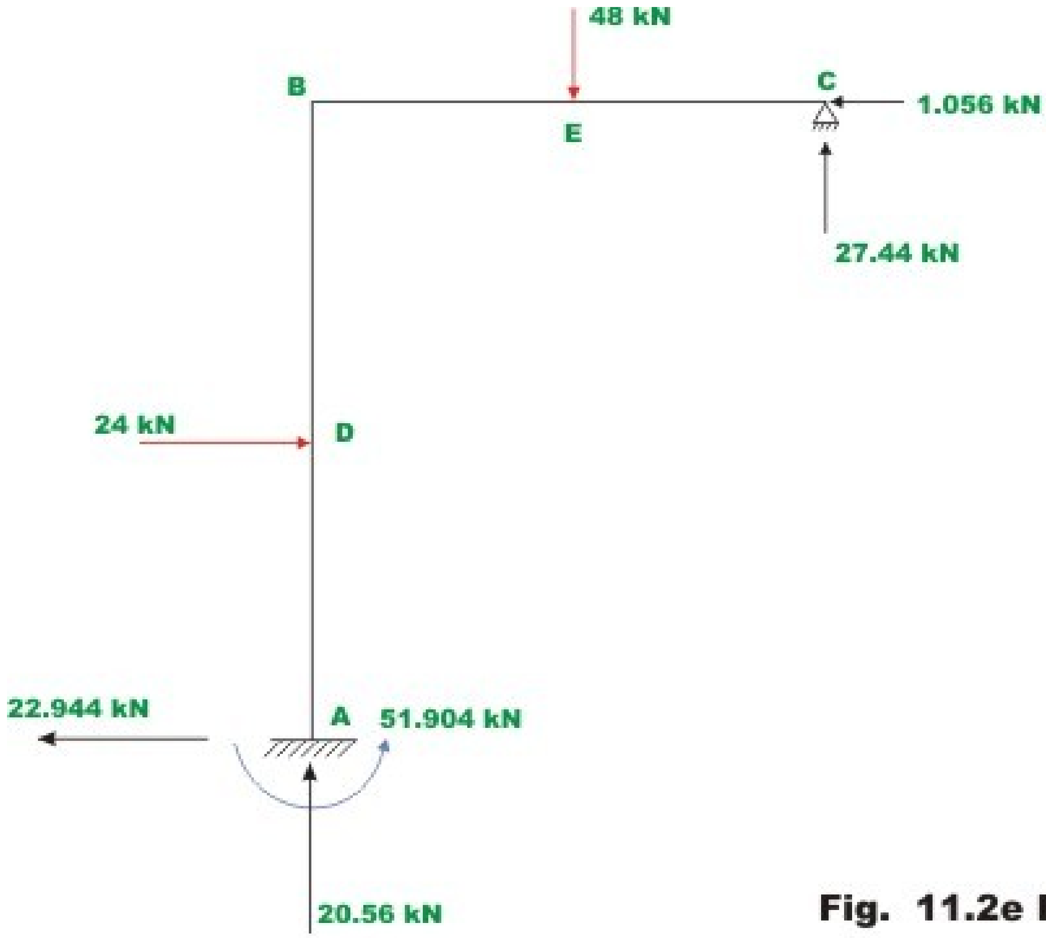


Fig. 11.2e Reaction

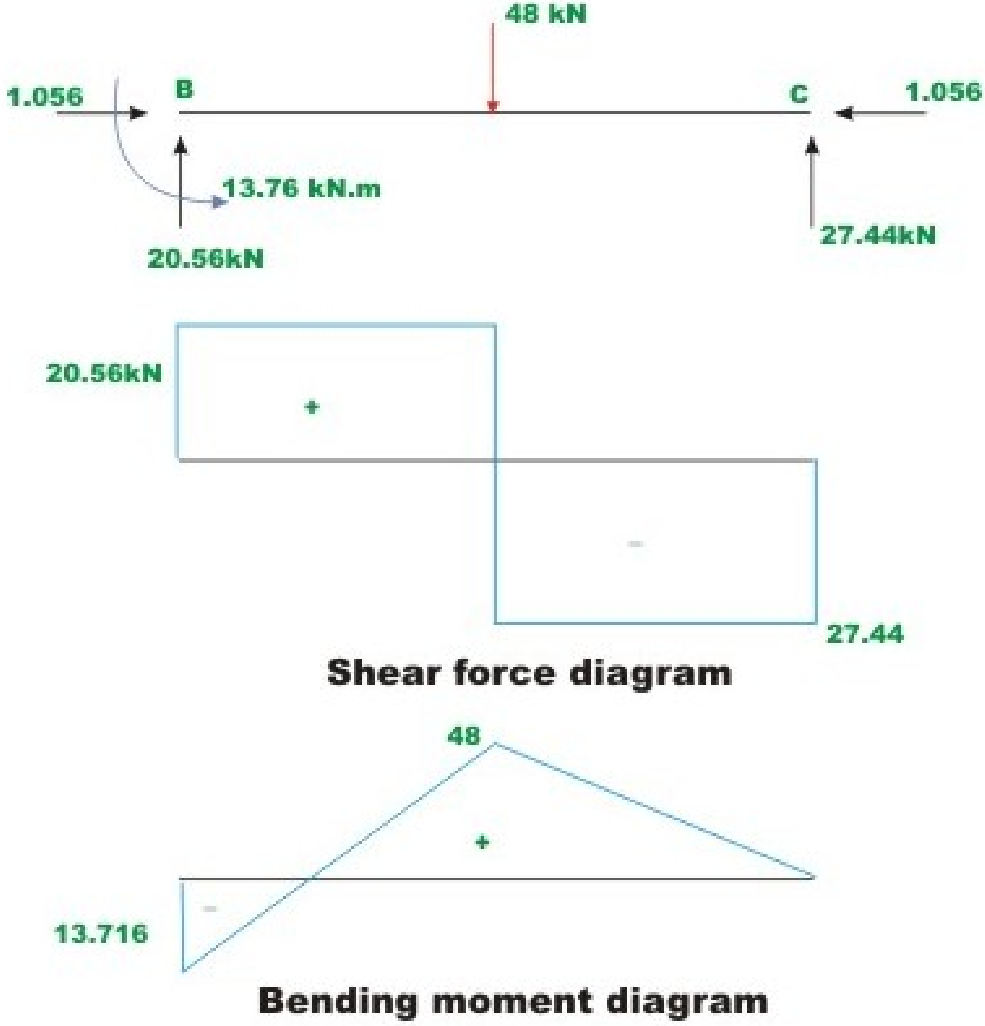


Fig. 11.2f

11.2 Support settlements

As discussed in the case of statically indeterminate beams, the reactions are induced in the case of indeterminate frame due to yielding of supports even when there are no external loads acting on it. The yielding of supports may be either linear displacements or rotations of supports (only in the case of fixed supports) .The compatibility condition is that the total displacement of the determinate frame (primary structure) due to external loading and that due to redundant reaction at a given support must be equal to the predicted amount of yielding at that support. If the support is unyielding then it must be equal to zero.

Example 11.3

A rigid frame ABC is loaded as shown in the Fig 11.3a, Compute the reactions if the support D settles by *10 mm.* vertically downwards. Assume EI to be constant for all members. Assume $E = 200\ GPa$ *and* $I = 10^{-4}\ m^2$.

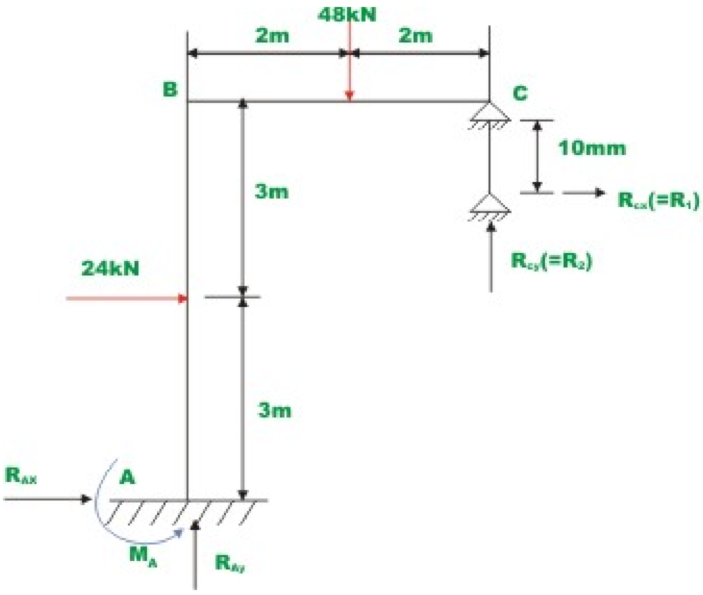
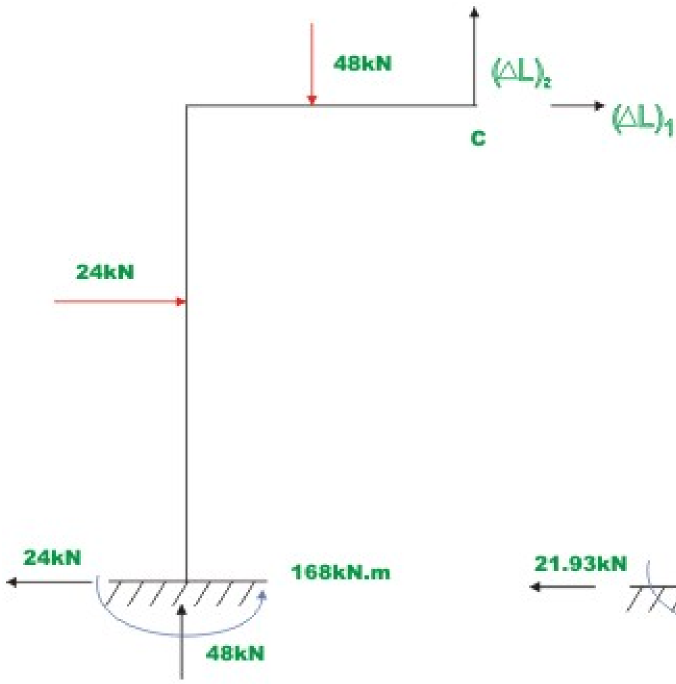


Fig 11.3(a) Example 11.3



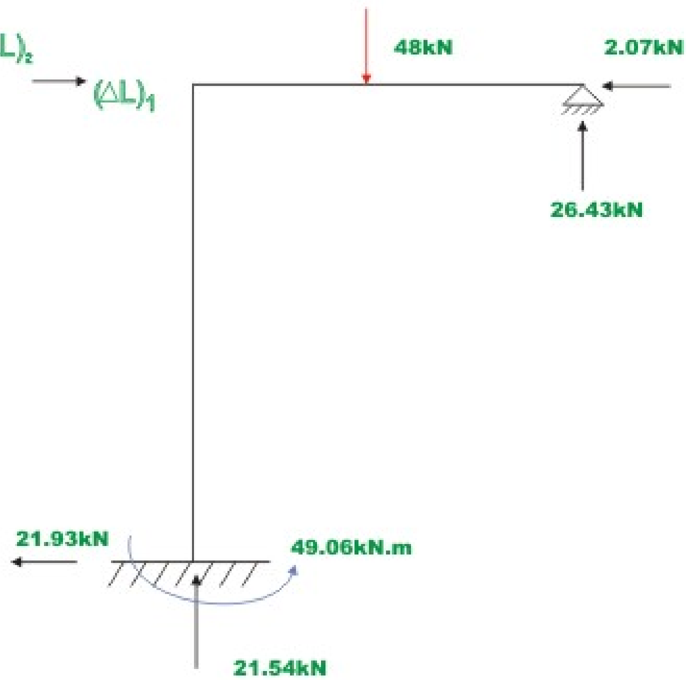


Fig 11.3 (b) Primary Structure **Fig 11.3 (c) Reactions**

This problem is similar to the previous example except for the support settlement .Hence only change will be in the compatibility equations. The released structure is as shown in Fig.11.3b .The deflections $(\Delta_L)_1$ and $(\Delta_L)_2$ at *C* in the primary structure due to external loading has already been computed in the previous example. Hence,

$$(\Delta_L)_1 = \frac{2052}{EI} \tag{1}$$

$$(\Delta_L)_2 = \frac{-3296}{EI} \quad (2)$$

Therefore,

$$(\Delta_L)_1 = 0.1026 \ m$$

$$(\Delta_L)_2 = -0.1635 \ m$$

The flexibility coefficients are,

$$a_{11} = \frac{72}{EI} \quad (3)$$

$$a_{12} = a_{21} = \frac{-72}{EI} \quad (4)$$

$$a_{22} = \frac{117.33}{EI} \quad (5)$$

Now, the compatibility equations may be written as,

$$(\Delta_L)_1 + a_{11}R_1 + a_{12}R_2 = 0$$

$$(\Delta_L)_2 + a_{21}R_1 + a_{22}R_2 = -10 \times 10^{-3} \quad (6)$$

Solving which,

$R_1 = -2.072$ kN *(towards left)*

$R_2 = +26.4$ kN *(upwards)* (7)

The reactions are shown in Fig.11.3c.

Example 11.4

Compute the reactions of the rigid frame shown in Fig.11.4a and draw bending moment diagram .Also sketch the deformed shape of the frame. Assume EI to be constant for all members.

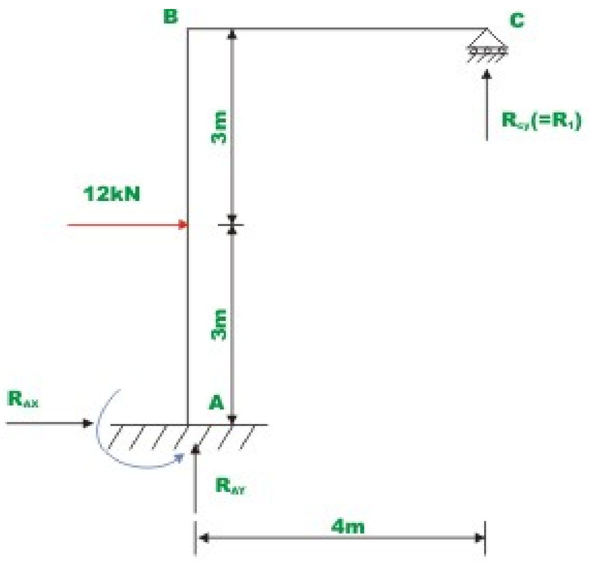
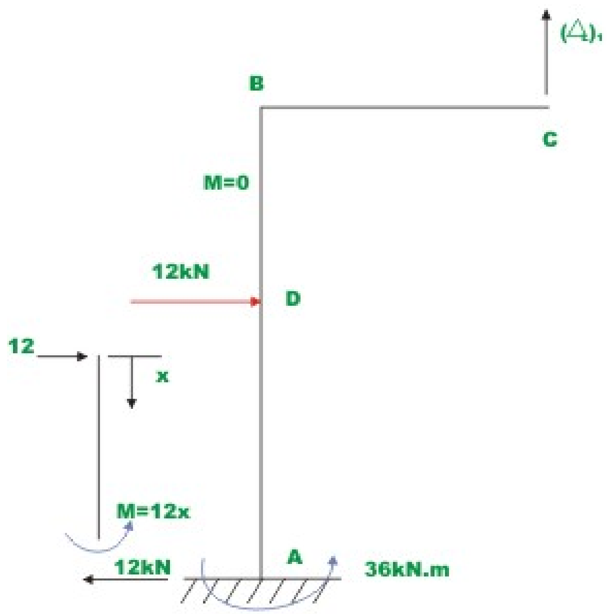


Fig 11.4 (a) Example 11.4



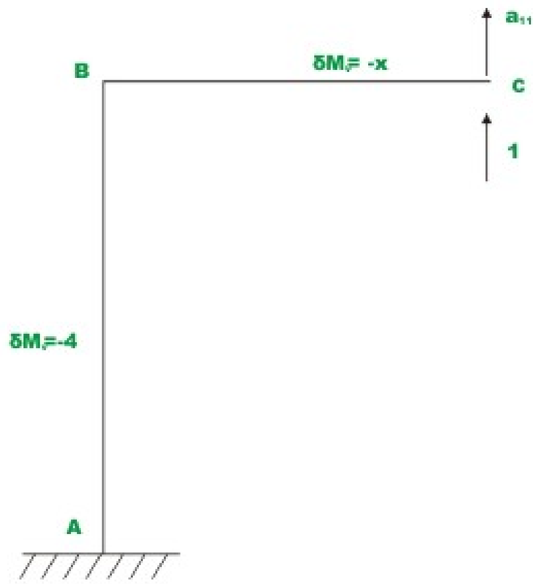


Fig 11.4 (b) Primary structure with external load **Fig 11.4 (c) Primary structure with unit load along R_1**

Select vertical reaction at C, R_1 as the redundant .Releasing constraint against redundant, the primary structure is obtained. It is shown in Fig.11.4b.

The deflection $(\Delta_L)_1$ in the primary structure due to external loading can be calculated from unit load method.

$$(\Delta_L)_1 = \int_0^3 \frac{(12x)(-4)dx}{EI}$$

(span DA, origin at D)

$$= \frac{-216}{EI} \; (\textit{Downwards}) \qquad (1)$$

Now, compute the flexibility coefficient,

$$a_{11} = \int_0^4 \frac{x^2}{EI} dx + \int_0^6 \frac{16}{EI} dx$$

$$= \frac{117.33}{EI} \qquad (2)$$

The compatibility condition at support *C* is that the displacement at *C* in the primary structure due to external loading plus the displacement at *C* due to redundant must vanish. Thus,

$$\frac{-216}{EI} + \frac{117.33}{EI} R_1 = 0 \qquad (3)$$

Solving,

$$R_1 = 1.84 \text{ kN} \qquad (4)$$

The remaining reactions are calculated from static equilibrium equations. They are shown in Fig.11.4d along with the bending moment diagram.

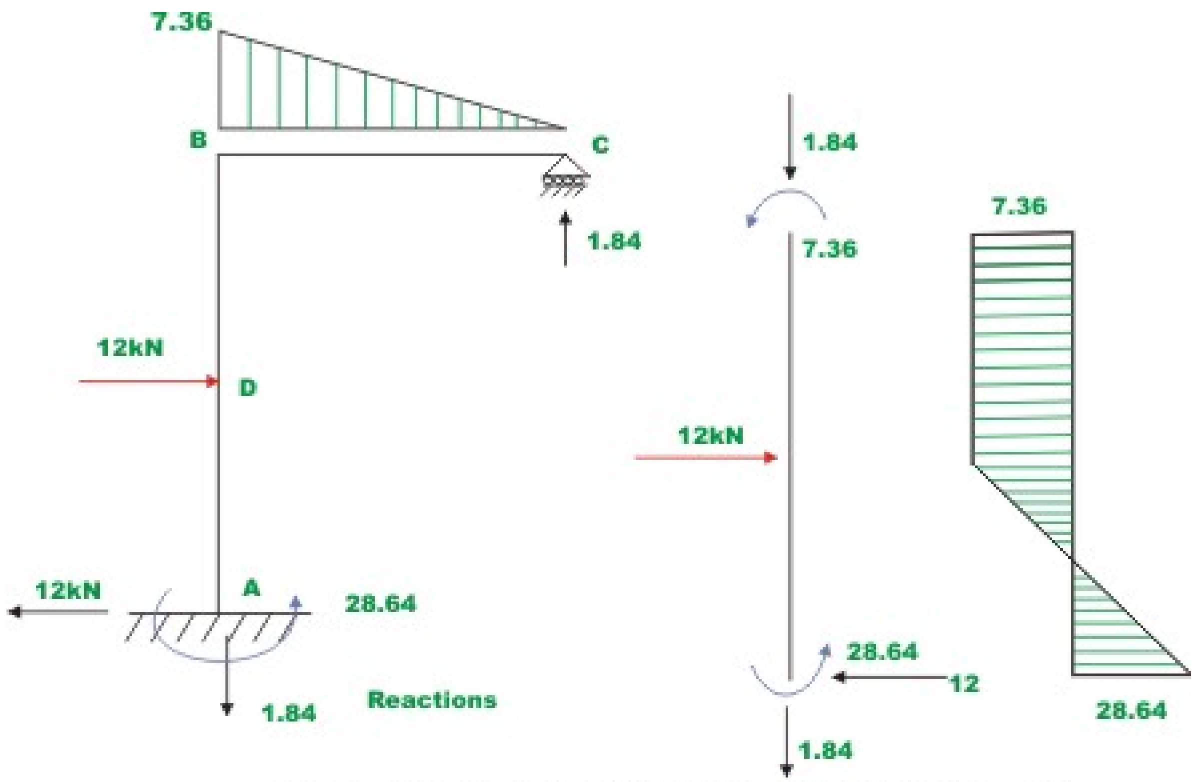


Fig 11.4 (d) bending moment diagram plotted on Compression side

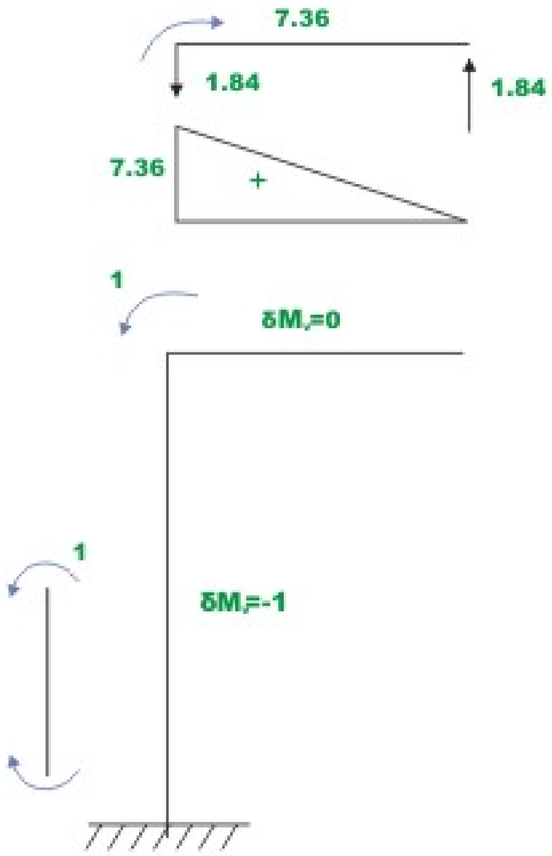


Fig 11.4(e)

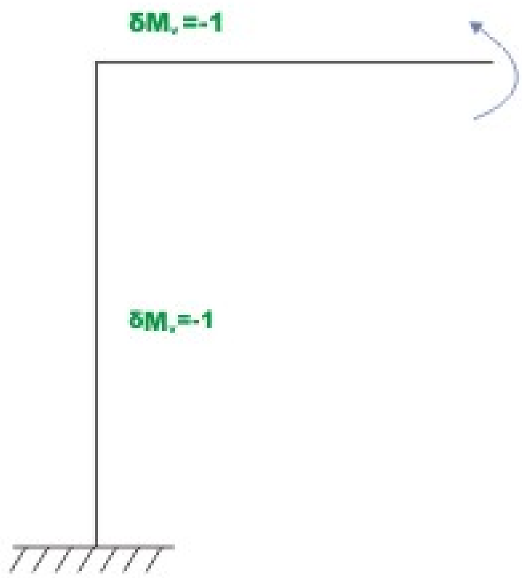


Fig 11.4(f)

To sketch the deformed shape/elastic curve of the frame, it is required to compute rotations of joints B and C and horizontal displacement of C. These joint rotations and displacements can also be calculated from the principle of superposition .The joint rotations are taken to be positive when clockwise. Towards this end first calculate joint rotations at $B\,(\theta_{BL})$ and $C\,(\theta_{CL})$ and horizontal displacement at C in the released structure (refer to Fig.11.4b).This can be evaluated by unit load method.

$$\theta_{BL} = \int_0^3 \frac{12(x)(-1)}{EI} dx = \frac{-54}{EI} \tag{5}$$

$$\theta_{CL} = \int_0^3 \frac{(12x)(-1)}{EI} dx = \frac{-54}{EI} \tag{6}$$

$$\Delta_{CL} = \int_0^3 \frac{12x(3+x)}{EI} dx = \frac{270}{EI} \tag{7}$$

Next, calculate the joint rotations and displacements when unit value of redundant is applied (Fig.11.4c). Let the joint rotations and displacements be θ_{BR}, θ_{CR} and Δ_{CR}.

$$\theta_{BR} = \int_0^6 \frac{4dx}{EI} = \frac{24}{EI} \tag{8}$$

$$\theta_{CR} = \int_0^4 \frac{(-x)(-1)}{EI} dx + \int_0^6 \frac{(-4)(-1)}{EI} dx = \frac{32}{EI} \tag{9}$$

$$\Delta_{CR} = \int_0^6 \frac{(-4)x}{EI} dx = \frac{-72}{EI} \tag{10}$$

Now using the principle of superposition, the actual rotations and displacements at the joints may be obtained.

$$\theta_B = \theta_{BL} + \theta_{BR} R_1 \tag{11}$$

$$= \frac{-54}{EI} + \frac{24 \times 1.84}{EI} = -\frac{9.84}{EI}$$

(Clockwise rotation)

$$\theta_C = \theta_{CL} + \theta_{CR} \times R_1 \tag{12}$$

$$= \frac{-54}{EI} + \frac{32 \times 1.84}{EI} = \frac{4.88}{EI}$$

(Counterclockwise rotation)

$$\Delta_C = \Delta_{CL} + \Delta_{CR} R_1 \tag{13}$$

$$= \frac{270}{EI} - \frac{72 \times 1.84}{EI} = \frac{137.52}{EI} \quad \textit{(towards right)}$$

The qualitative elastic curve is shown in Fig. 11.4h.

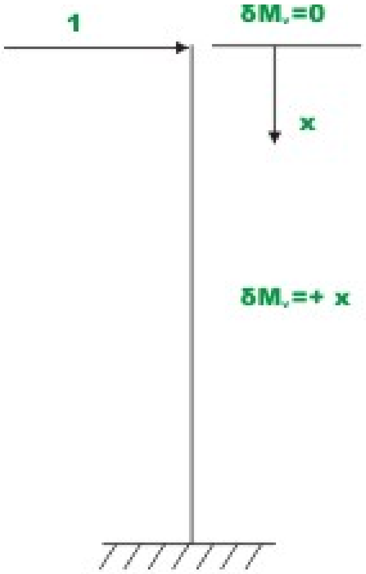


Fig 11.4 (g)

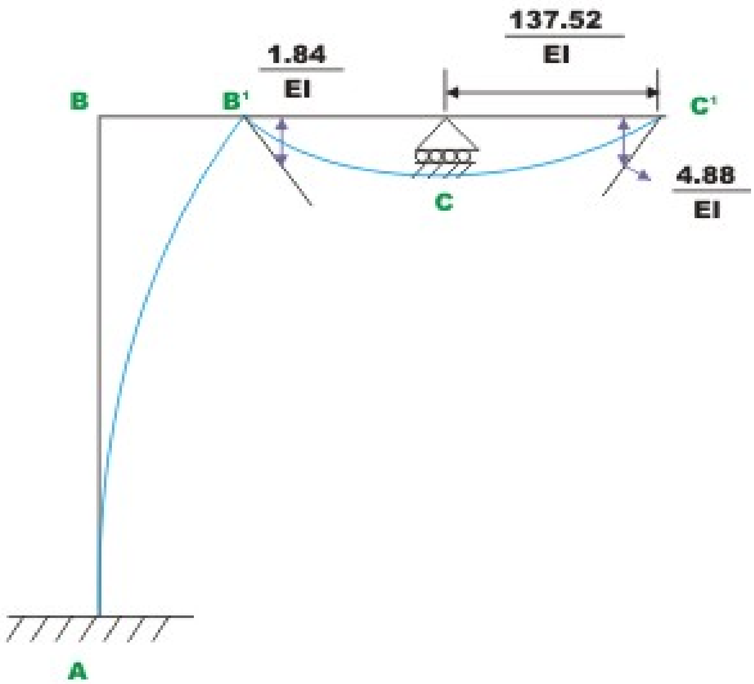


Fig 11.4 (h) Deformed shape

Summary

In this lesson, the statically indeterminate plane frames are analysed by force method. For the purpose of illustrations only bending deformations of the frame are considered as the axial deformations are very small. The problem of yielding of supports in the case of plane frames is also discussed. The procedure to draw qualitative elastic curve of the frame is illustrated with the help of typical example. The bending moment and shear force diagrams are also drawn for the case of plane frame.

Module 2

Analysis of Statically Indeterminate Structures by the Matrix Force Method

Lesson 12

The Three-Moment Equations-I

Instructional Objectives

After reading this chapter the student will be able to

1. Derive three-moment equations for a continuous beam with unyielding supports.
2. Write compatibility equations of a continuous beam in terms of three moments.
3. Compute reactions in statically indeterminate beams using three-moment equations.
4. Analyse continuous beams having different moments of inertia in different spans using three-moment equations.

12.1 Introduction

Beams that have more than one span are defined as continuous beams. Continuous beams are very common in bridge and building structures. Hence, one needs to analyze continuous beams subjected to transverse loads and support settlements quite often in design. When beam is continuous over many supports and moment of inertia of different spans is different, the force method of analysis becomes quite cumbersome if vertical components of reactions are taken as redundant reactions. However, the force method of analysis could be further simplified for this particular case (continuous beam) by choosing the unknown bending moments at the supports as unknowns. One compatibility equation is written at each intermediate support of a continuous beam in terms of the loads on the adjacent span and bending moment at left, center (the support where the compatibility equation is written) and rigid supports. Two consecutive spans of the continuous beam are considered at one time. Since the compatibility equation is written in terms of three moments, it is known as the equation of three moments. In this manner, each span is treated individually as a simply supported beam with external loads and two end support moments. For each intermediate support, one compatibility equation is written in terms of three moments. Thus, we get as many equations as there are unknowns. Each equation will have only three unknowns. It may be noted that, Clapeyron first proposed this method in 1857. In this lesson, three moment equations are derived for unyielding supports and in the next lesson the three moment equations are modified to consider support moments.

12.2 Three-moment equation

A continuous beam is shown in Fig.12.1a. Since, three moment equation relates moments at three successive supports to applied loading on adjacent spans, consider two adjacent spans of a continuous beam as shown in Fig.12.1b. M_L, M_C and M_R respectively denote support moments at left, center and right supports. The moments are taken to be positive when they cause tension at bottom fibers. The moment of inertia is taken to be different for different spans. In the present case I_L and I_R denote respectively moment of inertia of; left and right support and l_L and l_R are the left and right span respectively. It is assumed that supports are unyielding. The yielding of supports could be easily incorporated in three-moment equation, which will be discussed in the next lesson. Now it is required to derive a relation between M_L, M_C and M_R. This relationship is derived from the fact that the tangent to the elastic curve at C is horizontal. In other words the joint C may be considered rigid. Thus, the compatibility equation is written as,

$$\theta_{CL} + \theta_{CR} = 0 \qquad (12.1)$$

The rotation left of the support C, θ_{CL} and rotation right of the support C, θ_{CR} may be calculated from moment area method. Now,

$$\theta_{CL} = \frac{\text{Deflection of L from tangent drawn at C(LL}^{'}\text{)}}{l_L}$$

$$= \frac{\text{Moment of } {}^{\text{M}}\!/\!{}_{\text{EI}} \text{ diagram between C and L about L}}{l_L}$$

$$= \frac{1}{l_L}\left\{\left(\frac{A_L \bar{x}_L}{EI_L}\right) + \frac{1}{2}\left(\frac{M_L}{EI_L}\right) l_L \frac{1}{3} l_L + \frac{1}{2}\left(\frac{M_C}{EI_L}\right) l_L \frac{2}{3} l_L\right\}$$

$$\theta_{CL} = \frac{A_L \bar{x}_L}{EI_L l_L} + \frac{M_L l_L}{6EI_L} + \frac{M_C l_L}{3EI_L} \qquad (12.2)$$

Note that the actual moment diagram on span LC is broken into two parts (1) due to loads applied on span LC when it is considered as a simply supported beam and, (2) due to support moments. In the above equation A_L and A_R denote respectively area of the bending moment diagrams due to applied loads on left and right supports. x_L and x_R denote their respective C.G.(center of gravity) distances from the left and right support respectively. Similarly,

$$\theta_{CR} = \frac{\text{deflection of R from tangent drawn at C (RR')}}{l_R}$$

$$= \frac{\text{Moment of } {}^{\text{M}}\!/\!{}_{\text{EI}} \text{ diagram between C and R about R}}{l_R}$$

$$\theta_{CR} = \frac{A_R \bar{x}_R}{EI_R l_R} + \frac{M_R l_R}{6EI_R} + \frac{M_C l_R}{3EI_R} \tag{12.3}$$

Substituting the values of θ_{CL} and θ_{CR} in the compatibility equation (12.1),

$$\frac{A_L \bar{x}_L}{EI_L l_L} + \frac{M_L l_L}{6EI_L} + \frac{M_C l_L}{3EI_L} + \frac{A_R \bar{x}_R}{EI_R l_R} + \frac{M_R l_R}{6EI_R} + \frac{M_C l_R}{3EI_R} = 0 \tag{12.4}$$

which could be simplified to,

$$M_L \left(\frac{l_L}{I_L} \right) + 2M_C \left\{ \frac{l_L}{I_L} + \frac{l_R}{I_R} \right\} + M_R \left(\frac{l_R}{I_R} \right) = -\frac{6A_R \bar{x}_R}{I_R l_R} - \frac{6A_L \bar{x}_L}{I_L l_L} \tag{12.5}$$

The above equation (12.5) is known as the three-moment equation. It relates three support moments M_L, M_C and M_R with the applied loading on two adjacent spans. If in a span there are more than one type of loading (for example, uniformly distributed load and a concentrated load) then it is simpler to calculate moment diagram separately for each of loading and then to obtain moment diagram.

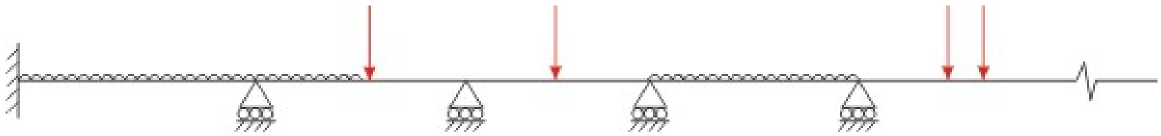
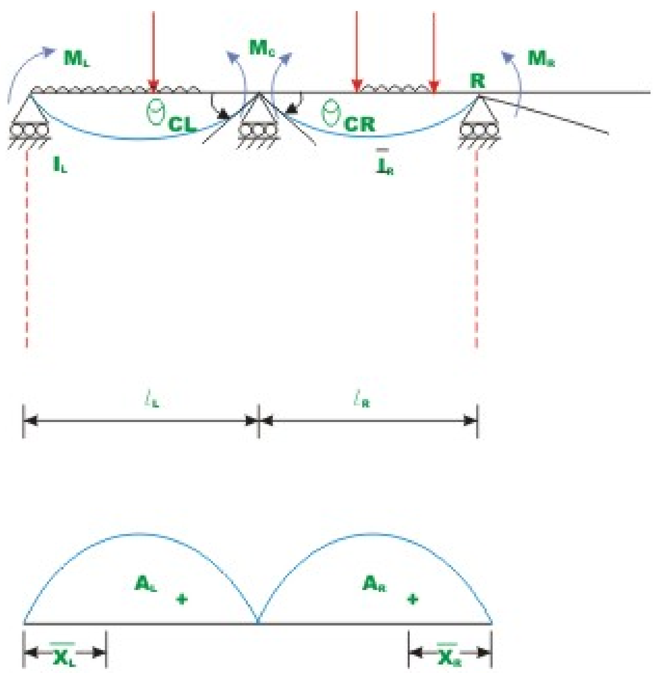
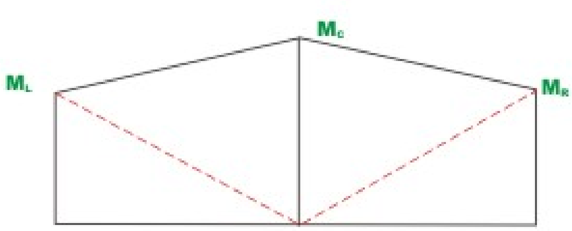


Fig. 12.1 (a) Continuous beam.



Bending moment diagram due to applied loading.



Bending moment diagram (B.M.D) due to support moments.

Fig. 12. 1(b) Two adjacent spans of a continuous beam.

12.3 Alternate derivation

The above three moment equations may also be derived by direct application of force method as follows. Now choose M_L, M_C and the M_R, the three support moments at left, centre and right supports respectively as the redundant moments. The primary determinate structure is obtained by releasing the constraint corresponding to redundant moments. In this particular case, inserting hinges at L, C and R, the primary structure is obtained as below (see Fig. 12.2)

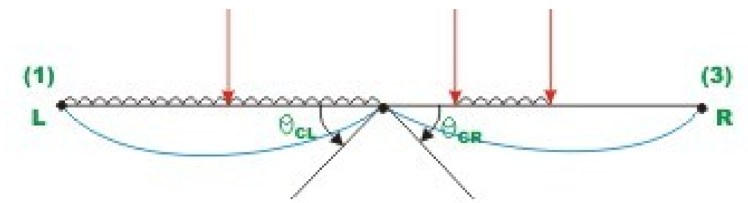


Fig. 12.2. Primary structure

Let displacement (in the primary case rotations) corresponding to rotation M_C be Δ_L, which is the sum of rotations θ_{CL} and θ_{CR}. Thus,

$$\Delta_L = \theta_{CL} + \theta_{CR} \tag{12.6}$$

It is observed that the rotations θ_{CL} and θ_{CR} are caused due to only applied loading as shown in Fig.12.2.This can be easily evaluated by moment area method as shown previously.

$$\Delta_L = \frac{A_L \bar{x}_L}{EI_L l_L} + \frac{A_R \bar{x}_R}{EI_R l_R} \tag{12.7}$$

In the next step, apply unit value of redundant moments at L, C and R and calculate rotation at C (i.e. flexibility coefficients).

$$\begin{aligned} a_{21} &= \frac{l_L}{6EI_L} \\ a_{22} &= \frac{l_L}{3EI_L} + \frac{l_R}{3EI_R} \\ a_{23} &= \frac{l_R}{6EI_R} \end{aligned} \tag{12.8}$$

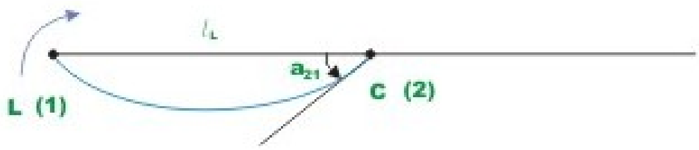


Fig. 12.3 (a) Unit redundant force applied at L (1)

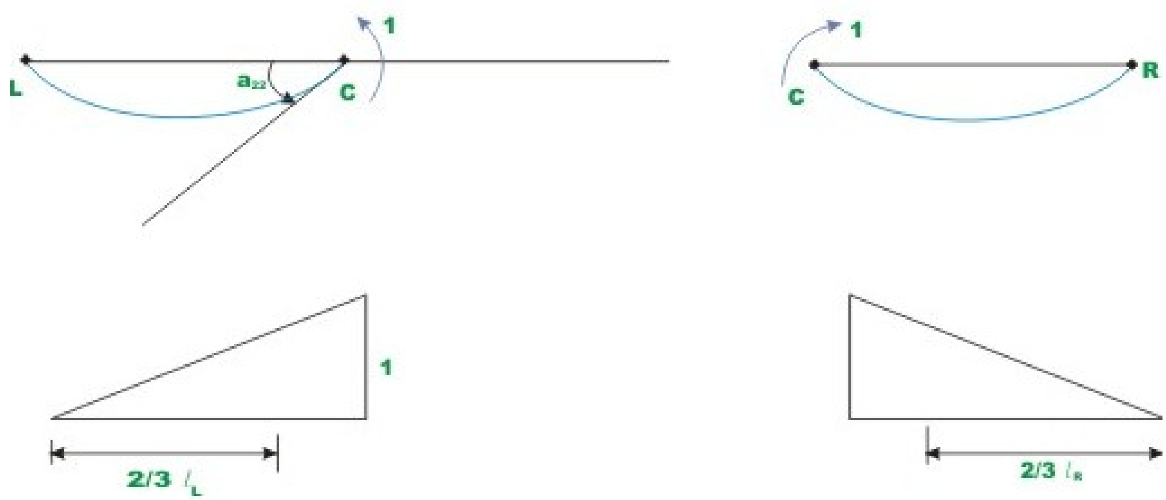


Fig. 12.3 (b) Unit redundant force applied at c.

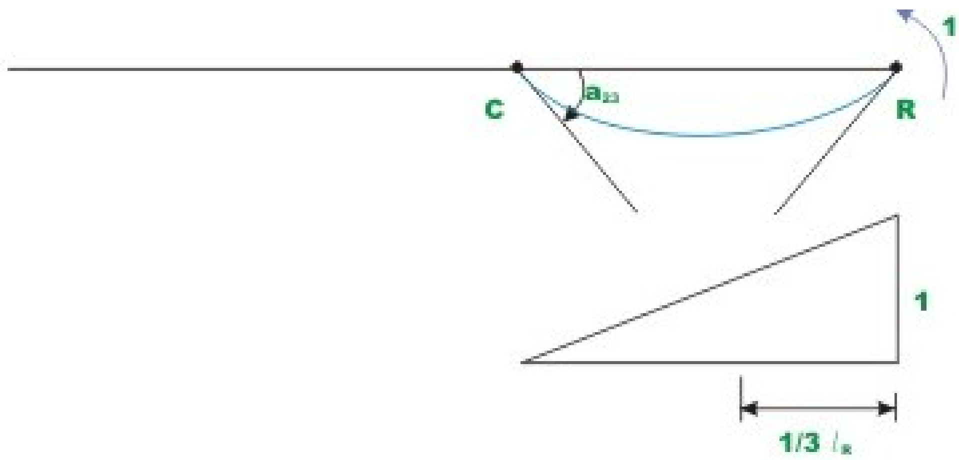


Fig. 12.3 (c) Unit moment applied at R

In the actual structure the relative rotation of both sides is zero. In other words the compatibility equation is written as,

$$\Delta_L + a_{21} M_L + a_{22} M_C + a_{23} M_R = 0 \qquad (12.9)$$

Substituting the values of flexibility coefficients and Δ_L in the above equation,

$$\frac{A_R \bar{x}_R}{EI_R l_R} + \frac{A_L \bar{x}_L}{EI_L l_L} + M_L \left(\frac{l_L}{6EI_L} \right) + M_C \left\{ \frac{l_L}{3EI_L} + \frac{l_R}{3EI_R} \right\} + M_R \left(\frac{l_R}{6EI_R} \right) = 0$$

Or,

$$M_L \left(\frac{l_L}{I_L} \right) + 2M_C \left\{ \frac{l_L}{I_L} + \frac{l_R}{I_R} \right\} + M_R \left(\frac{l_R}{I_R} \right) = -\frac{6A_R \bar{x}_R}{I_R l_R} - \frac{6A_L \bar{x}_L}{I_L l_L} \qquad (12.10)$$

when moment of inertia remains constant i.e. $I_R = I_L = I$,the above equation simplifies to,

$$M_L (l_L) + 2M_C \{ l_L + l_R \} + M_R (l_R) = -\frac{6A_R \bar{x}_R}{l_R} - \frac{6A_L \bar{x}_L}{l_L} \qquad (12.11)$$

Example 12.1

A continuous beam *ABCD* is carrying a uniformly distributed load of 1 kN/m over span ABC in addition to concentrated loads as shown in Fig.12.4a. Calculate support reactions. Also, draw bending moment and shear force diagram. Assume EI to be constant for all members.

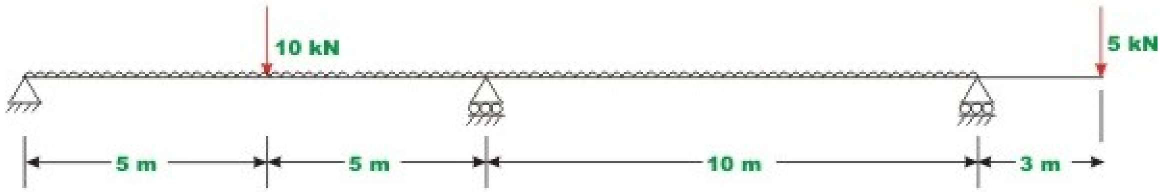


Fig. 12.4 (a) Continuous beam of Example 12 .1

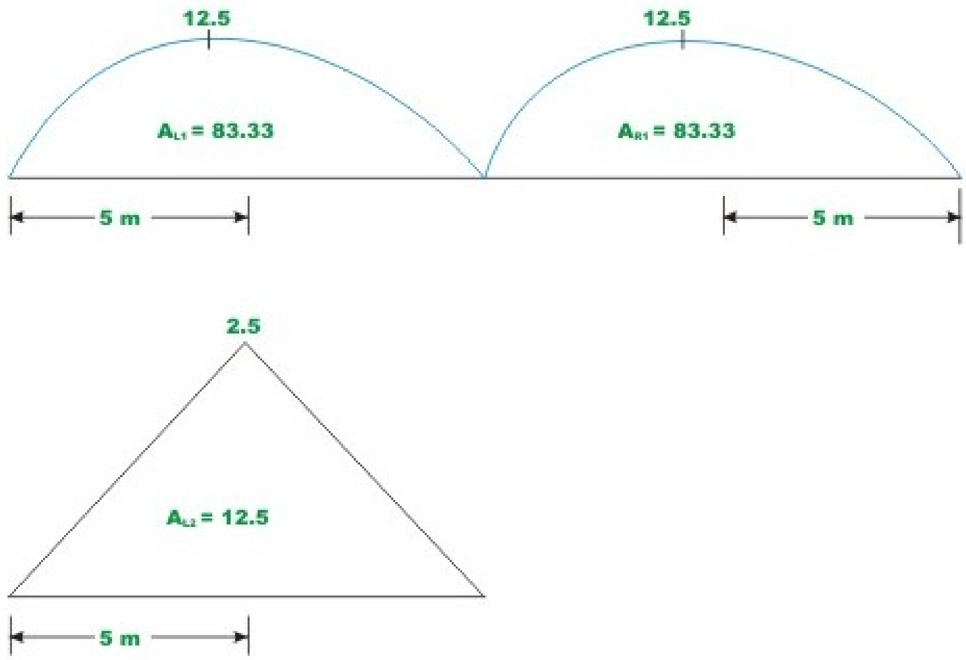


Fig. 12.4 (b) Bending moment diagram due to applied loading

From inspection, it is assumed that the support moments at A is zero and support moment at C,

$M_C = 15$ kN.m (negative because it causes compression at bottom at C)

Hence, only one redundant moment M_B needs to be evaluated. Applying three-moment equation to span ABC,

$$2M_C\{10+10\} + M_C(10) = -\frac{6A_R\overline{x}_R}{l_R} - \frac{6A_L\overline{x}_L}{l_L} \qquad (1)$$

The bending moment diagrams for each span due to applied uniformly distributed and concentrated load are shown in Fig.12.4b.

Equation (1) may be written as,

$$40M_B - 150 = -\frac{6 \times 83.33 \times 5}{10} - \frac{6 \times 125 \times 5}{10} - \frac{6 \times 83.33 \times 5}{10}$$

Thus,

$$M_B = -18.125 \quad \text{kN.m}$$

After determining the redundant moment, the reactions are evaluated by equations of static equilibrium. The reactions are shown in Fig.12.4c along with the external load and support bending moment.

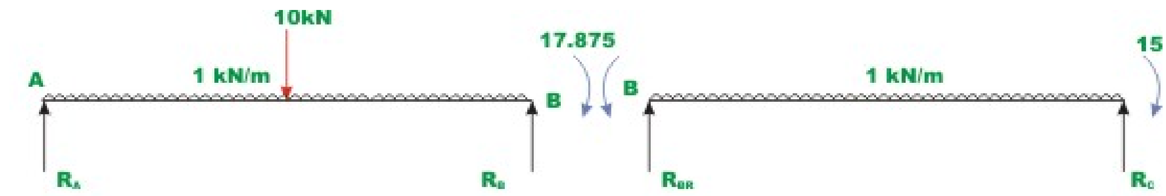
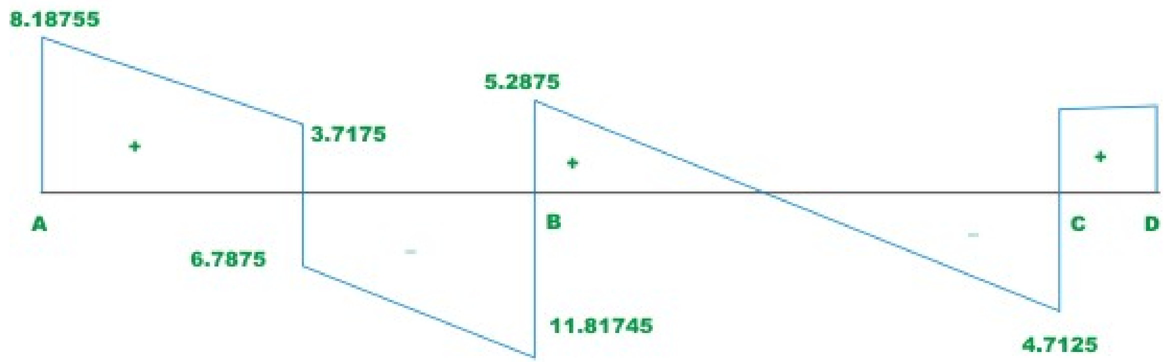
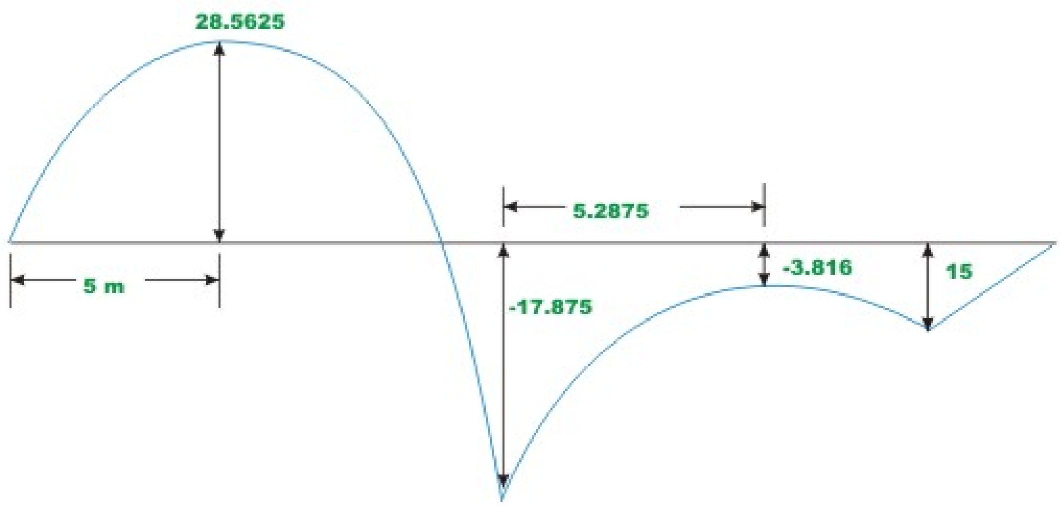


Fig. 12.4 (c) Free - body diagram of two members



Shear force diagram (S.F.D)



Bending moment diagram (B.M.D)

Fig. 12.4(d). SHEARE FORCE & BENDING MOMENT DIAGRAM.

In span AB, R_A can be calculated by the condition that $\sum M_B = 0$. Thus,

$$R_A \times 10 - 10 \times 5 - 10 \times 5 + 18.125 = 0$$

$$R_A = 8.1875 \quad \text{kN} \qquad (\uparrow)$$

$$R_{BL} = 11.8125 \quad \text{kN} \qquad (\uparrow)$$

Similarly from span BC,

$$R_C = 4.7125 \quad \text{kN} \qquad (\uparrow)$$

$$R_{BR} = 5.3125 \quad \text{kN} \qquad (\uparrow)$$

The shear force and bending moment diagrams are shown in Fig.12.4d.

Example 12.2

A continuous beam ABC is carrying uniformly distributed load of 2 kN/m as shown in Fig.12.5a.The moment of inertia of span AB is twice that of span BC. Evaluate reactions and draw bending moment and shear force diagrams.

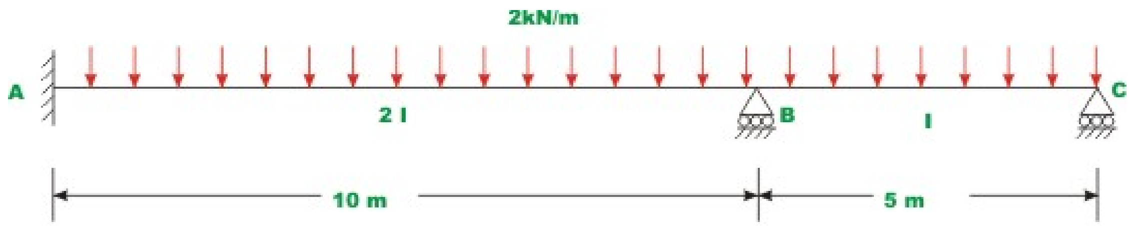


Fig. 12.5 (a) Example 12.2

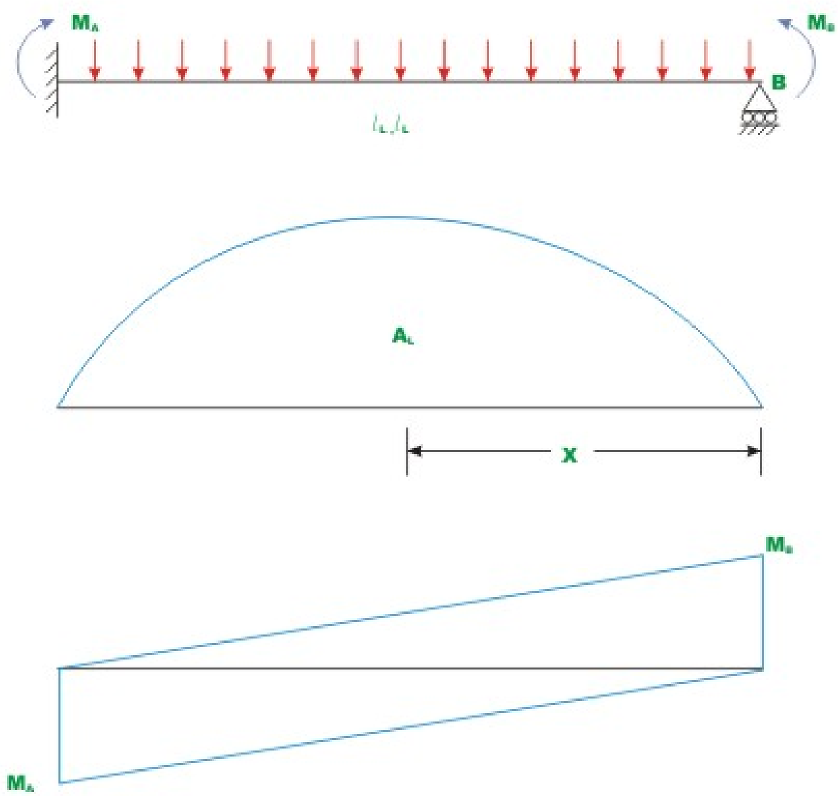


Fig. 12.5(b) Free body diagram of span AB

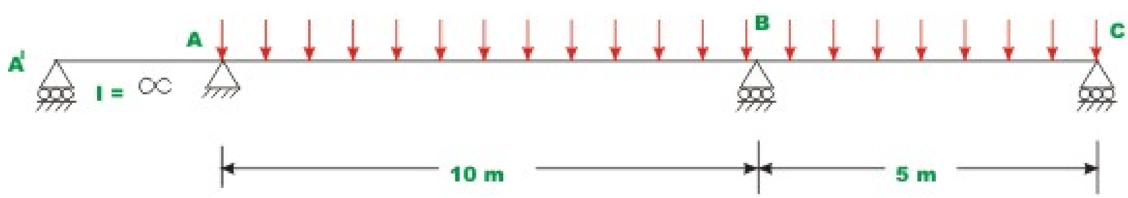


Fig. 12.5(c) Continuous beam within imaginary span AA'

By inspection it is seen that the moment at support C is zero. The support moment at A and B needs to be evaluated .For moment at B, the compatibility equation is written by noting that the tangent to the elastic curve at B is horizontal .The compatibility condition corresponding to redundant moment at A is written as follows. Consider span AB as shown in Fig.12.5b.

The slope at A, θ_A may be calculated from moment-area method. Thus,

$$\theta_A = \frac{M_B l_L}{6EI_L} + \frac{M_A l_L}{3EI_L} + \frac{A(\bar{x}_L)_R}{EIl_L} \tag{1}$$

Now, compatibility equation is,

$$\theta_A = 0 \tag{2}$$

It is observed that the tangent to elastic curve at A remains horizontal. This can also be achieved as follows. Assume an imaginary span AA' of length L' left of support A having a very high moment of inertia (see Fig. 12.5c). As the imaginary span has very high moment of inertia, it does not yield any imaginary span has very high moment of inertia it does not yield any M/EI diagram and hence no elastic curve. Hence, the tangent at A to elastic curve remains horizontal.
Now, consider the span $A'AB$, applying three-moment equation to support A,

$$2M_A\left\{\frac{L'}{\infty} + \frac{10}{2I}\right\} + M_B\left(\frac{10}{2I}\right) = -\frac{6A_R\bar{x}_R}{2I(10)} \tag{3}$$

The above equation is the same as the equation (2). The simply supported bending moment diagram is shown in Fig.12.5d.

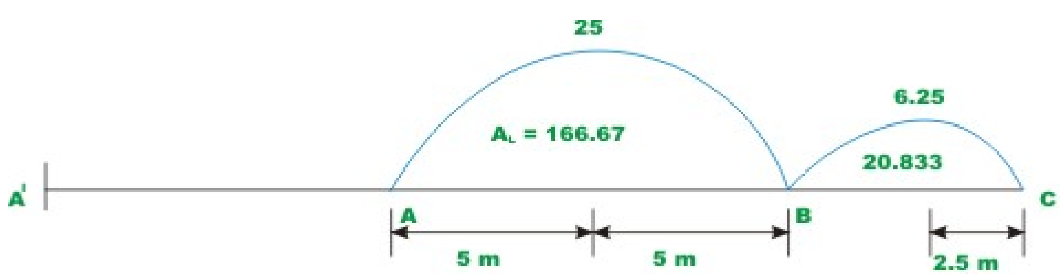


Fig. 12.5 (d) Bending moment diagram due to applied loading

Thus, equation (3) may be written as,

$$20M_A + M_B(10) = -\frac{6\times(166.67)\times 5}{10}$$

$$20M_A + 10M_B = -500 \qquad (4)$$

Now, consider span ABC, writing three moment equation for support B,

$$M_A\left\{\frac{10}{2I}\right\} + 2M_B\left\{\frac{10}{2I}+\frac{5}{I}\right\} = -\frac{6\times 166.67\times 5}{2I\times(10)} - \frac{6\times 20.837\times 2.5}{I\times(5)}$$

$$\begin{aligned} 5M_A + 20M_B &= -250 - 62.5 \\ &= -312.5 \end{aligned} \qquad (5)$$

Solving equation (4) and (5),

$$M_B = -6.25 \text{ kN.m}$$

$$M_A = -37.5 \text{ kN.m}$$

The remaining reactions are calculated by equilibrium equations (see Fig.12.5e)

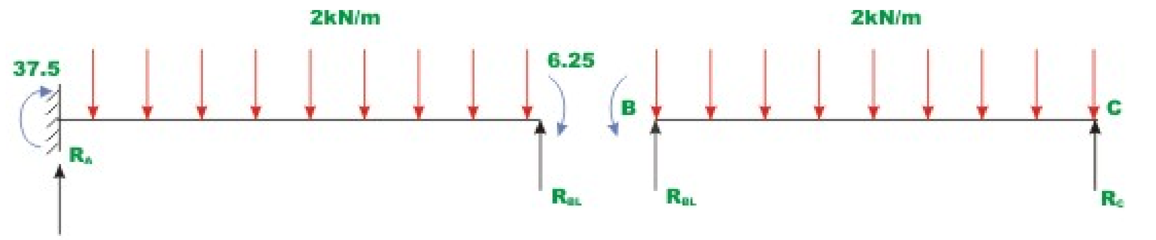
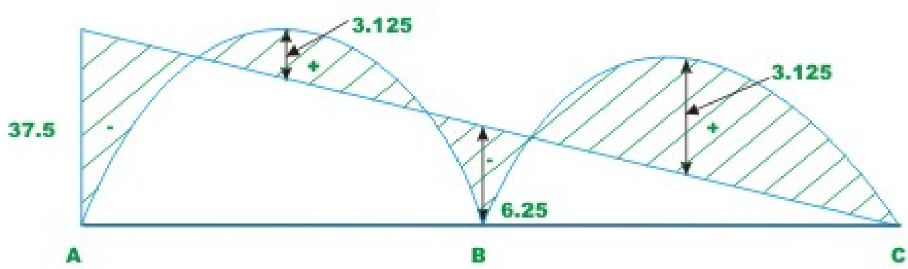


Fig. 12.5 (e) Free - body diagram of two members

S . F . D



B.M.D

Fig. 12.5 (f) Shear force and bending moment diagrams

In span AB, $\sum M_B = 0$

$$R_A \times 10 - 37.5 - 2 \times 10 \times 5 + 6.25 = 0$$

$$R_A = 13.125 \quad \text{kN} \qquad (\uparrow)$$

$$R_{BL} = 6.875 \quad \text{kN} \qquad (\uparrow)$$

Similarly from span BC ,

$$R_C = 3.75 \quad \text{kN} \qquad (\uparrow)$$

$$R_{BR} = 6.25 \quad \text{kN} \qquad (\uparrow)$$

The shear force and bending moment diagrams are shown in Fig. 12.5f.

Summary

In this lesson the continuous beam with unyielding supports is analysed by three-moment equations. The three-moment equations are derived for the case of a continuous beam having different moment of inertia in different spans. The three-moment equations also belong to force method of analysis and in this case, redundants are always taken as support moments. Hence, compatibility equations are derived in terms of three support moments. Few problems are solved to illustrate the procedure.

Module 2

Analysis of Statically Indeterminate Structures by the Matrix Force Method

Lesson 13

The Three-Moment Equations-Ii

Instructional Objectives

After reading this chapter the student will be able to
1. Derive three-moment equations for a continuous beam with yielding supports.
2. Write compatibility equations of a continuous beam in terms of three moments.
3. Compute reactions in statically indeterminate beams using three-moment equations.
4. Analyse continuous beams having different moments of inertia in different spans and undergoing support settlements using three-moment equations.

13.1 Introduction

In the last lesson, three-moment equations were developed for continuous beams with unyielding supports. As discussed earlier, the support may settle by unequal amount during the lifetime of the structure. Such future unequal settlement induces extra stresses in statically indeterminate beams. Hence, one needs to consider these settlements in the analysis. The three-moment equations developed in the pervious lesson could be easily extended to account for the support yielding. In the next section three-moment equations are derived considering the support settlements. In the end, few problems are solved to illustrate the method.

13.2 Derivation of Three-Moment Equation

Consider a two span of a continuous beam loaded as shown in Fig.13.1. Let M_L, M_C and M_R be the support moments at left, center and right supports respectively. As stated in the previous lesson, the moments are taken to be positive when they cause tension at the bottom fibers. I_L and I_R denote moment of inertia of left and right span respectively and l_L and l_R denote left and right spans respectively. Let δ_L, δ_C and δ_R be the support settlements of left, centre and right supports respectively. δ_L, δ_C and δ_R are taken as negative if the settlement is downwards. The tangent to the elastic curve at support C makes an angle θ_{CL} at left support and θ_{CR} at the right support as shown in Fig. 13.1. From the figure it is observed that,

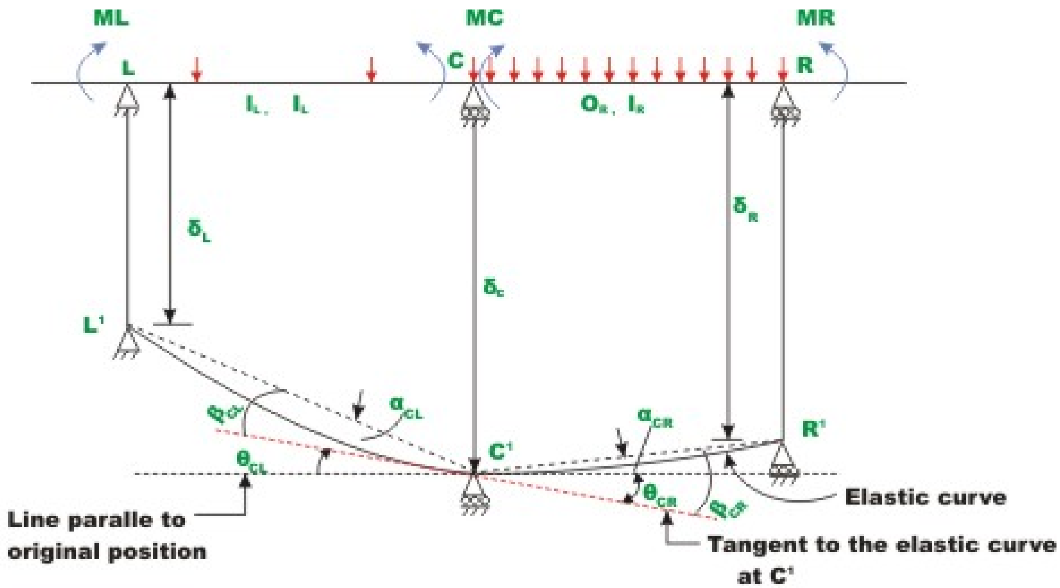


Fig. 13.1 Continuous beam with support settlement

$$\theta_{CL} = \theta_{CR} \tag{13.1}$$

The rotations β_{CL} and β_{CR} due to external loads and support moments are calculated from the M/EI diagram .They are (see lesson 12)

$$\beta_{CL} = \frac{A_L \bar{x}_L}{EI_L l_L} + \frac{M_L l_L}{6EI_L} + \frac{M_C l_L}{3EI_L} \tag{13.2a}$$

$$\beta_{CR} = \frac{A_R \bar{x}_R}{EI_R l_R} + \frac{M_R l_R}{6EI_R} + \frac{M_C l_R}{3EI_R} \tag{13.2b}$$

The rotations of the chord $L'C'$ and $C'R'$ from the original position is given by

$$\alpha_{CL} = \frac{\delta_L - \delta_C}{l_L} \tag{13.3a}$$

$$\alpha_{CR} = \frac{\delta_R - \delta_C}{l_R} \tag{13.3b}$$

From Fig. 13.1, one could write,

$$\theta_{CL} = \alpha_{CL} - \beta_{CL} \tag{13.4a}$$

$$\theta_{CR} = \beta_{CR} - \alpha_{CR} \tag{13.4b}$$

Thus, from equations (13.1) and (13.4), one could write,

$$\alpha_{CL} - \beta_{CL} = \beta_{CR} - \alpha_{CR} \tag{13.5}$$

Substituting the values of $\alpha_{CL}, \alpha_{CR}, \beta_{CL}$ and β_{CR} in the above equation,

$$M_L\left(\frac{l_L}{I_L}\right) + 2M_C\left\{\frac{l_L}{I_L} + \frac{l_R}{I_R}\right\} + M_R\left(\frac{l_R}{I_R}\right) = -\frac{6A_R\bar{x}_R}{I_R l_R} - \frac{6A_L\bar{x}_L}{I_L l_L} + 6E\left(\frac{\delta_L - \delta_C}{l_L}\right) + 6E\left(\frac{\delta_R - \delta_C}{l_R}\right)$$

This may be written as

$$M_L\left(\frac{l_L}{I_L}\right) + 2M_C\left\{\frac{l_L}{I_L} + \frac{l_R}{I_R}\right\} + M_R\left(\frac{l_R}{I_R}\right) = -\frac{6A_R\bar{x}_R}{I_R l_R} - \frac{6A_L\bar{x}_L}{I_L l_L} - 6E\left[\left(\frac{\delta_C - \delta_L}{l_L}\right) + \left(\frac{\delta_C - \delta_R}{l_R}\right)\right]$$
(13.6)

The above equation relates the redundant support moments at three successive spans with the applied loading on the adjacent spans and the support settlements.

Example 13.1

Draw the bending moment diagram of a continuous beam BC shown in Fig.13.2a by three moment equations. The support B settles by 5mm below A and C. Also evaluate reactions at A, B and C. Assume EI to be constant for all members and $E = 200 \text{ GPa}$, $I = 8 \times 10^6 \quad \text{mm}^4$

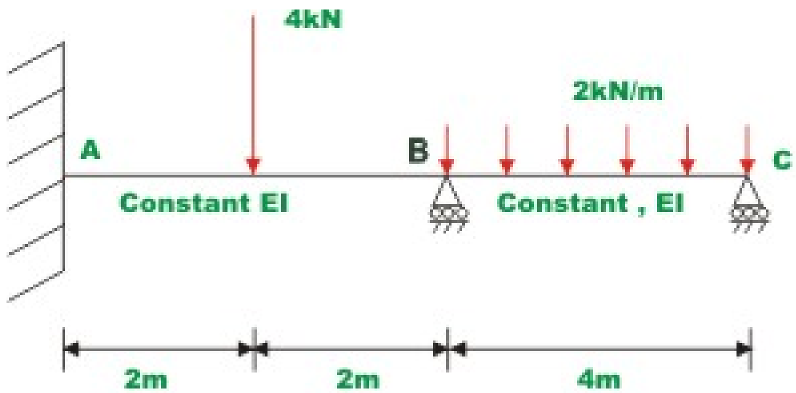


Fig. 13.2(a) Example 13.1

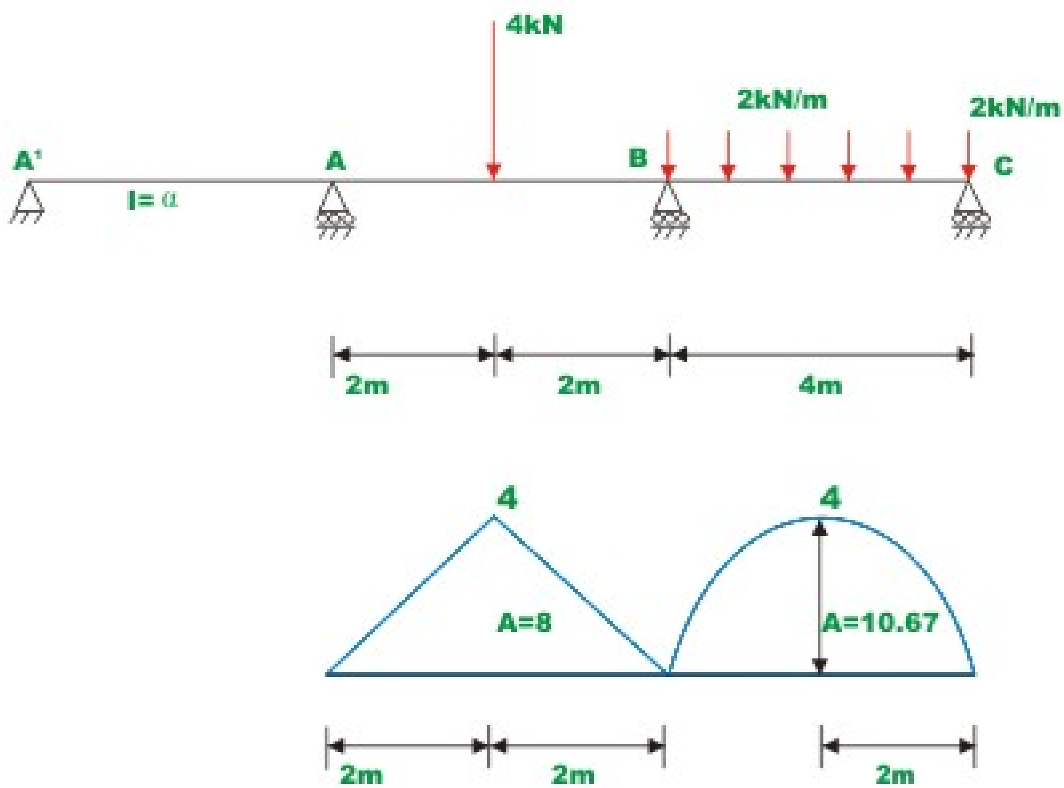


Fig. 13.2(b) Bending moment diagram due to applied loading

Assume an imaginary span having infinitely large moment of inertia and arbitrary span L' left of A as shown in Fig.13.2b .Also it is observed that moment at C is zero.

The given problem is statically indeterminate to the second degree. The moments M_A and M_B,the redundants need to be evaluated. Applying three moment equation to the span *A'AB*,

$$\delta_L = \delta_C = 0 \text{ and } \delta_R = -5\times10^{-3}\, m$$

$$M'_A\left(\frac{L'}{\infty}\right)+2M_A\left\{\frac{L'}{\infty}+\frac{4}{I}\right\}+M_B\left(\frac{4}{I}\right)=-\frac{6\times8\times2}{I(4)}-6E\left(0+\frac{0-(-5\times10^{-3}}{4}\right)$$

$$8M_A+4M_B=-24-6EI\times\frac{5\times10^{-3}}{4} \qquad (1)$$

Note that, $EI = 200\times10^9\times\dfrac{8\times10^6\times10^{-12}}{10^3}=1.6\times10^3\,\mathrm{kNm}^2$

Thus,

$$8M_A+4M_B=-24-6\times1.6\times10^3\times\frac{5\times10^{-3}}{4}$$

$$8M_A+4M_B=-36 \qquad (2)$$

Again applying three moment equation to span ABC the other equations is obtained. For this case, $\delta_L = 0$, $\delta_C = -5\times10^{-3}\, m$ (negative as the settlement is downwards) and $\delta_R = 0$.

$$M_A\left\{\frac{4}{I}\right\}+2M_B\left\{\frac{4}{I}+\frac{4}{I}\right\}=-\frac{24}{I}-\frac{6\times10.667\times2}{I\times4}-6E\left(\frac{-5\times10^{-3}}{4}-\frac{5\times10^{-3}}{4}\right)$$

$$4M_A+16M_B=-24-32+6\times1.6\times10^3\times\frac{10\times10^3}{4}$$

$$4M_A+16M_B=-32 \qquad (3)$$

Solving equations (2) and (3),

$$M_B=-1.0 \text{ kN.m}$$
$$M_A=-4.0 \text{ kN.m} \qquad (4)$$

Now, reactions are calculated from equations of static equilibrium (see Fig.13.2c).

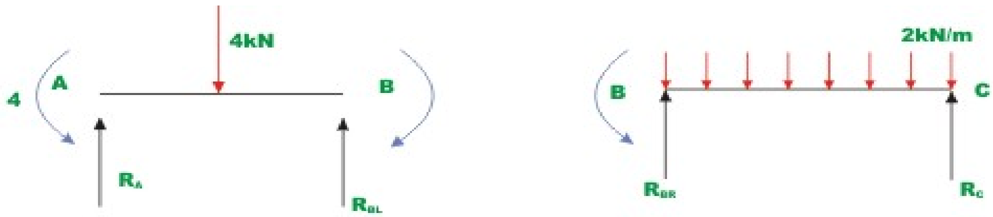
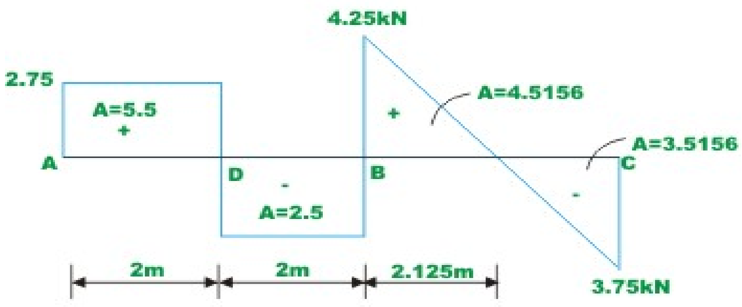
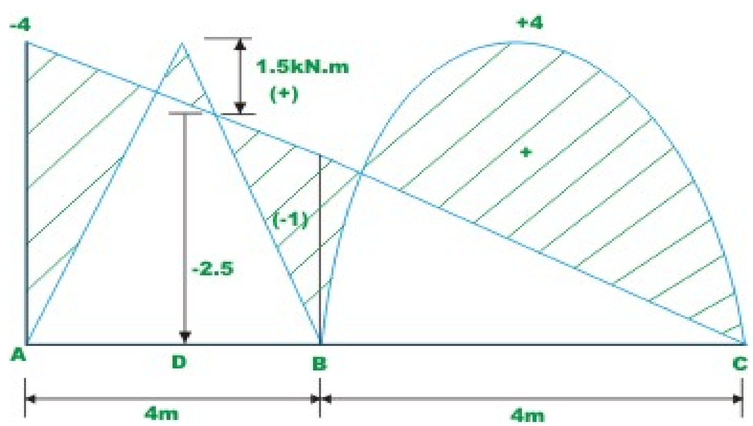


Fig.13.2 (c) Free - body diagram of two members



Shear force diagram



Bending moment diagram

Fig.13.2(d) Shear force and bending moment diagram

Thus,

$$R_A = 2.75\,\text{kN}\,(\uparrow)$$
$$R_{BL} = 1.25\,\text{kN}\,(\uparrow)$$
$$R_{BR} = 4.25\,\text{kN}\,(\uparrow)$$
$$R_C = 3.75\,\text{kN}\,(\uparrow)$$

The reactions at B,

$$R_B = R_{BR} + R_{BL} = 5.5\,\text{kN} \tag{5}$$

The area of each segment of the shear force diagram for the given continuous beam is also indicated in the above diagram. This could be used to verify the previously computed moments. For example, the area of the shear force diagram between A and B is $5.5\ \text{kN.m}$.This must be equal to the change in the bending moment between A and D, which is indeed the case ($-4-1.5=5.5\ \text{kN.m}$). Thus, moments previously calculated are correct.

Example 13.2

A continuous beam $ABCD$ is supported on springs at supports B and C as shown in Fig.13.3a. The loading is also shown in the figure. The stiffness of springs is $k_B = \frac{EI}{20}$ and $k_C = \frac{EI}{30}$ .Evaluate support reactions and draw bending moment diagram. Assume EI to be constant.

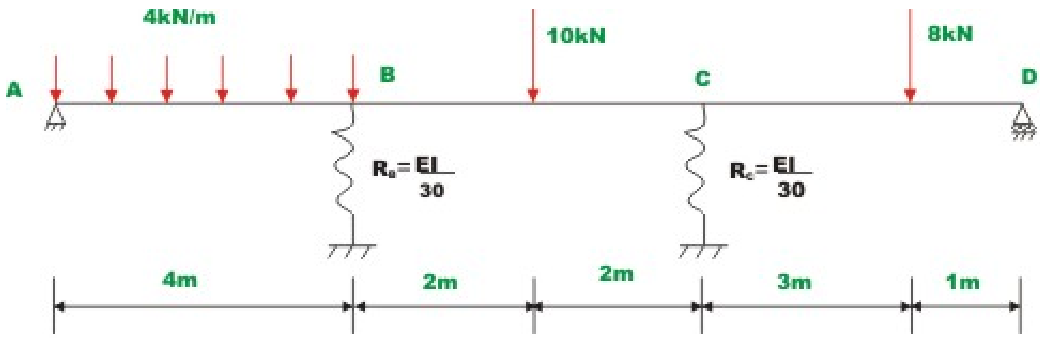


Fig.13.3(a) Continuous beam of Example 13.2

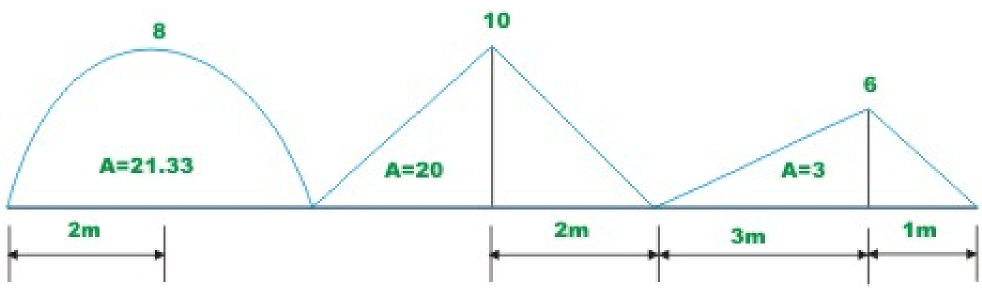


Fig.13.3(b) Bending moment diagram on simple spans due to applied loading

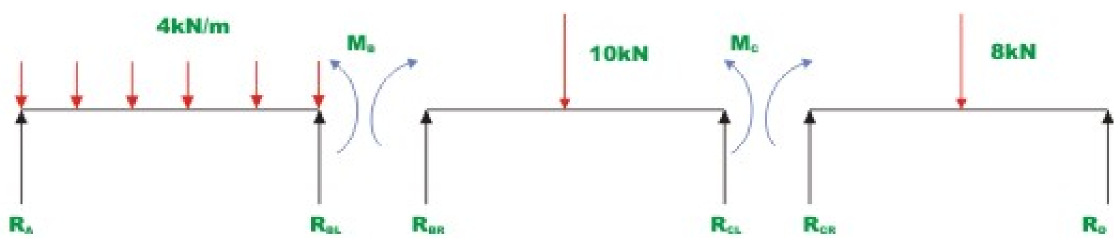


Fig.13.3(c) Computation of reactions

In the given problem it is required to evaluate bending moments at supports B and C. By inspection it is observed that the support moments at A and D are zero. Since the continuous beam is supported on springs at B and C, the support settles. Let R_B and R_C be the reactions at B and C respectively. Then the support settlement at B and C are $\frac{R_B}{k_B}$ and $\frac{R_C}{k_C}$ respectively. Both the settlements are negative and in other words they move downwards. Thus,

$$\delta_A = 0 \, , \delta_B = \frac{-20 R_B}{EI} \, , \delta_C = \frac{-30 R_C}{EI} \text{ and } \delta_D = 0 \qquad (1)$$

Now applying three moment equations to span ABC (see Fig.13.2b)

$$M_A\left\{\frac{4}{I}\right\}+2M_B\left\{\frac{4}{I}+\frac{4}{I}\right\}+M_C\left\{\frac{4}{I}\right\}=-\frac{6\times 21.33\times 2}{I\times 4}-\frac{6\times 20\times 2}{I\times 4}-6E\left[\frac{-20R_B}{4EI}+\frac{\frac{-20R_B}{EI}+\frac{30R_C}{EI}}{4}\right]$$

Simplifying,

$$16M_B+4M_C=-124+60R_B-45R_C \qquad (2)$$

Again applying three moment equation to adjacent spans BC and CD,

$$M_B\left\{\frac{4}{I}\right\}+2M_C\left\{\frac{4}{I}+\frac{4}{I}\right\}=-\frac{60}{I}-\frac{(6\times 9\times 2+6\times 3\times\frac{2}{3}\times 1)}{I\times 4}-6E\left[\frac{-\frac{30R_C}{EI}+\frac{20R_B}{EI}}{4}+\frac{-30R_C}{4EI}\right]$$

$$4M_B+16M_C=-90+90R_C-30R_B \qquad (3)$$

In equation (2) and (3) express R_B and R_C in terms of M_B and M_C (see Fig.13.2c)

$$\begin{aligned}
R_A &= 8+0.25M_B \quad (\uparrow)\\
R_{BL} &= 8-0.25M_B \quad (\uparrow)\\
R_{BR} &= 5+0.25M_C-0.25M_B \quad (\uparrow)\\
R_{CL} &= 5+0.25M_B-0.25M_C \quad (\uparrow)\\
R_{CR} &= 2-0.25M_C \quad (\uparrow)\\
R_D &= 6+0.25M_C \quad (\uparrow)
\end{aligned} \qquad (4)$$

Note that initially all reactions are assumed to act in the positive direction (i.e. upwards) .Now,

$$R_B=R_{BL}+R_{BR}=13-0.5M_B+0.25M_C$$

$$R_C=R_{CL}+R_{CR}=7+0.25M_B-0.5M_C \qquad (5)$$

Now substituting the values of R_B and R_C in equations (2) and (3),

$$16M_B+4M_C=-124+60(13-0.5M_B+0.25M_C)-45(7+0.25M_B-0.5M_C)$$

Or,

$$57.25M_B - 33.5M_C = 341 \tag{6}$$

And from equation 3,

$$4M_B + 16M_C = -90 + 90(7 + 0.25M_B - 0.5M_C) - 30(13 - 0.5M_B + 0.25M_C)$$

Simplifying,

$$-33.5M_B + 68.5M_C = 150 \tag{7}$$

Solving equations (6) and (7)

$$\begin{aligned} M_C &= 7.147 \text{ kN.m} \\ M_B &= 10.138 \text{ kN.m} \end{aligned} \tag{8}$$

Substituting the values of M_B and M_C in (4),reactions are obtained.

$$\begin{array}{ll} R_A = 10.535 \ \text{kN} \ (\uparrow) & R_{BL} = 5.465 \ \text{kN} \ (\uparrow) \\ R_{BR} = 4.252 \ \text{kN} \ (\uparrow) & R_{CL} = 5.748 \ \text{kN} \ (\uparrow) \\ R_{CR} = 0.213 \ \text{kN} \ (\uparrow) & R_D = 7.787 \ \text{kN} \ (\uparrow) \\ R_B = 9.717 \ \text{kN} \ (\uparrow) \ \ \textit{and} & R_C = 5.961 \ \text{kN} \ (\uparrow) \end{array}$$

The shear force and bending moment diagram are shown in Fig. 13.2d.

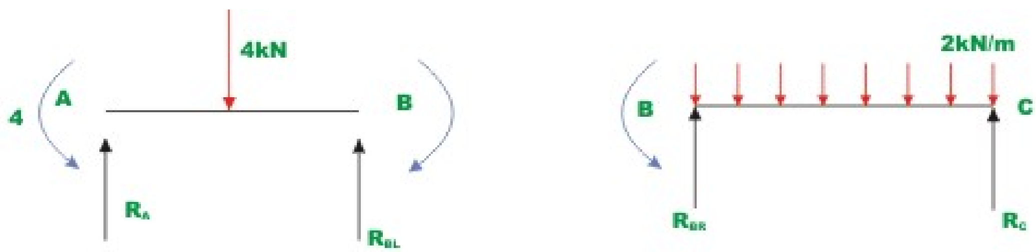
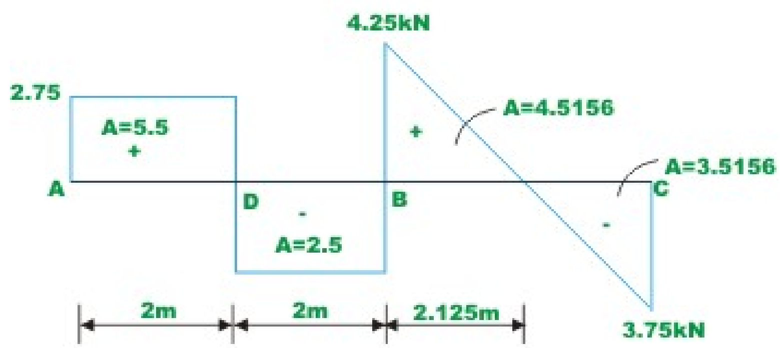
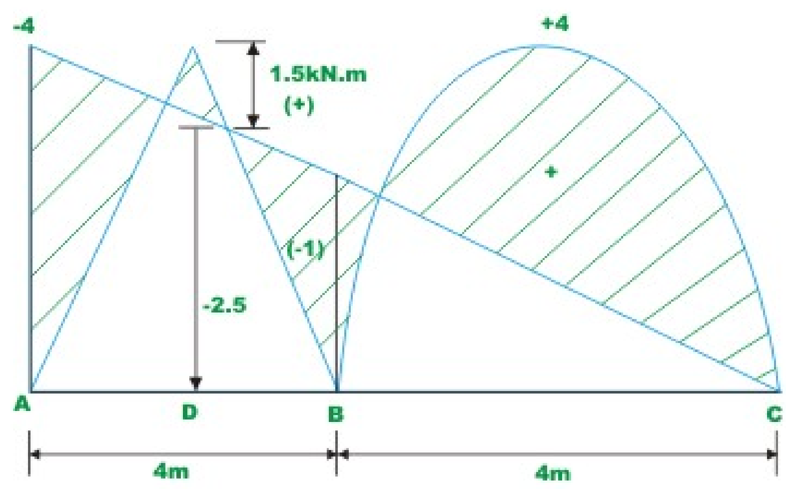


Fig.13.2 (c) Free - body diagram of two members



Shear force diagram



Bending moment diagram

Fig.13.2(d) Shear force and bending moment diagram

Example 13.3

Sketch the deflected shape of the continuous beam *ABC* of example 13.1.
The redundant moments M_A and M_B for this problem have already been computed in problem 13.1.They are,

$$M_B = -1.0 \text{ kN.m}$$
$$M_A = -4.0 \text{ kN.m}$$

The computed reactions are also shown in Fig.13.2c.Now to sketch the deformed shape of the beam it is required to compute rotations at *B* and *C*. These joints rotations are computed from equations (13.2) and (13.3).
For calculating θ_A, consider span *A'AB*

$$\theta_A = \beta_{AR} - \alpha_{AR}$$

$$= \frac{A_R \bar{x}_R}{EI_R l_R} + \frac{M_B l_R}{6EI_R} + \frac{M_A l_R}{3EI_R} - \left(\frac{\delta_B - \delta_A}{4}\right)$$

$$= \frac{6 \times 8 \times 2}{1.6 \times 10^3 \times 4} + \frac{M_B \times 4}{1.6 \times 10^3 \times 6} + \frac{M_A \times 4}{1.6 \times 10^3 \times 3} - \left(\frac{\delta_B - \delta_A}{4}\right)$$

$$= \frac{6 \times 8 \times 2}{1.6 \times 10^3 \times 4} + \frac{(-1) \times 4}{1.6 \times 10^3 \times 6} + \frac{(-4) \times 4}{1.6 \times 10^3 \times 3} + \left(\frac{5 \times 10^{-3}}{4}\right)$$

$$= 0 \qquad (1)$$

For calculating θ_{BL}, consider span *ABC*

$$\theta_{BL} = \alpha_{BL} - \beta_{BL}$$

$$= -\left(\frac{A_L \bar{x}_L}{EI_L l_L} + \frac{M_A l_L}{6EI_L} + \frac{M_B l_L}{3EI_L}\right) + \left(\frac{\delta_A - \delta_B}{l_L}\right)$$

$$= -\left(\frac{8 \times 2}{1.6 \times 10^3 \times 4} + \frac{(-4) \times 4}{1.6 \times 10^3 \times 6} + \frac{(-1) \times 4}{1.6 \times 10^3 \times 3}\right) + \left(\frac{5 \times 10^3}{4}\right)$$

$$= 1.25 \times 10^{-3} \text{ radians} \qquad (2)$$

For θ_{BR} consider span ABC

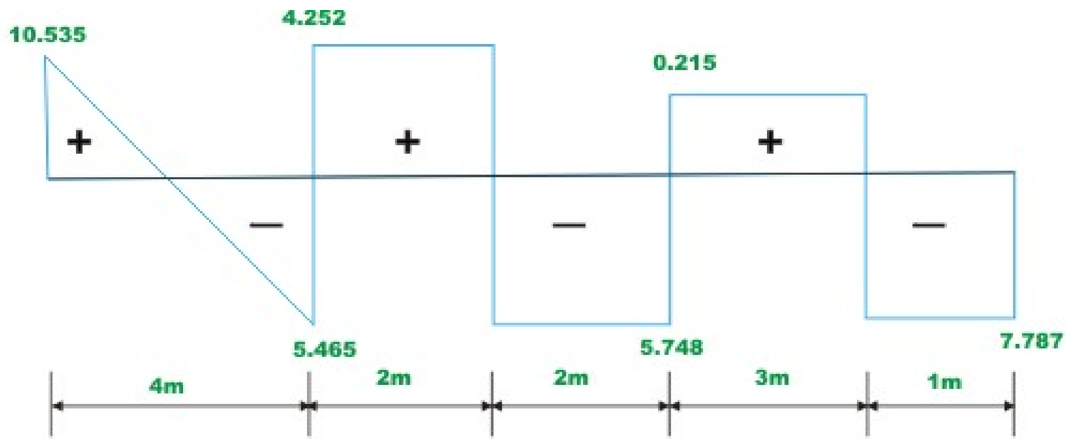
$$\theta_{BR} = \left(\frac{10.67 \times 2}{1.6 \times 10^3 \times 4} + \frac{(-1) \times 4}{1.6 \times 10^3 \times 3} \right) - \left(0 + \frac{5 \times 10^3}{4} \right)$$

$$= -1.25 \times 10^{-3} \text{ radians} \qquad (3)$$

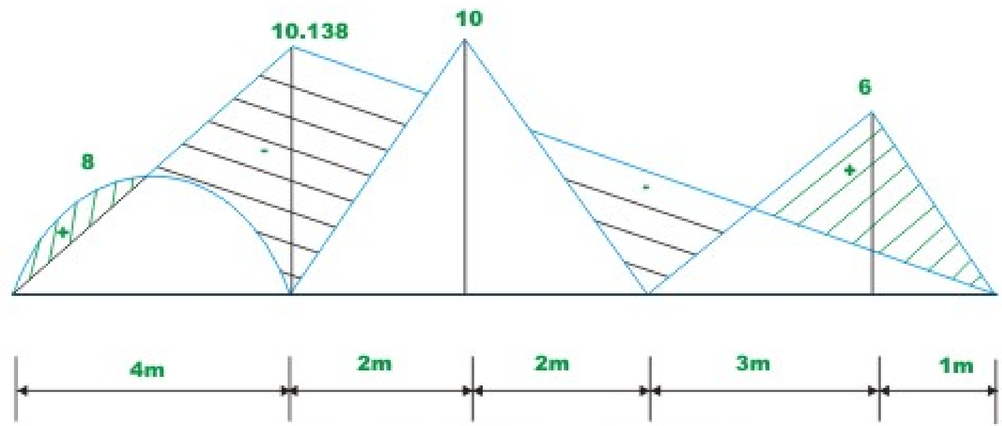
$$\theta_C = -\left(\frac{10.67 \times 2}{1.6 \times 10^3 \times 4} + \frac{(-1) \times 4}{1.6 \times 10^3 \times 3} \right) - \left(\frac{\delta_B - \delta_C}{4} \right)$$

$$= -3.75 \times 10^{-3} \text{ radians.} \qquad (4)$$

The deflected shape of the beam is shown in Fig. 13.4.



Shear force diagram



Bending moment diagram
Fig.13.3(d)

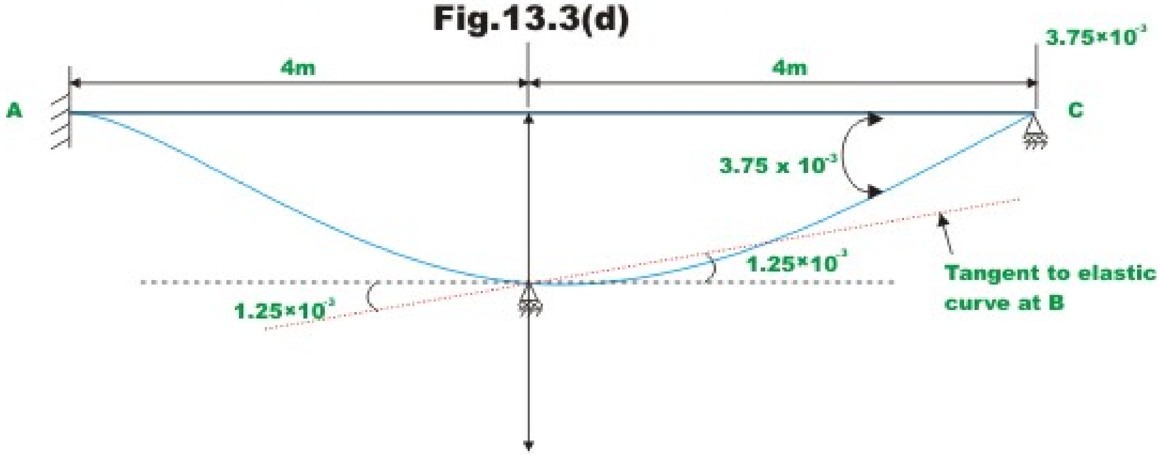


Fig.13.4(a) Elastic curve Example 13.3

Summary

The continuous beams with unyielding supports are analysed using three-moment equations in the last lesson. In this lesson, the three-moment-equations developed in the previous lesson are extended to account for the support settlements. The three-moment equations are derived for the case of a continuous beam having different moment of inertia in different spans. Few examples are derived to illustrate the procedure of analysing continuous beams undergoing support settlements using three-moment equations.

Module 3

Analysis of Statically Indeterminate Structures by the Displacement Method

Lesson
14
The Slope-Deflection Method: An Introduction

Introduction

As pointed out earlier, there are two distinct methods of analysis for statically indeterminate structures depending on how equations of equilibrium, load displacement and compatibility conditions are satisfied: 1) force method of analysis and (2) displacement method of analysis. In the last module, force method of analysis was discussed. In this module, the displacement method of analysis will be discussed. In the force method of analysis, primary unknowns are forces and compatibility of displacements is written in terms of pre-selected redundant reactions and flexibility coefficients using force displacement relations. Solving these equations, the unknown redundant reactions are evaluated. The remaining reactions are obtained from equations of equilibrium.
As the name itself suggests, in the displacement method of analysis, the primary unknowns are displacements. Once the structural model is defined for the problem, the unknowns are automatically chosen unlike the force method. Hence this method is more suitable for computer implementation. In the displacement method of analysis, first equilibrium equations are satisfied. The equilibrium of forces is written by expressing the unknown joint displacements in terms of load by using load displacement relations. These equilibrium equations are solved for unknown joint displacements. In the next step, the unknown reactions are computed from compatibility equations using force displacement relations. In displacement method, three methods which are closely related to each other will be discussed.

1) Slope-Deflection Method
2) Moment Distribution Method
3) Direct Stiffness Method

In this module first two methods are discussed and direct stiffness method is treated in the next module. All displacement methods follow the above general procedure. The Slope-deflection and moment distribution methods were extensively used for many years before the compute era. After the revolution occurred in the field of computing only direct stiffness method is preferred.

Degrees of freedom

In the displacement method of analysis, primary unknowns are joint displacements which are commonly referred to as the degrees of freedom of the structure. It is necessary to consider all the independent degrees of freedom while writing the equilibrium equations.These degrees of freedom are specified at supports, joints and at the free ends. For example, a propped cantilever beam (see Fig.14.01a) under the action of load P will undergo only rotation at B if axial deformation is neglected. In this case kinematic degree of freedom of the beam is only one i.e. θ_B as shown in the figure.

In Fig.14.01b, we have nodes at *A,B,C* and *D*. Under the action of lateral loads P_1, P_2 and P_3, this continuous beam deform as shown in the figure. Here axial deformations are neglected. For this beam we have five degrees of freedom $\theta_A, \theta_B, \theta_C$, θ_D and Δ_D as indicated in the figure. In Fig.14.02a, a symmetrical plane frame is loaded symmetrically. In this case we have only two degrees of freedom θ_B and θ_C. Now consider a frame as shown in Fig.14.02b. It has three degrees of freedom viz. θ_B, θ_C and Δ_D as shown. Under the action of horizontal and vertical load, the frame will be displaced as shown in the figure. It is observed that nodes at *B* and *C* undergo rotation and also get displaced horizontally by an equal amount.

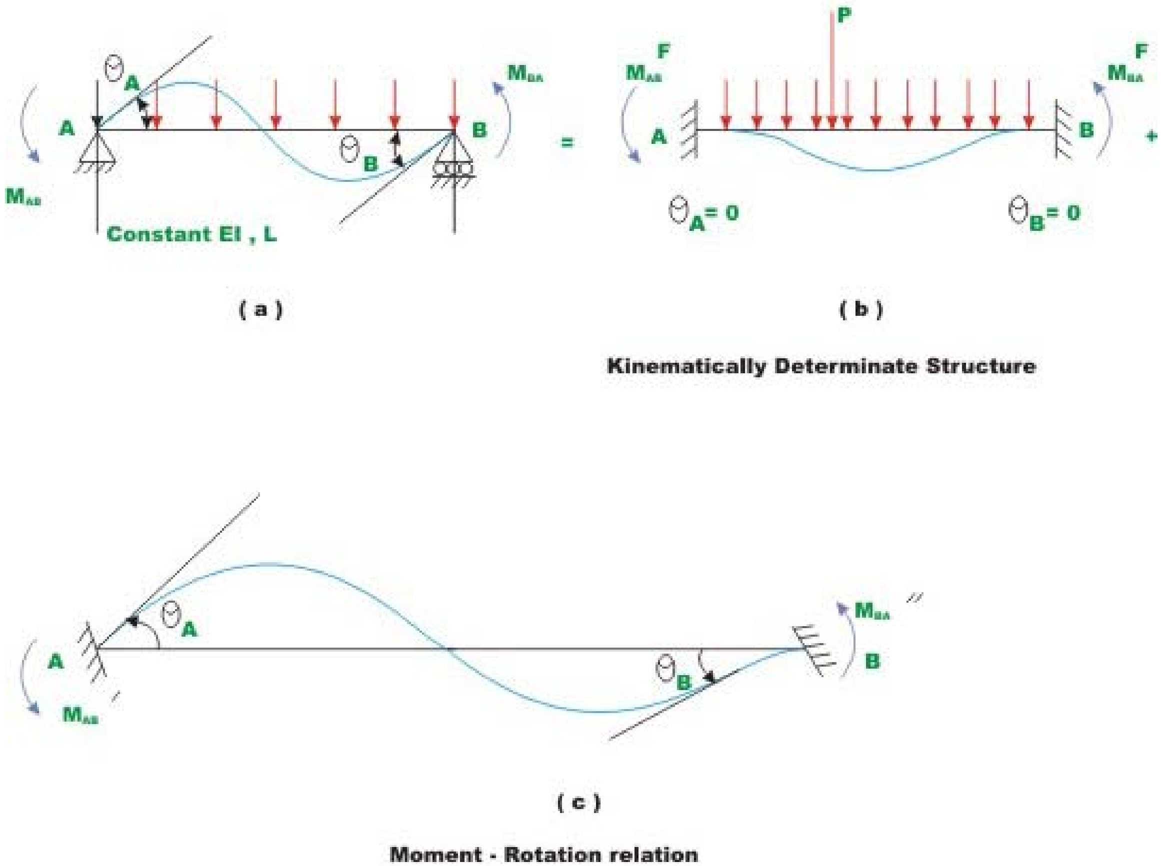


Fig.14.2 Derivation of slope - deflection equations

Hence in plane structures, each node can have at the most one linear displacement and one rotation. In this module first slope-deflection equations as applied to beams and rigid frames will be discussed.

Instructional Objectives

After reading this chapter the student will be able to
1. Calculate kinematic degrees of freedom of continuous beam.
2. Derive slope-deflection equations for the case beam with unyielding supports.
3. Differentiate between force method and displacement method of analyses.
4. State advantages of displacement method of analysis as compared to force method of analysis.
5. Analyse continuous beam using slope-deflection method.

14.1 Introduction

In this lesson the slope-deflection equations are derived for the case of a beam with unyielding supports .In this method, the unknown slopes and deflections at nodes are related to the applied loading on the structure. As introduced earlier, the slope-deflection method can be used to analyze statically determinate and indeterminate beams and frames. In this method it is assumed that all deformations are due to bending only. In other words deformations due to axial forces are neglected. As discussed earlier in the force method of analysis compatibility equations are written in terms of unknown reactions. It must be noted that all the unknown reactions appear in each of the compatibility equations making it difficult to solve resulting equations. The slope-deflection equations are not that lengthy in comparison.
The slope-deflection method was originally developed by Heinrich Manderla and Otto Mohr for computing secondary stresses in trusses. The method as used today was presented by G.A.Maney in 1915 for analyzing rigid jointed structures.

14.2 Slope-Deflection Equations

Consider a typical span of a continuous beam AB as shown in Fig.14.1.The beam has constant flexural rigidity EI and is subjected to uniformly distributed loading and concentrated loads as shown in the figure. The beam is kinematically indeterminate to second degree. In this lesson, the slope-deflection equations are derived for the simplest case i.e. for the case of continuous beams with unyielding supports. In the next lesson, the support settlements are included in the slope-deflection equations.

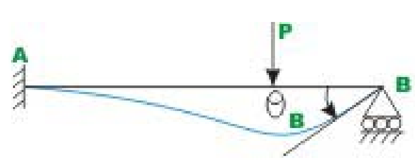


Fig. 14.01

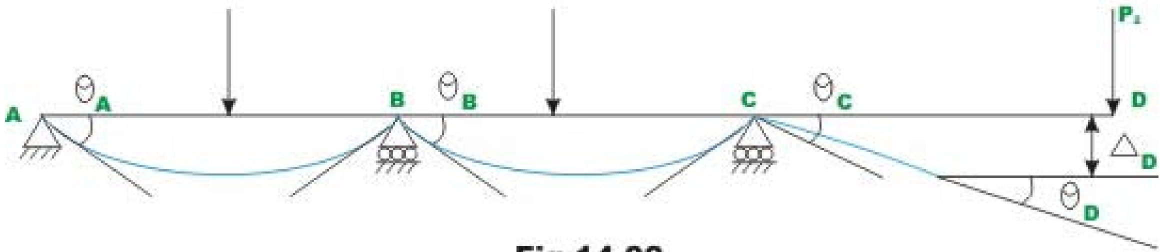


Fig.14.02

For this problem, it is required to derive relation between the joint end moments M_{AB} and M_{BA} in terms of joint rotations θ_A and θ_B and loads acting on the beam .Two subscripts are used to denote end moments. For example, end moments M_{AB} denote moment acting at joint *A* of the member *AB*. Rotations of the tangent to the elastic curve are denoted by one subscript. Thus, θ_A denotes the rotation of the tangent to the elastic curve at A. The following sign conventions are used in the slope-deflection equations (1) Moments acting at the ends of the member in counterclockwise direction are taken to be positive. (2) The rotation of the tangent to the elastic curve is taken to be positive when the tangent to the elastic curve has rotated in the counterclockwise direction from its original direction. The slope-deflection equations are derived by superimposing the end moments developed due to (1) applied loads (2) rotation θ_A (3) rotation θ_B. This is shown in Fig.14.2 (a)-(c). In Fig. 14.2(b) a kinematically determinate structure is obtained. This condition is obtained by modifying the support conditions to fixed so that the unknown joint rotations become zero. The structure shown in Fig.14.2 (b) is known as kinematically determinate structure or restrained structure. For this case, the end moments are denoted by M_{AB}^F and M_{BA}^F. The fixed end moments are evaluated by force–method of analysis as discussed in the previous module. For example for fixed- fixed beam subjected to uniformly distributed load, the fixed-end moments are shown in Fig.14.3.

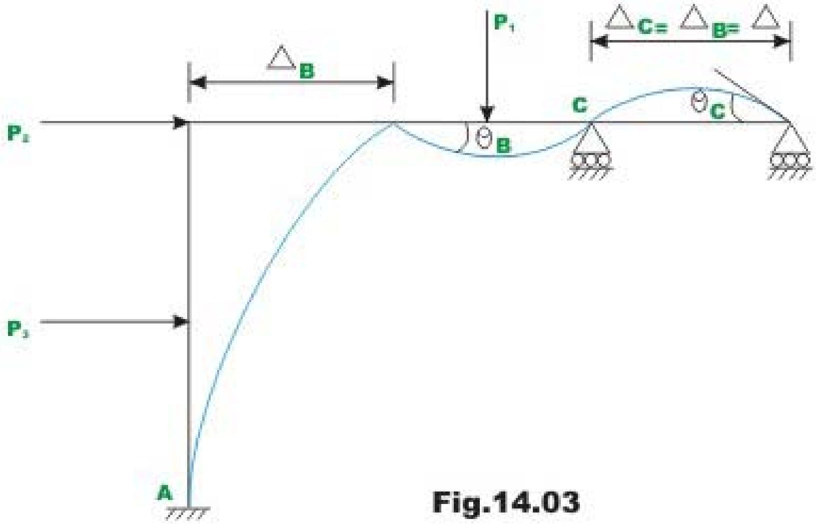


Fig.14.03

The fixed end moments are required for various load cases. For ease of calculations, fixed end forces for various load cases are given at the end of this lesson. In the actual structure end A rotates by θ_A and end B rotates by θ_B. Now it is required to derive a relation relating θ_A and θ_B with the end moments M'_{AB} and M'_{BA}. Towards this end, now consider a simply supported beam acted by moment M'_{AB} at A as shown in Fig. 14.4. The end moment M'_{AB} deflects the beam as shown in the figure. The rotations θ'_A and θ'_B are calculated from moment-area theorem.

$$\theta'_A = \frac{M'_{AB}L}{3EI} \qquad (14.1a)$$

$$\theta'_B = -\frac{M'_{AB}L}{6EI} \qquad (14.1b)$$

Now a similar relation may be derived if only M'_{BA} is acting at end B (see Fig. 14.4).

$$\theta''_B = \frac{M'_{BA}L}{3EI} \text{ and} \qquad (14.2a)$$

$$\theta''_A = -\frac{M'_{BA}L}{6EI} \qquad (14.2b)$$

Now combining these two relations, we could relate end moments acting at A and B to rotations produced at A and B as (see Fig. 14.2c)

$$\theta_A = \frac{M'_{AB}L}{3EI} - \frac{M'_{BA}L}{6EI} \qquad (14.3a)$$

$$\theta_B = \frac{M'_{BA}L}{3EI} - \frac{M'_{BA}L}{6EI} \qquad (14.3b)$$

Solving for M'_{AB} and M'_{BA} in terms of θ_A and θ_B,

$$M'_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B) \qquad (14.4)$$

$$M'_{BA} = \frac{2EI}{L}(2\theta_B + \theta_A) \qquad (14.5)$$

Now writing the equilibrium equation for joint moment at A (see Fig. 14.2).

$$M_{AB} = M^F_{AB} + M'_{AB} \qquad (14.6a)$$

Similarly writing equilibrium equation for joint B

$$M_{BA} = M^F_{BA} + M'_{BA} \qquad (14.6b)$$

Substituting the value of M'_{AB} from equation (14.4) in equation (14.6a) one obtains,

$$M_{AB} = M^F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B) \qquad (14.7a)$$

Similarly substituting M'_{BA} from equation (14.6b) in equation (14.6b) one obtains,

$$M_{BA} = M^F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A) \qquad (14.7b)$$

Sometimes one end is referred to as near end and the other end as the far end. In that case, the above equation may be stated as the internal moment at the near end of the span is equal to the fixed end moment at the near end due to external loads plus $\frac{2EI}{L}$ times the sum of twice the slope at the near end and the slope at the far end. The above two equations (14.7a) and (14.7b) simply referred to as slope–deflection equations. The slope-deflection equation is nothing but a load displacement relationship.

14.3 Application of Slope-Deflection Equations to Statically Indeterminate Beams.

The procedure is the same whether it is applied to beams or frames. It may be summarized as follows:

1. Identify all kinematic degrees of freedom for the given problem. This can be done by drawing the deflection shape of the structure. All degrees of freedom are treated as unknowns in slope-deflection method.
2. Determine the fixed end moments at each end of the span to applied load. The table given at the end of this lesson may be used for this purpose.
3. Express all internal end moments in terms of fixed end moments and near end, and far end joint rotations by slope-deflection equations.
4. Write down one equilibrium equation for each unknown joint rotation. For example, at a support in a continuous beam, the sum of all moments corresponding to an unknown joint rotation at that support must be zero. Write down as many equilibrium equations as there are unknown joint rotations.
5. Solve the above set of equilibrium equations for joint rotations.
6. Now substituting these joint rotations in the slope-deflection equations evaluate the end moments.
7. Determine all rotations.

Example 14.1

A continuous beam ABC is carrying uniformly distributed load of 2 kN/m in addition to a concentrated load of 20 kN as shown in Fig.14.5a. Draw bending moment and shear force diagrams. Assume *EI* to be constant.

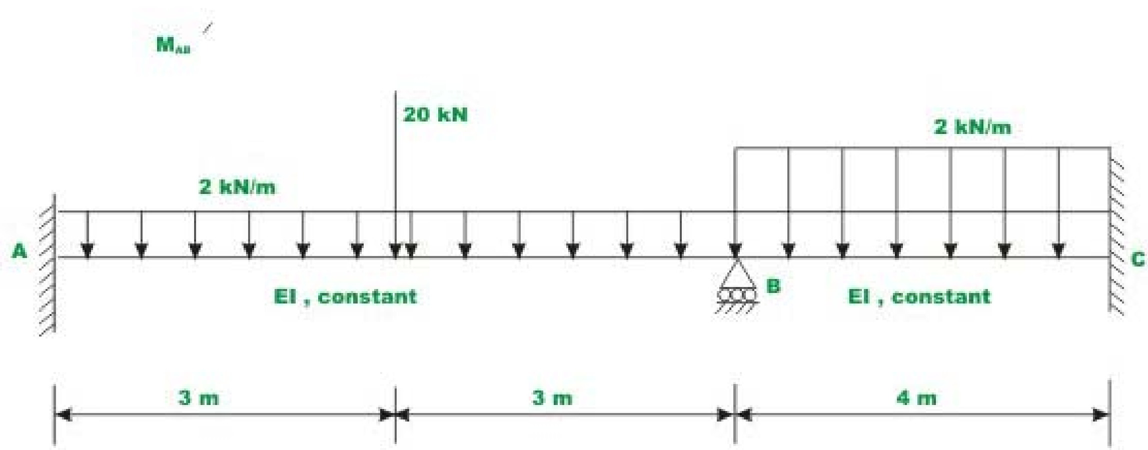


Fig. 14.5(a) Example 14.1

(a). Degrees of freedom

It is observed that the continuous beam is kinematically indeterminate to first degree as only one joint rotation θ_B is unknown. The deflected shape /elastic curve of the beam is drawn in Fig.14.5b in order to identify degrees of freedom. By fixing the support or restraining the support B against rotation, the fixed-fixed beams area obtained as shown in Fig.14.5c.

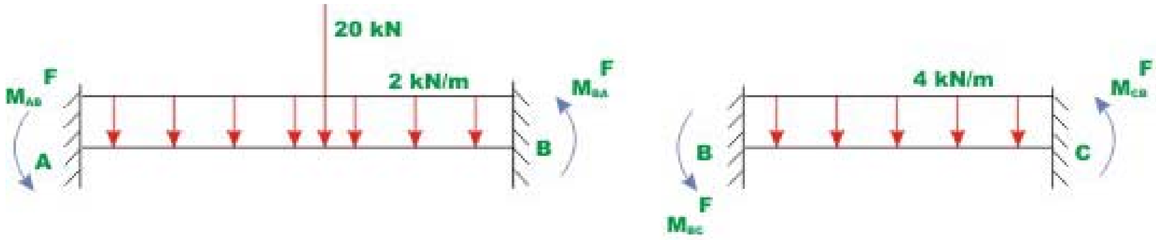


Fig. 14.5 (c) Restrained Structure.

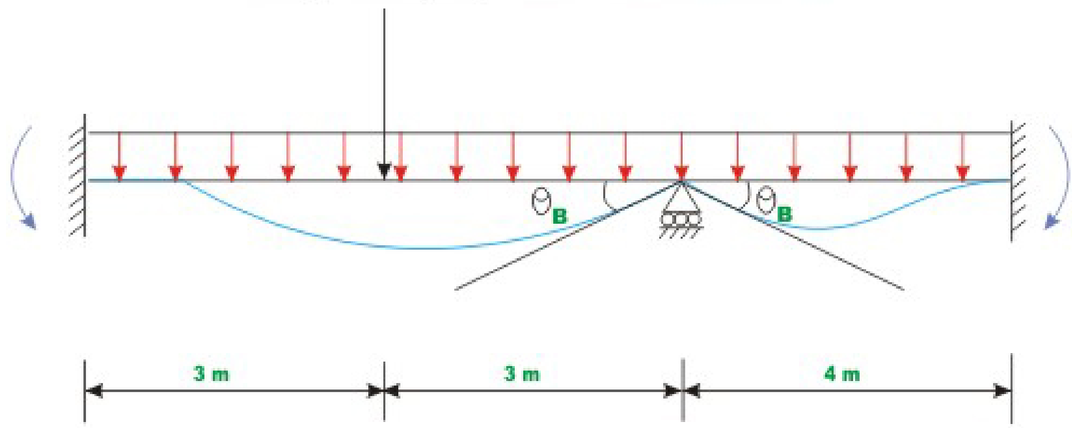


Fig. 14.5 (b) Elastic curve of the beam with unknown displacement component θ_B

(b). Fixed end moments $M^F_{AB}, M^F_{BA}, M^F_{BC}$ and M^F_{CB} are calculated referring to the Fig. 14. and following the sign conventions that counterclockwise moments are positive.

$$M^F_{AB} = \frac{2 \times 6^2}{12} + \frac{20 \times 3 \times 3^2}{6^2} = 21 \text{ kN.m}$$

$$M^F_{BA} = -21 \text{ kN.m}$$

$$M^F_{BC} = \frac{4 \times 4^2}{12} = 5.33 \text{ kN.m}$$

$$M^F_{CB} = -5.33 \text{ kN.m} \qquad (1)$$

(c) Slope-deflection equations

Since ends *A* and *C* are fixed, the rotation at the fixed supports is zero, $\theta_A = \theta_C = 0$. Only one non-zero rotation is to be evaluated for this problem. Now, write slope-deflection equations for span *AB* and *BC*.

$$M_{AB} = M^F_{AB} + \frac{2EI}{l}(2\theta_A + \theta_B)$$

$$M_{AB} = 21 + \frac{2EI}{6}\theta_B \qquad (2)$$

$$M_{BA} = -21 + \frac{2EI}{l}(2\theta_B + \theta_A)$$

$$M_{BA} = -21 + \frac{4EI}{6}\theta_B \qquad (3)$$

$$M_{BC} = 5.33 + EI\theta_B \qquad (4)$$

$$M_{CB} = -5.33 + 0.5EI\theta_B \qquad (5)$$

(d) Equilibrium equations

In the above four equations (2-5), the member end moments are expressed in terms of unknown rotation θ_B. Now, the required equation to solve for the rotation θ_B is the moment equilibrium equation at support B. The free body diagram of support B along with the support moments acting on it is shown in Fig. 14.5d. For, moment equilibrium at support B, one must have,

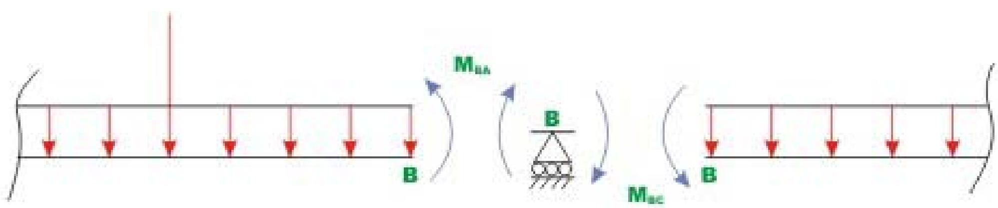


Fig. 14.5 d Free- body diagram of the joint B

$$\sum M_B = 0 \qquad M_{BA} + M_{BC} = 0 \qquad (6)$$

Substituting the values of M_{BA} and M_{BC} in the above equilibrium equation,

$$-21 + \frac{4EI}{6}\theta_B + 5.33 + EI\theta_B = 0$$

$$\Rightarrow 1.667\theta_B EI = 15.667$$

$$\theta_B = \frac{9.398}{EI} \cong \frac{9.40}{EI} \qquad (7)$$

(e) End moments

After evaluating θ_B, substitute it in equations (2-5) to evaluate beam end moments. Thus,

$$M_{AB} = 21 + \frac{EI}{3}\theta_B$$

$$M_{AB} = 21 + \frac{EI}{3} \times \frac{9.398}{EI} = 24.133 \text{kN.m}$$

$$M_{BA} = -21 + \frac{EI}{3}(2\theta_B)$$

$$M_{BA} = -21 + \frac{EI}{3} \times \frac{2 \times 9.4}{EI} = -14.733 \text{kN.m}$$

$$M_{BC} = 5.333 + \frac{9.4}{EI}EI = 14.733 \text{kN.m}$$

$$M_{CB} = -5.333 + \frac{9.4}{EI} \times \frac{EI}{2} = -0.63 \text{ kN.m} \qquad (8)$$

(f) Reactions
Now, reactions at supports are evaluated using equilibrium equations (vide Fig. 14.5e)

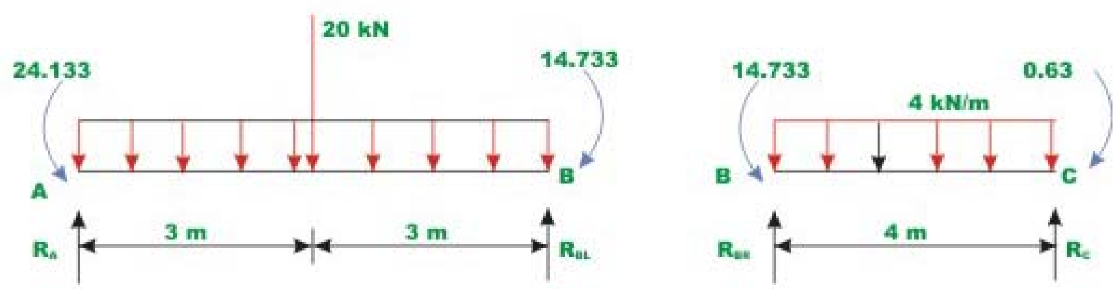
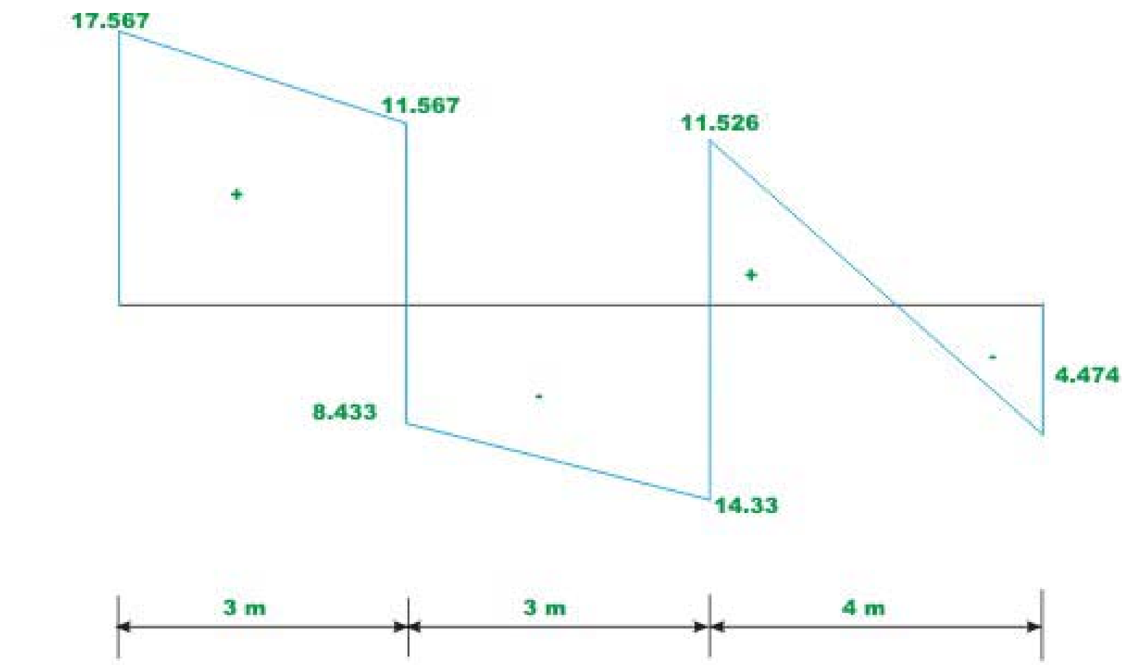


Fig. 14.5 (e) Free - body diagram of two members

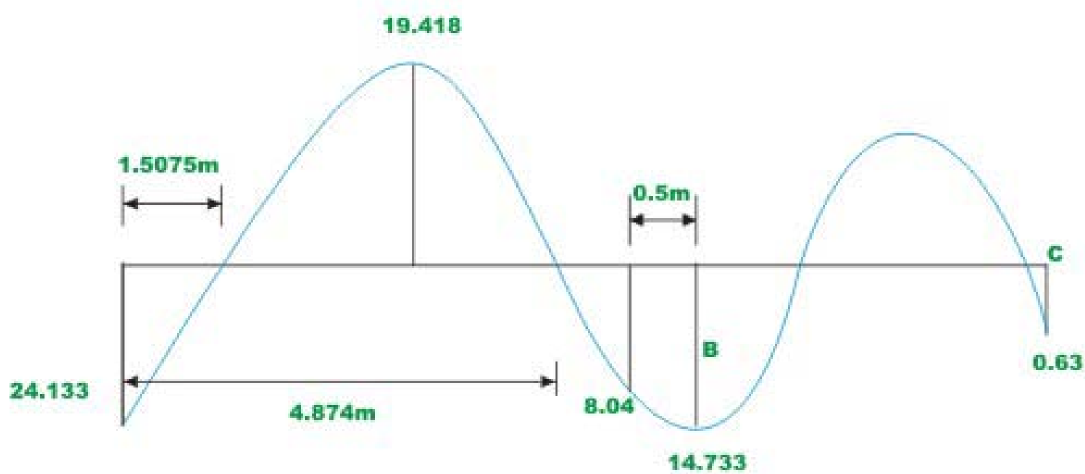
$$R_A \times 6 + 14.733 - 20 \times 3 - 2 \times 6 \times 3 - 24.133 = 0$$
$$R_A = 17.567 \text{ kN}(\uparrow)$$
$$R_{BL} = 16 - 1.567 = 14.433 \text{ kN}(\uparrow)$$
$$R_{BR} = 8 + \frac{14.733 - 0.63}{4} = 11.526 \text{ kN}(\uparrow)$$

$$R_C = 8 + 3.526 = 4.47 \text{ kN}(\uparrow) \qquad (9)$$

The shear force and bending moment diagrams are shown in Fig. 14.5f.



Shear force diagram



Bending Moment diagram

Fig. 14.5 f. Shear force and bending moment diagram of continuous beam ABC

Example 14.2

Draw shear force and bending moment diagram for the continuous beam $ABCD$ loaded as shown in Fig.14.6a.The relative stiffness of each span of the beam is also shown in the figure.

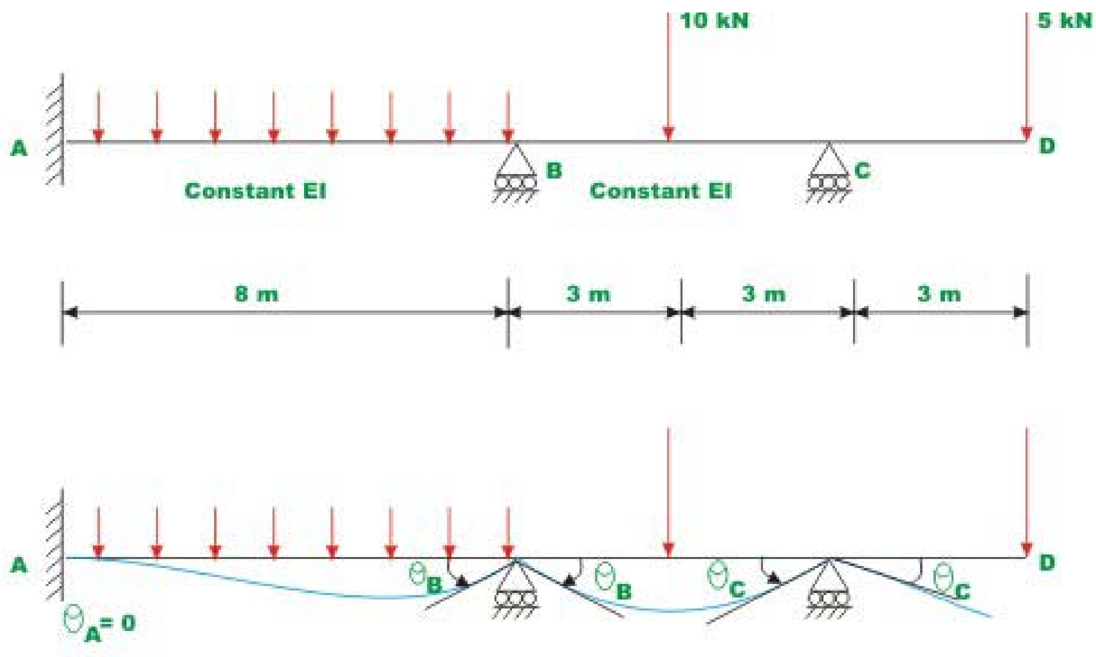


Fig. 14.6a Continuous beam of Example 14.2

For the cantilever beam portion *CD*, no slope-deflection equation need to be written as there is no internal moment at end *D*. First, fixing the supports at *B* and *C*, calculate the fixed end moments for span *AB* and *BC*. Thus,

$$M_{AB}^{F} = \frac{3 \times 8^2}{12} = 16 \text{ kN.m}$$

$$M_{BA}^{F} = -16 \text{ kN.m}$$

$$M_{BC}^{F} = \frac{10 \times 3 \times 3^2}{6^2} = 7.5 \text{ kN.m}$$

$$M_{CB}^{F} = -7.5 \text{ kN.m} \qquad (1)$$

In the next step write slope-deflection equation. There are two equations for each span of the continuous beam.

$$M_{AB} = 16 + \frac{2EI}{8}(\theta_B) = 16 + 0.25\theta_B EI$$
$$M_{BA} = -16 + 0.5\theta_B EI$$
$$M_{BC} = 7.5 + \frac{2 \times 2EI}{6}(2\theta_B + \theta_C) = 7.5 + 1.334EI\theta_B + 0.667EI\theta_C$$
$$M_{CB} = -7.5 + 1.334EI\theta_C + 0.667EI\theta_B \quad (2)$$

Equilibrium equations

The free body diagram of members AB, BC and joints B and C are shown in Fig.14.6b.One could write one equilibrium equation for each joint B and C.

Fig. 14.6 b Free - body diagrams of joints B and C along with members

Support B,

$$\sum M_B = 0 \qquad M_{BA} + M_{BC} = 0 \quad (3)$$

$$\sum M_C = 0 \qquad M_{CB} + M_{CD} = 0 \quad (4)$$

We know that $M_{CD} = 15$ kN.m (5)

$$\Rightarrow M_{CB} = -15 \text{ kN.m} \quad (6)$$

Substituting the values of M_{CB} and M_{CD} in the above equations for M_{AB}, M_{BA}, M_{BC} and M_{CB} we get,

$$\theta_B = \frac{24.5}{3.001} = 8.164$$

$$\theta_C = 9.704 \quad (7)$$

Substituting θ_B, θ_C in the slope-deflection equations, we get

$$M_{AB} = 16 + 0.25EI\theta_B = 16 + 0.25EI \times \frac{8.164}{EI} = 18.04 \text{ kN.m}$$

$$M_{BA} = -16 + 0.5EI\theta_B = -16 + 0.5EI \times \frac{8.164}{EI} = -11.918 \text{ kN.m}$$

$$M_{BC} = 7.5 + 1.334EI \times \frac{8.164}{EI} + 0.667EI(\frac{9.704}{EI}) = 11.918 \text{ kN.m}$$

$$M_{CB} = -7.5 + 0.667EI \times \frac{8.164}{EI} + 1.334EI(-\frac{9.704}{EI}) = -15 \text{ kN.m} \qquad (8)$$

Reactions are obtained from equilibrium equations (ref. Fig. 14.6c)

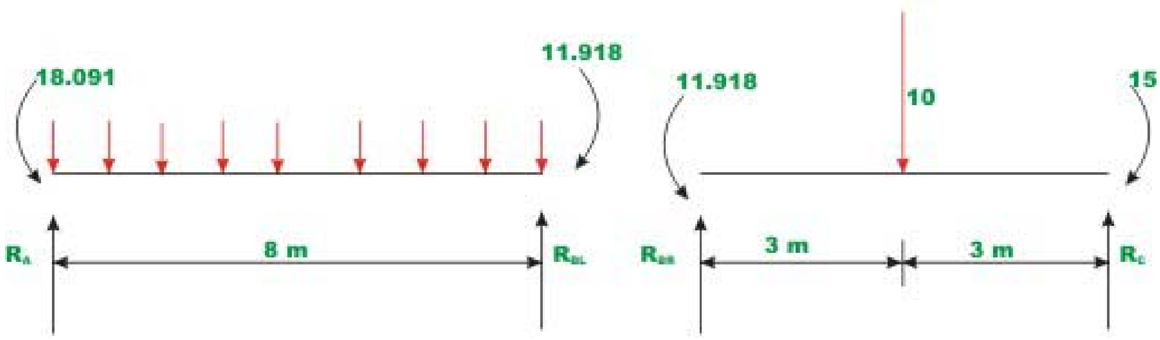


Fig. 14.6 c Computation of reactions

$$R_A \times 8 - 18.041 - 3 \times 8 \times 4 + 11.918 = 0$$

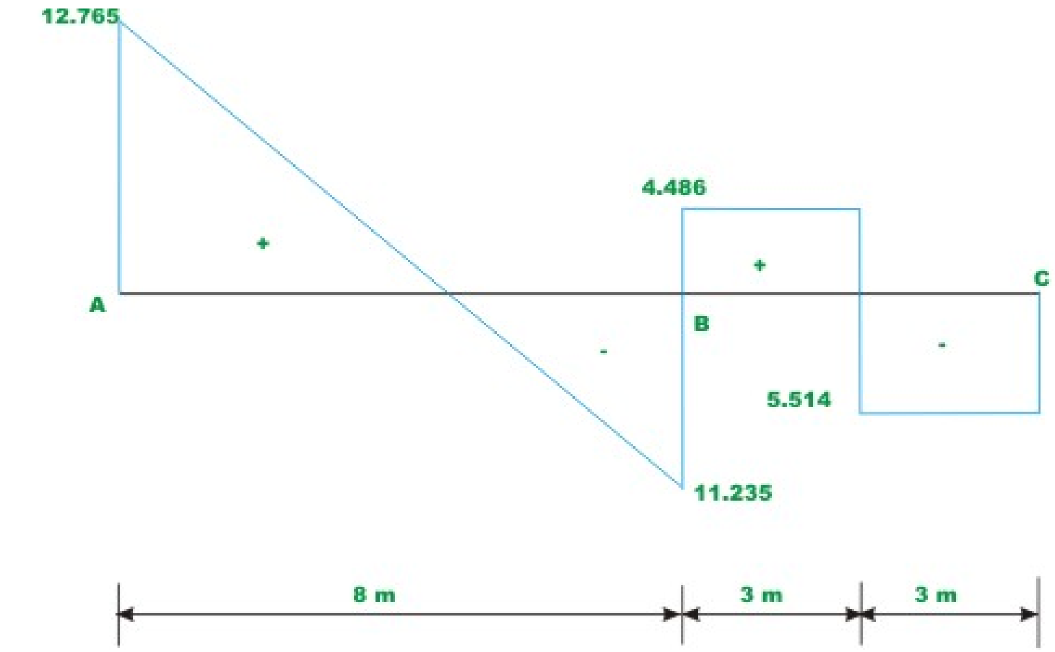
$$R_A = 12.765 \text{ kN}$$

$$R_{BR} = 5 - 0.514kN = 4.486 \text{ kN}$$

$$R_{BL} = 11.235 \text{ kN}$$

$$R_C = 5 + 0.514kN = 5.514 \text{ kN}$$

The shear force and bending moment diagrams are shown in Fig. 14.6d.



Shear force diagram

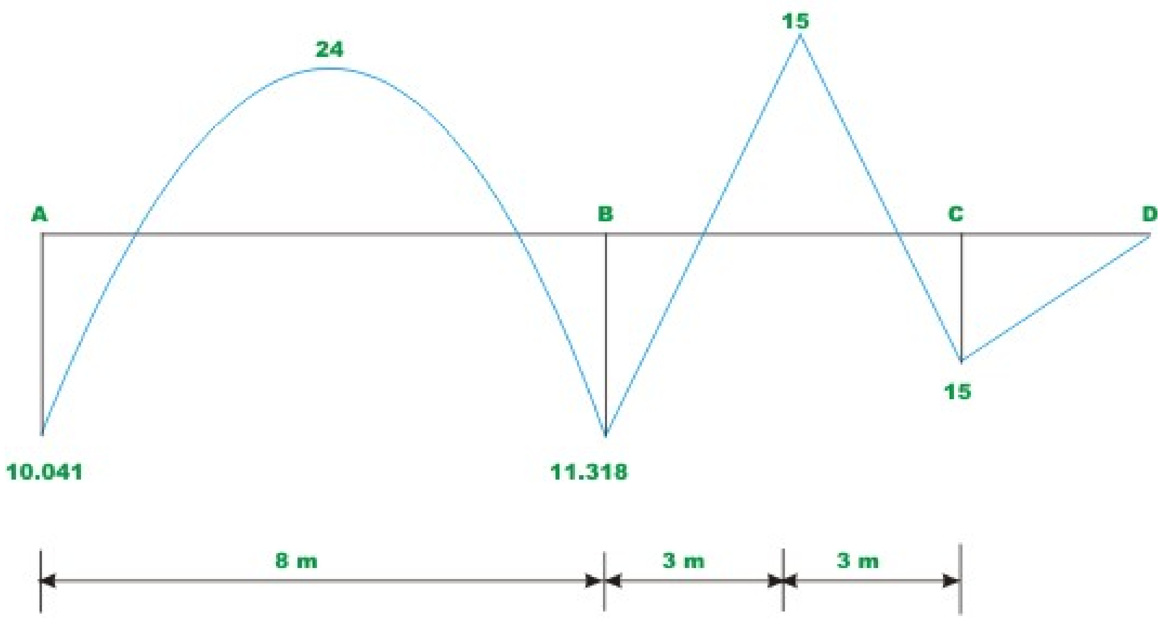


Fig. 14.6 (d) Shear force and bending moment diagram

For ease of calculations, fixed end forces for various load cases are given in Fig. 14.7.

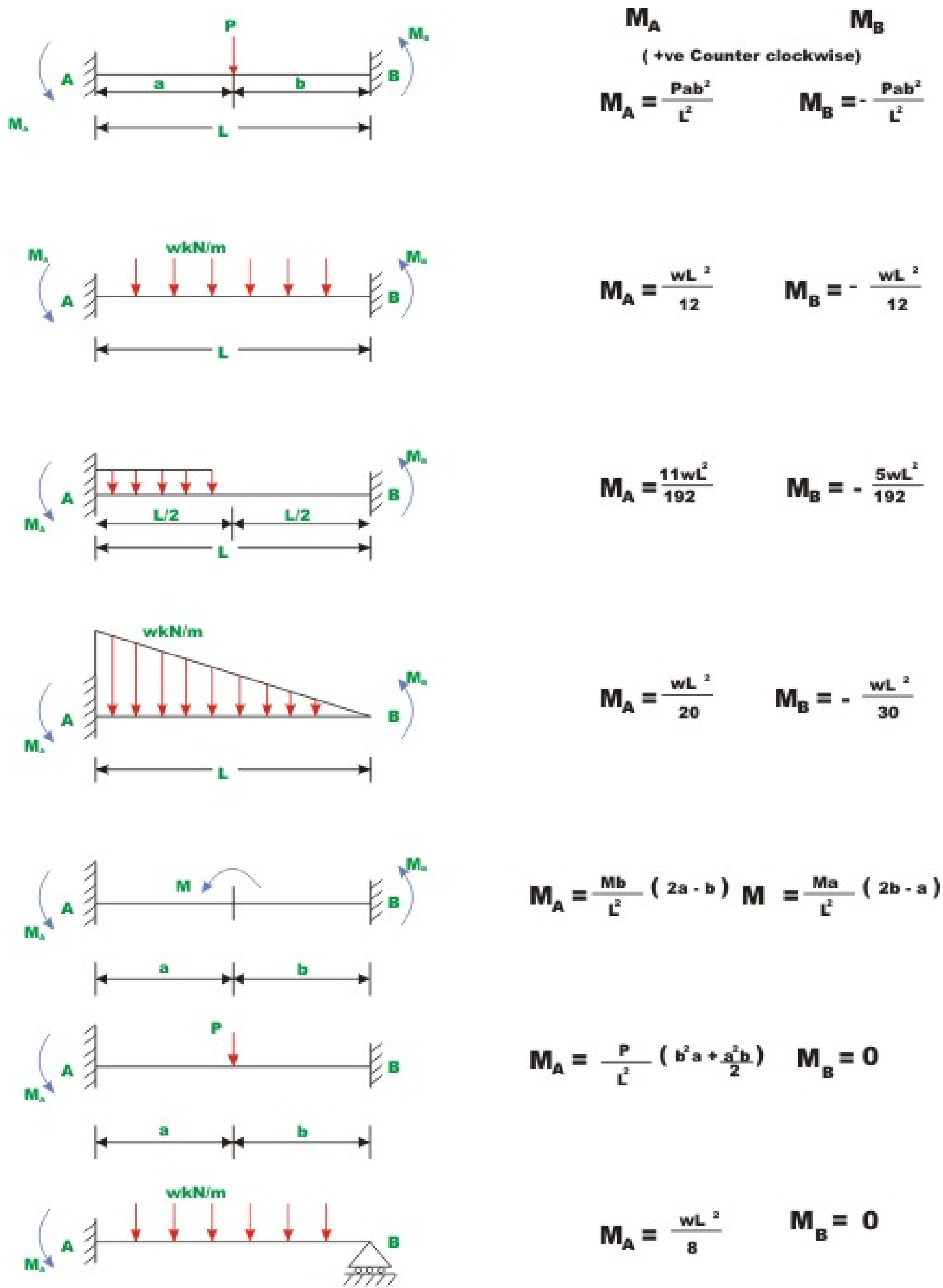


Fig. 14.7 Table of fixed end moments

Summary

In this lesson the slope-deflection equations are derived for beams with unyielding supports. The kinematically indeterminate beams are analysed by slope-deflection equations. The advantages of displacement method of analysis over force method of analysis are clearly brought out here. A couple of examples are solved to illustrate the slope-deflection equations.

Module 3

Analysis of Statically Indeterminate Structures by the Displacement Method

Lesson
15
The Slope-Deflection Method: Beams (Continued)

Instructional Objectives

After reading this chapter the student will be able to

1. Derive slope-deflection equations for the case beam with yielding supports.
2. Estimate the reactions induced in the beam due to support settlements.
3. Analyse the beam undergoing support settlements and subjected to external loads.
4. Write joint equilibrium equations in terms of moments.
5. Relate moments to joint rotations and support settlements.

15.1 Introduction

In the last lesson, slope-deflection equations were derived without considering the rotation of the beam axis. In this lesson, slope-deflection equations are derived considering the rotation of beam axis. In statically indeterminate structures, the beam axis rotates due to support yielding and this would in turn induce reactions and stresses in the structure. Hence, in this case the beam end moments are related to rotations, applied loads and beam axes rotation. After deriving the slope-deflection equation in section 15.2, few problems are solved to illustrate the procedure.

Consider a beam AB as shown in Fig.15.1.The support B is at a higher elevation compared to A by an amount Δ. Hence, the member axis has rotated by an amount ψ from the original direction as shown in the figure. Let L be the span of the beam and flexural rigidity of the beam EI, is assumed to be constant for the beam. The chord has rotated in the counterclockwise direction with respect to its original direction. The counterclockwise moment and rotations are assumed to be positive. As stated earlier, the slopes and rotations are derived by superposing the end moments developed due to

(1) Externally applied moments on beams.
(2) Displacements θ_A, θ_B and Δ (settlement)

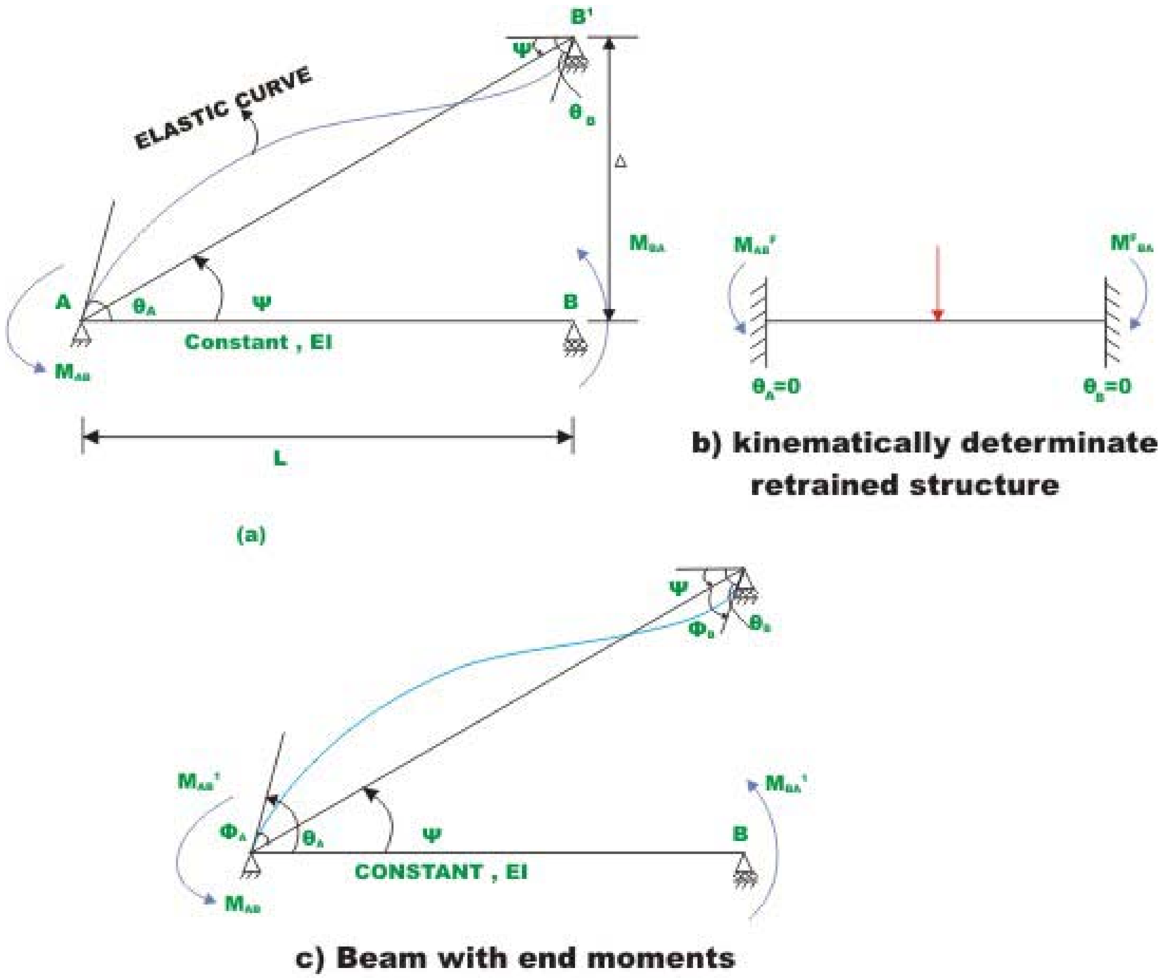


Figure 15.1

The given beam with initial support settlement may be thought of as superposition of two simple cases as shown in Fig.15.1 (b) and in Fig. 15.1(c). In Fig.15.1b, the kinematically determinate beam is shown with the applied load. For this case, the fixed end moments are calculated by force method. Let ϕ_A and ϕ_B be the end rotations of the elastic curve with respect to rotated beam axis AB' (see Fig.15.1c) that are caused by end moments $M^{'}_{AB}$ and $M^{'}_{BA}$. Assuming that rotations and displacements shown in Fig.15.1c are so small that

$$\tan \psi = \psi = \frac{\Delta}{l} \qquad (15.1)$$

Also, using the moment area theorem, ϕ_A and ϕ_B are written as

$$\phi_A = \theta_A - \psi = \frac{M_{AB}' L}{3EI} - \frac{M_{AB}' L}{6EI} \qquad (15.2a)$$

$$\phi_B = \theta_B - \psi = \frac{M_{BA}'L}{3EI} - \frac{M_{AB}'L}{6EI} \qquad (15.2b)$$

Now solving for $M_A^{'}$ and $M_B^{'}$ in terms of θ_A, θ_B and ψ ,

$$M_{AB}' = \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi) \qquad (15.3a)$$

$$M_{BA}' = \frac{2EI}{L}(2\theta_B + \theta_A - 3\psi) \qquad (15.3b)$$

Now superposing the fixed end moments due to external load and end moments due to displacements, the end moments in the actual structure is obtained .Thus (see Fig.15.1)

$$M_{AB} = M_{AB}^{F} + M_{AB}' \qquad (15.4a)$$

$$M_{BA} = M_{BA}^{F} + M_{BA}' \qquad (15.4b)$$

Substituting for $M_{AB}^{'}$ and $M_{BA}^{'}$ in equation (15.4a) and (15.4b), the slope-deflection equations for the general case are obtained. Thus,

$$M_{AB} = M_{AB}^{F} + \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi) \qquad (15.5a)$$

$$M_{BA} = M_{BA}^{F} + \frac{2EI}{L}(2\theta_B + \theta_A - 3\psi) \qquad (15.5b)$$

In the above equations, it is important to adopt consistent sign convention. In the above derivation Δ is taken to be negative for downward displacements.

Example 15.1

Calculate the support moments in the continuous beam ABC (see Fig.15.2a) having constant flexural rigidity EI throughout ,due to vertical settlement of the support B by 5mm. Assume E =200 GPa and $I = 4 \times 10^{-4} m^4$ .Also plot quantitative elastic curve.

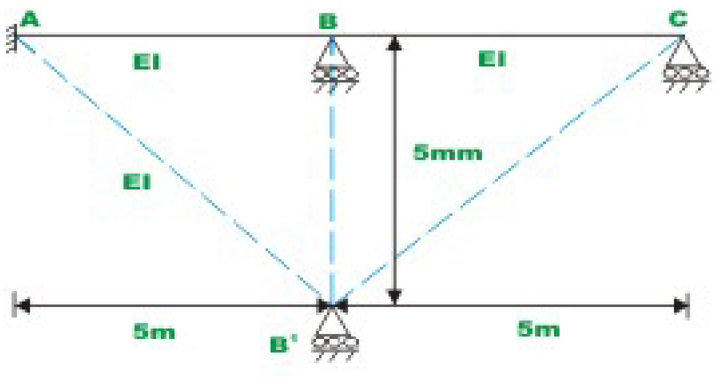


Figure 15.2 (a)

In the continuous beam ABC, two rotations θ_B and θ_C need to be evaluated. Hence, beam is kinematically indeterminate to second degree. As there is no external load on the beam, the fixed end moments in the restrained beam are zero (see Fig.15.2b).

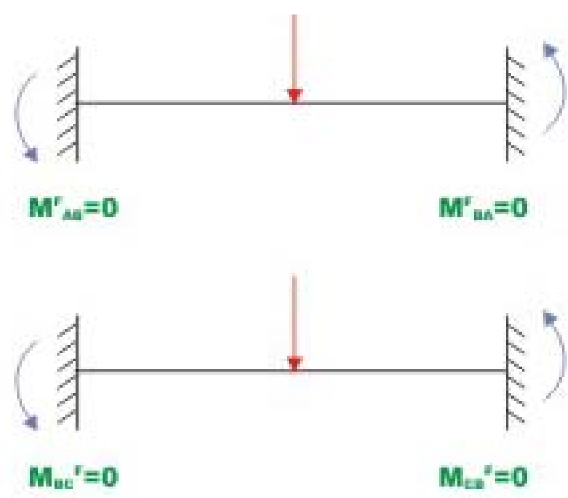


Figure 15.2 (b)

For each span, two slope-deflection equations need to be written. In span AB, B is below A. Hence, the chord AB rotates in clockwise direction. Thus, ψ_{AB} is taken as negative.

$$\psi_{AB} = \frac{-5\times10^{-3}}{5} = -1\times10^{-3} \qquad (1)$$

Writing slope-deflection equation for span AB,

$$M_{AB} = \frac{2EI}{L}\left(2\theta_A + \theta_B - 3\psi_{AB}\right)$$

For span AB, $\theta_A = 0,$ Hence,

$$M_{AB} = \frac{2EI}{5}\left(\theta_B + 3\times10^{-3}\right)$$

$$M_{AB} = O.4EI\theta_B + .0012EI \qquad (2)$$

Similarly, for beam-end moment at end B, in span AB

$$M_{BA} = 0.4EI\left(2\theta_B + 3\times10^{-3}\right)$$
$$M_{BA} = 0.8EI\theta_B + 0.0012EI \qquad (3)$$

In span BC, the support C is above support B, Hence the chord joining $B'C$ rotates in anticlockwise direction.

$$\psi_{BC} = \psi_{CB} = 1\times10^{-3} \qquad (4)$$

Writing slope-deflection equations for span BC,

$$M_{BC} = 0.8EI\theta_B + 0.4EI\theta_C - 1.2\times10^{-3}EI$$

$$M_{CB} = 0.8EI\theta_C + 0.4EI\theta_B - 1.2\times10^{-3}EI \qquad (5)$$

Now, consider the joint equilibrium of support B (see Fig.15.2c)

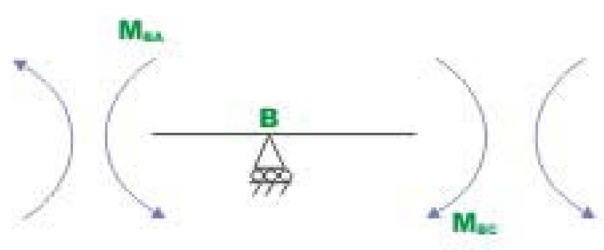


Fig 15.2c Free body diagram of joint B

$$M_{BA} + M_{BC} = 0 \quad (6)$$

Substituting the values of M_{BA} and M_{BC} in equation (6),

$$0.8EI\theta_B + 1.2\times 10^{-3} EI + 0.8EI\theta_B + 0.4EI\theta_C - 1.2\times 10^{-3} EI = 0$$

Simplifying,

$$1.6\theta_B + 0.4\theta_C = 1.2\times 10^{-3} \quad (7)$$

Also, the support C is simply supported and hence, $M_{CB} = 0$

$$M_{CB} = 0 = 0.8\theta_C + 0.4\theta_B - 1.2\times 10^{-3} EI$$

$$0.8\theta_C + 0.4\theta_B = 1.2\times 10^{-3} \quad (8)$$

We have two unknowns θ_B and θ_C and there are two equations in θ_B and θ_C. Solving equations (7) and (8)

$$\theta_B = -0.4286\times 10^{-3} \text{ radians}$$

$$\theta_C = 1.7143\times 10^{-3} \text{ radians} \quad (9)$$

Substituting the values of θ_B, θ_C and EI in slope-deflection equations,

$$M_{AB} = 82.285 \text{ kN.m}$$

$$M_{BA} = 68.570 \text{ kN.m}$$

$$M_{BC} = -68.573 \text{ kN.m}$$

$$M_{CB} = 0 \text{ kN.m} \quad (10)$$

Reactions are obtained from equations of static equilibrium (vide Fig.15.2d)

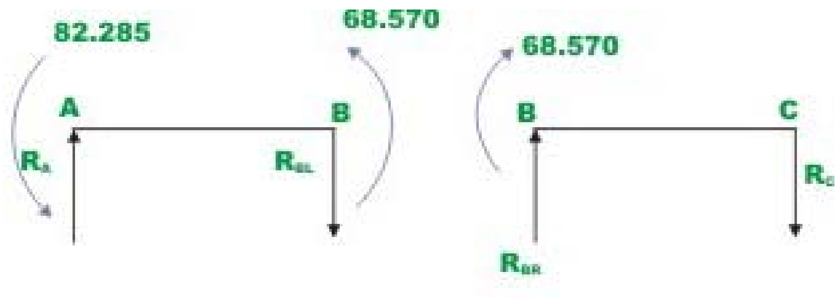


Fig 15.2d Computation of reactions

In beam AB ,

$$\sum M_B = 0\,,\ R_A = 30.171\,\mathrm{kN}(\uparrow)$$

$$R_{BL} = -30.171\,\mathrm{kN}(\downarrow)$$

$$R_{BR} = -13.714\,\mathrm{kN}(\downarrow)$$

$$R_C = 13.714\,\mathrm{kN}(\uparrow)$$

The shear force and bending moment diagram is shown in Fig.15.2e and elastic curve is shown in Fig.15.2f.

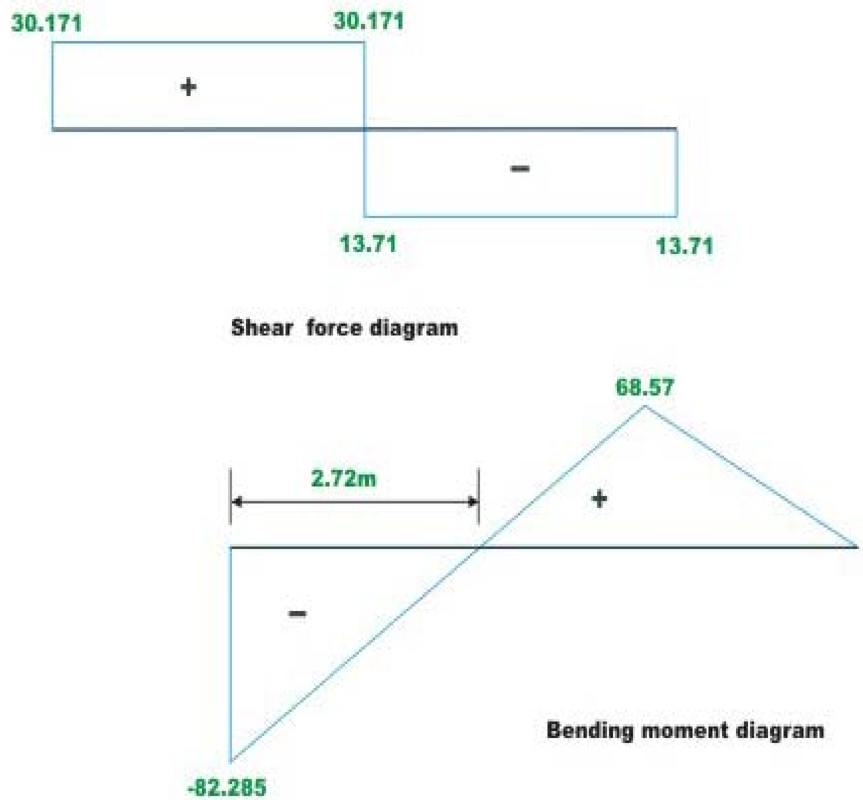
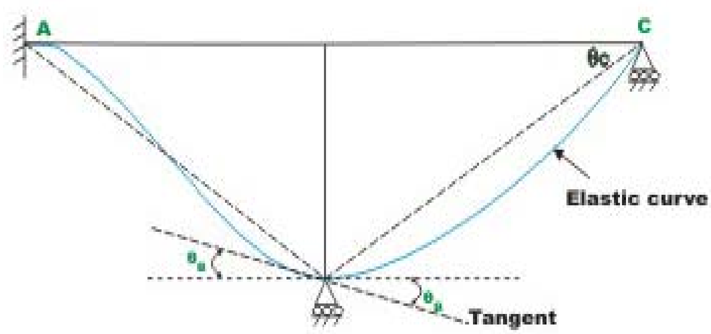


Figure 15.2e Shear force and bending moment diagram



15.2 f Elasctic curve

Example 15.2

A continuous beam *ABCD* is carrying a uniformly distributed load of 5 kN/m as shown in Fig.15.3a. Compute reactions and draw shear force and bending moment diagram due to following support settlements.

Support B 0.005m vertically downwards
Support C 0.01 m vertically downwards
Assume E =200 GPa, $I = 1.35 \times 10^{-3} m^4$

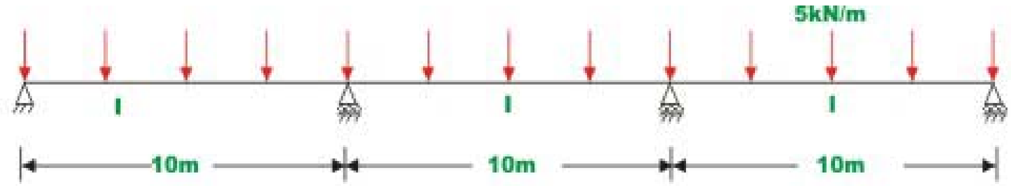


Fig 15.3a Continuous beam of Example 15.2

In the above continuous beam, four rotations $\theta_A, \theta_B, \theta_C$ and θ_D are to be evaluated. One equilibrium equation can be written at each support.Hence, solving the four equilibrium equations, the rotations are evaluated and hence the moments from slope-deflection equations. Now consider the kinematically restrained beam as shown in Fig.15.3b.

Referring to standard tables the fixed end moments may be evaluated .Otherwise one could obtain fixed end moments from force method of analysis. The fixed end moments in the present case are (vide fig.15.3b)

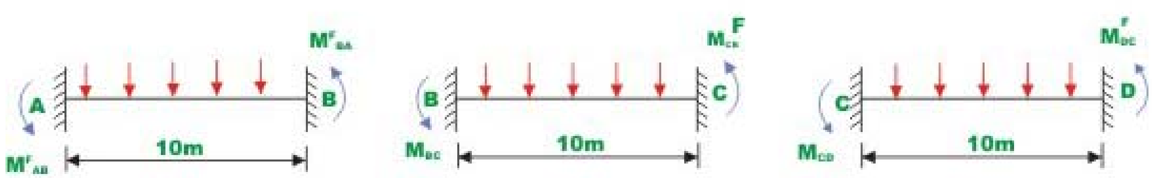


Fig 15.3b Kinematically restrained beam

$$M_{AB}^F = 41.667 \text{ kN.m}$$

$$M_{BA}^F = -41.667 \text{ kN.m} \text{ (clockwise)}$$

$$M_{BC}^F = 41.667 \text{ kN.m} \text{ (counterclockwise)}$$

$$M_{CB}^F = -41.667 \text{ kN.m} \text{ (clockwise)}$$

$$M_{CD}^{F} = 41.667 \text{ kN.m} \text{ (counterclockwise)}$$

$$M_{DC}^{F} = -41.667 \text{ kN.m} \text{ (clockwise)} \qquad (1)$$

In the next step, write slope-deflection equations for each span. In the span AB, B is below A and hence the chord joining AB' rotates in the clockwise direction (see Fig.15.3c)

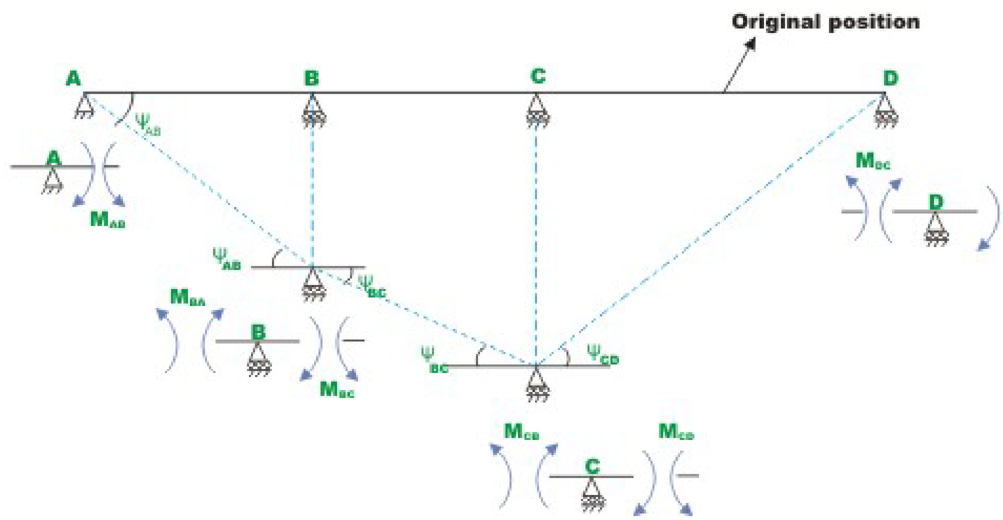


Fig 15.3c New support positions and free body diagrams of support

$\psi_{AB} = \dfrac{0 - 0.005}{10} = -0.0005$ radians (negative as the chord AB' rotates in the clockwise direction from the original direction)

$\psi_{BC} = -0.0005$ radians (negative as the chord $B'C'$ rotates in the clockwise direction)

$\psi_{CD} = \dfrac{0.01}{10} = 0.001$ radians (positive as the chord $C'D$ rotates in the counter clockwise direction from the original direction) (2)

Now, writing the expressions for the span end moments, for each of the spans,

$$M_{AB} = 41.667 + 0.2EI(2\theta_A + \theta_B + 0.0005)$$
$$M_{BA} = -41.667 + 0.2EI(2\theta_B + \theta_A + 0.0005)$$

$$M_{BC} = 41.667 + 0.2EI(2\theta_B + \theta_C + 0.0005)$$
$$M_{CB} = -41.667 + 0.2EI(2\theta_C + \theta_B + 0.0005)$$

$$M_{CD} = 41.667 + 0.2EI(2\theta_C + \theta_D - 0.001)$$
$$M_{DC} = -41.667 + 0.2EI(2\theta_D + \theta_C - 0.001) \qquad (3)$$

For the present problem, four joint equilibrium equations can be written, one each for each of the supports. They are

$$\sum M_A = 0 \Rightarrow M_{AB} = 0$$

$$\sum M_B = 0 \Rightarrow M_{BA} + M_{BC} = 0$$

$$\sum M_C = 0 \Rightarrow M_{CB} + M_{CD} = 0$$

$$\sum M_D = 0 \Rightarrow M_{DC} = 0 \qquad (4)$$

Substituting the values of beam end moments from equations (3) in equation (4), four equations are obtained in four unknown rotations $\theta_A, \theta_B, \theta_C$ and θ_D.They are,

$$(EI = 200 \times 10^3 \times 1.35 \times 10^{-6} = 270{,}000 \text{ kN.m}^2)$$

$$2\theta_A + \theta_B = -1.2716 \times 10^{-3}$$

$$\theta_A + 4\theta_B + \theta_C = -0.001$$

$$\theta_B + 4\theta_C + \theta_D = 0.0005$$

$$\theta_C + 2\theta_D = 1.7716 \times 10^{-3} \qquad (5)$$

Solving the above sets of simultaneous equations, values of $\theta_A, \theta_B, \theta_C$ and θ_D are evaluated.

$$\theta_A = -5.9629 \times 10^{-4} \quad \text{radians}$$

$$\theta_B = -7.9013 \times 10^{-5} \quad \text{radians}$$

$$\theta_C = -8.7653 \times 10^{-5} \quad \text{radians}$$

$$\theta_D = 9.2963 \times 10^{-4} \quad \text{radians} \qquad (6)$$

Substituting the values in slope-deflection equations the beam end moments are evaluated.

$$M_{AB} = 41.667 + 0.2\times 270{,}000\{2(-5.9629\times 10^{-4}) + (-7.9013\times 10^{-5}) + 0.0005)\} = 0$$

$$M_{BA} = -41.667 + 0.2\times 270{,}000\{2(-7.9013\times 10^{-5}) - 5.9629\times 10^{-4} + 0.0005\} = -55.40 \text{ kN.m}$$

$$M_{BC} = 41.667 + 0.2\times 270{,}000\{2(-7.9013\times 10^{-5}) + (-8.7653\times 10^{-5}) + 0.0005\} = 55.40 \text{ kN.m}$$

$$M_{CB} = -41.667 + 0.2\times 270{,}000\{2(-8.765\times 10^{-5}) - 7.9013\times 10^{-5} + 0.0005\} = -28.40 \text{ kN.m}$$

$$M_{CD} = 41.667 + 0.2\times 270{,}000\{2\times(-8.765\times 10^{-5}) + 9.2963\times 10^{-4} - 0.001\} = 28.40 \text{ kN.m}$$

$$M_{DC} = -41.667 + 0.2\times 270{,}000\{2\times 9.2963\times 10^{-4} - 8.7653\times 10^{-5} - 0.001\} = 0 \text{ kN.m} \quad (7)$$

Reactions are obtained from equilibrium equations. Now consider the free body diagram of the beam with end moments and external loads as shown in Fig.15.3d.

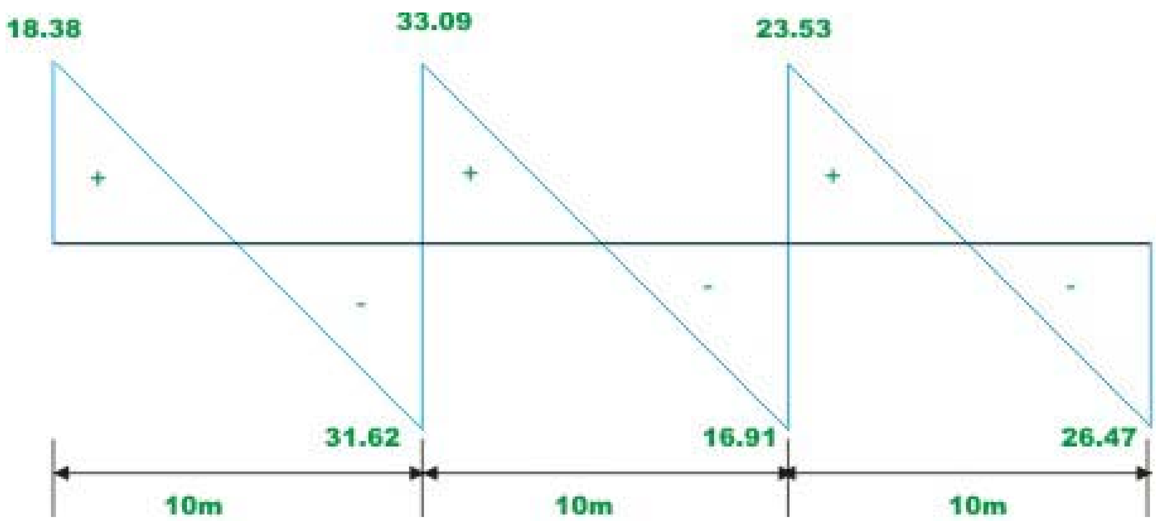


Fig 15.3d Shear force diagram

$$R_A = 19.46 \text{ kN}(\uparrow)$$

$$R_{BL} = 30.54 \text{ kN}(\uparrow)$$

$$R_{BR} = 27.7 \text{ kN}(\uparrow)$$

$$R_{CL} = 22.3 \text{ kN}(\uparrow)$$

$$R_{CR} = 27.84 \text{ kN}(\uparrow)$$

$$R_D = 22.16 \text{ kN}(\uparrow)$$

The shear force and bending moment diagrams are shown in Fig.15.5e.

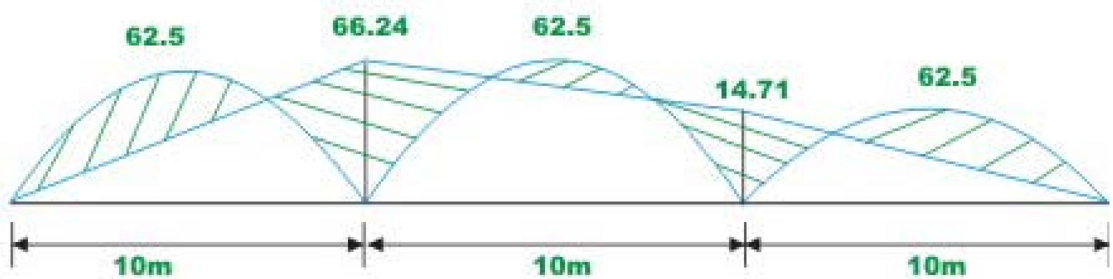


Fig. 15.3e Bending moment diagram

Summary

In this lesson, slope-deflection equations are derived for the case of beam with yielding supports. Moments developed at the ends are related to rotations and support settlements. The equilibrium equations are written at each support. The continuous beam is solved using slope-deflection equations. The deflected shape of the beam is sketched. The bending moment and shear force diagrams are drawn for the examples solved in this lesson.

Module 3

Analysis of Statically Indeterminate Structures by the Displacement Method

Lesson
16
The Slope-Deflection Method: Frames Without Sidesway

Instructional Objectives

After reading this chapter the student will be able to
1. State whether plane frames are restrained against sidesway or not.
2. Able to analyse plane frames restrained against sidesway by slope-deflection equations.
3. Draw bending moment and shear force diagrams for the plane frame.
4. Sketch the deflected shape of the plane frame.

16.1 Introduction

In this lesson, slope deflection equations are applied to solve the statically indeterminate frames without sidesway. In frames axial deformations are much smaller than the bending deformations and are neglected in the analysis. With this assumption the frames shown in Fig 16.1 will not sidesway. i.e. the frames will not be displaced to the right or left. The frames shown in Fig 16.1(a) and Fig 16.1(b) are properly restrained against sidesway. For example in Fig 16.1(a) the joint can't move to the right or left without support A also moving .This is true also for joint D. Frames shown in Fig 16.1 (c) and (d) are not restrained against sidesway. However the frames are symmetrical in geometry and in loading and hence these will not sidesway. In general, frames do not sidesway if

1) They are restrained against sidesway.
2) The frame geometry and loading is symmetrical

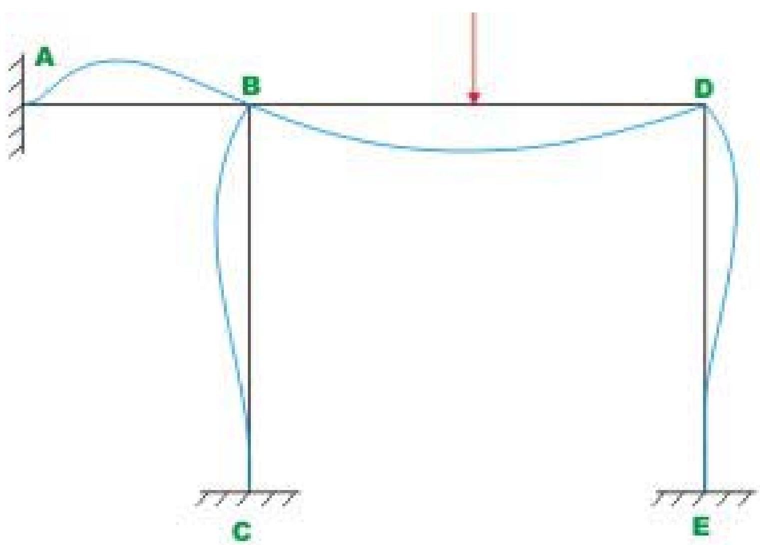


Fig- 16.1(a)

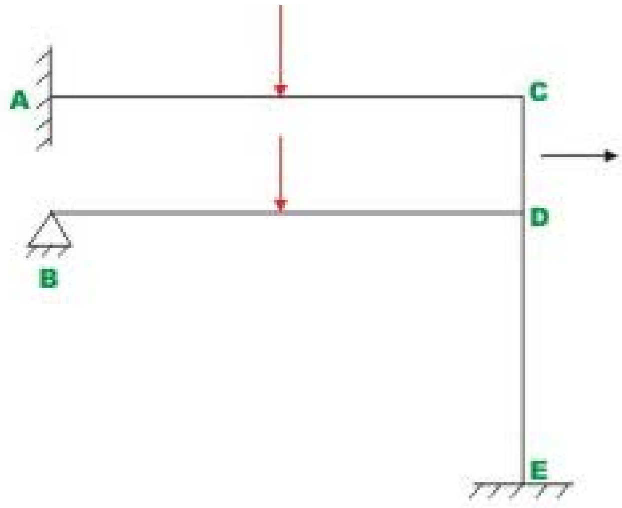


Fig- 16.1(b)

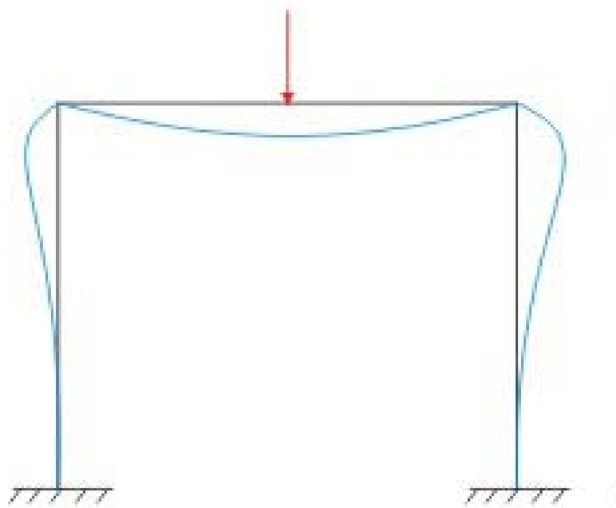


Fig- 16.1(c)

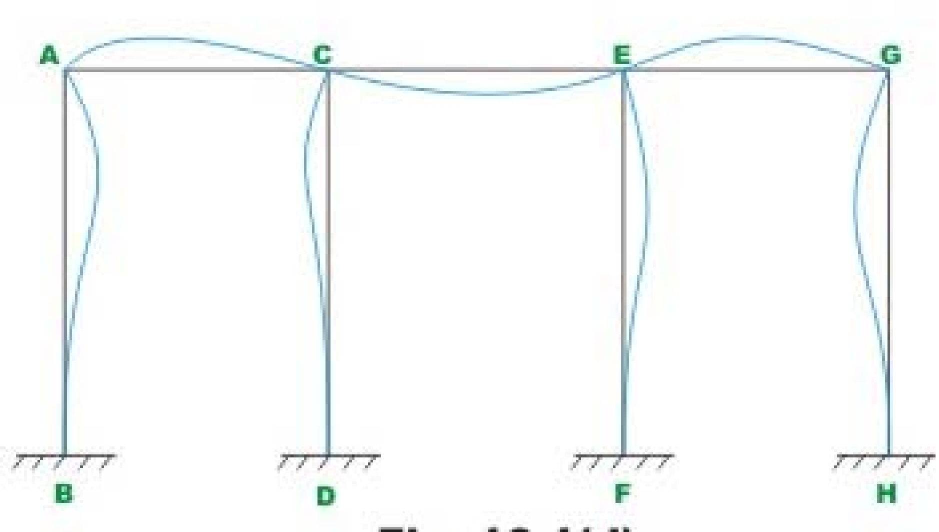


Fig- 16.1(d)

For the frames shown in Fig 16.1, the angle ψ in slope-deflection equation is zero. Hence the analysis of such rigid frames by slope deflection equation essentially follows the same steps as that of continuous beams without support settlements. However, there is a small difference. In the case of continuous beam, at a joint only two members meet. Whereas in the case of rigid frames two or more than two members meet at a joint. At joint C in the frame shown in Fig 16.1(d) three members meet. Now consider the free body diagram of joint C as shown in fig 16.2 .The equilibrium equation at joint C is

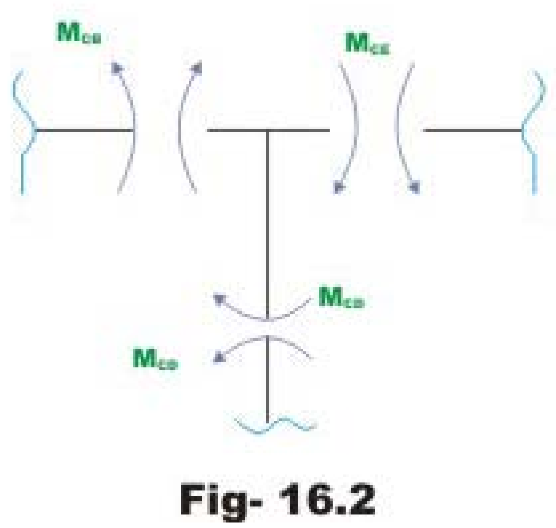


Fig- 16.2

$$\sum M_C = 0 \Rightarrow \qquad M_{CB} + M_{CE} + M_{CD} = 0$$

At each joint there is only one unknown as all the ends of members meeting at a joint rotate by the same amount. One would write as many equilibrium equations as the no of unknowns, and solving these equations joint rotations are evaluated. Substituting joint rotations in the slope–deflection equations member end moments are calculated. The whole procedure is illustrated by few examples. Frames undergoing sidesway will be considered in next lesson.

Example 16.1

Analyse the rigid frame shown in Fig 16.3 (a). Assume EI to be constant for all the members. Draw bending moment diagram and also sketch the elastic curve.

Solution

In this problem only one rotation needs to be determined i. e. θ_B. Thus the required equations to evaluate θ_B is obtained by considering the equilibrium of joint B. The moment in the cantilever portion is known. Hence this moment is applied on frame as shown in Fig 16.3 (b). Now, calculate the fixed-end moments by fixing the support B (vide Fig 16.3 c). Thus

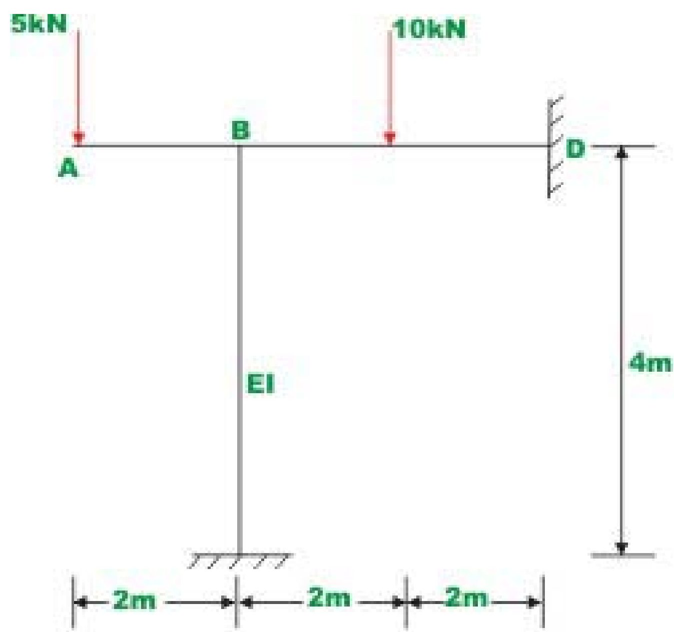


Fig- 16.3 a Example 16.1

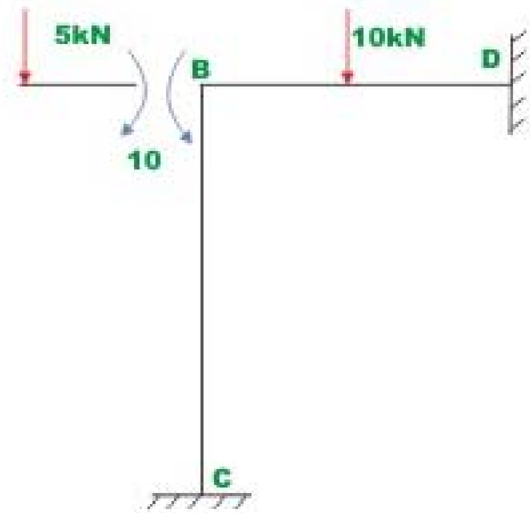


Fig- 16.3 b Moment at joint B due to overhang

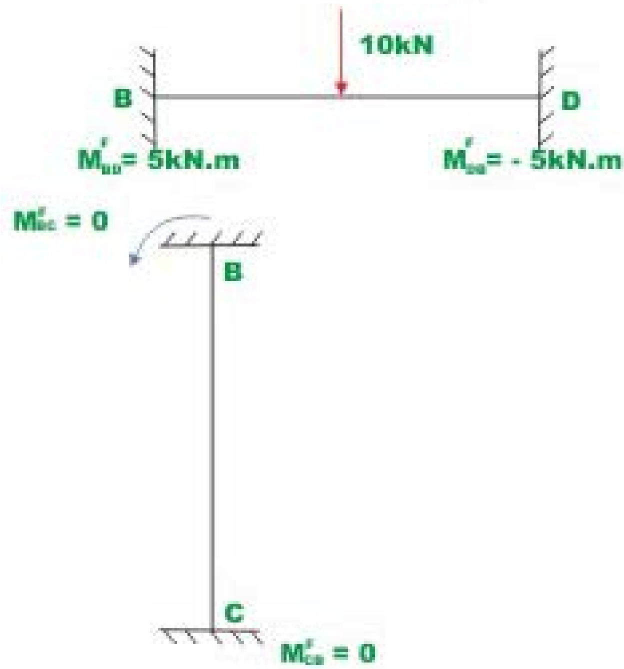


Fig- 16.3 © Kinematically restrained structure

$$M^F_{BD} = +5 \text{ kNm}$$

$$M^F_{DB} = -5 \text{ kNm}$$

$$M^F_{BC} = 0 \text{ kNm}$$

$$M^F_{BC} = 0 \text{ kNm}$$

For writing slope–deflection equations two spans must be considered, BC and BD. Since supports C and D are fixed $\theta_C = \theta_D = 0$. Also the frame is restrained against sidesway.

$$\begin{aligned}
M_{BD} &= 5 + \frac{2EI}{4}\left[2\theta_B\right] = 5 + EI\theta_B \\
M_{DB} &= 5 + \frac{2EI}{4}\left[\theta_B\right] = -5 + 0.5EI\theta_B \\
M_{BC} &= EI\theta_B \\
M_{CB} &= 0.5EI\theta_B
\end{aligned} \qquad (2)$$

Now consider the joint equilibrium of support B, (see Fig 16.3 d)

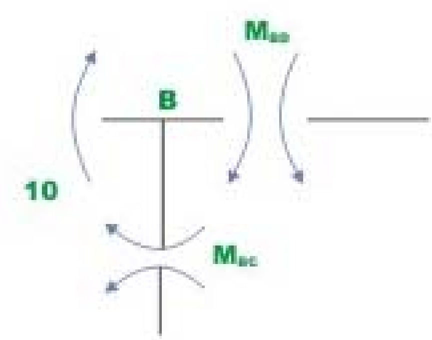


Fig- 16.3 (d) Free - body diagram of joint B

$$\sum M_B = 0 \quad \Rightarrow \quad M_{BD} + M_{BC} - 10 = 0 \qquad (3)$$

Substituting the value of M_{BD} and M_{BC} and from equation (2) in the above equation

$$5 + EI\theta_B + EI\theta_B - 10 = 0$$

$$\theta_B = \frac{2.5}{EI} \qquad (4)$$

Substituting the values of θ_B in equation (2), the beam end moments are calculated

$$M_{BD} = +7.5\,\mathrm{kN \cdot m}$$

$$M_{DB} = -3.75\,\mathrm{kN \cdot m}$$

$$M_{BC} = +2.5\,\mathrm{kN \cdot m}$$

$$M_{CB} = +1.25\,\mathrm{kN \cdot m} \qquad (5)$$

The reactions are evaluated from static equations of equilibrium. The free body diagram of each member of the frame with external load and end moments are shown in Fig 16.3 (e)

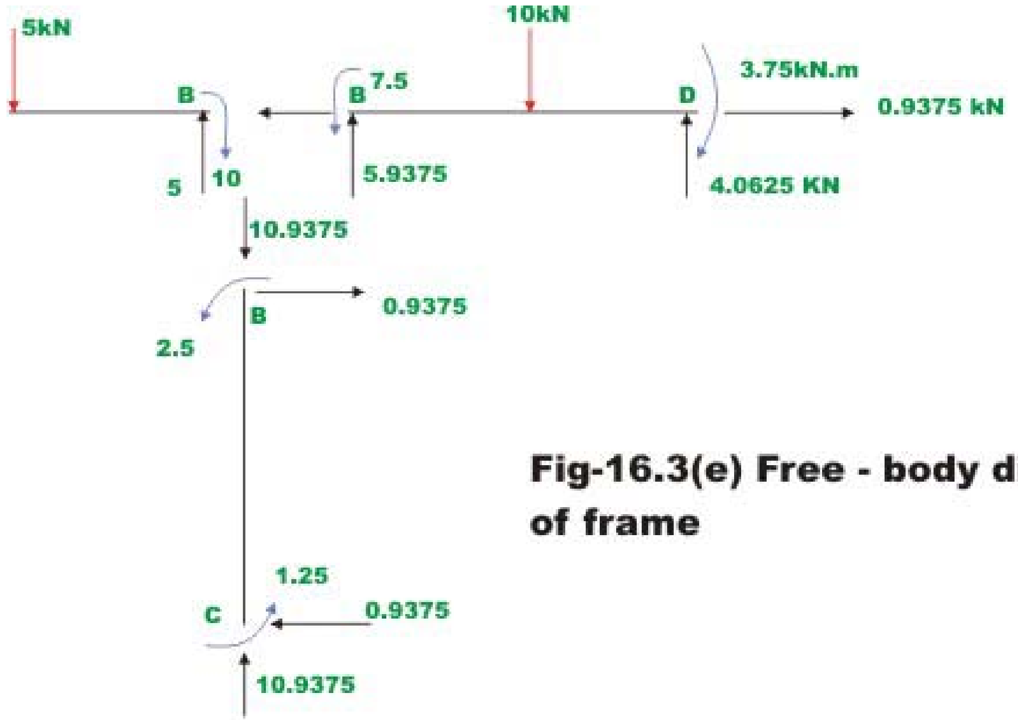


Fig-16.3(e) Free - body diagram of frame

$$R_{Cy} = 10.9375\,\text{kN}(\uparrow)$$

$$R_{Cx} = -0.9375\,\text{kN}(\leftarrow)$$

$$R_{Dy} = 4.0625\,\text{kN}(\uparrow)$$

$$R_{Dx} = 0.9375\,\text{kN}(\rightarrow) \qquad (6)$$

Bending moment diagram is shown in Fig 16.3(f)

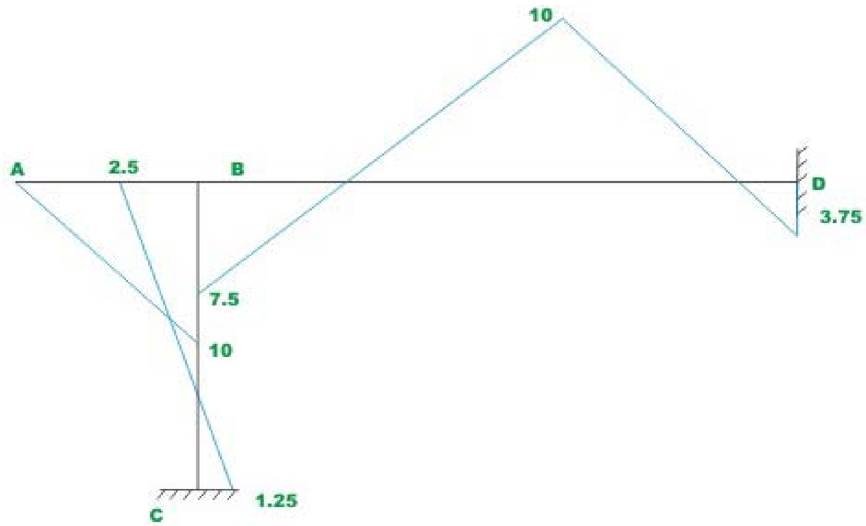


Fig-16.3(f) Bending moment diagram plotted on compression side

The vertical hatching is use to represent the bending moment diagram for the horizontal member (beams) and horizontal hatching is used for bending moment diagram for the vertical members.
The qualitative elastic curve is shown in Fig 16.3 (g).

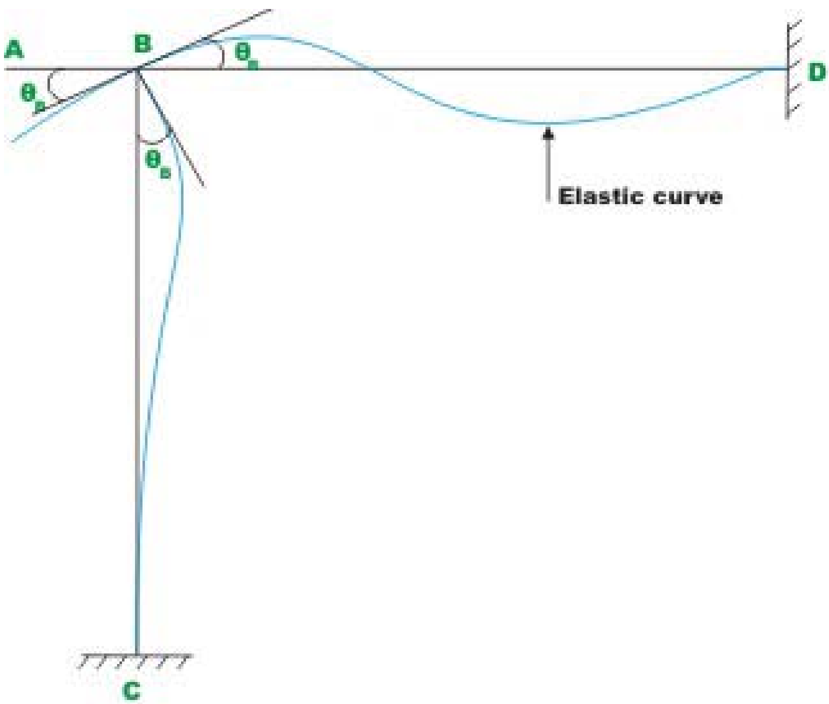


Fig-16.3(g) Elastic curve

Example 16.2

Compute reactions and beam end moments for the rigid frame shown in Fig 16.4 (a). Draw bending moment and shear force diagram for the frame and also sketch qualitative elastic curve.

Solution

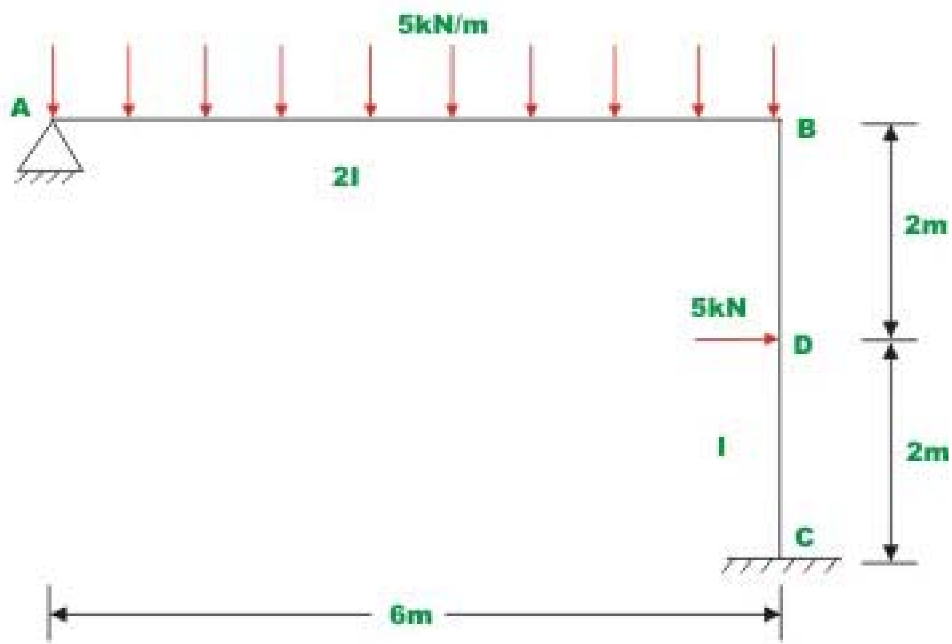


Fig-16.4(a) Example 16.2

In this frame rotations θ_A and θ_B are evaluated by considering the equilibrium of joint A and B. The given frame is kinematically indeterminate to second degree. Evaluate fixed end moments. This is done by considering the kinematically determinate structure. (Fig 16.4 b)

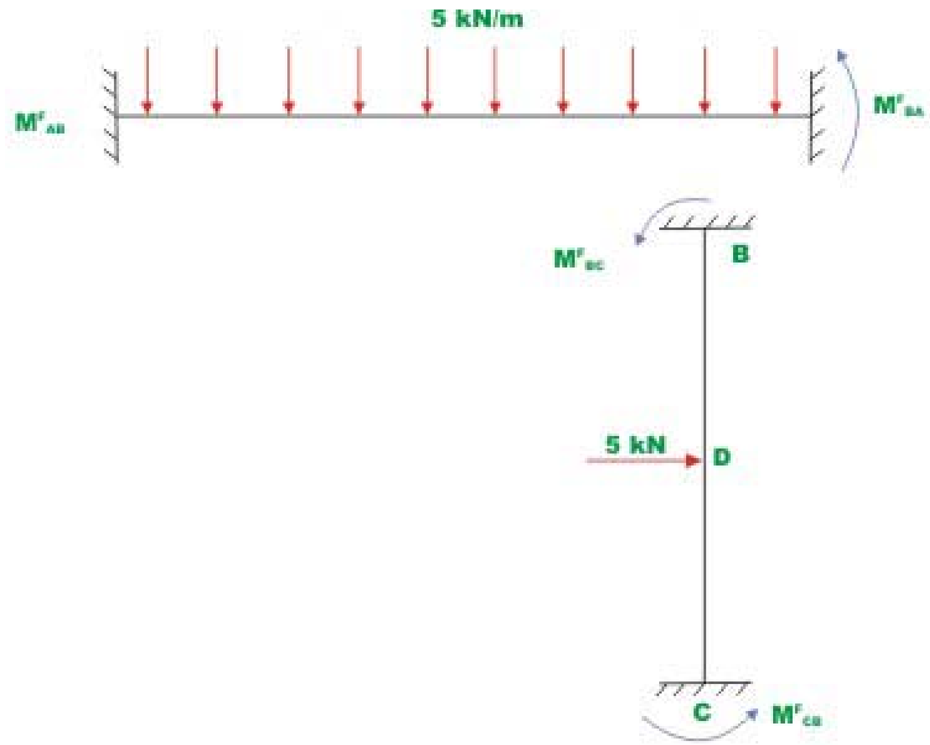


Fig-16.4(b) Kinematically restrained structure

$$M_{DB}^{F} = \frac{5 \times 6^2}{12} = 15\,\text{kN.m}$$

$$M_{BA}^{F} = \frac{-5 \times 6^2}{12} = -15\,\text{kN.m}$$

$$M_{BC}^{F} = \frac{5 \times 2 \times 2^2}{4^2} = 2.5\,\text{kN.m}$$

$$M_{CD}^{F} = \frac{-5 \times 2 \times 2^2}{4^2} = -2.5\,\text{kN.m} \qquad (1)$$

Note that the frame is restrained against sidesway. The spans must be considered for writing slope-deflection equations viz, A, B and AC. The beam end moments are related to unknown rotations θ_A and θ_B by following slope-deflection equations. (Force deflection equations). Support C is fixed and hence $\theta_C = 0$.

$$M_{AB} = M_{ABL}^{F} + \frac{2E(2I)}{L_{AB}}(2\theta_A + \theta_B)$$

$$M_{AB} = 15 - +1.333EI\theta_A + 0.667EI\theta_B$$

$$M_{BA} = -15 + 0.667EI\theta_A + 1.333EI\theta_B$$

$$M_{BC} = 2.5 + EI\theta_B + 0.5EI\theta_C$$

$$M_{CB} = -2.5 + 0.5EI\theta_B \qquad (2)$$

Consider the joint equilibrium of support *A* (See Fig 16.4 (c))

$$\sum M_A = 0$$

$$M_{AB} = 0 = 15 + 1.333EI\theta_A + 0.667EI\theta_B \qquad (3)$$

$$1.333EI\theta_A + +0.667EI\theta_B = -15$$

Or, $2\theta_A + \theta_B = \dfrac{-22.489}{EI}$

Equilibrium of joint B (Fig 16.4(d))

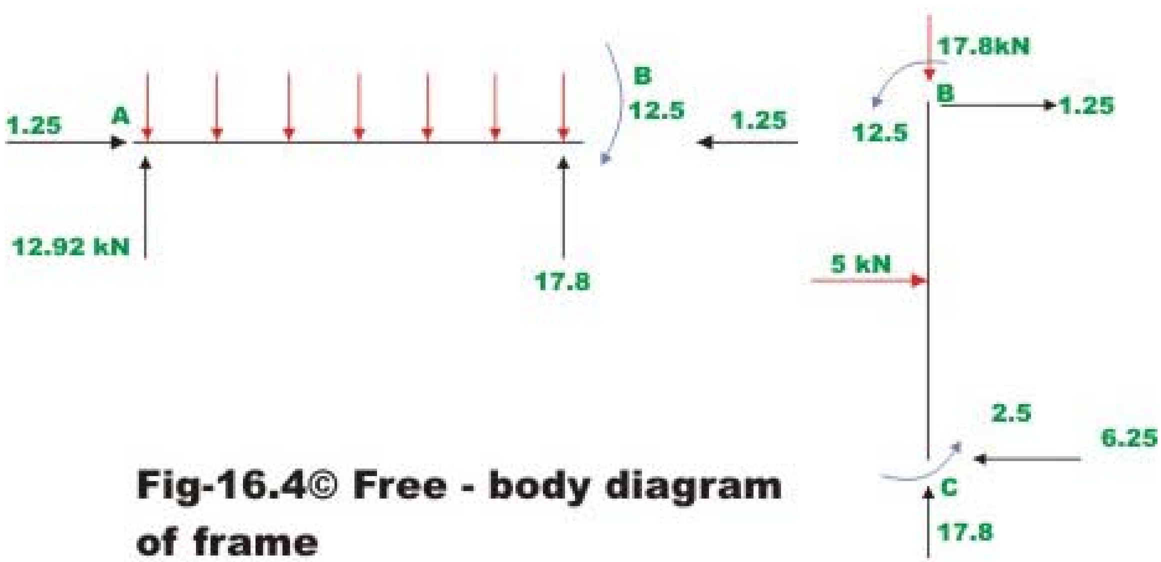


Fig-16.4© Free - body diagram of frame

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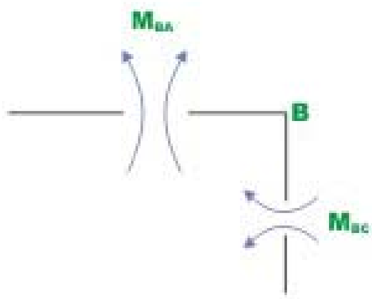


Fig-16.4(d) Free - body diagram of joint B

$$\sum M_B = 0 \qquad \Rightarrow \qquad M_{BC} + M_{BA} = 0 \tag{4}$$

Substituting the value of M_{BC} and M_{BA} in the above equation,

$$2.333EI\theta_B + 0.667EI\theta_A = 12.5 \tag{5}$$

Or, $$3.498\theta_B + \theta_A = \frac{18.741}{EI}$$

Solving equation (3) and (4)

$$\theta_B = \frac{10.002}{EI}(counterclockwise)$$
$$\theta_B = \frac{-16.245}{EI}(clockwise) \tag{6}$$

Substituting the value of θ_A and θ_B in equation (2) beam end moments are evaluated.

$$M_{AB} = 15 + 1.333EI\left(\frac{-16.245}{EI}\right) + 0.667EI\left(\frac{10.002}{EI}\right) = 0$$

$$M_{BA} = -15 + 0.667EI\left(\frac{-16.245}{EI}\right) + .1.33EI\left(\frac{10.002}{EI}\right) = -1$$

$$M_{BC} = 2.5 + EI\left(\frac{10.002}{EI}\right) = 12.5\,\text{kN.m}$$

$$M_{CB} = -2.5 + 0.5EI\left(\frac{10.002}{EI}\right) = 2.5\,\text{kN.m} \tag{7}$$

Using these results, reactions are evaluated from equilibrium equations as shown in Fig 16.4 (e)

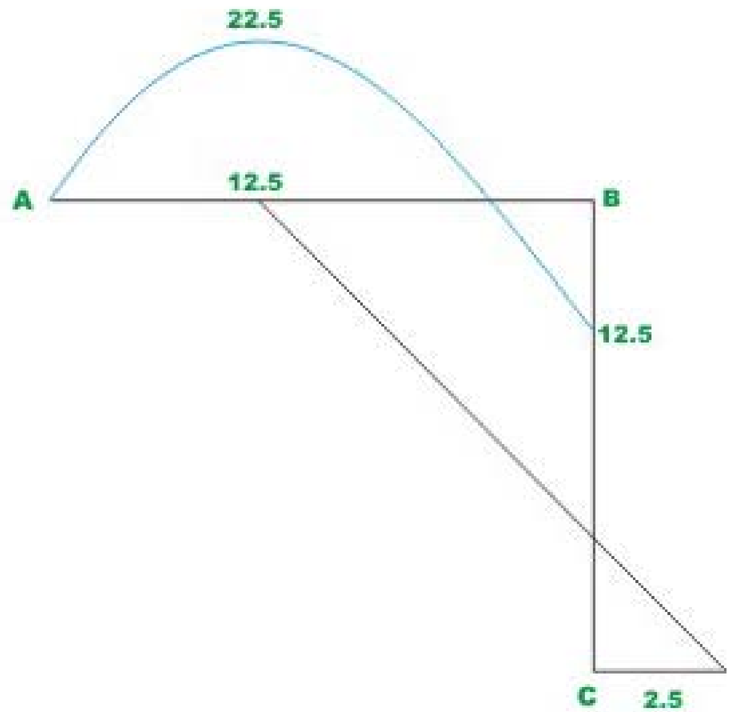


Fig-16.4(e) B.M.D

The shear force and bending moment diagrams are shown in Fig 16.4(g) and 16.4 h respectively. The qualitative elastic curve is shown in Fig 16.4 (h).

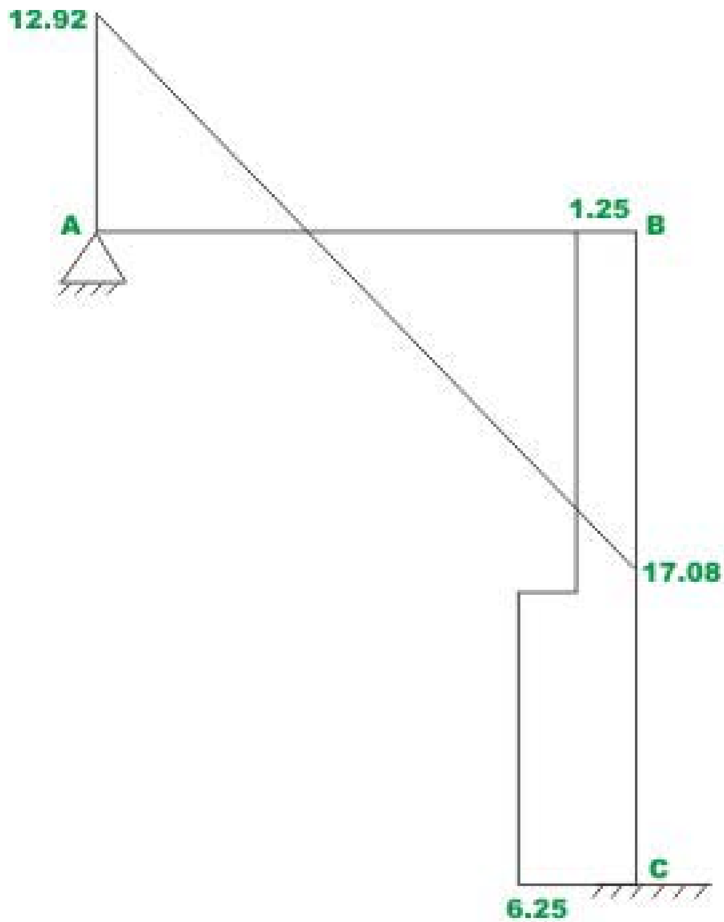


Fig-16.4(f) S.F.D

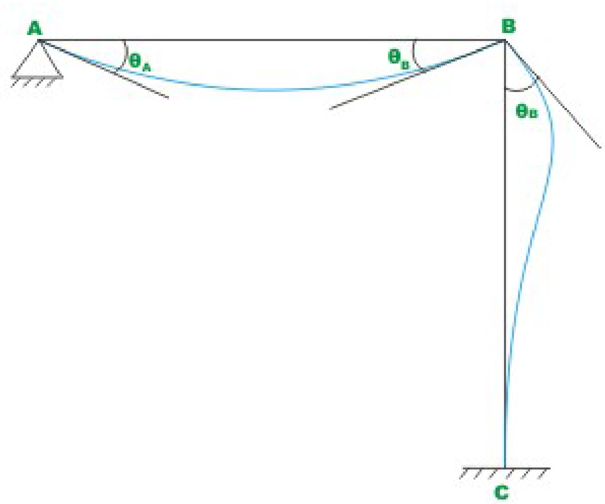


Fig.16.4 (g)Elastic Curve

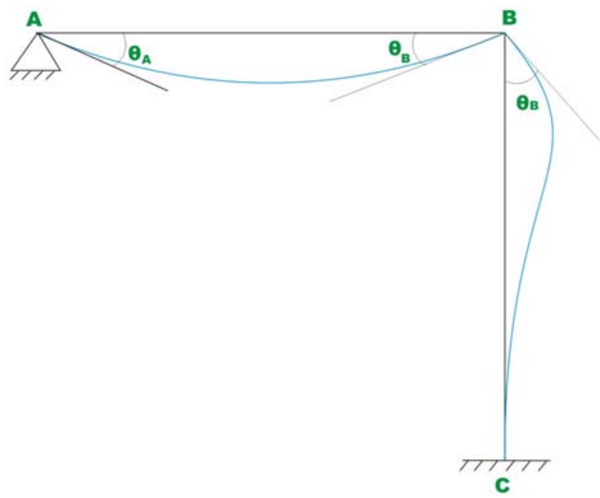


Fig- 16.4(h) Elastic Curve

Example 16.3

Compute reactions and beam end moments for the rigid frame shown in Fig 16.5(a). Draw bending moment diagram and sketch the elastic curve for the frame.

Solution

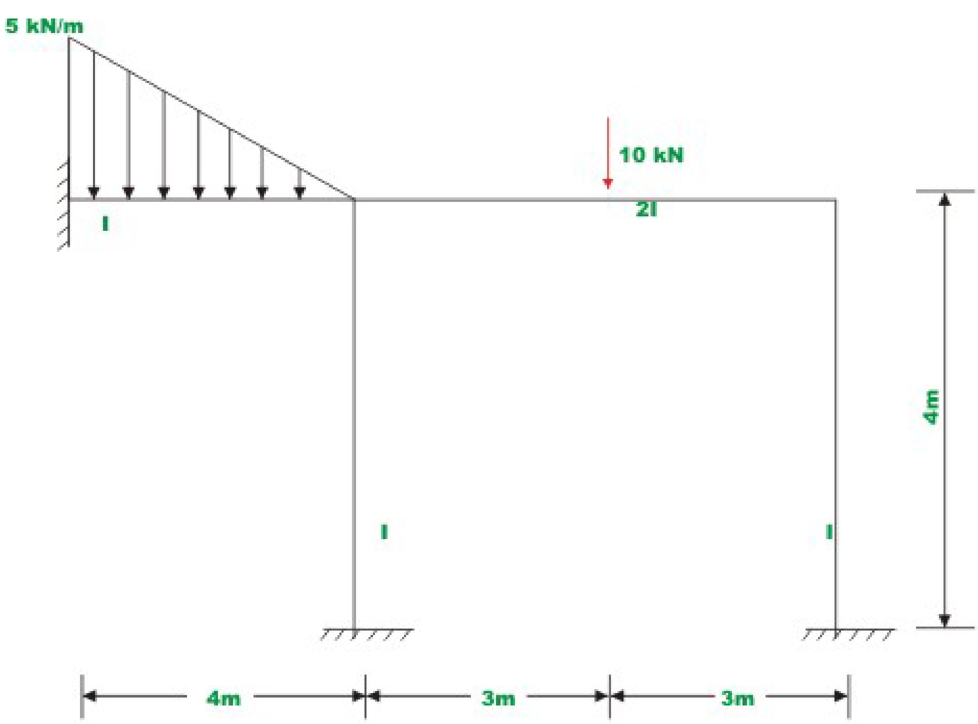


Fig-16.5(a) Example 16.3

The given frame is kinematically indeterminate to third degree so three rotations are to be calculated, θ_B, θ_C and θ_D. First calculate the fixed end moments (see Fig 16.5 b).

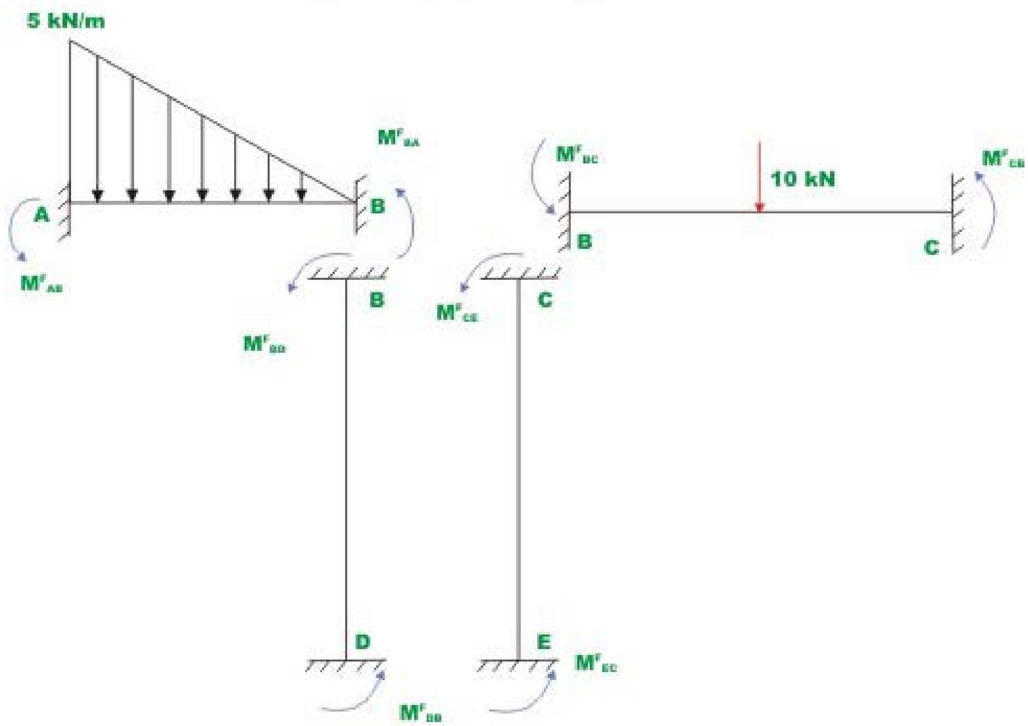


Fig.16.5b Kinematically restrained structure

$$M_{AB}^F = \frac{5 \times 4^2}{20} = 4 \text{ kN.m}$$

$$M_{BA}^F = \frac{-5 \times 4^2}{30} = -2.667 \text{ kN.m}$$

$$M_{BC}^F = \frac{10 \times 3 \times 3^2}{6^2} = 7.5 \text{ kN.m}$$

$$M_{CB}^F = \frac{-10 \times 3 \times 3^2}{6^2} = -7.5 \text{ kN.m}$$

$$M_{BD}^F = M_{DB}^F = M_{CE}^F = M_{EC}^F = 0 \qquad (1)$$

The frame is restrained against sidesway. Four spans must be considered for rotating slope – deflection equation: AB, BD, BC and CE. The beam end moments are related to unknown rotation at B, C, and D. Since the supports A and E are fixed. $\theta_A = \theta_E = 0$.

$$M_{AB} = 4 + \frac{2EI}{4}\left[2\theta_A + \theta_B\right]$$

$$M_{AB} = 4 + EI\theta_A + 0.5EI\theta_B = 4 + 0.5EI\theta_B$$

$$M_{BA} = -2.667 EI\theta_A + EI\theta_B = -2.667 + EI\theta_B$$

$$M_{BD} = EI\theta_B + 0.5EI\theta_D$$

$$M_{DB} = 0.5EI\theta_B + EI\theta_D$$

$$M_{BC} = 7.5 + \frac{2E(2I)}{6}\left[2\theta_B + \theta_C\right] = 7.5 + 1.333EI\theta_B + 0.667EI\theta_C$$

$$M_{CB} = -7.5 + .667EI\theta_B + 1.333EI\theta_C$$

$$M_{CE} = EI\theta_C + 0.5EI\theta_E = EI\theta_C$$

$$M_{EC} = 0.5EI\theta_C + 0.5EI\theta_E = 0.5EI\theta_C \qquad (2)$$

Consider the equilibrium of joints B, D, C (vide Fig. 16.5(c))

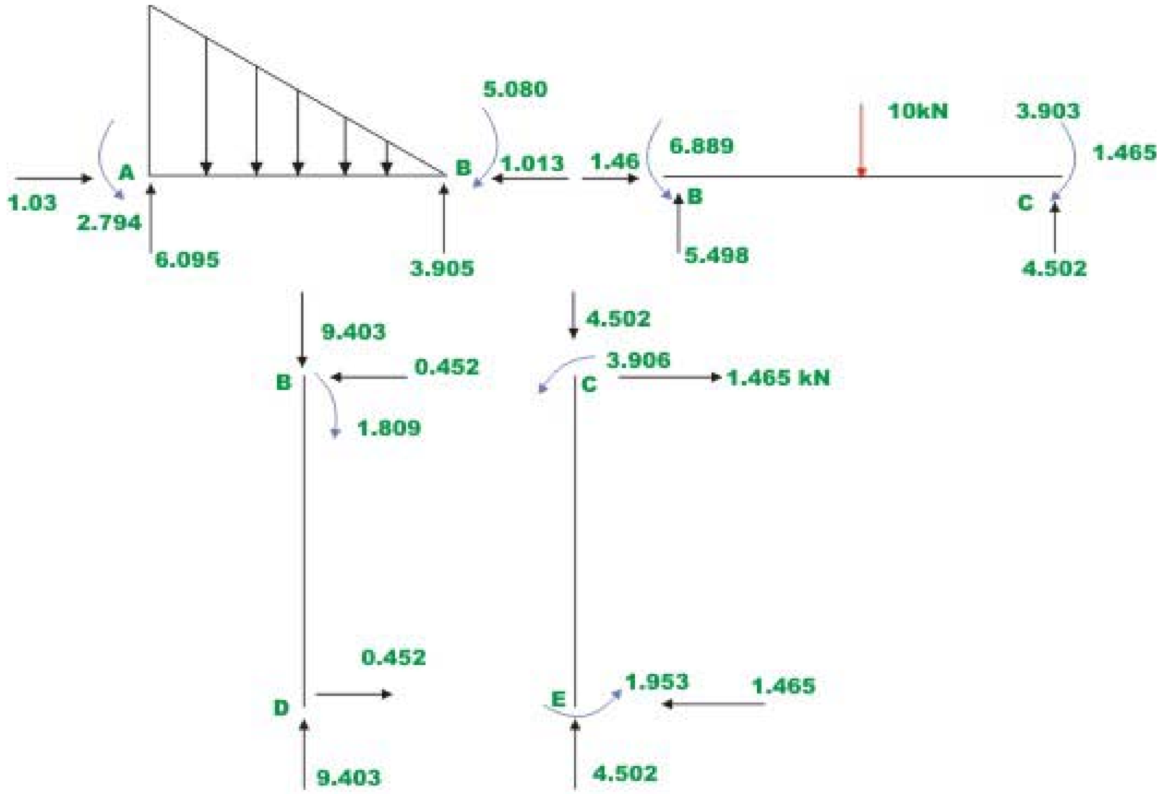


Fig-16.5 (c) Free - body diagram

$$\sum M_B = 0 \quad \Rightarrow \quad M_{BA} + M_{BC} + M_{BD} = 0 \qquad (3)$$

$$\sum M_D = 0 \quad \Rightarrow \quad M_{DB} = 0 \qquad (4)$$

$$\sum M_C = 0 \quad \Rightarrow \quad M_{CB} + M_{CE} = 0 \qquad (5)$$

Substituting the values of $M_{BA}, M_{BC}, M_{BD}, M_{DB}, M_{CB}$ and M_{CE} in the equations (3), (4), and (5)

$$3.333EI\theta_B + 0.667EI\theta_C + 0.5EI\theta_D = -4.833$$

$$0.5EI\theta_B + EI\theta_D = 0$$

$$2.333EI\theta_C + 0.667EI\theta_B = 7.5 \qquad (6)$$

Solving the above set of simultaneous equations, θ_B, θ_C and θ_D are evaluated.

$$EI\theta_B = -2.4125$$

$$EI\theta_C = 3.9057$$

$$EI\theta_D = 1.2063 \qquad (7)$$

Substituting the values of θ_B, θ_C and θ_D in (2), beam end moments are computed.

$$M_{AB} = 2.794 \text{ kN.m}$$

$$M_{BA} = -5.080 \text{ kN.m}$$

$$M_{BD} = -1.8094 \text{ kN.m}$$

$$M_{DB} = 0$$

$$M_{BC} = 6.859 \text{ kN.m}$$

$$M_{CB} = -3.9028 \text{ kN.m}$$

$$M_{CE} = 3.9057 \text{ kN.m}$$

$$M_{EC} = 1.953 \text{ kN.m} \qquad (8)$$

The reactions are computed in Fig 16.5(d), using equilibrium equations known beam-end moments and given loading.

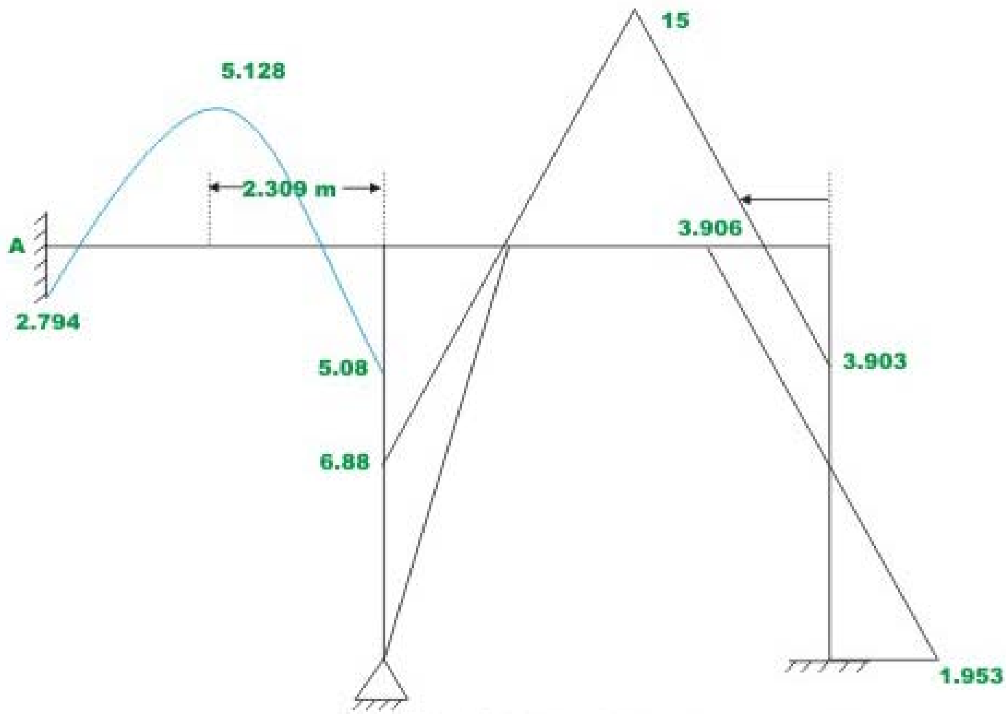


Fig-16.5(d) Bending moment diagram

$$R_{Ay} = 6.095 \text{ kN}(\uparrow)$$

$$R_{Dy} = 9.403 \text{ kN}(\uparrow)$$

$$R_{Ey} = 4.502 \text{ kN}(\uparrow)$$

$$R_{Ax} = 1.013 \text{ kN}(\rightarrow)$$

$$R_{Dx} = 0.542 \text{ kN}(\rightarrow)$$

$$R_{Ex} = -1.465 \text{ kN}(\leftarrow) \tag{9}$$

The bending moment diagram is shown in Fig 16.5.(e) and the elastic curve is shown in Fig 16.5(f).

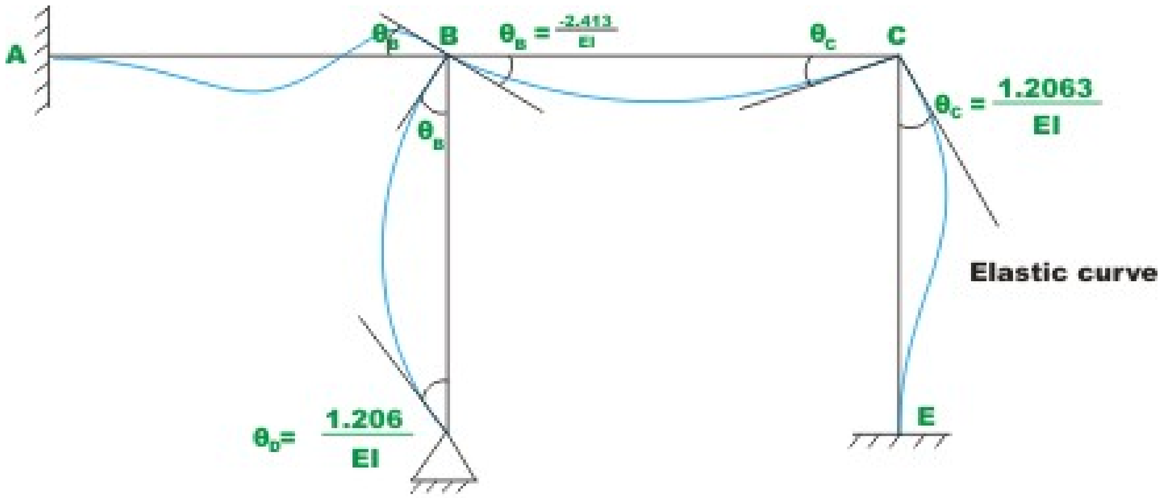


Fig-16.5(e)

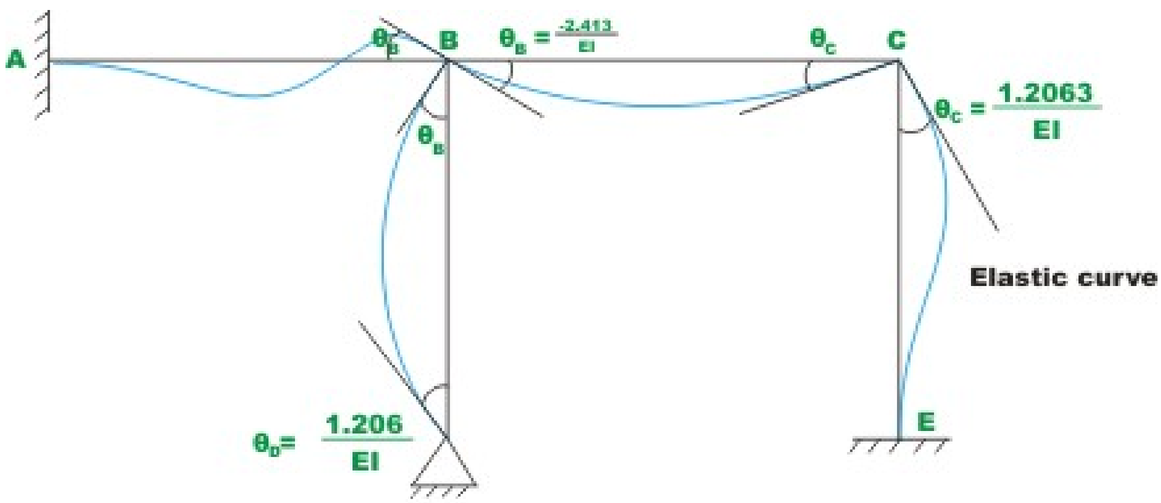


Fig-16.5(f)

Summary

In this lesson plane frames restrained against sidesway are analysed using slope-deflection equations. Equilibrium equations are written at each rigid joint of the frame and also at the support. Few problems are solved to illustrate the procedure. The shear force and bending moment diagrams are drawn for the plane frames.

Module
3
Analysis of Statically Indeterminate Structures by the Displacement Method

Lesson
17
The Slope-Deflection Method: Frames with Sidesway

Instructional Objectives

After reading this chapter the student will be able to
1. Derive slope-deflection equations for the frames undergoing sidesway.
2. Analyse plane frames undergoing sidesway.
3, Draw shear force and bending moment diagrams.
4. Sketch deflected shape of the plane frame not restrained against sidesway.

17.1 Introduction

In this lesson, slope-deflection equations are applied to analyse statically indeterminate frames undergoing sidesway. As stated earlier, the axial deformation of beams and columns are small and are neglected in the analysis. In the previous lesson, it was observed that sidesway in a frame will not occur if

1. They are restrained against sidesway.
2. If the frame geometry and the loading are symmetrical.

In general loading will never be symmetrical. Hence one could not avoid sidesway in frames.

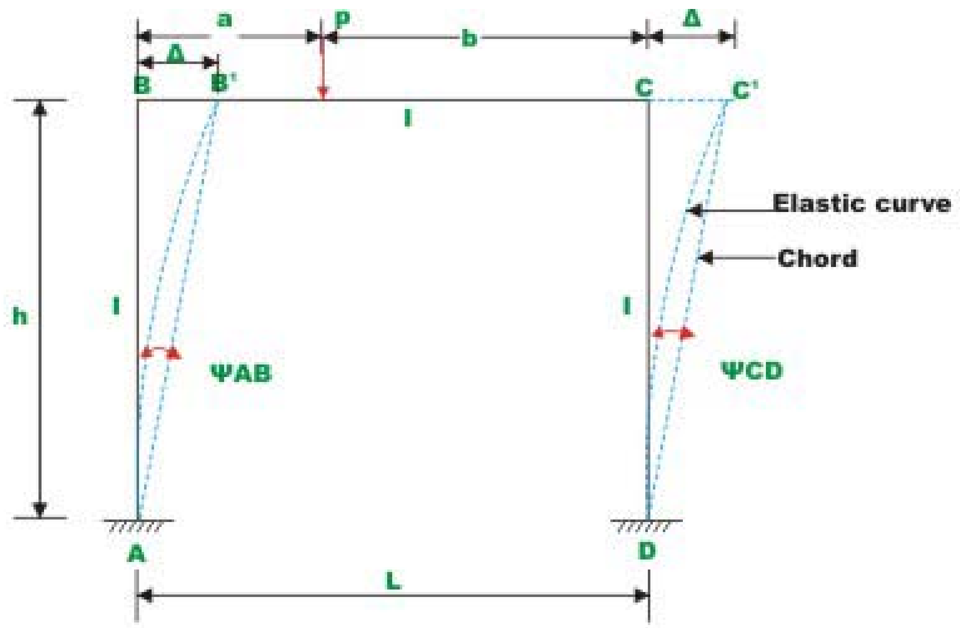


Fig.17.1 Plane frame undergoing sway

For example, consider the frame of Fig. 17.1. In this case the frame is symmetrical but not the loading. Due to unsymmetrical loading the beam end moments M_{BC} and M_{CB} are not equal. If b is greater than a, then $M_{BC} > M_{CB}$. In such a case joint B and C are displaced toward right as shown in the figure by an unknown amount Δ. Hence we have three unknown displacements in this frame: rotations θ_B, θ_C and the linear displacement Δ. The unknown joint rotations θ_B and θ_C are related to joint moments by the moment equilibrium equations. Similarly, when unknown linear displacement occurs, one needs to consider force-equilibrium equations. While applying slope-deflection equation to columns in the above frame, one must consider the column rotation $\psi\left(=\frac{\Delta}{h}\right)$ as unknowns. It is observed that in the column AB, the end B undergoes a linear displacement Δ with respect to end A. Hence the slope-deflection equation for column AB is similar to the one for beam undergoing support settlement. However, in this case Δ is unknown. For each of the members we can write the following slope-deflection equations.

$$M_{AB} = M_{AB}^F + \frac{2EI}{h}\left[2\theta_A + \theta_B - 3\psi_{AB}\right] \qquad \text{where } \psi_{AB} = -\frac{\Delta}{h}$$

ψ_{AB} is assumed to be negative as the chord to the elastic curve rotates in the clockwise directions.

$$M_{BA} = M_{BA}^F + \frac{2EI}{h}\left[2\theta_B + \theta_A - 3\psi_{AB}\right]$$

$$M_{BC} = M_{BC}^F + \frac{2EI}{h}\left[2\theta_B + \theta_C\right]$$

$$M_{CB} = M_{CB}^F + \frac{2EI}{h}\left[2\theta_C + \theta_B\right]$$

$$M_{CD} = M_{CD}^F + \frac{2EI}{h}\left[2\theta_C + \theta_D - 3\psi_{CD}\right] \qquad \psi_{CD} = -\frac{\Delta}{h}$$

$$M_{DC} = M_{DC}^F + \frac{2EI}{h}\left[2\theta_D + \theta_C - 3\psi_{CD}\right] \qquad (17.1)$$

As there are three unknowns (θ_B, θ_C and Δ), three equations are required to evaluate them. Two equations are obtained by considering the moment equilibrium of joint B and C respectively.

$$\sum M_B = 0 \qquad \Rightarrow \qquad M_{BA} + M_{BC} = 0 \qquad (17.2a)$$

$$\sum M_C = 0 \qquad \Rightarrow \qquad M_{CB} + M_{CD} = 0 \qquad (17.2b)$$

Now consider free body diagram of the frame as shown in Fig. 17.2. The horizontal shear force acting at A and B of the column AB is given by

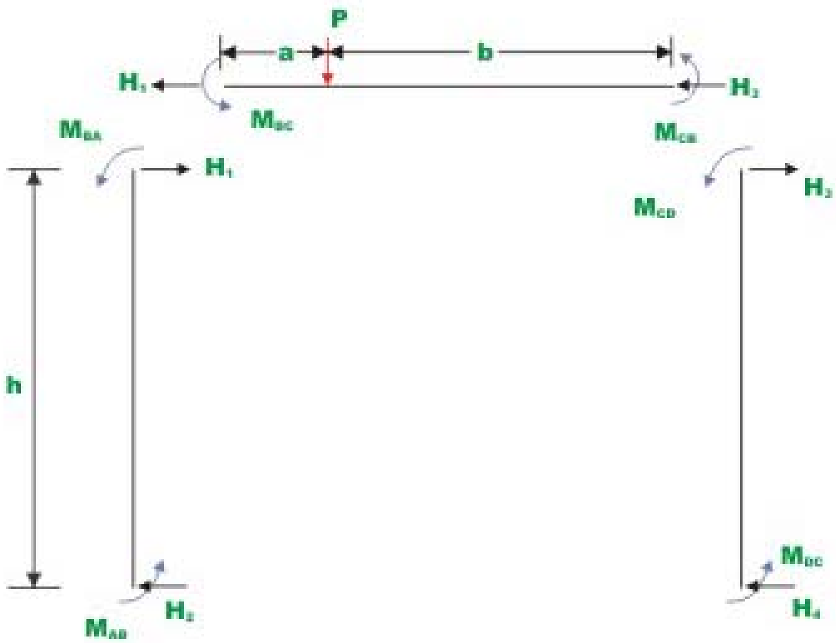


Fig.17.2 Free - body diagrams of columns and beams

$$H_1 = \frac{M_{BA} + M_{AB}}{h} \tag{17.3a}$$

Similarly for member CD, the shear force H_3 is given by

$$H_3 = \frac{M_{CD} + M_{DC}}{h} \tag{17.3b}$$

Now, the required third equation is obtained by considering the equilibrium of member BC,

$$\sum F_X = 0 \qquad \Rightarrow \qquad H_1 + H_3 = 0$$

$$\frac{M_{BA} + M_{AB}}{h} + \frac{M_{CD} + M_{DC}}{h} = 0 \tag{17.4}$$

Substituting the values of beam end moments from equation (17.1) in equations (17.2a), (17.2b) and (17.4), we get three simultaneous equations in three unknowns θ_B, θ_C and Δ, solving which joint rotations and translations are evaluated.

Knowing joint rotations and translations, beam end moments are calculated from slope-deflection equations. The complete procedure is explained with a few numerical examples.

Example 17.1

Analyse the rigid frame as shown in Fig. 17.3a. Assume EI to be constant for all members. Draw bending moment diagram and sketch qualitative elastic curve.

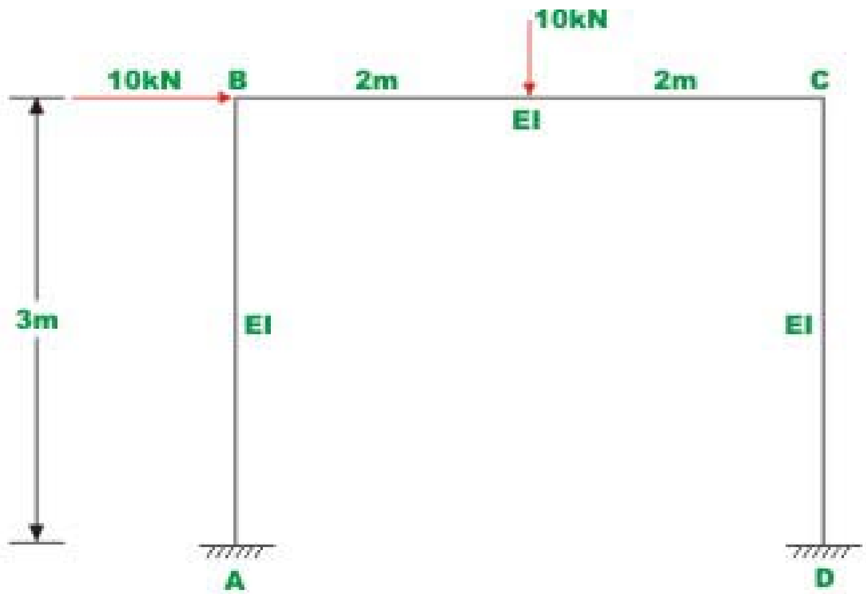


Fig.17.3 (a) Example 17.1

Solution

In the given problem, joints B and C rotate and also translate by an amount Δ. Hence, in this problem we have three unknown displacements (two rotations and one translation) to be evaluated. Considering the kinematically determinate structure, fixed end moments are evaluated. Thus,

$$M_{AB}^{F} = 0 \ ; M_{BA}^{F} = 0 \ ; M_{BC}^{F} = +10\,kN.m \ ; M_{CB}^{F} = -10kN.m \ ; M_{CD}^{F} = 0 \ ; M_{DC}^{F} = 0. \quad (1)$$

The ends A and D are fixed. Hence, $\theta_A = \theta_D = 0$. Joints B and C translate by the same amount Δ. Hence, chord to the elastic curve AB' and DC' rotates by an amount (see Fig. 17.3b)

$$\psi_{AB} = \psi_{CD} = -\frac{\Delta}{3} \quad (2)$$

Chords of the elastic curve AB' and DC' rotate in the clockwise direction; hence ψ_{AB} and ψ_{CD} are taken as negative.

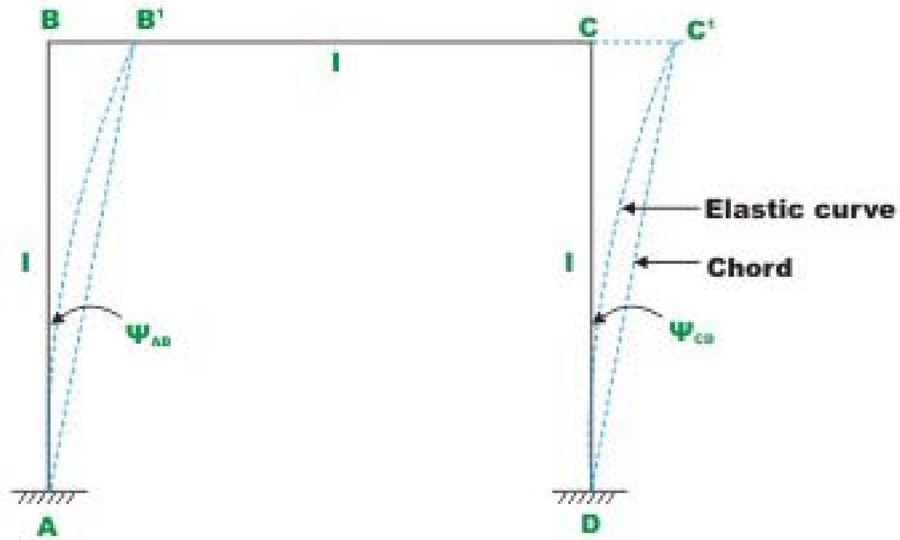


Fig.17.3b Column ratation

Now, writing the slope-deflection equations for the six beam end moments,

$$M_{AB} = M_{AB}^{F} + \frac{2EI}{3}\left[2\theta_A + \theta_B - 3\psi_{AB}\right]$$

$$M_{AB}^{F} = 0 \ ;\theta_A = 0 \ ;\psi_{AB} = -\frac{\Delta}{3}.$$

$$M_{AB} = \frac{2}{3}EI\theta_B + \frac{2}{3}EI\Delta$$

$$M_{BA} = \frac{4}{3}EI\theta_B + \frac{2}{3}EI\Delta$$

$$M_{BC} = 10 + EI\theta_B + \frac{1}{2}EI\theta_C$$

$$M_{CB} = -10 + \frac{1}{2}EI\theta_B + EI\theta_C$$

$$M_{CD} = \frac{4}{3}EI\theta_C + \frac{2}{3}EI\Delta$$

$$M_{DC} = \frac{2}{3} EI\theta_C + \frac{2}{3} EI\Delta \qquad (3)$$

Now, consider the joint equilibrium of B and C (vide Fig. 17.3c).

$$\sum M_B = 0 \qquad \Rightarrow \qquad M_{BA} + M_{BC} = 0 \qquad (4)$$

$$\sum M_C = 0 \qquad \Rightarrow \qquad M_{CB} + M_{CD} = 0 \qquad (5)$$

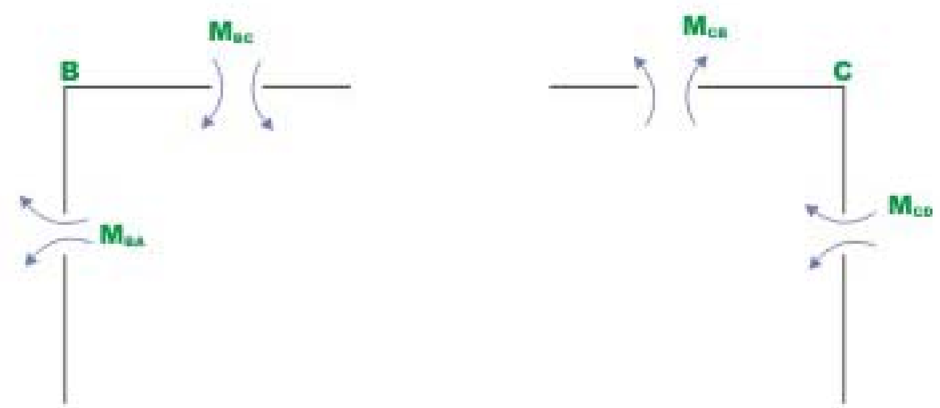


Fig.17.3c Free - body diagram of joints B and C

The required third equation is written considering the horizontal equilibrium of the entire frame *i.e.* $\sum F_X = 0$ (vide Fig. 17.3d).

$$-H_1 + 10 - H_2 = 0$$

$$\Rightarrow \qquad H_1 + H_2 = 10 . \qquad (6)$$

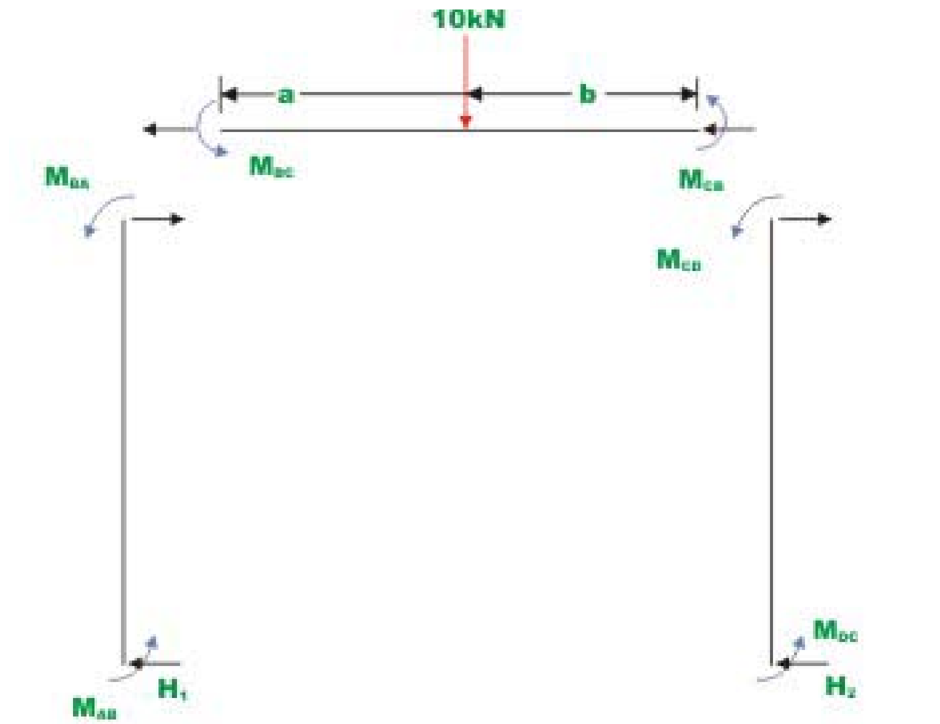


Fig.17.3d Free - body diagram of frame

Considering the equilibrium of the column AB and CD, yields

$$H_1 = \frac{M_{BA} + M_{AB}}{3}$$

and

$$H_2 = \frac{M_{CD} + M_{DC}}{3} \tag{7}$$

The equation (6) may be written as,

$$M_{BA} + M_{AB} + M_{CD} + M_{DC} = 30 \tag{8}$$

Substituting the beam end moments from equation (3) in equations (4), (5) and (6)

$$2.333EI\theta_B + 0.5EI\theta_C + 0.667EI\Delta = -10 \tag{9}$$

$$2.333EI\theta_C + 0.5EI\theta_B + 0.667EI\Delta = 10 \tag{10}$$

$$2EI\theta_B + 2EI\theta_C + \frac{8}{3}EI\Delta = 30 \qquad (11)$$

Equations (9), (10) and (11) indicate symmetry and this fact may be noted. This may be used as the check in deriving these equations.

Solving equations (9), (10) and (11),

$EI\theta_B = -9.572$; $EI\theta_C = 1.355$ and $EI\Delta = 17.417$.

Substituting the values of $EI\theta_B, EI\theta_C$ and $EI\Delta$ in the slope-deflection equation (3), one could calculate beam end moments. Thus,

$$M_{AB} = 5.23 \text{ kN.m} \quad \text{(counterclockwise)}$$

$$M_{BA} = -1.14 \text{ kN.m(clockwise)}$$

$$M_{BC} = 1.130 \text{ kN.m}$$

$$M_{CB} = -13.415 \text{ kN.m}$$

$$M_{CD} = 13.406 \text{ kN.m}$$

$$M_{DC} = 12.500 \text{ kN.m} .$$

The bending moment diagram for the frame is shown in Fig. 17.3 e. And the elastic curve is shown in Fig 17.3 f. the bending moment diagram is drawn on the compression side. Also note that the vertical hatching is used to represent bending moment diagram for the horizontal members (beams).

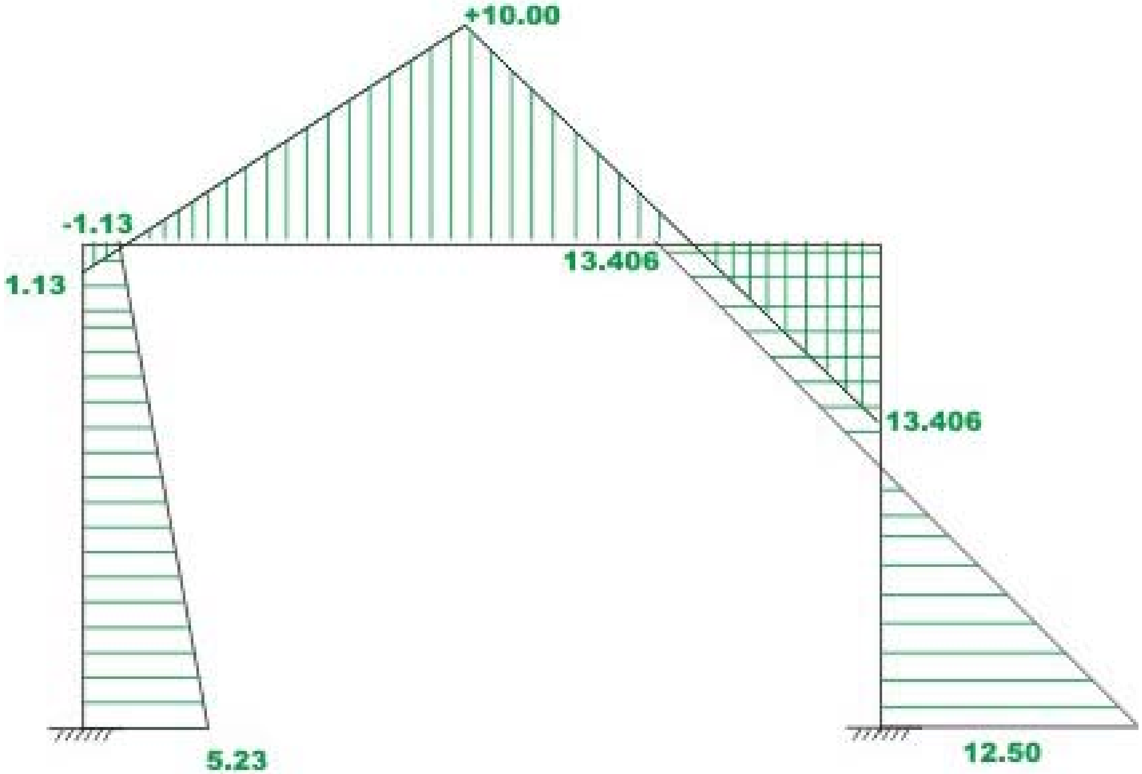


Fig.17.3e Bending moment diagram

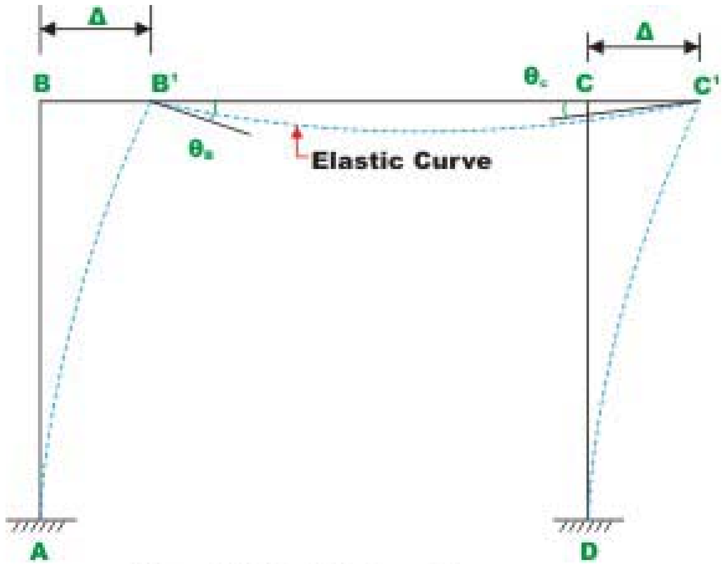


Fig.17.3f Elastic curve

Example 17.2

Analyse the rigid frame as shown in Fig. 17.4a and draw the bending moment diagram. The moment of inertia for all the members is shown in the figure. Neglect axial deformations.

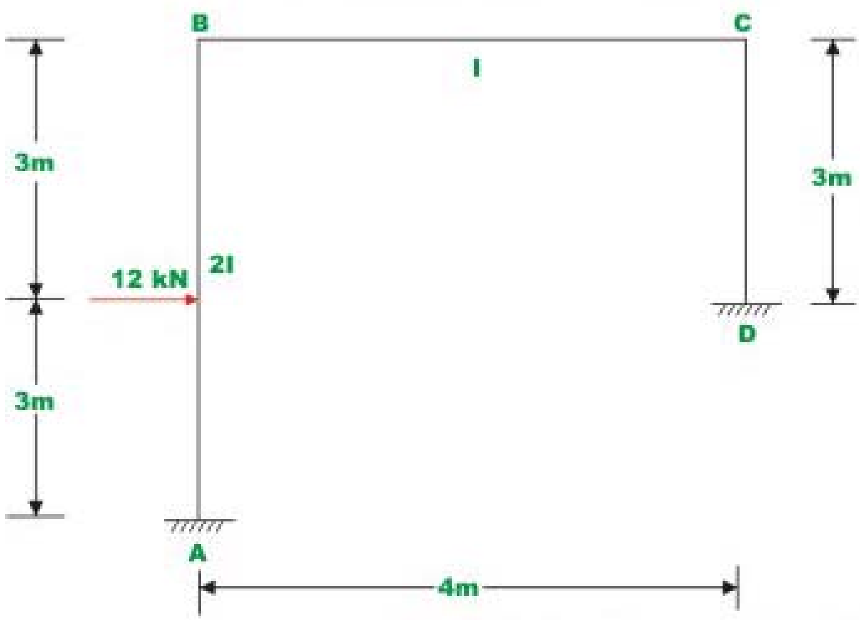


Fig.17.4a (Example 17.2)

Solution:

In this problem rotations and translations at joints B and C need to be evaluated. Hence, in this problem we have three unknown displacements: two rotations and one translation. Fixed end moments are

$$M_{AB}^{F} = \frac{12 \times 3 \times 9}{36} = 9\,kN.m\ ; M_{BA}^{F} = -9\,kN.m\ ;$$
$$M_{BC}^{F} = 0\ ; M_{CB}^{F} = 0\ ; M_{CD}^{F} = 0\ ; M_{DC}^{F} = 0. \qquad (1)$$

The joints B and C translate by the same amount Δ. Hence, the chord to the elastic curve rotates in the clockwise direction as shown in Fig. 17.3b.

$$\psi_{AB} = -\frac{\Delta}{6}$$

and

$$\psi_{CD} = -\frac{\Delta}{3} \qquad (2)$$

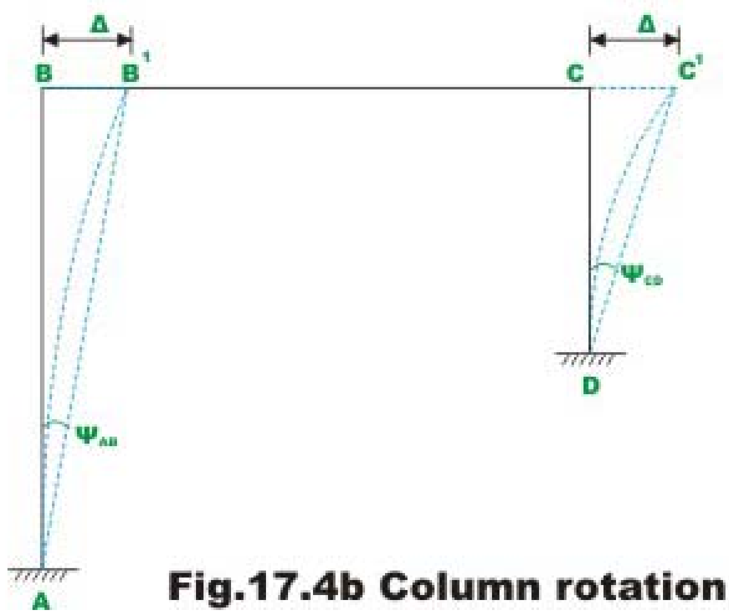


Fig.17.4b Column rotation due to sway

Now, writing the slope-deflection equations for six beam end moments,

$$M_{AB} = 9 + \frac{2(2EI)}{6}\left[\theta_B + \frac{\Delta}{2}\right]$$

$$M_{AB} = 9 + 0.667EI\theta_B + 0.333EI\Delta$$

$$M_{BA} = -9 + 1.333EI\theta_B + 0.333EI\Delta$$

$$M_{BC} = EI\theta_B + 0.5EI\theta_C$$

$$M_{CB} = 0.5EI\theta_B + EI\theta_C$$

$$M_{CD} = 1.333EI\theta_C + 0.667EI\Delta$$

$$M_{DC} = 0.667EI\theta_C + 0.667EI\Delta \tag{3}$$

Now, consider the joint equilibrium of B and C.

$$\sum M_B = 0 \quad \Rightarrow \quad M_{BA} + M_{BC} = 0 \tag{4}$$

$$\sum M_C = 0 \quad \Rightarrow \quad M_{CB} + M_{CD} = 0 \tag{5}$$

The required third equation is written considering the horizontal equilibrium of the entire frame. Considering the free body diagram of the member BC (vide Fig. 17.4c),

$H_1 + H_2 = 0$.

(6)

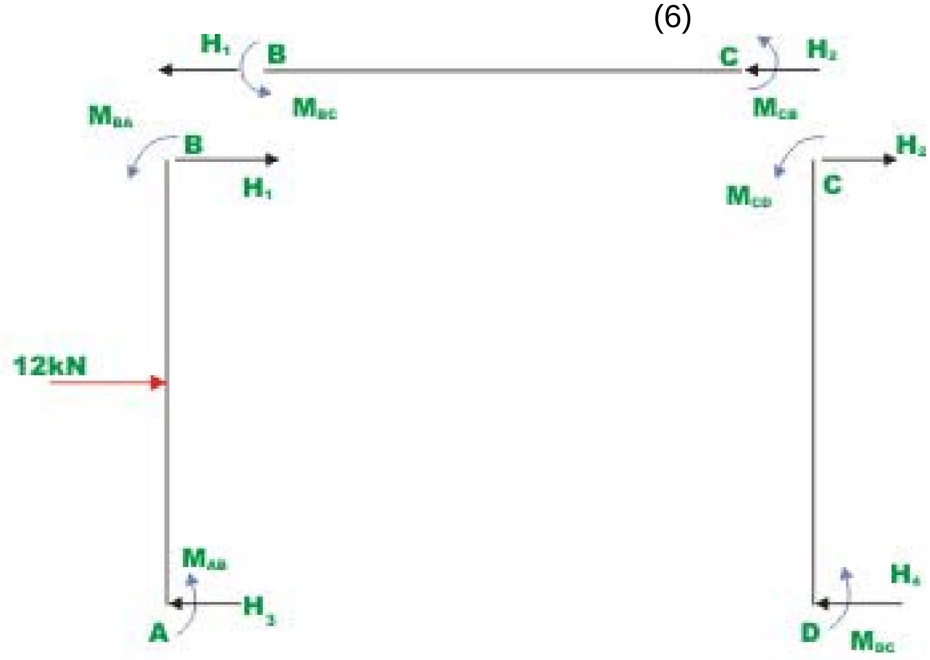


Fig.17.4c Free - body diagram

The forces H_1 and H_2 are calculated from the free body diagram of column AB and CD. Thus,

$$H_1 = -6 + \frac{M_{BA} + M_{AB}}{6}$$

and

$$H_2 = \frac{M_{CD} + M_{DC}}{3} \qquad (7)$$

Substituting the values of H_1 and H_2 into equation (6) yields,

$$M_{BA} + M_{AB} + 2M_{CD} + 2M_{DC} = 36 \qquad (8)$$

Substituting the beam end moments from equation (3) in equations (4), (5) and (8), yields

$$2.333EI\theta_B + 0.5EI\theta_C + 0.333EI\Delta = 9$$

$$2.333EI\theta_C + 0.5EI\theta_B + 0.667EI\Delta = 0$$

$$2EI\theta_B + 4EI\theta_C + 3.333EI\Delta = 36 \qquad (9)$$

Solving equations (9), (10) and (11),

$EI\theta_B = 2.76$; $EI\theta_C = -4.88$ and $EI\Delta = 15.00$.

Substituting the values of $EI\theta_B, EI\theta_C$ and $EI\Delta$ in the slope-deflection equation (3), one could calculate beam end moments. Thus,

$$M_{AB} = 15.835 \text{ kN.m} \quad \text{(counterclockwise)}$$
$$M_{BA} = -0.325 \text{ kN.m(clockwise)}$$
$$M_{BC} = 0.32 \text{ kN.m}$$
$$M_{CB} = -3.50 \text{ kN.m}$$
$$M_{CD} = 3.50 \text{ kN.m}$$
$$M_{DC} = 6.75 \text{ kN.m}.$$

The bending moment diagram for the frame is shown in Fig. 17.4 d.

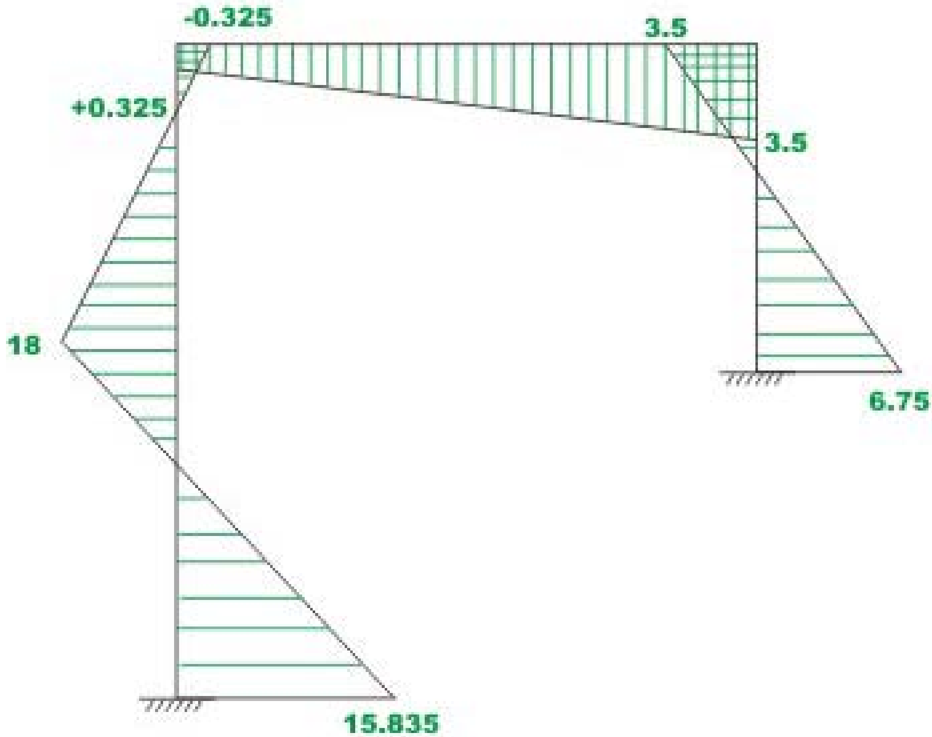


Fig.17.4d Bending moment diagram

Example 17.3

Analyse the rigid frame shown in Fig. 17.5 a. Moment of inertia of all the members are shown in the figure. Draw bending moment diagram.

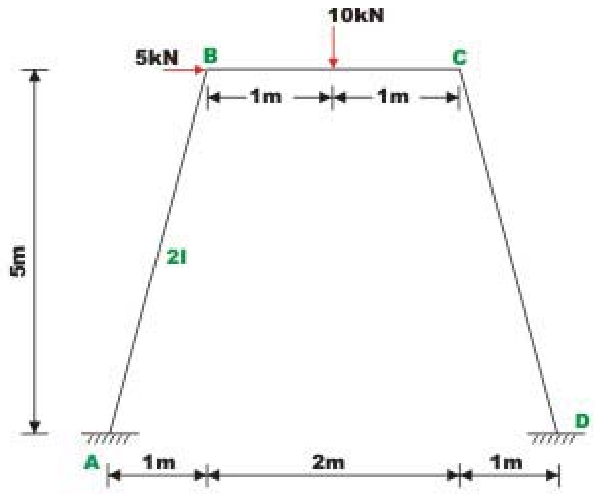


Fig.17.5a Example 17.3

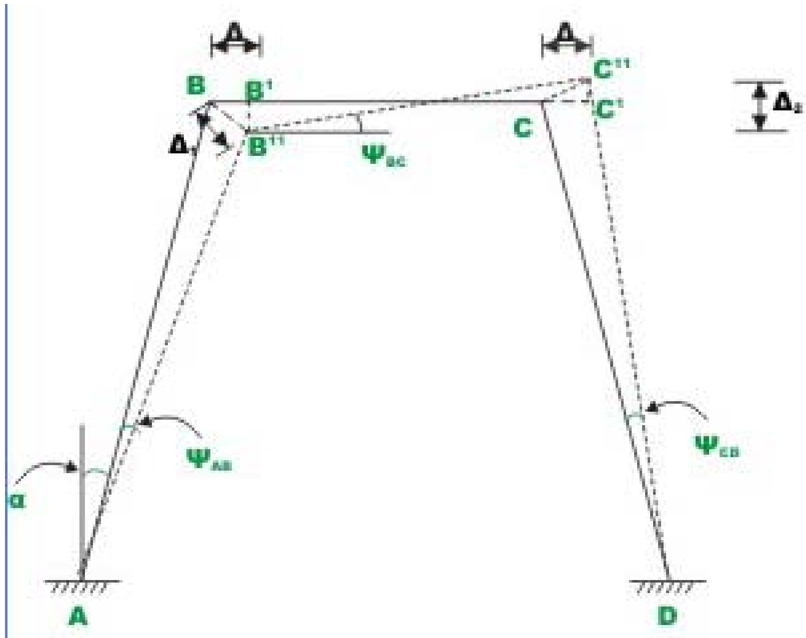


Fig.17.5b Rotation of Columns and beams

Under the action of external forces, the frame gets deformed as shown in Fig. 17.5b. In this figure, chord to the elastic curve are shown by dotted line. BB' is perpendicular to AB and CC'' is perpendicular to DC. The chords to the elastic curve AB'' rotates by an angle ψ_{AB}, $B''C''$ rotates by ψ_{BC} and DC rotates by ψ_{CD} as shown in figure. Due to symmetry, $\psi_{CD} = \psi_{AB}$. From the geometry of the figure,

$$\psi_{AB} = \frac{BB''}{L_{AB}} = -\frac{\Delta_1}{L_{AB}}$$

But

$$\Delta_1 = \frac{\Delta}{\cos\alpha}$$

Thus,

$$\psi_{AB} = -\frac{\Delta}{L_{AB}\cos\alpha} = -\frac{\Delta}{5}$$

$$\psi_{CD} = -\frac{\Delta}{5}$$

$$\psi_{BC} = \frac{\Delta_2}{2} = \frac{2\Delta\tan\alpha}{2} = \Delta\tan\alpha = \frac{\Delta}{5} \qquad (1)$$

We have three independent unknowns for this problem θ_B, θ_C and Δ. The ends A and D are fixed. Hence, $\theta_A = \theta_D = 0$. Fixed end moments are,

$$M_{AB}^F = 0 \;; M_{BA}^F = 0 \;; M_{BC}^F = +2.50\, kN.m \;; M_{CB}^F = -2.50 kN.m \;; M_{CD}^F = 0 \;; M_{DC}^F = 0.$$

Now, writing the slope-deflection equations for the six beam end moments,

$$M_{AB} = \frac{2E(2I)}{5.1}\left[\theta_A - 3\psi_{AB}\right]$$

$$M_{AB} = 0.784EI\theta_B + 0.471EI\Delta$$
$$M_{BA} = 1.568EI\theta_B + 0.471EI\Delta$$

$$M_{BC} = 2.5 + 2EI\theta_B + EI\theta_C - 0.6EI\Delta$$
$$M_{BC} = -2.5 + EI\theta_B + 2EI\theta_C - 0.6EI\Delta$$

$$M_{CD} = 1.568EI\theta_C + 0.471EI\Delta$$
$$M_{DC} = 0.784EI\theta_C + 0.471EI\Delta \qquad (2)$$

Now, considering the joint equilibrium of B and C, yields

$$\sum M_B = 0 \qquad \Rightarrow \qquad M_{BA} + M_{BC} = 0$$

$$3.568EI\theta_B + EI\theta_C - 0.129EI\Delta = -2.5 \tag{3}$$

$$\sum M_C = 0 \qquad \Rightarrow \qquad M_{CB} + M_{CD} = 0$$

$$3.568EI\theta_C + EI\theta_B - 0.129EI\Delta = 2.5 \tag{4}$$

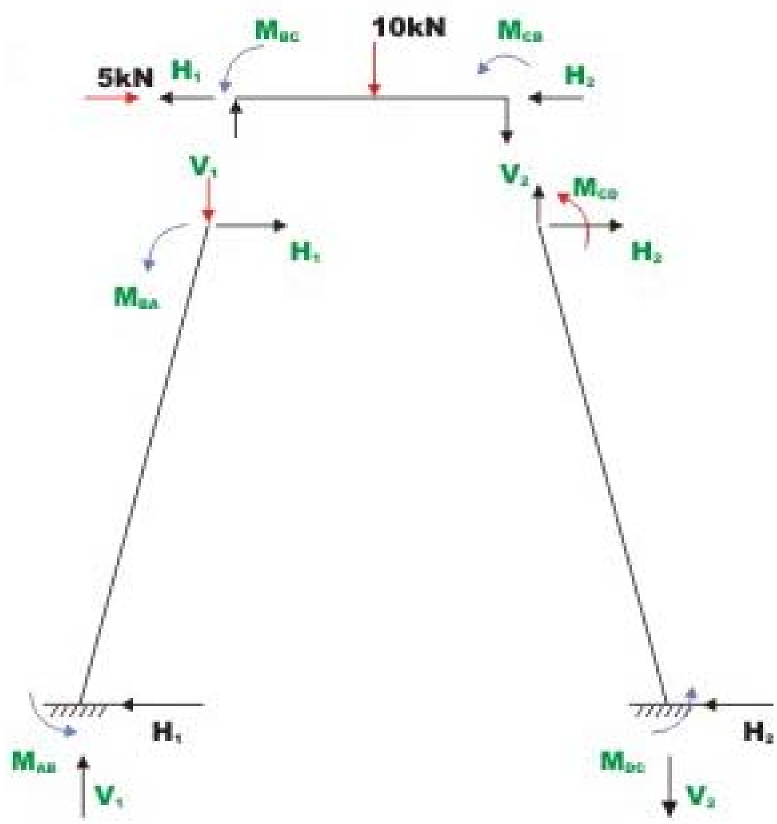


Fig.17.5c Free- body diagram

Shear equation for
Column AB

$$5H_1 - M_{AB} - M_{BA} + (1)V_1 = 0 \tag{5}$$

Column CD

$$5H_2 - M_{CD} - M_{DC} + (1)V_2 = 0 \tag{6}$$

Beam BC

$$\sum M_C = 0 \qquad 2V_1 - M_{BC} - M_{CB} - 10 = 0 \tag{7}$$

$$\sum F_X = 0 \qquad H_1 + H_2 = 5 \tag{8}$$

$$\sum F_Y = 0 \qquad V_1 - V_2 - 10 = 0 \tag{9}$$

From equation (7), $V_1 = \dfrac{M_{BC} + M_{CB} + 10}{2}$

From equation (8), $H_1 = 5 - H_2$

From equation (9), $V_2 = V_1 - 10 = \dfrac{M_{BC} + M_{CB} + 10}{2} - 10$

Substituting the values of V_1, H_1 and V_2 in equations (5) and (6),

$$60 - 10H_2 - 2M_{AB} - 2M_{BA} + M_{BC} + M_{CB} = 0 \tag{10}$$
$$-10 + 10H_2 - 2M_{CD} - 2M_{DC} + M_{BC} + M_{CB} = 0 \tag{11}$$

Eliminating H_2 in equation (10) and (11),

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} - M_{BC} - M_{CB} = 25 \tag{12}$$

Substituting the values of $M_{AB}, M_{BA}, M_{CD}, M_{DC}$ in (12) we get the required third equation. Thus,

$$0.784EI\theta_B + 0.471EI\Delta + 1.568EI\theta_B + 0.471EI\Delta + 1.568EI\theta_C + 0.471EI\Delta +$$
$$0.784EI\theta_C + 0.471EI\Delta \text{ -}(2.5 + 2EI\theta_B + EI\theta_C - 0.6EI\Delta)\text{-}$$
$$(-2.5 + EI\theta_B + 2EI\theta_C - 0.6EI\Delta) = 25$$

Simplifying,

$$-0.648EI\theta_C - 0.648EI\theta_B + 3.084EI\Delta = 25 \tag{13}$$

Solving simultaneously equations (3) (4) and (13), yields

$$EI\theta_B = -0.741; \qquad EI\theta_C = 1.205 \quad \text{and} \quad EI\Delta = 8.204.$$

Substituting the values of $EI\theta_B, EI\theta_C$ and $EI\Delta$ in the slope-deflection equation (3), one could calculate beam end moments. Thus,

$$M_{AB} = 3.28 \text{ kN.m}$$

$$M_{BA} = 2.70 \text{ kN.m}$$

$$M_{BC} = -2.70 \text{ kN.m}$$

$$M_{CB} = -5.75 \text{ kN.m}$$

$$M_{CD} = 5.75 \text{ kN.m}$$

$$M_{DC} = 4.81 \text{ kN.m} . \qquad (14)$$

The bending moment diagram for the frame is shown in Fig. 17.5 d.

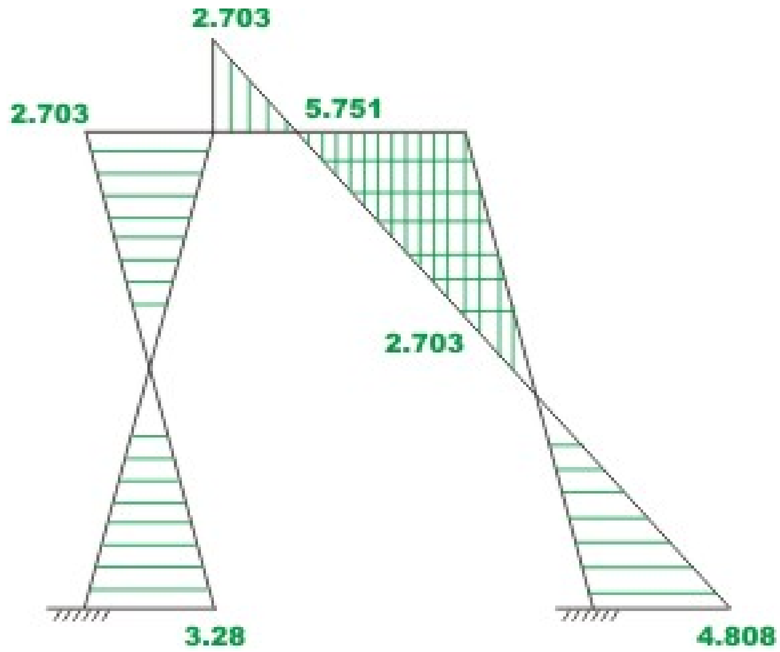


Fig.17.5d Bending moment diagram

Summary

In this lesson, slope-deflection equations are derived for the plane frame undergoing sidesway. Using these equations, plane frames with sidesway are analysed. The reactions are calculated from static equilibrium equations. A couple of problems are solved to make things clear. In each numerical example, the bending moment diagram is drawn and deflected shape is sketched for the plane frame.

Module 3

Analysis of Statically Indeterminate Structures by the Displacement Method

Lesson 18

The Moment-Distribution Method: Introduction

Instructional Objectives

After reading this chapter the student will be able to

1. Calculate stiffness factors and distribution factors for various members in a continuous beam.
2. Define unbalanced moment at a rigid joint.
3. Compute distribution moment and carry-over moment.
4. Derive expressions for distribution moment, carry-over moments.
5. Analyse continuous beam by the moment-distribution method.

18.1 Introduction

In the previous lesson we discussed the slope-deflection method. In slope-deflection analysis, the unknown displacements (rotations and translations) are related to the applied loading on the structure. The slope-deflection method results in a set of simultaneous equations of unknown displacements. The number of simultaneous equations will be equal to the number of unknowns to be evaluated. Thus one needs to solve these simultaneous equations to obtain displacements and beam end moments. Today, simultaneous equations could be solved very easily using a computer. Before the advent of electronic computing, this really posed a problem as the number of equations in the case of multistory building is quite large. The moment-distribution method proposed by Hardy Cross in 1932, actually solves these equations by the method of successive approximations. In this method, the results may be obtained to any desired degree of accuracy. Until recently, the moment-distribution method was very popular among engineers. It is very simple and is being used even today for preliminary analysis of small structures. It is still being taught in the classroom for the simplicity and physical insight it gives to the analyst even though stiffness method is being used more and more. Had the computers not emerged on the scene, the moment-distribution method could have turned out to be a very popular method. In this lesson, first moment-distribution method is developed for continuous beams with unyielding supports.

18.2 Basic Concepts

In moment-distribution method, counterclockwise beam end moments are taken as positive. The counterclockwise beam end moments produce clockwise moments on the joint Consider a continuous beam *ABCD* as shown in Fig.18.1a. In this beam, ends *A* and *D* are fixed and hence, $\theta_A = \theta_D = 0$ .Thus, the deformation of this beam is completely defined by rotations θ_B and θ_C at joints *B* and *C* respectively. The required equation to evaluate θ_B and θ_C is obtained by considering equilibrium of joints *B* and *C*. Hence,

$$\sum M_B = 0 \qquad \Rightarrow M_{BA} + M_{BC} = 0 \qquad (18.1a)$$
$$\sum M_C = 0 \qquad \Rightarrow M_{CB} + M_{CD} = 0 \qquad (18.1b)$$

According to slope-deflection equation, the beam end moments are written as

$$M_{BA} = M_{BA}^F + \frac{2EI_{AB}}{L_{AB}}(2\theta_B)$$

$\frac{4EI_{AB}}{L_{AB}}$ is known as stiffness factor for the beam *AB* and it is denoted by k_{AB} . M_{BA}^F is the fixed end moment at joint *B* of beam *AB* when joint *B* is fixed. Thus,

$$M_{BA} = M_{BA}^F + K_{AB}\theta_B$$

$$M_{BC} = M_{BC}^F + K_{BC}\left(\theta_B + \frac{\theta_C}{2}\right)$$

$$M_{CB} = M_{CB}^F + K_{CB}\left(\theta_C + \frac{\theta_B}{2}\right)$$

$$M_{CD} = M_{CD}^F + K_{CD}\theta_C \qquad (18.2)$$

In Fig.18.1b, the counterclockwise beam-end moments M_{BA} and M_{BC} produce a clockwise moment M_B on the joint as shown in Fig.18.1b. To start with, in moment-distribution method, it is assumed that joints are locked i.e. joints are prevented from rotating. In such a case (vide Fig.18.1b), $\theta_B = \theta_C = 0$, and hence

$$M_{BA} = M_{BA}^F$$
$$M_{BC} = M_{BC}^F$$
$$M_{CB} = M_{CB}^F$$
$$M_{CD} = M_{CD}^F \qquad (18.3)$$

Since joints *B* and *C* are artificially held locked, the resultant moment at joints *B* and *C* will not be equal to zero. This moment is denoted by M_B and is known as the unbalanced moment.

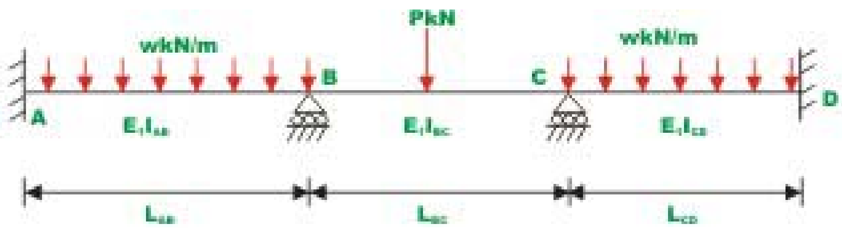


Fig. 18.1a Continuous Beam

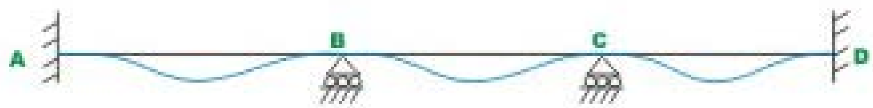


Fig. 18.1b Continuous beam with fixed joints.

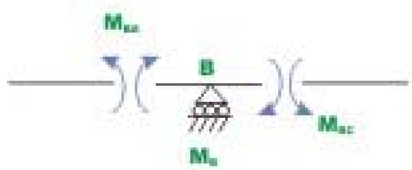


Fig. 18.1c Free - body diagram of joints B

Thus,

$$M_B = M_{BA}^F + M_{BC}^F$$

In reality joints are not locked. Joints *B* and *C* do rotate under external loads. When the joint *B* is unlocked, it will rotate under the action of unbalanced moment M_B. Let the joint B rotate by an angle θ_{B1}, under the action of M_B. This will deform the structure as shown in Fig.18.1d and introduces distributed moment M_{BA}^d, M_{BC}^d in the span BA and BC respectively as shown in the figure. The unknown distributed moments are assumed to be positive and hence act in counterclockwise direction. The unbalanced moment is the algebraic sum of the fixed end moments and act on the joint in the clockwise direction. The unbalanced moment restores the equilibrium of the joint B. Thus,

$$\sum M_B = 0, \qquad M_{BA}^d + M_{BC}^d + M_B = 0 \tag{18.4}$$

The distributed moments are related to the rotation θ_{B1} by the slope-deflection equation.

$$M_{BA}^{d} = K_{BA}\theta_{B1}$$

$$M_{BC}^{d} = K_{BC}\theta_{B1} \tag{18.5}$$

Substituting equation (18.5) in (18.4), yields

$$\theta_{B1}(K_{BA} + K_{BC}) = -M_B$$

$$\theta_{B1} = -\frac{M_B}{K_{BA} + K_{BC}}$$

In general,

$$\theta_{B1} = -\frac{M_B}{\sum K} \tag{18.6}$$

where summation is taken over all the members meeting at that particular joint. Substituting the value of θ_{B1} in equation (18.5), distributed moments are calculated. Thus,

$$M_{BA}^{d} = -\frac{K_{BA}}{\sum K} M_B$$

$$M_{BC}^{d} = -\frac{K_{BC}}{\sum K} M_B \tag{18.7}$$

The ratio $\frac{K_{BA}}{\sum K}$ is known as the distribution factor and is represented by DF_{BA}.

Thus,

$$M_{BA}^{d} = -DF_{BA.}M_B$$

$$M_{BC}^{d} = -DF_{BC.}M_B \tag{18.8}$$

The distribution moments developed in a member meeting at *B*, when the joint *B* is unlocked and allowed to rotate under the action of unbalanced moment M_B is equal to a distribution factor times the unbalanced moment with its sign reversed.

As the joint *B* rotates under the action of the unbalanced moment, beam end moments are developed at ends of members meeting at that joint and are known as distributed moments. As the joint *B* rotates, it bends the beam and beam end moments at the far ends (i.e. at *A* and *C*) are developed. They are known as carry over moments. Now consider the beam *BC* of continuous beam *ABCD*.

When the joint B is unlocked, joint C is locked .The joint B rotates by θ_{B1} under the action of unbalanced moment M_B (vide Fig. 18.1e). Now from slope-deflection equations

$$M^d_{BC} = K_{BC}\theta_B$$

$$M_{BC} = \frac{1}{2}K_{BC}\theta_B$$

$$M_{CB} = \frac{1}{2}M^d_{BC} \tag{18.9}$$

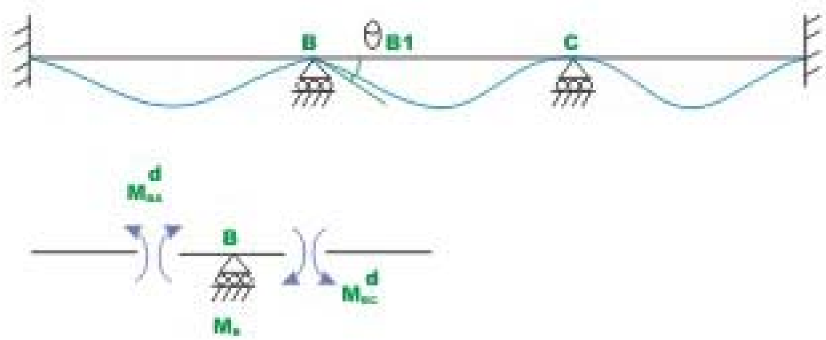


Fig. 18.1d Joint B is unlocked keeping C locked.

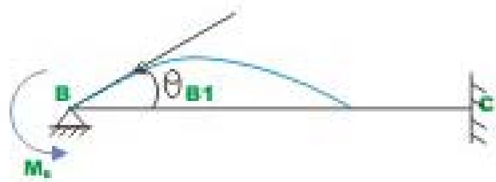


Fig.18.1e Carry - over moment

The carry over moment is one half of the distributed moment and has the same sign. With the above discussion, we are in a position to apply moment-distribution method to statically indeterminate beam. Few problems are solved here to illustrate the procedure. Carefully go through the first problem, wherein the moment-distribution method is explained in detail.

Example 18.1

A continuous prismatic beam ABC (see Fig.18.2a) of constant moment of inertia is carrying a uniformly distributed load of 2 kN/m in addition to a concentrated load of 10 kN. Draw bending moment diagram. Assume that supports are unyielding.

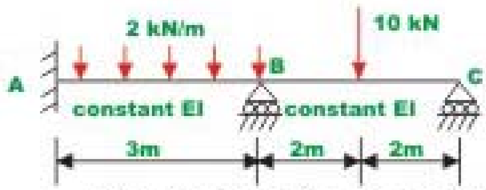


Fig. 18.2a Example 18.1

Solution

Assuming that supports *B* and C are locked, calculate fixed end moments developed in the beam due to externally applied load. Note that counterclockwise moments are taken as positive.

$$M_{AB}^{F} = \frac{wL_{AB}^{2}}{12} = \frac{2 \times 9}{12} = 1.5 \text{ kN.m}$$

$$M_{BA}^{F} = -\frac{wL_{AB}^{2}}{12} = -\frac{2 \times 9}{12} = -1.5 \text{ kN.m}$$

$$M_{BC}^{F} = \frac{Pab^{2}}{L_{BC}^{2}} = \frac{10 \times 2 \times 4}{16} = 5 \text{ kN.m}$$

$$M_{CB}^{F} = -\frac{Pa^{2}b}{L_{BC}^{2}} = -\frac{10 \times 2 \times 4}{16} = -5 \text{ kN.m} \tag{1}$$

Before we start analyzing the beam by moment-distribution method, it is required to calculate stiffness and distribution factors.

$$K_{BA} = \frac{4EI}{3}$$

$$K_{BC} = \frac{4EI}{4}$$

At *B*: $\sum K = 2.333EI$

$$DF_{BA} = \frac{1.333EI}{2.333EI} = 0.571$$

$$DF_{BC} = \frac{EI}{2.333EI} = 0.429$$

At C: $\sum K = EI$

$DF_{CB} = 1.0$

Note that distribution factor is dimensionless. The sum of distribution factor at a joint, except when it is fixed is always equal to one. The distribution moments are developed only when the joints rotate under the action of unbalanced moment. In the case of fixed joint, it does not rotate and hence no distribution moments are developed and consequently distribution factor is equal to zero.
In Fig.18.2b the fixed end moments and distribution factors are shown on a working diagram. In this diagram *B* and *C* are assumed to be locked.

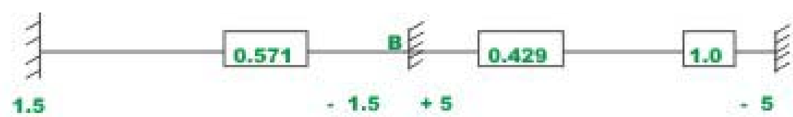


Fig. 18.2b

Now unlock the joint *C*. Note that joint *C* starts rotating under the unbalanced moment of 5 kN.m (counterclockwise) till a moment of -5 kN.m is developed (clockwise) at the joint. This in turn develops a beam end moment of +5 kN.m (M_{CB}). This is the distributed moment and thus restores equilibrium. Now joint C is relocked and a line is drawn below +5 kN.m to indicate equilibrium. When joint *C* rotates, a carry over moment of +2.5 kN.m is developed at the B end of member *BC*.These are shown in Fig.18.2c.

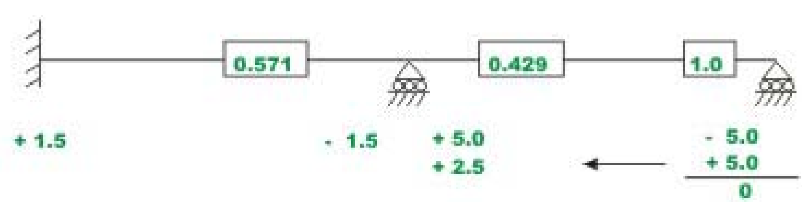


Fig. 18.2c

When joint B is unlocked, it will rotate under an unbalanced moment equal to algebraic sum of the fixed end moments(+5.0 and -1.5 kN.m) and a carry over moment of +2.5 kN.m till distributed moments are developed to restore equilibrium. The unbalanced moment is 6 kN.m. Now the distributed moments M_{BC} and M_{BA} are obtained by multiplying the unbalanced moment with the corresponding distribution factors and reversing the sign. Thus,

$M_{BC} = -2.574$ kN.m and $M_{BA} = -3.426$ kN.m. These distributed moments restore the equilibrium of joint *B*. Lock the joint *B*. This is shown in Fig.18.2d along with the carry over moments.

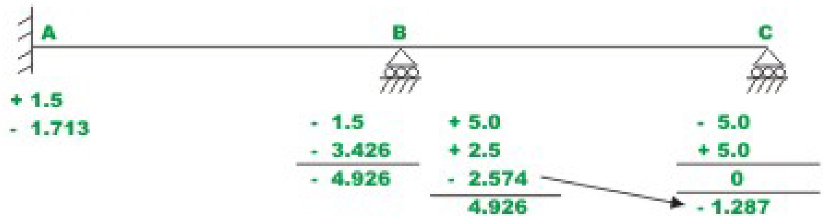


Fig. 18.2d

Now, it is seen that joint *B* is balanced. However joint *C* is not balanced due to the carry over moment -1.287 kN.m that is developed when the joint *B* is allowed to rotate. The whole procedure of locking and unlocking the joints *C* and *B* successively has to be continued till both joints B and C are balanced simultaneously. The complete procedure is shown in Fig.18.2e.

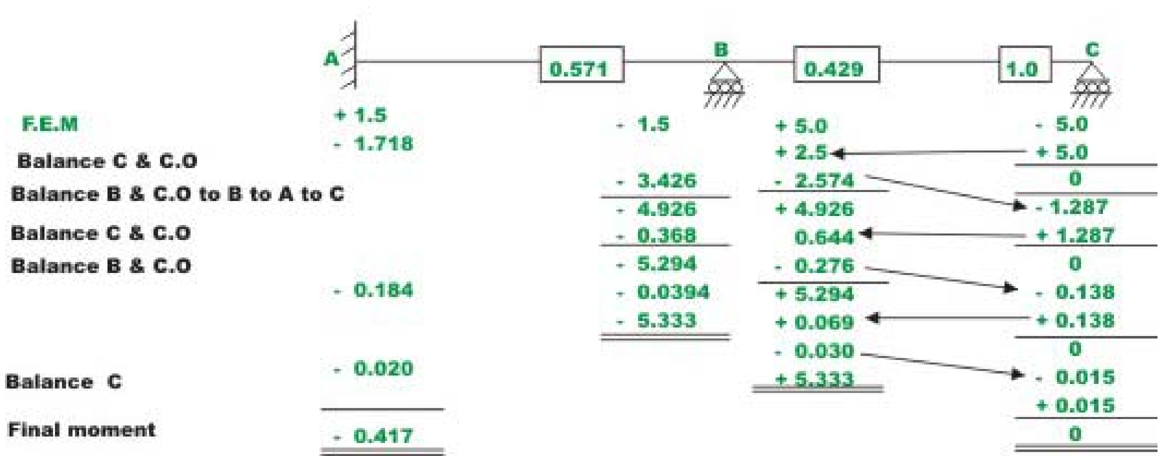


Fig. 18.2e Moment - distribution method : Computation

The iteration procedure is terminated when the change in beam end moments is less than say 1%. In the above problem the convergence may be improved if we leave the hinged end *C* unlocked after the first cycle. This will be discussed in the next section. In such a case the stiffness of beam BC gets modified. The above calculations can also be done conveniently in a tabular form as shown in Table 18.1. However the above working method is preferred in this course.

Table 18.1 Moment-distribution for continuous beam ABC

Joint	A	B		C
Member	AB	BA	BC	CB
Stiffness	1.333EI	1.333EI	EI	EI
Distribution factor		0.571	0.429	1.0
FEM in kN.m	+1.5	-1.5	+5.0	-5.0
Balance joints C ,B and C.O.	-1.713	-3.426	+2.5 -2.579	+5.0 0
		-4.926	+4.926	-1.287
Balance C and C.O.			+0.644	1.287
Balance B and C.O.		-0.368	-0.276	-0.138
Balance C	-0.184	-5.294	+5.294	0.138
C.O.			+0.069	0
Balance B and C.O.	-0.02	-0.039	-0.030	-0.015
Balance C				+0.015
Balanced moments in kN.m	-0.417	-5.333	+5.333	0

Modified stiffness factor when the far end is hinged

As mentioned in the previous example, alternate unlocking and locking at the hinged joint slows down the convergence of moment-distribution method. At the hinged end the moment is zero and hence we could allow the hinged joint C in the previous example to rotate freely after unlocking it first time. This necessitates certain changes in the stiffness parameters. Now consider beam *ABC* as shown in Fig.18.2a. Now if joint *C* is left unlocked then the stiffness of member *BC* changes. When joint *B* is unlocked, it will rotate by θ_{B1} under the action of unbalanced moment M_B.The support *C* will also rotate by θ_{C1} as it is free to rotate. However, moment $M_{CB} = 0$. Thus

$$M_{CB} = K_{BC}\theta_C + \frac{K_{BC}}{2}\theta_B \tag{18.7}$$

But, $M_{CB} = 0$

$$\Rightarrow \theta_C = -\frac{\theta_B}{2} \tag{18.8}$$

Now,

$$M_{BC} = K_{BC}\theta_B + \frac{K_{BC}}{2}\theta_C \tag{18.9}$$

Substituting the value of θ_C in eqn. (18.9),

$$M_{BC} = K_{BC}\theta_B - \frac{K_{BC}}{4}\theta_B = \frac{3}{4}K_{BC}\theta_B \tag{18.10}$$

$$M_{BC} = K_{BC}^R\theta_B \tag{18.11}$$

The K_{BC}^R is known as the reduced stiffness factor and is equal to $\frac{3}{4}K_{BC}$ .Accordingly distribution factors also get modified. It must be noted that there is no carry over to joint *C* as it was left unlocked.

Example 18.2

Solve the previous example by making the necessary modification for hinged end *C*.

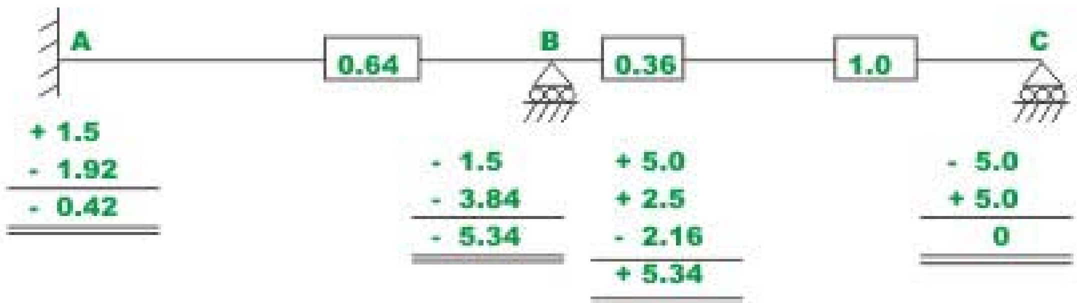


Fig. 18.3 Example 18.2

Fixed end moments are the same. Now calculate stiffness and distribution factors.

$$K_{BA} = 1.333EI, K_{BC} = \frac{3}{4}EI = 0.75EI$$

Joint B: $\sum K = 2.083,\quad D_{BA}^F = 0.64,\quad D_{BC}^F = 0.36$

Joint C: $\sum K = 0.75EI,\quad D_{CB}^F = 1.0$

All the calculations are shown in Fig.18.3a

Please note that the same results as obtained in the previous example are obtained here in only one cycle. All joints are in equilibrium when they are unlocked. Hence we could stop moment-distribution iteration, as there is no unbalanced moment anywhere.

Example 18.3

Draw the bending moment diagram for the continuous beam *ABCD* loaded as shown in Fig.18.4a.The relative moment of inertia of each span of the beam is also shown in the figure.

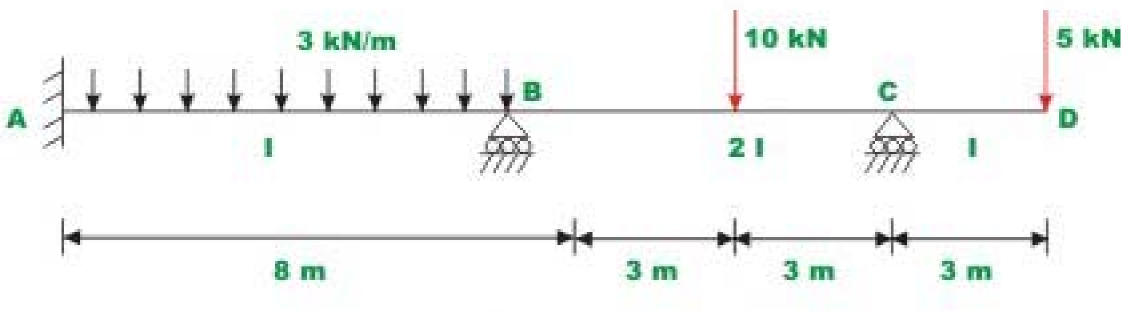


Fig. 18.4a Example 18.3

Solution

Note that joint *C* is hinged and hence stiffness factor *BC* gets modified. Assuming that the supports are locked, calculate fixed end moments. They are

$$M_{AB}^{F} = 16 \text{ kN.m}$$

$$M_{BA}^{F} = -16 \text{ kN.m}$$

$$M_{BC}^{F} = 7.5 \text{ kN.m}$$

$$M_{CB}^{F} = -7.5 \text{ kN.m}$$, and

$$M_{CD}^{F} = 15 \text{ kN.m}$$

In the next step calculate stiffness and distribution factors

$$K_{BA} = \frac{4EI}{8}$$

$$K_{BC} = \frac{3}{4}\frac{8EI}{6}$$

$$K_{CB} = \frac{8EI}{6}$$

At joint B:

$$\sum K = 0.5EI + 1.0EI = 1.5EI$$

$$D_{BA}^{F} = \frac{0.5EI}{1.5EI} = 0.333$$

$$D_{BC}^{F} = \frac{1.0EI}{1.5EI} = 0.667$$

At *C*:

$$\sum K = EI, D_{CB}^{F} = 1.0$$

Now all the calculations are shown in Fig.18.4b

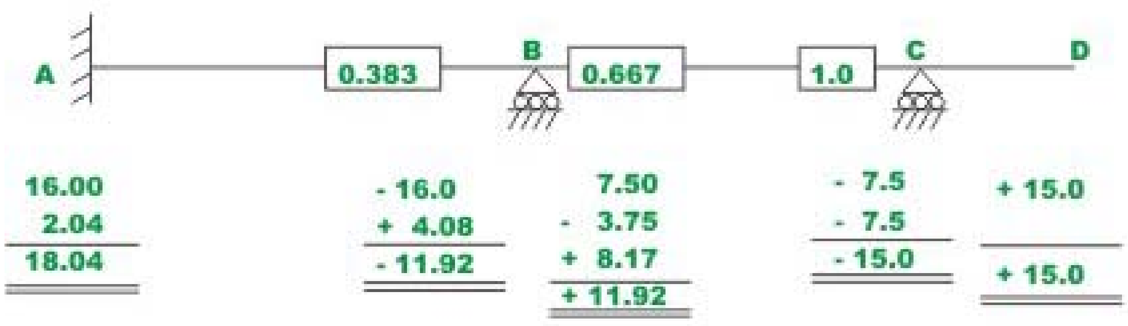


Fig. 18.4b Computation

This problem has also been solved by slope-deflection method (see example 14.2).The bending moment diagram is shown in Fig.18.4c.

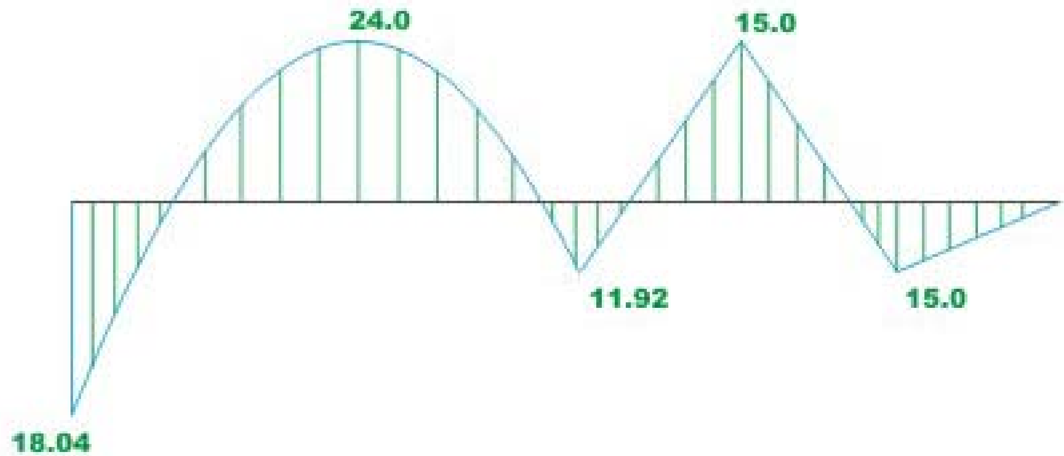


Fig. 18.4c Bending - moment diagram

Summary

An introduction to the moment-distribution method is given here. The moment-distribution method actually solves these equations by the method of successive approximations. Various terms such as stiffness factor, distribution factor, unbalanced moment, distributing moment and carry-over-moment are defined in this lesson. Few problems are solved to illustrate the moment-distribution method as applied to continuous beams with unyielding supports.

MODULE 3

ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES BY THE DISPLACEMENT METHOD

LESSON 19

THE MOMENT-DISTRIBUTION METHOD: STATICALLY INDETERMINATE BEAMS WITH SUPPORT SETTLEMENTS

Instructional Objectives

After reading this chapter the student will be able to

1. Solve continuous beam with support settlements by the moment-distribution method.
2. Compute reactions at the supports.
3. Draw bending moment and shear force diagrams.
4. Draw the deflected shape of the continuous beam.

19.1 Introduction

In the previous lesson, moment-distribution method was discussed in the context of statically indeterminate beams with unyielding supports. It is very well known that support may settle by unequal amount during the lifetime of the structure. Such support settlements induce fixed end moments in the beams so as to hold the end slopes of the members as zero (see Fig. 19.1).

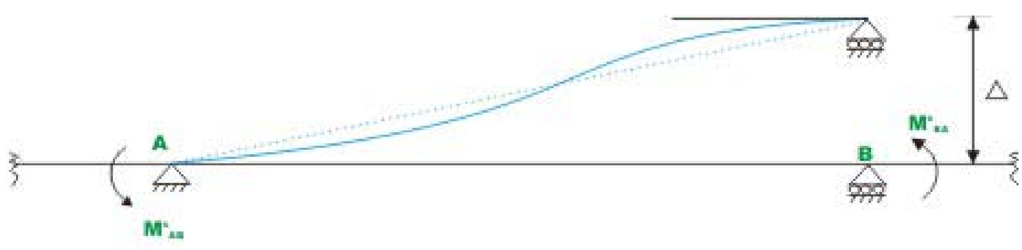


Fig . 19.1 Support settlement without ratation

In lesson 15, an expression (equation 15.5) for beam end moments were derived by superposing the end moments developed due to

1. Externally applied loads on beams
2. Due to displacements θ_A, θ_B and Δ (settlements).

The required equations are,

$$M_{AB} = M_{AB}^{F} + \frac{2EI_{AB}}{L_{AB}}\left[2\theta_A + \theta_B - \frac{3\Delta}{L_{AB}}\right] \qquad (19.1a)$$

$$M_{BA} = M_{BA}^{F} + \frac{2EI_{AB}}{L_{AB}}\left[2\theta_B + \theta_A - \frac{3\Delta}{L_{AB}}\right] \tag{19.1b}$$

This may be written as,

$$M_{AB} = M_{AB}^{F} + 2K_{AB}\left[2\theta_A + \theta_B\right] + M_{AB}^{S} \tag{19.2a}$$

$$M_{BA} = M_{BA}^{F} + 2K_{AB}\left[2\theta_B + \theta_A\right] + M_{BA}^{S} \tag{19.2b}$$

where $K_{AB} = \frac{EI_{AB}}{L_{AB}}$ is the stiffness factor for the beam *AB*. The coefficient 4 has been dropped since only relative values are required in calculating distribution factors.

Note that $M_{AB}^{S} = M_{BA}^{S} = -\frac{6EI_{AB}\Delta}{L_{AB}^{2}}$ (19.3)

M_{AB}^{S} is the beam end moments due to support settlement and is negative (clockwise) for positive support settlements (upwards). In the moment-distribution method, the support moments M_{AB}^{S} and M_{BA}^{S} due to uneven support settlements are distributed in a similar manner as the fixed end moments, which were described in details in lesson 18.

It is important to follow consistent sign convention. Here counterclockwise beam end moments are taken as positive and counterclockwise chord rotation $\left(\frac{\Delta}{L}\right)$ is taken as positive. The moment-distribution method as applied to statically indeterminate beams undergoing uneven support settlements is illustrated with a few examples.

Example 19.1

Calculate the support moments of the continuous beam ABC (Fig. 19.2a) having constant flexural rigidity EI throughout, due to vertical settlement of support B by 5mm. Assume $E = 200$ GPa ; and $I = 4 \times 10^{-4}\,\mathrm{m}^4$.

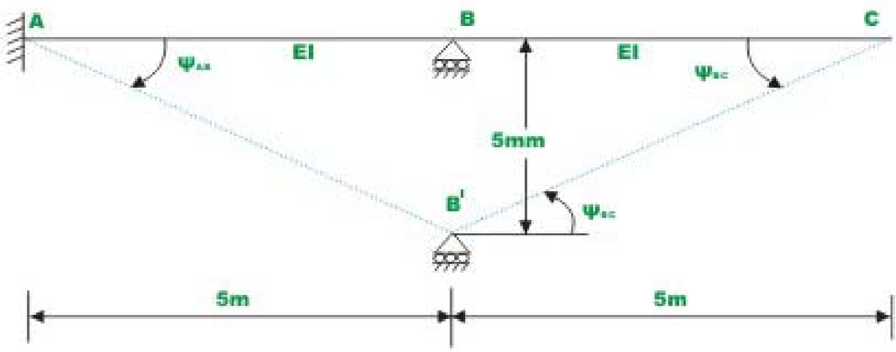


Fig . 19.2a Chord rotation due to support settlement (Example 19.1)

Solution

There is no load on the beam and hence fixed end moments are zero. However, fixed end moments are developed due to support settlement of B by 5mm. In the span AB , the chord rotates by ψ_{AB} in clockwise direction. Thus,

$$\psi_{AB} = -\frac{5 \times 10^{-3}}{5}$$

$$M_{AB}^{S} = M_{BA}^{S} = -\frac{6EI_{AB}}{L_{AB}}\psi_{AB} = -\frac{6 \times 200 \times 10^{9} \times 4 \times 10^{-4}}{5}\left(-\frac{5 \times 10^{-3}}{5}\right)$$

$$= 96000 \quad \mathrm{Nm} = 96 \quad \mathrm{kNm}. \qquad (1)$$

In the span BC , the chord rotates by ψ_{BC} in the counterclockwise direction and hence taken as positive.

$$\psi_{BC} = \frac{5 \times 10^{-3}}{5}$$

$$M_{BC}^{S} = M_{CB}^{S} = -\frac{6EI_{BC}}{L_{BC}}\psi_{BC} = -\frac{6\times 200\times 10^{9}\times 4\times 10^{-4}}{5}\left(\frac{5\times 10^{-3}}{5}\right)$$

$$= -96000\ Nm = -96\ kNm. \qquad (2)$$

Now calculate stiffness and distribution factors.

$$K_{BA} = \frac{EI_{AB}}{L_{AB}} = 0.2EI \quad \text{and} \quad K_{BC} = \frac{3}{4}\frac{EI_{BC}}{L_{BC}} = 0.15EI \qquad (3)$$

Note that, while calculating stiffness factor, the coefficient 4 has been dropped since only relative values are required in calculating the distribution factors. For span BC, reduced stiffness factor has been taken as support C is hinged.

At B:

$$\sum K = 0.35EI$$

$$DF_{BA} = \frac{0.2EI}{0.35EI} = 0.571$$

$$DF_{BC} = \frac{0.15EI}{0.35EI} = 0.429 \qquad (4)$$

At support C:

$$\sum K = 0.15EI\ ; \qquad DF_{CB} = 1.0\ .$$

Now joint moments are balanced as discussed previously by unlocking and locking each joint in succession and distributing the unbalanced moments till the joints have rotated to their final positions. The complete procedure is shown in Fig. 19.2b and also in Table 19.1.

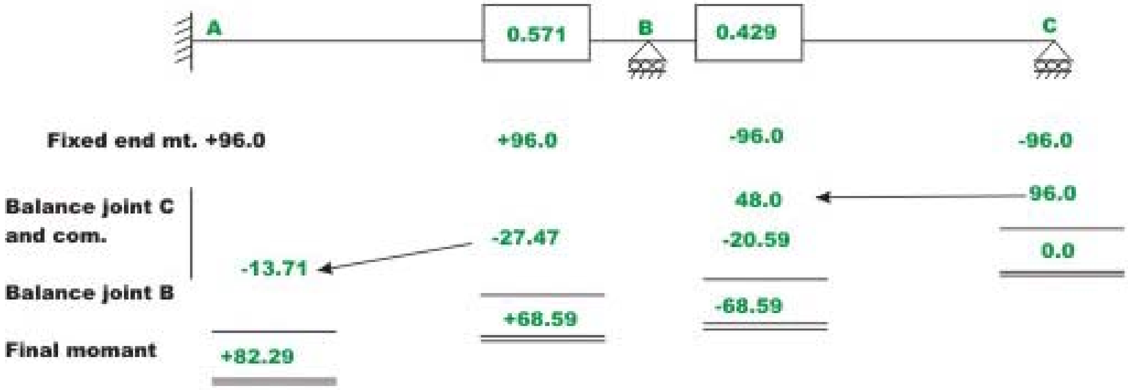


Fig. 19.2b Computation

Table 19.1 Moment-distribution for continuous beam ABC

Joint	A	B		C
Member		BA	BC	CB
Stiffness factor		0.2EI	0.15EI	0.15EI
Distribution Factor		0.571	0.429	1.000
Fixd End Moments (kN.m)	96.000	96.000	-96.000	-96.000
Balance joint C and C.O. to B			48.00	96.000
Balance joint B and C.O. to A	-13,704	-27.408	-20.592	
Final Moments (kN.m)	82.296	68.592	-68.592	0.000

Note that there is no carry over to joint C as it was left unlocked.

Example 19.2

A continuous beam $ABCD$ is carrying uniformly distributed load $5\,kN/m$ as shown in Fig. 19.3a. Compute reactions and draw shear force and bending moment diagram due to following support settlements.

Support B, 0.005m vertically downwards.

Support C, .0100m vertically downwards.

Assume $E = 200GPa$; $I = 1.35 \times 10^{-3} m^4$.

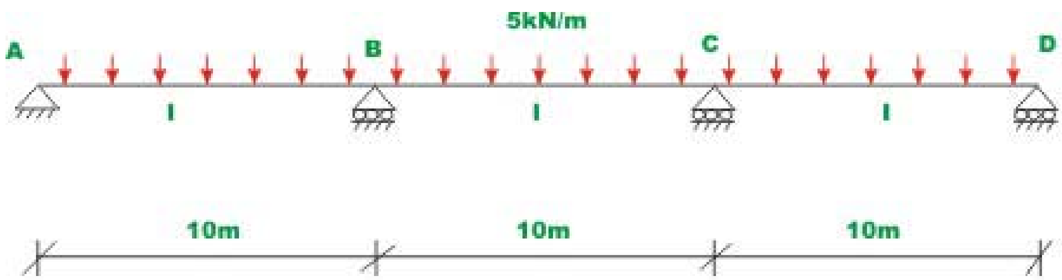


Fig .19.3a Continuous beam of Example 19.2

Solution:

Assume that supports A, B, C and D are locked and calculate fixed end moments due to externally applied load and support settlements. The fixed end beam moments due to externally applied loads are,

$$M_{AB}^{F} = \frac{5 \times 100}{12} = 41.67 \text{ kN.m}; \; M_{BA}^{F} = -41.67 \text{ kN.m}$$

$$M_{BC}^{F} = +41.67 \text{ kN.m}; \qquad M_{BC}^{F} = -41.67 \text{ kN.m}$$

$$M_{CD}^{F} = +41.67 \text{ kN.m}; \qquad M_{DC}^{F} = -41.67 \text{ kN.m} \qquad (1)$$

In the span AB, the chord joining joints A and B rotates in the clockwise direction as B moves vertical downwards with respect to A (see Fig. 19.3b).

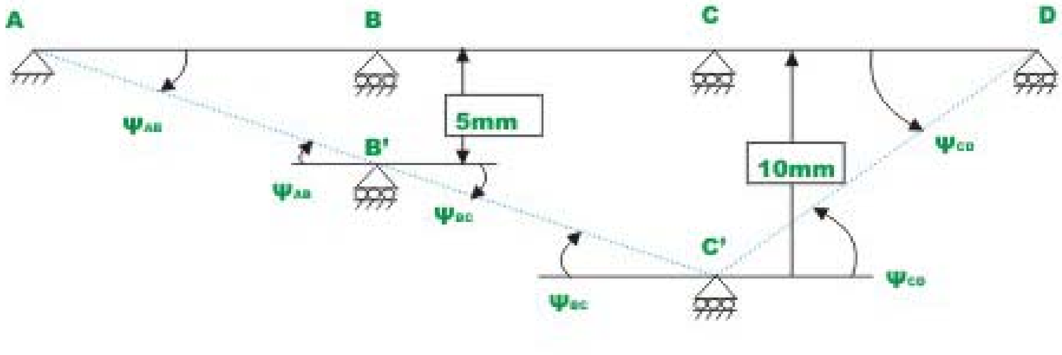


Fig. 19.3b Member rotation due to support settlement

$\psi_{AB} = -0.0005$ radians (negative as chord AB' rotates in the clockwise direction from its original position)

$\psi_{BC} = -0.0005$ radians

$\psi_{CD} = 0.001$ radians (positive as chord $C'D$ rotates in the counterclockwise direction).

Now the fixed end beam moments due to support settlements are,

$$M^S_{AB} = -\frac{6EI_{AB}}{L_{AB}}\psi_{AB} = -\frac{6 \times 200 \times 10^9 \times 1.35 \times 10^{-3}}{10}(-0.0005)$$

$$= 81000 \text{ N.m} = 81.00 \text{ kN.m}$$

$$M^S_{BA} = 81.00 \text{ kN.m}$$

$$M^S_{BC} = M^S_{CB} = 81.00 \text{ kN.m}$$

$$M^S_{CD} = M^S_{DC} = -162.00 \text{ kN.m} \qquad (3)$$

In the next step, calculate stiffness and distribution factors. For span AB and CD modified stiffness factors are used as supports A and D are hinged. Stiffness factors are,

$$K_{BA} = \frac{3}{4}\frac{EI}{10} = 0.075EI\,; \qquad K_{BC} = \frac{EI}{10} = 0.10EI$$

$$K_{CB} = \frac{EI}{10} = 0.10EI\,; \qquad K_{CD} = \frac{3}{4}\frac{EI}{10} = 0.075EI \qquad (4)$$

At joint A : $\sum K = 0.075EI$; $DF_{AB} = 1.0$

At joint B : $\sum K = 0.175EI$; $DF_{BA} = 0.429$; $DF_{BC} = 0.571$

At joint C : $\sum K = 0.175EI$; $DF_{CB} = 0.571$; $DF_{CD} = 0.429$

At joint D : $\sum K = 0.075EI$; $DF_{DC} = 1.0$

The complete procedure of successively unlocking the joints, balancing them and locking them is shown in a working diagram in Fig.19.3c. In the first row, the distribution factors are entered. Then fixed end moments due to applied loads and support settlements are entered. In the first step, release joints A and D. The unbalanced moments at A and D are 122.67 kN.m, -203.67 kN.m respectively. Hence balancing moments at A and D are -122.67 kN.m, 203.67 kN.m respectively. (Note that we are dealing with beam end moments and not joint moments). The joint moments are negative of the beam end moments. Further leave A and D unlocked as they are hinged joints. Now carry over moments -61.34 kN.m and 101.84 kN.m to joint B and C respectively. In the next cycle, balance joints B and C. The unbalanced moment at joint B is 100.66 kN.m. Hence balancing moment for beam BA is -43.19 (-100.66×0.429) and for BC is -57.48 kN.m (-100.66 x 0.571). The balancing moment on BC gives a carry over moment of -26.74 kN.m to joint C. The whole procedure is shown in Fig. 19.3c and in Table 19.2. It must be noted that there is no carryover to joints A and D as they were left unlocked.

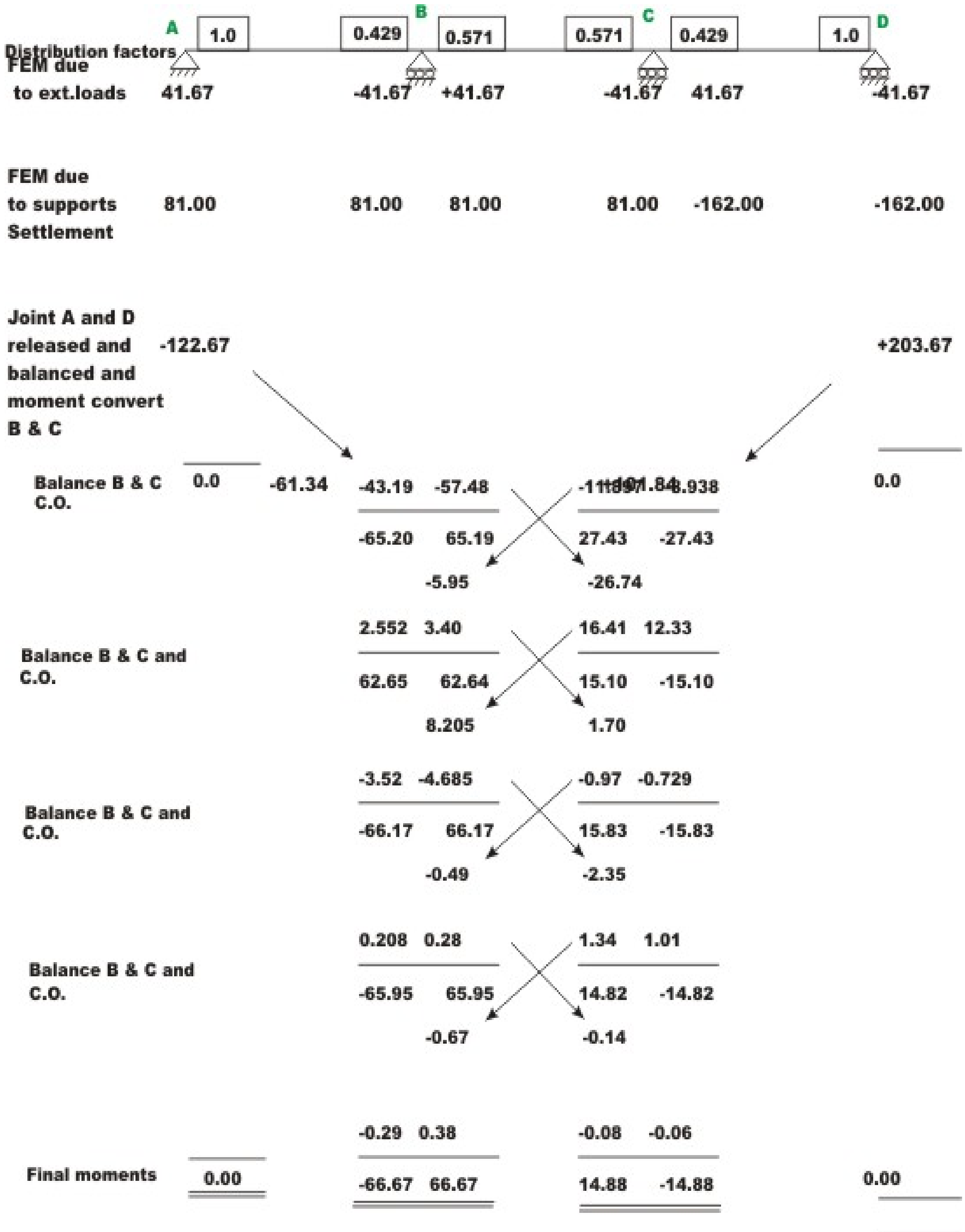


Fig. 19.3 © Computation

Table 19.2 Moment-distribution for continuous beam ABCD

Joint	A	B		C		D
Members	AB	BA	BC	CB	CD	DC
Stiffness factors	0.075 EI	0.075 EI	0.1 EI	0.1 EI	0.075 EI	0.075 EI
Distribution Factors	1.000	0.429	0.571	0.571	0.429	1.000
FEM due to externally applied loads	41.670	-41.670	41.670	-41.670	41.670	-41.670
FEM due to support settlements	81.000	81.000	81.000	81.000	-162.000	-162.000
Total	122.670	39.330	122.670	39.330	-120.330	-203.670
Balance A and D released	-122.670					203.670
Carry over		-61.335			101.835	
Balance B and C		-43.185	-57.480	-11.897	-8.94	
Carry over			-5.95	-26.740		
Balance B and C		2.552	3.40	16.410	12.33	
Carry over to B and C			8.21	1.70		
Balance B and C		-3.52	-4.69	-0.97	-0.73	
C.O. to B and C			-0.49	-2.33		
Balance B and C		0.21	0.28	1.34	1.01	
Carry over			0.67	0.14		
Balance B and C		-0.29	-0.38	-0.08	-0.06	
Final Moments	0.000	-66.67	66.67	14.88	-14.88	0.000

Example 19.3

Analyse the continuous beam ABC shown in Fig. 19.4a by moment-distribution method. The support B settles by $5mm$ below A and C. Assume EI to be constant for all members $E = 200GPa$; and $I = 8 \times 10^6\, mm^4$.

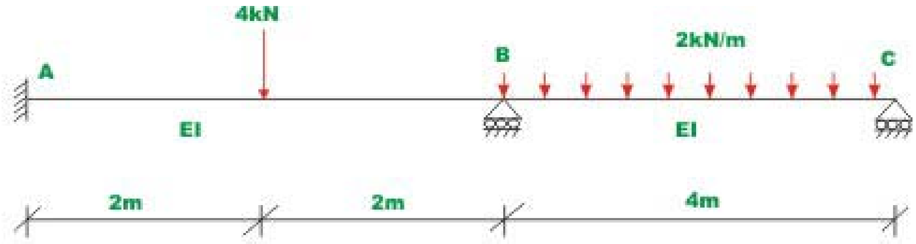


Fig. 19.4 (a) Example 19.4a

Solution:

Calculate fixed end beam moments due to externally applied loads assuming that support B and C are locked.

$$M_{AB}^{F} = +2\ kN.m\ ; \qquad M_{BA}^{F} = -2\ kN.m$$
$$M_{BC}^{F} = +2.67\ kN.m\ ; \qquad M_{CB}^{F} = -2.67\ kN.m \qquad (1)$$

In the next step calculate fixed end moments due to support settlements. In the span AB, the chord AB' rotates in the clockwise direction and in span BC, the chord $B'C$ rotates in the counterclockwise direction (Fig. 19.4b).

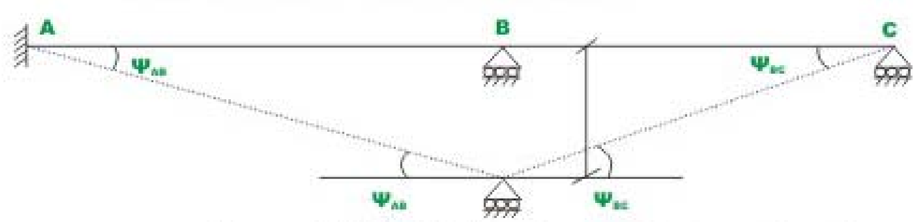


Fig. 19.4 (b) Member rotation due to support settlement

$$\psi_{AB} = -\frac{5\times10^{-3}}{4} = -1.25\times10^{-3}\, radians$$

$$\psi_{BC} = \frac{5\times10^{-3}}{4} = 1.25\times10^{-3}\, radians \qquad (2)$$

$$M_{AB}^{S} = M_{BA}^{S} = -\frac{6EI_{AB}}{L_{AB}}\psi_{AB} = -\frac{6\times200\times10^{9}\times8\times10^{-6}}{4}\left(-\frac{5\times10^{-3}}{4}\right)$$

$$= 3000\ Nm = 3\,kNm. \qquad (3)$$

$$M_{BC}^{S} = M_{CB}^{S} = -3.0\,kN.m$$

In the next step, calculate stiffness and distribution factors.

$$K_{AB} = K_{BA} = 0.25EI$$
$$K_{BC} = \frac{3}{4}0.25EI = 0.1875EI \qquad (4)$$

At joint B : $\sum K = 0.4375EI$; $DF_{BA} = 0.571$; $DF_{BC} = 0.429$

At joint C : $\sum K = 0.1875EI$; $DF_{CB} = 1.0$

At fixed joint, the joint does not rotate and hence no distribution moments are developed and consequently distribution factor is equal to zero. The complete moment-distribution procedure is shown in Fig. 19.4c and Table 19.3. The diagram is self explanatory. In this particular case results are obtained in two cycles. In the first cycle joint C is balanced and carry over moment is taken to joint B. In the next cycle , joint B is balanced and carry over moment is taken to joint A. The bending moment diagram is shown in fig. 19.4d.

Table 19.3 Moment-distribution for continuous beam ABC

Joints	A	B		C
Member	AB	BA	BC	CB
Stiffness factor	0.25 EI	0.25 EI	0.1875 EI	0.1875 EI
Distribution Factor		0.571	0.429	1.000
Fixed End Moments due to applied loads (kN.m)	2.000	-2.000	2.667	-2.667
Fixed End Moments due to support settlements (kN.m)	3.000	3.000	-3.000	-3.000
Total	5.000	1.000	-0.333	-5.667
Balance joint C and C.O.			2.835	5.667
Total	5.000	1.000	2.502	0.000
Balance joint B and C.O. to A	-1.00	-2.000	-1.502	
Final Moments (kN.m)	4.000	-1.000	1.000	0.000

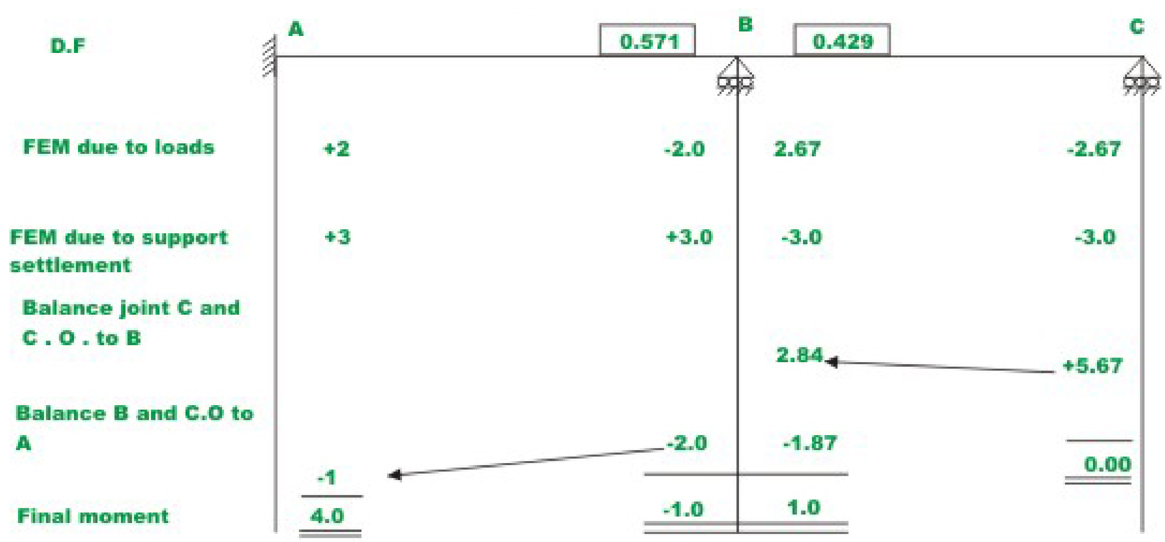


Fig. 19.4 (c) Computation

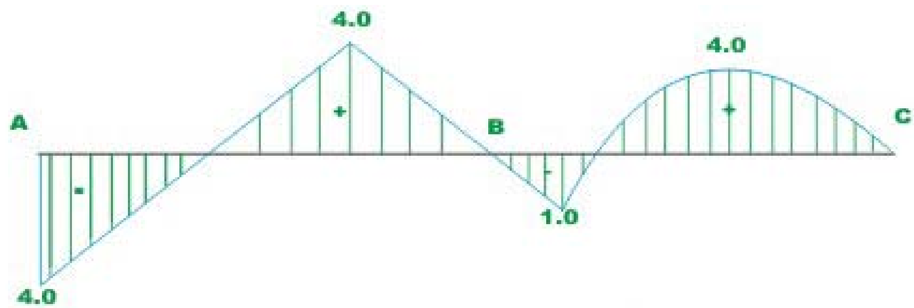


FIG. 19.4 (d) B.M.D

Summary

The moment-distribution method is applied to analyse continuous beam having support settlements. Each step in the numerical example is explained in detail. All calculations are shown at appropriate locations. The deflected shape of the continuous beam is sketched. Also, wherever required, the bending moment diagram is drawn. The numerical examples are explained with the help of free-body diagrams.

Module 3

Analysis of Statically Indeterminate Structures by the Displacement Method

Lesson
20
The Moment-Distribution Method: Frames without Sidesway

Instructional Objectives

After reading this chapter the student will be able to
1. Solve plane frame restrained against sidesway by the moment-distribution method.
2. Compute reactions at the supports.
3. Draw bending moment and shear force diagrams.
4. Draw the deflected shape of the plane frame.

20.1 Introduction

In this lesson, the statically indeterminate rigid frames properly restrained against sidesway are analysed using moment-distribution method. Analysis of rigid frames by moment-distribution method is very similar to that of continuous beams described in lesson 18. As pointed out earlier, in the case of continuous beams, at a joint only two members meet, where as in case of rigid frames two or more than two members meet at a joint. At such joints (for example joint C in Fig. 20.1) where more than two members meet, the unbalanced moment at the beginning of each cycle is the algebraic sum of fixed end beam moments (in the first cycle) or the carry over moments (in the subsequent cycles) of the beam meeting at C. The unbalanced moment is distributed to members CB, CD and CE according to their distribution factors. Few examples are solved to explain procedure. The moment-distribution method is carried out on a working diagram.

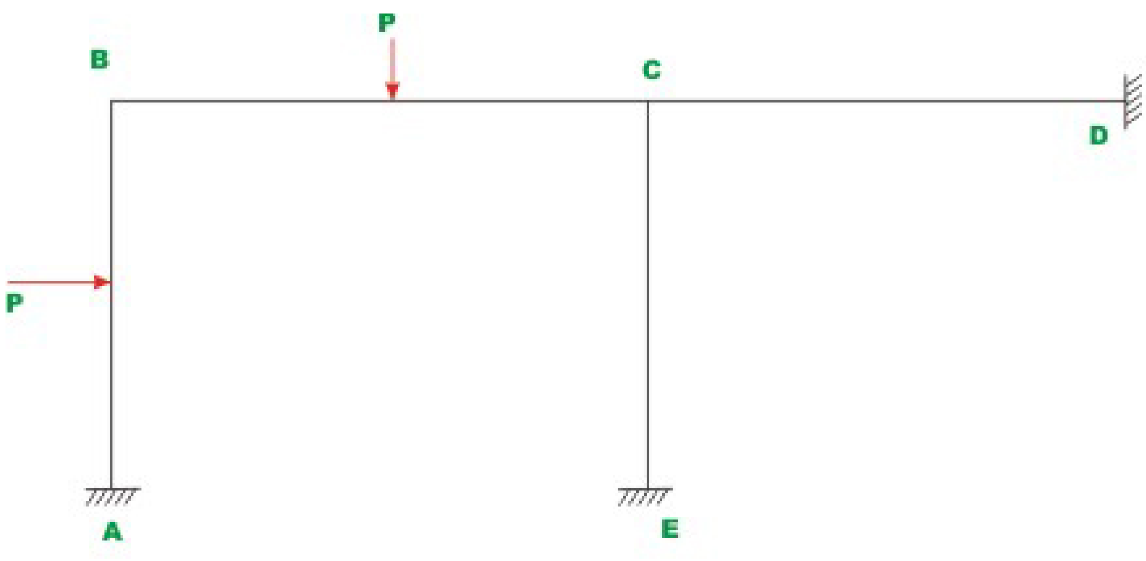


Fig. 20.1 Plane frame

Example 20.1

Calculate reactions and beam end moments for the rigid frame shown in Fig. 20.2a. Draw bending moment diagram for the frame. Assume EI to be constant for all the members.

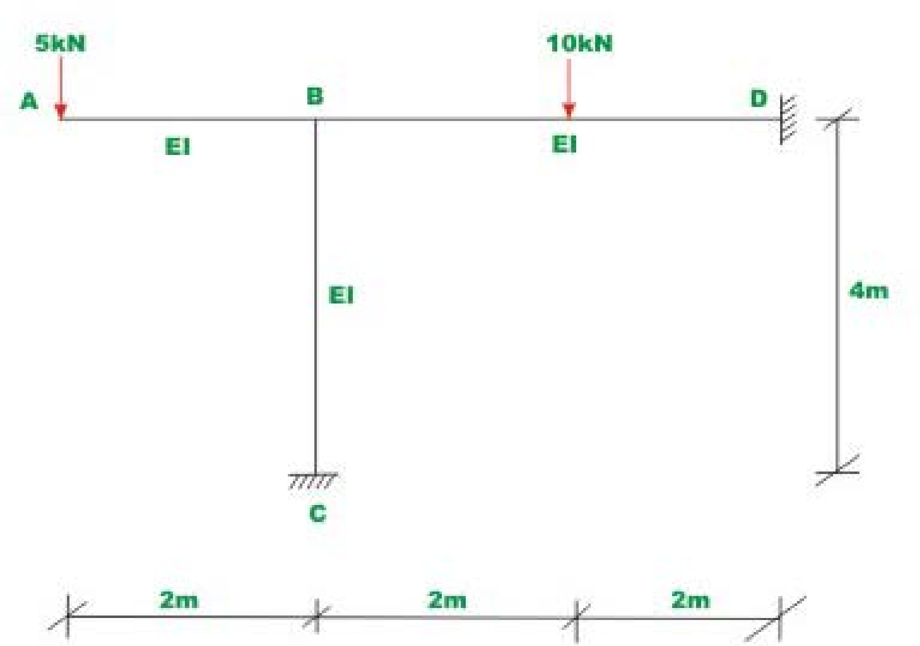


Fig. 20.2a Rigid plane frame of Example 20.1

Solution

In the first step, calculate fixed end moments.

$$M_{BD}^{F} = 5.0\ kN.m$$

$$M_{DB}^{F} = -5.0\ kN.m$$

$$M_{BC}^{F} = 0.0\ kN.m \tag{1}$$

$$M_{CB}^{F} = 0.0\ kN.m$$

Also, the fixed end moment acting at B on BA is clockwise.

$$M_{BA}^{F} = -10.0\ kN.m$$

In the next step calculate stiffness and distribution factors.

$$K_{BD} = \frac{EI}{4} = 0.25EI \quad \text{and} \quad K_{BC} = \frac{EI}{4} = 0.25EI$$

At joint B :

$$\sum K = 0.50EI$$

$$DF_{BD} = \frac{0.25EI}{0.5EI} = 0.5 \; ; \qquad DF_{BC} = 0.5 \qquad (2)$$

All the calculations are shown in Fig. 20.2b. Please note that cantilever member does not have any restraining effect on the joint B from rotation. In addition its stiffness factor is zero. Hence unbalanced moment is distributed between members BC and BD only.

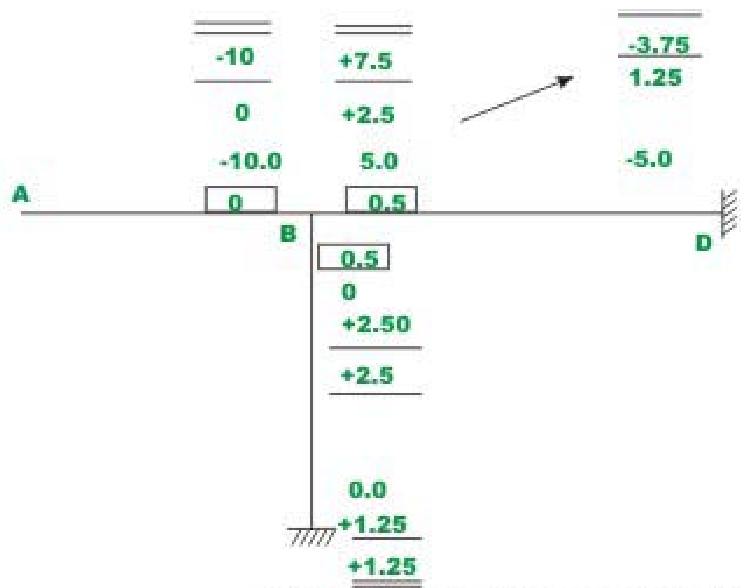


Fig. 20.2b Moment distribution

In this problem the moment-distribution method is completed in only one cycle, as equilibrium of only one joint needs to be considered. In other words, there is only one equation that needs to be solved for the unknown θ_B in this problem. This problem has already been solved by slop- deflection method wherein reactions are computed from equations of statics. The free body diagram of each member of the frame with external load and beam end moments are again reproduced here in Fig. 20.2c for easy reference. The bending moment diagram is shown in Fig. 20.2d.

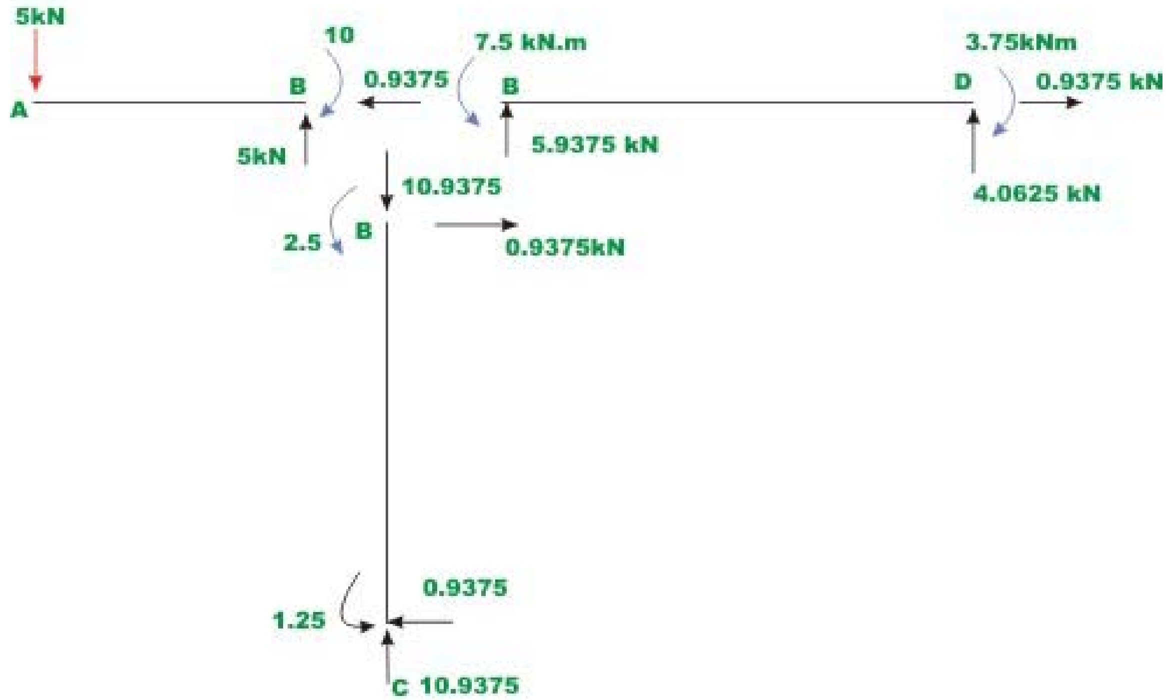


Fig. 20.2c Reactions

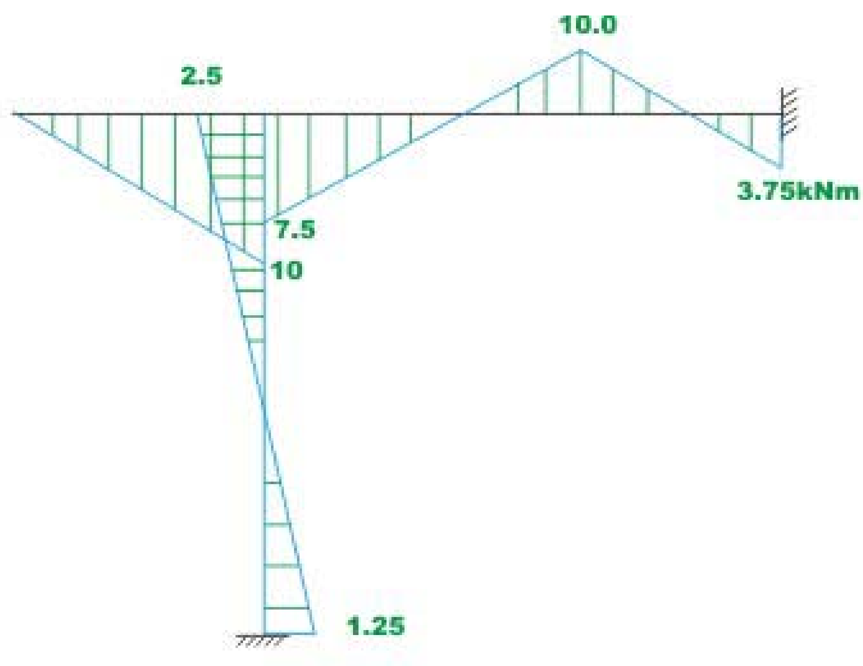


Fig. 20.2 (d) Bending moment diagram

Example 20.2

Analyse the rigid frame shown in Fig. 20.3a by moment-distribution method. Moment of inertia of different members are shown in the diagram.

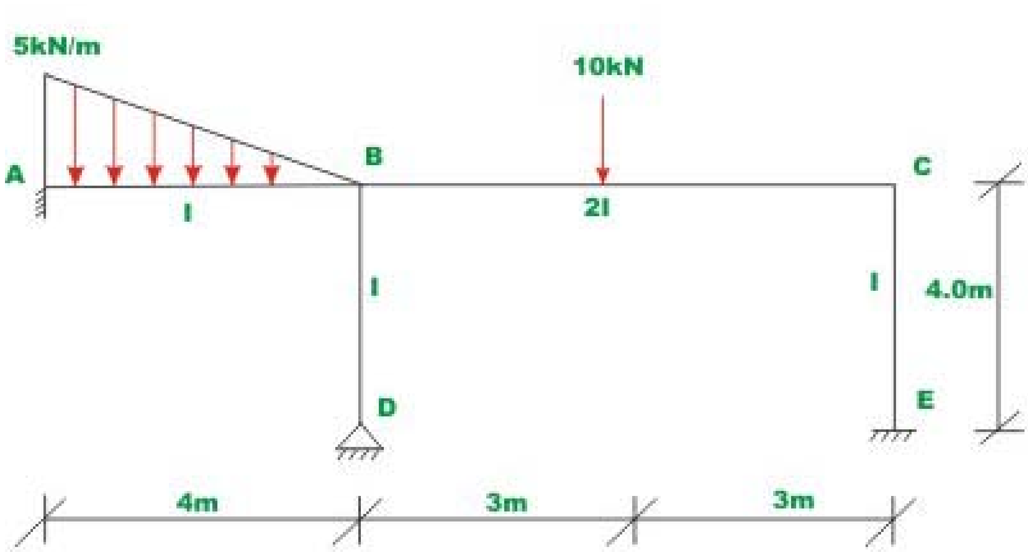


Fig. 20.3 (a) Example 20.2

Solution:

Calculate fixed end moments by locking the joints A, B, C, D and E

$$M_{AB}^{F} = \frac{5 \times 4^2}{20} = 4.0 \text{ kN.m}$$

$$M_{BA}^{F} = -2.667 \text{ kN.m}$$

$$M_{BC}^{F} = 7.5 \text{ kN.m}$$

$$M_{CB}^{F} = -7.5 \text{ kN.m}$$

$$M_{BD}^{F} = M_{DB}^{F} = M_{CE}^{F} = M_{EC}^{F} = 0 \tag{1}$$

The frame is restrained against sidesway. In the next step calculate stiffness and distribution factors.

$$K_{BA} = 0.25EI \quad \text{and} \quad K_{BC} = \frac{2EI}{6} = 0.333EI$$

$$K_{BD} = \frac{3}{4}\frac{EI}{4} = 0.1875EI \;; \qquad K_{CE} = 0.25EI \qquad (2)$$

At joint B:

$$\sum K = K_{BA} + K_{BC} + K_{BD}$$
$$= 0.7705EI$$

$$DF_{BA} = 0.325\;; \qquad DF_{BC} = 0.432$$

$$DF_{BD} = 0.243 \qquad (3)$$

At joint C:

$$\sum K = 0.583EI$$

$$DF_{CB} = 0.571\;; \qquad DF_{CD} = 0.429$$

In Fig. 20.3b, the complete procedure is shown on a working diagram. The moment-distribution method is started from joint C. When joint C is unlocked, it will rotate under the action of unbalanced moment of 7.5 kN.m. Hence the 7.5 kN.m is distributed among members CB and CE according to their distribution factors. Now joint C is balanced. To indicate that the joint C is balanced a horizontal line is drawn. This balancing moment in turn developed moments $+2.141 \text{ kN.m}$ at BC and $+1.61 \text{ kN.m}$ at EC. Now unlock joint B. The joint B is unbalanced and the unbalanced moment is $-(7.5 + 2.141 - 2.67) = -6.971 \text{ kN.m}$. This moment is distributed among three members meeting at B in proportion to their distribution factors. Also there is no carry over to joint D from beam end moment BD as it was left unlocked. For member BD, modified stiffness factor is used as the end D is hinged.

Example 20.3

Analyse the rigid frame shown in Fig. 20.4a by moment-distribution method. Draw bending moment diagram for the rigid frame. The flexural rigidities of the members are shown in the figure.

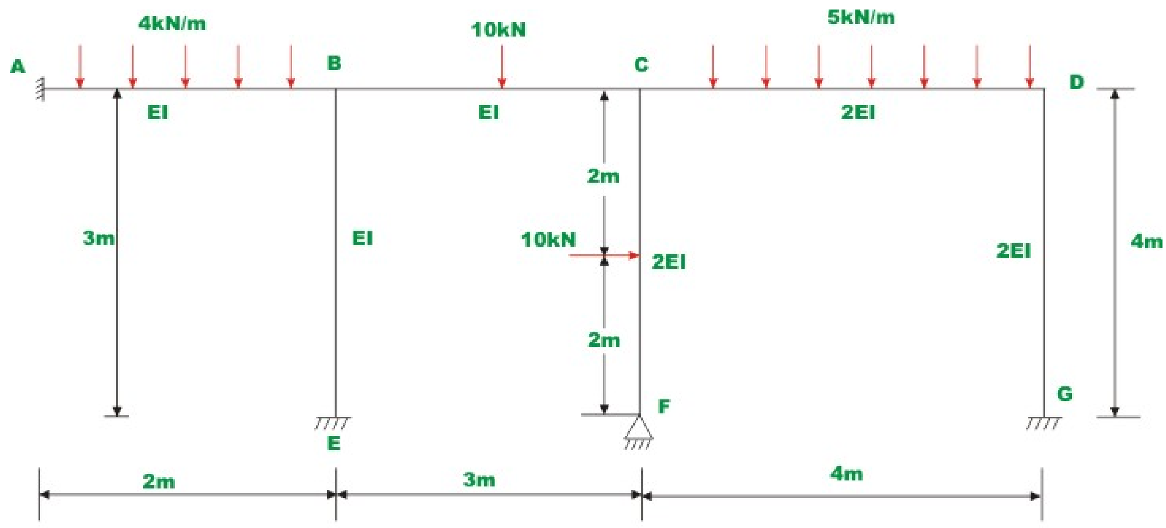


Fig. 20.4a Example 20.3

Solution:

Assuming that the joints are locked, calculate fixed end moments.

$$M_{AB}^{F} = 1.333 \quad \text{kN.m} \quad ; \quad M_{BA}^{F} = -1.333 \quad \text{kN.m}$$

$$M_{BC}^{F} = 4.444 \quad \text{kN.m}; \quad M_{CB}^{F} = -2.222 \quad \text{kN.m}$$

$$M_{CD}^{F} = 6.667 \quad \text{kN.m} \quad ; \quad M_{DC}^{F} = -6.667 \quad \text{kN.m}$$

$$M_{BE}^{F} = 0.0 \quad \text{kN.m} \quad ; \quad M_{EB}^{F} = 0.0 \quad \text{kN.m}$$

$$M_{CF}^{F} = 5.0 \quad \text{kN.m} \quad ; \quad M_{FC}^{F} = -5.0 \quad \text{kN.m} \tag{1}$$

The frame is restrained against sidesway. Calculate stiffness and distribution factors.

$$K_{BA} = 0.5EI \; ; \qquad K_{BC} = 0.333EI \; ; \qquad K_{BE} = 0.333EI$$

$$K_{CB} = 0.333EI \; ; \qquad K_{CD} = 0.5EI \; ; \qquad K_{CF} = \frac{3}{4}\frac{2EI}{4} = 0.375EI$$

$$K_{DC} = 0.5EI \; ; \qquad K_{DG} = 0.5EI \tag{2}$$

Joint B**:**

$$\sum K = 0.5EI + 0.333EI + 0.333EI = 1.166EI$$

$DF_{BA} = 0.428$; $DF_{BC} = 0.286$

$DF_{BE} = 0.286$

Joint C:

$$\sum K = 0.333EI + 0.5EI + 0.375EI = 1.208EI$$

$DF_{CB} = 0.276$; $DF_{CD} = 0.414$

$DF_{CF} = 0.31$

Joint D:

$$\sum K = 1.0EI$$

$DF_{DC} = 0.50$; $DF_{DG} = 0.50$ (3)

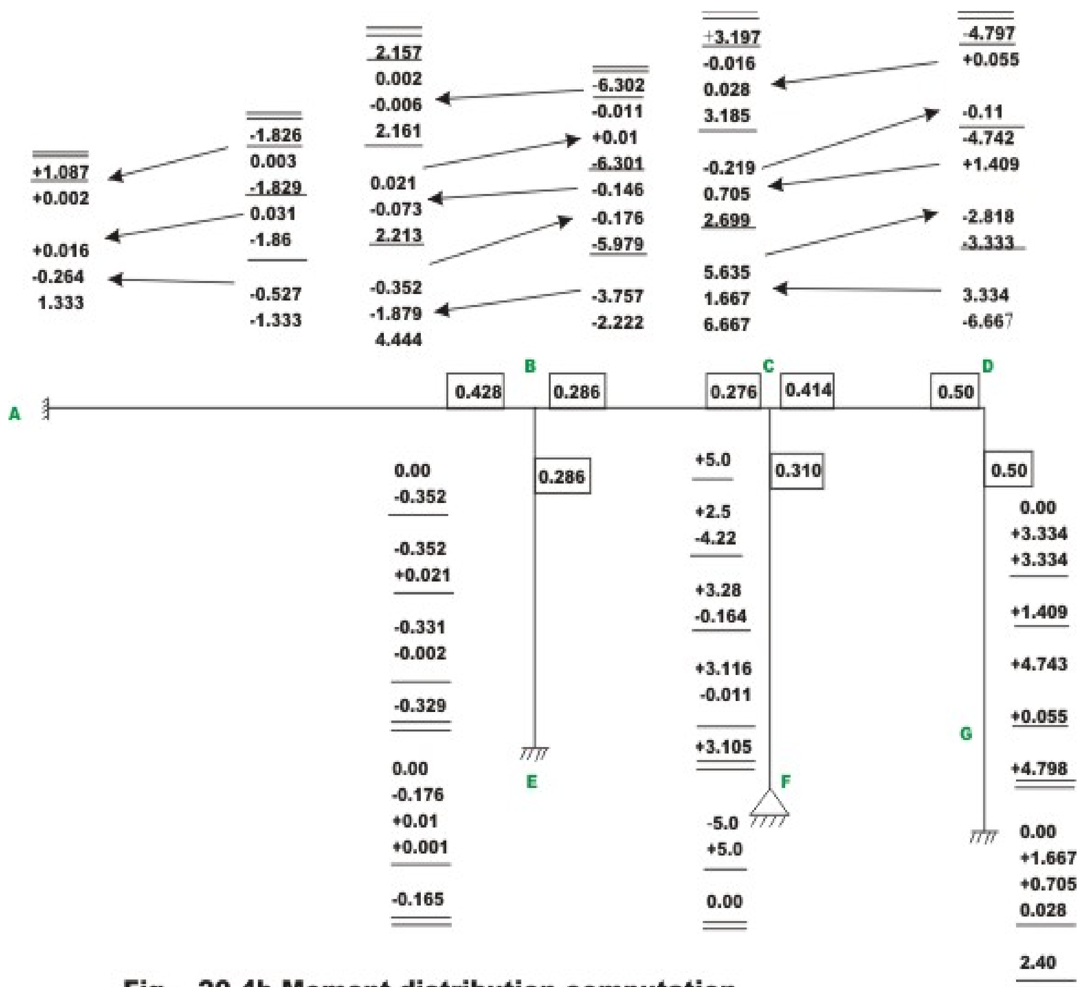


Fig. 20.4b Moment distribution computation

The complete moment-distribution method is shown in Fig. 20.4b. The moment-distribution is stopped after three cycles. The moment-distribution is started by releasing and balancing joint D. This is repeated for joints C and B respectively in that order. After balancing joint F, it is left unlocked throughout as it is a hinged joint. After balancing each joint a horizontal line is drawn to indicate that joint has been balanced and locked. When moment-distribution method is finally stopped all joints except fixed joints will be left unlocked.

Summary

In this lesson plane frames which are restrained against sidesway are analysed by the moment-distribution method. As many equilibrium equations are written as there are unknown displacements. The reactions of the frames are computed from equations of static equilibrium. The bending moment diagram is drawn for the frame. A few problems are solved to illustrate the procedure. Free-body diagrams are drawn wherever required.

Module 3

Analysis of Statically Indeterminate Structures by the Displacement Method

Lesson
21
The Moment-Distribution Method: Frames with Sidesway

Instructional Objectives

After reading this chapter the student will be able to

1. Extend moment-distribution method for frames undergoing sidesway.
2. Draw free-body diagrams of plane frame.
3. Analyse plane frames undergoing sidesway by the moment-distribution method.
4. Draw shear force and bending moment diagrams.
5. Sketch deflected shape of the plane frame not restrained against sidesway.

21.1 Introduction

In the previous lesson, rigid frames restrained against sidesway are analyzed using moment-distribution method. It has been pointed in lesson 17, that frames which are unsymmetrical or frames which are loaded unsymmetrically usually get displaced either to the right or to the left. In other words, in such frames apart from evaluating joint rotations, one also needs to evaluate joint translations (sidesway). For example in frame shown in Fig 21.1, the loading is symmetrical but the geometry of frame is unsymmetrical and hence sidesway needs to be considered in the analysis. The number of unknowns is this case are: joint rotations θ_B and θ_C and member rotation ψ. Joint B and C get translated by the same amount as axial deformations are not considered and hence only one independent member rotation need to be considered. The procedure to analyze rigid frames undergoing lateral displacement using moment-distribution method is explained in section 21.2 using an example.

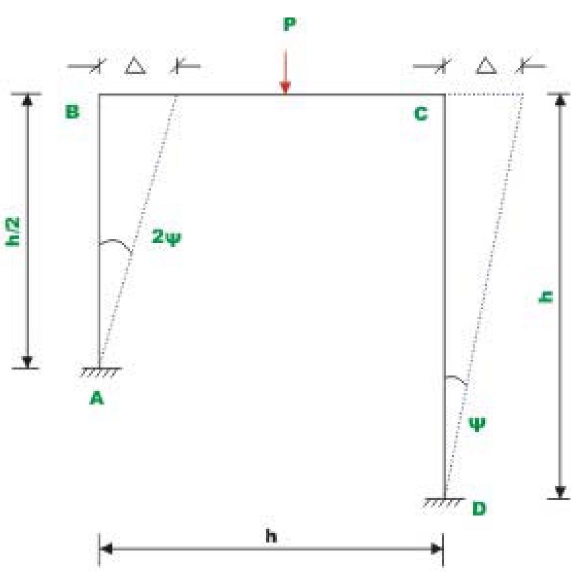


Fig 21.1 Rigid frame

21.2 Procedure

A special procedure is required to analyze frames with sidesway using moment-distribution method. In the first step, identify the number of independent rotations (ψ) in the structure. The procedure to calculate independent rotations is explained in lesson 22. For analyzing frames with sidesway, the method of superposition is used. The structure shown in Fig. 21.2a is expressed as the sum of two systems: Fig. 21.2b and Fig. 21.2c. The systems shown in figures 21.2b and 21.2c are analyzed separately and superposed to obtain the final answer. In system 21.2b, sidesway is prevented by artificial support at C. Apply all the external loads on frame shown in Fig. 21.2b. Since for the frame, sidesway is prevented, moment-distribution method as discussed in the previous lesson is applied and beam end moments are calculated. Let $M'_{AB}, M'_{BA}, M'_{BC}, M'_{CB}, M'_{CD}$ and M'_{DC} be the balanced moments obtained by distributing fixed end moments due to applied loads while allowing only joint rotations (θ_B and θ_C) and preventing sidesway.

Now, calculate reactions H_{A1} and H_{D1} (ref. Fig 21.3a).they are ,

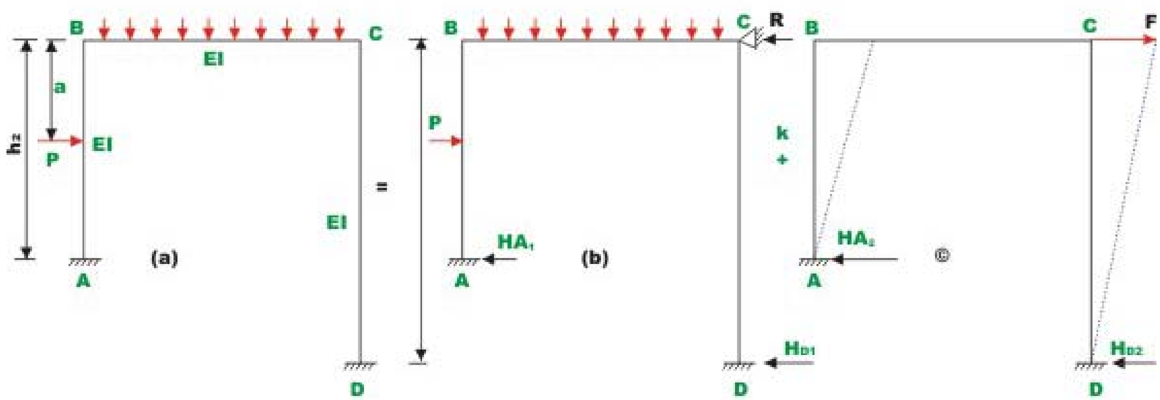


Fig 21.2 Frame with sidesway

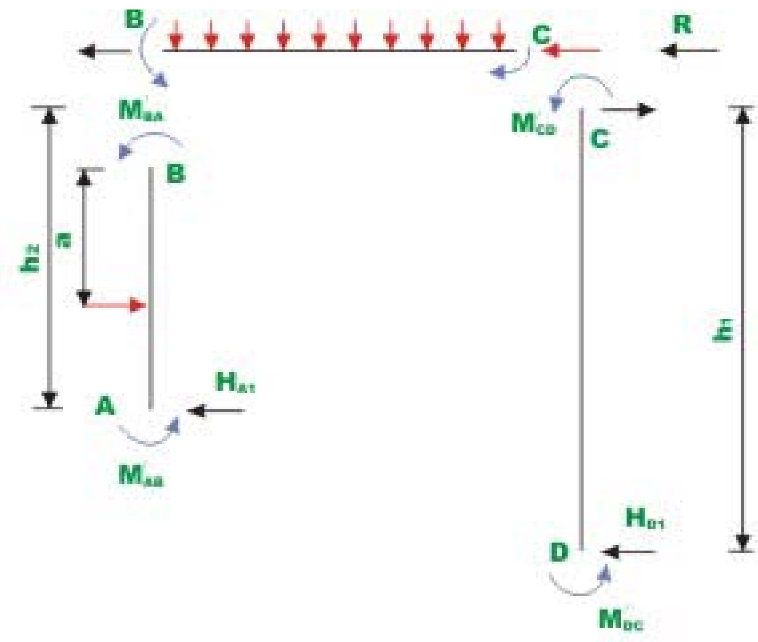


Fig.21.3a Free body diagram

$$H_{A1} = \frac{M'_{AB} + M'_{BA}}{h_2} + \frac{Pa}{h_2}$$

$$H_{D1} = \frac{M'_{CD} + M'_{DC}}{h_1} \tag{21.1}$$

again, $$R = P - (H_{A1} + H_{D1}) \tag{21.2}$$

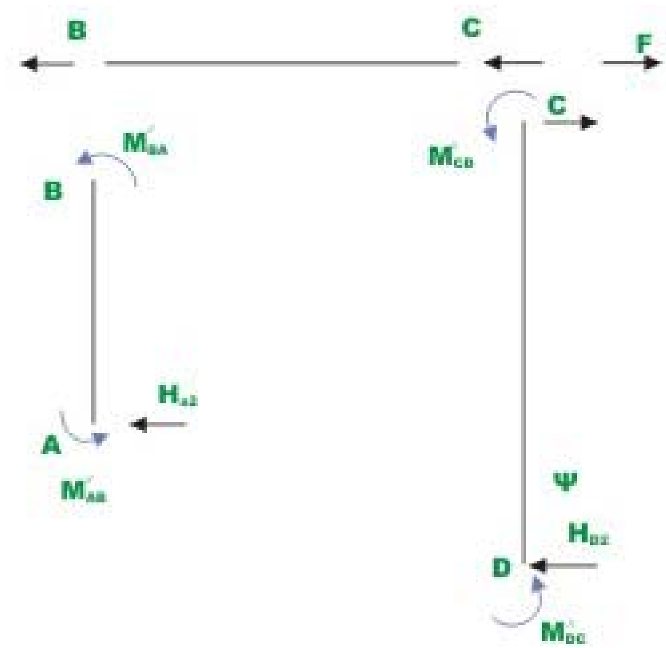


Fig.21.3b Free body diagram of frame

In Fig 21.2c apply a horizontal force F in the opposite direction of R. Now $kF = R$, then the superposition of beam end moments of system (b) and k times (c) gives the results for the original structure. However, there is no way one could analyze the frame for horizontal force F, by moment-distribution method as sway comes in to picture. Instead of applying F, apply arbitrary known displacement / sidesway Δ' as shown in the figure. Calculate the fixed end beam moments in the column AB and CD for the imposed horizontal displacement. Since joint displacement is known beforehand, one could use moment-distribution method to analyse this frame. In this case, member rotations ψ are related to joint translation which is known. Let $M''_{AB}, M''_{BA}, M''_{BC}, M''_{CB}, M''_{CD}$ and M''_{DC} are the balanced moment obtained by distributing the fixed end moments due to assumed sidesway Δ' at joints B and C. Now, from statics calculate horizontal force F due to arbitrary sidesway Δ'.

$$H_{A2} = \frac{M''_{AB} + M''_{BA}}{h_2}$$

$$H_{D2} = \frac{M''_{CD} + M''_{DC}}{h_1} \tag{21.3}$$

$$F = (H_{A2} + H_{D2}) \tag{21.4}$$

In Fig 21.2, by method of superposition

$kF = R$ or $k = R/F$

Substituting the values of R and F from equations (21.2) and (21.4),

$$k = \frac{P - (H_{A1} + H_{D1})}{(H_{A2} + H_{D2})} \tag{21.5}$$

Now substituting the values of H_{A1}, H_{A2}, H_{D1} and H_{D2} in 21.5,

$$k = \frac{P - \left(\frac{M'_{AB} + M'_{BA}}{h_2} + \frac{Pa}{h_2}\right) + \frac{M'_{CD} + M'_{DC}}{h_1}}{\left(\frac{M''_{AB+} M''_{BA}}{h_2} + \frac{M''_{CD} + M''_{DC}}{h_1}\right)} \tag{21.6}$$

Hence, beam end moment in the original structure is obtained as,

$$M_{original} = M_{system(b)} + kM_{system(c)}$$

If there is more than one independent member rotation, then the above procedure needs to be modified and is discussed in the next lesson.

Example 21.1

Analyse the rigid frame shown in Fig 21.4a. Assume EI to be constant for all members. Also sketch elastic curve.

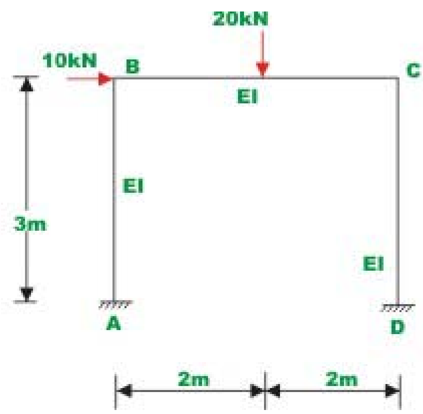


Fig. 21.4a Rigid frame of Example 21.1

Solution

In the given problem, joint C can also rotate and also translate by an unknown amount Δ. This problem has to be solved in two steps. In the first step, evaluate the beam-end moment by preventing the sidesway.

In the second step calculate beam end moments by moment-distribution method for known translation (see Fig 21.4b). By appropriately superposing the two results, the beam end moment of the original structure is obtained.

a) Calculate stiffness and distribution factors

$$K_{BA} = 0.333EI \;\; ; \;\; K_{BC} = 0.25EI \; ;$$

$$K_{CB} = 0.25EI \;\; ; K_{CD} = 0.333EI$$

Joint B: $\sum K = 0.583EI$

$$DF_{BA} = 0.571 ; \;\; DF_{BC} = 0.429$$

Joint C: $\sum K = 0.583EI$

$$DF_{CB} = 0.429\,; \; DF_{CD} = 0.571\,. \qquad (1)$$

b) Calculate fixed end moment due to applied loading.

$$M^F_{AB} = 0\,; \qquad M^F_{BA} = 0 \text{ kN.m}$$

$$M^F_{BC} = +10 \text{ kN.m}\,; \quad M^F_{CB} = -10 \text{ kN.m}$$

$$M^F_{CD} = 0 \text{ kN.m} \quad ; \quad M^F_{DC} = 0 \text{ kN.m}\,. \qquad (2)$$

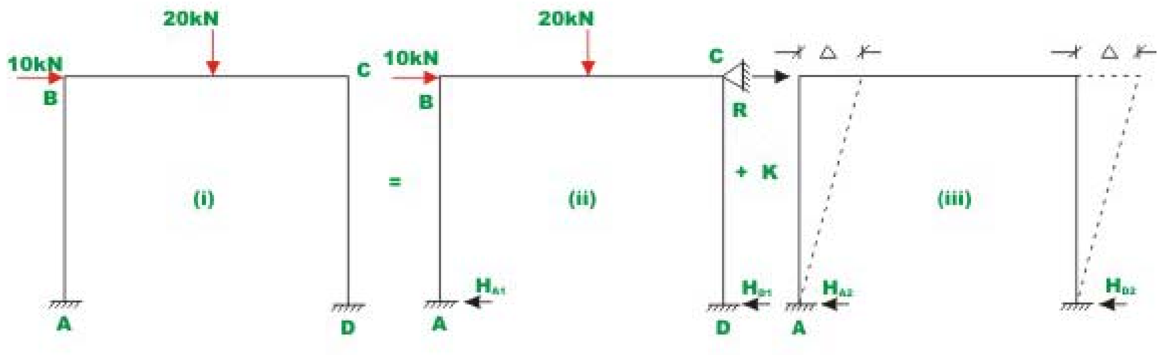


Fig. 21.4b Frame with side - sway

Now the frame is prevented from sidesway by providing a support at C as shown in Fig 21.4b (ii). The moment-distribution for this frame is shown in Fig 21.4c. Let $M'_{AB}, M'_{BA}, M'_{CD}$ and M'_{DC} be the balanced end moments. Now calculate horizontal reactions at A and D from equations of statics.

$$H_{A1} = \frac{M'_{AB} + M'_{BA}}{3}$$

$$= \frac{-3.635 + 7.268}{3}$$
$$= -3.635 \; KN(\rightarrow)\,.$$

$$H_{D1} = \frac{3.636 - 17.269}{3} = 3.635 \text{ kN}(\leftarrow)\,.$$

$$R = 10 - (-3.635 + 3.635) = -10 \ \text{kN}(\rightarrow) \qquad (3)$$

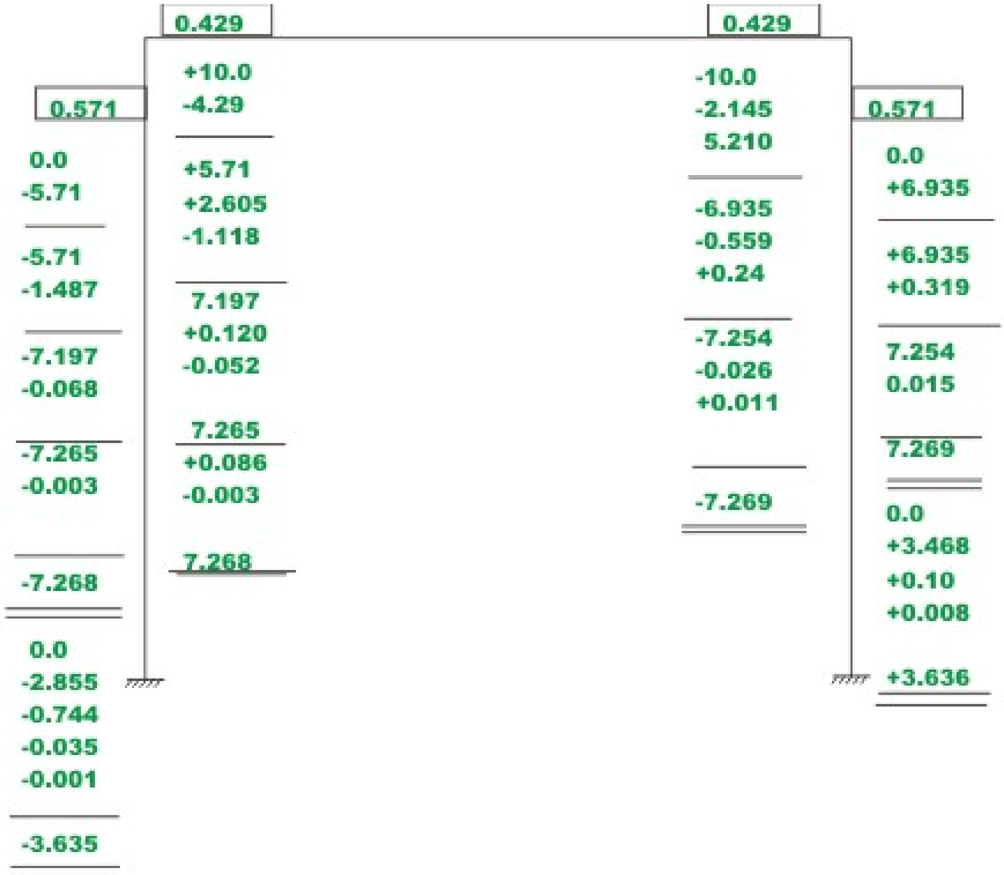


Fig. 21.4c Moment distribution with sidesway prevented

d) Moment-distribution for arbitrary known sidesway Δ' .

Since Δ' is arbitrary, Choose any convenient value. Let $\Delta' = \dfrac{150}{EI}$ Now calculate fixed end beam moments for this arbitrary sidesway.

$$M_{AB}^{F} = -\frac{6EI\psi}{L} = -\frac{6EI}{3} \times \left(-\frac{150}{3EI}\right) = 100 \ \text{kN.m}$$

$$M_{BA}^{F} = 100 \ \text{kN.m}$$

$$M_{CD}^{F} = M_{DC}^{F} = +100 \, \text{kN.m} \qquad (4)$$

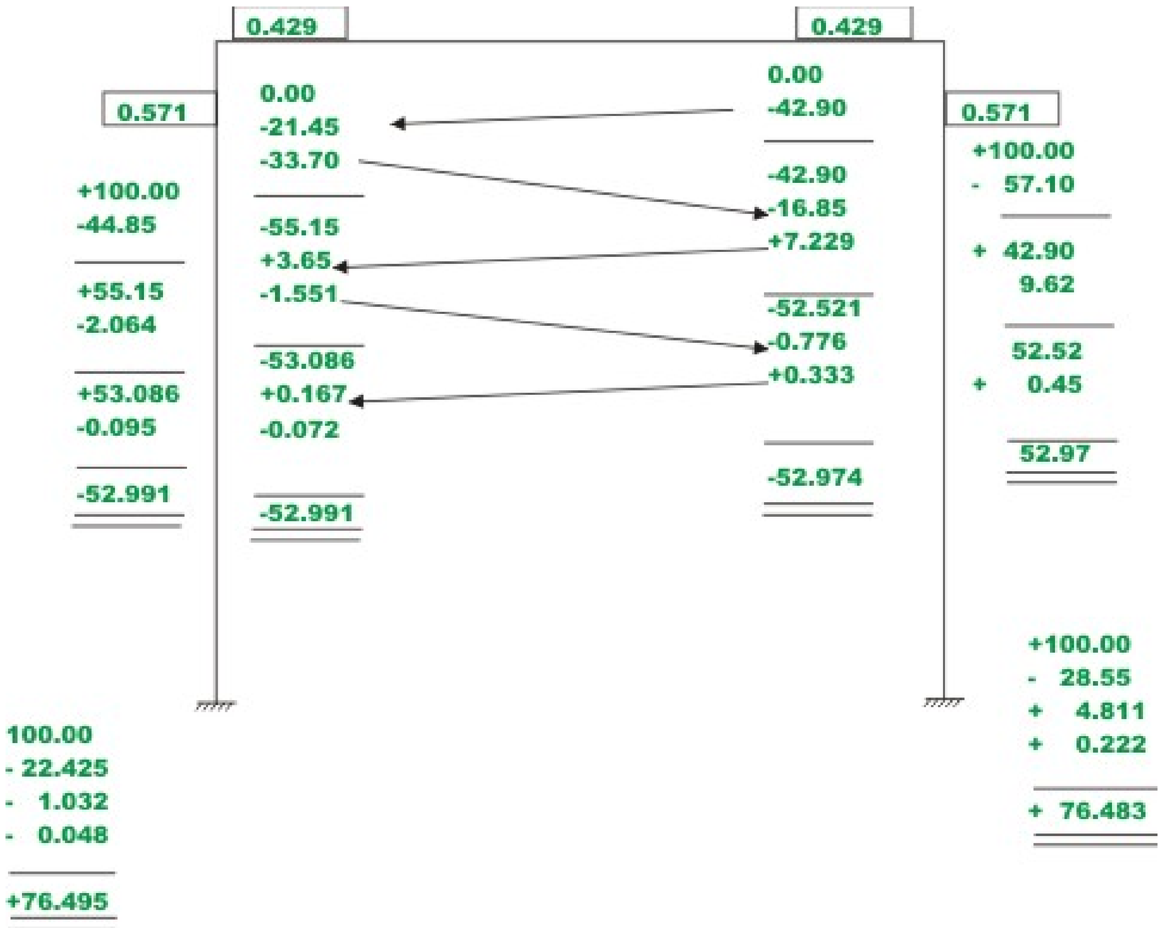


Fig. 21.4d Moment distribution for sidesway

The moment-distribution for this case is shown in Fig 24.4d. Now calculate horizontal reactions H_{A2} and H_{D2}.

$$H_{A2} = \frac{52.98 + 76.48}{3} = 43.15 \text{ kN}(\leftarrow)$$

$$H_{D2} = \frac{52.97 + 76.49}{3} = 43.15 \text{ kN}(\leftarrow)$$

$$F = -86.30 \text{ kN}(\rightarrow)$$

Let k be a factor by which the solution of case (iii) needs to be multiplied. Now actual moments in the frame is obtained by superposing the solution (ii) on the solution obtained by multiplying case (iii) by k. Thus kF cancel out the holding force R such that final result is for the frame without holding force.

Thus, $k\,F = R$.

$$k = \frac{-10}{-86.13} = 0.1161 \tag{5}$$

Now the actual end moments in the frame are,

$$M_{AB} = M'_{AB} + k\,M''_{AB}$$

$$M_{AB} = -3.635 + 0.1161(+76.48) = +5.244 \quad \text{kN.m}$$

$$M_{BA} = -7.268 + 0.1161(+52.98) = -1.117 \quad \text{kN.m}$$

$$M_{BC} = +7.268 + 0.1161(-52.98) = +1.117 \quad \text{kN.m}$$

$$M_{CB} = -7.269 + 0.1161(-52.97) = -13.419 \quad \text{kN.m}$$

$$M_{CD} = +7.268 + 0.1161(+52.97) = +13.418 \quad \text{kN.m}$$

$$M_{DC} = +3.636 + 0.1161(+76.49) = +12.517 \quad \text{kN.m}$$

The actual sway is computed as,

$$\Delta = k\Delta' = 0.1161 \times \frac{150}{EI}$$

$$= \frac{17.415}{EI}$$

The joint rotations can be calculated using slope-deflection equations.

$$M_{AB} = M^F_{AB} + \frac{2EI}{L}\left[2\theta_A + \theta_B - 3\psi_{AB}\right] \qquad \text{where } \psi_{AB} = -\frac{\Delta}{L}$$

$$M_{BA} = M^F_{BA} + \frac{2EI}{L}\left[2\theta_B + \theta_A - 3\psi_{AB}\right]$$

In the above equation, except θ_A and θ_B all other quantities are known. Solving for θ_A and θ_B,

$$\theta_A = 0\,; \qquad \theta_B = \frac{-9.55}{EI}.$$

The elastic curve is shown in Fig. 21.4e.

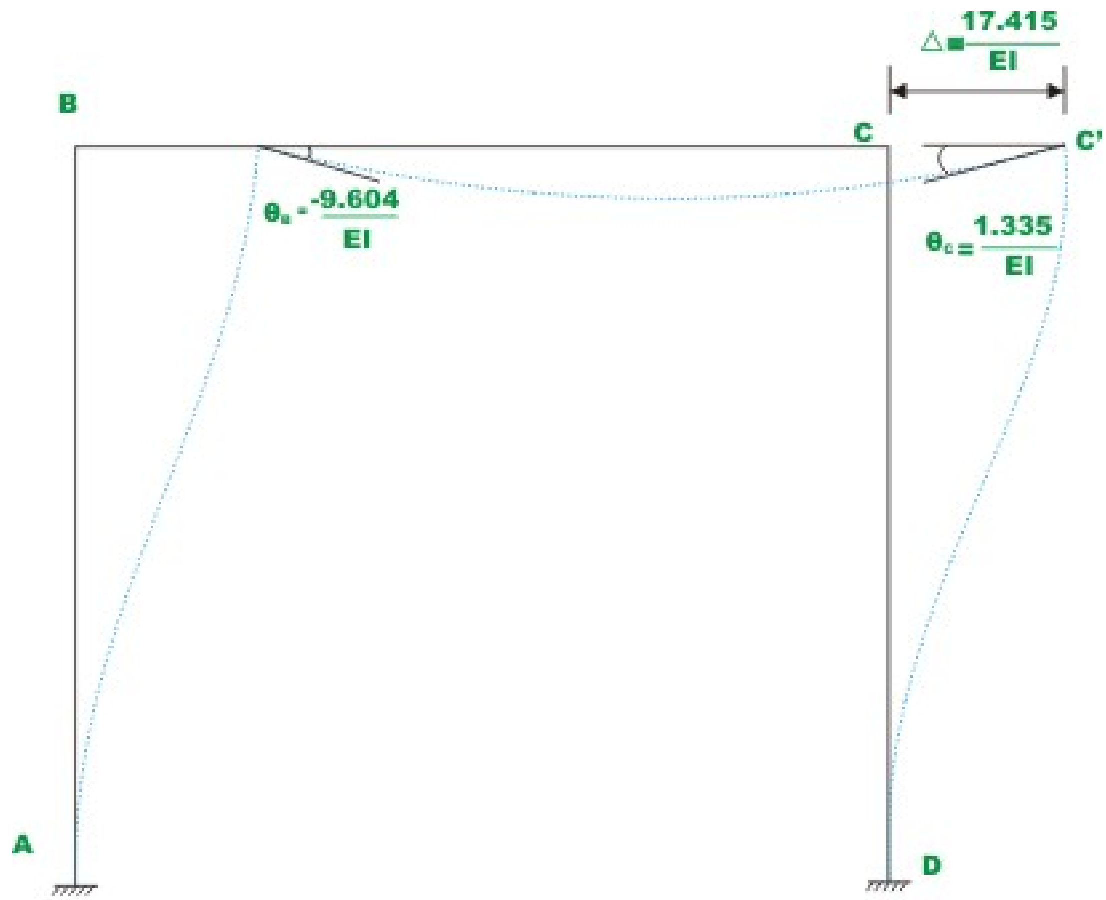


Fig.21.4e Elastic curve

Example 21.2

Analyse the rigid frame shown in Fig. 21.5a by moment-distribution method. The moment of inertia of all the members is shown in the figure. Neglect axial deformations.

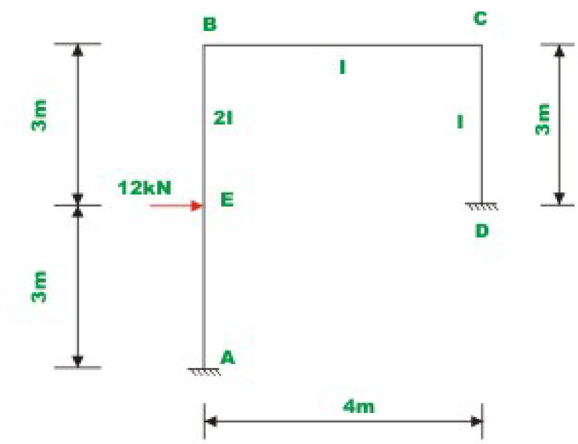


Fig. 21.5a Example 21.2

Solution:

In this frame joint rotations B and C and translation of joint B and C need to be evaluated.

a) Calculate stiffness and distribution factors.

$$K_{BA} = 0.333EI \; ; \qquad K_{BC} = 0.25EI$$

$$K_{CB} = 0.25EI \; ; \qquad K_{CD} = 0.333EI$$

At joint B :

$$\sum K = 0.583EI$$

$$DF_{BA} = 0.571 \; ; \qquad DF_{BC} = 0.429$$

At joint C :

$$\sum K = 0.583EI$$

$$DF_{CB} = 0.429 ; \qquad DF_{CD} = 0.571$$

b) Calculate fixed end moments due to applied loading.

$$M_{AB}^{F} = \frac{12 \times 3 \times 3^2}{6^2} = 9.0 \quad \text{kN.m} ; M_{BA}^{F} = -9.0 \quad \text{kN.m}$$

$$M_{BC}^{F} = 0 \quad \text{kN.m} ; \qquad M_{CB}^{F} = 0 \quad \text{kN.m}$$

$$M_{CD}^{F} = 0 \quad \text{kN.m} ; \qquad M_{DC}^{F} = 0 \quad \text{kN.m}$$

c) Prevent sidesway by providing artificial support at C. Carry out moment-distribution (*i.e.* Case A in Fig. 21.5b). The moment-distribution for this case is shown in Fig. 21.5c.

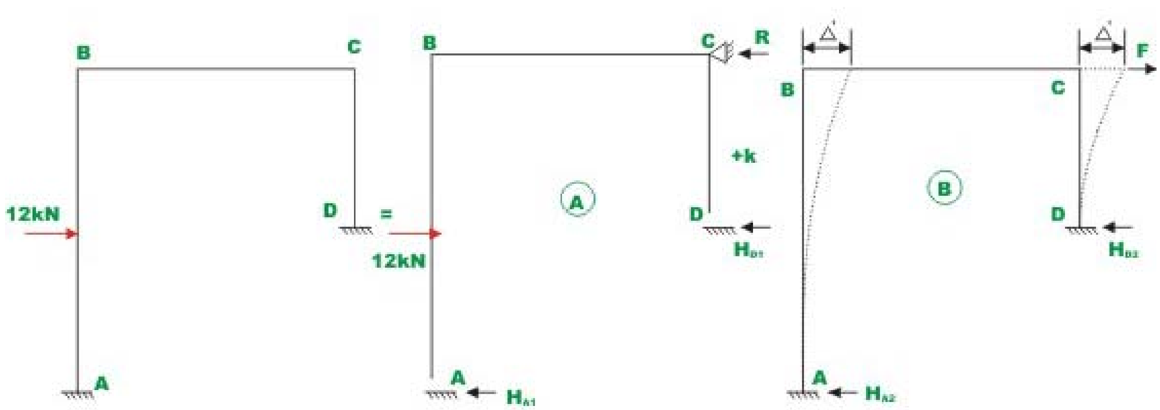


Fig. 21.5 b Frame with sidesway

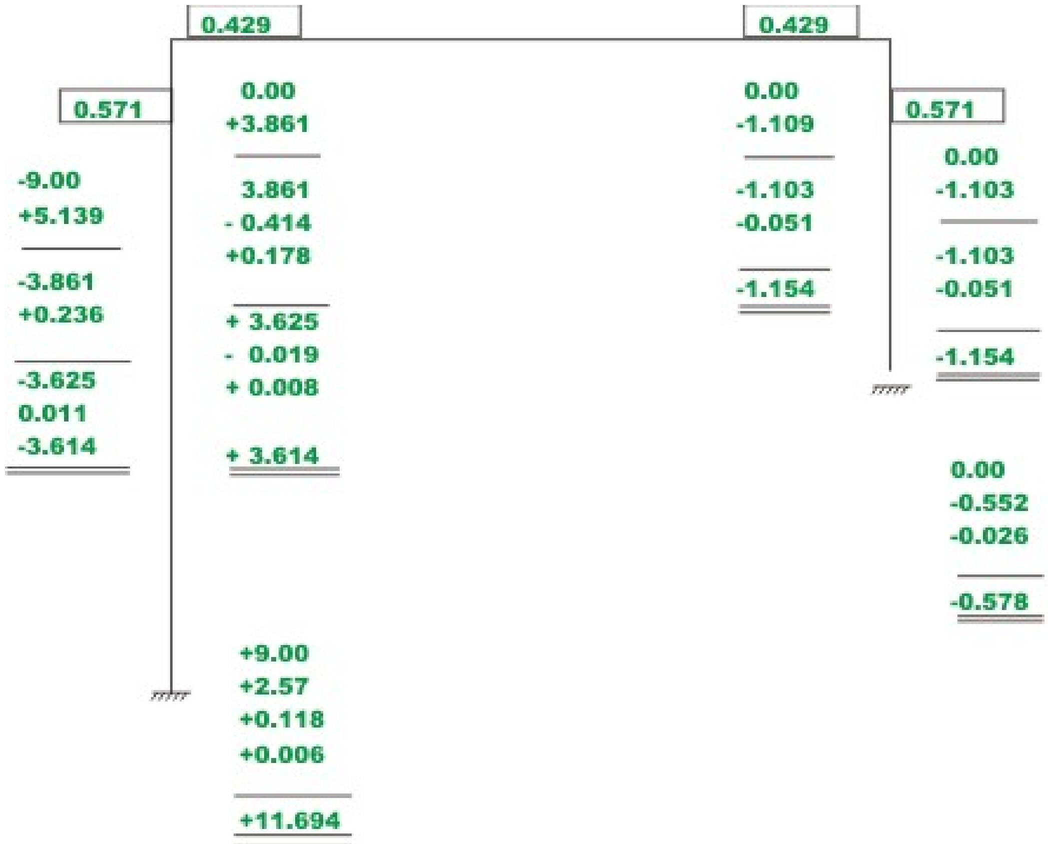


Fig. 21.5c Moment distribution with sidesway prevented

Now calculate horizontal reaction at A and D from equations of statics.

$$H_{A1} = \frac{11.694 - 3.614}{6} + 6 = 7.347 \quad \text{kN}(\leftarrow)$$

$$H_{D1} = \frac{-1.154 - 0.578}{3} = -0.577 \quad \text{kN}(\rightarrow)$$

$$R = 12 - (7.347 - 0.577) = -5.23 \quad \text{kN}(\rightarrow)$$

d) Moment-distribution for arbitrary sidesway Δ'(case B, Fig. 21.5c)

Calculate fixed end moments for the arbitrary sidesway of $\Delta' = \dfrac{150}{EI}$.

$$M_{AB}^{F} = -\frac{6E(2I)}{L}\psi = \frac{12EI}{6} \times \left(-\frac{150}{6EI}\right) = +50 \quad \text{kN.m} \quad ; \qquad M_{BA}^{F} = +50 \quad \text{kN.m} \quad ;$$

$$M_{CD}^F = -\frac{6E(I)}{L}\psi = -\frac{6EI}{3}\times(-\frac{150}{3EI}) = +100 \quad \text{kN.m} \quad ; \qquad M_{DC}^F = +100 \quad \text{kN.m} \quad ;$$

The moment-distribution for this case is shown in Fig. 21.5d. Using equations of static equilibrium, calculate reactions H_{A2} and H_{D2}.

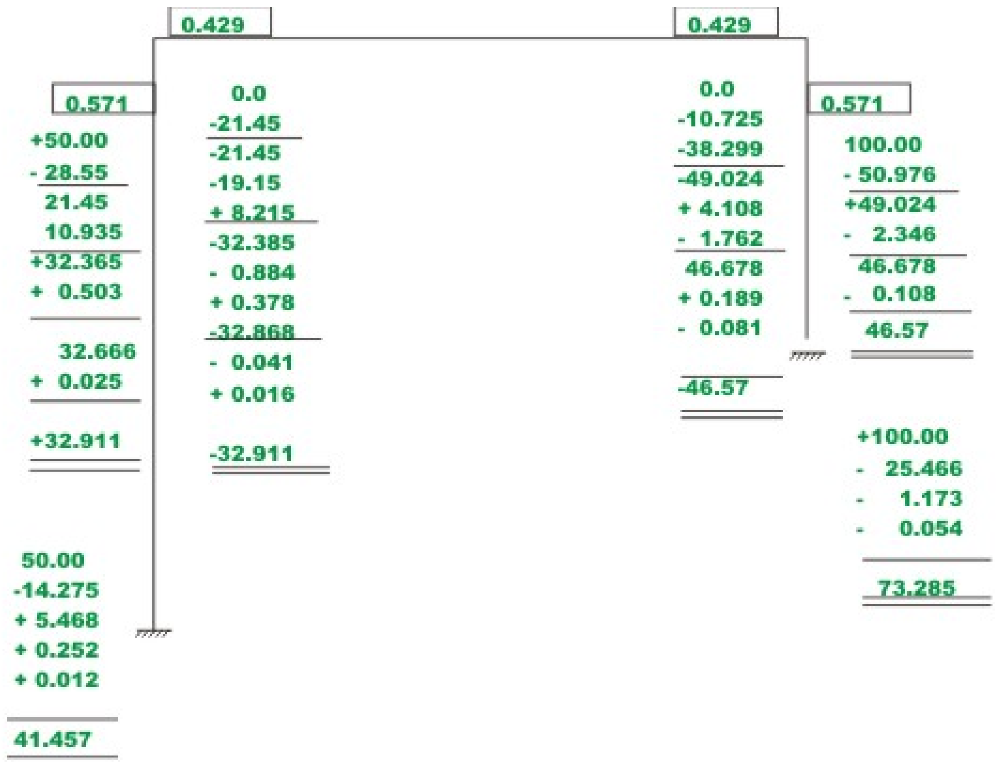


Fig. 21.5d Moment Distribution for arbitrary known sidesway

$$H_{A2} = \frac{32.911 + 41.457}{6} = 12.395\ kN(\leftarrow)$$

$$H_{D2} = \frac{46.57 + 73.285}{3} = 39.952\ kN(\leftarrow)$$

$$F = -(12.395 + 39.952) = -52.347\ kN(\rightarrow)$$

e) Final results

Now, the shear condition for the frame is (vide Fig. 21.5b)

$$(H_{A1} + H_{D1}) + k(H_{A2} + H_{D2}) = 12$$

$$(7.344 - 0.577) + k(12.395 + 39.952) = 12$$

$$k = 0.129$$

Now the actual end moments in the frame are,

$$M_{AB} = M'_{AB} + k\,M''_{AB}$$

$$M_{AB} = 11.694 + 0.129(+41.457) = +17.039 \quad \text{kN.m}$$

$$M_{BA} = -3.614 + 0.129(+32.911) = 0.629 \quad \text{kN.m}$$

$$M_{BC} = 3.614 + 0.129(-32.911) = -0.629 \quad \text{kN.m}$$

$$M_{CB} = -1.154 + 0.129(-46.457) = -4.853 \quad \text{kN.m}$$

$$M_{CD} = -1.154 + 0.129(+46.457) = +4.853 \quad \text{kN.m}$$

$$M_{DC} = -0.578 + 0.129(+73.285) = +8.876 \quad \text{kN.m}$$

The actual sway

$$\Delta = k\,\Delta' = 0.129 \times \frac{150}{EI}$$

$$= \frac{19.35}{EI}$$

The joint rotations can be calculated using slope-deflection equations.

$$M_{AB} - M_{AB}^{F} = +\frac{2E(2I)}{L}\left[2\theta_A + \theta_B - 3\psi\ \right]$$

or

$$\left[2\theta_A + \theta_B\right] = \frac{L}{4EI}\left[M_{AB} - M_{AB}^{F} + \frac{12EI\psi}{L}\right] = \frac{L}{4EI}\left[M_{AB} - \left(M_{AB}^{F} - \frac{12EI\psi}{L}\right)\right]$$

$$\left[2\theta_B + \theta_A\right] = \frac{L}{4EI}\left[M_{BA} - M_{BA}^{F} + \frac{12EI\psi}{L}\right] = \frac{L}{4EI}\left[M_{BA} - \left(M_{BA}^{F} - \frac{12EI\psi}{L}\right)\right]$$

$$M_{AB} = +17.039 \quad \text{kN.m}$$

$$M_{BA} = 0.629 \quad \text{kN.m}$$

$$\left(M_{AB}^{F}\right) = 9 + 0.129(50) = 15.45 \quad \text{kN.m}$$

$$\left(M_{BA}^{F}\right) = -9 + 0.129(50) = -2.55 \quad \text{kN.m}$$

$$\theta_A = \frac{\text{change in near end} + \left(-\frac{1}{2}\right)\text{change in far end}}{3EI/L}$$

$$= \frac{(17.039 - 15.45) + \left(-\frac{1}{2}\right)(0.629 + 2.55)}{3EI/6} = 0.0$$

$$\theta_B = \frac{4.769}{EI}$$

Example 21.3

Analyse the rigid frame shown in Fig. 21.6a. The moment of inertia of all the members are shown in the figure.

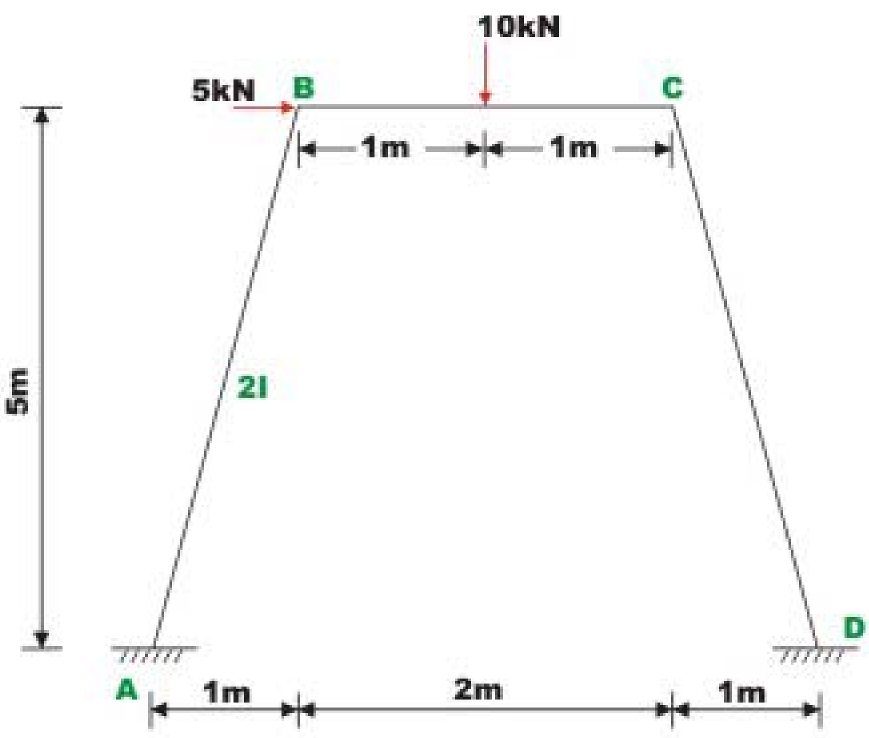


Fig.21.6a Example 21.3

Solution:

a) Calculate stiffness and distribution factors

$$K_{BA} = \frac{2EI}{5.1} = 0.392EI \ ; \qquad K_{BC} = 0.50EI$$

$$K_{CB} = 0.50EI \ ; \qquad K_{CD} = 0.392EI$$

At joint B:

$$\sum K = 0.892EI$$

$$DF_{BA} = 0.439 \ ; \qquad DF_{BC} = 0.561$$

At joint C:

$$\sum K = 0.892EI$$

$$DF_{CB} = 0.561 \ ; \qquad DF_{CD} = 0.439 \qquad (1)$$

b) Calculate fixed end moments due to applied loading.

$$M_{AB}^F = M_{BA}^F = M_{CD}^F = M_{DC}^F = 0 \quad \text{kN.m}$$

$$M_{BC}^F = 2.50 \quad \text{kN.m}$$

$$M_{CB}^F = -2.50 \quad \text{kN.m} \qquad (2)$$

c) Prevent sidesway by providing artificial support at C. Carry out moment-distribution for this case as shown in Fig. 21.6b.

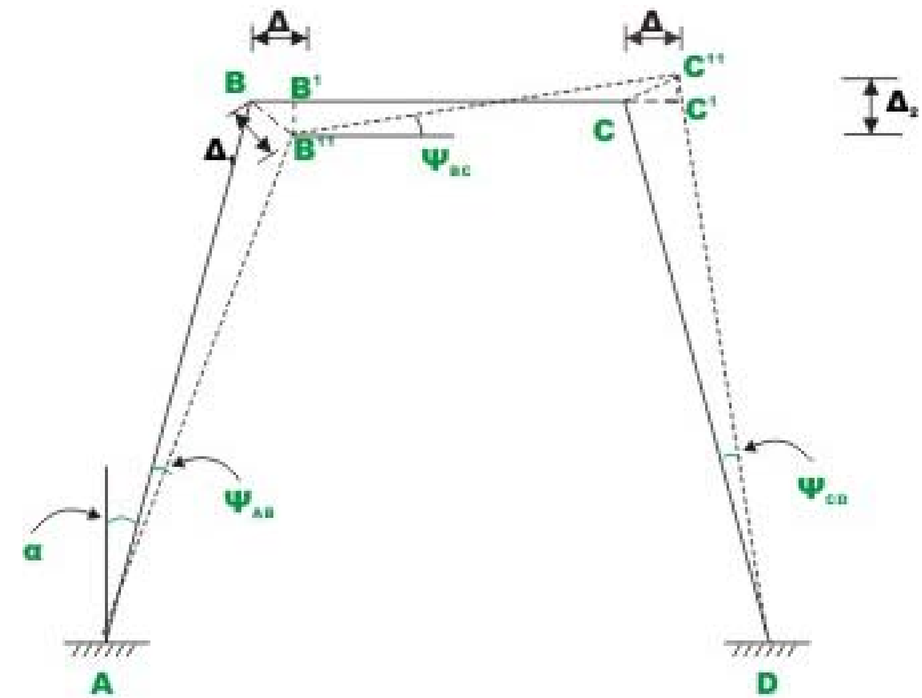


Fig.21.6b Rotation of Columns and beams

Now calculate reactions from free body diagram shown in Fig. 21.5d.

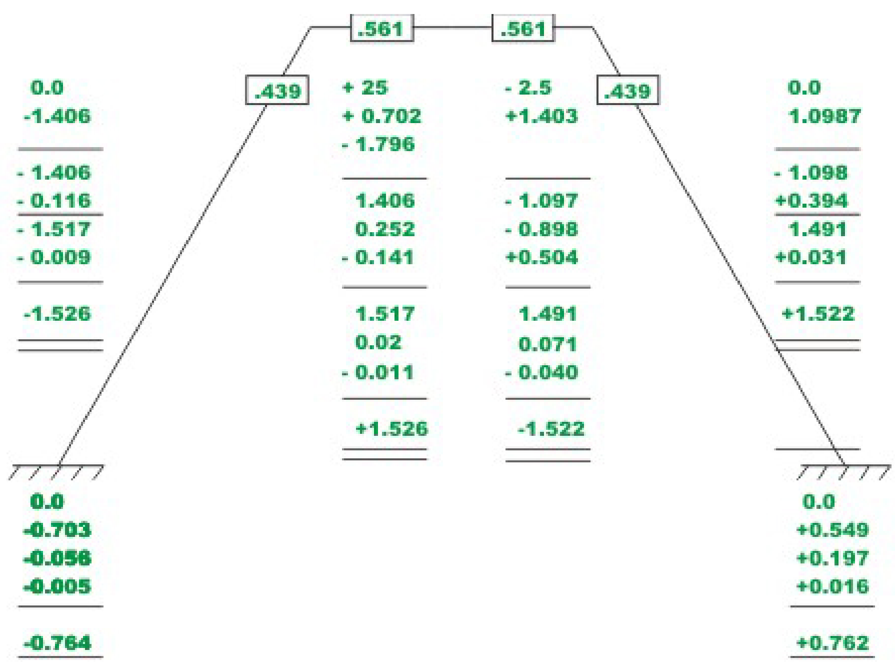


Fig. 21.6 © Moment distribution for applied loading

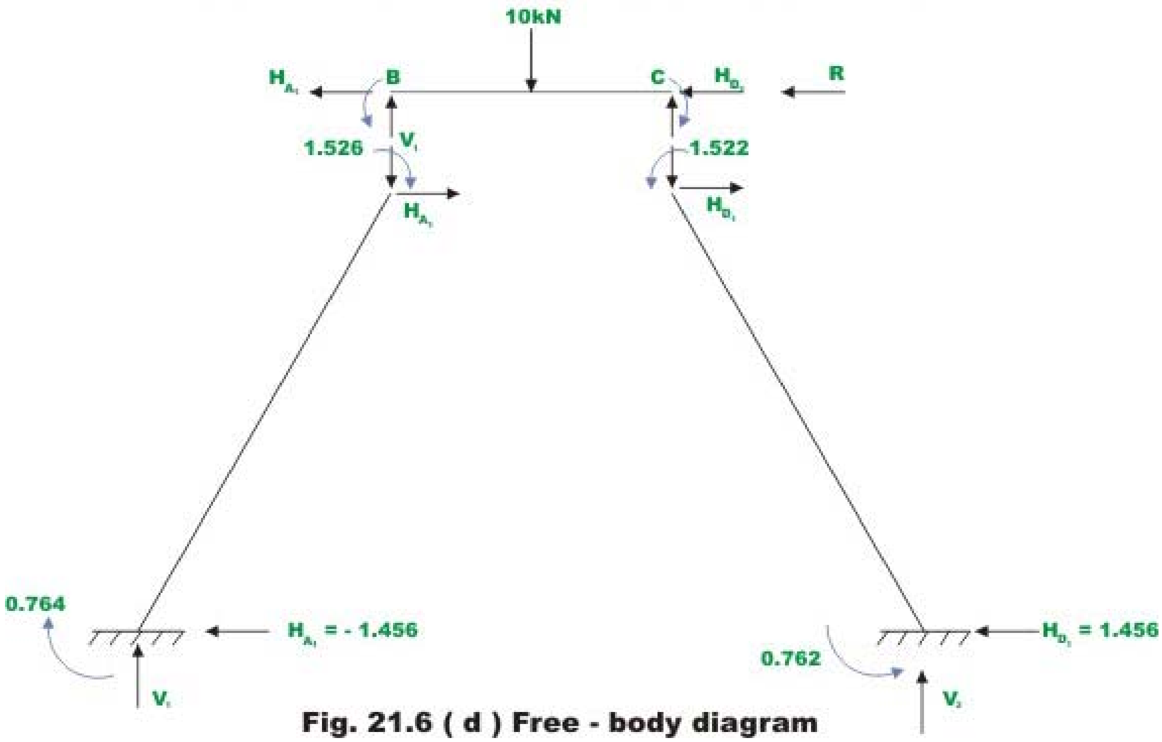


Fig. 21.6 (d) Free - body diagram

Column *AB*

$$\sum M_A = 0 \Rightarrow 5H_{A1} + 1.526 + 0.764 + V_1 = 0$$

$$5H_{A1} + V_1 = -2.29 \tag{3}$$

Column *CD*

$$\sum M_D = 0 \Rightarrow 5H_{D1} - 1.522 - 0.762 - V_2 = 0$$

$$5H_{D1} - V_2 = 2.284 \tag{4}$$

Beam *BC*

$$\sum M_C = 0 \Rightarrow 2V_1 + 1.522 - 1.526 - 10 \times 1 = 0$$

$$V_1 = 5.002 \quad \text{kN}(\uparrow)$$

$$V_2 = 4.998 \quad \text{kN}(\uparrow) \tag{5}$$

Thus from (3) $H_{A1} = -1.458 \quad \text{kN}(\rightarrow)$

from (4) $H_{D1} = 1.456 \quad \text{kN}(\leftarrow)$ (6)

$$\sum F_X = 0 \qquad H_{A1} + H_{D1} + R - 5 = 0$$
$$R = +5.002 \quad \text{kN}(\leftarrow) \tag{7}$$

d) Moment-distribution for arbitrary sidesway Δ'.

Calculate fixed end beam moments for arbitrary sidesway of

$$\Delta' = \frac{12.75}{EI}$$

The member rotations for this arbitrary sidesway is shown in Fig. 21.6e.

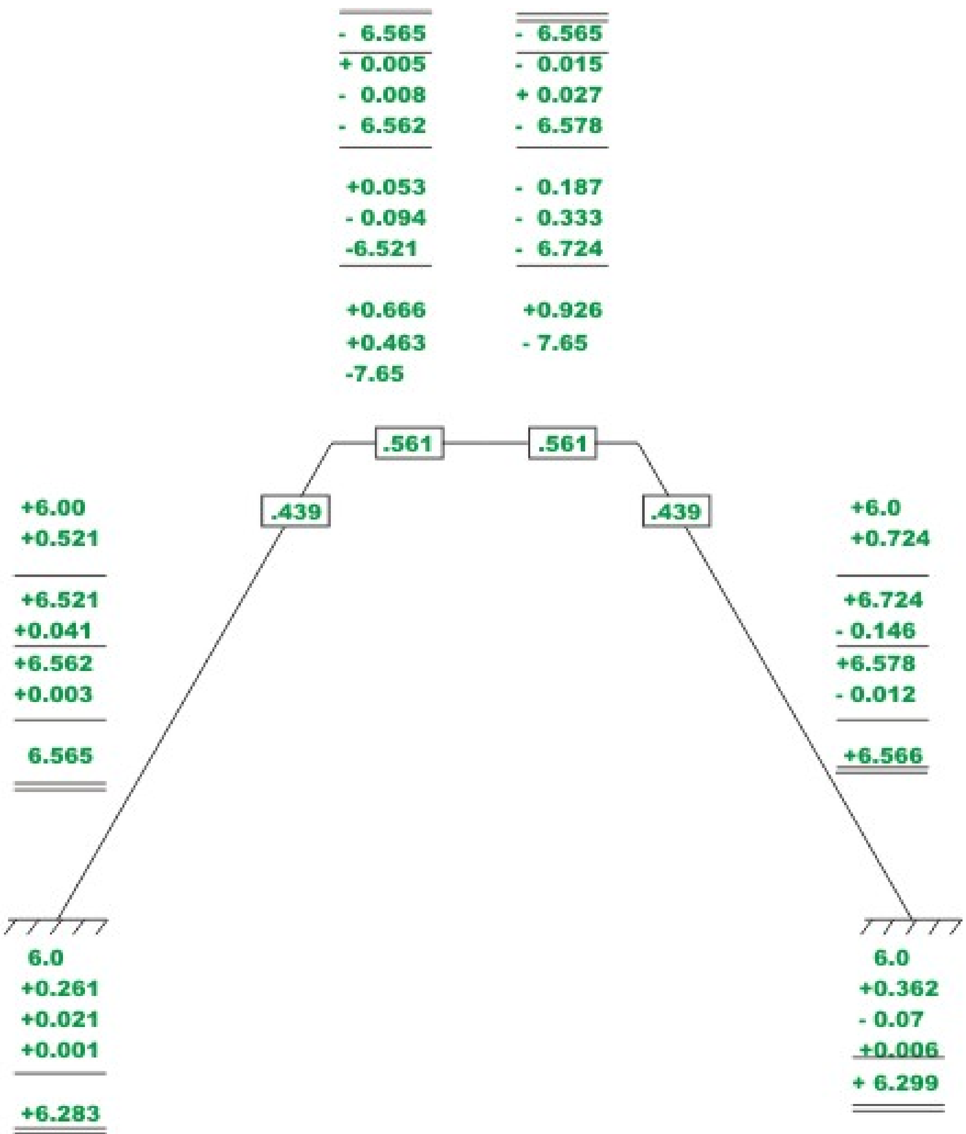


Fig. 21.6 (e) Moment distribution of arbitrary known sidesway

$$\psi_{AB} = \frac{BB''}{L_{AB}} = -\frac{\Delta_1}{L_{AB}} \quad ; \qquad \Delta_1 = \frac{\Delta'}{\cos\alpha} = \frac{5.1\Delta'}{5}$$

$$\Delta_2 = \frac{2\Delta'}{5} = 0.4\Delta'$$

$$\psi_{AB} = -\frac{\Delta'}{5}(clockwise)\,;\psi_{CD} = -\frac{\Delta'}{5}(clockwise)$$

$$\psi_{BC} = \frac{\Delta_2}{2} = \frac{2\Delta'\tan\alpha}{2} = \frac{\Delta'}{5}(counterclockwise)$$

$$M_{AB}^{F} = -\frac{6EI_{AB}}{L_{AB}}\psi_{AB} = -\frac{6E(2I)}{5.1}\left(-\frac{12.75}{5EI}\right) = +6.0 \text{ kN.m}$$

$$M_{BA}^{F} = +6.0 \quad \text{kN.m}$$

$$M_{BC}^{F} = -\frac{6EI_{BC}}{L_{BC}}\psi_{BC} = -\frac{6E(I)}{2}\left(\frac{12.75}{5EI}\right) = -7.65 \quad \text{kN.m}$$

$$M_{CB}^{F} = -7.65 \quad \text{kN.m}$$

$$M_{CD}^{F} = -\frac{6EI_{CD}}{L_{CD}}\psi_{CD} = -\frac{6E(2I)}{5.1}\left(-\frac{12.75}{5EI}\right) = +6.0 \quad \text{kN.m}$$

$$M_{DC}^{F} = +6.0 \quad \text{kN.m}$$

The moment-distribution for the arbitrary sway is shown in Fig. 21.6f. Now reactions can be calculated from statics.

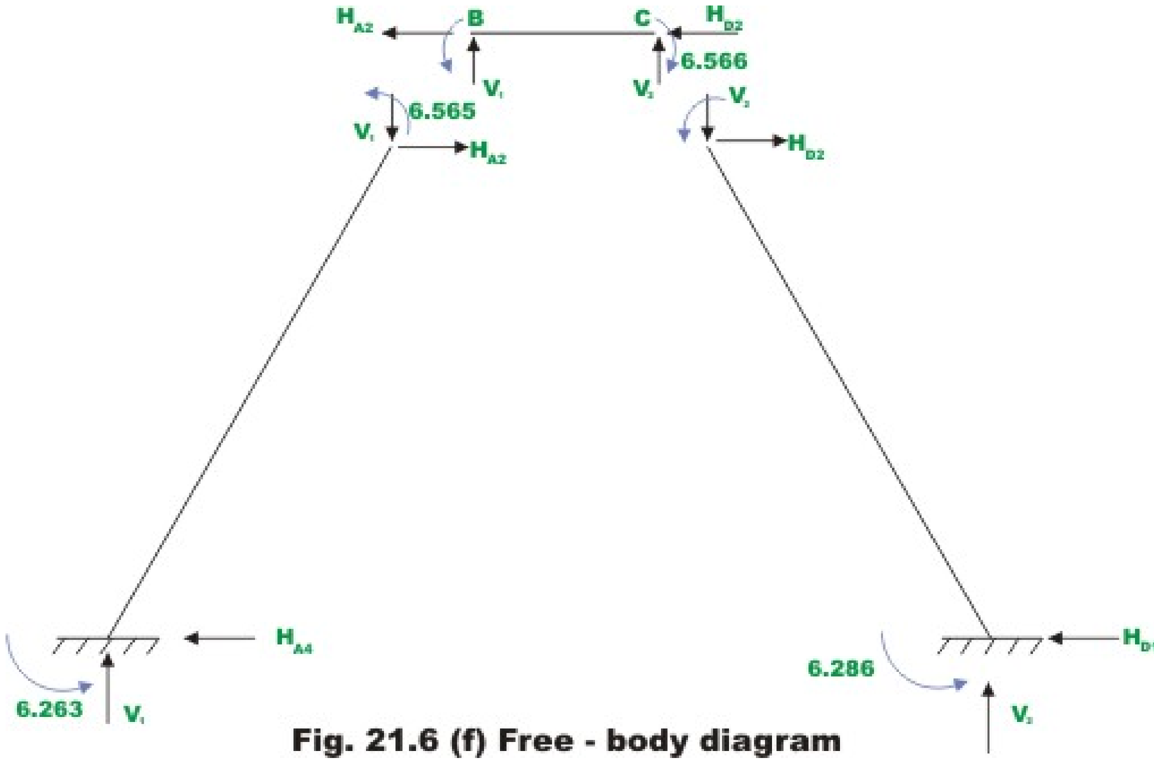


Fig. 21.6 (f) Free - body diagram

Column AB

$$\sum M_A = 0 \Rightarrow 5H_{A2} - 6.283 - 6.567 + V_1 = 0$$

$$5H_{A1} + V_1 = 12.85 \qquad (3)$$

Column CD

$$\sum M_D = 0 \Rightarrow 5H_{D2} - 6.567 - 6.283 - V_2 = 0$$

$$5H_{D1} - V_2 = 12.85 \qquad (4)$$

Beam BC

$$\sum M_C = 0 \Rightarrow 2V_1 + 6.567 + 6.567 = 0$$

$$V_1 = -6.567 \quad kN(\downarrow); V_2 = +6.567 \quad \mathrm{kN}(\uparrow) \qquad (5)$$

Thus from 3 $H_{A2} = +3.883 \quad \mathrm{kN}(\leftarrow)$

from 4 $H_{D2} = 3.883\ \mathrm{kN}(\leftarrow)$ (6)

$$F = 7.766 \quad \text{kN}(\leftarrow) \tag{7}$$

e) **Final results**

$$k\,F = R$$

$$k = \frac{5.002}{7.766} = 0.644$$

Now the actual end moments in the frame are,

$$M_{AB} = M'_{AB} + k\,M''_{AB}$$

$$M_{AB} = -0.764 + 0.644(+6.283) = +3.282 \quad \text{kN.m}$$

$$M_{BA} = -1.526 + 0.644(+6.567) = 2.703 \quad \text{kN.m}$$

$$M_{BC} = 1.526 + 0.644(-6.567) = -2.703 \quad \text{kN.m}$$

$$M_{CB} = -1.522 + 0.644(-6.567) = --5.751 \quad \text{kN.m}$$

$$M_{CD} = 1.522 + 0.644(6.567) = 5.751 \quad \text{kN.m}$$

$$M_{DC} = 0.762 + 0.644(6.283) = 4.808 \quad \text{kN.m}$$

The actual sway

$$\Delta = k\,\Delta' = 0.644 \times \frac{12.75}{EI}$$

$$= \frac{8.212}{EI}$$

Summary

In this lesson, the frames which are not restrained against sidesway are identified and solved by the moment-distribution method. The moment-distribution method is applied in two steps: in the first step, the frame prevented from sidesway but subjected to external loads is analysed and subsequently, the frame which is undergoing an arbitrary but known sidesway is analysed. Using shear equation for the frame, the moments in the frame is obtained. The numerical examples are explained with the help of free-body diagrams. The deflected shape of the frame is sketched to understand its deformation under external loads.

Module 3

Analysis of Statically Indeterminate Structures by the Displacement Method

Lesson 22

The Multistory Frames with Sidesway

Instructional Objectives

After reading this chapter the student will be able to

1. Identify the number of independent rotational degrees of freedom of a rigid frame.
2. Write appropriate number of equilibrium equations to solve rigid frame having more than one rotational degree of freedom.
3. Draw free-body diagram of multistory frames.
4. Analyse multistory frames with sidesway by the slope-deflection method.
5. Analyse multistory frames with sidesway by the moment-distribution method.

22.1 Introduction

In lessons 17 and 21, rigid frames having single independent member rotational ($\psi\left(=\frac{\Delta}{h}\right)$) degree of freedom (or joint translation Δ) is solved using slope-deflection and moment-distribution method respectively. However multistory frames usually have more than one independent rotational degree of freedom. Such frames can also be analysed by slope-deflection and moment-distribution methods. Usually number of independent member rotations can be evaluated by inspection. However if the structure is complex the following method may be adopted. Consider the structure shown in Fig. 22.1a. Temporarily replace all rigid joints of the frame by pinned joint and fixed supports by hinged supports as shown in Fig. 22.1b. Now inspect the stability of the modified structure. If one or more joints are free to translate without any resistance then the structure is geometrically unstable. Now introduce forces in appropriate directions to the structure so as to make it stable. The number of such externally applied forces represents the number of independent member rotations in the structure.

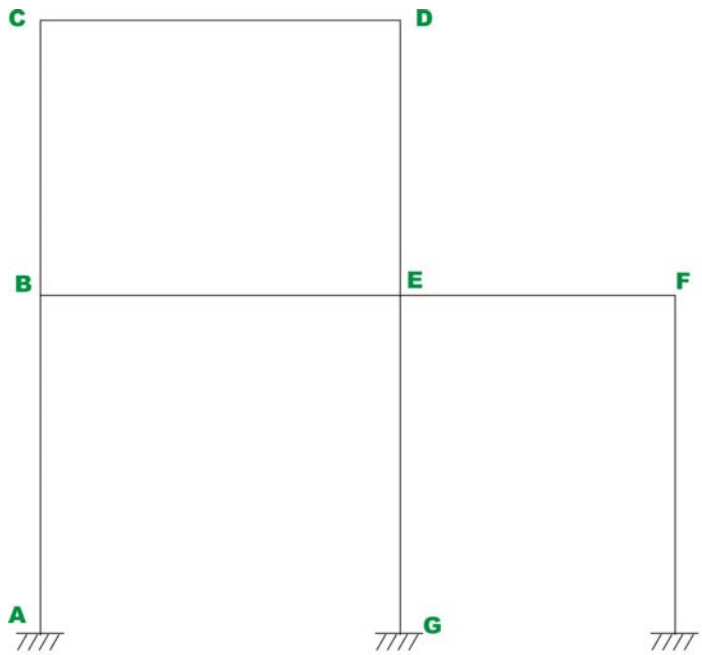


Fig. 22.1a Rigid frame

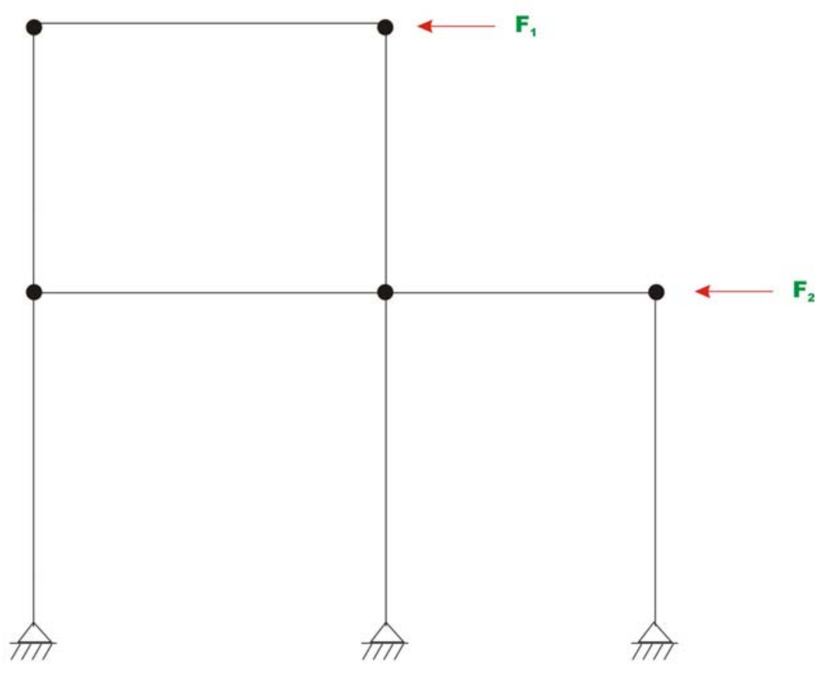


Fig. 22.1b Modified structure

In the modified structure Fig. 22.1b, two forces are required to be applied at level CD and level BF for stability of the structure. Hence there are two independent member rotations (ψ) that need to be considered apart from joint rotations in the analysis.

The number of independent rotations to be considered for the frame shown in Fig. 22.2a is three and is clear from the modified structure shown in Fig. 22.2b.

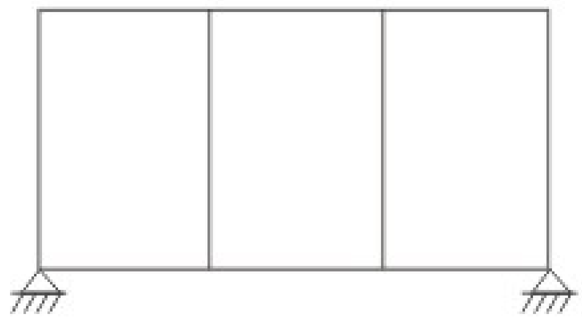


Figure 22.2a Rigid frame

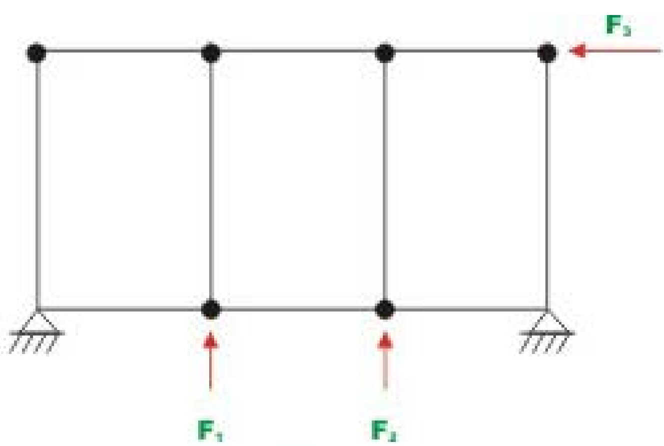


Figure 22.2b Modified structure

From the above procedure it is clear that the frame shown in Fig. 22.3a has three independent member rotations and frame shown in Fig 22.4a has two independent member rotations.

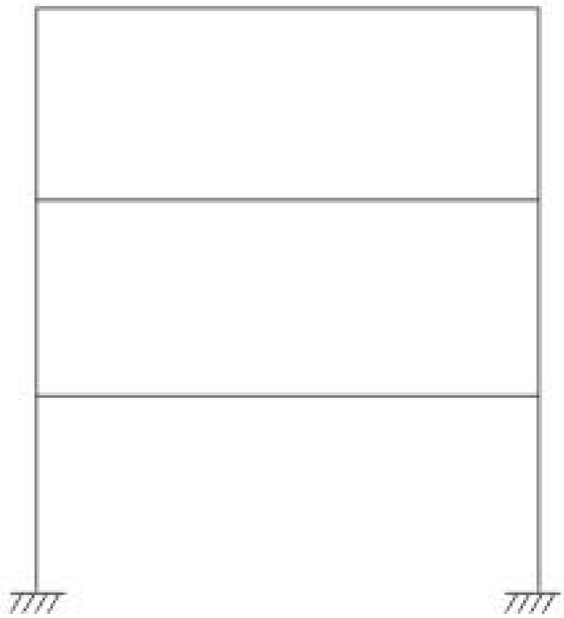


Figure 22.3a Rigid frame

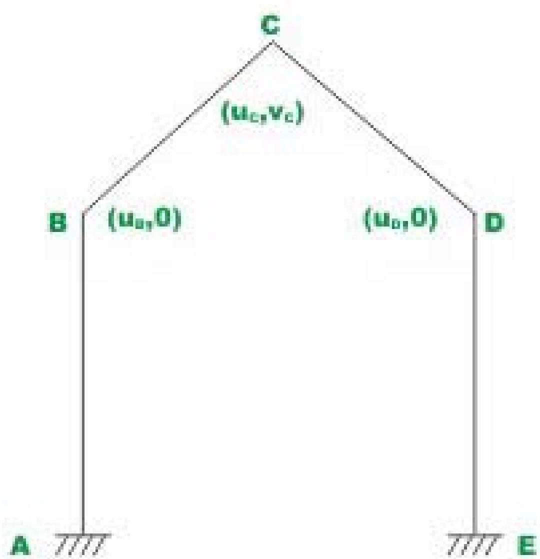


Figure 22.4a Gable frame

For the gable frame shown in Fig. 22.4a, the possible displacements at each joint are also shown. Horizontal displacement is denoted by u and vertical displacement is denoted by v. Recall that in the analysis, we are not considering the axial deformation. Hence at B and D only horizontal deformation is possible and joint C can have both horizontal and vertical deformation. The displacements u_B, u_C, u_D and u_D should be such that the lengths BC and CD must not change as the axial deformation is not considered. Hence we can have only two independent translations. In the next section slope-deflection method as applied to multistoried frame is discussed.

22.2 Slope-deflection method

For the two story frame shown in Fig. 22.5, there are four joint rotations $(\theta_B, \theta_C, \theta_D \text{ and } \theta_E)$ and two independent joint translations (sidesway) Δ_1 at the level of CD and Δ_2 at the level of BE.

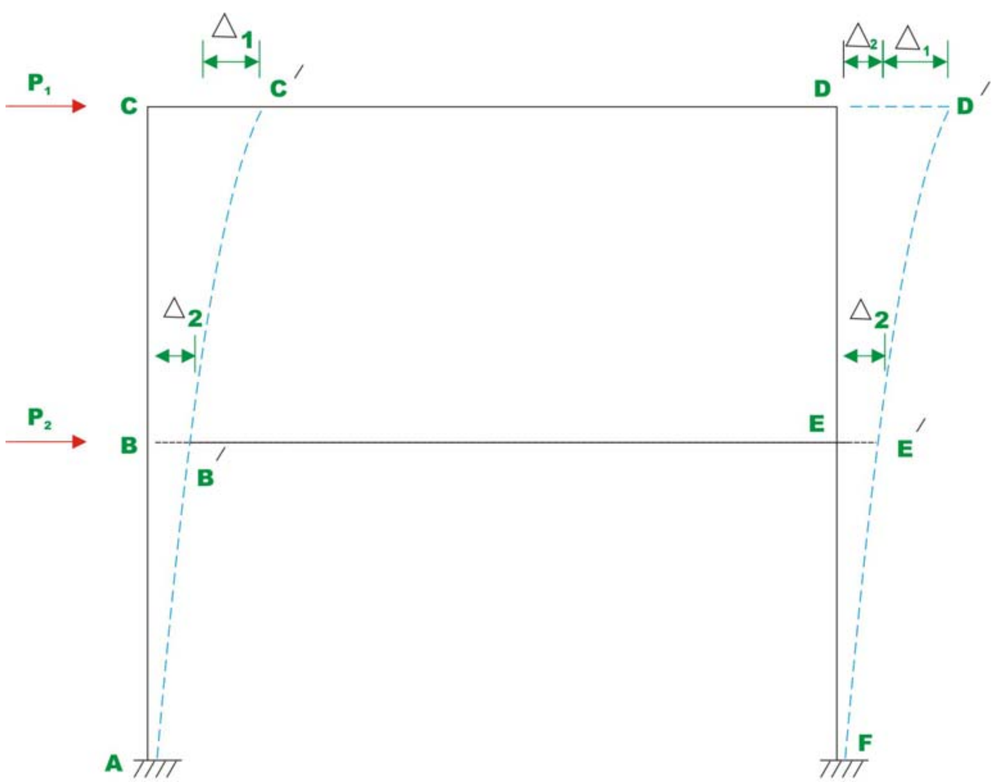


Fig.22.5 Two story frame.

Six simultaneous equations are required to evaluate the six unknowns (four rotations and two translations). For each of the member one could write two slope-deflection equations relating beam end moments to (i) externally applied loads and (ii) displacements (rotations and translations). Four of the required six equations are obtained by considering the moment equilibrium of joint B, C, D and E respectively. For example,

$$\sum M_B = 0 \qquad \Rightarrow M_{BA} + M_{BC} + M_{BE} = 0 \tag{22.1}$$

The other two equations are obtained by considering the force equilibrium of the members. Thus, the shear at the base of all columns for any story must be equal to applied load. Thus $\sum F_X = 0$ at the base of top story gives (ref. Fig. 22.6)

$$P_1 - H_C - H_D = 0 \tag{22.2}$$

Similarly $\sum F_X = 0$ at the base of frame results in

$$P_1 + P_2 - H_A - H_F = 0 \tag{22.3}$$

Thus we get six equations in six unknowns. Solving the above six equations all the unknowns are evaluated. The above procedure is explained in example 22.1.

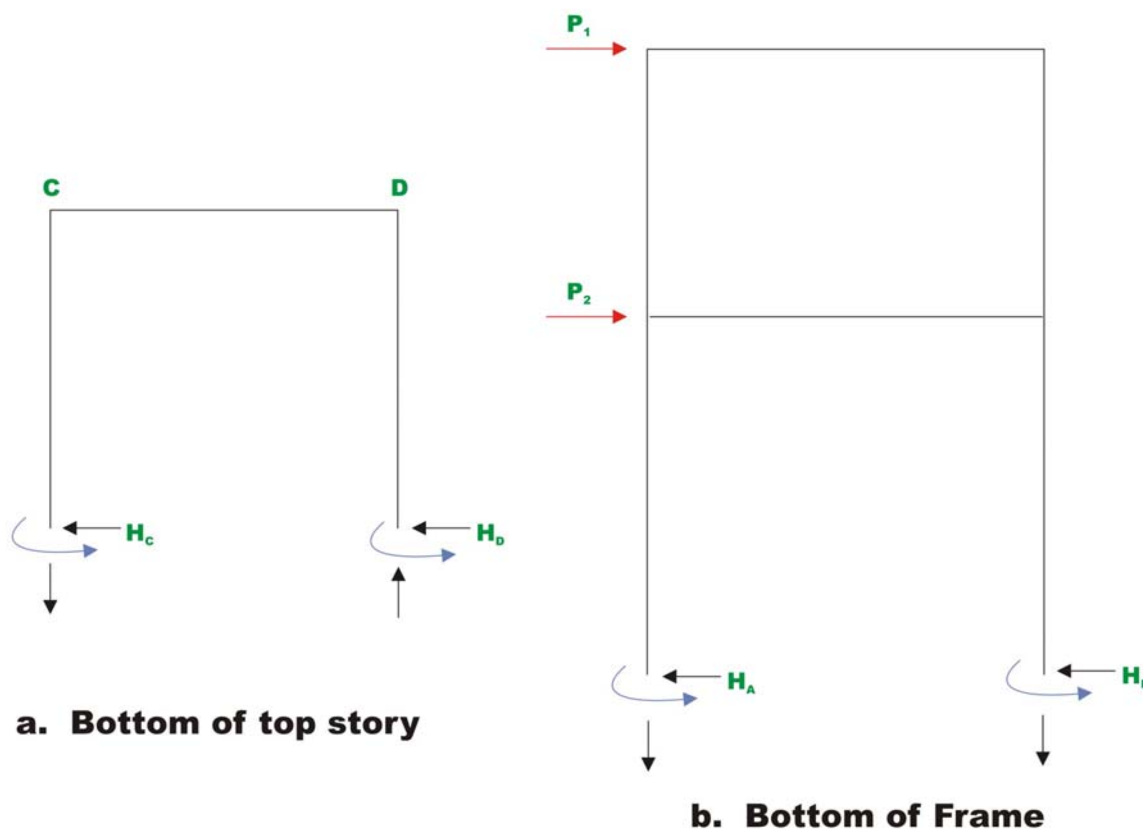


Fig. 22.6

Example 22.1

Analyse the two story rigid frame shown in Fig. 22.7a by the slope-deflection method. Assume EI to be constant for all members.

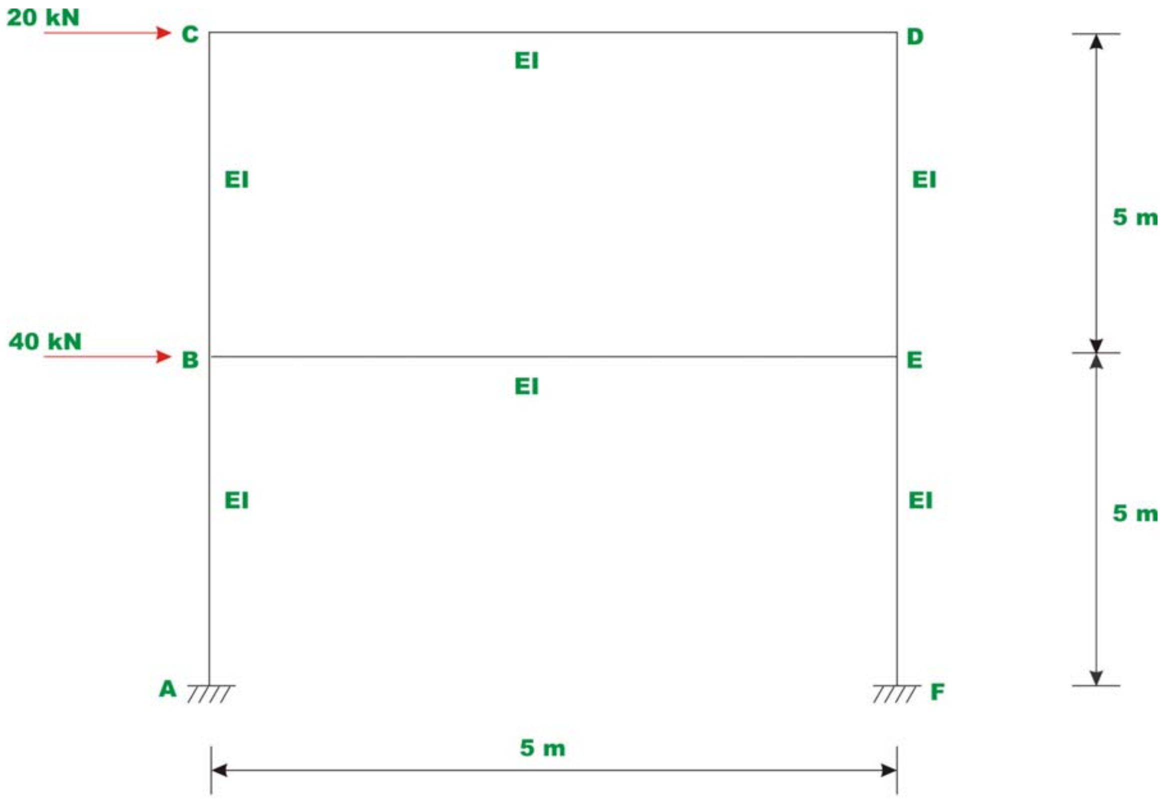


Fig.22.7a Example 22.1

In this case all the fixed end moments are zero. The members AB and EF undergo rotations $\psi_2 = -\frac{\Delta_2}{5}$ (negative as it is clockwise) and member BC and ED undergo rotations $\psi_1 = -\frac{\Delta_1}{5}$. Now writing slope-deflection equations for 12 beam end moments.

$$M_{AB} = 0 + \frac{2EI}{5}\left[2\theta_A + \theta_B - 3\psi_2\right] \qquad \theta_A = 0; \qquad \psi_2 = -\frac{\Delta_2}{5}$$

$$M_{AB} = 0.4EI\theta_B + 0.24EI\Delta_2$$

$$M_{BA} = 0.8EI\theta_B + 0.24EI\Delta_2$$

$$M_{BC} = 0.8EI\theta_B + 0.4EI\theta_C + 0.24EI\Delta_1$$

$$M_{CB} = 0.8EI\theta_C + 0.4EI\theta_B + 0.24EI\Delta_1$$

$$M_{BE} = 0.8EI\theta_B + 0.4EI\theta_E$$

$$M_{EB} = 0.8EI\theta_E + 0.4EI\theta_B$$

$$M_{CD} = 0.8EI\theta_C + 0.4EI\theta_D$$

$$M_{DC} = 0.8EI\theta_D + 0.4EI\theta_C$$

$$M_{DE} = 0.8EI\theta_D + 0.4EI\theta_E + 0.24EI\Delta_1$$

$$M_{ED} = 0.8EI\theta_E + 0.4EI\theta_D + 0.24EI\Delta_1$$

$$M_{EF} = 0.8EI\theta_E + 0.24EI\Delta_2$$

$$M_{FE} = 0.4EI\theta_E + 0.24EI\Delta_2 \qquad (1)$$

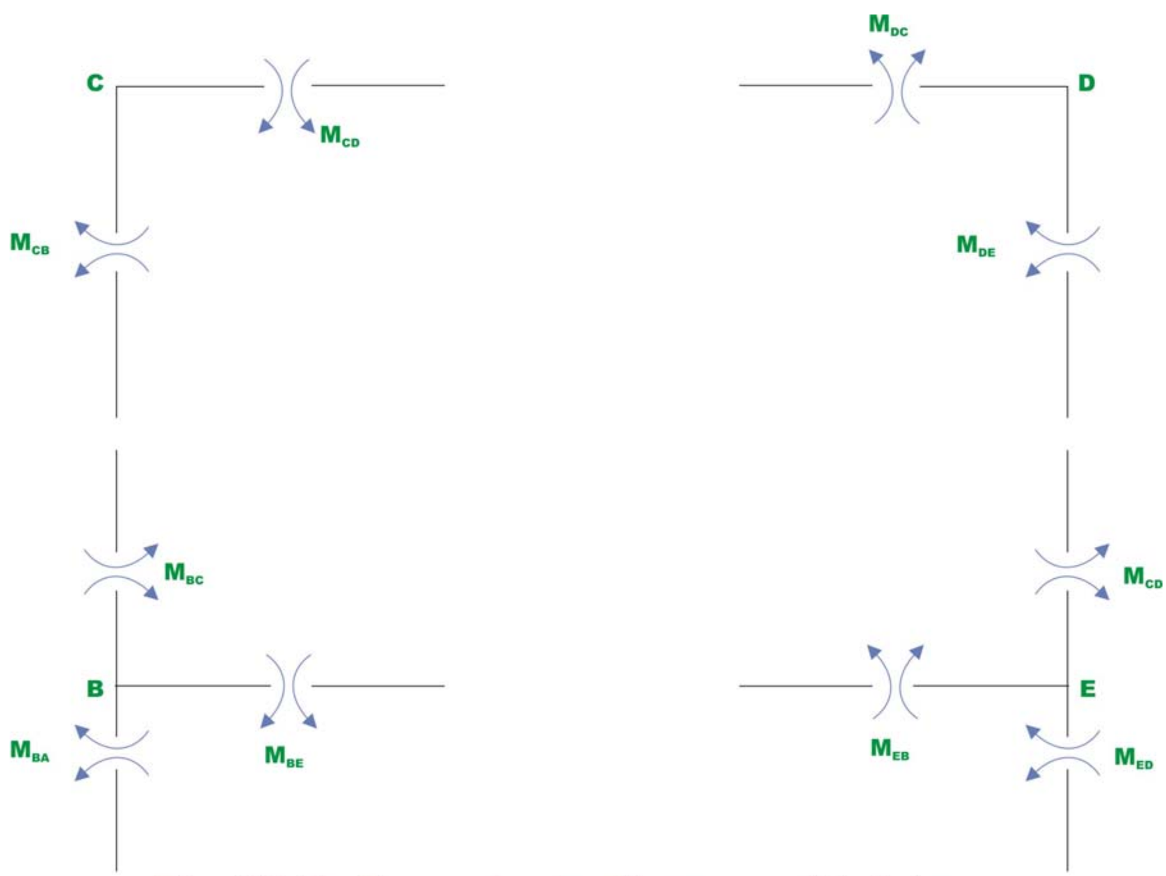


Fig. 22.7c Free - body diagram of joints

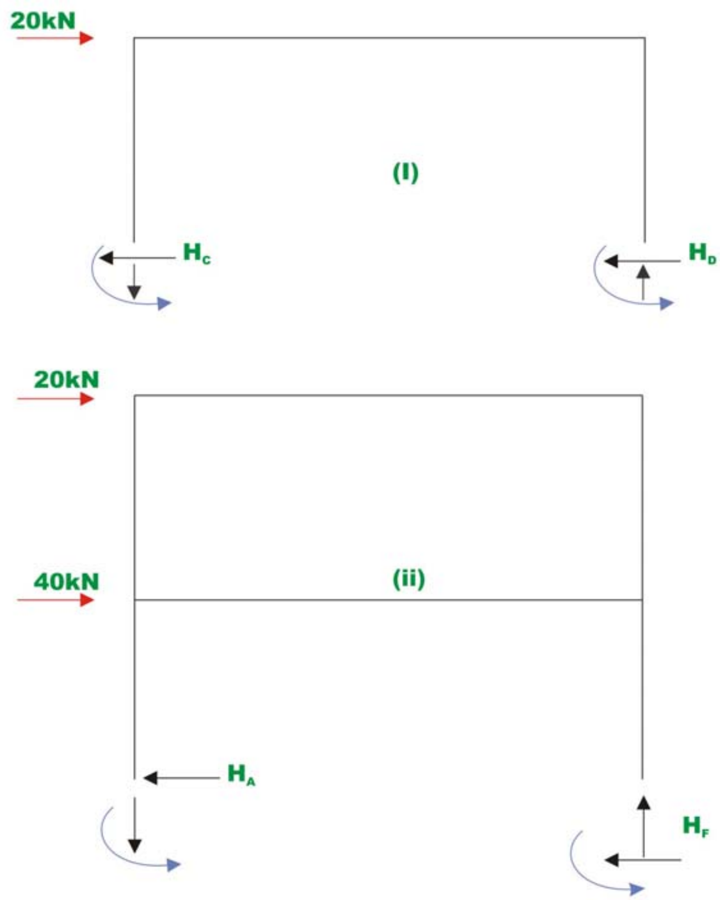


Fig.22.7d Free - body diagram

Moment equilibrium of joint B, C, D and E requires that (vide Fig. 22.7c).

$$
\begin{aligned}
&M_{BA} + M_{BC} + M_{BE} = 0 \\
&M_{CB} + M_{CD} = 0 \\
&M_{DC} + M_{DE} = 0 \\
&M_{EB} + M_{ED} + M_{EF} = 0
\end{aligned} \tag{2}
$$

The required two more equations are written considering the horizontal equilibrium at each story level. *i.e.* $\sum F_X = 0$ (vide., Fig. 22.7d). Thus,

$$H_C + H_D = 20$$

$$H_A + H_F = 60 \tag{3}$$

Considering the equilibrium of column AB, EF, BC and ED, we get (vide 22.7c)

$$H_C = \frac{M_{BC} + M_{CB}}{5}$$

$$H_D = \frac{M_{DE} + M_{ED}}{5}$$

$$H_A = \frac{M_{AB} + M_{BA}}{5}$$

$$H_F = \frac{M_{EF} + M_{FE}}{5} \qquad (4)$$

Using equation (4), equation (3) may be written as,

$$M_{BC} + M_{CB} + M_{DE} + M_{ED} = 100$$

$$M_{AB} + M_{BA} + M_{EF} + M_{FE} = 300 \qquad (5)$$

Substituting the beam end moments from equation (1) in (2) and (5) the required equations are obtained. Thus,

$$2.4\theta_B + 0.4\theta_C + 0.4\theta_E + 0.24\Delta_1 + 0.24\Delta_2 = 0$$

$$1.6\theta_C + 0.4\theta_D + 0.4\theta_B + 0.24\Delta_1 = 0$$

$$1.6\theta_D + 0.4\theta_C + 0.4\theta_E + 0.24\Delta_1 = 0$$

$$2.4\theta_E + 0.4\theta_B + 0.4\theta_D + 0.24\Delta_1 + 0.24\Delta_2 = 0$$

$$1.2\theta_B + 1.2\theta_C + 1.2\theta_D + 1.2\theta_E + 0.96\Delta_1 = 100$$

$$1.2\theta_B + 1.2\theta_E + +0.96\Delta_2 = 300 \qquad (6)$$

Solving above equations, yields

$$\theta_B = \frac{-65.909}{EI}; \quad \theta_C = \frac{-27.273}{EI}; \quad \theta_E = \frac{-27.273}{EI}; \quad \theta_D = \frac{-65.909}{EI};$$

$$\Delta_1 = \frac{337.12}{EI}; \quad \Delta_2 = \frac{477.27}{EI} \qquad (7)$$

Substituting the above values of rotations and translations in equation (1) beam end moments are evaluated. They are,

$$M_{AB} = 88.18 \ \text{kN.m} \ ; \ M_{BA} = 61.81 \ \text{kN.m}$$

$$M_{BC} = 17.27 \ \text{kN.m} \ ; \ M_{CB} = 32.72. \ \text{kN.m}$$

$$M_{BE} = -79.09 \ \text{kN.m} \ ; \ M_{EB} = -79.09 \ \text{kN.m}$$

$$M_{CD} = -32.72 \ \text{kN.m} \ ; \ M_{DC} = -32.72 \ \text{kN.m}$$

$$M_{DE} = 32.72 \ \text{kN.m} \ ; \ M_{ED} = 17.27 \ \text{kN.m}$$

$$M_{EF} = 61.81 \ \text{kN.m} \ ; \ M_{FE} = 88.18 \ \text{kN.m}$$

22.3 Moment-distribution method

The two-story frame shown in Fig. 22.8a has two independent sidesways or member rotations. Invoking the method of superposition, the structure shown in Fig. 22.8a is expressed as the sum of three systems;

1) The system shown in Fig. 22.8b, where in the sidesway is completely prevented by introducing two supports at E and D. All external loads are applied on this frame.
2) System shown in Fig. 22.8c, wherein the support E is locked against sidesway and joint C and D are allowed to displace horizontally. Apply arbitrary sidesway Δ'_1 and calculate fixed end moments in column BC and DE. Using moment-distribution method, calculate beam end moments.
3) Structure shown in Fig. 22.8d, the support D is locked against sidesway and joints B and E are allowed to displace horizontally by removing the support at E. Calculate fixed end moments in column AB and EF for an arbitrary sidesway Δ'_2 as shown the in figure. Since joint displacement as known beforehand, one could use the moment-distribution method to analyse the frame.

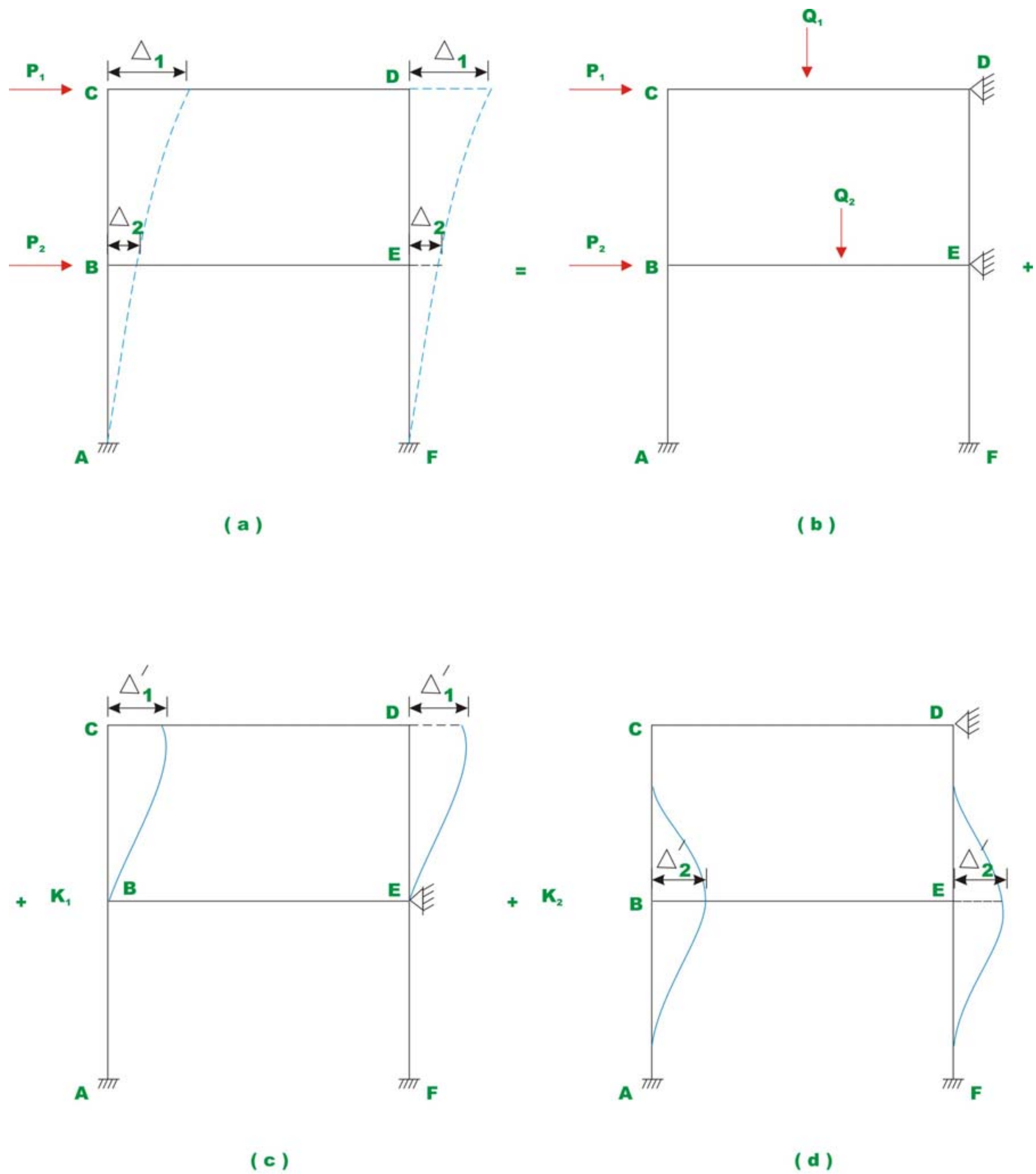


Fig.22.8 Two - story frame

All three systems are analysed separately and superposed to obtain the final answer. Since structures 22.8c and 22.8d are analysed for arbitrary sidesway Δ'_1 and Δ'_2 respectively, the end moments and the displacements of these two analyses are to be multiplied by constants k_1 and k_2 before superposing with the results obtained in Fig. 22.8b. The constants k_1 and k_2 must be such that

$$k_1\,\Delta'_1 = \Delta_1 \text{ and } k_2\,\Delta'_2 = \Delta_2 \,. \tag{22.4}$$

The constants k_1 and k_2 are evaluated by solving shear equations. From Fig. 22.9, it is clear that the horizontal forces developed at the beam level CD in Fig. 22.9c and 22.9d must be equal and opposite to the restraining force applied at the restraining support at D in Fig. 22.9b. Thus,

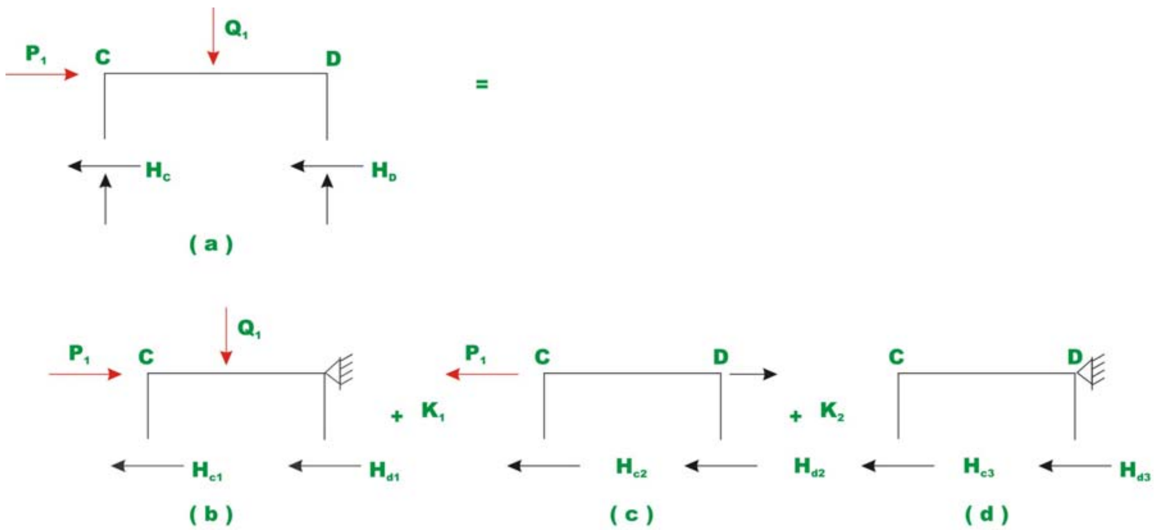


Fig.22.9 The free body diagram at top story.

$$k_1\left(H_{C2} + H_{D2}\right) + k_2\left(H_{C3} + H_{D3}\right) = P_1 \tag{23.5}$$

From similar reasoning, from Fig. 22.10, one could write,

$$k_1\left(H_{A2} + H_{F2}\right) + k_2\left(H_{A3} + H_{F3}\right) = P_2 \tag{23.6}$$

Solving the above two equations, k_1 and k_2 are calculated.

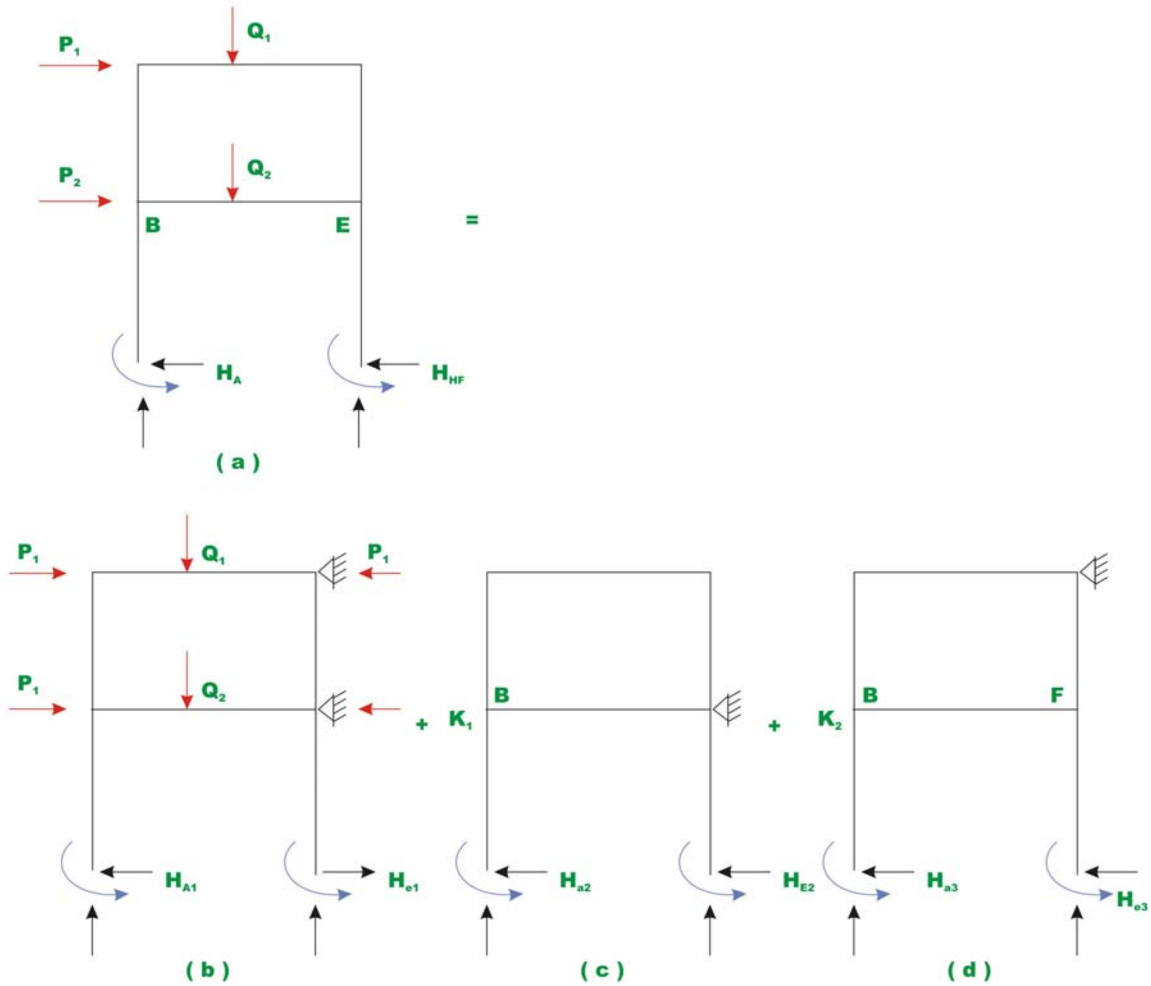


Fig.22.10 Free body diagram at the base of Frame.

Example 22.2

Analyse the rigid frame of example 22.1 by the moment-distribution method.

Solution:

First calculate stiffness and distribution factors for all the six members.

$$\begin{array}{lll} K_{BA} = 0.20EI; & K_{BC} = 0.20EI; & K_{BE} = 0.20EI; \\ K_{CB} = 0.20EI; & K_{CD} = 0.20EI; & \\ K_{DC} = 0.20EI; & K_{DE} = 0.20EI; & \\ K_{EB} = 0.20EI; & K_{ED} = 0.20EI; & K_{EF} = 0.20EI \end{array} \tag{1}$$

Joint B: $\sum K = 0.60EI$

$$DF_{BA} = 0.333; \qquad DF_{BC} = 0.333; \qquad DF_{BE} = 0.333$$

Joint C : $\sum K = 0.40EI$

$DF_{CB} = 0.50$; $DF_{CD} = 0.50$

Joint D : $\sum K = 0.40EI$

$DF_{DC} = 0.50$; $DF_{DE} = 0.50$

Joint E : $\sum K = 0.60EI$

$DF_{EB} = 0.333$; $DF_{ED} = 0.333$; $DF_{EF} = 0.333$ (2)

The frame has two independent sidesways: Δ_1 to the right of CD and Δ_2 to the right of BE. The given problem may be broken in to three systems as shown in Fig.22.11a.

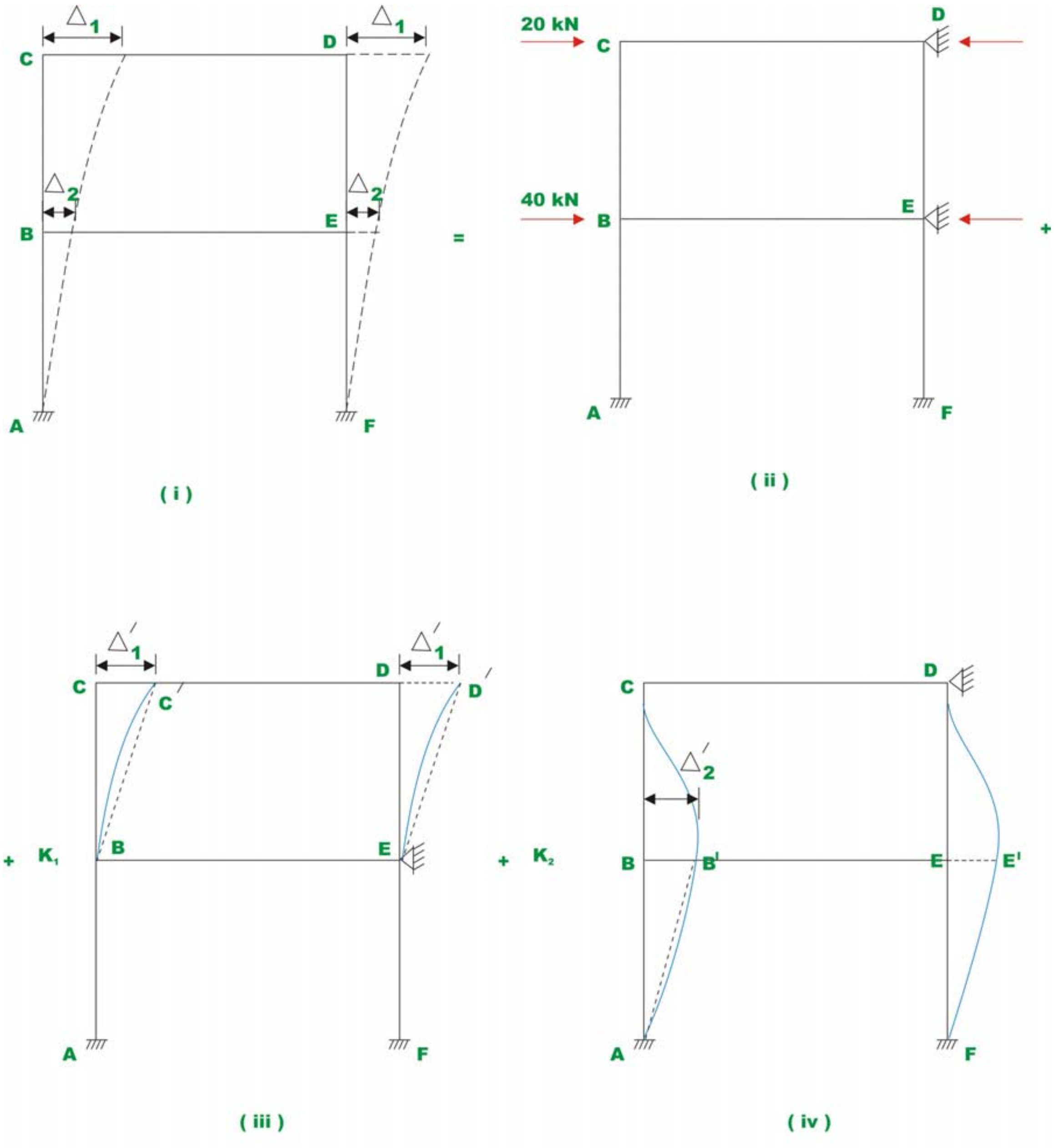


Fig. 22.11a Example 22.2

In the first case, when the sidesway is prevented [Fig. 22.10a (ii)], the only internal forces induced in the structure being $20\,kN$ and $40\,kN$ axial forces in member CD and BE respectively. No bending moment is induced in the structure. Thus we need to analyse only (iii) and (iv).

Case I :

Moment-distribution for sidesway Δ'_1 at beam CD [Fig. 22.1qa (iii)]. Let the arbitrary sidesway be $\Delta'_1 = \dfrac{25}{EI}$. Thus the fixed end moment in column CB and DE due to this arbitrary sidesway is

$$M_{BC}^{F} = M_{CB}^{F} = \frac{6EI\Delta'_1}{L^2} = \frac{6EI}{25} \times \frac{25}{EI} = +6.0 \quad \text{kN.m}$$

$$M_{ED}^{F} = M_{DE}^{F} = +6.0 \quad \text{kN.m} \tag{3}$$

Now moment-distribution is carried out to obtain the balanced end moments. The whole procedure is shown in Fig. 22.11b. Successively joint D, C, B and E are released and balanced.

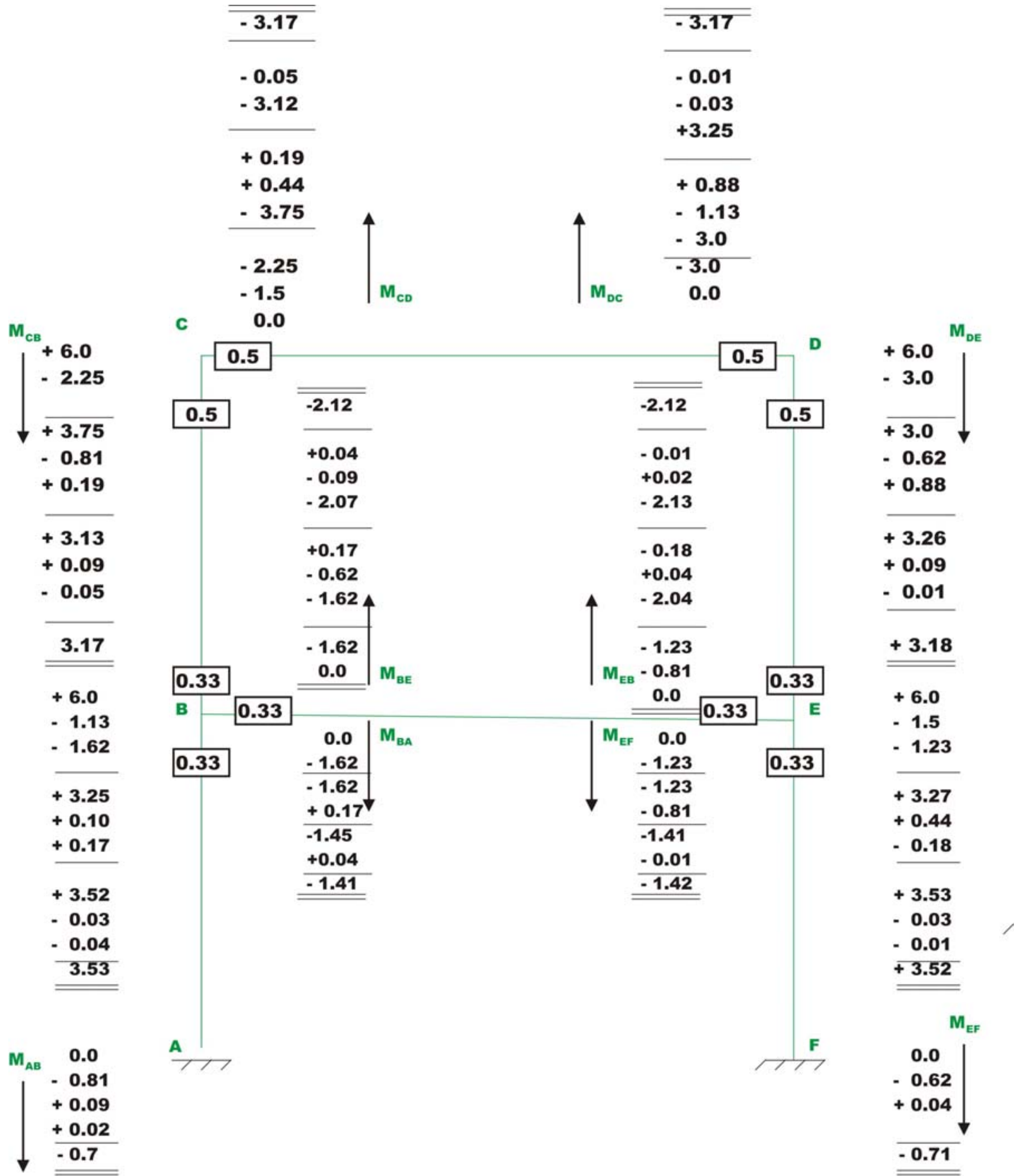


Fig. 22.11b Moment distribution for known sidesway at top story

From the free body diagram of the column shown in Fig. 22.11c, the horizontal forces are calculated. Thus,

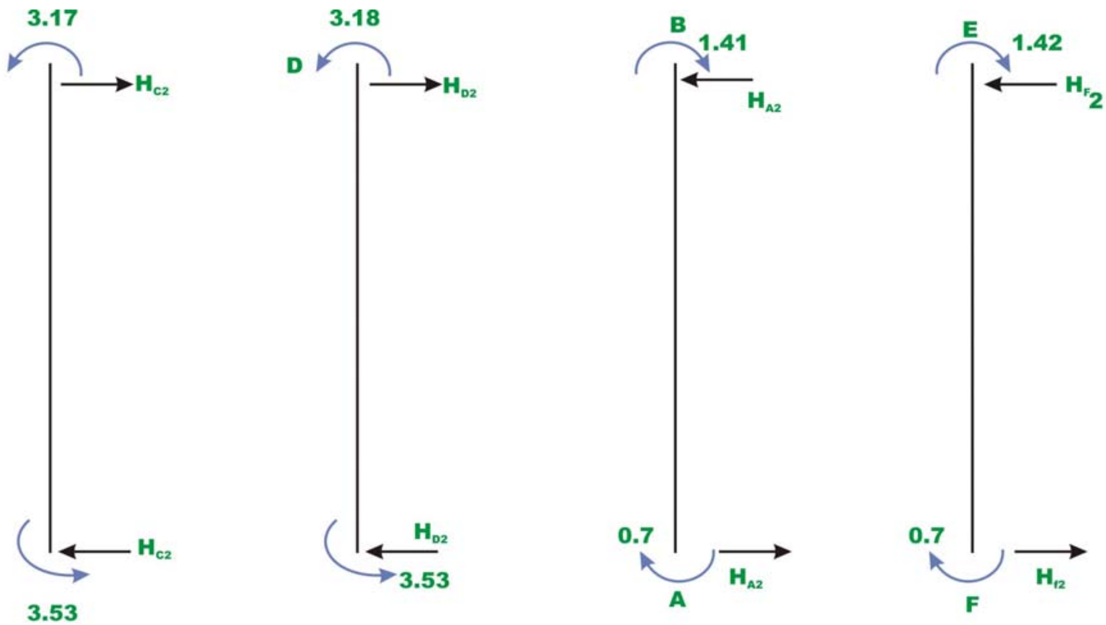


Fig.22.11c Free - body diagrams of Columns for applied load

$$H_{C2} = \frac{3.53 + 3.17}{5} = 1.34 \quad \text{kN}; \qquad H_{D2} = 1.34 \quad \text{kN}$$

$$H_{A2} = \frac{-0.70 - 1.41}{5} = -0.42 \quad \text{kN}; \qquad H_{F2} = -0.42 \quad \text{kN} \qquad (4)$$

Case II **:**

Moment-distribution for sidesway Δ'_2 at beam BE [Fig. 22.11a (iv)]. Let the arbitrary sidesway be $\Delta'_2 = \frac{25}{EI}$

Thus the fixed end moment in column AB and EF due to this arbitrary sidesway is

$$M_{AB}^F = M_{BA}^F = \frac{6EI\Delta'_2}{L^2} = \frac{6EI}{25} \times \frac{25}{EI} = +6.0 \quad \text{kN.m}$$

$$M_{FE}^F = M_{EF}^F = +6.0 \quad \text{kN.m} \qquad (5)$$

Moment-distribution is carried out to obtain the balanced end moments as shown in Fig. 22.11d. The whole procedure is shown in Fig. 22.10b. Successively joint D, C, B and E are released and balanced.

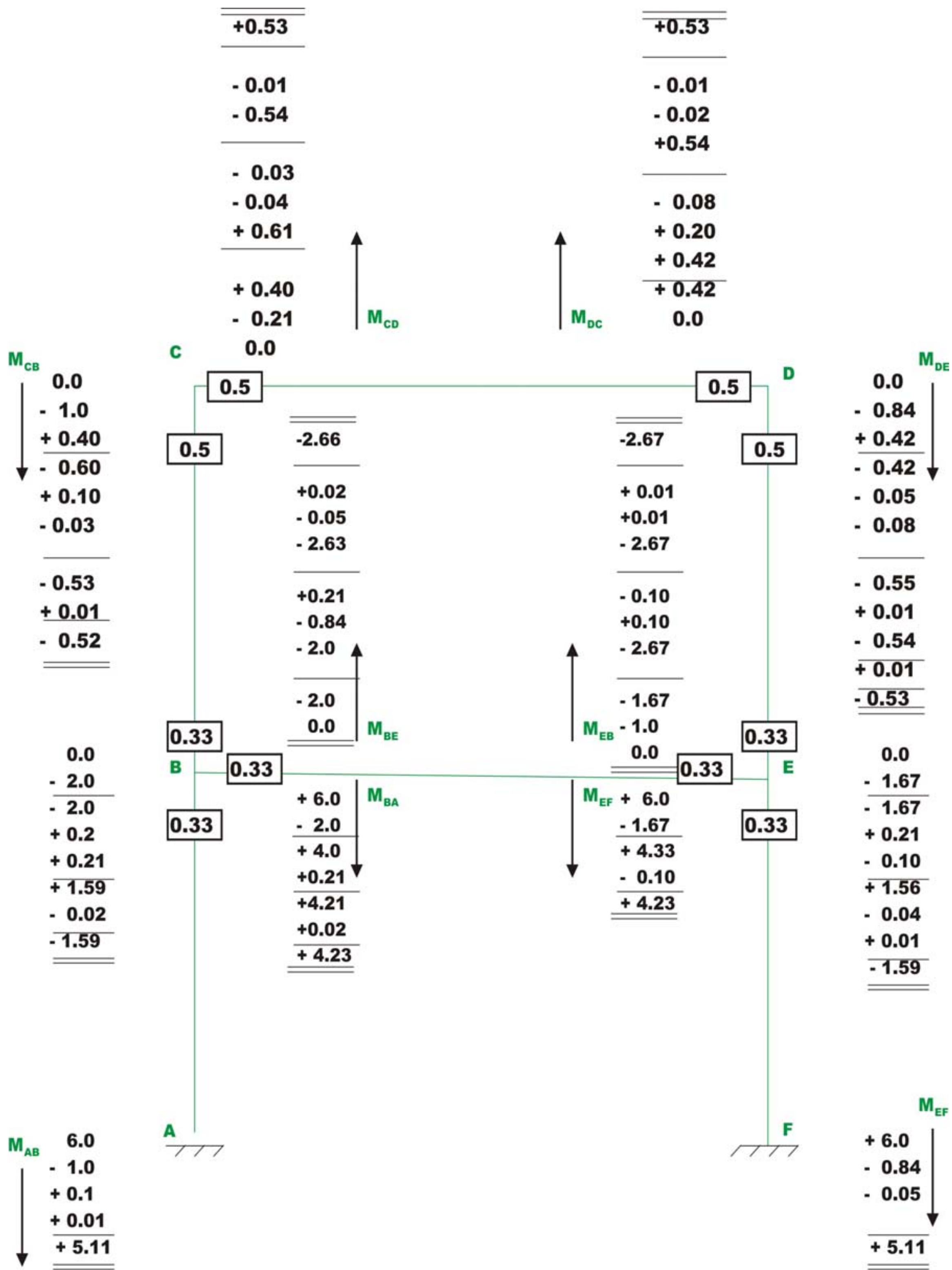


Fig. 22.11d Moment distribution for known sidesway at bottom story

From the free body diagram of the column shown in Fig. 22.11e, the horizontal forces are calculated. Thus,

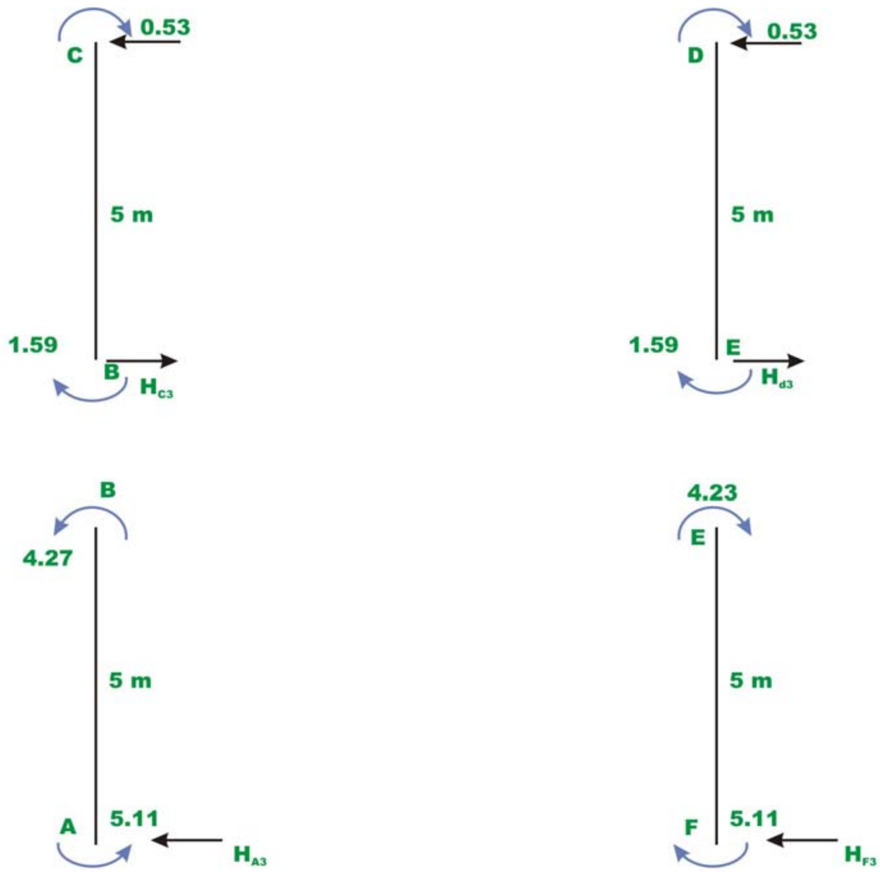


Figure 22.11e Free - body diagrams of Columns for arbitrary known sidesway

$$H_{C3} = \frac{-1.59 - 0.53}{5} = -0.42 \quad \text{kN}; \qquad H_{D3} = -0.42 \quad \text{kN}$$
$$H_{A3} = \frac{5.11 + 4.23}{5} = 1.86 \quad \text{kN}; \qquad H_{F3} = 1.86 \quad \text{kN} \tag{6}$$

For evaluating constants k_1 and k_2, we could write, (see Fig. 22.11a, 22.11c and 22.11d).

$$k_1(H_{C2} + H_{D2}) + k_2(H_{C3} + H_{D3}) = 20$$
$$k_1(H_{A2} + H_{F2}) + k_2(H_{A3} + H_{F3}) = 60$$
$$k_1(1.34 + 1.34) + k_2(-0.42 - 0.42) = 20$$
$$k_1(-0.42 - 0.42) + k_2(1.86 + 1.86) = 60$$

$$k_1(1.34) + k_2(-0.42) = 10$$
$$k_1(-0.42) + k_2(1.86) = 30$$

Solving which, $k_1 = 13.47$ $\qquad$ $k_2 = 19.17$ (7)

Thus the final moments are,

$$M_{AB} = 88.52 \text{ kN.m} \; ; \; M_{BA} = 62.09 \text{ kN.m}$$

$$M_{BC} = 17.06 \text{ kN.m} \; ; \; M_{CB} = 32.54. \text{ kN.m}$$

$$M_{BE} = -79.54 \text{ kN.m} \; ; M_{EB} = -79.54 \text{ kN.m}$$

$$M_{CD} = -32.54 \text{ kN.m} \; ; M_{DC} = -32.54 \text{ kN.m}$$

$$M_{DE} = 32.54 \text{ kN.m} \; ; \; M_{ED} = 17.06 \text{ kN.m}$$

$$M_{EF} = 62.09 \text{ kN.m} \; ; \; M_{FE} = 88.52 \text{ kN.m} \qquad (8)$$

Summary

A procedure to identify the number of independent rotational degrees of freedom of a rigid frame is given. The slope-deflection method and the moment-distribution method are extended in this lesson to solve rigid multistory frames having more than one independent rotational degrees of freedom. A multistory frames having side sway is analysed by the slope-deflection method and the moment-distribution method. Appropriate number of equilibrium equations is written to evaluate all unknowns. Numerical examples are explained with the help of free-body diagrams.

Module 4

Analysis of Statically Indeterminate Structures by the Direct Stiffness Method

Lesson
23
The Direct Stiffness Method: An Introduction

Instructional Objectives:

After reading this chapter the student will be able to
1. Differentiate between the direct stiffness method and the displacement method.
2. Formulate flexibility matrix of member.
3. Define stiffness matrix.
4. Construct stiffness matrix of a member.
5. Analyse simple structures by the direct stiffness matrix.

23.1 Introduction

All known methods of structural analysis are classified into two distinct groups:-

(i) force method of analysis and
(ii) displacement method of analysis.

In module 2, the force method of analysis or the method of consistent deformation is discussed. An introduction to the displacement method of analysis is given in module 3, where in slope-deflection method and moment- distribution method are discussed. In this module the direct stiffness method is discussed. In the displacement method of analysis the equilibrium equations are written by expressing the unknown joint displacements in terms of loads by using load-displacement relations. The unknown joint displacements (the degrees of freedom of the structure) are calculated by solving equilibrium equations. The slope-deflection and moment-distribution methods were extensively used before the high speed computing era. After the revolution in computer industry, only direct stiffness method is used.

The displacement method follows essentially the same steps for both statically determinate and indeterminate structures. In displacement /stiffness method of analysis, once the structural model is defined, the unknowns (joint rotations and translations) are automatically chosen unlike the force method of analysis. Hence, displacement method of analysis is preferred to computer implementation. The method follows a rather a set procedure. The direct stiffness method is closely related to slope-deflection equations.

The general method of analyzing indeterminate structures by displacement method may be traced to Navier (1785-1836). For example consider a four member truss as shown in Fig.23.1.The given truss is statically indeterminate to second degree as there are four bar forces but we have only two equations of equilibrium. Denote each member by a number, for example (1), (2), (3) and (4). Let α_i be the angle, the *i*-th member makes with the horizontal. Under the action of external loads P_x and P_y, the joint *E* displaces to *E'*. Let *u* and *v* be its vertical and horizontal displacements. Navier solved this problem as follows.

In the displacement method of analysis *u* and *v* are the only two unknowns for this structure. The elongation of individual truss members can be expressed in terms of these two unknown joint displacements. Next, calculate bar forces in the members by using force–displacement relation. Now at *E*, two equilibrium equations can be written viz., $\sum F_x = 0$ and $\sum F_y = 0$ by summing all forces in *x* and *y* directions. The unknown displacements may be calculated by solving the equilibrium equations. In displacement method of analysis, there will be exactly as many equilibrium equations as there are unknowns.

Let an elastic body is acted by a force F and the corresponding displacement be *u* in the direction of force. In module 1, we have discussed force- displacement relationship. The force (*F*) is related to the displacement (*u*) for the linear elastic material by the relation

$$F = ku \tag{23.1}$$

where the constant of proportionality *k* is defined as the stiffness of the structure and it has units of force per unit elongation. The above equation may also be written as

$$u = aF \tag{23.2}$$

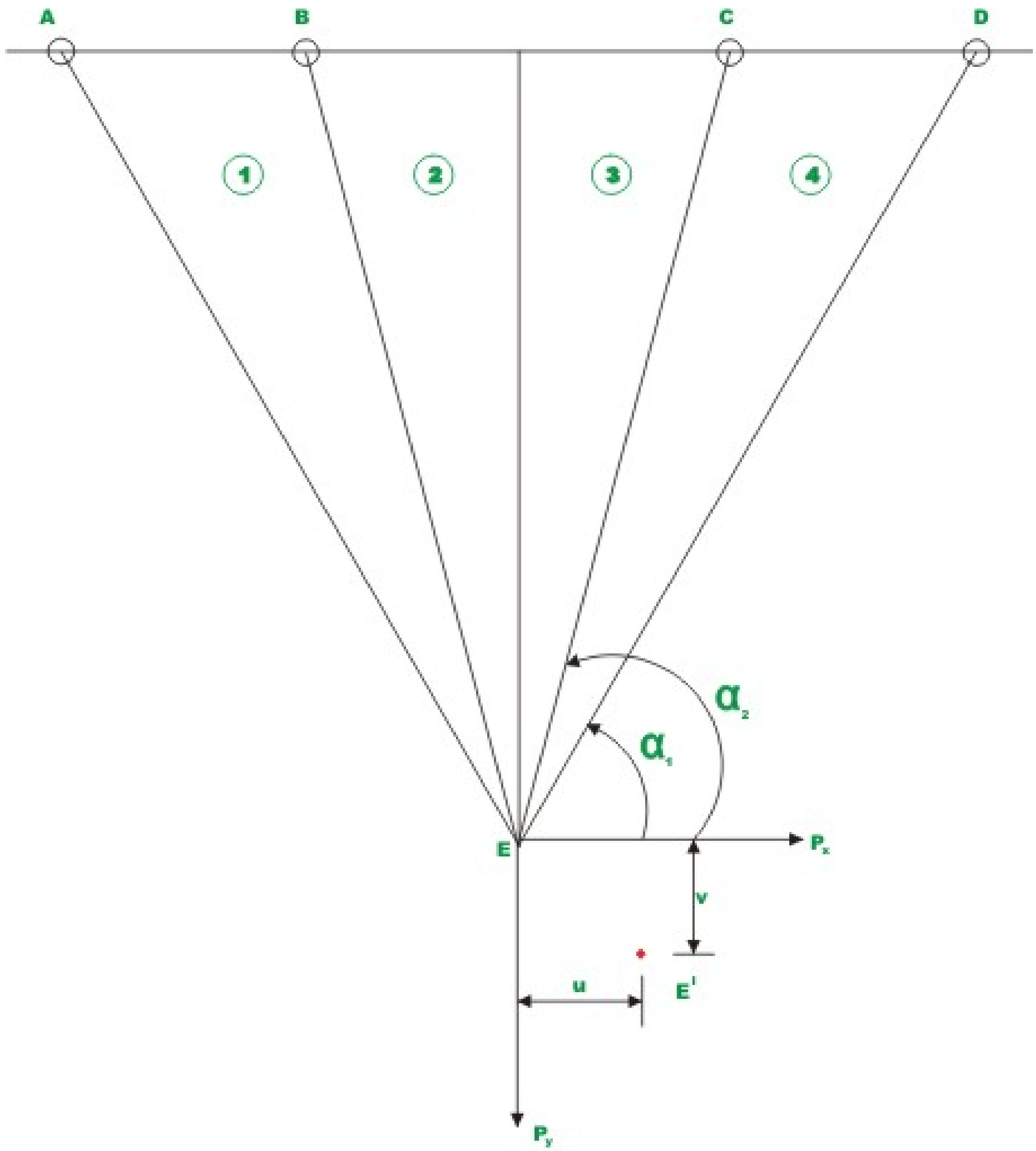


Fig. 23.1 Four member truss

The constant a is known as flexibility of the structure and it has a unit of displacement per unit force. In general the structures are subjected to n forces at n different locations on the structure. In such a case, to relate displacement at i to load at j, it is required to use flexibility coefficients with subscripts. Thus the flexibility coefficient a_{ij} is the deflection at i due to unit value of force applied at j. Similarly the stiffness coefficient k_{ij} is defined as the force generated at i

due to unit displacement at j with all other displacements kept at zero. To illustrate this definition, consider a cantilever beam which is loaded as shown in Fig.23.2. The two degrees of freedom for this problem are vertical displacement at *B* and rotation at *B*. Let them be denoted by u_1 and u_2 ($=\theta_1$). Denote the vertical force *P* by P_1 and the tip moment *M* by P_2. Now apply a unit vertical force along P_1 and calculate deflection u_1 and u_2.The vertical deflection is denoted by flexibility coefficient a_{11} and rotation is denoted by flexibility coefficient a_{21}. Similarly, by applying a unit force along P_1, one could calculate flexibility coefficient a_{12} and a_{22}. Thus a_{12} is the deflection at 1 corresponding to P_1 due to unit force applied at 2 in the direction of P_2. By using the principle of superposition, the displacements u_1 and u_2 are expressed as the sum of displacements due to loads P_1 and P_2 acting separately on the beam. Thus,

$$
\begin{aligned}
u_1 &= a_{11}P_1 + a_{12}P_2 \\
u_2 &= a_{21}P_1 + a_{22}P_2
\end{aligned}
\tag{23.3a}
$$

The above equation may be written in matrix notation as

$$\{u\} = [a]\{P\}$$

where, $\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$; $\{a\} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$; and $\{P\} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$

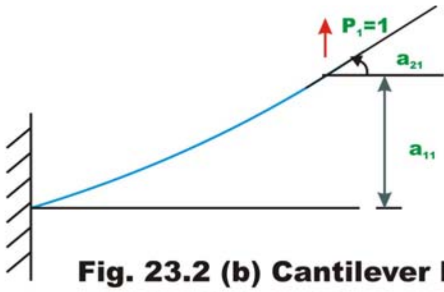


Fig. 23.2 (b) Cantilever beam with unit load along P_1

Fig.23.2(a) Cantilever beam

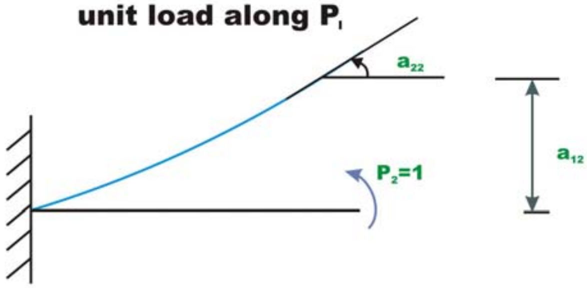


Fig. 23.2 © Cantilever beam with unit moment along P_2

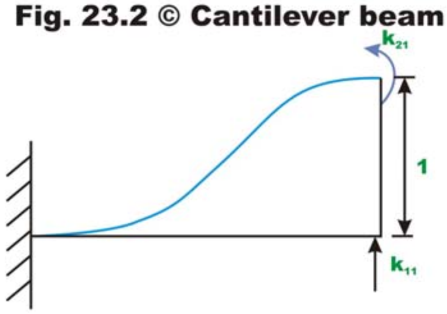


Fig. 23.2(d) Cantilever beam with unit displacement along u_1

The forces can also be related to displacements using stiffness coefficients. Apply a unit displacement along u_1 (see Fig.23.2d) keeping displacement u_2 as zero. Calculate the required forces for this case as k_{11} and k_{21}.Here, k_{21} represents force developed along P_2 when a unit displacement along u_1 is introduced keeping u_2 =0. Apply a unit rotation along u_2 (vide Fig.23.2c) ,keeping $u_1 = 0$. Calculate the required forces for this configuration k_{12} and k_{22}. Invoking the principle of superposition, the forces P_1 and P_2 are expressed as the sum of forces developed due to displacements u_1 and u_2 acting separately on the beam. Thus,

$$P_1 = k_{11}u_1 + k_{12}u_2$$

$$P_2 = k_{21}u_1 + k_{22}u_2 \tag{23.4}$$

$$\{P\} = [k]\{u\}$$

where, $\{P\}=\begin{Bmatrix}P_1\\P_2\end{Bmatrix}$; $\{k\}=\begin{bmatrix}k_{11} & k_{12}\\k_{21} & k_{22}\end{bmatrix}$;and $\{u\}=\begin{Bmatrix}u_1\\u_2\end{Bmatrix}$.

$[k]$ is defined as the stiffness matrix of the beam.
In this lesson, using stiffness method a few problems will be solved. However this approach is very rudimentary and is suited for hand computation. A more formal approach of the stiffness method will be presented in the next lesson.

23.2 A simple example with one degree of freedom

Consider a fixed–simply supported beam of constant flexural rigidity *EI* and span *L* which is carrying a uniformly distributed load of w kN/m as shown in Fig.23.3a.

If the axial deformation is neglected, then this beam is kinematically indeterminate to first degree. The only unknown joint displacement is θ_B.Thus the degrees of freedom for this structure is one (for a brief discussion on degrees of freedom, please see introduction to module 3).The analysis of above structure by stiffness method is accomplished in following steps:

1. Recall that in the flexibility /force method the redundants are released (i.e. made zero) to obtain a statically determinate structure. A similar operation in the stiffness method is to make all the unknown displacements equal to zero by altering the boundary conditions. Such an altered structure is known as kinematically determinate structure as all joint displacements are known in this case. In the present case the restrained structure is obtained by preventing the rotation at B as shown in Fig.23.3b. Apply all the external loads on the kinematically determinate structure. Due to restraint at *B*, a moment M_B is developed at *B*. In the stiffness method we adopt the following sign convention. Counterclockwise moments and counterclockwise rotations are taken as positive, upward forces and displacements are taken as positive. Thus,

$$M_B=-\frac{wl^2}{12} \qquad \text{(-ve as } M_B \text{ is clockwise)} \qquad (23.5)$$

The fixed end moment may be obtained from the table given at the end of lesson 14.

2. In actual structure there is no moment at *B*. Hence apply an equal and opposite moment M_B at *B* as shown in Fig.23.3c. Under the action of (-M_B) the joint rotates in the clockwise direction by an unknown amount. It is observed that superposition of above two cases (Fig.23.3b and Fig.23.3c) gives the forces in the actual structure. Thus the rotation of joint

B must be θ_B which is unknown .The relation between $-M_B$ and θ_B is established as follows. Apply a unit rotation at *B* and calculate the moment. (k_{BB}) caused by it. That is given by the relation

$$k_{BB} = \frac{4EI}{L} \tag{23.6}$$

where k_{BB} is the stiffness coefficient and is defined as the force at joint *B* due to unit displacement at joint *B*. Now, moment caused by θ_B rotation is

$$M_B = k_{BB}\theta_B \tag{23.7}$$

3. Now, write the equilibrium equation for joint *B*. The total moment at *B* is $M_B + k_{BB}\theta_B$, but in the actual structure the moment at *B* is zero as support *B* is hinged. Hence,

$$M_B + k_{BB}\theta_B = 0 \tag{23.8}$$

$$\theta_B = -\frac{M_B}{k_{BB}}$$

$$\theta_B = \frac{wl^3}{48EI} \tag{23.9}$$

The relation $M_B = \frac{4EI}{L}\theta_B$ has already been derived in slope –deflection method in lesson 14. Please note that exactly the same steps are followed in slope-deflection method.

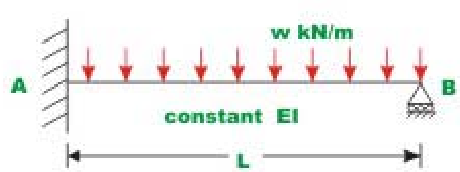


Fig. 23.3(a) Propped - Cantilever beam : one - degree freedom system

Fig.23.3(a) Cantilever beam

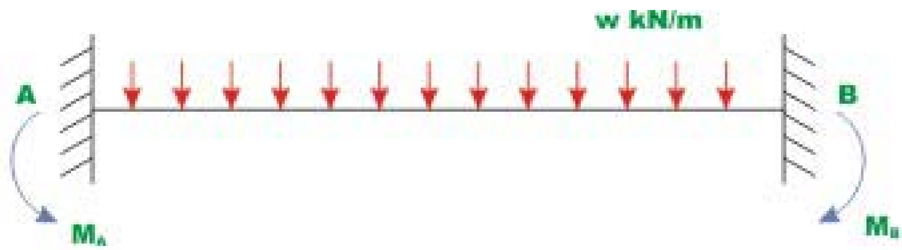


Fig. 23.3b Kinematically determinate beam

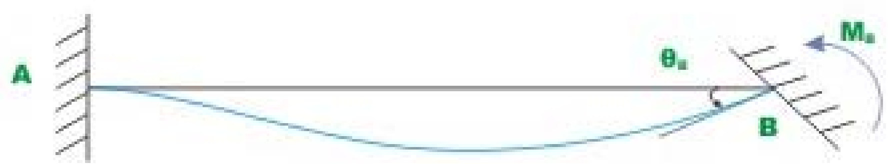


Fig. 23.3 ©

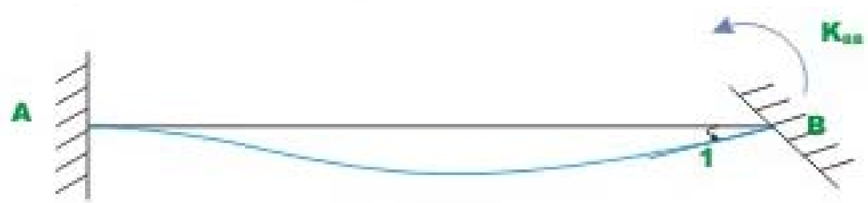


Fig. 23.3 (d) Computation of stiffness co-efficients

23.3 Two degrees of freedom structure

Consider a plane truss as shown in Fig.23.4a.There is four members in the truss and they meet at the common point at *E*. The truss is subjected to external loads P_1 and P_2 acting at *E*. In the analysis, neglect the self weight of members. There are two unknown displacements at joint *E* and are denoted by u_1 and u_2.Thus the structure is kinematically indeterminate to second degree. The applied forces and unknown joint displacements are shown in the positive directions. The members are numbered from (1), (2), (3) and (4) as shown in the figure. The length and axial rigidity of *i*-th member is l_i and EA_i respectively. Now it is sought to evaluate u_1 and u_2 by stiffness method. This is done in following steps:

1. In the first step, make all the unknown displacements equal to zero by altering the boundary conditions as shown in Fig.23.4b. On this restrained /kinematically determinate structure, apply all the external loads except the joint loads and calculate the reactions corresponding to unknown joint displacements u_1 and u_2. Since, in the present case, there are no external loads other than the joint loads, the reactions $(R_L)_1$ and $(R_L)_2$ will be equal to zero. Thus,

$$\begin{Bmatrix} (R_L)_1 \\ (R_L)_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{23.10}$$

2. In the next step, calculate stiffness coefficients k_{11}, k_{21}, k_{12} and k_{22}.This is done as follows. First give a unit displacement along u_1 holding displacement along u_2 to zero and calculate reactions at *E* corresponding to unknown displacements u_1 and u_2 in the kinematically determinate structure. They are denoted by k_{11}, k_{21}. The joint stiffness k_{11}, k_{21} of the whole truss is composed of individual member stiffness of the truss. This is shown in Fig.23.4c. Now consider the member AE . Under the action of unit displacement along u_1 , the joint E displaces to E'. Obviously the new length is not equal to length AE . Let us denote the new length of the members by $l_1 + \Delta l_1$, where Δl , is the change in length of the member AE'. The member AE' also makes an angle θ_1 with the horizontal. This is justified as Δl_1 is small. From the geometry, the change in length of the members AE' is

$$\Delta l_1 = \cos\theta_1 \tag{23.11a}$$

The elongation Δl_1 is related to the force in the member AE', $F_{AE'}$ by

$$\Delta l_1 = \frac{F_{AE'} l_1}{A_1 E} \tag{23.11b}$$

Thus from (23.11a) and (23.11b), the force in the members AE' is

$$F'_{AE} = \frac{EA_1}{l_1}\cos\theta_1 \tag{23.11c}$$

This force acts along the member axis. This force may be resolved along u_1 and u_2 directions. Thus, horizontal component of force F'_{AE} is $\frac{EA_1}{l_1}\cos^2\theta_1$ (23.11d)

and vertical component of force F'_{AE} is $\frac{EA_1}{l_1}\cos\theta_1 \sin\theta_1$ (23.11e)

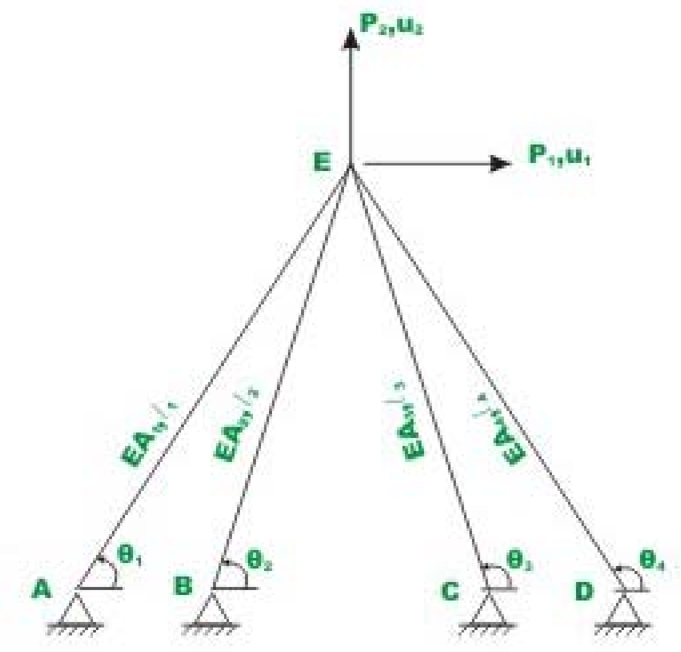


Fig 23.4a A four - member truss

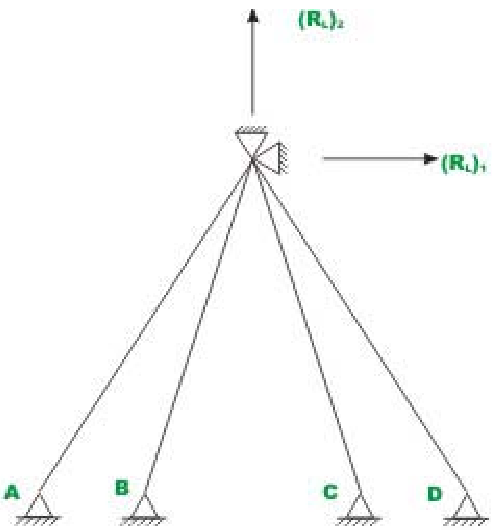


Fig. 23.4b Kinematically determinate structure

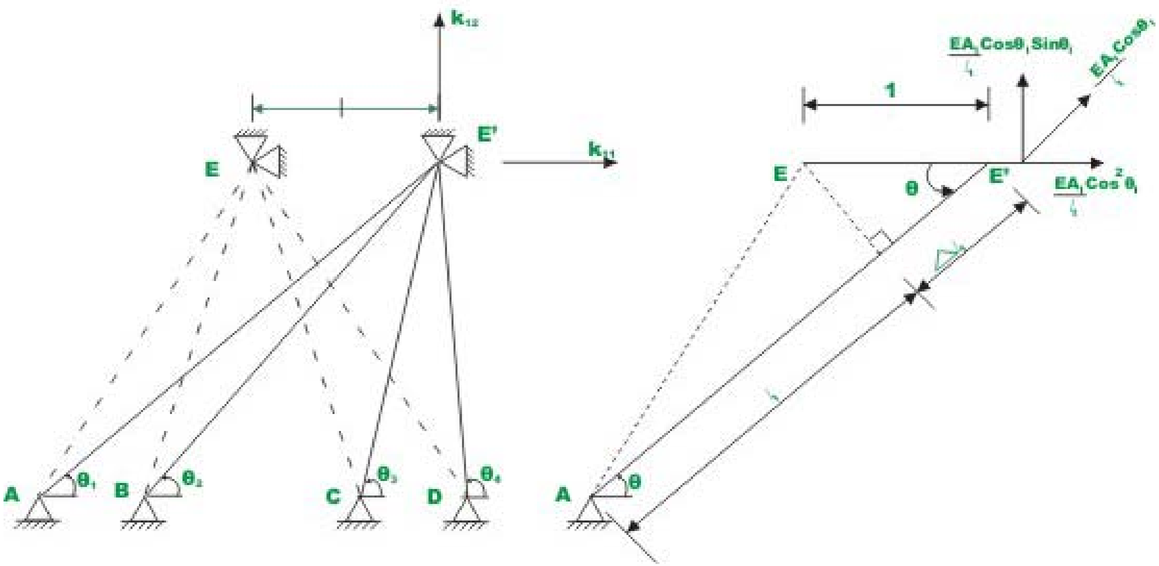


Fig. 23.4c Unit displacement along u_1

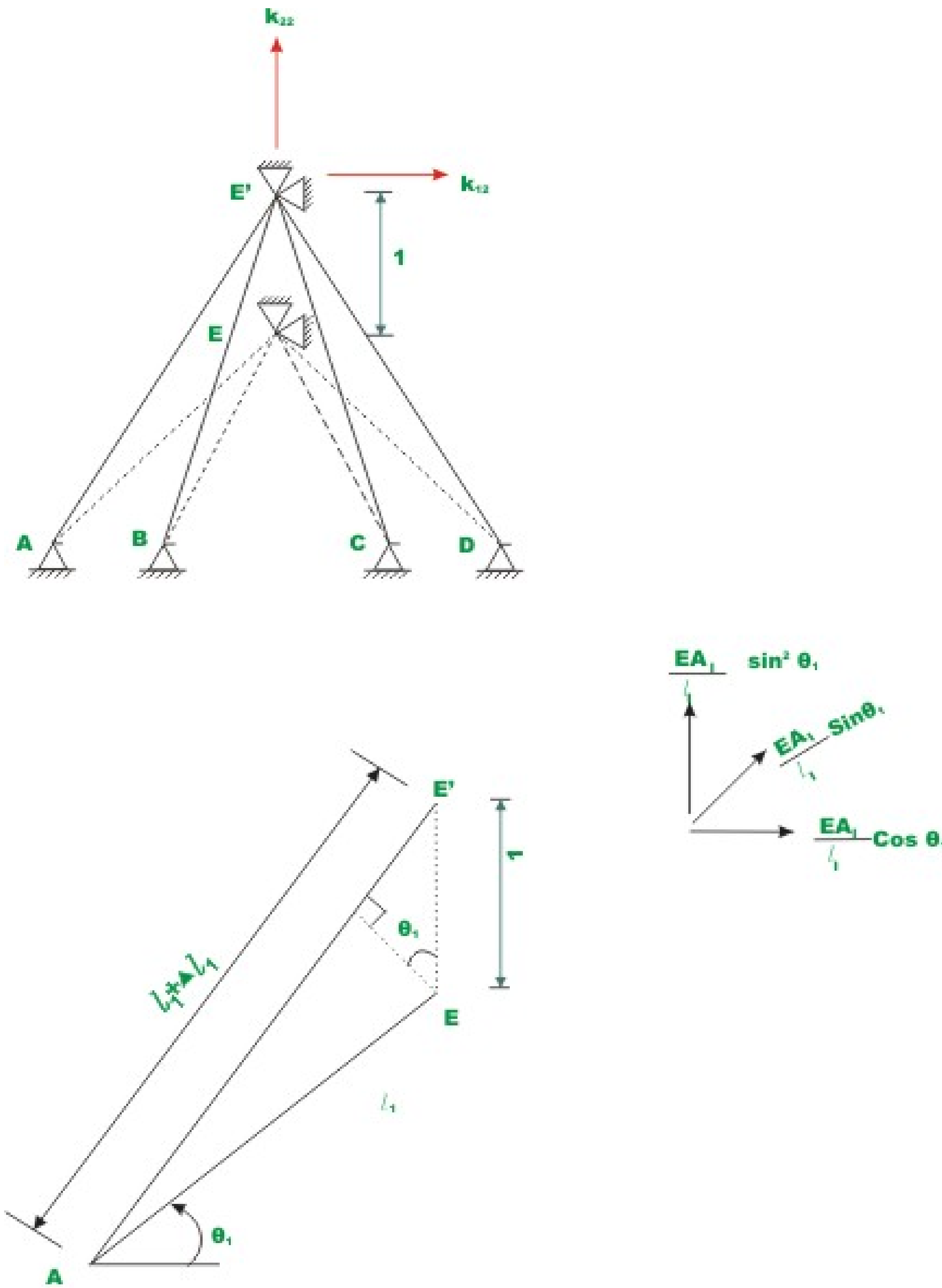


Fig.23.4d Unit displacement along u_2

Expressions of similar form as above may be obtained for all members. The sum of all horizontal components of individual forces gives us the stiffness coefficient k_{11} and sum of all vertical component of forces give us the required stiffness coefficient k_{21}.

$$k_{11} = \frac{EA_1}{l_1}\cos^2\theta_1 + \frac{EA_2}{l_2}\cos^2\theta_2 + \frac{EA_3}{l_3}\cos^2\theta_3 + \frac{EA_4}{l_4}\cos^2\theta_4$$

$$k_{11} = \sum_{i=1}^{4}\frac{EA_i}{l_i}\cos^2\theta_i \qquad (23.12)$$

$$k_{21} = \sum \frac{EA_i}{l_i}\cos\theta_i \sin\theta_i \qquad (23.13)$$

In the next step, give a unit displacement along u_2 holding displacement along u_1 equal to zero and calculate reactions at E corresponding to unknown displacements u_1 and u_2 in the kinematically determinate structure. The corresponding reactions are denoted by k_{12} and k_{22} as shown in Fig.23.4d. The joint E gets displaced to E' when a unit vertical displacement is given to the joint as shown in the figure. Thus, the new length of the member AE' is $l_1 + \Delta l_1$. From the geometry, the elongation Δl_1 is given by

$$\Delta l_1 = \sin\theta_1 \qquad (23.14a)$$

Thus axial force in the member along its centroidal axis is $\frac{EA_1}{l_1}\sin\theta_1$ (23.14b)

Resolve the axial force in the member along u_1 and u_2 directions. Thus, horizontal component of force in the member AE' is $\frac{EA_1}{l_1}\sin\theta_1\cos\theta_1$ (23.14c)

and vertical component of force in the member AE' is $\frac{EA_1}{l_1}\sin^2\theta_1$ (23.14d)

In order to evaluate k_{22}, we need to sum vertical components of forces in all the members meeting at joint E .Thus,

$$k_{22} = \sum_{i=1}^{4} \frac{EA_i}{l_i} \sin^2 \theta_i \tag{23.15}$$

Similarly, $$k_{12} = \sum_{i=1}^{4} \frac{EA_i}{l_i} \sin\theta_i \cos\theta_i \tag{23.16}$$

3. Joint forces in the original structure corresponding to unknown displacements u_1 and u_2 are

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \tag{23.17}$$

Now the equilibrium equations at joint E states that the forces in the original structure are equal to the superposition of (i) reactions in the kinematically restrained structure corresponding to unknown joint displacements and (ii) reactions in the restrained structure due to unknown displacements themselves. This may be expressed as,

$$\begin{aligned} F_1 &= (R_L)_1 + k_{11}u_1 + k_{12}u_2 \\ F_2 &= (R_L)_2 + k_{21}u_1 + k_{22}u_2 \end{aligned} \tag{23.18}$$

This may be written compactly as

$$\{F\} = \{R_i\} + [k]\{u\} \tag{23.19}$$

where,

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix};$$

$$\{R_L\} = \begin{Bmatrix} (R_L)_1 \\ (R_L)_2 \end{Bmatrix}$$

$$[k] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{23.20}$$

For example take $P_1 = P_2 = P$, $L_i = \dfrac{L}{\sin\theta_i}$, $A_1 = A_2 = A_3 = A_4 = A$ and $\theta_1 = 35°$, $\theta_2 = 70°$, $\theta_3 = 105°$ and $\theta_4 = 140°$

Then.

$$\{F\} = \begin{Bmatrix} P \\ P \end{Bmatrix} \tag{23.21}$$

$$\{R_L\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$k_{11} = \sum \frac{EA}{L}\cos^2\theta_i \sin\theta_i = 0.9367\frac{EA}{L}$$

$$k_{12} = \sum \frac{EA}{L}\sin^2\theta_i \cos\theta_i = 0.0135\frac{EA}{L}$$

$$k_{21} = \sum \frac{EA}{L}\sin^2\theta_i \cos\theta_i = 0.0135\frac{EA}{L}$$

$$k_{22} = \sum \frac{EA}{L}\sin^3\theta_i = 2.1853\frac{EA}{L} \tag{23.22}$$

$$\begin{Bmatrix} P \\ P \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 0.9367 & 0.0135 \\ 0.0135 & 2.1853 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Solving which, yields

$$u_1 = 1.0611\frac{L}{EA}$$

$$u_2 = 0.451\frac{L}{EA}$$

Example 23.1

Analyze the plane frame shown in Fig.23.5a by the direct stiffness method. Assume that the flexural rigidity for all members is the same .Neglect axial displacements.

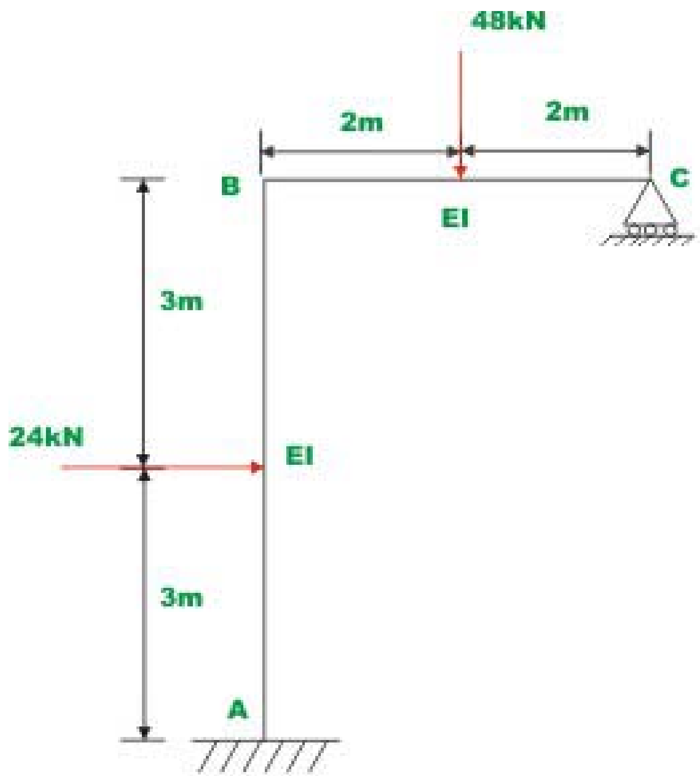


Fig 23.5a Plane - frame of Example 23.1

Solution

In the first step identify the degrees of freedom of the frame .The given frame has three degrees of freedom (see Fig.23.5b):

(i) Two rotations as indicated by u_1 and u_2 and

(ii) One horizontal displacement of joint B and C as indicated by u_3 .

In the next step make all the displacements equal to zero by fixing joints B and C as shown in Fig.23.5c. On this kinematically determinate structure apply all the external loads and calculate reactions corresponding to unknown joint displacements .Thus,

$$\left(R_D^F\right)_1 = \frac{48\times 2\times 4}{16} + \left(-\frac{24\times 3\times 9}{36}\right)$$ (1)

$$= 24 - 18 = 6 \text{ kN.m}$$

$$\left(R_D^F\right)_2 = -24 \text{ kN.m}$$

$$\left(R_D^F\right)_3 = 12 \text{ kN.m} \qquad (2)$$

Thus,

$$\begin{Bmatrix} \left(R_D^F\right)_1 \\ \left(R_D^F\right)_2 \\ \left(R_D^F\right)_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -24 \\ 12 \end{Bmatrix} \qquad (3)$$

Next calculate stiffness coefficients. Apply unit rotation along u_1 and calculate reactions corresponding to the unknown joint displacements in the kinematically determinate structure (vide Fig.23.5d)

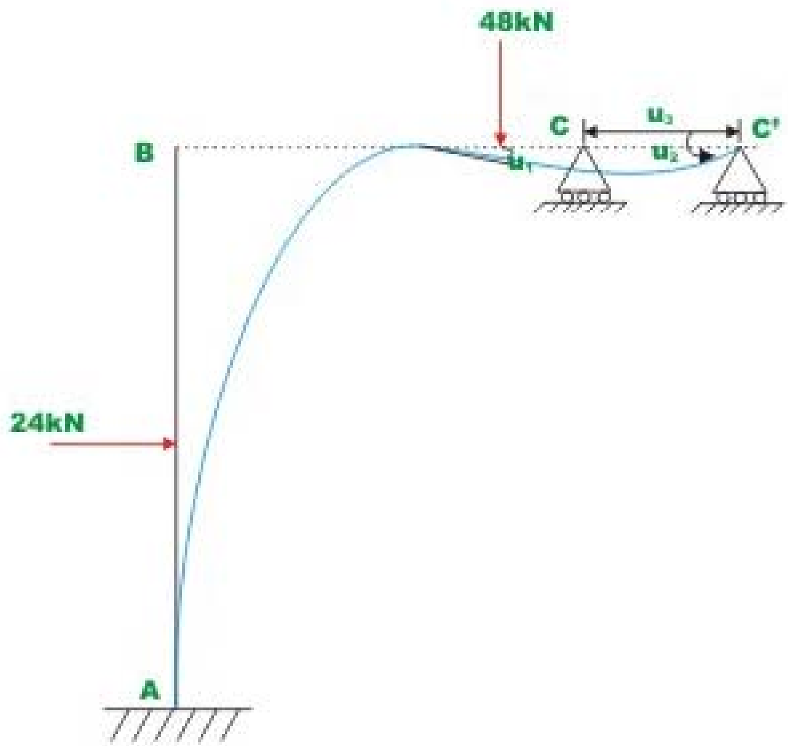


Fig 23.5b Approximate deflected shape

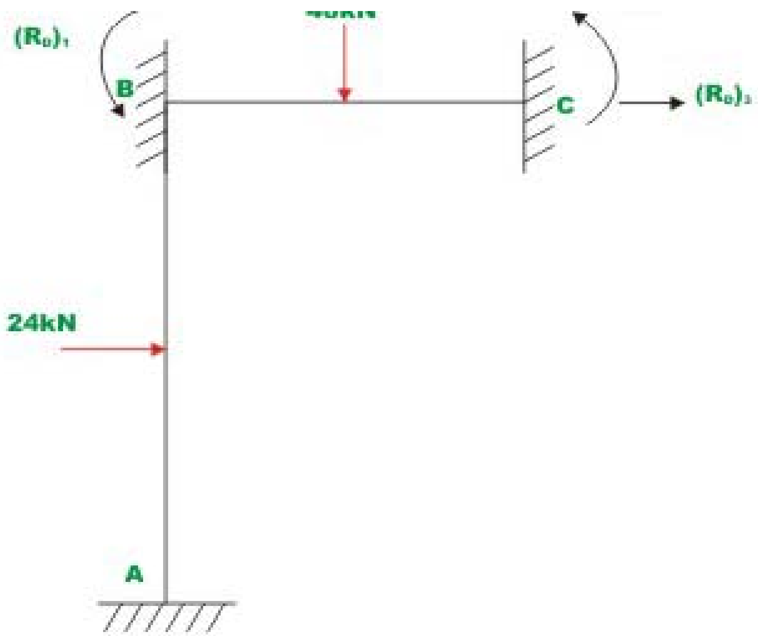


Fig 23.5c Kinematically restrained structure

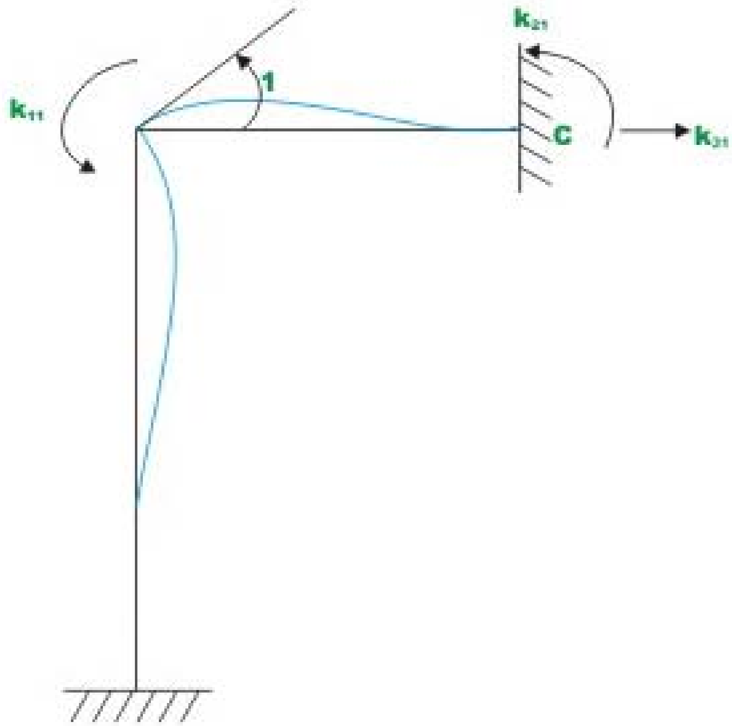


Fig.23.5d Unit displacement along u_1

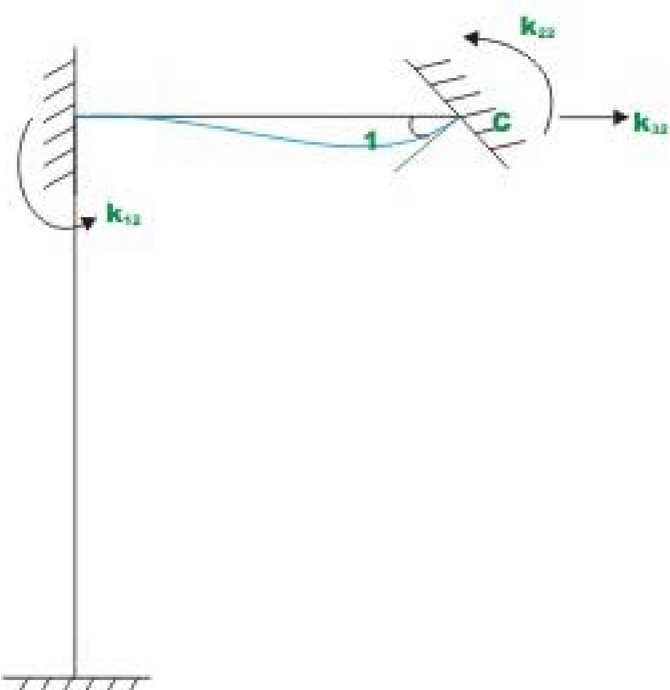


Fig 23.5e Unit displacement along u_2

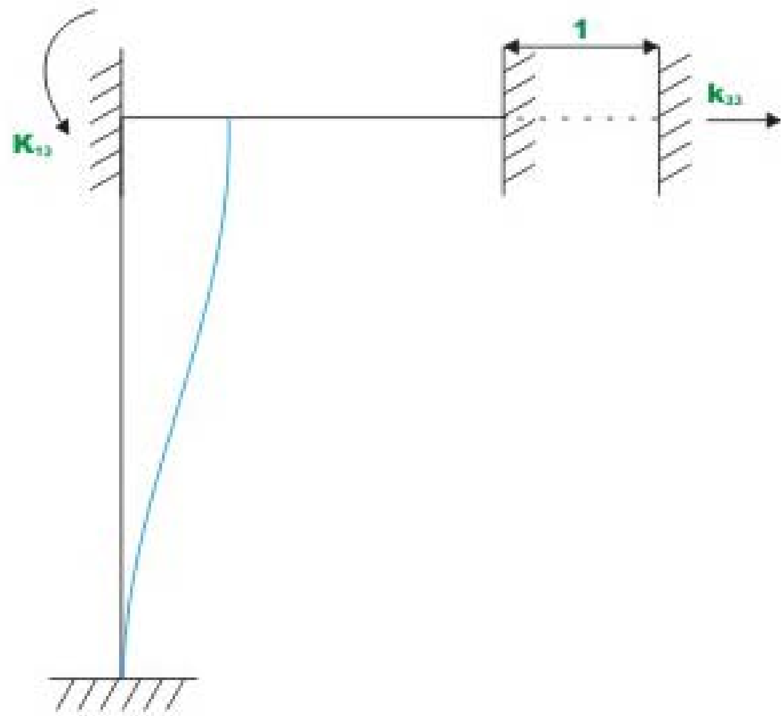


Fig. 23.5f Unit displacement along u_3

$$k_{11} = \frac{4EI}{4} + \frac{4EI}{6} = 1.667$$

$$k_{21} = \frac{2EI}{4} = 0.5EI$$

$$k_{31} = -\frac{6EI}{6 \times 6} = -0.166EI \qquad (4)$$

Similarly, apply a unit rotation along u_2 and calculate reactions corresponding to three degrees of freedom (see Fig.23.5e)

$$k_{12} = 0.5EI$$

$$k_{22} = EI$$

$$k_{32} = 0 \qquad (5)$$

Apply a unit displacement along u_3 and calculate joint reactions corresponding to unknown displacements in the kinematically determinate structure.

$$k_{13} = -\frac{6EI}{L^2} = -0.166E$$

$$k_{23} = 0$$

$$k_{33} = \frac{12EI}{6^3} = 0.056EI \tag{6}$$

Finally applying the principle of superposition of joint forces, yields

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -24 \\ 12 \end{Bmatrix} + EI \begin{Bmatrix} 1.667 & 0.5 & -0.166 \\ 0.5 & 1 & 0 \\ -0.166 & 0 & 0.056 \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Now,

$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ as there are no loads applied along u_1, u_2 and u_3 .Thus the

unknown displacements are,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = -\frac{1}{EI} \begin{bmatrix} 1 & 0.5 & -0.166 \\ 0.5 & 1 & 0 \\ -0.166 & 0 & 0.056 \end{bmatrix}^{-1} \begin{Bmatrix} 6 \\ -24 \\ -24 \end{Bmatrix} \tag{7}$$

Solving

$$u_1 = \frac{18.996}{EI}$$

$$u_2 = \frac{14.502}{EI}$$

$$u_3 = -\frac{270.587}{EI} \tag{8}$$

Summary

The flexibility coefficient and stiffness coefficients are defined in this section. Construction of stiffness matrix for a simple member is explained. A few simple problems are solved by the direct stiffness method. The difference between the slope-deflection method and the direct stiffness method is clearly brought out.

Module 4

Analysis of Statically Indeterminate Structures by the Direct Stiffness Method

Lesson 24

The Direct Stiffness Method: Truss Analysis

Instructional Objectives

After reading this chapter the student will be able to
1. Derive member stiffness matrix of a truss member.
2. Define local and global co-ordinate system.
3. Transform displacements from local co-ordinate system to global co-ordinate system.
4. Transform forces from local to global co-ordinate system.
5. Transform member stiffness matrix from local to global co-ordinate system.
6. Assemble member stiffness matrices to obtain the global stiffness matrix.
7. Analyse plane truss by the direct stiffness matrix.

24.1 Introduction

An introduction to the stiffness method was given in the previous chapter. The basic principles involved in the analysis of beams, trusses were discussed. The problems were solved with hand computation by the direct application of the basic principles. The procedure discussed in the previous chapter though enlightening are not suitable for computer programming. It is necessary to keep hand computation to a minimum while implementing this procedure on the computer. In this chapter a formal approach has been discussed which may be readily programmed on a computer. In this lesson the direct stiffness method as applied to planar truss structure is discussed.

Plane trusses are made up of short thin members interconnected at hinges to form triangulated patterns. A hinge connection can only transmit forces from one member to another member but not the moment. For analysis purpose, the truss is loaded at the joints. Hence, a truss member is subjected to only axial forces and the forces remain constant along the length of the member. The forces in the member at its two ends must be of the same magnitude but act in the opposite directions for equilibrium as shown in Fig. 24.1.

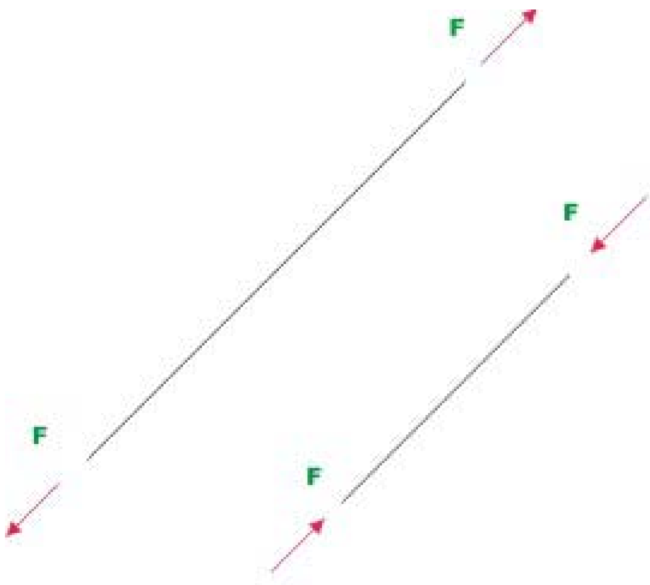


Fig 24.1 Truss member in equilibrium

Now consider a truss member having cross sectional area A, Young's modulus of material E, and length of the member L. Let the member be subjected to axial tensile force F as shown in Fig. 24.2. Under the action of constant axial force F, applied at each end, the member gets elongated by u as shown in Fig. 24.2.

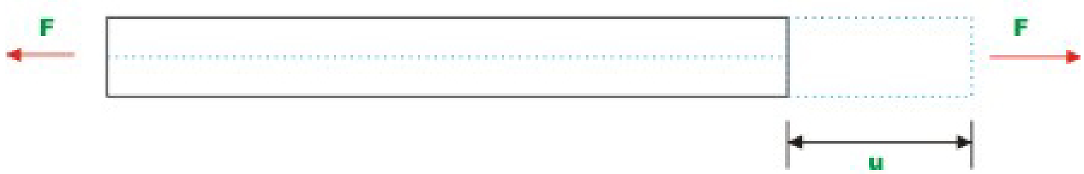


Fig 24.2 Force - displacement relationship

The elongation u may be calculated by (vide lesson 2, module 1).

$$u = \frac{FL}{AE} \quad (24.1)$$

Now the force-displacement relation for the truss member may be written as,

$$F = \frac{AE}{L} u \quad (24.2)$$

$$F = ku \quad (24.3)$$

where $k = \dfrac{AE}{L}$ is the stiffness of the truss member and is defined as the force required for unit deformation of the structure. The above relation (24.3) is true along the centroidal axis of the truss member. But in reality there are many members in a truss. For example consider a planer truss shown in Fig. 24.3. For each member of the truss we could write one equation of the type $F = ku$ along its axial direction (which is called as local co-ordinate system). Each member has different local co ordinate system. To analyse the planer truss shown in Fig. 24.3, it is required to write force-displacement relation for the complete truss in a co ordinate system common to all members. Such a co-ordinate system is referred to as global co ordinate system.

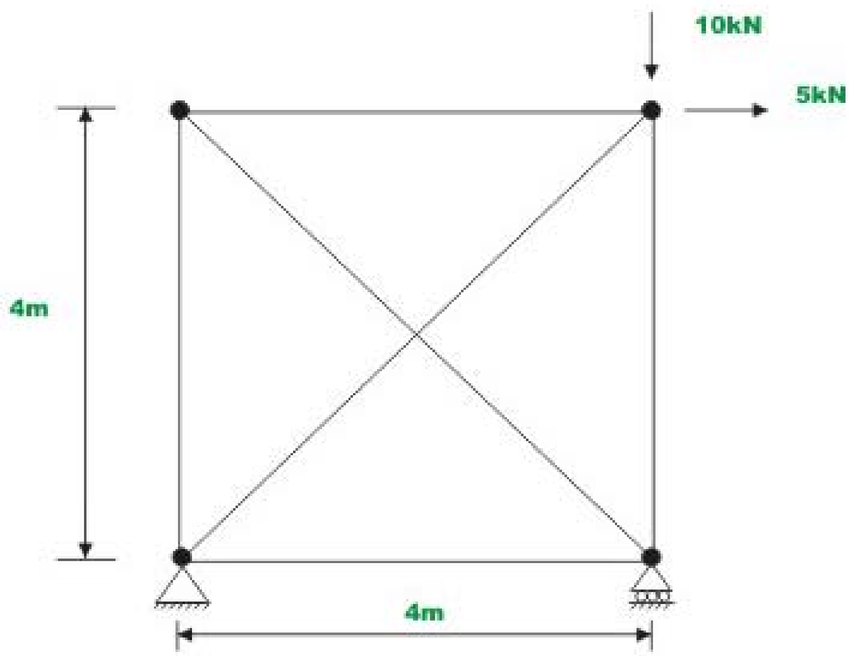


Fig 24.3 Plane truss

24.2 Local and Global Co-ordinate System

Loads and displacements are vector quantities and hence a proper coordinate system is required to specify their correct sense of direction. Consider a planar truss as shown in Fig. 24.4. In this truss each node is identified by a number and each member is identified by a number enclosed in a circle. The displacements and loads acting on the truss are defined with respect to global co-ordinate system xyz. The same co ordinate system is used to define each of the loads and displacements of all loads. In a global co-ordinate system, each node of a planer truss can have only two displacements: one along x-axis and another along y-axis. The truss shown in figure has eight displacements. Each displacement (degree of freedom) in a truss is shown by a number in the figure at the joint. The direction of the displacements is shown by an arrow at the node. However out of eight displacements, five are unknown. The displacements indicated by numbers 6,7 and 8 are zero due to support conditions. The displacements denoted by numbers 1-5 are known as unconstrained degrees of freedom of the truss and displacements denoted by 6-8 represent constrained degrees of freedom. In this course, unknown displacements are denoted by lower numbers and the known displacements are denoted by higher code numbers.

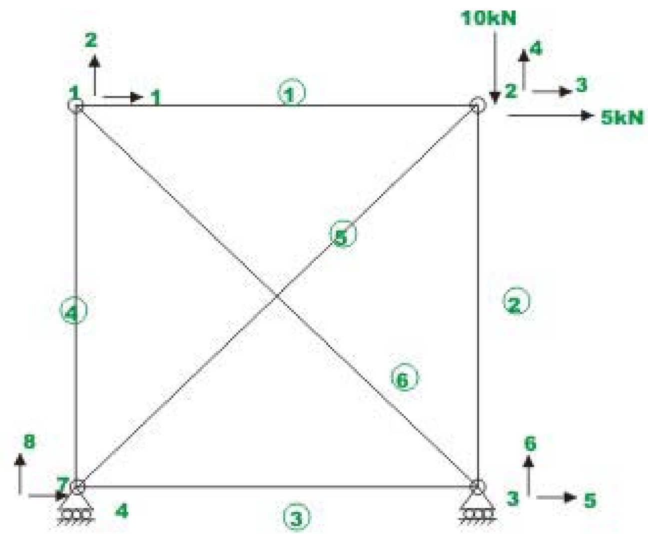


Fig 24.4 Node and members numbering

To analyse the truss shown in Fig. 24.4, the structural stiffness matrix K need to be evaluated for the given truss. This may be achieved by suitably adding all the member stiffness matrices k', which is used to express the force-displacement relation of the member in local co-ordinate system. Since all members are oriented at different directions, it is required to transform member displacements and forces from the local co-ordinate system to global co-ordinate system so that a global load-displacement relation may be written for the complete truss.

24.3 Member Stiffness Matrix

Consider a member of the truss as shown in Fig. 24.5a in local co-ordinate system $x'y'$. As the loads are applied along the centroidal axis, only possible displacements will be along x'-axis. Let the u'_1 and u'_2 be the displacements of truss members in local co-ordinate system *i.e.* along x'-axis. Here subscript 1 refers to node 1 of the truss member and subscript 2 refers to node 2 of the truss member. Give displacement u'_1 at node 1 of the member in the positive x' direction, keeping all other displacements to zero. This displacement in turn induces a compressive force of magnitude $\frac{EA}{L}u'_1$ in the member. Thus, $q'_1 = \frac{EA}{L}u'_1$ and $q'_2 = -\frac{EA}{L}u'_1$ (24.4a) ($-ve$ as it acts in the $-ve$ direction for equilibrium). Similarly by giving positive displacements of u'_2 at end 2 of the member, tensile force of magnitude $\frac{EA}{L}u'_2$ is induced in the member. Thus,

$$q''_1 = -\frac{EA}{L}u'_2 \text{ and } q''_2 = \frac{EA}{L}u'_2 \tag{24.4b}$$

Now the forces developed at the ends of the member when both the displacements are imposed at nodes 1 and 2 respectively may be obtained by method of superposition. Thus (vide Fig. 24.5d)

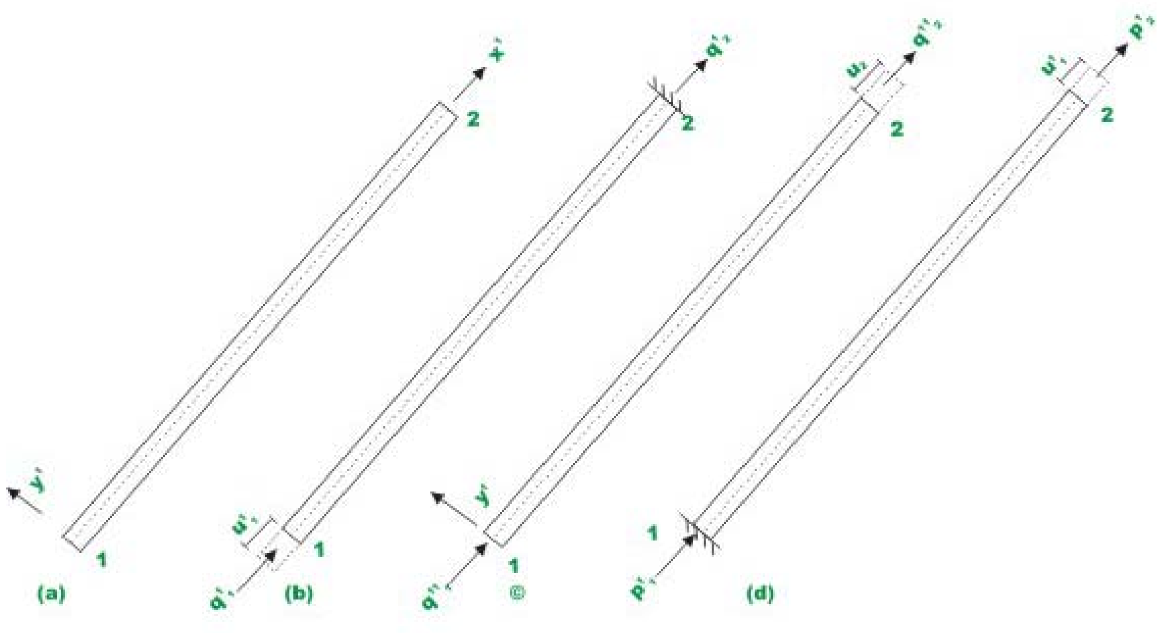


Fig 24.5 Force displacement reaction in load co-ordinate

$$p'_1 = \frac{EA}{L}u'_1 - \frac{EA}{L}u'_2 \tag{24.5a}$$

$$p'_2 = \frac{EA}{L}u'_2 - \frac{EA}{L}u'_1 \tag{24.5b}$$

Or we can write

$$\begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} \tag{24.6a}$$

$$\{p'\} = [k']\{u'\} \tag{24.6b}$$

Thus the member stiffness matrix is

$$k' = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{24.7}$$

This may also be obtained by giving unit displacement at node 1 and holding displacement at node 2 to zero and calculating forces developed at two ends. This will generate the first column of stiffness matrix. Similarly the second column of stiffness matrix is obtained by giving unit displacement at 2 and holding displacement at node 1 to zero and calculating the forces developed at both ends.

24.4 Transformation from Local to Global Co-ordinate System.

Displacement Transformation Matrix

A truss member is shown in local and global co ordinate system in Fig. 24.6. Let $x'y'z'$ be in local co ordinate system and xyz be the global co ordinate system.

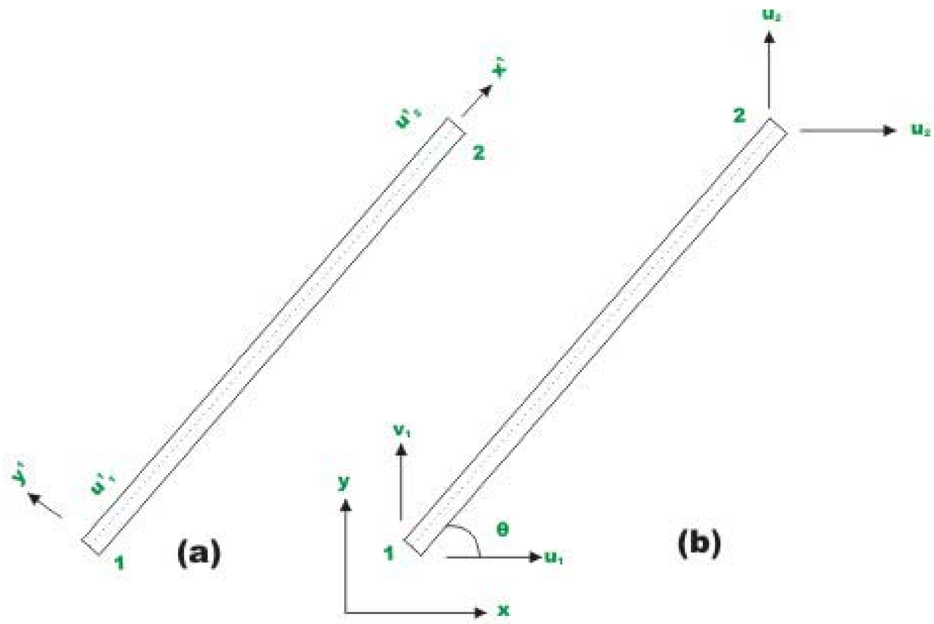


Fig 24.6 Truss element (a) local co-ordinate system (b) global co-ordinate system

The nodes of the truss member be identified by 1 and 2. Let u'_1 and u'_2 be the displacement of nodes 1 and 2 in local co ordinate system. In global co ordinate system, each node has two degrees of freedom. Thus, u_1, v_1 and u_2, v_2 are the nodal displacements at nodes 1 and 2 respectively along x- and y- directions. Let the truss member be inclined to x axis by θ as shown in figure. It is observed from the figure that u'_1 is equal to the projection of u_1 on x' axis plus projection of v_1 on x'-axis. Thus, (vide Fig. 24.7)

$$u'_1 = u_1 \cos\theta + v_1 \sin\theta \tag{24.8a}$$

$$u'_2 = u_2 \cos\theta + v_2 \sin\theta \tag{24.8b}$$

This may be written as

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \tag{24.9}$$

Introducing direction cosines $l = \cos\theta\ ; m = \sin\theta$; the above equation is written as

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \tag{24.10a}$$

Or, $$\{u'\} = [T]\,\{u\} \tag{24.10b}$$

In the above equation $[T]$ is the displacement transformation matrix which transforms the four global displacement components to two displacement component in local coordinate system.

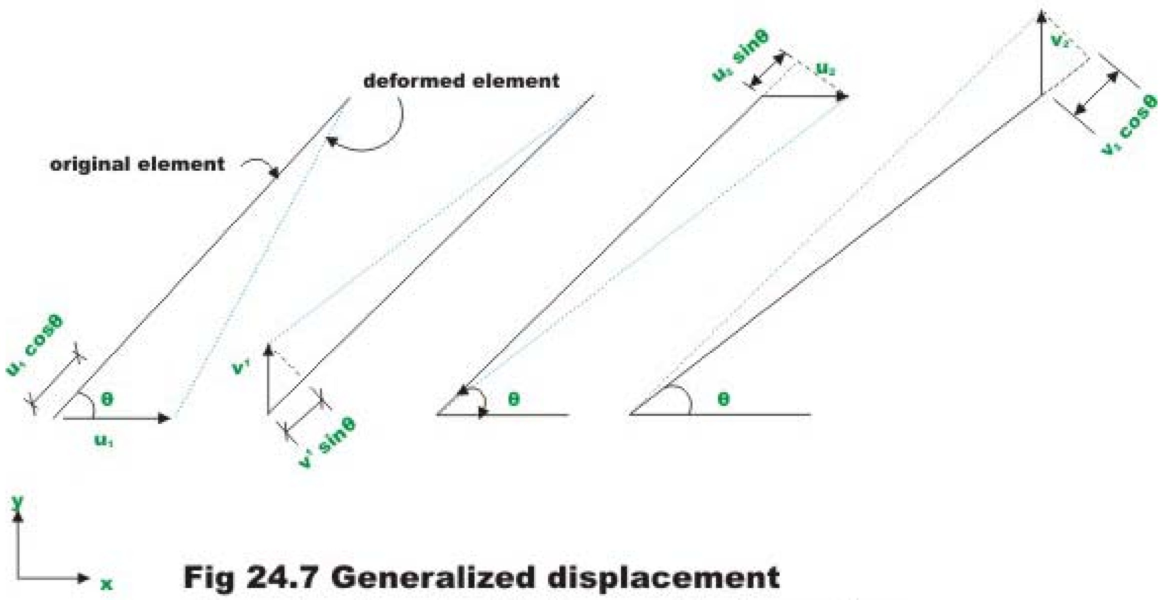


Fig 24.7 Generalized displacement along local and global co-ordiante system

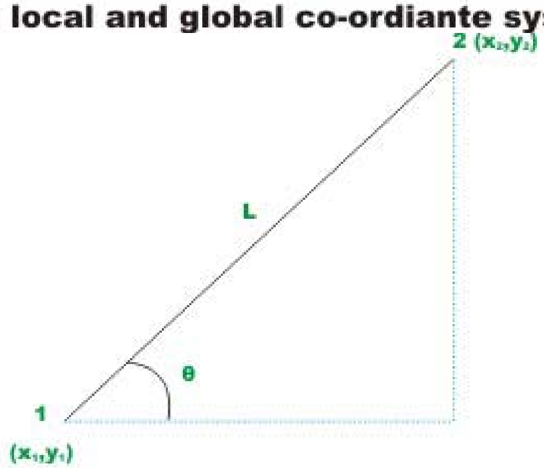


Fig 24.8 A typical truss member

Let co-ordinates of node 1 be (x_1, y_1) and node 2 be (x_2, y_2). Now from Fig. 24.8,

$$l = \cos\theta = \frac{x_2 - x_1}{L} \tag{24.11a}$$

$$m = \sin\theta = \frac{y_2 - y_1}{L} \tag{24.11b}$$

and $$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{24.11c}$$

Force transformation matrix

Let p'_1, p'_2 be the forces in a truss member at node 1 and 2 respectively producing displacements u'_1 and u'_2 in the local co-ordinate system and p_1, p_2, p_3, p_4 be the force in global co-ordinate system at node 1 and 2 respectively producing displacements u_1, v_1 and u_2, v_2 (refer Fig. 24.9a-d).

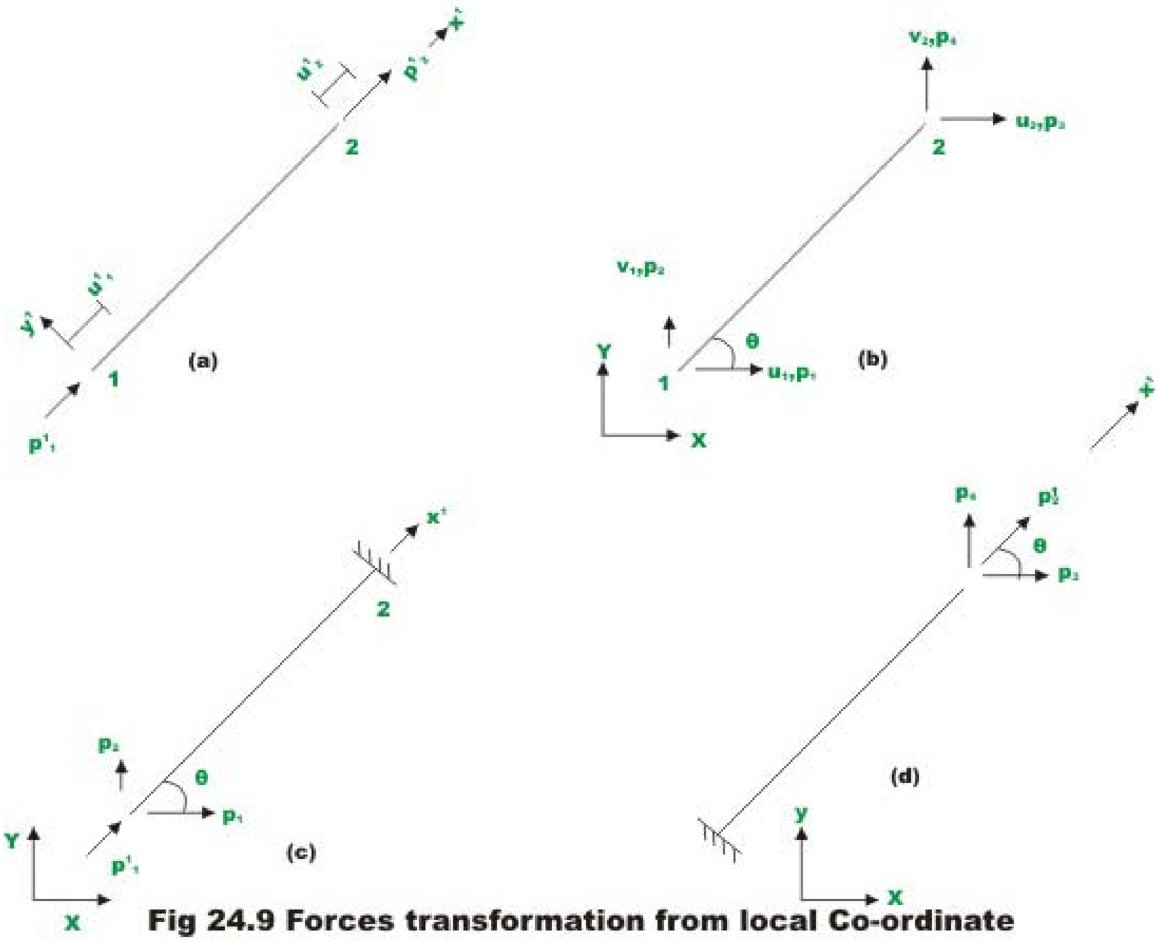


Fig 24.9 Forces transformation from local Co-ordinate system to global Co-ordinate system

Referring to fig. 24.9c, the relation between p'_1 and p_1, may be written as,

$$p_1 = p'_1 \cos\theta \tag{24.12a}$$

$$p_2 = p'_1 \sin\theta \tag{24.12b}$$

Similarly referring to Fig. 24.9d, yields

$$p_3 = p'_2 \cos\theta \tag{24.12c}$$

$$p_4 = p'_2 \sin\theta \tag{24.12d}$$

Now the relation between forces in the global and local co-ordinate system may be written as

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} \tag{24.13}$$

$$\{p\} = [T]^T \{p'\} \tag{24.14}$$

where matrix $\{p\}$ stands for global components of force and matrix $\{p'\}$ are the components of forces in the local co-ordinate system. The superscript T stands for the transpose of the matrix. The equation (24.14) transforms the forces in the local co-ordinate system to the forces in global co-ordinate system. This is accomplished by force transformation matrix $[T]^T$. Force transformation matrix is the transpose of displacement transformation matrix.

Member Global Stiffness Matrix

From equation (24.6b) we have,

$$\{p'\} = [k'] \{u'\}$$

Substituting for $\{p'\}$ in equation (24.14), we get

$$\{p\} = [T]^T [k'] \{u'\} \tag{24.15}$$

Making use of the equation (24.10b), the above equation may be written as

$$\{p\} = [T]^T [k'][T]\{u\} \tag{24.16}$$

$$\{p\} = [k]\{u\} \tag{24.17}$$

Equation (24.17) represents the member load displacement relation in global co-ordinates and thus $[k]$ is the member global stiffness matrix. Thus,

$$\{k\} = [T]^T [k'][T] \tag{24.18}$$

$$[k] = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$[k] = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \tag{24.19}$$

Each component k_{ij} of the member stiffness matrix $[k]$ in global co-ordinates represents the force in x-or y-directions at the end i required to cause a unit displacement along x– or y–directions at end j.

We obtained the member stiffness matrix in the global co-ordinates by transforming the member stiffness matrix in the local co-ordinates. The member stiffness matrix in global co-ordinates can also be derived from basic principles in a direct method. Now give a unit displacement along x-direction at node 1 of the truss member. Due to this unit displacement (see Fig. 24.10) the member length gets changed in the axial direction by an amount equal to $\Delta l_1 = \cos\theta$. This axial change in length is related to the force in the member in two axial directions by

$$F_{1'2'} = \frac{EA}{L}\cos\theta \tag{24.20a}$$

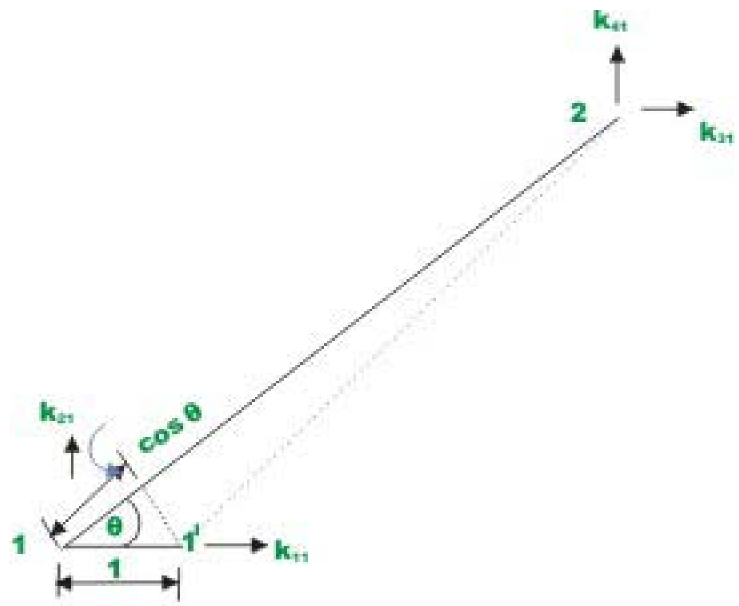


Fig 24.10

This force may be resolved along u_1 and v_1 directions. Thus horizontal component of force $F_{1'2'}$ is $k_{11} = \frac{EA}{L}\cos^2\theta$ (24.20b)

Vertical component of force $F_{1'2'}$ is $k_{21} = \frac{EA}{L}\cos\theta\sin\theta$ (24.20c)

The forces at the node 2 are readily found from static equilibrium. Thus,

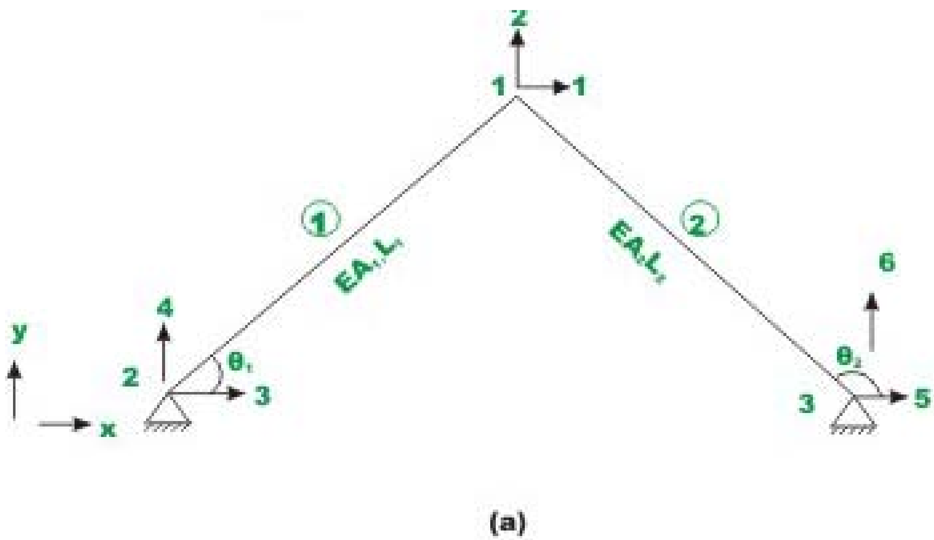
$$k_{31} = -k_{11} = -\frac{EA}{L}\cos^2\theta \qquad (24.20d)$$

$$k_{41} = -k_{21} = \frac{EA}{L}\cos\theta\sin\theta \qquad (24.20e)$$

The above four stiffness coefficients constitute the first column of a stiffness matrix in the global co-ordinate system. Similarly, remaining columns of the stiffness matrix may be obtained.

24.5 Analysis of plane truss.

Number all the joints and members of a plane truss. Also indicate the degrees of freedom at each node. In a plane truss at each node, we can have two displacements. Denote unknown displacements by lower numbers and known displacements by higher numbers as shown in Fig. 24.4. In the next step evaluate member stiffness matrix of all the members in the global co ordinate system. Assemble all the stiffness matrices in a particular order, the stiffness matrix K for the entire truss is found. The assembling procedure is best explained by considering a simple example. For this purpose consider a two member truss as shown in Fig. 24.11. In the figure, joint numbers, member numbers and possible displacements of the joints are shown.



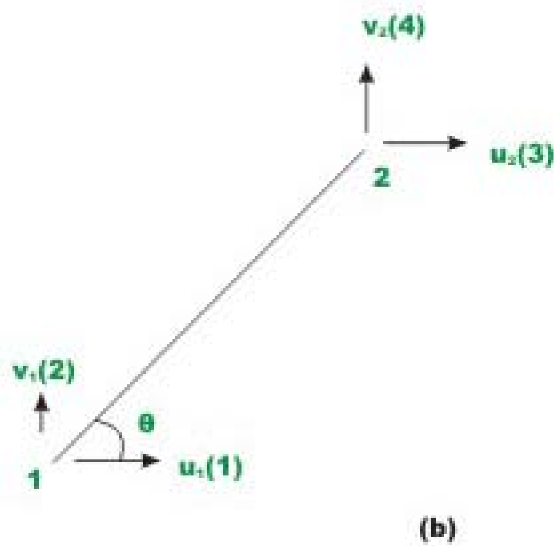


Fig 24.11 Analysis of plane - truss

The area of cross-section of the members, its length and its inclination with the x- axis are also shown. Now the member stiffness matrix in the global co-ordinate system for both the members are given by

$$\begin{array}{c|cccc} Global & 3 & 4 & 1 & 2 \\ Memb & 1 & 2 & 3 & 4 \end{array}$$

$$\left[k^1\right] = \frac{EA_1}{L_1}\begin{bmatrix} l_1^2 & l_1 m_1 & -l_1^2 & -l_1 m_1 \\ l_1 m_1 & m_1^2 & -l_1 m_1 & -m_1^2 \\ -l_1^2 & -l_1 m_1 & l_1^2 & l_1 m_1 \\ -l_1 m_1 & -m_1^2 & l_1 m_1 & m_1^2 \end{bmatrix} \qquad (24.21a)$$

On the member stiffness matrix the corresponding member degrees of freedom and global degrees of freedom are also shown.

$$\left[k^2\right] = \frac{EA_2}{L_2}\begin{bmatrix} l_2^2 & l_2 m_2 & -l_2^2 & -l_2 m_2 \\ l_2 m_2 & m_2^2 & -l_2 m_2 & -m_2^2 \\ -l_2^2 & -l_2 m_2 & l_2^2 & l_2 m_2 \\ -l_2 m_2 & -m_2^2 & l_2 m_2 & m_2^2 \end{bmatrix} \qquad (24.21b)$$

Note that the member stiffness matrix in global co-ordinate system is derived referring to Fig. 24.11b. The node 1 and node 2 remain same for all the members. However in the truss, for member 1, the same node ($i.e.$ node 1 and 2 in Fig. 24.11b) are referred by 2 and 1 respectively. Similarly for member 2, the nodes 1 and 2 are referred by nodes 3 and 4 in the truss. The member stiffness matrix is of the order 4×4. However the truss has six possible displacements and hence truss stiffness matrix is of the order 6×6. Now it is required to put elements of the member stiffness matrix of the entire truss. The stiffness matrix of the entire truss is known as assembled stiffness matrix. It is also known as structure stiffness matrix; as overall stiffness matrix. Thus, it is clear that by algebraically adding the above two stiffness matrix we get global stiffness matrix. For example the element $k^1{}_{11}$ of the member stiffness matrix of member 1 must go to location $(3,3)$ in the global stiffness matrix. Similarly $k^2{}_{11}$ must go to location $(3,3)$ in the global stiffness matrix. The above procedure may be symbolically written as,

$$K = \sum_{i=0}^{n} k^i \qquad (24.22)$$

$$= \frac{EA_1}{L_1}\begin{bmatrix} l_1^2 & l_1 m_1 & -l_1^2 & -l_1 m_1 \\ l_1 m_1 & m_1^2 & -l_1 m_1 & -m_1^2 \\ -l_1^2 & -l_1 m_1 & l_1^2 & l_1 m_1 \\ -l_1 m_1 & -m_1^2 & l_1 m_1 & m_1^2 \end{bmatrix} + \frac{EA_2}{L_2}\begin{bmatrix} l_2^2 & l_2 m_2 & -l_2^2 & -l_2 m_2 \\ l_2 m_2 & m_2^2 & -l_2 m_2 & -m_2^2 \\ -l_2^2 & -l_2 m_2 & l_2^2 & l_2 m_2 \\ -l_2 m_2 & -m_2^2 & l_2 m_2 & m_2^2 \end{bmatrix}$$

(24.23a)

The assembled stiffness matrix is of the order 6×6. Hence, it is easy to visualize assembly if we expand the member stiffness matrix to 6×6 size. The missing columns and rows in matrices k^1 and k^2 are filled with zeroes. Thus,

$$K = \frac{EA_1}{L_1}\begin{bmatrix} l_1^{\;2} & l_1 m_1 & -l_1^{\;2} & -l_1 m_1 & 0 & 0 \\ l_1 m_1 & m_1^{\;2} & -l_1 m_1 & -m_1^{\;2} & 0 & 0 \\ -l_1^{\;2} & -l_1 m_1 & l_1^{\;2} & l_1 m_1 & 0 & 0 \\ -l_1 m_1 & -m_1^{\;2} & l_1 m_1 & m_1^{\;2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+\frac{EA_2}{L_2}\begin{bmatrix} l_2^{\;2} & l_2 m_2 & -l_2^{\;2} & -l_2 m_2 & 0 & 0 \\ l_2 m_2 & m_2^{\;2} & -l_2 m_2 & -m_2^{\;2} & 0 & 0 \\ -l_2^{\;2} & -l_2 m_2 & l_2^{\;2} & l_2 m_2 & 0 & 0 \\ -l_2 m_2 & -m_2^{\;2} & l_2 m_2 & m_2^{\;2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(24.24)

Adding appropriate elements of first matrix with the appropriate elements of the second matrix,

$$K = \begin{bmatrix} \frac{EA_1}{L_1} l_1^2 + \frac{EA_2}{L_2} l_2^2 & \frac{EA_1}{L_1} l_1 m_1 + \frac{EA_2}{L_2} l_2 m_2 & -\frac{EA_1}{L_1} l_1^2 & -\frac{EA_1}{L_1} l_1 m_1 & -\frac{EA_2}{L_2} l_2^2 & -\frac{EA_2}{L_2} l_2 m_2 \\ \frac{EA_1}{L_1} l_1 m_1 + \frac{EA_2}{L_2} l_2 m_2 & \frac{EA_1}{L_1} m_1^2 + \frac{EA_2}{L_2} m_2^2 & -\frac{EA_1}{L_1} l_1 m_1 & -\frac{EA_1}{L_1} m_1^2 & -\frac{EA_2}{L_2} l_2 m_2 & -\frac{EA_2}{L_2} m_2^2 \\ -\frac{EA_1}{L_1} l_1^2 & -\frac{EA_1}{L_1} l_1 m_1 & \frac{EA_1}{L_1} l_1^2 & \frac{EA_1}{L_1} l_1 m_1 & 0 & 0 \\ -\frac{EA_1}{L_1} l_1 m_1 & -\frac{EA_1}{L_1} m_1^2 & \frac{EA_1}{L_1} l_1 m_1 & \frac{EA_1}{L_1} m_1^2 & 0 & 0 \\ -\frac{EA_2}{L_2} l_2^2 & -\frac{EA_2}{L_2} l_2 m_2 & 0 & 0 & \frac{EA_2}{L_2} l_2^2 & \frac{EA_2}{L_2} l_2 m_2 \\ -\frac{EA_2}{L_2} l_2 m_2 & -\frac{EA_2}{L_2} m_2^2 & 0 & 0 & \frac{EA_2}{L_2} l_2 m_2 & \frac{EA_2}{L_2} m_2^2 \end{bmatrix}$$

If more than one member meet at a joint then the stiffness coefficients of member stiffness matrix corresponding to that joint are added.

After evaluating global stiffness matrix of the truss, the load displacement equation for the truss is written as,

$$\{p\} = [K] \quad \{u\} \tag{24.26}$$

where $\{p\}$ is the vector of joint loads acting on the truss, $\{u\}$ is the vector of joint displacements and $[K]$ is the global stiffness matrix. The above equation is known as the equilibrium equation. It is observed that some joint loads are known and some are unknown. Also some displacements are known due to support conditions and some displacements are unknown. Hence the above equation may be partitioned and written as,

$$\begin{Bmatrix} \{p_k\} \\ \{p_u\} \end{Bmatrix} = \begin{bmatrix} [k_{11}] & [k_{12}] \\ [k_{21}] & [k_{22}] \end{bmatrix} \begin{Bmatrix} \{u_u\} \\ \{u_k\} \end{Bmatrix} \tag{24.27}$$

where $\{p_k\},\{u_k\}$ denote vector of known forces and known displacements respectively. And $\{p_u\},\{u_u\}$ denote vector of unknown forces and unknown displacements respectively.

Expanding equation 24.27,

$$\{p_k\} = [k_{11}]\{u_u\} + [k_{12}]\{u_k\} \tag{24.28a}$$

$$\{p_u\} = [k_{21}]\{u_u\} + [k_{22}]\{u_k\} \qquad (24.28b)$$

In the present case (vide Fig. 24.11a) the known displacements are u_3, u_4, u_5 and u_6. The known displacements are zero due to boundary conditions. Thus,

$\{u_k\} = \{0\}$. And from equation (24.28a),

$$\{p_k\} = [k_{11}]\{u_u\} \qquad (24.29)$$

Solving $\{u_u\} = [k_{11}]^{-1}\{p_k\}$

where $[k_{11}]$ corresponding to stiffness matrix of the truss corresponding to unconstrained degrees of freedom. Now the support reactions are evaluated from equation (24.28b).

$$\{p_u\} = [k_{21}]\{u_u\} \qquad (24.30)$$

The member forces are evaluated as follows. Substituting equation (24.10b) $\{u'\} = [T]\{u\}$ in equation (24.6b) $\{p'\} = [k']\{u'\}$, one obtains

$$\{p'\} = [k'][T]\{u\} \qquad (24.31)$$

Expanding this equation,

$$\begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \qquad (24.32)$$

Example 24.1

Analyse the two member truss shown in Fig. 24.12a. Assume EA to be constant for all members. The length of each member is $5m$.

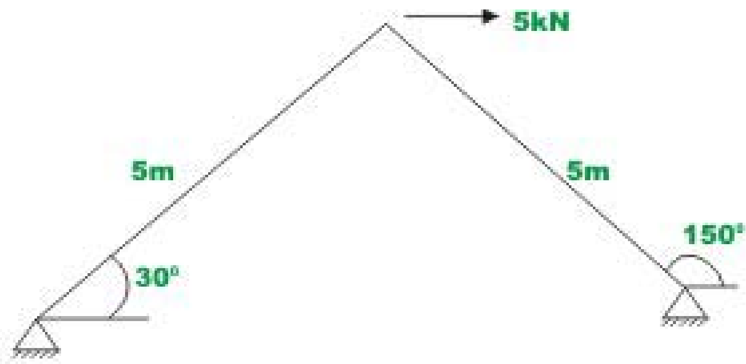


Fig 24.12(a) Example 24.1

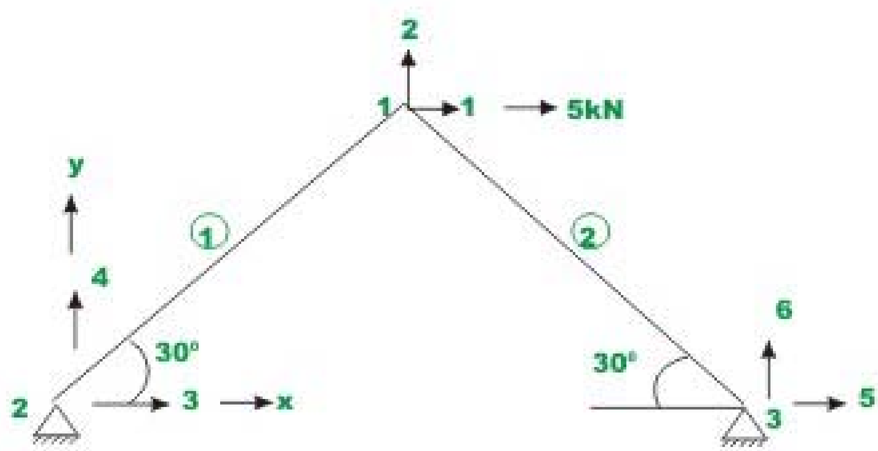


Fig 24.12(b) Members and node numbering

The co-ordinate axes, the number of nodes and members are shown in Fig.24.12b. The degrees of freedom at each node are also shown. By inspection it is clear that the displacement $u_3 = u_4 = u_5 = u_6 = 0$. Also the external loads are

$$p_1 = 5 \quad \text{kN} \quad ; p_2 = 0 \quad \text{kN}. \tag{1}$$

Now member stiffness matrix for each member in global co-ordinate system is $(\theta_1 = 30°)$.

$$[k^1] = \frac{EA}{5}\begin{bmatrix} 0.75 & 0.433 & -0.75 & -0.433 \\ 0.433 & 0.25 & -0.433 & -0.25 \\ -0.75 & -0.433 & 0.75 & 0.433 \\ -0.433 & -0.25 & 0.433 & 0.25 \end{bmatrix} \tag{2}$$

$$[k^2] = \frac{EA}{5}\begin{bmatrix} 0.75 & -0.433 & -0.75 & 0.433 \\ -0.433 & 0.25 & 0.433 & -0.25 \\ -0.75 & 0.433 & 0.75 & -0.433 \\ 0.433 & -0.25 & -0.433 & 0.25 \end{bmatrix} \tag{3}$$

The global stiffness matrix of the truss can be obtained by assembling the two member stiffness matrices. Thus,

$$[K] = \frac{EA}{5}\left[\begin{array}{cc|cccc} 1.5 & 0 & -0.75 & -0.433 & -0.75 & 0.433 \\ 0 & 0.5 & -0.433 & -0.25 & 0.433 & -0.25 \\ \hline -0.75 & -0.433 & 0.75 & 0.433 & 0 & 0 \\ -0.433 & -0.25 & 0.433 & 0.25 & 0 & 0 \\ -0.75 & 0.433 & 0 & 0 & 0.75 & -0.433 \\ 0.433 & -0.25 & 0 & 0 & -0.433 & 0.25 \end{array}\right] \tag{4}$$

Again stiffness matrix for the unconstrained degrees of freedom is,

$$[K] = \frac{EA}{5}\begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix} \tag{5}$$

Writing the load displacement-relation for the truss for the unconstrained degrees of freedom

$$\{p_k\} = [k_{11}]\{u_u\} \tag{6}$$

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \frac{EA}{5}\begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{7}$$

$$\begin{Bmatrix} 5 \\ 0 \end{Bmatrix} = \frac{EA}{5}\begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u_1 = \frac{16.667}{EA} \quad ; u_2 = 0 \tag{8}$$

Support reactions are evaluated using equation (24.30).

$$\{p_u\} = [k_{21}]\{u_u\} \tag{9}$$

Substituting appropriate values in equation (9),

$$\{p_u\} = \frac{EA}{5}\begin{bmatrix} -0.75 & -0.433 \\ -0.433 & -0.25 \\ -0.75 & 0.433 \\ 0.433 & -0.25 \end{bmatrix}\frac{1}{AE}\begin{Bmatrix} 16.667 \\ 0 \end{Bmatrix} \tag{10}$$

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{pmatrix} -2.5 \\ -1.443 \\ -2.5 \\ 1.443 \end{pmatrix} \tag{11}$$

The answer can be verified by equilibrium of joint 1. Also,

$$p_3 + p_5 + 5 = 0$$

Now force in each member is calculated as follows,

Member 1: $l = 0.866$; $m = 0.5$; $L = 5\,m$.

$$\{p'\} = [k']\{u'\}$$

$$= [k'][T]\{u\}$$

$$\begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}\begin{Bmatrix} u_3 \\ v_4 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{L}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} u_3 \\ v_4 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{L}[-0.866]\left\{\frac{16.667}{AE}\right\} = -2.88 \quad \text{kN}$$

Member 2: $l = -0.866$; $m = 0.5$; $L = 5m$.

$$\begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}\begin{Bmatrix} u_5 \\ v_6 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{L}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} u_3 \\ v_4 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{5}[-0.866]\left\{\frac{16.667}{AE}\right\} = -2.88 \quad \text{kN}$$

Summary

The member stiffness matrix of a truss member in local co-ordinate system is defined. Suitable transformation matrices are derived to transform displacements and forces from the local to global co-ordinate system. The member stiffness matrix of truss member is obtained in global co-ordinate system by suitable transformation. The system stiffness matrix of a plane truss is obtained by assembling member matrices of individual members in global co-ordinate system. In the end, a few plane truss problems are solved using the direct stiffness matrix approach.

Module 4

Analysis of Statically Indeterminate Structures by the Direct Stiffness Method

Lesson 25

The Direct Stiffness Method: Truss Analysis (Continued)

Instructional Objectives

After reading this chapter the student will be able to
1. Transform member stiffness matrix from local to global co-ordinate system.
2. Assemble member stiffness matrices to obtain the global stiffness matrix.
3. Analyse plane truss by the direct stiffness matrix.
4. Analyse plane truss supported on inclined roller supports.

25.1 Introduction

In the previous lesson, the direct stiffness method as applied to trusses was discussed. The transformation of force and displacement from local co-ordinate system to global co-ordinate system were accomplished by single transformation matrix. Also assembly of the member stiffness matrices was discussed. In this lesson few plane trusses are analysed using the direct stiffness method. Also the problem of inclined support will be discussed.

Example 25.1

Analyse the truss shown in Fig. 25.1a and evaluate reactions. Assume EA to be constant for all the members.

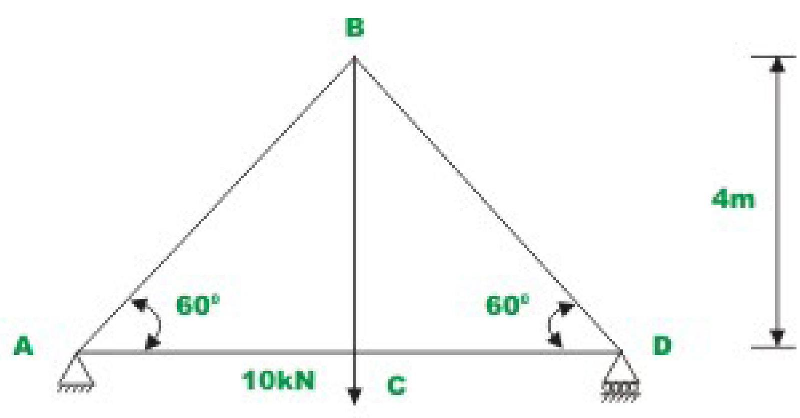


Fig.25.1(a) Plane truss of Example 25.1

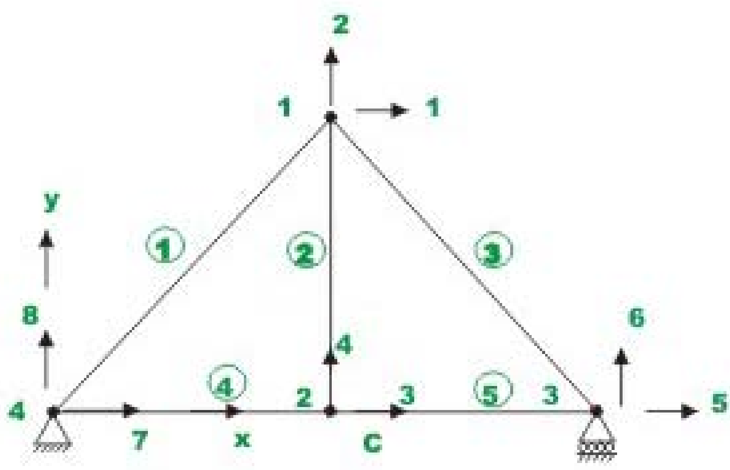


Fig. 25.1(b) Node and member numbering

The numbering of joints and members are shown in Fig. 25.1b. Also, the possible displacements (degrees of freedom) at each node are indicated. Here lower numbers are used to indicate unconstrained degrees of freedom and higher numbers are used for constrained degrees of freedom. Thus displacements 6,7 and 8 are zero due to boundary conditions.

First write down stiffness matrix of each member in global co-ordinate system and assemble them to obtain global stiffness matrix.

Element 1: $\theta = 60°, \quad L = 4.619\ m.$ Nodal points 4-1

$$[k^1] = \frac{EA}{4.619}\begin{bmatrix} 0.25 & 0.433 & -0.25 & -0.433 \\ 0.433 & 0.75 & -0.433 & -0.75 \\ -0.25 & -0.433 & 0.25 & 0.433 \\ -0.433 & -0.75 & 0.433 & 0.75 \end{bmatrix} \quad (1)$$

Element 2: $\theta = 90°, \quad L = 4.00\ m.$ Nodal points 2-1

$$\left[k^2\right]=\frac{EA}{4.0}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\end{bmatrix} \qquad (2)$$

Element 3: $\theta = 120°, \quad L = 4.619\ m.$ Nodal points 3-1

$$\left[k^3\right]=\frac{EA}{4.619}\begin{bmatrix}0.25 & -0.433 & -0.25 & 0.433\\ -0.433 & 0.75 & 0.433 & -0.75\\ -0.25 & 0.433 & 0.25 & -0.433\\ 0.433 & -0.75 & -0.433 & 0.75\end{bmatrix} \qquad (3)$$

Element 4: $\theta = 0°, \quad L = 2.31\ m.$ Nodal points 4-2

$$\left[k^4\right]=\frac{EA}{2.31}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad (4)$$

Element 5: $\theta = 0°, \quad L = 2.31\ m.$ Nodal points 2-3

$$\left[k^5\right]=\frac{EA}{2.31}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad (5)$$

The assembled global stiffness matrix of the truss is of the order 8×8. Now assemble the global stiffness matrix. Note that the element k_{11}^1 of the member stiffness matrix of truss member 1 goes to location $(7,7)$ of global stiffness matrix. On the member stiffness matrix the corresponding global degrees of freedom are indicated to facilitate assembling. Thus,

$$
[K] = EA \begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\left[\begin{array}{ccccc|ccc}
0.108 & 0 & 0 & 0 & -0.054 & 0.094 & -0.054 & -0.094 \\
0 & 0.575 & 0 & -0.25 & 0.094 & -0.162 & -0.094 & -0.162 \\
0 & 0 & 0.866 & 0 & -0.433 & 0 & -0.433 & 0 \\
0 & -0.25 & 0 & 0.25 & 0 & 0 & 0 & 0 \\
-0.054 & 0.094 & -0.433 & 0 & 0.487 & -0.094 & 0 & 0 \\
\hline
0.094 & -0.162 & 0 & 0 & -0.094 & 0.162 & 0 & 0 \\
-0.054 & -0.094 & -0.433 & 0 & 0 & 0 & 0.487 & 0.0934 \\
-0.094 & -0.162 & 0 & 0 & 0 & 0 & 0.0934 & 0.162
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\tag{6}
$$

Writing the load-displacement relation for the truss, yields

$$
\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = EA
\left[\begin{array}{ccccc|ccc}
0.108 & 0 & 0 & 0 & -0.054 & 0.094 & -0.054 & -0.094 \\
0 & 0.575 & 0 & -0.25 & 0.094 & -0.162 & -0.094 & -0.162 \\
0 & 0 & 0.866 & 0 & -0.433 & 0 & -0.433 & 0 \\
0 & -0.25 & 0 & 0.25 & 0 & 0 & 0 & 0 \\
-0.054 & 0.094 & -0.433 & 0 & 0.487 & -0.094 & 0 & 0 \\
\hline
0.094 & -0.162 & 0 & 0 & -0.094 & 0.162 & 0 & 0 \\
-0.054 & -0.094 & -0.433 & 0 & 0 & 0 & 0.487 & 0.0934 \\
-0.094 & -0.162 & 0 & 0 & 0 & 0 & 0.0934 & 0.162
\end{array}\right]
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}
\tag{7}
$$

The displacements u_1 to u_5 are unknown. The displacements $u_6 = u_7 = u_8 = 0$.

Also $p_1 = p_2 = p_3 = p_5 = 0$. But $p_4 = -10 \text{ kN}$.

$$
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -10 \\ 0 \end{Bmatrix} = EA
\begin{bmatrix}
0.108 & 0 & 0 & 0 & -0.054 \\
0 & 0.575 & 0 & -0.25 & 0.094 \\
0 & 0 & 0.866 & 0 & -0.433 \\
0 & -0.25 & 0 & 0.25 & 0 \\
-0.054 & 0.094 & -0.433 & 0 & 0.487
\end{bmatrix}
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}
\tag{8}
$$

Solving which, the unknown displacements are evaluated. Thus,

$$u_1 = \frac{6.668}{AE}\ ;\ u_2 = \frac{-34.64}{AE};\ u_3 = \frac{6.668}{AE};\ u_4 = \frac{-74.642}{AE};\ u_5 = \frac{13.334}{AE} \qquad (9)$$

Now reactions are evaluated from equation,

$$\begin{Bmatrix} p_6 \\ p_7 \\ p_8 \end{Bmatrix} = EA \begin{bmatrix} 0.094 & -0.162 & 0 & 0 & -0.094 \\ -0.054 & -0.094 & -0.433 & 0 & 0 \\ -0.094 & -0.162 & 0 & 0 & 0 \end{bmatrix} \frac{1}{EA} \begin{Bmatrix} 6.668 \\ -34.64 \\ 6.668 \\ -74.642 \\ 13.334 \end{Bmatrix} \qquad (10)$$

Thus,

$$p_6 = 5.00 \quad \text{kN} \quad ; \quad p_7 = \ 0 \ ; \quad p_8 = 5.00 \quad \text{kN}\ . \qquad (11)$$

Now calculate individual member forces.

Member 1: $l = 0.50\ ;\ m = 0.866\ ; L = 4.619\, m$.

$$\{p'_1\} = \frac{AE}{4.619}\begin{bmatrix} l & m & -l & -m \end{bmatrix} \begin{Bmatrix} u_7 \\ u_8 \\ u_1 \\ u_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{4.619}\begin{bmatrix} -0.5 & -0.866 \end{bmatrix} \frac{1}{AE} \begin{Bmatrix} 6.667 \\ -34.64 \end{Bmatrix} = 5.77 \quad \text{kN} \qquad (12)$$

Member 2: $l = 0\ ;\ m = 1.0\ ; L = 4.0\, m$.

$$\{p'_1\} = \frac{AE}{4.0}\begin{bmatrix} l & m & -l & -m \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{4.619}\begin{bmatrix} 1 & -1 \end{bmatrix} \frac{1}{AE} \begin{Bmatrix} -74.642 \\ -34.64 \end{Bmatrix} = -10.0 \quad \text{kN} \qquad (13)$$

Member 3: $l = -0.50\ ;\ m = 0.866\ ; L = 4.619\, m$.

$$\{p'_1\} = \frac{AE}{4.619}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{4.619}\begin{bmatrix} -0.5 & 0.5 & -0.866 \end{bmatrix}\frac{1}{AE}\begin{Bmatrix} 13.334 \\ 6.667 \\ -34.64 \end{Bmatrix} = 5.77 \quad \text{kN} \tag{14}$$

Member 4: $l = 1.0$; $m = 0$; $L = 2.31.0\,m$.

$$\{p'_1\} = \frac{AE}{2.31}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} u_7 \\ u_8 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{2.31}\begin{bmatrix} 1 & -1 \end{bmatrix}\frac{1}{AE}\begin{Bmatrix} 0 \\ 6.667 \end{Bmatrix} = -2.88 \quad \text{kN} \tag{15}$$

Member 5: $l = 1.0$; $m = 0$; $L = 2.31.0\,m$.

$$\{p'_1\} = \frac{AE}{2.31}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{2.31}\begin{bmatrix} 1 & -1 \end{bmatrix}\frac{1}{AE}\begin{Bmatrix} 6.667 \\ 13.334 \end{Bmatrix} = -2.88 \quad \text{kN} \tag{16}$$

Example 25.2

Determine the forces in the truss shown in Fig. 25.2a by the direct stiffness method. Assume that all members have the same axial rigidity.

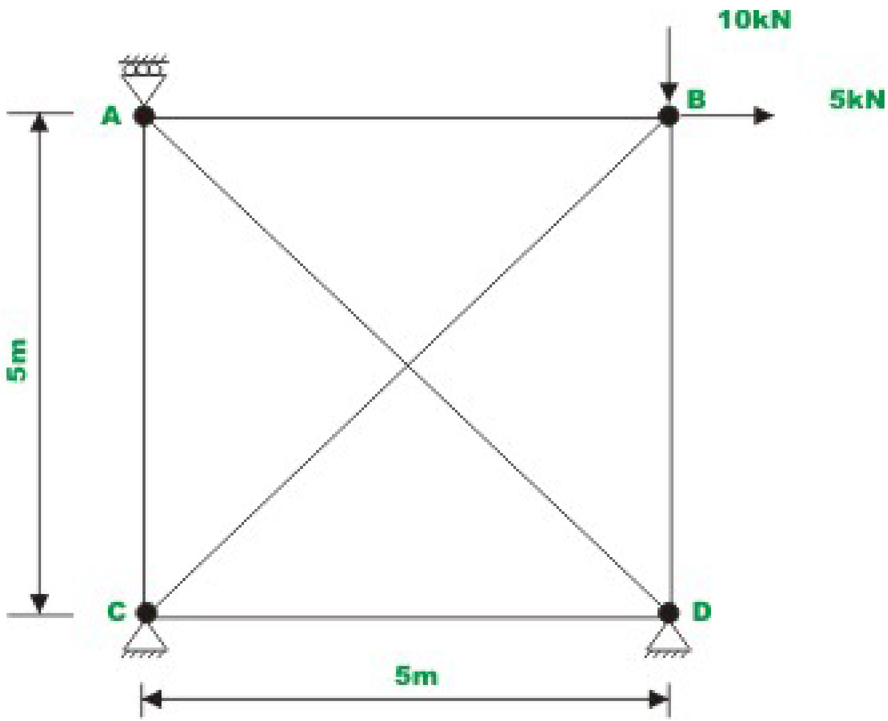


Fig. 25.2a Example 25.2

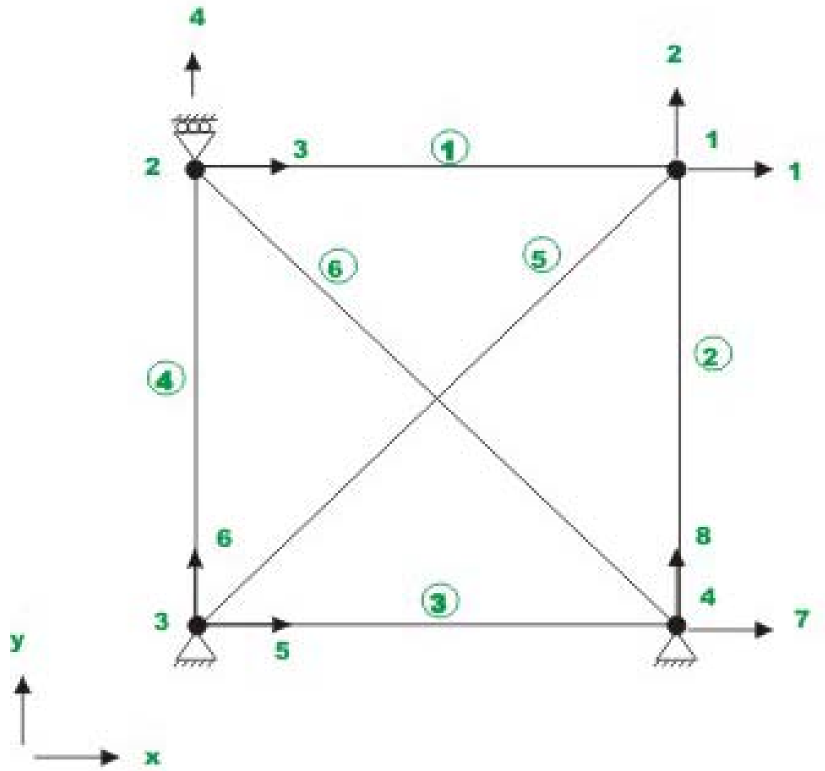


Fig.25.2b Node and member numbering

The joint and member numbers are indicated in Fig. 25.2b. The possible degree of freedom are also shown in Fig. 25.2b. In the given problem u_1, u_2 and u_3 represent unconstrained degrees of freedom and $u_4 = u_5 = u_6 = u_7 = u_8 = 0$ due to boundary condition. First let us generate stiffness matrix for each of the six members in global co-ordinate system.

Element 1: $\theta = 0°, \quad L = 5.00\ m.$ Nodal points 2-1

$$[k^1] = \frac{EA}{5.0} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array} \quad (1)$$

Element 2: $\theta = 90°, \quad L = 5.00\ m.$ Nodal points 4-1

$$
\begin{array}{c} \\ \left[k^{2}\right]=\dfrac{EA}{5.0} \end{array}
\begin{array}{cccc|c} 7 & 8 & 1 & 2 & \\ \hline \end{array}
$$

$$
\left[k^{2}\right]=\frac{EA}{5.0}\begin{array}{c} \begin{array}{cccc} 7 & 8 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 7 \\ 8 \\ 1 \\ 2 \end{array} \qquad (2)
$$

Element 3: $\theta = 0°, \quad L = 5.00\ m.$ Nodal points 3-4

$$
\left[k^{3}\right]=\frac{EA}{5.0}\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \qquad (3)
$$

Element 4: $\theta = 90°, \quad L = 5.00\ m.$ Nodal points 3-2

$$
\left[k^{4}\right]=\frac{EA}{5.0}\begin{array}{c} \begin{array}{cccc} 5 & 6 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 3 \\ 4 \end{array} \qquad (4)
$$

Element 5: $\theta = 45°, \quad L = 7.07\ m.$ Nodal points 3-1

$$
\left[k^{5}\right]=\frac{EA}{7.07}\begin{array}{c} \begin{array}{cccc} 5 & 6 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 1 \\ 2 \end{array} \qquad (5)
$$

Element 6: $\theta = 135°, \quad L = 7.07\ m.$ Nodal points 4-2

$$
\begin{array}{c}
\\
[k^6] = \dfrac{EA}{7.07}
\end{array}
\begin{array}{c}
\begin{array}{cccc} \quad 7 & \quad\quad 8 & \quad\quad 3 & \quad\quad 4 \end{array} \\
\begin{bmatrix}
0.5 & -0.5 & -0.5 & 0.5 \\
-0.5 & 0.5 & 0.5 & -0.5 \\
-0.5 & 0.5 & 0.5 & -0.5 \\
0.5 & -0.5 & -0.5 & 0.5
\end{bmatrix}
\end{array}
\begin{array}{c} \\ 7 \\ 8 \\ 3 \\ 4 \end{array}
\tag{6}
$$

There are eight possible global degrees of freedom for the truss shown in the figure. Hence the global stiffness matrix is of the order (8×8). On the member stiffness matrix, the corresponding global degrees of freedom are indicated to facilitate assembly. Thus the global stiffness matrix is,

$$
[K] = AE \left[\begin{array}{ccc|ccccc}
0.271 & 0.071 & -0.2 & 0 & -0.071 & -0.071 & 0 & 0 \\
0.071 & 0.271 & 0 & 0 & -0.071 & -0.071 & 0 & -0.20 \\
-0.20 & 0 & 0.271 & -0.071 & 0 & 0 & -0.071 & 0.071 \\
\hline
0 & 0 & -0.071 & 0.271 & 0.071 & -0.20 & 0.071 & -0.071 \\
-0.071 & -0.071 & 0 & 0 & 0.271 & 0.071 & -0.20 & 0 \\
-0.071 & -0.071 & 0 & -0.20 & 0.071 & 0.271 & 0 & 0 \\
0 & 0 & -0.071 & 0.071 & -0.20 & 0 & 0.271 & -0.071 \\
0 & -0.20 & 0.071 & -0.071 & 0 & 0 & -0.071 & 0.271
\end{array}\right] \tag{7}
$$

The force-displacement relation for the truss is,

$$
\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix}
= EA \left[\begin{array}{ccc|ccccc}
0.271 & 0.071 & -0.2 & 0 & -0.071 & -0.071 & 0 & 0 \\
0.071 & 0.271 & 0 & 0 & -0.071 & -0.071 & 0 & -0.20 \\
-0.20 & 0 & 0.271 & -0.071 & 0 & 0 & -0.071 & 0.071 \\
\hline
0 & 0 & -0.071 & 0.271 & 0.071 & -0.20 & 0.071 & -0.071 \\
-0.071 & -0.071 & 0 & 0 & 0.271 & 0.071 & -0.20 & 0 \\
-0.071 & -0.071 & 0 & -0.20 & 0.071 & 0.271 & 0 & 0 \\
0 & 0 & -0.071 & 0.071 & -0.20 & 0 & 0.271 & -0.071 \\
0 & -0.20 & 0.071 & -0.071 & 0 & 0 & -0.071 & 0.271
\end{array}\right]
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} \tag{8}
$$

The displacements u_1, u_2 and u_3 are unknowns.

Here, $p_1 = 5 \quad \text{kN} \quad ; p_2 = -10 \quad ; p_3 = 0$ and $u_4 = u_5 = u_6 = u_7 = u_8 = 0$.

$$\begin{Bmatrix} 5 \\ -10 \\ 0 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = EA \left[\begin{array}{ccc:ccccc} 0.271 & 0.071 & -0.2 & 0 & -0.071 & -0.071 & 0 & 0 \\ 0.071 & 0.271 & 0 & 0 & -0.071 & -0.071 & 0 & -0.20 \\ -0.20 & 0 & 0.271 & -0.071 & 0 & 0 & -0.071 & 0.071 \\ \hdashline 0 & 0 & -0.071 & 0.271 & 0.071 & -0.20 & 0.071 & -0.071 \\ -0.071 & -0.071 & 0 & 0 & 0.271 & 0.071 & -0.20 & 0 \\ -0.071 & -0.071 & 0 & -0.20 & 0.071 & 0.271 & 0 & 0 \\ 0 & 0 & -0.071 & 0.071 & -0.20 & 0 & 0.271 & -0.071 \\ 0 & -0.20 & 0.071 & -0.071 & 0 & 0 & -0.071 & 0.271 \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{9}$$

Thus,

$$\begin{Bmatrix} 5 \\ -10 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0.271 & 0.071 & -0.20 \\ 0.071 & 0.271 & 0 \\ -0.20 & 0 & 0.271 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{10}$$

Solving which, yields

$$u_1 = \frac{72.855}{AE}\ ;\ u_2 = \frac{-55.97}{AE};\ u_3 = \frac{53.825}{AE}$$

Now reactions are evaluated from the equation,

$$\begin{Bmatrix} p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & -0.071 \\ -0.071 & -0.071 & 0 \\ -0.071 & -0.071 & 0 \\ 0 & 0 & -0.071 \\ 0 & -0.20 & 0.071 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{11}$$

$p_4 = -3.80$ kN ; $p_5 = -1.19$ kN ; $p_6 = -1.19$ kN ; $p_7 = 3.8\ 0\ kN$; $p_8 = 15.00$ kN

In the next step evaluate forces in members.

Element 1: $\theta = 0°, \quad L = 5.00\ m.$ Nodal points 2-1

$$\{p'_1\} = \frac{AE}{5.0}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{5.0}\begin{bmatrix} 1 & -1 \end{bmatrix}\frac{1}{AE}\begin{Bmatrix} 53.825 \\ 72.855 \end{Bmatrix} = -3.80 \ \ \text{kN} \tag{12}$$

Element 2: $\theta = 90°, \quad L = 5.00 \ m.$ Nodal points 4-1

$$\{p'_1\} = \frac{AE}{5}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} u_7 \\ u_8 \\ u_1 \\ u_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{5}\begin{bmatrix} 1 & -1 \end{bmatrix}\frac{1}{AE}\begin{Bmatrix} 0 \\ -55.97 \end{Bmatrix} = 11.19\text{kN} \tag{13}$$

Element 3: $\theta = 0°, \quad L = 5.00 \ m.$ Nodal points 3-4

$$\{p'_1\} = \frac{AE}{5}[0]\{0\} = 0 \tag{14}$$

Element 4: $\theta = 90°, \quad L = 5.00 \ m.$ Nodal points 3-2

$$\{p'_1\} = \frac{AE}{5}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} u_5 \\ u_6 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{5}[0]\frac{1}{AE}\{53.825\} = 0 \tag{15}$$

Element 5: $\theta = 45°, \quad L = 7.07 \ m.$ Nodal points 3-1

$$\{p'_1\} = \frac{AE}{7.07}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{7.07}[-0.707 \quad -0.707]\frac{1}{AE}\begin{Bmatrix} 72.855 \\ -55.97 \end{Bmatrix} = -1.688 \quad \text{kN} \tag{16}$$

Element 6: $\theta = 135°, \quad L = 7.07 \; m.$ Nodal points 4-2

$$\{p'_1\} = \frac{AE}{7.07}[l \quad m \quad -l \quad -m]\begin{Bmatrix} u_7 \\ u_8 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{7.07}[0.707]\frac{1}{AE}\{53.825\} = 5.38 \quad \text{kN} \tag{17}$$

25.2 Inclined supports

Sometimes the truss is supported on a roller placed on an oblique plane (vide Fig. 25.3a). At a roller support, the displacement perpendicular to roller support is zero. $i.e.$ displacement along y'' is zero in the present case.

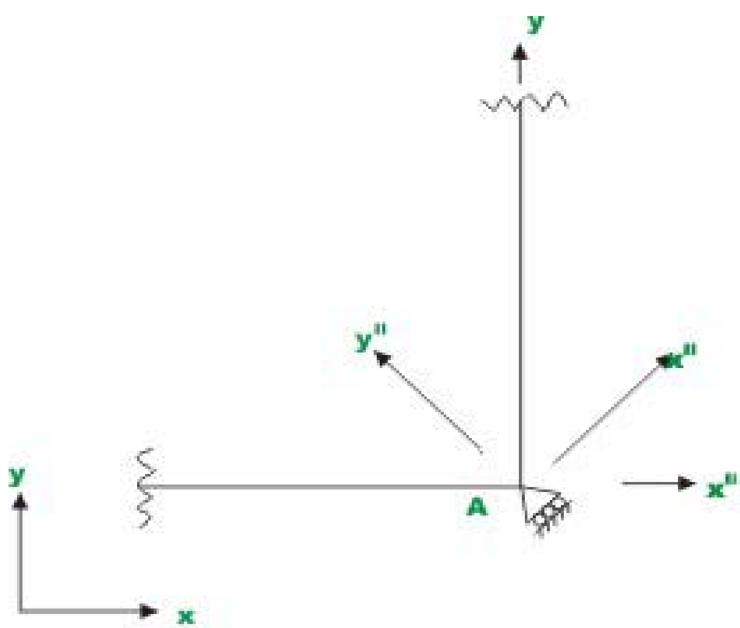


Fig.25.3(a) Inclined support

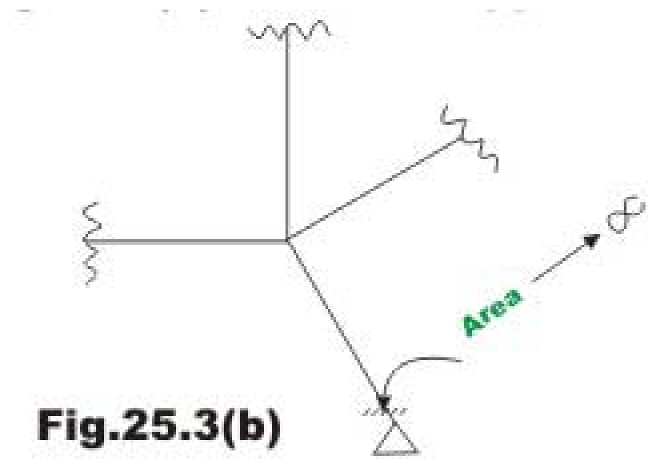


Fig.25.3(b)

If the stiffness matrix of the entire truss is formulated in global co-ordinate system then the displacements along y are not zero at the oblique support. So, a special procedure has to be adopted for incorporating the inclined support in the analysis of truss just described. One way to handle inclined support is to replace the inclined support by a member having large cross sectional area as shown in Fig. 25.3b but having the length comparable with other members meeting at that joint. The inclined member is so placed that its centroidal axis is perpendicular to the inclined plane. Since the area of cross section of this new member is very high, it does not allow any displacement along its centroidal axis of the joint A. Another method of incorporating inclined support in the analysis is to suitably modify the member stiffness matrix of all the members meeting at the inclined support.

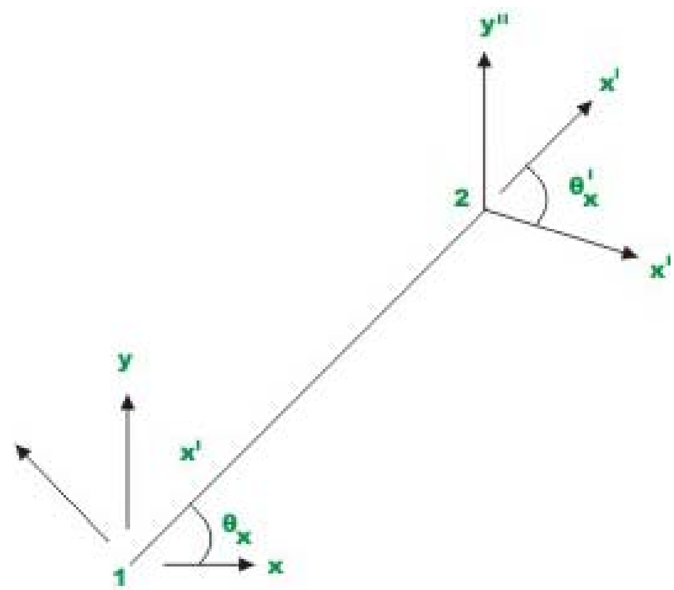


Fig.25.4 Truss member in global and local co-ordinate system

Consider a truss member as shown in Fig. 25.4. The nodes are numbered as 1 and 2. At 2, it is connected to a inclined support. Let $x'y'$ be the local co-ordinate axes of the member. At node 1, the global co-ordinate system xy is also shown. At node 2, consider nodal co-ordinate system as $x''y''$, where y'' is perpendicular to oblique support. Let u'_1 and u'_2 be the displacements of nodes 1 and 2 in the local co-ordinate system. Let u_1, v_1 be the nodal displacements of node 1 in global co-ordinate system xy. Let u''_2, v''_2 be the nodal displacements along x''- and y''- are in the local co-ordinate system $x''y''$ at node 2. Then from Fig. 25.4,

$$u'_1 = u_1 \cos\theta_x + v_1 \sin\theta_x$$

$$u'_2 = u''_2 \cos\theta_{x''} + v''_2 \sin\theta_{x''} \tag{25.1}$$

This may be written as

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta_x & \sin\theta_x & 0 & 0 \\ 0 & 0 & \cos\theta_{x''} & \sin\theta_{x''} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u''_2 \\ v''_2 \end{Bmatrix} \tag{25.2}$$

Denoting $l = \cos\theta_x \ ; m = \sin\theta_x \ ; l'' = \cos\theta_{x''} \ ; m'' = \sin\theta_{x''}$

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l'' & m'' \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u''_2 \\ v''_2 \end{Bmatrix} \tag{25.3a}$$

or $\{u'\} = [T']\{u\}$

where $[T']$ is the displacement transformation matrix.

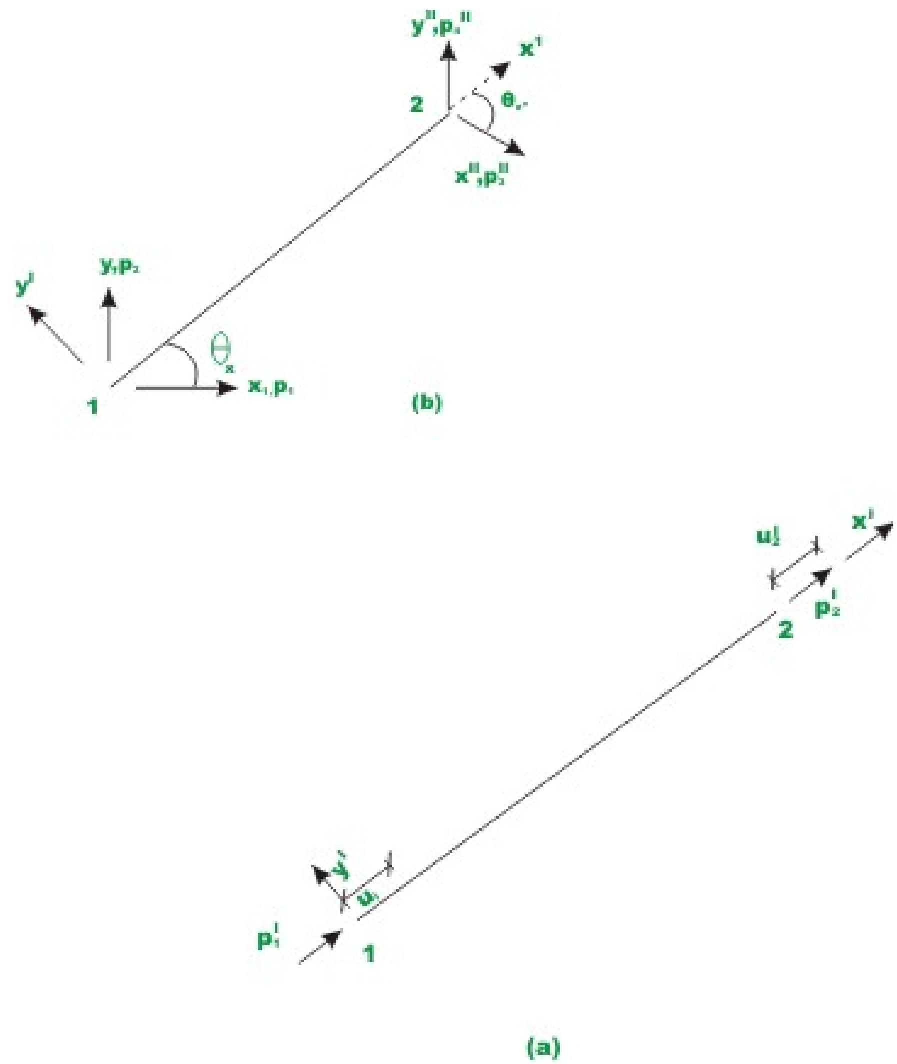


Fig.25.5 Displacement and force transformation

Similarly referring to Fig. 25.5, the force p'_1 has components along x and y axes. Hence

$$p_1 = p'_1 \cos\theta_x \tag{25.4a}$$

$$p_2 = p'_1 \sin\theta_x \tag{25.4b}$$

Similarly, at node 2, the force p'_2 has components along x'' and y'' axes.

$$p''_3 = p'_2 \cos\theta''_x \tag{25.5a}$$

$$p''_4 = p'_2 \sin\theta''_x \tag{25.5b}$$

The relation between forces in the global and local co-ordinate system may be written as,

$$\begin{Bmatrix} p_1 \\ p_2 \\ p''_3 \\ p''_4 \end{Bmatrix} = \begin{bmatrix} \cos\theta_x & 0 \\ \sin\theta_x & 0 \\ 0 & \cos\theta''_x \\ 0 & \sin\theta''_x \end{bmatrix} \begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} \tag{25.6}$$

$$\{p\} = [T']^T \{p'\} \tag{25.7}$$

Using displacement and force transformation matrices, the stiffness matrix for member having inclined support is obtained.

$$[k] = [T']^T [k'][T']$$

$$[k] = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l'' \\ 0 & m'' \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l'' & m'' \end{bmatrix} \tag{25.8}$$

Simplifying,

$$[k] = \frac{EA}{L} \begin{bmatrix} l^2 & lm & -ll'' & -lm'' \\ lm & m^2 & -ml'' & -mm'' \\ -ll'' & -ml'' & l''^2 & l''m'' \\ -lm'' & -mm'' & l''m'' & m''^2 \end{bmatrix} \tag{25.9}$$

If we use this stiffness matrix, then it is easy to incorporate the condition of zero displacement perpendicular to the inclined support in the stiffness matrix. This is shown by a simple example.

Example 25.3

Analyse the truss shown in Fig. 25.6a by stiffness method. Assume axial rigidity EA to be constant for all members.

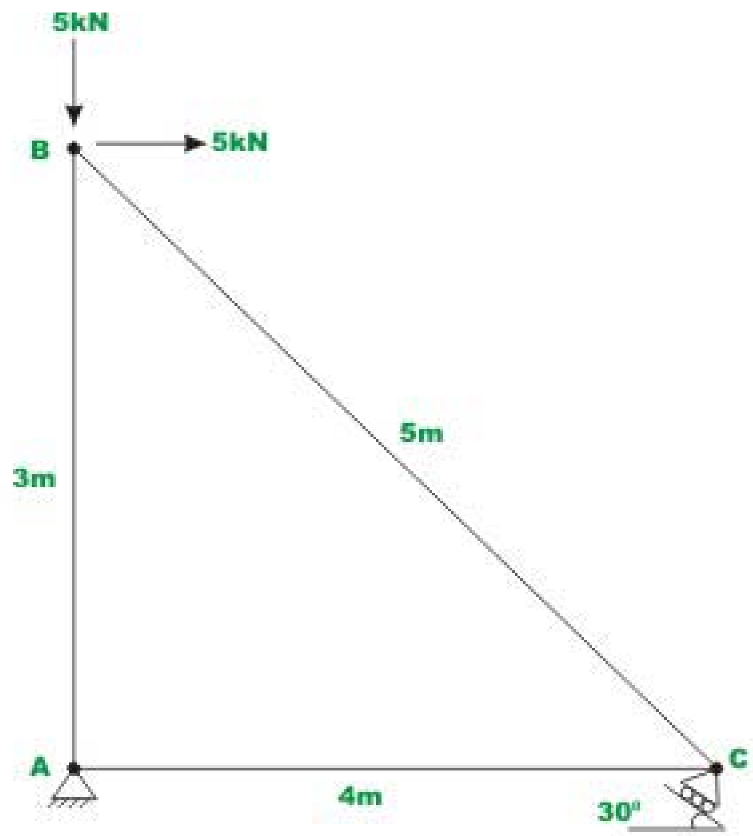


Fig.25.6(a) Plane truss with inclined support

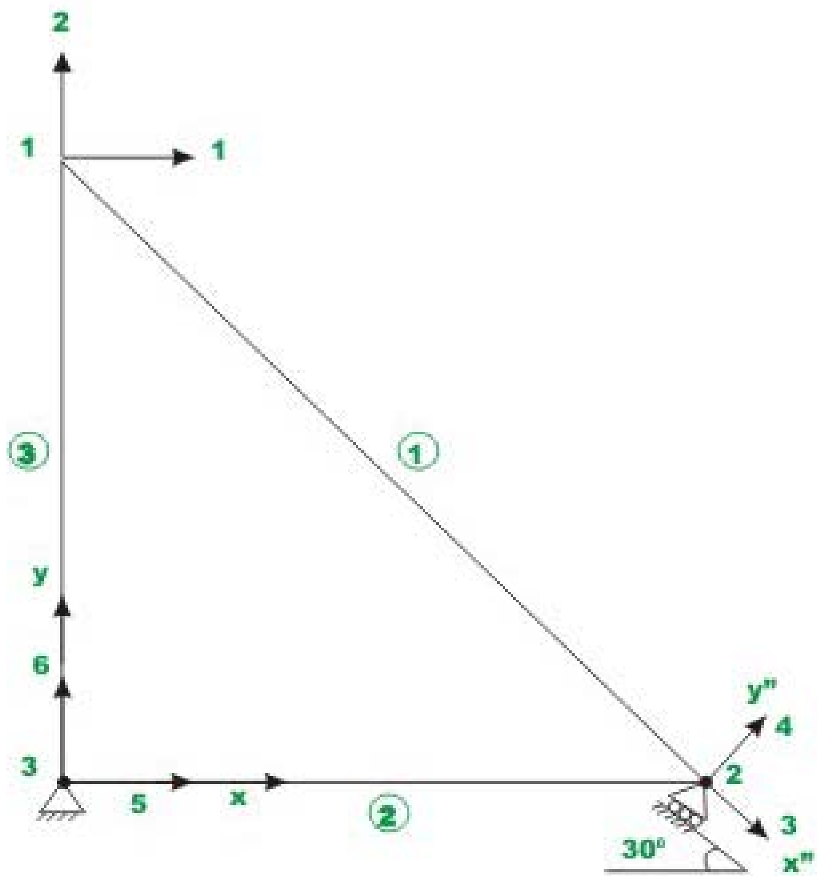


Fig. 25.6(b) Member and node numbering

The nodes and members are numbered in Fig. 25.6b. The global co-ordinate axes are shown at node 3. At node 2, roller is supported on inclined support. Hence it is required to use nodal co-ordinates $x''-y''$ at node 2 so that u_4 could be set to zero. All the possible displacement degrees of freedom are also shown in the figure. In the first step calculate member stiffness matrix.

Member 1: $\theta_x = 143.13°, \theta_{x''} = 6.87°, \quad L = 5.00 \ m.$ Nodal points 1-2

$l = -0.80 \ ; \quad m = 0.6 \ ; \quad l'' = 0.993 \ ; \quad m'' = 0.12$.

$$[k^1] = \frac{EA}{5.0} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.64 & -0.48 & 0.794 & 0.096 \\ -0.48 & 0.36 & -0.596 & -0.072 \\ 0.794 & -0.596 & 0.986 & 0.119 \\ 0.096 & -0.072 & 0.119 & 0.014 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad (1)$$

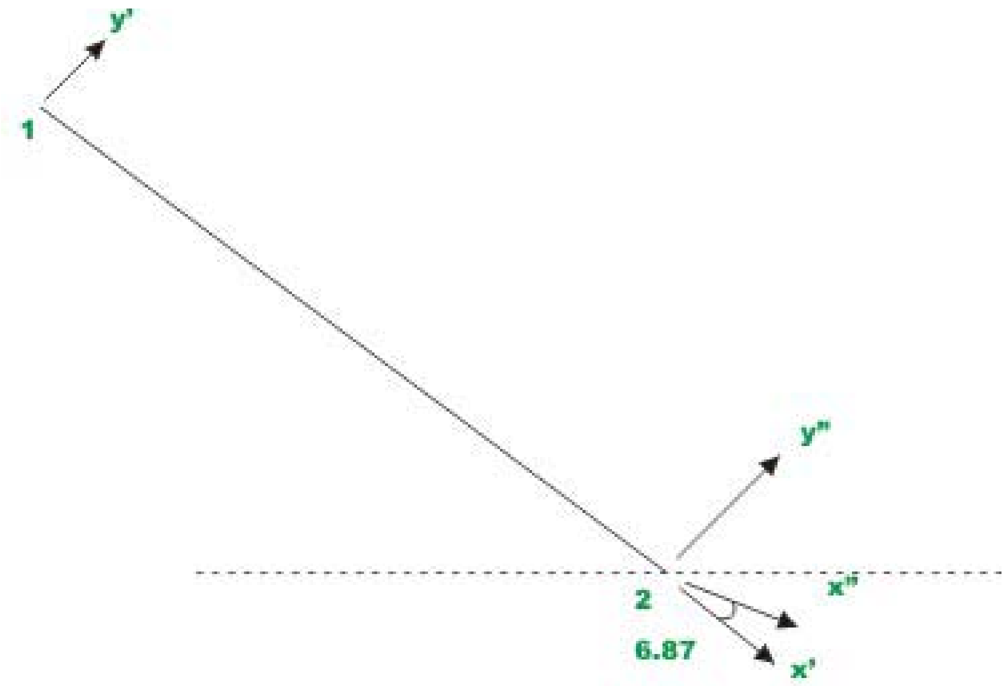


Fig.25.6c Member ①

Member 2: $\theta_x = 0°, \theta_{x''} = 30°, \quad L = 4.00\ m.$ Nodal points 2-3

$l = 1\ ; \quad m = 0 \quad ; \quad l'' = 0.866 \ ; \quad m'' = 0.50$.

Fig.25.6(d) Member ②

$$
\begin{array}{cccccc}
 & 5 & 6 & 3 & 4 & \\
\left[k^2\right]=\dfrac{EA}{4.0} & \multicolumn{4}{c}{\left[\begin{array}{cccc} 0.64 & -0.48 & 0.794 & 0.096 \\ -0.48 & 0.36 & -0.596 & -0.072 \\ 0.794 & -0.596 & 0.986 & 0.119 \\ 0.096 & -0.072 & 0.119 & 0.014 \end{array}\right]} & \begin{array}{c} 5 \\ 6 \\ 3 \\ 4 \end{array}
\end{array}
\tag{2}
$$

Member 3: $\theta_x = 90°, \quad L = 3.00\ m., l = 0\ ; \quad m = 1$ Nodal points 3-1

.

$$
\begin{array}{cccccc}
 & 5 & 6 & 1 & 2 & \\
\left[k^3\right]=\dfrac{EA}{3.0} & \multicolumn{4}{c}{\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right]} & \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array}
\end{array}
\tag{3}
$$

For the present problem, the global stiffness matrix is of the order (6×6). The global stiffness matrix for the entire truss is.

$$
\begin{array}{cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & \\
[k] = EA & \multicolumn{6}{c}{\left[\begin{array}{ccc:ccc} 0.128 & -0.096 & 0.159 & 0.019 & 0 & 0 \\ -0.096 & 0.405 & -0.119 & -0.014 & 0 & -0.333 \\ 0.159 & -0.119 & 0.385 & 0.132 & -0.217 & 0 \\ \hdashline 0.019 & .0.014 & 0.132 & 0.065 & -0.125 & 0 \\ 0 & 0 & -0.217 & -0.125 & 0.25 & 0 \\ 0 & -0.333 & 0 & 0 & 0 & 0.333 \end{array}\right]} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
\tag{4}
$$

Writing load-displacement equation for the truss for unconstrained degrees of freedom,

$$
\begin{Bmatrix} -5 \\ 5 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0.128 & -0.096 & 0.159 \\ -0.096 & 0.405 & -0.119 \\ 0.159 & -0.119 & 0.385 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}
\tag{5}
$$

Solving ,

$$
u_1 = \frac{-77.408}{AE}\ ;\ u_2 = \frac{3.728}{AE};\ u_3 = \frac{33.12}{AE}
\tag{6}
$$

Now reactions are evaluated from the equation

$$\begin{Bmatrix} p_4 \\ p_5 \\ p_6 \end{Bmatrix} = AE \begin{bmatrix} 0.019 & .0.014 & 0.132 \\ 0 & 0 & -0.217 \\ 0 & -0.333 & 0 \end{bmatrix} \frac{1}{AE} \begin{Bmatrix} -77.40 \\ 3.728 \\ 33.12 \end{Bmatrix} \qquad (6)$$

$$p_4 = 2.85 \quad \text{kN} \ ; \quad p_5 = -7.19 \quad \text{kN} \quad ; p_6 = -1.24 \quad \text{kN}$$

Summary

Sometimes the truss is supported on a roller placed on an oblique plane. In such situations, the direct stiffness method as discussed in the previous lesson needs to be properly modified to make the displacement perpendicular to the roller support as zero. In the present approach, the inclined support is handled in the analysis by suitably modifying the member stiffness matrices of all members meeting at the inclined support. A few problems are solved to illustrate the procedure.

Module 4

Analysis of Statically Indeterminate Structures by the Direct Stiffness Method

Lesson 26

The Direct Stiffness Method: Temperature Changes and Fabrication Errors in Truss Analysis

Instructional Objectives

After reading this chapter the student will be able to

1. Compute stresses developed in the truss members due to temperature changes.
2. Compute stresses developed in truss members due to fabrication members.
3. Compute reactions in plane truss due to temperature changes and fabrication errors.

26.1 Introduction

In the last four lessons, the direct stiffness method as applied to the truss analysis was discussed. Assembly of member stiffness matrices, imposition of boundary conditions, and the problem of inclined supports were discussed. Due to the change in temperature the truss members either expand or shrink. However, in the case of statically indeterminate trusses, the length of the members is prevented from either expansion or contraction. Thus, the stresses are developed in the members due to changes in temperature. Similarly the error in fabricating truss members also produces additional stresses in the trusses. Both these effects can be easily accounted for in the stiffness analysis.

26.2 Temperature Effects and Fabrication Errors

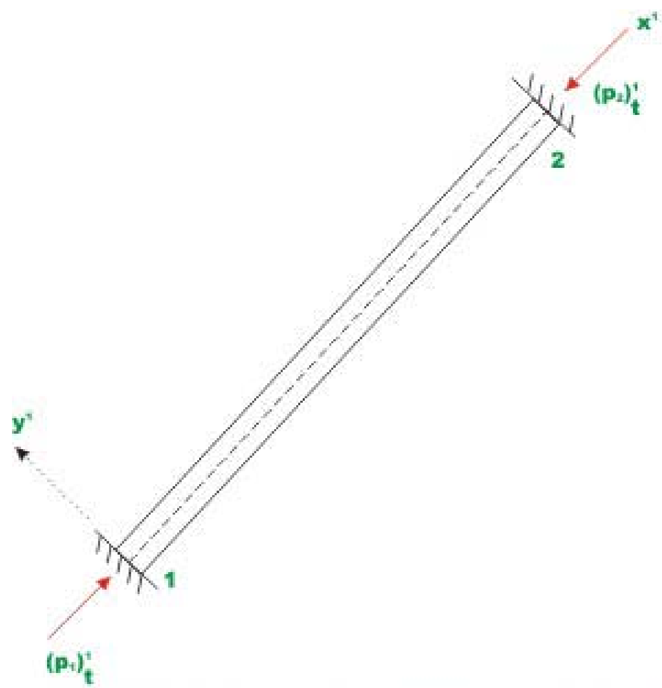


Fig.26.1 Truss member subjected to temperature loads

Consider truss member of length *L*, area of cross section *A* as shown in Fig.26.1.The change in length Δl is given by

$$\Delta l = \alpha L \Delta T \tag{26.1}$$

where α is the coefficient of thermal expansion of the material considered. If the member is not allowed to change its length (as in the case of statically indeterminate truss) the change in temperature will induce additional forces in the member. As the truss element is a one dimensional element in the local coordinate system, the thermal load can be easily calculated in global co-ordinate system by

$$(p_1')_t = AE\Delta L \tag{26.2a}$$

$$(p_2')_t = -AE\Delta L \tag{26.2b}$$

or

$$\left\{(p')_t\right\} = AE\Delta L \begin{Bmatrix} +1 \\ -1 \end{Bmatrix} \tag{26.3}$$

The equation (26.3) can also be used to calculate forces developed in the truss member in the local coordinate system due to fabrication error. ΔL will be considered positive if the member is too long. The forces in the local coordinate system can be transformed to global coordinate system by using the equation,

$$\begin{Bmatrix} (p_1)_t \\ (p_2)_t \\ (p_3)_t \\ (p_4)_t \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{Bmatrix} (p_1')_t \\ (p_2')_t \end{Bmatrix} \tag{26.4a}$$

where $(p_1)_t, (p_2)_t$ and $(p_3)_t, (p_4)_t$ are the forces in the global coordinate system at nodes 1 and 2 of the truss member respectively Using equation (26.3), the equation (26.4a) may be written as,

$$\begin{Bmatrix} (p_1)_t \\ (p_2)_t \\ (p_3)_t \\ (p_4)_t \end{Bmatrix} = AE\Delta L \begin{Bmatrix} \cos\theta \\ \sin\theta \\ -\cos\theta \\ -\sin\theta \end{Bmatrix} \tag{26.4b}$$

The force displacement equation for the entire truss may be written as,

$$\{p\} = [k]\{u\} + \{(p)_t\} \tag{26.5}$$

where , $\{p\}$ is the vector of external joint loads applied on the truss and $\{(p)_t\}$ is the vector of joint loads developed in the truss due to change in temperature/fabrication error of one or more members. As pointed out earlier. in the truss analysis, some joint displacements are known due to boundary conditions and some joint loads are known as they are applied externally.Thus,one could partition the above equation as,

$$\begin{Bmatrix} p_k \\ p_u \end{Bmatrix} = \begin{bmatrix} [k_{11}] & [k_{12}] \\ [k_{21}] & [k_{22}] \end{bmatrix} \begin{Bmatrix} \{u_u\} \\ \{u_k\} \end{Bmatrix} + \begin{Bmatrix} (p_k)_t \\ (p_u)_t \end{Bmatrix} \tag{26.6}$$

where subscript u is used to denote unknown quantities and subscript k is used to denote known quantities of forces and displacements. Expanding equation (26.6),

$$\{p_k\} = [k_{11}]\{u_u\} + [k_{12}]\{u_k\} + \{(p_k)_t\} \tag{26.7a}$$

$$\{p_u\} = [k_{21}]\{u_u\} + [k_{22}]\{u_k\} + \{(p_u)_t\} \tag{26.7b}$$

If the known displacement vector $\{u_k\} = \{0\}$ then using equation (26.2a) the unknown displacements can be calculated as

$$\{u_u\} = [k_{11}]^{-1}\left(\{p_k\} - \{(p_k)_t\}\right) \tag{26.8a}$$

If $\{u_k\} \neq 0$ then

$$\{u_u\} = [k_u]^{-1}\left(\{p_k\} - [k_{12}]\{u_k\} - \{(p_k)_t\}\right) \tag{26.8b}$$

After evaluating unknown displacements, the unknown force vectors are calculated using equation (26.7b).After evaluating displacements, the member forces in the local coordinate system for each member are evaluated by,

$$\{p'\} = [k'][T]\{u\} + \{p'\}_t \tag{26.9a}$$

or

$$\begin{Bmatrix} p_1' \\ p_2' \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} + \begin{Bmatrix} (p_1')_t \\ (p_2')_t \end{Bmatrix}$$

Expanding the above equation, yields

$$\{p_1'\} = \frac{AE}{L}\{\cos\theta \quad \sin\theta \quad -\cos\theta \quad -\sin\theta\}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} + AE\Delta L \qquad (26.10a)$$

And,

$$\{p_2'\} = \frac{AE}{L}\{-\cos\theta \quad -\sin\theta \quad \cos\theta \quad \sin\theta\}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} - AE\Delta L \qquad (26.10b)$$

Few problems are solved to illustrate the application of the above procedure to calculate thermal effects /fabrication errors in the truss analysis:-

Example 26.1

Analyze the truss shown in Fig.26.2a, if the temperature of the member (2) is raised by $40^\circ C$.The sectional areas of members in square centimeters are shown in the figure. Assume $E = 2\times10^5 N/mm^2$ and $\alpha = 1/75,000$ per $^\circ C$.

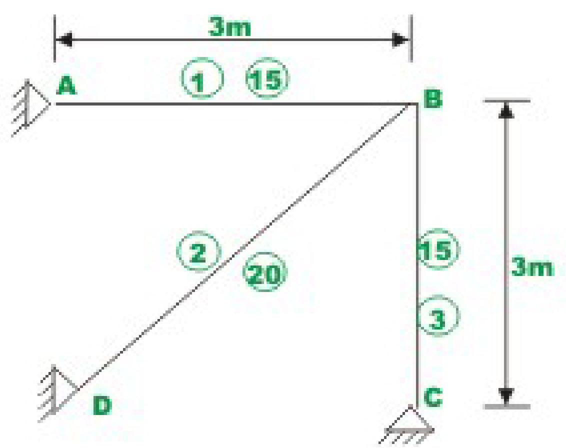


Fig 26.2a Example 26.2

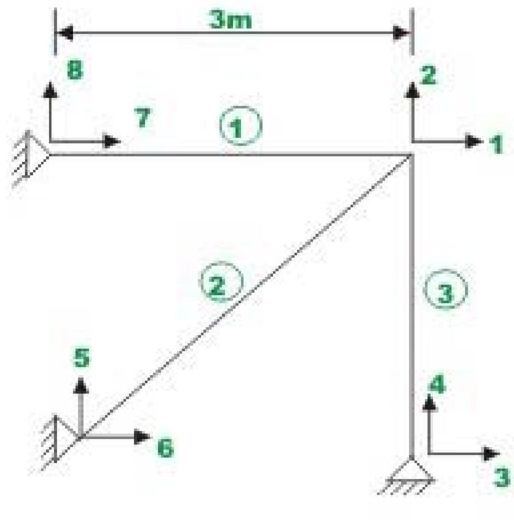


Fig 26.2b Node and Members numbering

The numbering of joints and members are shown in Fig.26.2b. The possible global displacement degrees of freedom are also shown in the figure. Note that lower numbers are used to indicate unconstrained degrees of freedom. From the figure it is obvious that the displacements $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$ due to boundary conditions.

The temperature of the member (2) has been raised by $40^\circ C$. Thus,

$$\Delta L = \alpha L \Delta T$$

$$\Delta L = \frac{1}{75000}\left(3\sqrt{2}\right)(40) = 2.2627 \times 10^{-3} \text{ m} \qquad (1)$$

The forces in member (2) due to rise in temperature in global coordinate system can be calculated using equation (26.4b).Thus,

$$\begin{Bmatrix} (p_5)_t \\ (p_6)_t \\ (p_1)_t \\ (p_2)_t \end{Bmatrix} = AE\Delta L \begin{Bmatrix} \cos\theta \\ \sin\theta \\ -\cos\theta \\ -\sin\theta \end{Bmatrix} \qquad (2)$$

For member (2),

$$A = 20cm^2 = 20 \times 10^{-4} m^2 \text{ and } \theta = 45^\circ$$

$$\begin{Bmatrix} (p_5)_t \\ (p_6)_t \\ (p_1)_t \\ (p_2)_t \end{Bmatrix} = 20\times10^{-4}\times 2\times10^{11}\times 2.2627\times10^{-3}/10^3 \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} (p_5)_t \\ (p_6)_t \\ (p_1)_t \\ (p_2)_t \end{Bmatrix} = 150.82 \begin{Bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{Bmatrix} kN \tag{4}$$

In the next step, write stiffness matrix of each member in global coordinate system and assemble them to obtain global stiffness matrix

Element (1): $\theta = 0^0, L = 3m, A = 15\times10^{-4} m^2$,nodal points 4-1

$$\left[k^{'}\right] = \frac{15\times10^{-4}\times 2\times10^{11}}{3\times10^3}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

Member (2): $\theta = 45^\circ, L = 3\sqrt{2}m, A = 20\times10^{-4} m^2$, nodal points 3-1

$$\left[k^2\right] = \frac{20\times10^{-4}\times 2\times10^{11}}{3\sqrt{2}}\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \tag{6}$$

Member (3): $\theta = 90^\circ, A = 15\times10^{-4} m^2, L = 30m,$ nodal points 2-1

$$\left[k^3\right] = \frac{15\times10^{-4}\times 2\times10^{11}}{3 \text{ x } 10^3\times10^3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{7}$$

The global stiffness matrix is of the order 8×8 ,assembling the three member stiffness matrices, one gets

$$[k] = 10^3\begin{bmatrix} 147.14 & 47.14 & 0 & 0 & -47.14 & -47.14 & -100 & 0 \\ 47.14 & 147.14 & 0 & -100 & -47.14 & -47.14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -100 & 0 & 100 & 0 & 0 & 0 & 0 \\ -47.14 & -47.14 & 0 & 0 & 47.14 & 47.14 & 0 & 0 \\ -47.14 & -47.14 & 0 & 0 & 47.14 & 47.14 & 0 & 0 \\ -100 & 0 & 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

Writing the load displacement equation for the truss

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = 10^3\begin{bmatrix} 147.14 & 47.14 & 0 & 0 & -47.14 & -47.14 & -100 & 0 \\ 47.14 & 147.14 & 0 & -100 & -47.14 & -47.14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -100 & 0 & 100 & 0 & 0 & 0 & 0 \\ -47.14 & -47.14 & 0 & 0 & 47.14 & 47.14 & 0 & 0 \\ -47.14 & -47.14 & 0 & 0 & 47.14 & 47.14 & 0 & 0 \\ -100 & 0 & 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} + 640\begin{Bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \tag{9}$$

In the present case, the displacements u_1 and u_2 are not known. All other displacements are zero. Also $p_1 = p_2 = 0$ (as no joint loads are applied).Thus,

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = \begin{bmatrix} 147.14 & 47.14 & 0 & 0 & -47.14 & -47.14 & -100 & 0 \\ 47.14 & 147.14 & 0 & -100 & -47.14 & -47.14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -100 & 0 & 100 & 0 & 0 & 0 & 0 \\ -47.14 & -47.14 & 0 & 0 & 47.14 & 47.14 & 0 & 0 \\ -47.14 & -47.14 & 0 & 0 & 47.14 & 47.14 & 0 & 0 \\ -100 & 0 & 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} + 640 \begin{Bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad (10)$$

Thus unknown displacements are

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{10^3} \begin{bmatrix} 147.14 & 47.14 \\ 47.14 & 147.14 \end{bmatrix}^{-1} \left(\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - 150.82 \begin{Bmatrix} -1 \\ -1 \end{Bmatrix} \right) \quad (11)$$

$$u_1 = 7.763 \times 10^{-4} m$$

$$u_2 = 7.763 \times 10^{-4} m$$

Now reactions are calculated as

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = 10^3 \begin{bmatrix} 0 & 0 \\ 0 & -100 \\ -47.14 & -47.14 \\ -47.14 & -47.14 \\ -100 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 47.14 & 47.14 & 0 & 0 \\ 0 & 0 & 47.14 & 47.14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + 640 \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -77.63 \\ 77.63 \\ 77.63 \\ -77.63 \\ 0 \end{Bmatrix} \text{kN} \quad (12)$$

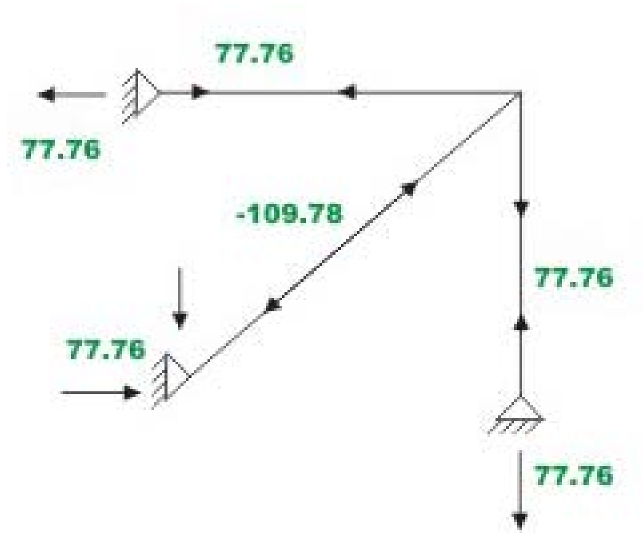


Fig 26.2c Force in members

The support reactions are shown in Fig.26.2c.The member forces can be easily calculated from reactions. The member end forces can also be calculated by using equation (26.10a) and (26.10b). For example, for member (1),

$\theta = 0^\circ$

$$p_2' = 10^3 \times 100 \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 7.763 \times 10^{-4} \\ 7.763 \times 10^{-4} \end{Bmatrix} \tag{13}$$

= 77.763 kN. Thus the member (1) is in tension.

Member (2)

$\theta = 45^\circ$

$$p_2' = 10^3 \times 94.281 [\text{-0.707 -0.707 0.707 0.707}] \begin{Bmatrix} 0 \\ 0 \\ 3.2942 \times 10^{-3} \\ 3.2942 \times 10^{-3} \end{Bmatrix}$$

$$p_2' = -109.78 \, \text{kN.}$$

Thus member (2) is in compression

Example 26.2

Analyze the truss shown in Fig.26.3a, if the member BC is made 0.01m too short before placing it in the truss. Assume *AE=300* kN for all members.

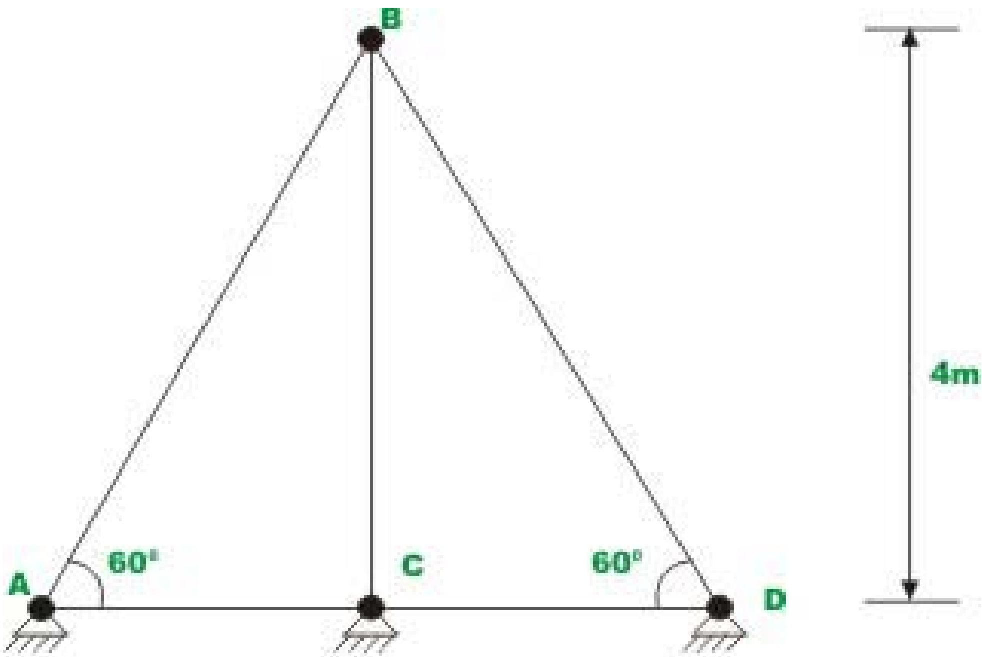


Fig 26.3a (Example 26.2)

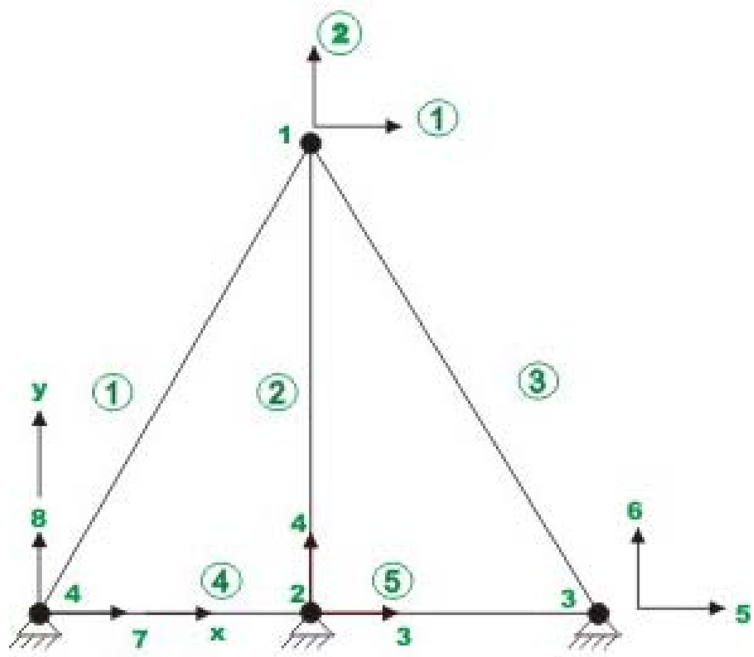


Fig. 26.3b Node and member numbering

Solution

A similar truss with different boundary conditions has already been solved in example 25.1. For the sake of completeness the member of nodes and members are shown in Fig.26.3b.The displacements u_3, u_4, u_5, u_6, u_7 and u_8 are zero due to boundary conditions. For the present problem the unconstrained degrees of freedom are u_1 and u_2.The assembled stiffness matrix is of the order 8×8 and is available in example 25.1.

In the given problem the member (2) is short by 0.01m.The forces developed in member (2) in the global coordinate system due to fabrication error is

$$\begin{Bmatrix} (p_3)_0 \\ (p_4)_0 \\ (p_1)_0 \\ (p_2)_0 \end{Bmatrix} = \frac{AE(-0.01)}{4} \begin{Bmatrix} \cos\theta \\ \sin\theta \\ -\cos\theta \\ -\sin\theta \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ -0.75 \\ 0 \\ 0.75 \end{Bmatrix} \text{kN} \qquad (1)$$

Now force-displacement relations for the truss are

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = AE \begin{bmatrix} 0.108 & 0 & 0 & 0 & -0.054 & 0.094 & -0.054 & -0.094 \\ 0 & 0.575 & 0 & -0.25 & 0.094 & -0.162 & -0.094 & -0.162 \\ 0 & 0 & 0.866 & 0 & -0.433 & 0 & -0.433 & 0 \\ 0 & -0.25 & 0 & 0.25 & 0 & 0 & 0 & 0 \\ -0.054 & 0.094 & -0.433 & 0 & 0.487 & -0.094 & 0 & 0 \\ 0.094 & -0.162 & 0 & 0 & -0.094 & 0.162 & 0 & 0 \\ -0.054 & -0.094 & -0.433 & 0 & 0 & 0 & 0.487 & 0.0934 \\ -0.094 & -0.162 & 0 & 0 & 0 & 0 & 0.0934 & 0.162 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0.75 \\ 0 \\ -0.75 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2}$$

Note that $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$
Thus, solving

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{AE} \begin{bmatrix} 0.108 & 0 \\ 0 & 0.575 \end{bmatrix}^{-1} \left(\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0.75 \end{Bmatrix} \right) \tag{3}$$

$u_1 = 0$

and, $u_2 = -4.3478 \times 10^{-3}\, m$ (4)

Reactions are calculated as,

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = AE \begin{bmatrix} 0 & 0 \\ 0 & -0.25 \\ -0.054 & 0.094 \\ 0.094 & -0.162 \\ -0.054 & -0.094 \\ -0.094 & -0.162 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -0.75 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{5}$$

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.424 \\ -0.123 \\ 0.211 \\ 0.123 \\ 0.211 \end{Bmatrix} \tag{6}$$

The reactions and member forces are shown in Fig.26.3c. The member forces can also be calculated by equation (26.10a) and (26.10b). For example, for member (2),

$\theta = 90^\circ$

$$p_2^{'} = \frac{300}{4}\,[0\ -1\ 0\ 1]\begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix} - \frac{AE\Delta L}{L}$$

$$= \frac{300}{4}\left(-4.3478 \times 10^{-3}\right) - \frac{300(-0.01)}{4}$$

$$= 0.4239 \cong 0.424 \text{ kN} \tag{7}$$

Example 26.3

Evaluate the member forces of truss shown in Fig.26.4a.The temperature of the member BC is raised by $40^\circ C$ and member BD is raised by $50^\circ C$.Assume AE=300KN for all members and $\alpha = \frac{1}{75000}$ per $^\circ$C.

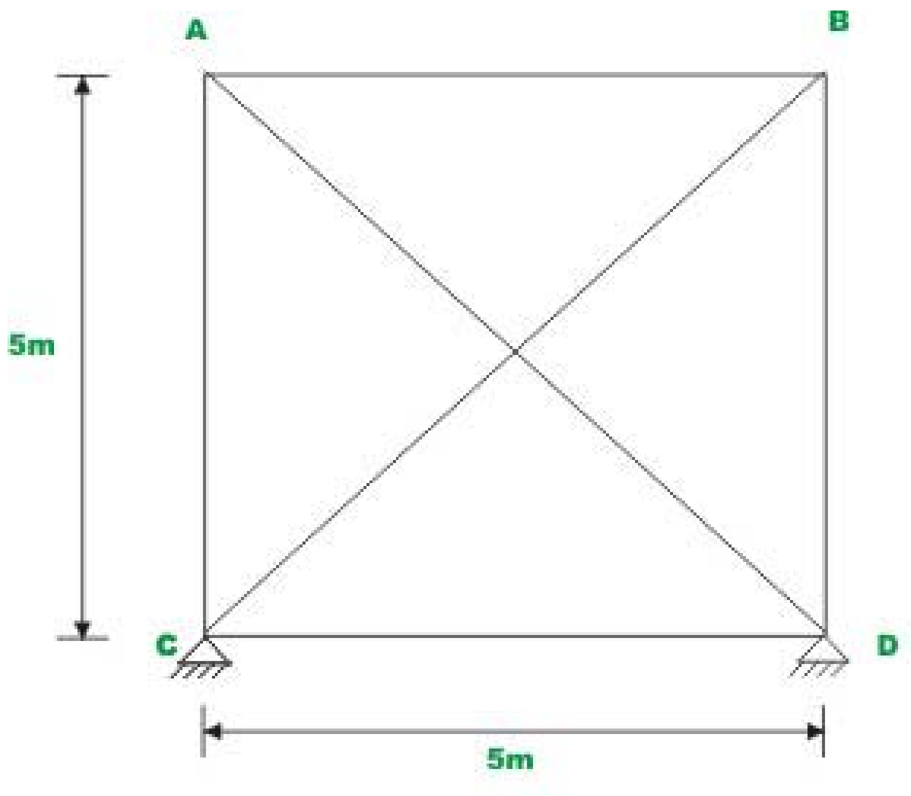


Fig 26.4a (Example 26.1)

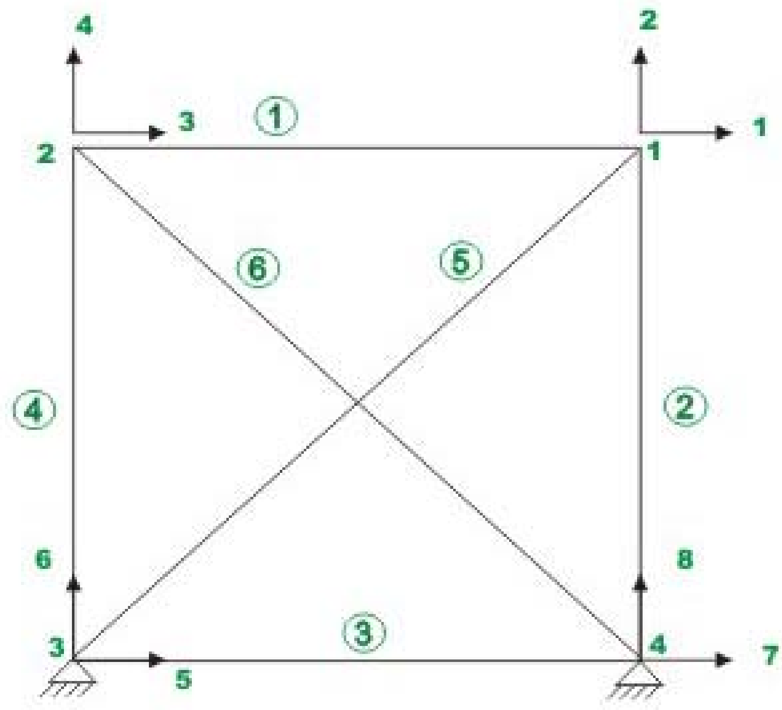


Fig 26.4b Node and member numbering

Solution

For this problem assembled stiffness matrix is available in Fig.26.4b.The joints and members are numbered as shown in Fig.26.4b. In the given problem u_1, u_2, u_3, u_4 and u_5 represent unconstrained degrees of freedom. Due to support conditions, $u_6 = u_7 = u_8 = 0$.

The temperature of the member (2) is raised by 50° C.Thus,

$$\Delta L^2 = \alpha L \Delta T = \frac{1}{75000} \times 5 \times 50 = 3.333 \times 10^{-3} m \qquad (1)$$

The forces are developed in member (2), as it was prevented from expansion.

$$\begin{Bmatrix} (p_7)_f \\ (p_8)_f \\ (p_1)_f \\ (p_2)_f \end{Bmatrix} = 300 \times 3.333 \times 10^{-3} \begin{Bmatrix} \cos\theta \\ \sin\theta \\ -\cos\theta \\ -\sin\theta \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix} \tag{2}$$

The displacement of the member (5) was raised by $40^\circ C$. Thus,

$$\Delta L^5 = \alpha L \Delta T = \frac{1}{75{,}000} \times 5\sqrt{2} \times 40 = 3.771 \times 10^{-3}\, m$$

The forces developed in member (5) as it was not allowed to expand is

$$\begin{Bmatrix} (p_5)_t \\ (p_6)_t \\ (p_7)_t \\ (p_8)_t \end{Bmatrix} = 300 \times 3.771 \times 10^{-3} \begin{Bmatrix} 0.707 \\ 0.707 \\ -0.707 \\ -0.707 \end{Bmatrix}$$

$$= 0.8 \begin{Bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{Bmatrix} \tag{3}$$

The global force vector due to thermal load is

$$\begin{Bmatrix} (p_1)_t \\ (p_2)_t \\ (p_3)_t \\ (p_4)_t \\ (p_5)_t \\ (p_6)_t \\ (p_7)_t \\ (p_8)_t \end{Bmatrix} = \begin{Bmatrix} -0.8 \\ -1.8 \\ 0 \\ 0 \\ 0.8 \\ 0.8 \\ 0 \\ 1 \end{Bmatrix} \tag{4}$$

Writing the load-displacement relation for the entire truss is given below.

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = AE \begin{bmatrix} 0.271 & 0.071 & -0.20 & 0 & -0.071 & -0.071 & 0 & 0 \\ 0.071 & 0.271 & 0 & 0 & -0.071 & -0.071 & 0 & -0.2 \\ -0.20 & 0 & 0.271 & -0.071 & 0 & 0 & -0.071 & 0.071 \\ 0 & 0 & -0.071 & 0.129 & 0 & -0.2 & 0.071 & 0.071 \\ -0.071 & -0.071 & 0 & 0 & 0.271 & 0.071 & -0.2 & 0 \\ -0.071 & -0.071 & 0 & -0.2 & 0.071 & 0.271 & 0 & 0 \\ 0 & 0 & -0.071 & 0.071 & -0.2 & 0 & 0.271 & -0.071 \\ 0 & -0.2 & 0.071 & -0.071 & 0 & 0 & -0.071 & 0.271 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} + \begin{Bmatrix} -0.8 \\ -1.8 \\ 0 \\ 0 \\ 0.8 \\ 0.8 \\ 0 \\ 1 \end{Bmatrix} \quad (5)$$

In the above problem $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = p_8 = 0$ and $u_6 = u_7 = u_8 = 0$.

Thus solving for unknown displacements,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \frac{1}{AE} \begin{bmatrix} 0.271 & 0.071 & -0.2 & 0 & -0.071 \\ 0.071 & 0.271 & 0 & 0 & -0.071 \\ -0.20 & 0 & 0.271 & -0.071 & 0 \\ 0 & 0 & -0.071 & 0.129 & 0 \\ -0.071 & -0.071 & 0 & 0 & 0.271 \end{bmatrix} \left(\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -0.8 \\ -1.8 \\ 0 \\ 0 \\ 0.8 \end{Bmatrix} \right) \quad (5)$$

Solving equation (5), the unknown displacements are calculated as

$$\begin{aligned} &u_1 = 0.0013m, u_2 = 0.0020m, u_3 = -0.0005m, u_4 = 0 \\ &u_5 = -0.0013m \end{aligned} \quad (6)$$

Now, reactions are computed as,

$$\begin{Bmatrix} p_6 \\ p_7 \\ p_8 \end{Bmatrix} = \begin{bmatrix} -0.071 & -0.071 & 0 & -0.2 & 0.071 \\ 0 & 0 & -0.071 & 0.071 & -0.2 \\ 0 & -0.2 & 0.071 & -0.071 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} + \begin{Bmatrix} 0.8 \\ 0 \\ 1 \end{Bmatrix} \quad (7)$$

All reactions are zero as truss is externally determinate and hence change in temperature does not induce any reaction. Now member forces are calculated by using equation (26.10b)

Member (1): L=5m, $\theta = 0^\circ$

$$p_2' = \frac{AE}{5}[-1\ 0\ 1\ 0]\begin{Bmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{Bmatrix} \tag{8}$$

$p_2' =$ 0.1080 Kn

Member 2: L=5m, $\theta = 90^\circ$,nodal points 4-1

$$p_2' = \frac{AE}{5}[0\ -1\ 0\ 1]\begin{Bmatrix} u_7 \\ u_8 \\ u_1 \\ u_2 \end{Bmatrix} - 300 \times 3.771 \times 10^{-5} \tag{9}$$

=0.1087 kN

Member (3): L=5m, $\theta = 0^\circ$,nodal points 3-4

$$p_2' = \frac{300}{5}[-1\ 0\ 1\ 0]\begin{Bmatrix} u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} \tag{10}$$

=0.0780kN

Member (4): $\theta = 90^\circ, L = 5m,$ nodal points 3-2

$$p_2' = \frac{300}{5}[0\ -1\ 0\ 1]\begin{Bmatrix} u_5 \\ u_6 \\ u_3 \\ u_4 \end{Bmatrix} = 0 \tag{11}$$

Member (5): $\theta = 45^\circ, L = 5\sqrt{2}$,nodal points 3-1

$$p_2' = \frac{300}{5\sqrt{2}}[-0.707\ -0.707\ 0.707\ 0.707]\begin{Bmatrix} u_5 \\ u_6 \\ u_1 \\ u_2 \end{Bmatrix} - 300 \times 3.333 \times 10^{-3} \tag{12}$$

=-0.8619 kN

Member (6) : $\theta = 135^\circ, L = 5\sqrt{2}$,nodal points 4-2

$$p_2^{'} = \frac{300}{5\sqrt{2}} [0.707\ -0.707\ -0.707\ 0.707] \begin{Bmatrix} u_7 \\ u_8 \\ u_3 \\ u_4 \end{Bmatrix} = 0.0150 \text{ kN.} \qquad (13)$$

Summary

In the last four lessons, the direct stiffness method as applied to the truss analysis was discussed. Assembly of member stiffness matrices, imposition of boundary conditions, and the problem of inclined supports were discussed. Due to the change in temperature the truss members either expand or shrink. However, in the case of statically indeterminate trusses, the length of the members is prevented from either expansion or contraction. Thus, the stresses are developed in the members due to changes in temperature. Similarly the errors in fabricating truss members also produce additional stresses in the trusses. In this lesson, these effects are accounted for in the stiffness analysis. A couple of problems are solved.

Module 4

Analysis of Statically Indeterminate Structures by the Direct Stiffness Method

Lesson
27
The Direct Stiffness Method: Beams

Instructional Objectives

After reading this chapter the student will be able to
1. Derive member stiffness matrix of a beam element.
2. Assemble member stiffness matrices to obtain the global stiffness matrix for a beam.
3. Write down global load vector for the beam problem.
4. Write the global load-displacement relation for the beam.

27.1 Introduction.

In chapter 23, a few problems were solved using stiffness method from fundamentals. The procedure adopted therein is not suitable for computer implementation. In fact the load displacement relation for the entire structure was derived from fundamentals. This procedure runs into trouble when the structure is large and complex. However this can be much simplified provided we follow the procedure adopted for trusses. In the case of truss, the stiffness matrix of the entire truss was obtained by assembling the member stiffness matrices of individual members.
In a similar way, one could obtain the global stiffness matrix of a continuous beam from assembling member stiffness matrix of individual beam elements. Towards this end, we break the given beam into a number of beam elements. The stiffness matrix of each individual beam element can be written very easily. For example, consider a continuous beam $ABCD$ as shown in Fig. 27.1a. The given continuous beam is divided into three beam elements as shown in Fig. 27.1b. It is noticed that, in this case, nodes are located at the supports. Thus each span is treated as an individual beam. However sometimes it is required to consider a node between support points. This is done whenever the cross sectional area changes suddenly or if it is required to calculate vertical or rotational displacements at an intermediate point. Such a division is shown in Fig. 27.1c. If the axial deformations are neglected then each node of the beam will have two degrees of freedom: a vertical displacement (corresponding to shear) and a rotation (corresponding to bending moment). In Fig. 27.1b, numbers enclosed in a circle represents beam numbers. The beam $ABCD$ is divided into three beam members. Hence, there are four nodes and eight degrees of freedom. The possible displacement degrees of freedom of the beam are also shown in the figure. Let us use lower numbers to denote unknown degrees of freedom (unconstrained degrees of freedom) and higher numbers to denote known (constrained) degrees of freedom. Such a method of identification is adopted in this course for the ease of imposing boundary conditions directly on the structure stiffness matrix. However, one could number sequentially as shown in Fig. 27.1d. This is preferred while solving the problem on a computer.

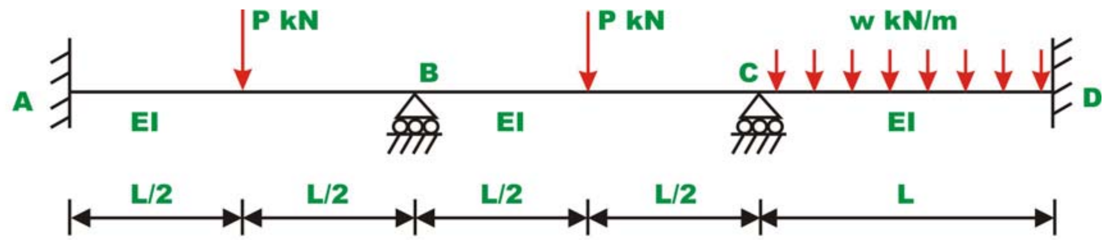


Fig 27.1a Continuous beam

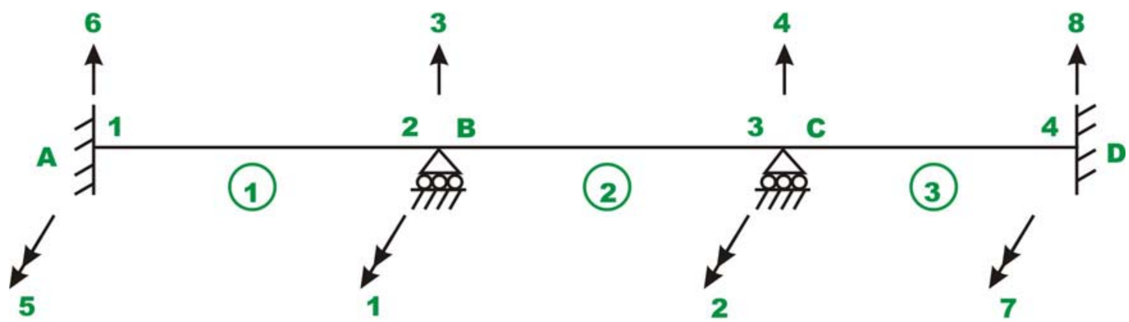


Fig. 27.1b Member and node numbering

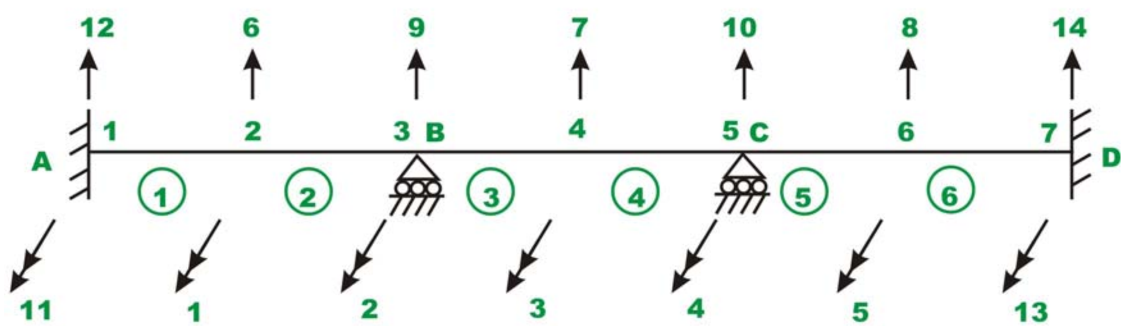


Fig. 27.1c Member and node numbering

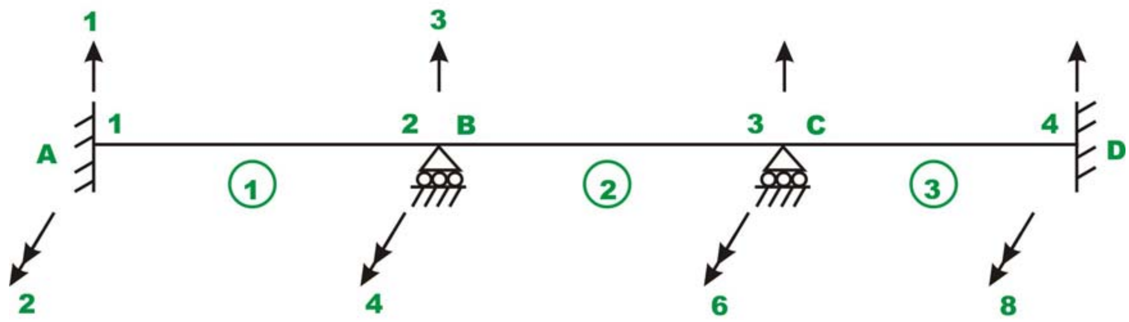


Fig 27.1d Member and node numbering

In the above figures, single headed arrows are used to indicate translational and double headed arrows are used to indicate rotational degrees of freedom.

27.2 Beam Stiffness Matrix.

Fig. 27.2 shows a prismatic beam of a constant cross section that is fully restrained at ends in local orthogonal co-ordinate system $x'y'z'$. The beam ends are denoted by nodes j and k. The x' axis coincides with the centroidal axis of the member with the positive sense being defined from j to k. Let L be the length of the member, A area of cross section of the member and I_{zz} is the moment of inertia about z' axis.

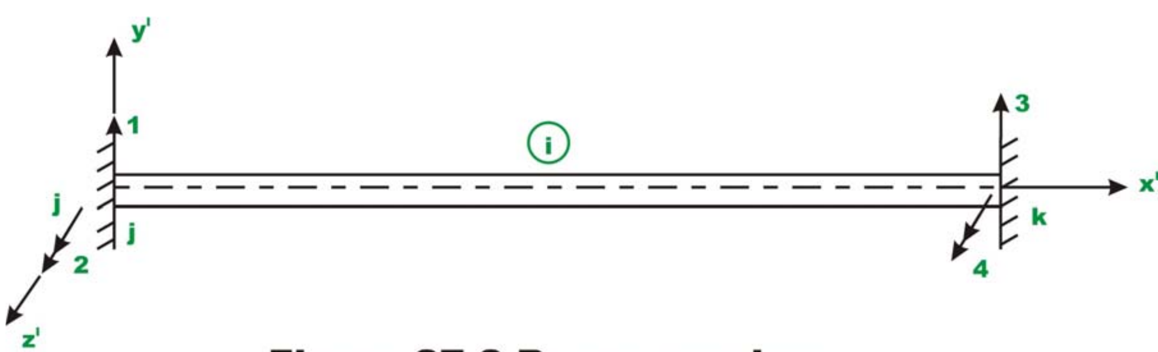
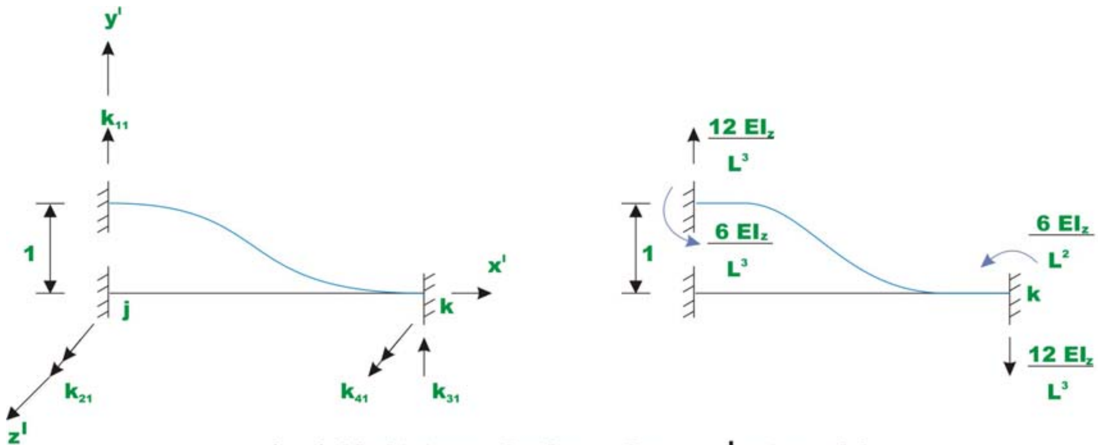


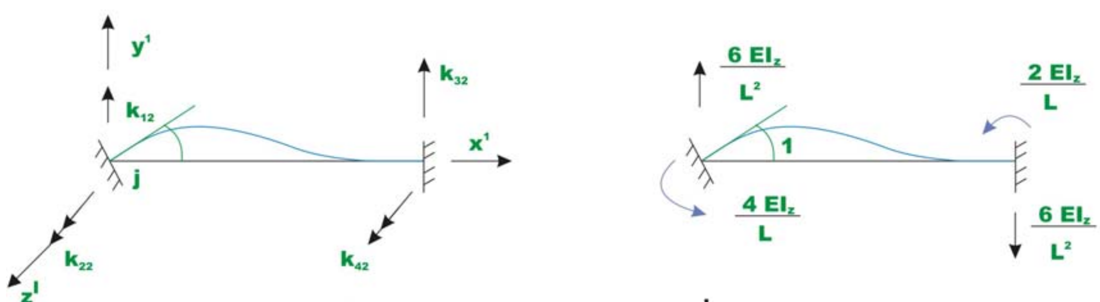
Figure 27.2 Beam member

Two degrees of freedom (one translation and one rotation) are considered at each end of the member. Hence, there are four possible degrees of freedom for this member and hence the resulting stiffness matrix is of the order 4×4. In this method counterclockwise moments and counterclockwise rotations are taken as positive. The positive sense of the translation and rotation are also shown in the figure. Displacements are considered as positive in the direction of the co-ordinate axis. The elements of the stiffness matrix indicate the forces exerted on the member by the restraints at the ends of the member when unit displacements are imposed at each end of the member. Let us calculate the forces developed in the above beam member when unit displacement is imposed along each degree of freedom holding all other displacements to zero. Now impose a unit displacement along y' axis at j end of the member while holding all other displacements to zero as shown in Fig. 27.3a. This displacement causes both shear and moment in the beam. The restraint actions are also shown in the figure. By definition they are elements of the member stiffness matrix. In particular they form the first column of element stiffness matrix.

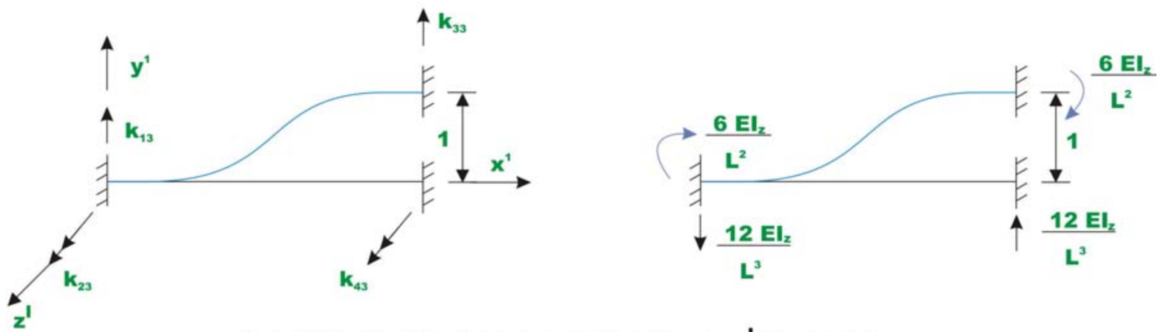
In Fig. 27.3b, the unit rotation in the positive sense is imposed at j end of the beam while holding all other displacements to zero. The restraint actions are shown in the figure. The restraint actions at ends are calculated referring to tables given in lesson …



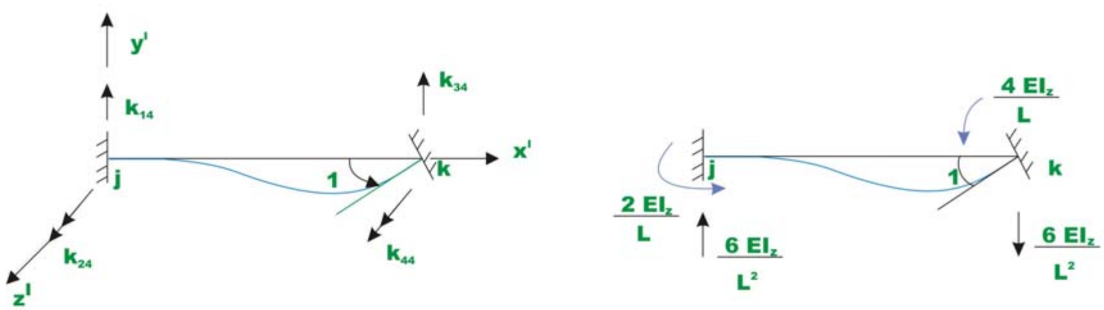
(a) Unit translation along y' at end j



(b) Unit rotation about z' at end j



(c) Unit displacement along y' at end k



(d) Unit rotation about z' at end k

Fig. 27.3 Computation of beam stiffness matrix

In Fig. 27.3c, unit displacement along y' axis at end k is imposed and corresponding restraint actions are calculated. Similarly in Fig. 27.3d, unit rotation about z' axis at end k is imposed and corresponding stiffness coefficients are calculated. Hence the member stiffness matrix for the beam member is

$$
[k] = \begin{array}{c c}
\begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad \end{array} & \\
\left[\begin{array}{cc|cc}
\dfrac{12EI_z}{L^3} & \dfrac{6EI_z}{L^2} & -\dfrac{12EI_z}{L^3} & \dfrac{6EI_z}{L^2} \\
\dfrac{6EI_z}{L^2} & \dfrac{4EI_z}{L} & -\dfrac{6EI_z}{L^2} & \dfrac{2EI_z}{L} \\
\hline
-\dfrac{12EI_z}{L^3} & -\dfrac{6EI_z}{L^2} & \dfrac{12EI_z}{L^3} & -\dfrac{6EI_z}{L^2} \\
\dfrac{6EI_z}{L^2} & \dfrac{2EI_z}{L} & -\dfrac{6EI_z}{L^2} & \dfrac{4EI_z}{L}
\end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array} \qquad (27.1)
$$

The stiffness matrix is symmetrical. The stiffness matrix is partitioned to separate the actions associated with two ends of the member. For continuous beam problem, if the supports are unyielding, then only rotational degree of freedom shown in Fig. 27.4 is possible. In such a case the first and the third rows and columns will be deleted. The reduced stiffness matrix will be,

$$[k]=\left[\begin{array}{c|c}\frac{4EI_z}{L} & \frac{2EI_z}{L} \\ \hline \frac{2EI_z}{L} & \frac{4EI_z}{L}\end{array}\right] \tag{27.2}$$

Instead of imposing unit displacement along y' at j end of the member in Fig. 27.3a, apply a displacement u'_1 along y' at j end of the member as shown in Fig. 27.5a, holding all other displacements to zero. Let the restraining forces developed be denoted by q_{11}, q_{21}, q_{31} and q_{41}.

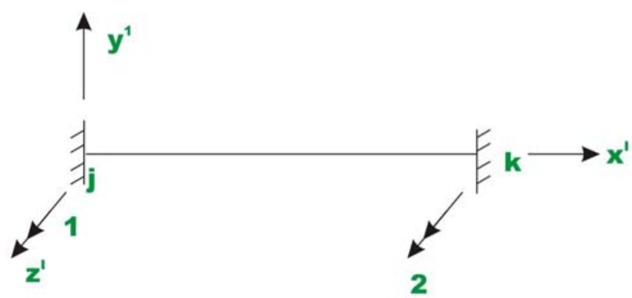


Fig. 27.4

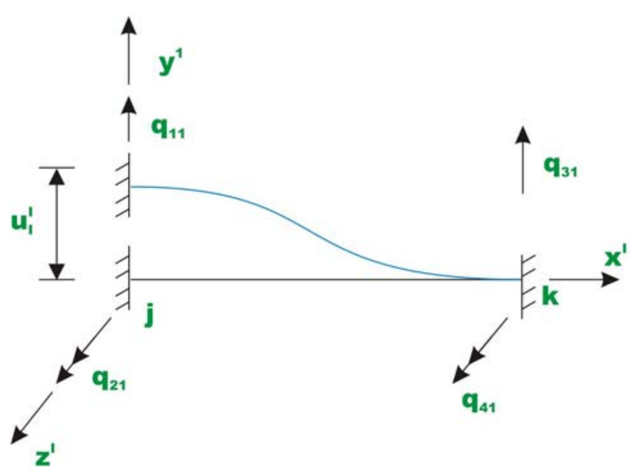


Fig. 27.5 (a)

The forces are equal to,

$$q_{11} = k_{11} u'_1 \quad ; \quad q_{21} = k_{21} u'_1 \quad ; \quad q_{31} = k_{31} u'_1 \quad ; \quad q_{41} = k_{41} u'_1 \tag{27.3}$$

Now, give displacements u'_1, u'_2, u'_3 and u'_4 simultaneously along displacement degrees of freedom $1,2,3$ and 4 respectively. Let the restraining forces developed at member ends be q_1, q_2, q_3 and q_4 respectively as shown in Fig. 27.5b along respective degrees of freedom. Then by the principle of superposition, the force displacement relationship can be written as,

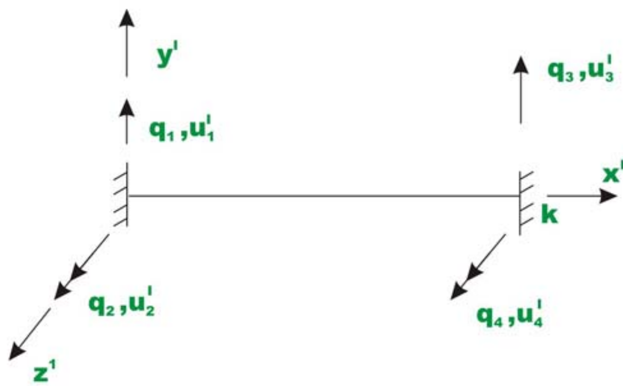


Fig. 27.5 (b) Force - displacement relation

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \end{bmatrix} \tag{27.4}$$

This may also be written in compact form as,

$$\{q\} = [k]\{u'\} \tag{27.5}$$

27.3 Beam (global) Stiffness Matrix.

The formation of structure (beam) stiffness matrix from its member stiffness matrices is explained with help of two span continuous beam shown in Fig. 27.6a. Note that no loading is shown on the beam. The orthogonal co-ordinate system xyz denotes the global co-ordinate system.

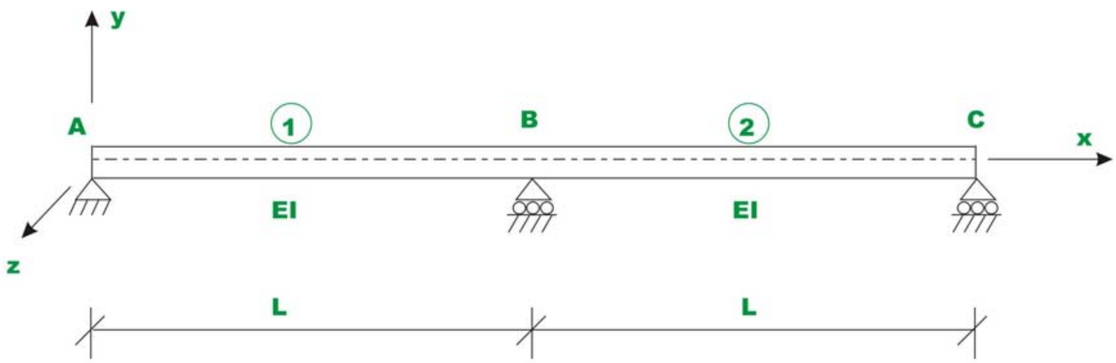


Fig. 27.6a Continuous beam

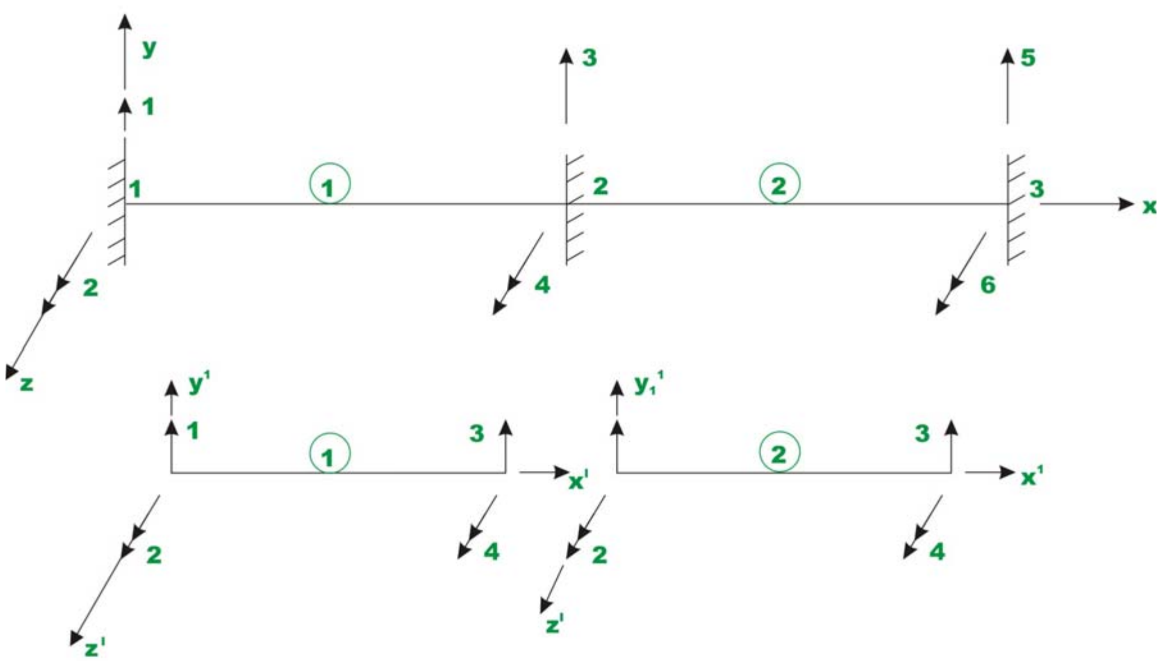


Fig. 27.6 b

For the case of continuous beam, the x- and x'- axes are collinear and other axes (y and y', z and z') are parallel to each other. Hence it is not required to transform member stiffness matrix from local co-ordinate system to global co ordinate system as done in the case of trusses. For obtaining the global stiffness matrix, first assume that all joints are restrained. The node and member numbering for the possible degrees of freedom are shown in Fig 27.6b. The continuous beam is divided into two beam members. For this member there are six possible degrees of freedom. Also in the figure, each beam member with its displacement degrees of freedom (in local co ordinate system) is also shown. Since the continuous beam has the same moment of inertia and span, the member stiffness matrix of element 1 and 2 are the same. They are,

$$
\begin{array}{lcccccc}
Global\ d.o.f & 1 & 2 & 3 & 4 & & \\
Local\ d.o.f & 1 & 2 & 3 & 4 & & \\
\end{array}
$$

$$
[k'] = \begin{bmatrix} k'_{11} & k'_{12} & k'_{13} & k'_{14} \\ k'_{21} & k'_{22} & k'_{23} & k'_{24} \\ k'_{31} & k'_{32} & k'_{33} & k'_{34} \\ k'_{41} & k'_{42} & k'_{43} & k'_{44} \end{bmatrix} \begin{array}{cc} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{array} \qquad (27.6a)
$$

$$
\begin{array}{lcccc}
Global\ d.o.f & 3 & 4 & 5 & 6 \\
Local\ d.o.f & 1 & 2 & 3 & 4 \\
\end{array}
$$

$$
[k^2] = \begin{bmatrix} k^2_{11} & k^2_{12} & k^2_{13} & k^2_{14} \\ k^2_{21} & k^2_{22} & k^2_{23} & k^2_{24} \\ k^2_{31} & k^2_{32} & k^2_{33} & k^2_{34} \\ k^2_{41} & k^2_{42} & k^2_{43} & k^2_{44} \end{bmatrix} \begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 3 & 5 \\ 4 & 6 \end{array} \qquad (27.6b)
$$

The local and the global degrees of freedom are also indicated on the top and side of the element stiffness matrix. This will help us to place the elements of the element stiffness matrix at the appropriate locations of the global stiffness matrix. The continuous beam has six degrees of freedom and hence the stiffness matrix is of the order 6×6. Let $[K]$ denotes the continuous beam stiffness matrix of order 6×6. From Fig. 27.6b, $[K]$ may be written as,

$$\text{Member } AB \ (1)$$

$$[K] = \left[\begin{array}{cc|cc|cc}
k_{11}^{1} & k_{12}^{1} & k_{13}^{1} & k_{14}^{1} & 0 & 0 \\
k_{21}^{1} & k_{22}^{1} & k_{23}^{1} & k_{24}^{1} & 0 & 0 \\
\hline
k_{31}^{1} & k_{32}^{1} & k_{33}^{1}+k_{11}^{2} & k_{34}^{1}+k_{12}^{2} & k_{13}^{2} & k_{14}^{2} \\
k_{41}^{1} & k_{42}^{1} & k_{43}^{1}+k_{21}^{2} & k_{44}^{1}+k_{22}^{2} & k_{23}^{2} & k_{24}^{2} \\
\hline
0 & 0 & k_{31}^{2} & k_{32}^{2} & k_{33}^{2} & k_{34}^{2} \\
0 & 0 & k_{41}^{2} & k_{42}^{2} & k_{43}^{2} & k_{44}^{2}
\end{array}\right] \qquad (27.7)$$

$$\text{Member } BC \ (2)$$

The 4×4 upper left hand section receives contribution from member 1 (AB) and 4×4 lower right hand section of global stiffness matrix receives contribution from member 2. The element of the global stiffness matrix corresponding to global degrees of freedom 3 and 4 [overlapping portion of equation (27.7)] receives element from both members 1 and 2.

27.4 Formation of load vector.

Consider a continuous beam ABC as shown in Fig. 27.7.

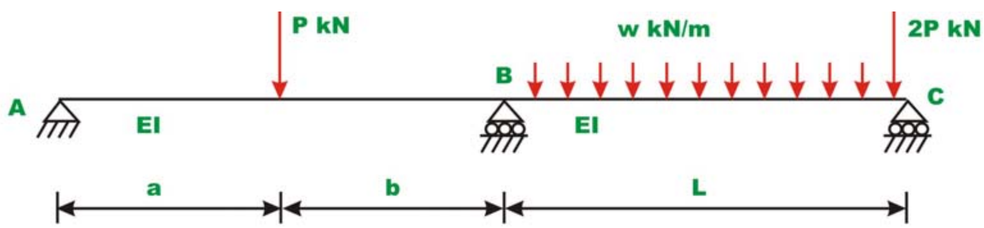
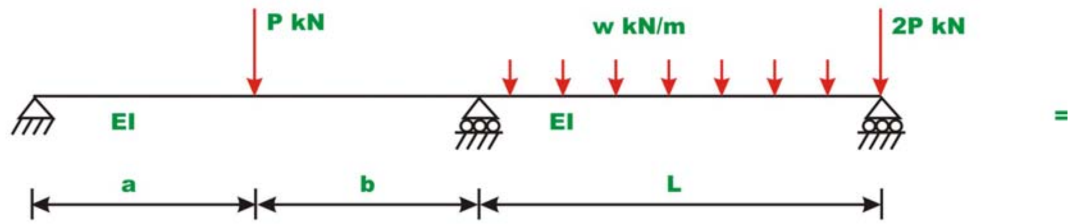
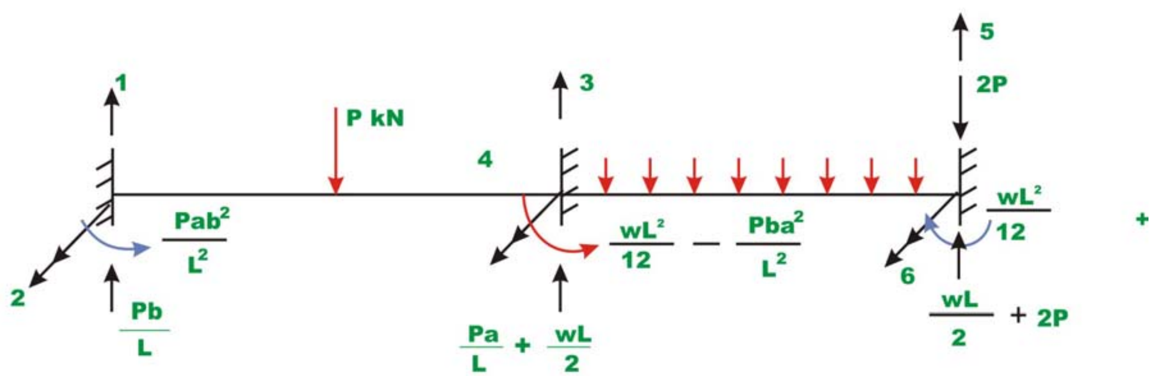


Fig.27.7

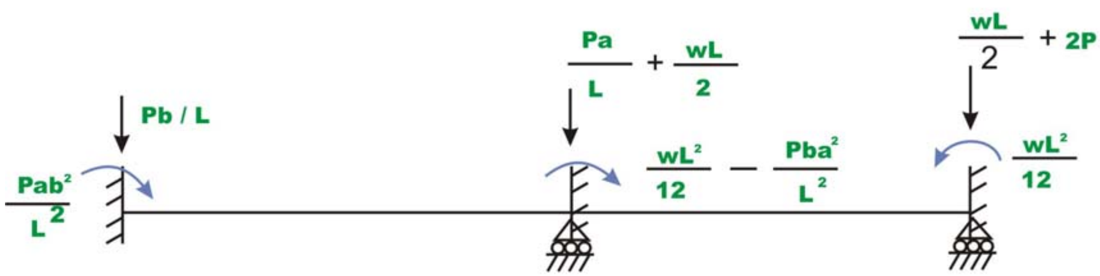
We have two types of load: member loads and joint loads. Joint loads could be handled very easily as done in case of trusses. Note that stiffness matrix of each member was developed for end loading only. Thus it is required to replace the member loads by equivalent joint loads. The equivalent joint loads must be evaluated such that the displacements produced by them in the beam should be the same as the displacements produced by the actual loading on the beam. This is evaluated by invoking the method of superposition.



(a) Actual beam with loading



(b) Reaction in the restrained beam



(c) Equivalent joint loads

Fig. 27.8

The loading on the beam shown in Fig. 27.8(a), is equal to the sum of Fig. 27.8(b) and Fig. 27.8(c). In Fig. 27.8(c), the joints are restrained against displacements and fixed end forces are calculated. In Fig. 27.8(c) these fixed end actions are shown in reverse direction on the actual beam without any load. Since the beam in Fig. 27.8(b) is restrained (fixed) against any displacement, the displacements produced by the joint loads in Fig. 27.8(c) must be equal to the displacement produced by the actual beam in Fig. 27.8(a). Thus the loads shown in Fig. 27.8(c) are the equivalent joint loads .Let, p_1, p_2, p_3, p_4, p_5 and p_6 be the equivalent joint loads acting on the continuous beam along displacement degrees of freedom $1,2,3,4,5$ and 6 respectively as shown in Fig. 27.8(b). Thus the global load vector is,

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{Pb}{L} \\ -\dfrac{Pab^2}{L^2} \\ -\left(\dfrac{Pa}{L}+\dfrac{wL}{2}\right) \\ -\left(\dfrac{wL^2}{12}-\dfrac{Pba^2}{L^2}\right) \\ -\left(\dfrac{wL}{2}+2P\right) \\ \dfrac{wL^2}{12} \end{Bmatrix} \tag{27.8}$$

27.5 Solution of equilibrium equations

After establishing the global stiffness matrix and load vector of the beam, the load displacement relationship for the beam can be written as,

$$\{P\} = [K]\{u\} \tag{27.9}$$

where $\{P\}$ is the global load vector, $\{u\}$ is displacement vector and $[K]$ is the global stiffness matrix. This equation is solved exactly in the similar manner as discussed in the lesson 24. In the above equation some joint displacements are known from support conditions. The above equation may be written as

$$\begin{Bmatrix} \{p_k\} \\ \{p_u\} \end{Bmatrix} = \begin{bmatrix} [k_{11}] & [k_{12}] \\ [k_{21}] & [k_{22}] \end{bmatrix} \begin{Bmatrix} \{u_u\} \\ \{u_k\} \end{Bmatrix} \tag{27.10}$$

where $\{p_k\}$ and $\{u_k\}$ denote respectively vector of known forces and known displacements. And $\{p_u\}$, $\{u_u\}$ denote respectively vector of unknown forces and unknown displacements respectively. Now expanding equation (27.10),

$$\{p_k\} = [k_{11}]\{u_u\} + [k_{12}]\{u_k\} \qquad (27.11a)$$
$$\{p_u\} = [k_{21}]\{u_u\} + [k_{22}]\{u_k\} \qquad (27.11b)$$

Since $\{u_k\}$ is known, from equation 27.11(a), the unknown joint displacements can be evaluated. And support reactions are evaluated from equation (27.11b), after evaluating unknown displacement vector.

Let R_1, R_3 and R_5 be the reactions along the constrained degrees of freedom as shown in Fig. 27.9a. Since equivalent joint loads are directly applied at the supports, they also need to be considered while calculating the actual reactions. Thus,

$$\begin{Bmatrix} R_1 \\ R_3 \\ R_5 \end{Bmatrix} = -\begin{Bmatrix} p_1 \\ p_3 \\ p_5 \end{Bmatrix} + [K_{21}]\{u_u\} \qquad (27.12)$$

The reactions may be calculated as follows. The reactions of the beam shown in Fig. 27.9a are equal to the sum of reactions shown in Fig. 27.9b, Fig. 27.9c and Fig. 27.9d.

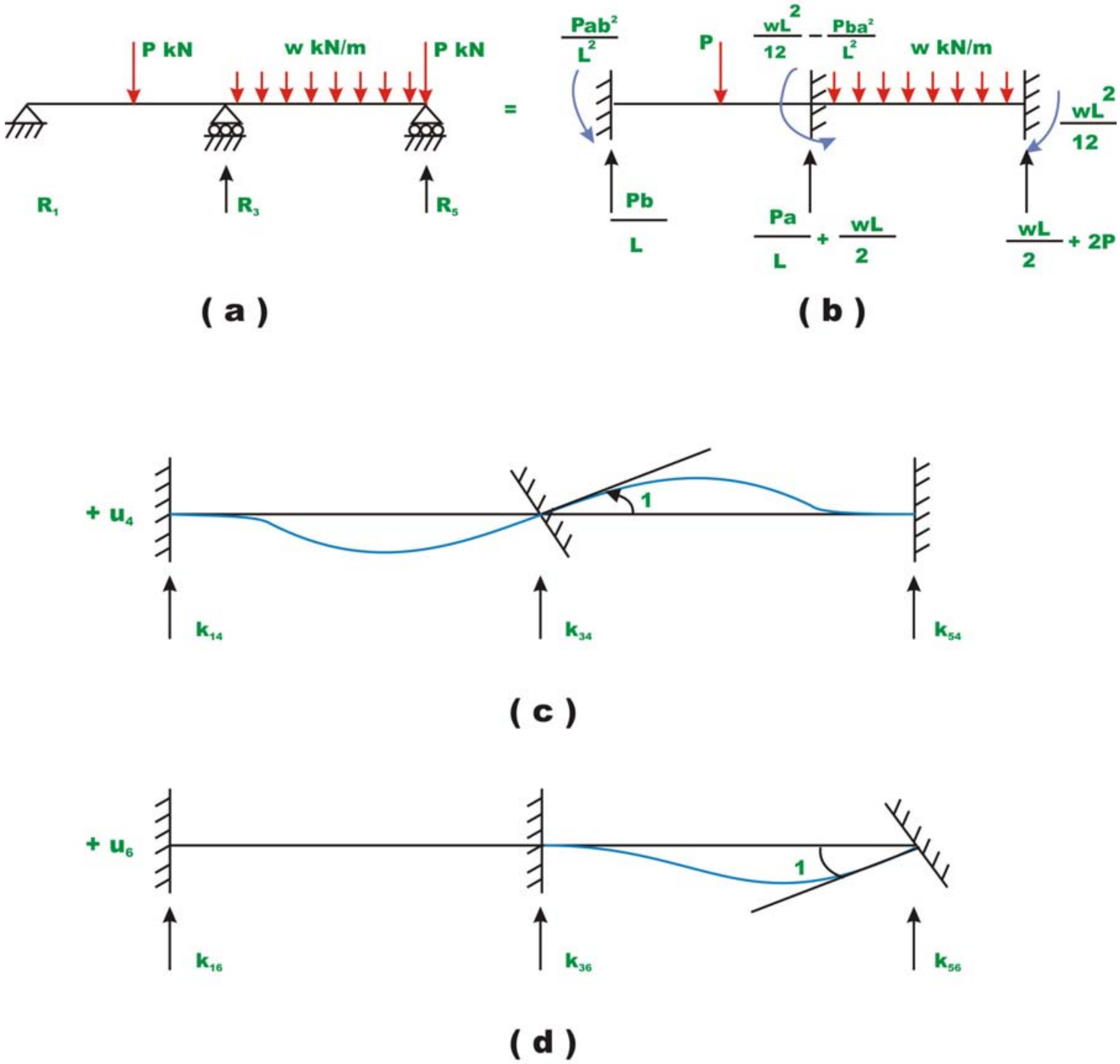


Fig. 27.9

From the method of superposition,

$$R_1 = \frac{Pb}{L} + K_{14}u_4 + K_{16}u_6 \tag{27.13a}$$

$$R_3 = \frac{Pa}{L} + K_{34}u_4 + K_{36}u_6 \tag{27.13b}$$

$$R_5 = \frac{wL}{2} + 2P + K_{54}u_4 + K_{56}u_6 \tag{27.13c}$$

or

$$\begin{Bmatrix} R_1 \\ R_3 \\ R_5 \end{Bmatrix} = \begin{Bmatrix} Pb/L \\ Pa/L \\ \dfrac{wl}{2} + 2P \end{Bmatrix} + \begin{bmatrix} K_{14} & K_{16} \\ K_{34} & K_{36} \\ K_{54} & K_{56} \end{bmatrix} \begin{Bmatrix} u_4 \\ u_6 \end{Bmatrix} \tag{27.14a}$$

Equation (27.14a) may be written as,

$$\begin{Bmatrix} R_1 \\ R_3 \\ R_5 \end{Bmatrix} = -\begin{Bmatrix} Pb/L \\ Pa/L \\ \dfrac{wl}{2} + 2P \end{Bmatrix} + \begin{bmatrix} K_{14} & K_{16} \\ K_{34} & K_{36} \\ K_{54} & K_{56} \end{bmatrix} \begin{Bmatrix} u_4 \\ u_6 \end{Bmatrix} \tag{27.14b}$$

Member end actions q_1, q_2, q_3, q_4 are calculated as follows. For example consider the first element 1.

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} \dfrac{Pb}{L} \\ \dfrac{Pab^2}{L^2} \\ \dfrac{Pa}{L} \\ -\dfrac{Pa^2b}{L^2} \end{Bmatrix} + [K]_{element1} \begin{Bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \end{Bmatrix} \tag{27.16}$$

In the next lesson few problems are solved to illustrate the method so far discussed.

Summary

In this lesson the beam element stiffness matrix is derived from fundamentals. Assembling member stiffness matrices, the global stiffness matrix is generated. The global load vector is defined. The global load-displacemet relation is written for the complete beam structure. The procedure to impose boundary conditions on the load-displacement relation is discussed. With this background, one could analyse continuous beam by the direct stiffness method.

Module 4

Analysis of Statically Indeterminate Structures by the Direct Stiffness Method

Lesson
28
The Direct Stiffness Method: Beams (Continued)

Instructional Objectives

After reading this chapter the student will be able to
1. Derive member stiffness matrix of a beam element.
2. Assemble member stiffness matrices to obtain the global stiffness matrix for a beam.
3. Write the global load-displacement relation for the beam.
4. Impose boundary conditions on the load-displacement relation of the beam.
5. Analyse continuous beams by the direct stiffness method.

28.1 Introduction

In the last lesson, the procedure to analyse beams by direct stiffness method has been discussed. No numerical problems are given in that lesson. In this lesson, few continuous beam problems are solved numerically by direct stiffness method.

Example 28.1

Analyse the continuous beam shown in Fig. 28.1a. Assume that the supports are unyielding. Also assume that EI is constant for all members.

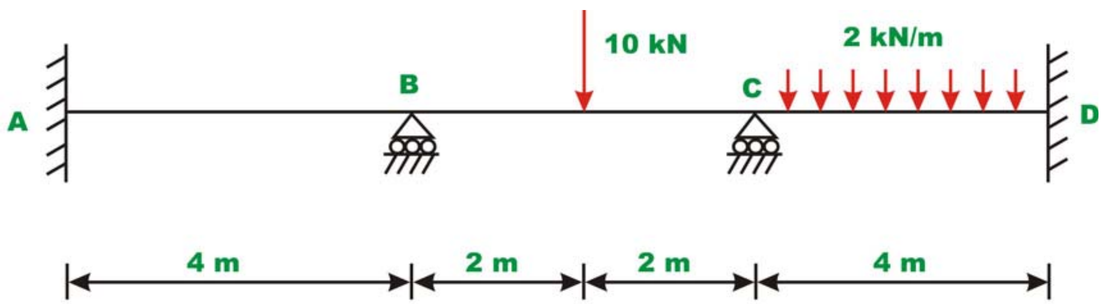


Fig. 28.1a

The numbering of joints and members are shown in Fig. 28.1b. The possible global degrees of freedom are shown in the figure. Numbers are put for the unconstrained degrees of freedom first and then that for constrained displacements.

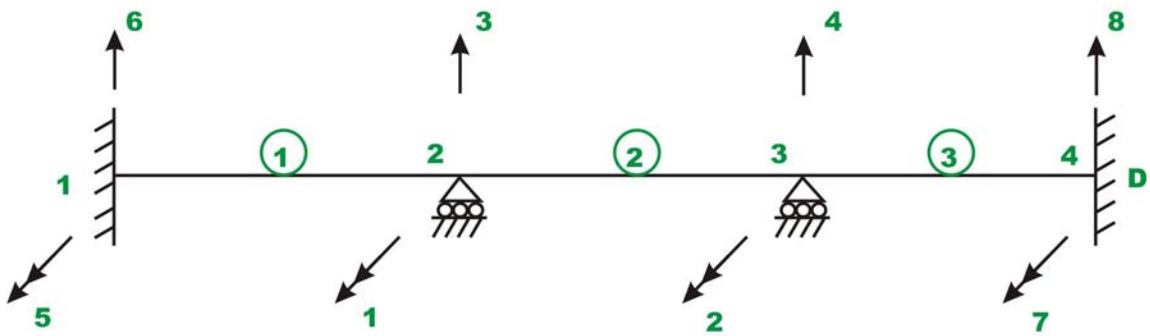


Figure 28.1b

The given continuous beam is divided into three beam elements Two degrees of freedom (one translation and one rotation) are considered at each end of the member. In the above figure, double headed arrows denote rotations and single headed arrow represents translations. In the given problem some displacements are zero, i.e., $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$ from support conditions.

In the case of beams, it is not required to transform member stiffness matrix from local co-ordinate system to global co-ordinate system, as the two co-ordinate system are parallel to each other.

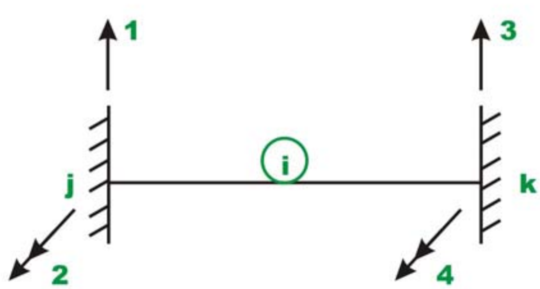


Figure 28.1c

First construct the member stiffness matrix for each member. This may be done from the fundamentals. However, one could use directly the equation (27.1) given in the previous lesson and reproduced below for the sake convenience.

$$
[k] = \left[\begin{array}{cc:cc}
\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\
\frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\
\hdashline
-\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\
\frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L}
\end{array}\right] \tag{1}
$$

The degrees of freedom of a typical beam member are shown in Fig. 28.1c. Here equation (1) is used to generate element stiffness matrix.

Member 1: $L = 4\,m$, node points 1-2.

The member stiffness matrix for all the members are the same, as the length and flexural rigidity of all members is the same.

$$
\begin{array}{cc}
\text{Global } d.o.f & \begin{array}{cccc} \quad 6 \quad & \quad 5 \quad & \quad 3 \quad & \quad 1 \quad \end{array} \\
[k'] = EI_{zz} & \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \begin{array}{c} 6 \\ 5 \\ 3 \\ 1 \end{array}
\end{array} \tag{2}
$$

On the member stiffness matrix, the corresponding global degrees of freedom are indicated to facilitate assembling.

Member 2: $L = 4\,m$, node points 2-3.

$$
\begin{array}{cc}
\text{Global } d.o.f & \begin{array}{cccc} \quad 3 \quad & \quad 1 \quad & \quad 4 \quad & \quad 2 \quad \end{array} \\
[k^2] = EI_{zz} & \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \begin{array}{c} 3 \\ 1 \\ 4 \\ 2 \end{array}
\end{array} \tag{3}
$$

Member 3: $L = 4\,m$, node points 3-4.

$$
\begin{array}{c}
\text{Global d.o.f} \\
\\
\left[k^3\right] = EI_{zz}
\end{array}
\begin{array}{cc}
\begin{array}{cccc} 4 & 2 & 8 & 7 \end{array} & \\
\begin{bmatrix}
0.1875 & 0.375 & -0.1875 & 0.375 \\
0.375 & 1.0 & -0.375 & 0.5 \\
-0.1875 & -0.375 & 0.1875 & -0.375 \\
0.375 & 0.5 & -0.375 & 1.0
\end{bmatrix} &
\begin{array}{c} 4 \\ 2 \\ 8 \\ 7 \end{array}
\end{array}
\qquad (4)
$$

The assembled global stiffness matrix of the continuous beam is of the order 8×8. The assembled global stiffness matrix may be written as,

$$
[K] = EI_{zz}
\begin{bmatrix}
2.0 & 0.5 & 0.0 & -0.375 & 0.5 & 0.375 & 0 & 0 \\
0.5 & 2.0 & 0.375 & 0 & 0 & 0 & 0.5 & -0.375 \\
0 & 0.375 & 0.375 & -0.1875 & -0.375 & -0.1875 & 0 & 0 \\
-0.375 & 0 & -0.1875 & 0.375 & 0 & 0 & 0.375 & -0.1875 \\
0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 & 0 & 0 \\
0.375 & 0 & -0.1875 & 0 & 0.375 & 0.1875 & 0 & 0 \\
0 & 0.5 & 0 & 0.375 & 0 & 0 & 1.0 & -0.375 \\
0 & -0.375 & 0 & -0.1875 & 0 & 0 & -0.375 & 0.1875
\end{bmatrix}
\qquad (5)
$$

Now it is required to replace the given members loads by equivalent joint loads. The equivalent loads for the present case is shown in Fig. 28.1d. The displacement degrees of freedom are also shown in Fig. 28.1d.

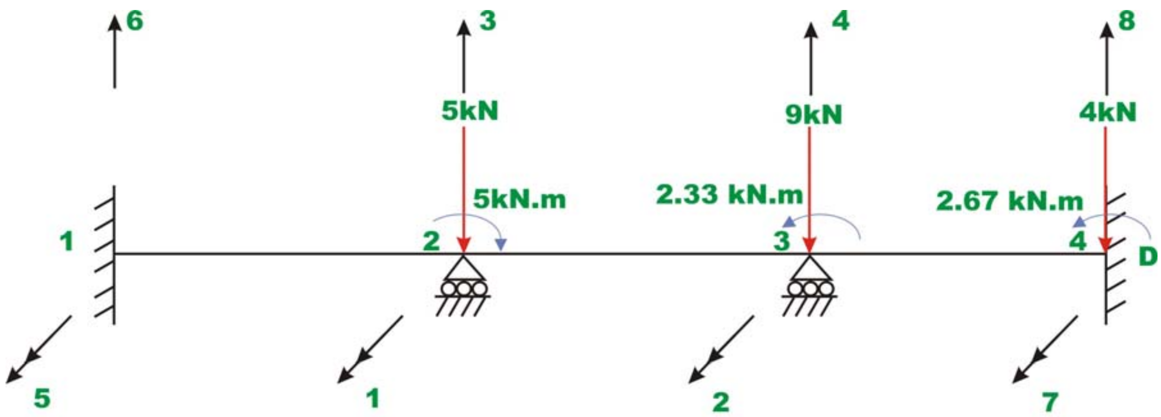


Fig. 28.1 (d) Equivalent joint loads

Thus the global load vector corresponding to unconstrained degree of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} -5 \\ 2.33 \end{Bmatrix} \tag{6}$$

Writing the load displacement relation for the entire continuous beam,

$$\begin{Bmatrix} -5 \\ 2.33 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = EI_{zz} \left[\begin{array}{cc:cccccc} 2.0 & 0.5 & 0.0 & -0.375 & 0.5 & 0.375 & 0 & 0 \\ 0.5 & 2.0 & 0.375 & 0 & 0 & 0 & 0.5 & -0.375 \\ \hdashline 0 & 0.375 & 0.375 & -0.187 & -0.375 & -0.187 & 0 & 0 \\ -0.375 & 0 & -0.187 & 0.375 & 0 & 0 & 0.375 & -0.187 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 & 0 & 0 \\ 0.375 & 0 & -0.187 & 0 & 0.375 & 0.187 & 0 & 0 \\ 0 & 0.5 & 0 & 0.375 & 0 & 0 & 1.0 & -0.375 \\ 0 & -0.375 & 0 & -0.187 & 0 & 0 & -0.375 & 0.187 \end{array} \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} \tag{7}$$

where $\{p\}$ is the joint load vector, $\{u\}$ is displacement vector.

We know that $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$. Thus solving for unknowns u_1 and u_2, yields

$$\begin{Bmatrix} -5 \\ 2.33 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 2.0 & 0.5 \\ 0.5 & 2.0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{8}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{3.75 EI_{zz}} \begin{bmatrix} 2.0 & -0.5 \\ -0.5 & 2.0 \end{bmatrix} \begin{Bmatrix} -5 \\ 2.333 \end{Bmatrix} \tag{9}$$

$$= \frac{1}{EI_{zz}} \begin{Bmatrix} -2.977 \\ 1.909 \end{Bmatrix}$$

Thus displacements are,

$$u_1 = \frac{-2.977}{EI_{zz}} \quad \text{and} \quad u_2 = \frac{1.909}{EI_{zz}} \tag{10}$$

The unknown joint loads are given by,

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 0 & 0.375 \\ -0.375 & 0 \\ 0.5 & 0 \\ 0.375 & 0 \\ 0 & 0.5 \\ 0 & -0.375 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} -2.977 \\ 1.909 \end{Bmatrix} \tag{11}$$

$$= \begin{Bmatrix} 0.715 \\ 1.116 \\ -1.488 \\ -1.116 \\ 0.955 \\ -0.715 \end{Bmatrix}$$

The actual reactions at the supports are calculated as,

$$\begin{Bmatrix} R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} p_3^F \\ p_4^F \\ p_5^F \\ p_6^F \\ p_7^F \\ p_8^F \end{Bmatrix} + \begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 9 \\ 0 \\ 0 \\ -2.67 \\ 4 \end{Bmatrix} + \begin{Bmatrix} 0.715 \\ 1.116 \\ -1.488 \\ -1.116 \\ 0.955 \\ -0.715 \end{Bmatrix} = \begin{Bmatrix} 5.716 \\ 10.116 \\ -1.489 \\ -1.116 \\ -1.715 \\ 3.284 \end{Bmatrix} \tag{12}$$

Member end actions for element 1

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2.977 \end{Bmatrix}$$

$$= \begin{Bmatrix} -1.116 \\ -1.488 \\ 1.116 \\ -2.977 \end{Bmatrix} \tag{13}$$

Member end actions for element 2

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = +EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ -2.977 \\ 0 \\ 1.909 \end{Bmatrix}$$

$$= \begin{Bmatrix} 4.6 \\ 2.98 \\ 5.4 \\ -4.58 \end{Bmatrix} \tag{14}$$

Member end actions for element 3

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 4.0 \\ 2.67 \\ 4.0 \\ -2.67 \end{Bmatrix} + EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ 1.909 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 4.72 \\ 4.58 \\ 3.28 \\ -1.72 \end{Bmatrix} \qquad (15)$$

Example 28.2

Analyse the continuous beam shown in Fig. 28.2a. Assume that the supports are unyielding. Assume EI to be constant for all members.

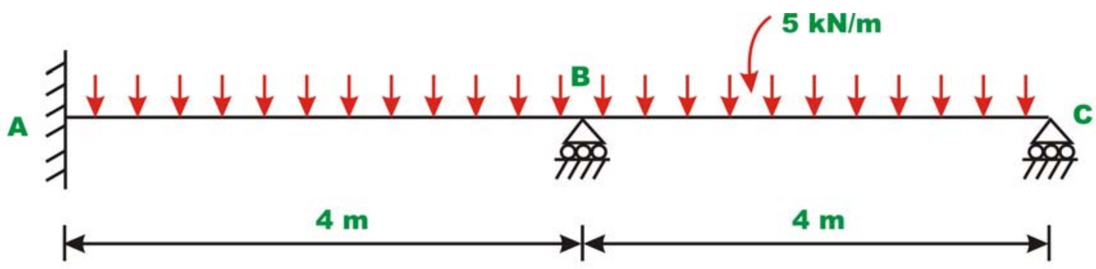


Fig. 28.2a

The numbering of joints and members are shown in Fig. 28.2b. The global degrees of freedom are also shown in the figure.

The given continuous beam is divided into two beam elements. Two degrees of freedom (one translation and one rotation) are considered at each end of the member. In the above figure, double headed arrows denote rotations and single headed arrow represents translations. Also it is observed that displacements $u_3 = u_4 = u_5 = u_6 = 0$ from support conditions.
First construct the member stiffness matrix for each member.

Member 1: $L = 4\,m$, node points 1-2.

The member stiffness matrix for all the members are the same, as the length and flexural rigidity of all members is the same.

$$\begin{array}{ll} \text{Global } d.o.f & \begin{array}{cccc} \quad 6 \quad & \quad 5 \quad & \quad 3 \quad & \quad 1 \quad \end{array} \\ [k'] = EI_{zz} & \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \begin{array}{c} 6 \\ 5 \\ 3 \\ 1 \end{array} \end{array} \tag{1}$$

On the member stiffness matrix, the corresponding global degrees of freedom are indicated to facilitate assembling.

Member 2: $L = 4m$, node points 2-3.

$$\begin{array}{ll} \text{Global } d.o.f & \begin{array}{cccc} \quad 3 \quad & \quad 1 \quad & \quad 4 \quad & \quad 2 \quad \end{array} \\ [k^2] = EI_{zz} & \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \begin{array}{c} 3 \\ 1 \\ 4 \\ 2 \end{array} \end{array} \tag{2}$$

The assembled global stiffness matrix of the continuous beam is of order 6×6. The assembled global stiffness matrix may be written as,

$$[K] = EI_{zz} \begin{bmatrix} 2 & 0.5 & 0 & -0.375 & 0.5 & 0.375 \\ 0.5 & 1.0 & 0.375 & -0.375 & 0 & 0 \\ 0 & 0.375 & 0.375 & -0.1875 & -0.375 & -0.1875 \\ -0.375 & -0.375 & -0.1875 & 0.1875 & 0 & 0 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 \\ 0.375 & 0 & -0.1875 & 0 & 0.375 & 0.1875 \end{bmatrix} \tag{3}$$

Now it is required to replace the given members loads by equivalent joint loads. The equivalent loads for the present case is shown in Fig. 28.2c. The displacement degrees of freedom are also shown in figure.

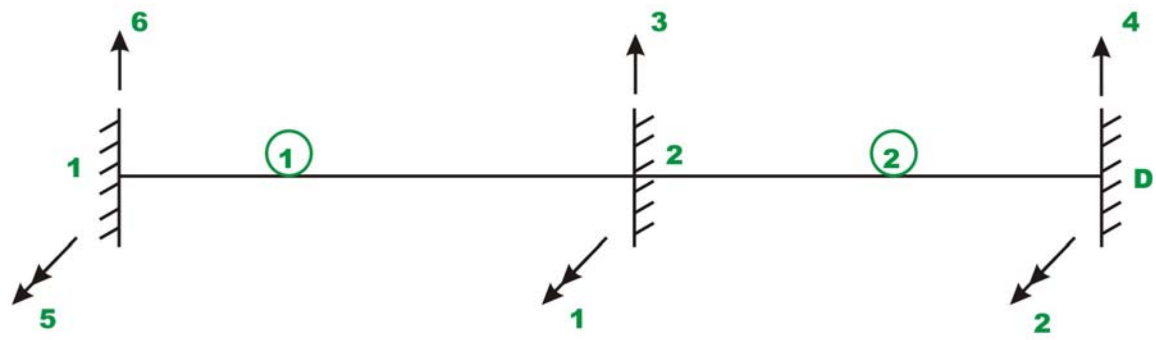


Fig. 28.2b Node and member

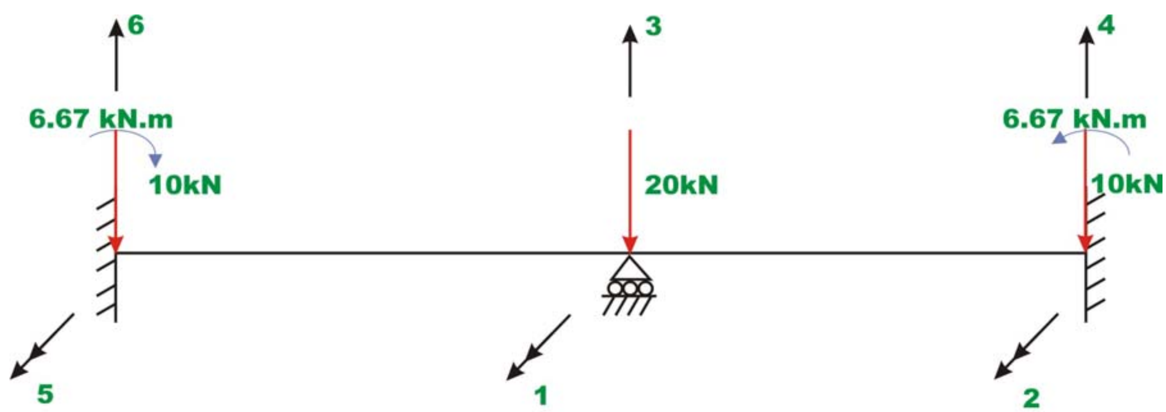


Fig. 28.2c Equivalent joint loads

Thus the global load vector corresponding to unconstrained degree of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 6.67 \end{Bmatrix} \tag{4}$$

Writing the load displacement relation for the entire continuous beam,

$$\begin{Bmatrix} 0 \\ 6.67 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 2 & 0.5 & 0 & -0.375 & 0.5 & 0.375 \\ 0.5 & 1.0 & 0.375 & -0.375 & 0 & 0 \\ 0 & 0.375 & 0.375 & -0.1875 & -0.375 & -0.1875 \\ -0.375 & -0.375 & -0.1875 & 0.1875 & 0 & 0 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 \\ 0.375 & 0 & -0.1875 & 0 & 0.375 & 0.1875 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \quad (5)$$

We know that $u_3 = u_4 = u_5 = u_6 = 0$. Thus solving for unknowns u_1 and u_2, yields

$$\begin{Bmatrix} 0 \\ 6.67 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 2.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad (6)$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{1.75 EI_{zz}} \begin{bmatrix} 1.0 & -0.5 \\ -0.5 & 2.0 \end{bmatrix} \begin{Bmatrix} 0 \\ 6.67 \end{Bmatrix}$$

$$= \frac{1}{EI_{zz}} \begin{Bmatrix} -1.905 \\ 7.62 \end{Bmatrix} \quad (7)$$

Thus displacements are,

$$u_1 = \frac{-1.905}{EI_{zz}} \quad \text{and} \quad u_2 = \frac{7.62}{EI_{zz}}$$

The unknown joint loads are given by,

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 0 & 0.375 \\ -0.375 & -0.375 \\ 0.5 & 0 \\ 0.375 & 0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} -1.905 \\ 7.62 \end{Bmatrix}$$

$$= \begin{Bmatrix} 2.857 \\ -2.14 \\ -0.95 \\ -0.714 \end{Bmatrix} \tag{8}$$

The actual support reactions are,

$$\begin{Bmatrix} R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 10 \\ -6.67 \\ 10 \end{Bmatrix} + \begin{Bmatrix} 2.857 \\ -2.14 \\ -0.95 \\ -0.714 \end{Bmatrix} = \begin{Bmatrix} 22.857 \\ 7.86 \\ -7.62 \\ 9.286 \end{Bmatrix} \tag{9}$$

Member end actions for element 1

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 6.66 \\ 10 \\ -6.66 \end{Bmatrix} + EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1.905 \end{Bmatrix}$$

$$= \begin{Bmatrix} 9.285 \\ 5.707 \\ 10.714 \\ -8.565 \end{Bmatrix} \tag{10}$$

Member end actions for element 2

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 6.66 \\ 10 \\ -6.66 \end{Bmatrix} + EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ -1.905 \\ 0 \\ 7.62 \end{Bmatrix}$$

$$= \begin{Bmatrix} 12.14 \\ 8.565 \\ 7.856 \\ 0 \end{Bmatrix}$$

Summary

In the previous lesson the beam element stiffness matrix is derived from fundamentals. Assembling member stiffness matrices, the global stiffness matrix is generated. The procedure to impose boundary conditions on the load-displacement relation is discussed. In this lesson, a few continuous beam problems are analysed by the direct stiffness method.

Module 4

Analysis of Statically Indeterminate Structures by the Direct Stiffness Method

Lesson 29

The Direct Stiffness Method: Beams (Continued)

Instructional Objectives

After reading this chapter the student will be able to

1. Compute moments developed in the continuous beam due to support settlements.
2. Compute moments developed in statically indeterminate beams due to temperature changes.
3. Analyse continuous beam subjected to temperature changes and support settlements.

29.1 Introduction

In the last two lessons, the analysis of continuous beam by direct stiffness matrix method is discussed. It is assumed in the analysis that the supports are unyielding and the temperature is maintained constant. However, support settlements can never be prevented altogether and hence it is necessary to make provisions in design for future unequal vertical settlements of supports and probable rotations of fixed supports. The effect of temperature changes and support settlements can easily be incorporated in the direct stiffness method and is discussed in this lesson. Both temperature changes and support settlements induce fixed end actions in the restrained beams. These fixed end forces are handled in the same way as those due to loads on the members in the analysis. In other words, the global load vector is formulated by considering fixed end actions due to both support settlements and external loads. At the end, a few problems are solved to illustrate the procedure.

29.2 Support settlements

Consider continuous beam ABC as shown in Fig. 29.1a. Assume that the flexural rigidity of the continuous beam is constant throughout. Let the support B settles by an amount Δ as shown in the figure. The fixed end actions due to loads are shown in Fig. 29.1b. The support settlements also induce fixed end actions and are shown in Fig. 29.1c. In Fig. 29.1d, the equivalent joint loads are shown. Since the beam is restrained against displacement in Fig. 29.1b and Fig. 29.1c, the displacements produced in the beam by the joint loads in Fig. 29.1d must be equal to the displacement produced in the beam by the actual loads in Fig. 29.1a. Thus to incorporate the effect of support settlement in the analysis it is required to modify the load vector by considering the negative of the fixed end actions acting on the restrained beam.

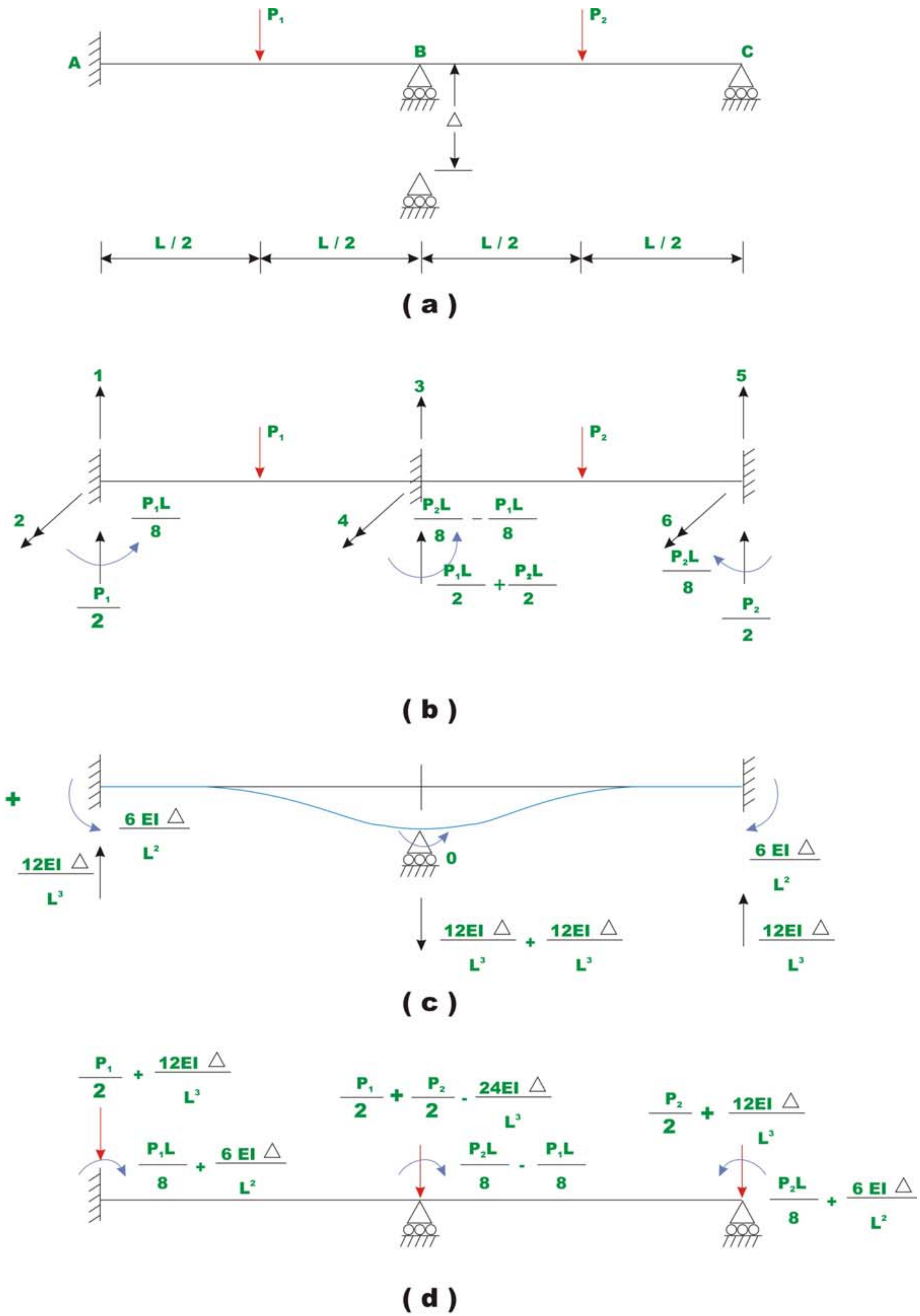

A
B
C
P_1
P_2
Δ
L / 2
L / 2
L / 2
L / 2
(a)
1
3
5
2
4
6
$\frac{P_1 L}{8}$
$\frac{P_2 L}{8} - \frac{P_1 L}{8}$
$\frac{P_1 L}{2} + \frac{P_2 L}{2}$
$\frac{P_2 L}{8}$
$\frac{P_1}{2}$
$\frac{P_2}{2}$
(b)
+
$\frac{6 EI \Delta}{L^2}$
$\frac{12EI \Delta}{L^3}$
0
$\frac{12EI \Delta}{L^3} + \frac{12EI \Delta}{L^3}$
$\frac{6 EI \Delta}{L^2}$
$\frac{12EI \Delta}{L^3}$
(c)
$\frac{P_1}{2} + \frac{12EI \Delta}{L^3}$
$\frac{P_1}{2} + \frac{P_2}{2} - \frac{24EI \Delta}{L^3}$
$\frac{P_2}{2} + \frac{12EI \Delta}{L^3}$
$\frac{P_1 L}{8} + \frac{6 EI \Delta}{L^2}$
$\frac{P_2 L}{8} - \frac{P_1 L}{8}$
$\frac{P_2 L}{8} + \frac{6 EI \Delta}{L^2}$
(d)

Fig. 29.1

29.3 Effect of temperature change

The effect of temperature on the statically indeterminate beams has already been discussed in lesson 9 of module 2 in connection with the flexibility matrix method. Consider the continuous beam ABC as shown in Fig. 29.2a, in which span BC is subjected to a differential temperature T_1 at top and T_2 at the bottom of the beam. Let temperature in span AB be constant. Let d be the depth of beam and EI be the flexural rigidity. As the cross section of the member remains plane after bending, the relative angle of rotation $d\theta$ between two cross sections at a distance dx apart is given by

$$d\theta = \alpha \frac{(T_1 - T_2)}{d} dx \tag{29.1}$$

where α is the co-efficient of the thermal expansion of the material. When beam is restrained, the temperature change induces fixed end moments in the beam as shown in Fig. 29.2b. The fixed end moments developed are,

$$M_1^T = -M_2^T = \alpha \ EI \ \frac{(T_1 - T_2)}{d} \tag{29.2}$$

Corresponding to the above fixed end moments; the equivalent joint loads can easily be constructed. Also due to differential temperatures there will not be any vertical forces/reactions in the beam.

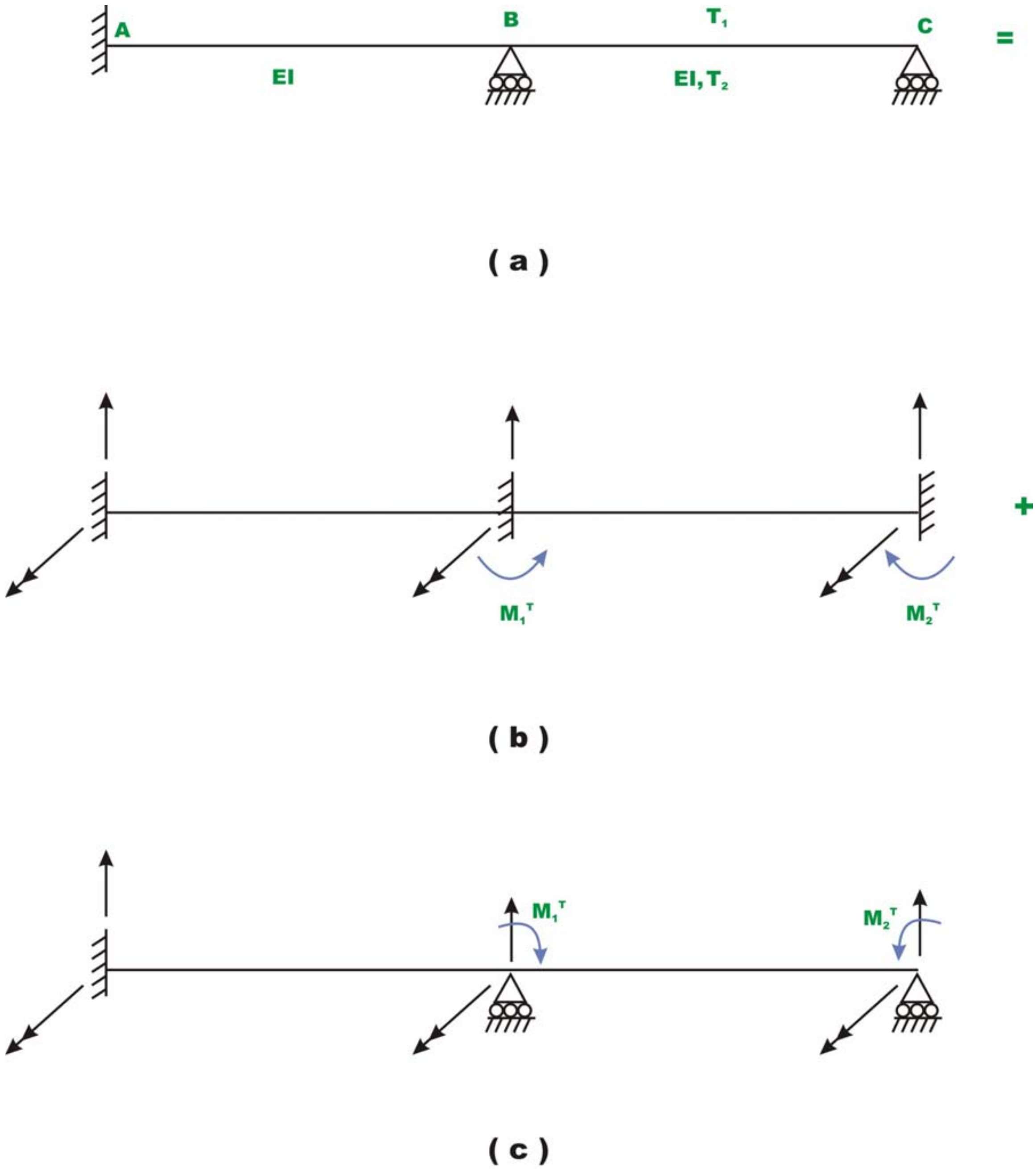
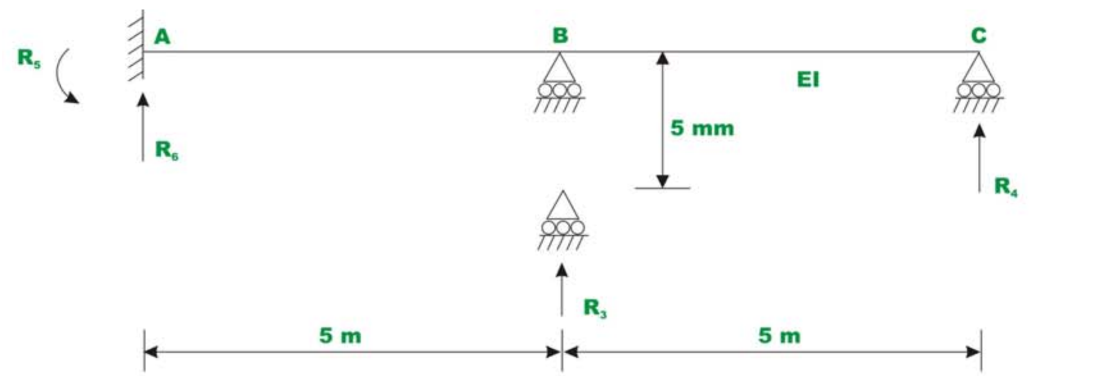


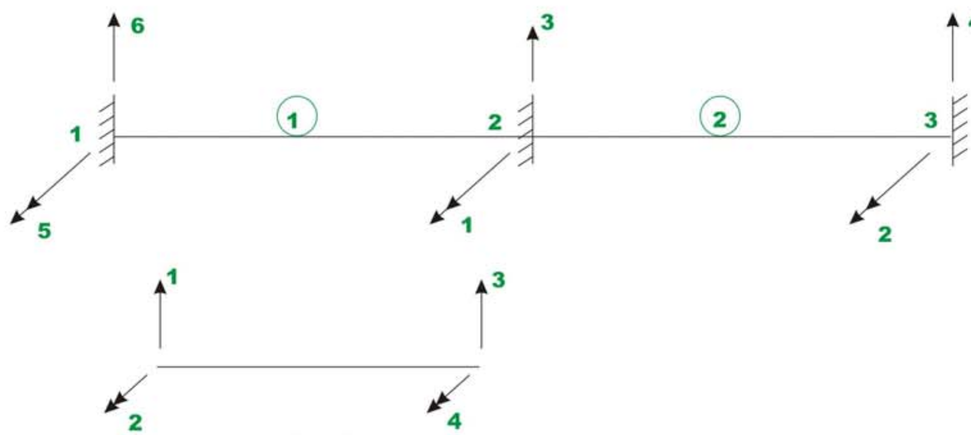
Fig. 29.2

Example 29.1

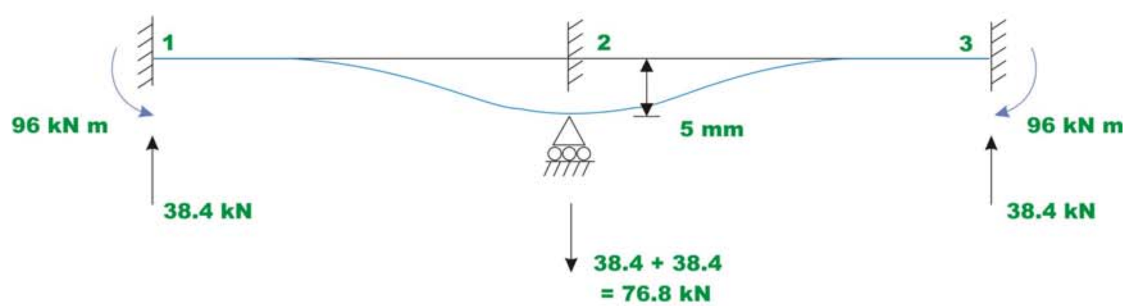
Calculate support reactions in the continuous beam ABC (vide Fig. 29.3a) having constant flexural rigidity EI, throughout due to vertical settlement of support B, by $5\,mm$ as shown in the figure. Assume $E = 200\ GPa$ and $I = 4 \times 10^{-4}\ m^4$.



(a) Continuous beam



(b) Node and member numbering



(c) Fixed end actions due to support settelement

(d) Equivalent joint loads

Fig. 29.3 Example 29.1

The continuous beam considered is divided into two beam elements. The numbering of the joints and members are shown in Fig. 29.3b. The possible global degrees of freedom are also shown in the figure. A typical beam element with two degrees of freedom at each node is also shown in the figure. For this problem, the unconstrained degrees of freedom are u_1 and u_2. The fixed end actions due to support settlement are,

$$M_{AB}^{F} = \frac{6EI\Delta}{L^2} = 96 \text{ kN.m}; \qquad M_{BA}^{F} = 96 \text{ kN.m}$$

$$M_{BC}^{F} = -96 \text{ kN.m} \quad ; M_{CB}^{F} = -96 \text{ kN.m} \tag{1}$$

The fixed-end moments due to support settlements are shown in Fig. 29.3c.

The equivalent joint loads due to support settlement are shown in Fig. 29.3d. In the next step, let us construct member stiffness matrix for each member.

Member 1: $L = 5m$, node points 1-2.

$$\begin{array}{ll} \text{Global } d.o.f & \begin{array}{cccc} \quad 6 \quad & \quad 5 \quad & \quad 3 \quad & \quad 1 \quad \end{array} \\ [k^1] = EI_{zz} & \begin{bmatrix} 0.096 & 0.24 & -0.096 & 0.24 \\ 0.24 & 0.80 & -0.24 & 0.40 \\ -0.096 & -0.24 & 0.096 & -0.24 \\ 0.24 & 0.40 & -0.24 & 0.80 \end{bmatrix} \begin{array}{c} 6 \\ 5 \\ 3 \\ 1 \end{array} \end{array} \tag{2}$$

Member 2: $L = 5m$, node points 2-3.

$$\begin{array}{ll} \text{Global } d.o.f & \begin{array}{cccc} \quad 3 \quad & \quad 1 \quad & \quad 4 \quad & \quad 2 \quad \end{array} \\ [k^2] = EI_{zz} & \begin{bmatrix} 0.096 & 0.24 & -0.096 & 0.24 \\ 0.24 & 0.80 & -0.24 & 0.40 \\ -0.096 & -0.24 & 0.096 & -0.24 \\ 0.24 & 0.40 & -0.24 & 0.80 \end{bmatrix} \begin{array}{c} 3 \\ 1 \\ 4 \\ 2 \end{array} \end{array} \tag{3}$$

On the member stiffness matrix, the corresponding global degrees of freedom are indicated to facilitate assembling. The assembled global stiffness matrix is of order 6×6. Assembled stiffness matrix $[K]$ is given by,

$$[K] = EI_{zz}\begin{bmatrix} & & & - & & \\ & & & - & & \\ & & & - & - & - \\ - & - & - & & & \\ & & - & & & \\ & & - & & & \end{bmatrix} \qquad (4)$$

Thus the global load vector corresponding to unconstrained degrees of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 96 \end{Bmatrix} \qquad (5)$$

Thus the load displacement relation for the entire continuous beam is,

$$\begin{Bmatrix} 0 \\ 96 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = EI_{zz}\left[\begin{array}{cc|cccc} 1.6 & 0.4 & 0 & -0.24 & 0.4 & 0.24 \\ 0.4 & 0.8 & 0.24 & -0.24 & 0 & 0 \\ \hline 0 & 0.24 & 0.192 & -0.096 & -0.24 & -0.096 \\ -0.24 & -0.24 & -0.096 & 0.096 & 0 & 0 \\ 0.4 & 0 & -0.24 & 0 & 0.8 & 0.24 \\ 0.24 & 0 & -0.096 & 0 & 0.24 & 0.096 \end{array}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \qquad (6)$$

Since, $u_3 = u_4 = u_5 = u_6 = 0$ due to support conditions. We get,

$$\begin{Bmatrix} 0 \\ 96 \end{Bmatrix} = EI_{zz}\begin{bmatrix} 1.6 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Thus solving for unknowns u_1 and u_2,

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{1.12EI_{zz}}\begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 1.6 \end{bmatrix}\begin{Bmatrix} 0 \\ 96 \end{Bmatrix}$$

$$= \frac{1}{EI_{zz}}\begin{Bmatrix} -34.285 \\ 137.14 \end{Bmatrix} = \begin{Bmatrix} -0.429 \times 10^{-3} \\ 1.714 \times 10^{-3} \end{Bmatrix}$$

$$u_1 = -0.429 \times 10^{-3} \text{ radians}; \qquad u_2 = 1.714 \times 10^{-3} \text{ radians} \qquad (7)$$

Now, unknown joint loads are calculated by,

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 0 & 0.24 \\ -0.24 & -0.24 \\ 0.4 & 0 \\ 0.24 & 0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} -34.285 \\ 137.14 \end{Bmatrix} \tag{8}$$

$$= \begin{Bmatrix} 32.91 \\ -24.68 \\ -13.71 \\ -8.23 \end{Bmatrix}$$

Now the actual support reactions R_3, R_4, R_5 and R_6 must include the fixed end support reactions. Thus,

$$\begin{Bmatrix} R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = \begin{Bmatrix} -76.8 \\ 38.4 \\ 96 \\ 38.4 \end{Bmatrix} + \begin{Bmatrix} 32.91 \\ -24.68 \\ -13.71 \\ -8.23 \end{Bmatrix} = \begin{Bmatrix} -43.88 \\ 13.72 \\ 82.29 \\ 30.17 \end{Bmatrix} \tag{9}$$

$$R_3 = -43.88 \text{ kN}; \quad R_4 = 13.72 \text{ kN}; \quad R_5 = 82.29 \text{ kN.m}; \quad R_6 = 30.17 \text{ kN} \tag{10}$$

Example 29.2

A continuous beam $ABCD$ is carrying a uniformly distributed load of $5\,kN/m$ as shown in Fig. 29.4a. Compute reactions due to following support settlements.

Support B $0.005\,m$ vertically downwards.

Support C $0.010\,m$ vertically downwards.

Assume $E = 200\ GPa$ and $I = 4 \times 10^{-4}\ m^4$.

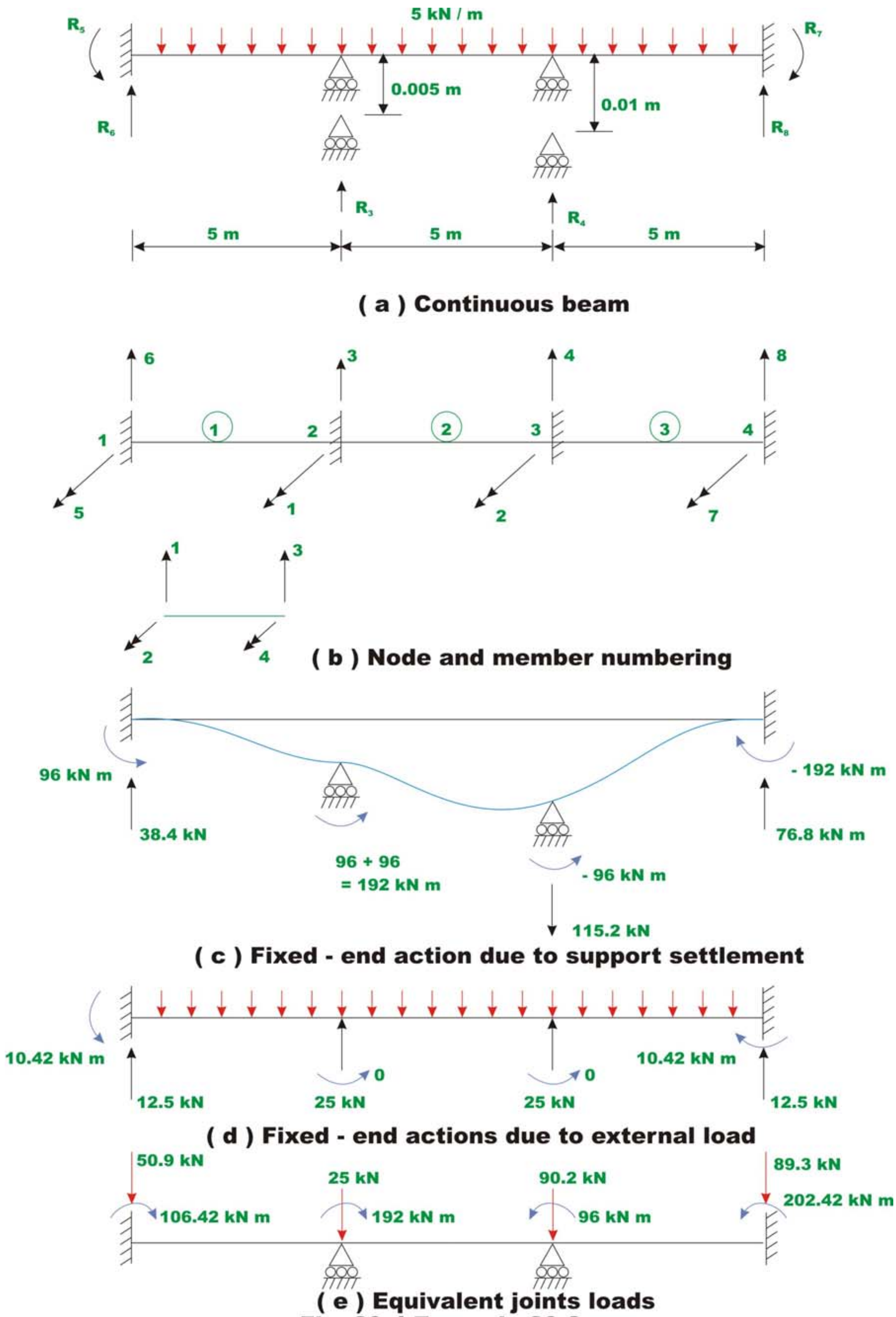


Fig. 29.4 Example 29.2

Solution

The node and member numbering are shown in Fig. 29.4(b), wherein the continuous beam is divided into three beam elements. It is observed from the figure that the unconstrained degrees of freedom are u_1 and u_2. The fixed end actions due to support settlements are shown in Fig. 29.4(c). and fixed end moments due to external loads are shown in Fig. 29.4(d). The equivalent joint loads due to support settlement and external loading are shown in Fig. 29.4(e). The fixed end actions due to support settlement are,

$M_A^F = -\frac{6EI}{L}(\psi)$ where ψ is the chord rotation and is taken $+ve$ if the rotation is counterclockwise.

Substituting the appropriate values in the above equation,

$$M_A^F = -\frac{6 \times 200 \times 10^9 \times 4 \times 10^{-4}}{5 \times 10^3}\left(-\frac{0.005}{5}\right) = 96 \quad \text{kN.m.}$$

$$M_B^F = 96 + 96 = 192 \quad \text{kN.m.}$$

$$M_C^F = 96 - 192 = -96 \quad \text{kN.m.}$$

$$M_D^F = -192 \quad \text{kN.m.} \qquad (1)$$

The vertical reactions are calculated from equations of equilibrium. The fixed end actions due to external loading are,

$$M_A^F = \frac{w \quad L^2}{12} = 10.42 \quad \text{kN.m.}$$

$$M_B^F = 10.42 - 10.42 = 0 \quad \text{kN.m.}$$

$$M_C^F = 0$$

$$M_D^F = -10.42 \quad \text{kN.m.} \qquad (2)$$

In the next step, construct member stiffness matrix for each member.

Member 1, $L = 5m$, node points 1-2.

$$\begin{array}{lccccc} Global\,d.o.f & 6 & 5 & 3 & 1 & \\ [k^1] = EI_{zz} & \multicolumn{4}{c}{\left[\begin{array}{cccc} 0.096 & 0.24 & -0.096 & 0.24 \\ 0.24 & 0.80 & -0.24 & 0.40 \\ -0.096 & -0.24 & 0.096 & -0.24 \\ 0.24 & 0.40 & -0.24 & 0.80 \end{array}\right]} & \begin{array}{c} 6 \\ 5 \\ 3 \\ 1 \end{array} \end{array} \quad (3)$$

Member 2, $L = 5m$, node points 2-3.

$$\begin{array}{lccccc} Global\,d.o.f & 3 & 1 & 4 & 2 & \\ [k^2] = EI_{zz} & \multicolumn{4}{c}{\left[\begin{array}{cccc} 0.096 & 0.24 & -0.096 & 0.24 \\ 0.24 & 0.80 & -0.24 & 0.40 \\ -0.096 & -0.24 & 0.096 & -0.24 \\ 0.24 & 0.40 & -0.24 & 0.80 \end{array}\right]} & \begin{array}{c} 3 \\ 1 \\ 4 \\ 2 \end{array} \end{array} \quad (4)$$

Member 3, $L = 5m$, node points 3-4.

$$\begin{array}{lccccc} Global\,d.o.f & 4 & 2 & 8 & 7 & \\ [k^3] = EI_{zz} & \multicolumn{4}{c}{\left[\begin{array}{cccc} 0.096 & 0.24 & -0.096 & 0.24 \\ 0.24 & 0.80 & -0.24 & 0.40 \\ -0.096 & -0.24 & 0.096 & -0.24 \\ 0.24 & 0.40 & -0.24 & 0.80 \end{array}\right]} & \begin{array}{c} 4 \\ 2 \\ 8 \\ 7 \end{array} \end{array} \quad (5)$$

On the member stiffness matrix, the corresponding global degrees of freedom are indicated to facilitate assembling. The assembled global stiffness matrix is of the order 8×8. Assembled stiffness matrix $[K]$ is,

$$[K] = EI_{zz}\left[\begin{array}{cc|cccccc} 1.60 & 0.40 & 0.0 & -0.24 & 0.40 & 0.24 & 0 & 0 \\ 0.40 & 1.60 & 0.24 & 0 & 0 & 0 & 0.40 & -0.24 \\ \hline 0 & 0.24 & 0.192 & -0.096 & -0.24 & -0.096 & 0 & 0 \\ -0.24 & 0 & -0.096 & 0.192 & 0 & 0 & 0.24 & -0.096 \\ 0.40 & 0 & -0.24 & 0 & 0.80 & 0.24 & 0 & 0 \\ 0.24 & 0 & -0.096 & 0 & 0.24 & 0.096 & 0 & 0 \\ 0 & 0.40 & 0 & 0.24 & 0 & 0 & 0.80 & -0.24 \\ 0 & -0.24 & 0 & -0.096 & 0 & 0 & -0.24 & 0.096 \end{array}\right] \tag{6}$$

The global load vector corresponding to unconstrained degree of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} -192 \\ 96 \end{Bmatrix} \tag{7}$$

Writing the load displacement relation for the entire continuous beam,

$$\begin{Bmatrix} -192 \\ 96 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = EI_{zz}\left[\begin{array}{cc|cccccc} 1.60 & 0.40 & 0.0 & -0.24 & 0.40 & 0.375 & 0 & 0 \\ 0.40 & 1.60 & 0.24 & 0 & 0 & 0 & 0.40 & -0.24 \\ \hline 0 & 0.24 & 0.192 & -0.096 & -0.24 & -0.096 & 0 & 0 \\ -0.24 & 0 & -0.096 & 0.192 & 0 & 0 & 0.24 & -0.096 \\ 0.40 & 0 & -0.24 & 0 & 0.80 & 0.24 & 0 & 0 \\ 0.24 & 0 & -0.096 & 0 & 0.24 & 0.096 & 0 & 0 \\ 0 & 0.40 & 0 & 0.24 & 0 & 0 & 0.80 & -0.24 \\ 0 & -0.24 & 0 & -0.096 & 0 & 0 & -0.24 & 0.096 \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} \tag{8}$$

We know that $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$. Thus solving for unknowns displacements u_1 and u_2 from equation,

$$\begin{Bmatrix} -192 \\ 96 \end{Bmatrix} = EI_{zz}\begin{bmatrix} 1.60 & 0.40 \\ 0.40 & 1.60 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{9}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{2.4(80\times10^3)} \begin{bmatrix} 1.60 & -0.40 \\ -0.40 & 1.60 \end{bmatrix} \begin{Bmatrix} -192 \\ 96 \end{Bmatrix}$$

$$= \begin{Bmatrix} -1.80\times10^{-3} \\ 1.20\times10^{-3} \end{Bmatrix} \tag{10}$$

$$u_1 = -1.80\times10^{-3} \text{ radians}; \qquad u_2 = 1.20\times10^{-3} \text{ radians} \tag{11}$$

The unknown joint loads are calculated as,

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = \left(80\times10^3\right) \begin{bmatrix} 0 & 0.24 \\ -0.24 & 0 \\ 0.40 & 0 \\ 0.24 & 0 \\ 0 & 0.40 \\ 0 & -0.24 \end{bmatrix} \begin{Bmatrix} -1.80\times10^{-3} \\ 1.20\times10^{-3} \end{Bmatrix}$$

$$= \begin{Bmatrix} 23.04 \\ 34.56 \\ -57.60 \\ -34.56 \\ 38.40 \\ -23.04 \end{Bmatrix} \tag{12}$$

Now the actual support reactions R_3, R_4, R_5, R_6, R_7 and R_8 must include the fixed end support reactions. Thus,

$$\begin{Bmatrix} R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} p_3^F \\ p_4^F \\ p_5^F \\ p_6^F \\ p_7^F \\ p_8^F \end{Bmatrix} + \begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = \begin{Bmatrix} 25 \\ -90.2 \\ 106.42 \\ 50.9 \\ -202.42 \\ 89.3 \end{Bmatrix} + \begin{Bmatrix} 23.04 \\ 34.56 \\ -57.60 \\ -34.56 \\ 38.40 \\ -23.04 \end{Bmatrix} = \begin{Bmatrix} 48.04 \\ -55.64 \\ 48.82 \\ 16.34 \\ -164.02 \\ 66.26 \end{Bmatrix} \tag{13}$$

$$\begin{aligned} &R_3 = 48.04 \text{ kN}; \quad R_4 = -55.64 \text{ kN}; \quad R_5 = 48.82 \text{ kN.m}; \\ &R_6 = 16.34 \text{ kN}; \quad R_7 = -164.02 \text{ kN.m}; \quad R_8 = 66.26 \text{ kN} \end{aligned} \tag{14}$$

Summary

The effect of temperature changes and support settlements can easily be incorporated in the direct stiffness method and is discussed in the present lesson. Both temperature changes and support settlements induce fixed end actions in the restrained beams. These fixed end forces are handled in the same way as those due to loads on the members in the analysis. In other words, the global load vector is formulated by considering fixed end actions due to both support settlements and external loads. At the end, a few problems are solved to illustrate the procedure.

Module
4
Analysis of Statically Indeterminate Structures by the Direct Stiffness Method

Lesson 30

The Direct Stiffness Method: Plane Frames

Instructional Objectives

After reading this chapter the student will be able to
1. Derive plane frame member stiffness matrix in local co-ordinate system.
2. Transform plane frame member stiffness matrix from local to global co-ordinate system.
3. Assemble member stiffness matrices to obtain the global stiffness matrix of the plane frame.
4. Write the global load-displacement relation for the plane frame.
5. Impose boundary conditions on the load-displacement relation.
6. Analyse plane frames by the direct stiffness matrix method.

30.1 Introduction

In the case of plane frame, all the members lie in the same plane and are interconnected by rigid joints. The internal stress resultants at a cross-section of a plane frame member consist of bending moment, shear force and an axial force. The significant deformations in the plane frame are only flexural and axial. In this lesson, the analysis of plane frame by direct stiffness matrix method is discussed. Initially, the stiffness matrix of the plane frame member is derived in its local co-ordinate axes and then it is transformed to global co-ordinate system. In the case of plane frames, members are oriented in different directions and hence before forming the global stiffness matrix it is necessary to refer all the member stiffness matrices to the same set of axes. This is achieved by transformation of forces and displacements to global co-ordinate system.

30.2 Member Stiffness Matrix

Consider a member of a plane frame as shown in Fig. 30.1a in the member co-ordinate system $x'y'z'$. The global orthogonal set of axes xyz is also shown in the figure. The frame lies in the xy plane. The member is assumed to have uniform flexural rigidity EI and uniform axial rigidity EA for sake of simplicity. The axial deformation of member will be considered in the analysis. The possible displacements at each node of the member are: translation in x'- and y'-direction and rotation about z'- axis.

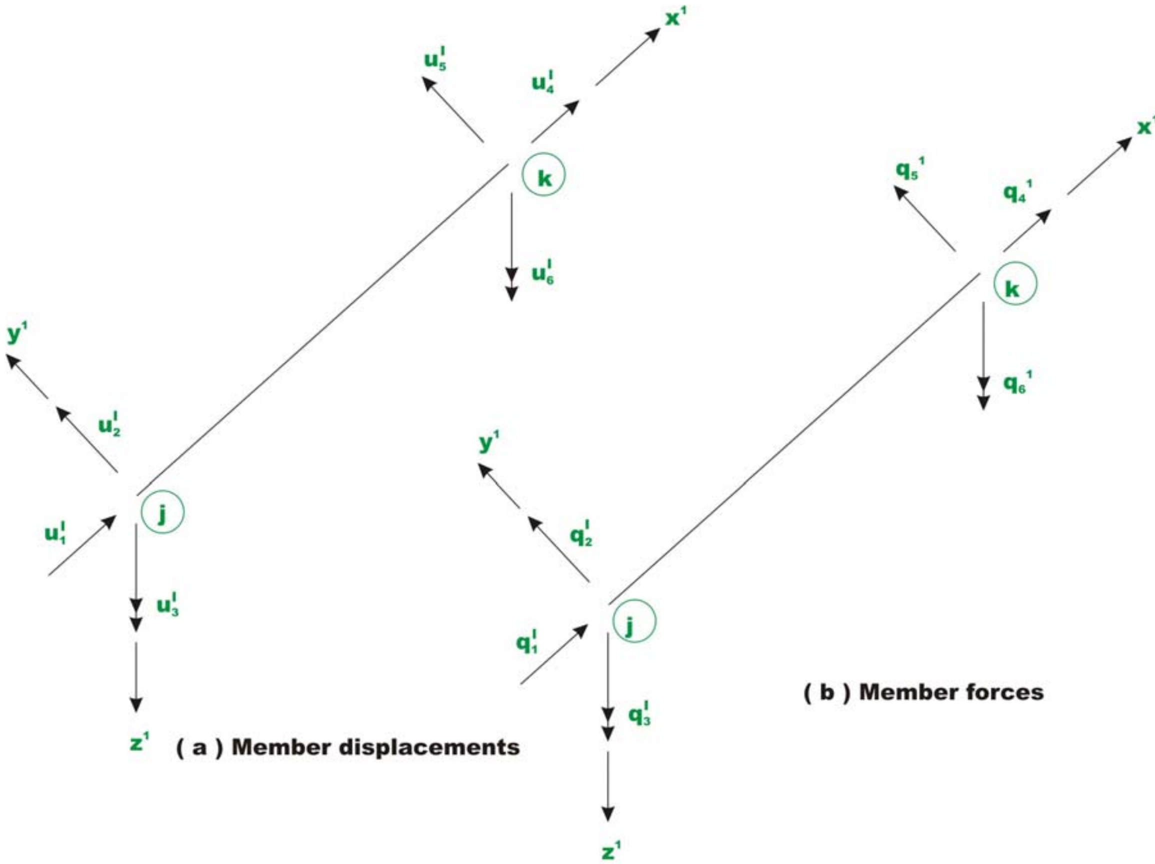


Fig. 30.1 Frame member in local co-ordinate system

Thus the frame members have six (6) degrees of freedom and are shown in Fig.30.1a. The forces acting on the member at end j and k are shown in Fig. 30.1b. The relation between axial displacement and axial forces is derived in chapter 24. Similarly the relation between shear force, bending moment with translation along y' axis and rotation about z' axis are given in lesson 27. Combining them, we could write the load-displacement relation in the local co-ordinate axes for the plane frame as shown in *Fig 30.1a, b* as,

$$
\begin{Bmatrix} q'_1 \\ q'_2 \\ q'_3 \\ q'_4 \\ q'_5 \\ q'_6 \end{Bmatrix} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \\ u'_5 \\ u'_6 \end{Bmatrix} \qquad (30.1a)
$$

This may be succinctly written as

$$
\{q'\} = [k']\{u'\} \qquad (30.1b)
$$

where $[k']$ is the member stiffness matrix. The member stiffness matrix can also be generated by giving unit displacement along each possible displacement degree of freedom one at a time and calculating resulting restraint actions.

30.3 Transformation from local to global co-ordinate system

30.3.1 *Displacement transformation matrix*

In plane frame the members are oriented in different directions and hence it is necessary to transform stiffness matrix of individual members from local to global co-ordinate system before formulating the global stiffness matrix by assembly. In Fig. 30.2a the plane frame member is shown in local coordinate axes $x'y'z'$ and in Fig. 30.2b, the plane frame is shown in global coordinate axes xyz. Two ends of the plane frame member are identified by j and k. Let u'_1, u'_2, u'_3 and u'_4, u'_5, u'_6 be respectively displacements of ends j and k of the member in local coordinate system $x'y'z'$. Similarly u_1, u_2, u_3 and u_4, u_5, u_6 respectively are displacements of ends j and k of the member in global co-ordinate system.

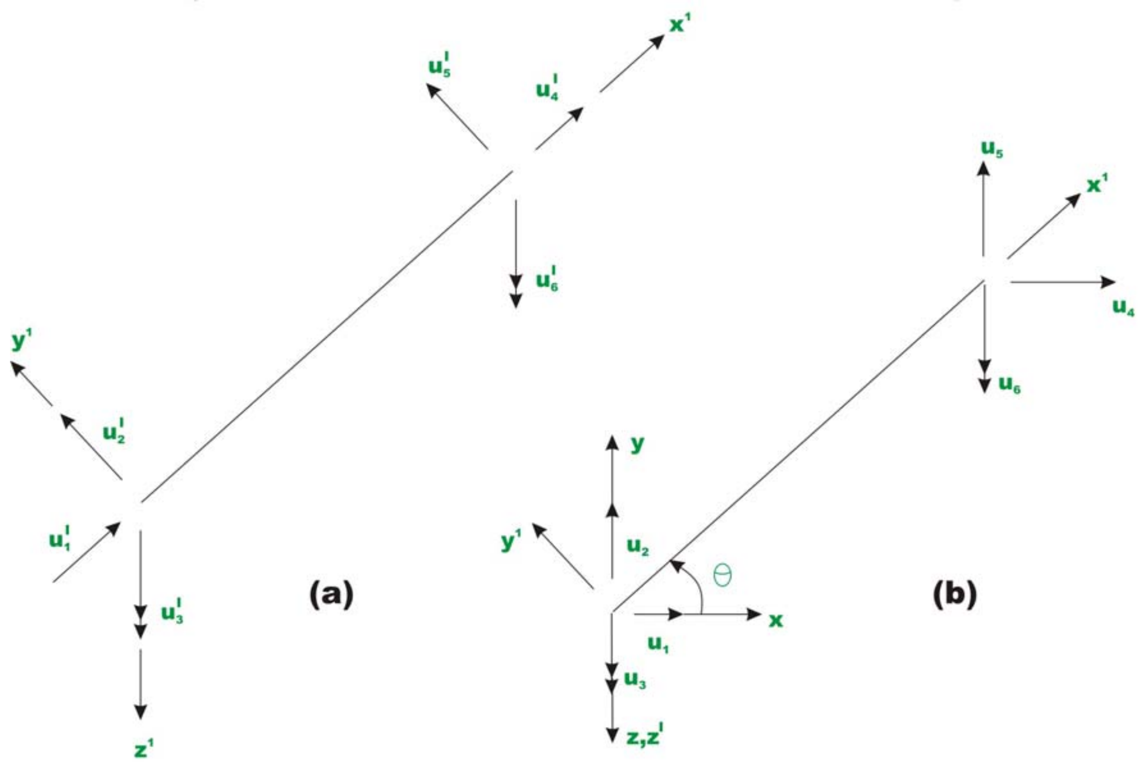


Fig. 30.2 Plane frame member in
(a) Local co-ordinate system
(b) Global co-ordinate system.

Let θ be the angle by which the member is inclined to global x-axis. From Fig.30.2a and b, one could relate u'_1, u'_2, u'_3 to u_1, u_2, u_3 as,

$$u'_1 = u_1 \cos\theta + u_2 \sin\theta \tag{30.2a}$$

$$u'_2 = -u_1 \sin\theta + u_2 \cos\theta \tag{30.2b}$$

$$u'_3 = u_3 \tag{30.2c}$$

This may be written as,

$$\begin{Bmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \\ u'_5 \\ u'_6 \end{Bmatrix} = \left[\begin{array}{ccc|ccc} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \qquad (30.3a)$$

Where, $l = \cos\theta$ and $m = \sin\theta$.

This may be written in compact form as,

$$\{u'\} = [T]\{u\} \qquad (30.3b)$$

In the above equation, $[T]$ is defined as the displacement transformation matrix and it transforms the six global displacement components to six displacement components in local co-ordinate axes. Again, if the coordinate of node j is (x_1, y_1) and coordinate of node k are (x_2, y_2), then,

$$l = \cos\theta = \frac{x_2 - x_1}{L} \quad \text{and} \quad m = \sin\theta = \frac{y_2 - y_1}{L}.$$

Where $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (30.4)

30.3.2 *Force displacement matrix*

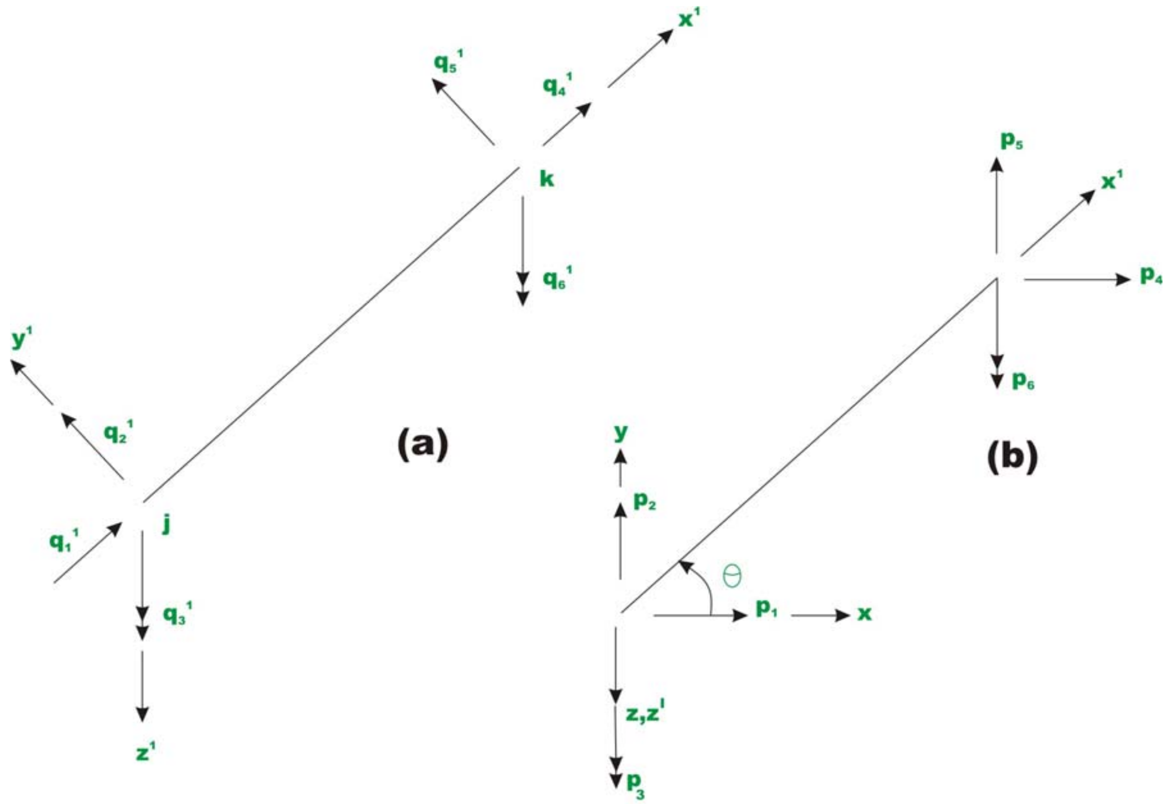


Fig. 30.3 Plane frame member in
(a) Local co-ordinate axes and
(b) In global co-ordinate system

Let q'_1, q'_2, q'_3 and q'_4, q'_5, q'_6 be respectively the forces in member at nodes j and k as shown in Fig. 30.3a in local coordinate system. p_1, p_2, p_3 and p_4, p_5, p_6 are the forces in members at node j and k respectively as shown in Fig. 30.3b in the global coordinate system. Now from Fig 30.3a and b,

$$p_1 = q'_1 \cos\theta - q'_2 \sin\theta \tag{30.5a}$$

$$p_2 = q'_1 \sin\theta + q'_2 \cos\theta \tag{30.5b}$$

$$p_3 = q'_3 \tag{30.5c}$$

Thus the forces in global coordinate system can be related to forces in local coordinate system by

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \left[\begin{array}{ccc|ccc} l & -m & 0 & 0 & 0 & 0 \\ m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & l & -m & 0 \\ 0 & 0 & 0 & m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} q'_1 \\ q'_2 \\ q'_3 \\ q'_4 \\ q'_5 \\ q'_6 \end{Bmatrix} \tag{30.6a}$$

Where, $l = \cos\theta$ and $m = \sin\theta$.

This may be compactly written as,

$$\{p\} = [T]^T \{q'\} \tag{30.6b}$$

30.3.3 *Member global stiffness matrix*

From equation (30.1b), we have

$$\{q'\} = [k']\{u'\}$$

Substituting the above value of $\{q'\}$ in equation (30.6b) results in,

$$\{p\} = [T]^T [k']\{u'\} \tag{30.7}$$

Making use of equation (30.3b), the above equation may be written as

$$\{p\} = [T]^T [k'][T]\{u\} \tag{30.8}$$

or

$$\{p\} = [k]\{u\} \tag{30.9}$$

The equation (30.9) represents the member load-displacement relation in global coordinate system. The global member stiffness matrix $[k]$ is given by,

$$[k] = [T]^T [k'][T] \tag{30.10}$$

After transformation, the assembly of member stiffness matrices is carried out in a similar procedure as discussed for truss. Finally the global load-displacement equation is written as in the case of continuous beam. Few numerical problems are solved by direct stiffness method to illustrate the procedure discussed.

Example 30.1

Analyze the rigid frame shown in Fig 30.4a by direct stiffness matrix method. Assume $E = 200 GPa$; $I_{ZZ} = 1.33 \times 10^{-4}\ m^4$ and $A = 0.04 m^2$. The flexural rigidity EI and axial rigidity EA are the same for both the beams.

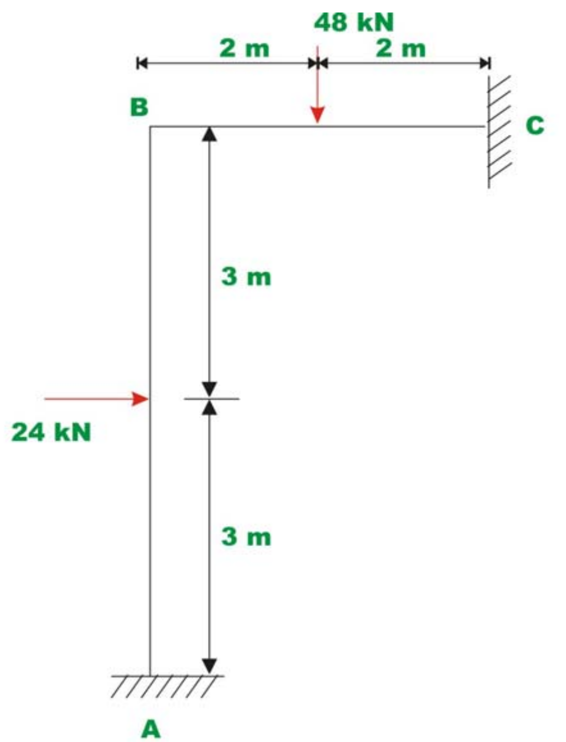


Fig. 30.4a Rigid Frame.

Solution:

The plane frame is divided in to two beam elements as shown in Fig. 30.4b. The numbering of joints and members are also shown in Fig. 30.3b. Each node has three degrees of freedom. Degrees of freedom at all nodes are also shown in the figure. Also the local degrees of freedom of beam element are shown in the figure as inset.

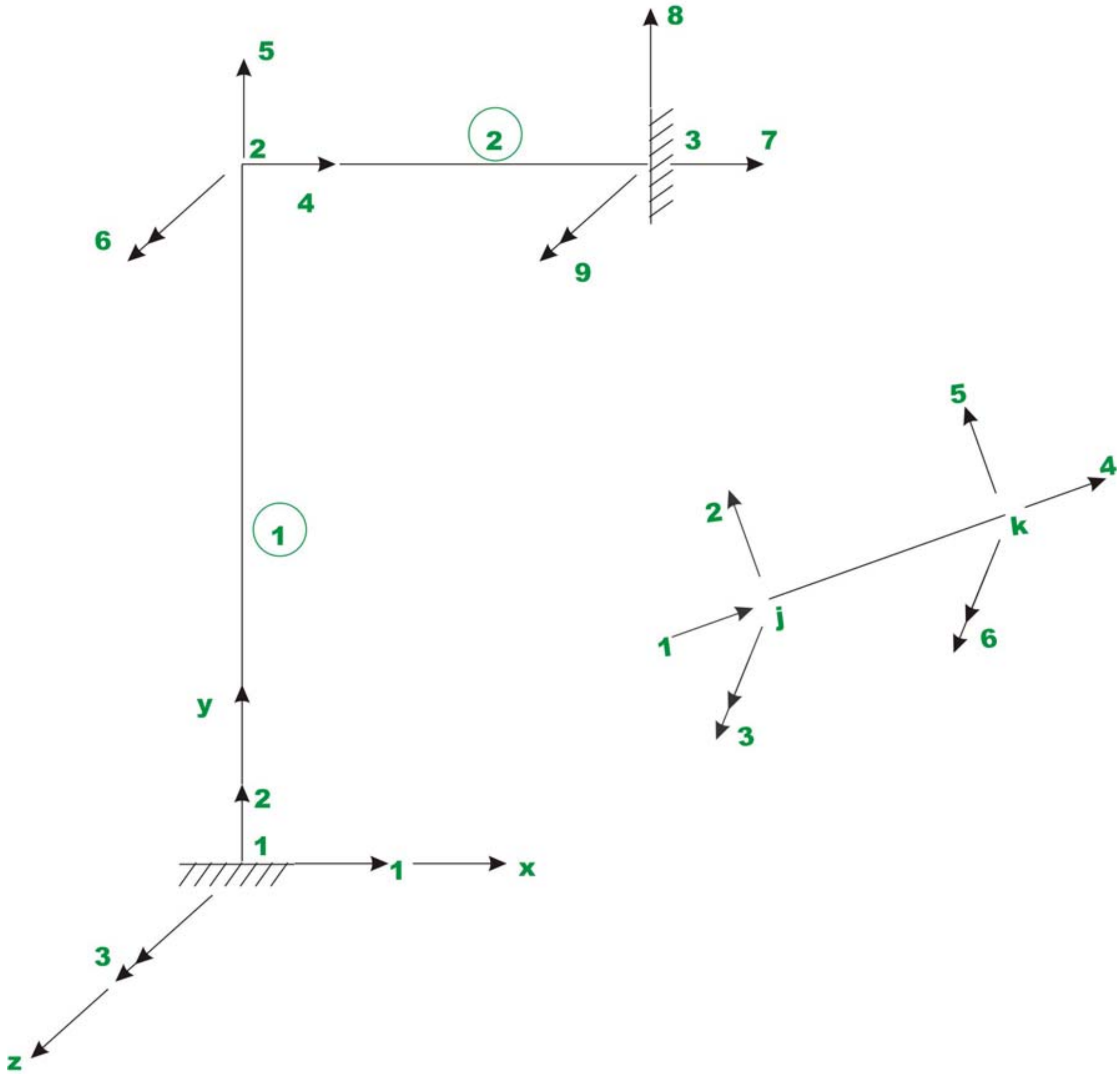


Fig. 30.4b Node and member numbering.

Formulate the element stiffness matrix in local co-ordinate system and then transform it to global co-ordinate system. The origin of the global co-ordinate system is taken at node 1. Here the element stiffness matrix in global co-ordinates is only given.

Member 1: $L = 6\ m$; $\theta = 90°$ node points 1-2; $l = 0$ and $m = 1$.

$$\left[k^1\right]=\left[T\right]^T\left[k'\right]\ \left[T\right]$$

$$\begin{array}{c}
\\ \left[k^1\right]=
\end{array}
\begin{array}{cc}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\
\begin{bmatrix}
1.48\times10^3 & 0 & 4.44\times10^3 & 1.48\times10^3 & 0 & 4.44\times10^3 \\
0 & 1.333\times10^6 & 0 & 0 & -1.333\times10^6 & 0 \\
4.44\times10^3 & 0 & 17.78\times10^3 & 4.44\times10^3 & 0 & 8.88\times10^3 \\
1.48\times10^3 & 0 & 4.44\times10^3 & 1.48\times10^3 & 0 & 4.44\times10^3 \\
0 & -1.333\times10^6 & 0 & 0 & 1.333\times10^6 & 0 \\
4.44\times10^3 & 0 & 8.88\times10^3 & 4.44\times10^3 & 0 & 17.78\times10^3
\end{bmatrix} &
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}$$

(1)

Member 2: $L=4\ m$; $\theta=0°$; node points 2-3 ; $l=1$ and $m=0$.

$$\left[k^2\right]=\left[T\right]^T\left[k'\right]\left[T\right]$$

$$= \begin{array}{cc}
\begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} & \\
\begin{bmatrix}
2.0\times10^6 & 0 & 0 & -2.0\times10^6 & 0 & 0 \\
0 & 5\times10^3 & 10\times10^3 & 0 & -5\times10^3 & 10\times10^3 \\
0 & 10\times10^3 & 26.66\times10^3 & 0 & -10\times10^3 & 8.88\times10^3 \\
-2.0\times10^6 & 0 & 0 & 2.0\times10^6 & 0 & 0 \\
0 & -5\times10^3 & -10\times10^3 & 0 & 5\times10^3 & -10\times10^3 \\
0 & 10\times10^3 & 8.88\times10^3 & 0 & -10\times10^3 & 26.66\times10^3
\end{bmatrix} &
\begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
\end{array}$$

(2)

The assembled global stiffness matrix $\left[K\right]$ is of the order 9×9. Carrying out assembly in the usual manner, we get,

$$[K]=\left[\begin{array}{ccc|ccc|ccc} 1.48 & 0 & -4.44 & -1.48 & 0 & -4.44 & 0 & 0 & 0 \\ 0 & 1333.3 & 0 & 0 & -1333.3 & 0 & 0 & 0 & 0 \\ -4.44 & 0 & 17.77 & 4.44 & 0 & 8.88 & 0 & 0 & 0 \\ \hline -1.48 & 0 & 4.44 & 2001.5 & 0 & 4.44 & -2000 & 0 & 0 \\ 0 & -1333.3 & 0 & 0 & 1338.3 & 10 & 0 & -5 & 10 \\ -4.44 & 0 & 8.88 & 4.44 & 10 & 44.44 & 0 & -10 & 13.33 \\ \hline 0 & 0 & 0 & -2000 & 0 & 0 & 2000 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & -10 & 0 & 5 & -10 \\ 0 & 0 & 0 & 0 & 10 & 13.33 & 0 & -10 & 26.66 \end{array}\right] \quad (3)$$

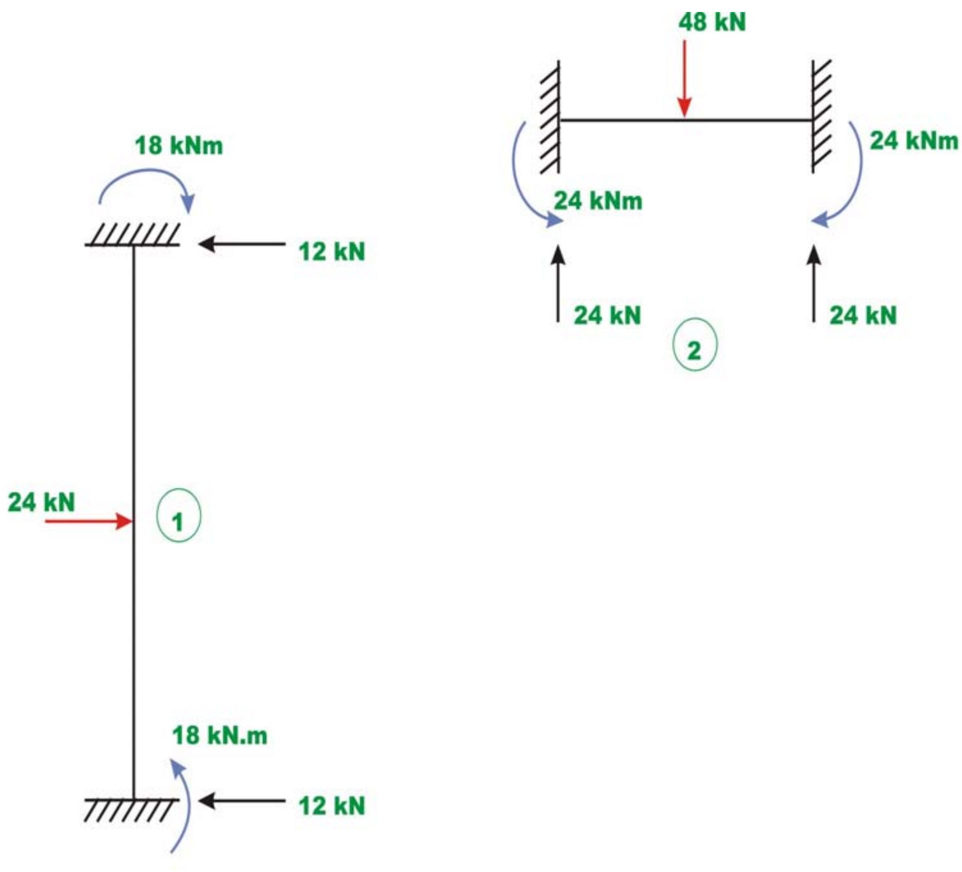


Fig. 30.4c Fixed end action due to external load in element (1) and (2)

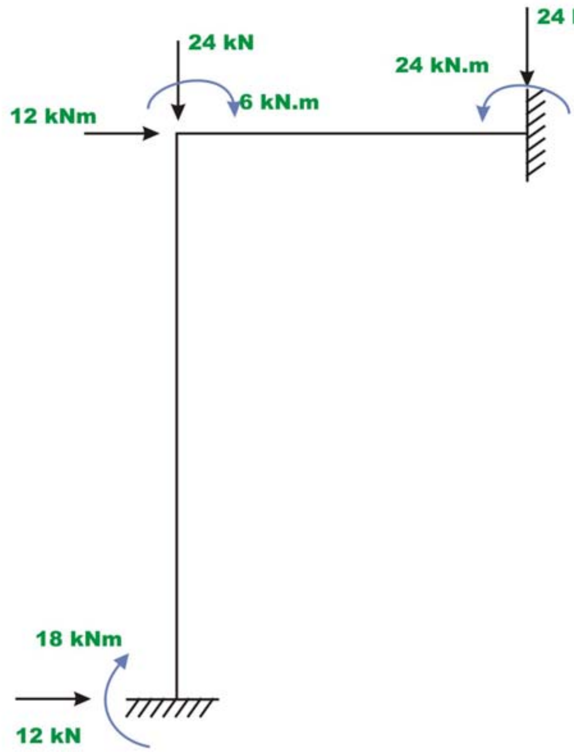


Fig. 30.4d Equivalent joint loads.

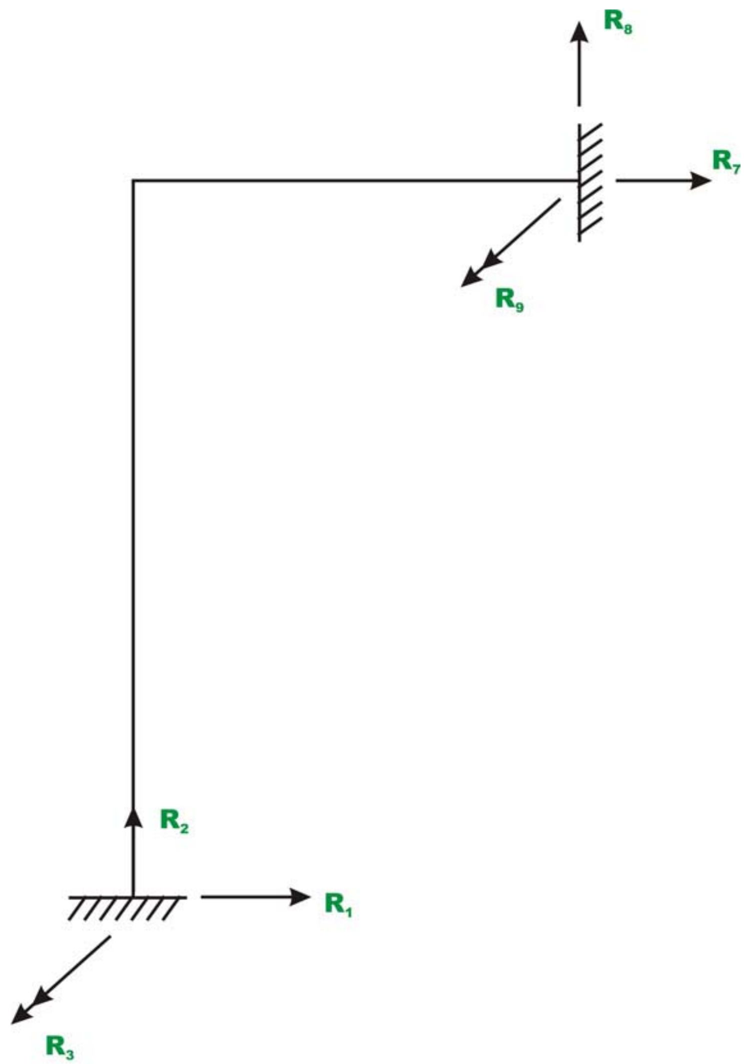


Fig. 30.4e Support Reactions.

The load vector corresponding to unconstrained degrees of freedom is (vide 30.4d),

$$\{p_k\} = \begin{Bmatrix} p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{Bmatrix} 12 \\ -24 \\ -6 \end{Bmatrix} \qquad (4)$$

In the given frame constraint degrees of freedom are $u_1, u_2, u_3, u_7, u_8, u_9$. Eliminating rows and columns corresponding to constrained degrees of freedom from global stiffness matrix and writing load-displacement relationship for only unconstrained degree of freedom,

$$\begin{Bmatrix} 12 \\ -24 \\ -6 \end{Bmatrix} = 10^3 \begin{bmatrix} 2001.5 & 0 & 4.44 \\ 0 & 1338.3 & 10 \\ 4.44 & 10 & 44.44 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} \tag{5}$$

Solving we get,

$$\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 6.28\times 10^{-6} \\ -1.695\times 10^{-5} \\ -0.13\times 10^{-3} \end{Bmatrix} \tag{6}$$

$u_4 = 6.28\times 10^{-6}$ m., $\quad u_5 = -1.695\times 10^{-5}$

Let $R_1, R_2, R_3, R_7, R_8, R_9$ be the support reactions along degrees of freedom $1, 2, 3, 7, 8, 9$ respectively (vide Fig. 30.4e). Support reactions are calculated by

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_7 \\ R_8 \\ R_9 \end{Bmatrix} = \begin{Bmatrix} p_1^F \\ p_2^F \\ p_3^F \\ p_7^F \\ p_8^F \\ p_9^F \end{Bmatrix} + 10^3 \begin{array}{c} \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1.48 & 0 & -4.44 \\ - & -1333.3 & 0 \\ 4.44 & 0 & 8.88 \\ -2000 & 0 & 0 \\ 0 & -5 & -10 \\ 0 & 10 & 13.33 \end{bmatrix} \end{array} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_7 \\ R_8 \\ R_9 \end{Bmatrix} = \begin{Bmatrix} -12 \\ 0 \\ 18 \\ 0 \\ 24 \\ -24 \end{Bmatrix} + \begin{Bmatrix} 0.57 \\ 22.59 \\ -1.14 \\ -12.57 \\ 1.40 \\ -1.92 \end{Bmatrix} = \begin{Bmatrix} -11.42 \\ 22.59 \\ 16.85 \\ -12.57 \\ 25.40 \\ -25.92 \end{Bmatrix} \tag{7}$$

Example 30.2

Analyse the rigid frame shown in Fig 30.5a by direct stiffness matrix method. Assume $E = 200$ GPa ; $I_{ZZ} = 1.33 \times 10^{-5}$ m^4 and $A = 0.01 m^2$. The flexural rigidity EI and axial rigidity EA are the same for all beams.

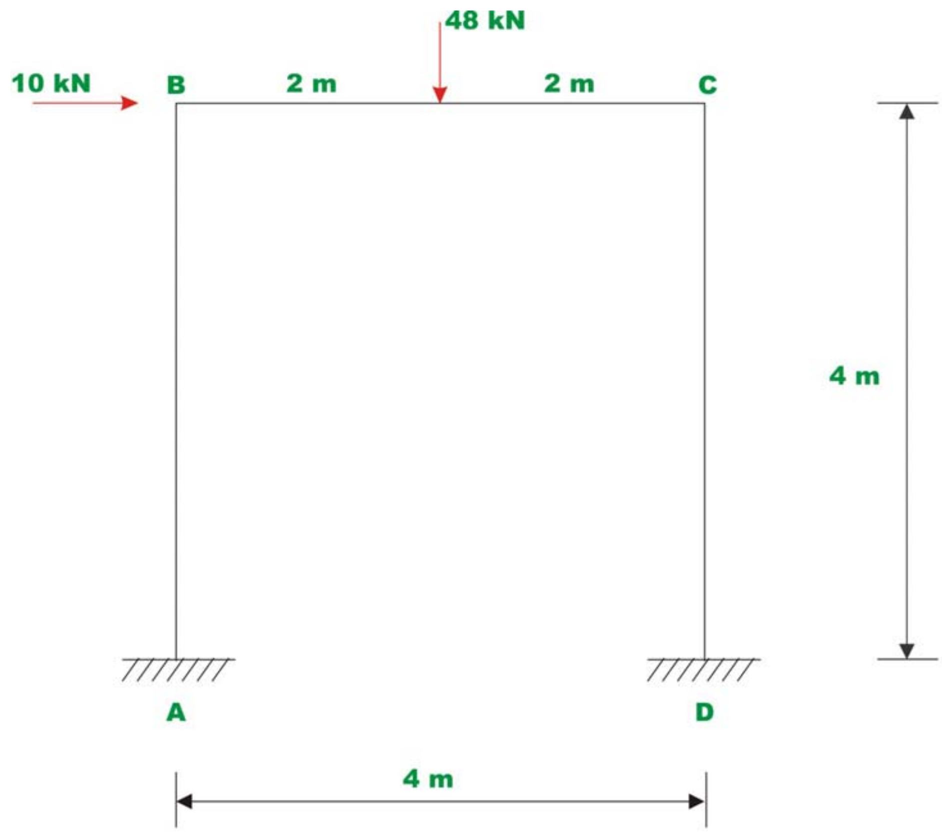


Fig. 30.5a Rigid Frame of Example 30.2

Solution:

The plane frame is divided in to three beam elements as shown in Fig. 30.5b. The numbering of joints and members are also shown in Fig. 30.5b. The possible degrees of freedom at nodes are also shown in the figure. The origin of the global co- ordinate system is taken at A (node 1).

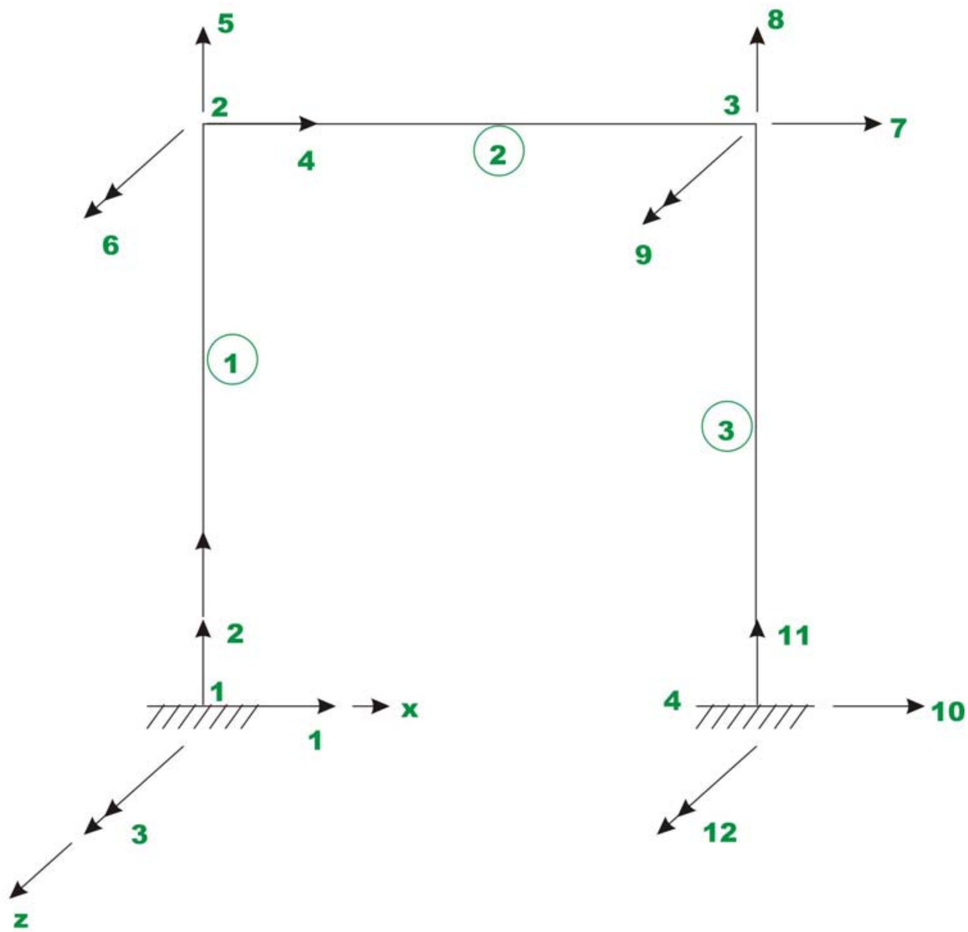


Fig. 30.5b Node and Member numbering.

Now formulate the element stiffness matrix in local co-ordinate system and then transform it to global co-ordinate system. In the present case three degrees of freedom are considered at each node.

Member 1: $L = 4\ m$; $\theta = 90°$; node points 1-2 ; $l = \dfrac{x_2 - x_1}{L} = 0$ and $m = \dfrac{y_2 - y_1}{L} = 1$.

The following terms are common for all elements.

$$\frac{AE}{L} = 5 \times 10^5 \text{ kN/m}; \qquad \frac{6EI}{L^2} = 9.998 \times 10^2 \text{kN}$$

$$\frac{12EI}{L^3} = 4.999 \times 10^2 \text{ kN/m}; \qquad \frac{4EI}{L} = 2.666 \times 10^3 \text{kN.m}$$

$$[k^1] = [T]^T [k'][T]$$

$$
= \begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
0.50\times10^3 & 0 & -1\times10^3 & -0.50\times10^3 & 0 & -1\times10^3 \\
0 & 5\times10^5 & 0 & 0 & -5\times10^5 & 0 \\
-1\times10^3 & 0 & 2.66\times10^3 & 1\times10^3 & 0 & 1.33\times10^3 \\
-0.50\times10^3 & 0 & 1\times10^3 & 0.50\times10^3 & 0 & 1\times10^3 \\
0 & -5\times10^5 & 0 & 0 & 5\times10^5 & 0 \\
-1\times10^3 & 0 & 1.33\times10^3 & 1\times10^3 & 0 & 2.66\times10^3
\end{bmatrix}
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

(1)

Member 2: $L = 4\ m$; $\theta = 0°$ node points 2-3; $l = 1$ and $m = 0$.

$$[k^2] = [T]^T [k'][T]$$

$$
= \begin{array}{c}
\begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{bmatrix}
5.0\times10^5 & 0 & 0 & -5.0\times10^6 & 0 & 0 \\
0 & 0.5\times10^3 & 1\times10^3 & 0 & -0.5\times10^3 & 1\times10^3 \\
0 & 1\times10^3 & 2.666\times10^3 & 0 & -1\times10^3 & 1.33\times10^3 \\
-5.0\times10^6 & 0 & 0 & 5.0\times10^6 & 0 & 0 \\
0 & -0.5\times10^3 & -1\times10^3 & 0 & 0.5\times10^3 & -1\times10^3 \\
0 & 1\times10^3 & 1.33\times10^3 & 0 & -1\times10^3 & 2.666\times10^3
\end{bmatrix}
\end{array}
\begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
$$

(2)

Member 3: $L = 4\ m$; $\theta = 270°$; node points 3-4 ; $l = \frac{x_2 - x_1}{L} = 0$ and $m = \frac{y_2 - y_1}{L} = -1$.

$$[k^3] = [T]^T [k'][T]$$

$$
= \begin{array}{c c}
\begin{array}{cccccc} 7 & 8 & 9 & 10 & 11 & 12 \end{array} & \\
\begin{bmatrix}
0.50\times10^3 & 0 & 1\times10^3 & -0.50\times10^3 & 0 & 1\times10^3 \\
0 & 5\times10^5 & 0 & 0 & -5\times10^5 & 0 \\
1\times10^3 & 0 & 2.66\times10^3 & -1\times10^3 & 0 & 1.33\times10^3 \\
-0.50\times10^3 & 0 & -1\times10^3 & 0.50\times10^3 & 0 & -1\times10^3 \\
0 & -5\times10^5 & 0 & 0 & 5\times10^5 & 0 \\
1\times10^3 & 0 & 1.33\times10^3 & -1\times10^3 & 0 & 2.66\times10^3
\end{bmatrix} & \begin{array}{c} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}
\end{array}
$$

(3)

The assembled global stiffness matrix $[K]$ is of the order 12×12. Carrying out assembly in the usual manner, we get,

$$
[K] = 10^3 \times \left[\begin{array}{ccc|cccccc|ccc}
0.50 & 0 & -1.0 & -0.50 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 500 & 0 & 0 & -500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1.0 & 0 & 2.66 & 1.0 & 0 & 1.33 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-0.50 & 0 & 1.0 & 500.5 & 0 & 1.0 & -500 & 0 & 0 & 0 & 0 & 0 \\
0 & -500 & 0 & 0 & 500.5 & 1.0 & 0 & -0.50 & 1.0 & 0 & 0 & 0 \\
-1.0 & 0 & 1.33 & 1.0 & 1.0 & 5.33 & 0 & -1.0 & 1.33 & 0 & 0 & 0 \\
0 & 0 & 0 & -500 & 0 & 0 & 500.5 & 0 & 1.0 & -0.5 & 0 & 1.0 \\
0 & 0 & 0 & 0 & -0.5 & -1.0 & 0 & 500.5 & -1.0 & 0 & -500 & 0 \\
0 & 0 & 0 & 0 & 1.0 & 1.33 & 1.0 & -1.0 & 5.33 & -1.0 & 0 & 1.33 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 & -1.0 & 0.5 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -500 & 0 & 0 & 500 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.0 & 0 & 1.33 & -1 & 0 & 2.66
\end{array}\right]
$$

(4)

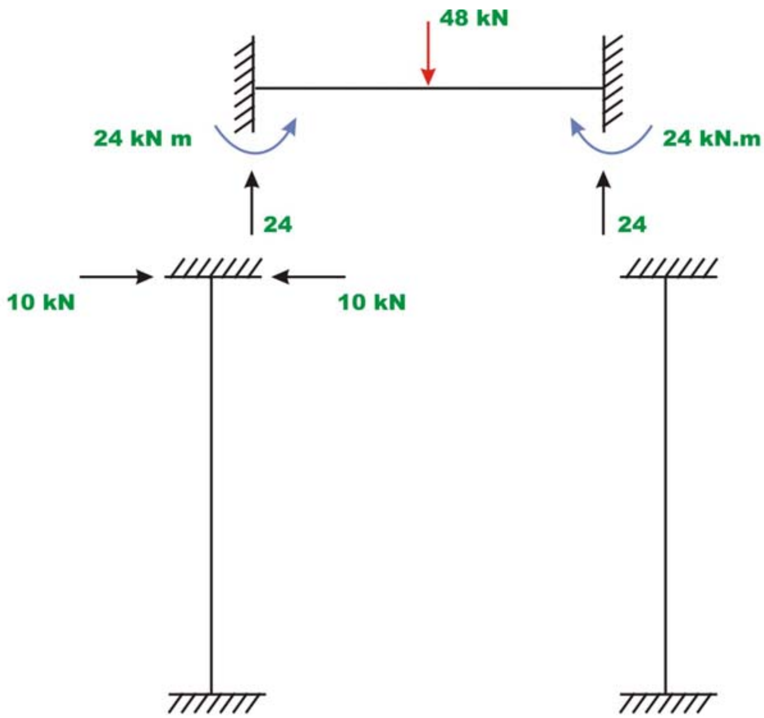


Fig . 30.5c Fixed end action due to external load.

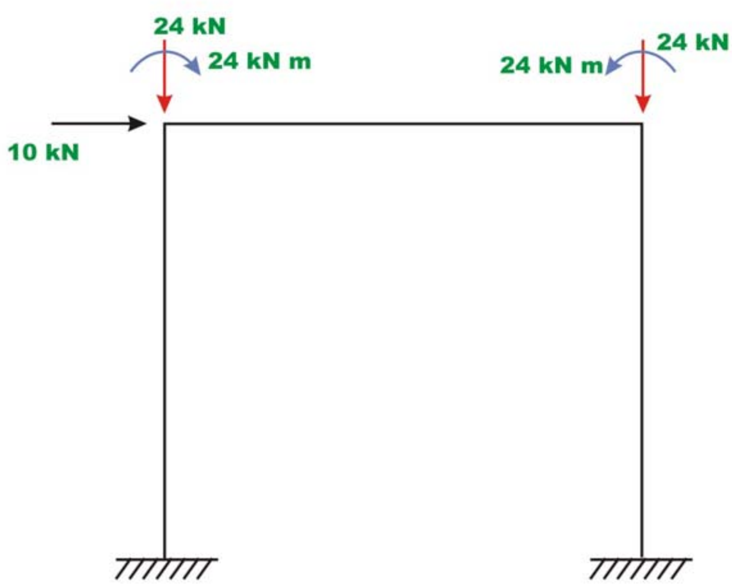


Fig. 30.5d Equivalent joint loads.

The load vector corresponding to unconstrained degrees of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} 10 \\ -24 \\ -24 \\ 0 \\ -24 \\ 24 \end{Bmatrix} \tag{5}$$

In the given frame, constraint (known) degrees of freedom are $u_1, u_2, u_3, u_{10}, u_{11}, u_{12}$. Eliminating rows and columns corresponding to constrained degrees of freedom from global stiffness matrix and writing load displacement relationship,

$$\begin{Bmatrix} 10 \\ -24 \\ -24 \\ 0 \\ -24 \\ 24 \end{Bmatrix} = 10^3 \begin{bmatrix} 500.5 & 0 & 1.0 & -500 & 0 & 0 \\ 0 & 500.5 & 1.0 & 0 & -0.5 & 1.0 \\ 1.0 & 1.0 & 5.33 & 0 & -1.0 & 1.33 \\ -500 & 0 & 0 & 500.5 & 0 & 1 \\ 0 & -0.5 & -1 & 0 & 500.5 & -1 \\ 0 & 1 & 1.33 & 1 & -1 & 5.33 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} \tag{6}$$

Solving we get,

$$\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{Bmatrix} 1.43 \times 10^{-2} \\ -3.84 \times 10^{-5} \\ -8.14 \times 10^{-3} \\ 1.43 \times 10^{-2} \\ -5.65 \times 10^{-5} \\ 3.85 \times 10^{-3} \end{Bmatrix} \tag{7}$$

Let $R_1, R_2, R_3, R_{10}, R_{11}, R_{12}$ be the support reactions along degrees of freedom $1, 2, 3, 10, 11, 12$ respectively. Support reactions are calculated by

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_{10} \\ R_{11} \\ R_{12} \end{Bmatrix} = \begin{Bmatrix} p_1^{\ F} \\ p_2^{\ F} \\ p_3^{\ F} \\ p_{10}^{\ F} \\ p_{11}^{\ F} \\ p_{12}^{\ F} \end{Bmatrix} + 10^3 \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} -0.5 & 0 & -1.0 & 0 & 0 & 0 \\ 0 & -500 & 0 & 0 & 0 & 0 \\ 1.0 & 0 & 1.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5 & 0 & -1.0 \\ 0 & 0 & 0 & 0 & -500 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 1.33 \end{bmatrix} \end{array} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_{10} \\ R_{11} \\ R_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0.99 \\ 19.71 \\ 3.43 \\ -10.99 \\ 28.28 \\ 19.42 \end{Bmatrix} = \begin{Bmatrix} 0.99 \\ 19.71 \\ 3.43 \\ -10.99 \\ 28.28 \\ 19.42 \end{Bmatrix} \qquad (8)$$

Summary

In this lesson, the analysis of plane frame by the direct stiffness matrix method is discussed. Initially, the stiffness matrix of the plane frame member is derived in its local co-ordinate axes and then it is transformed to global co-ordinate system. In the case of plane frames, members are oriented in different directions and hence before forming the global stiffness matrix it is necessary to refer all the member stiffness matrices to the same set of axes. This is achieved by transformation of forces and displacements to global co-ordinate system. In the end, a few problems are solved to illustrate the methodology.

Module 5

Cables and Arches

Lesson 31

Cables

Instructional Objectives:

After reading this chapter the student will be able to

1. Differentiate between rigid and deformable structures.
2. Define funicular structure.
3. State the type stress in a cable.
4. Analyse cables subjected to uniformly distributed load.
5. Analyse cables subjected to concentrated loads.

31.1 Introduction

Cables and arches are closely related to each other and hence they are grouped in this course in the same module. For long span structures (for e.g. in case bridges) engineers commonly use cable or arch construction due to their efficiency. In the first lesson of this module, cables subjected to uniform and concentrated loads are discussed. In the second lesson, arches in general and three hinged arches in particular along with illustrative examples are explained. In the last two lessons of this module, two hinged arch and hingeless arches are considered.

Structure may be classified into rigid and deformable structures depending on change in geometry of the structure while supporting the load. Rigid structures support externally applied loads without appreciable change in their shape (geometry). Beams trusses and frames are examples of rigid structures. Unlike rigid structures, deformable structures undergo changes in their shape according to externally applied loads. However, it should be noted that deformations are still small. Cables and fabric structures are deformable structures. Cables are mainly used to support suspension roofs, bridges and cable car system. They are also used in electrical transmission lines and for structures supporting radio antennas. In the following sections, cables subjected to concentrated load and cables subjected to uniform loads are considered.

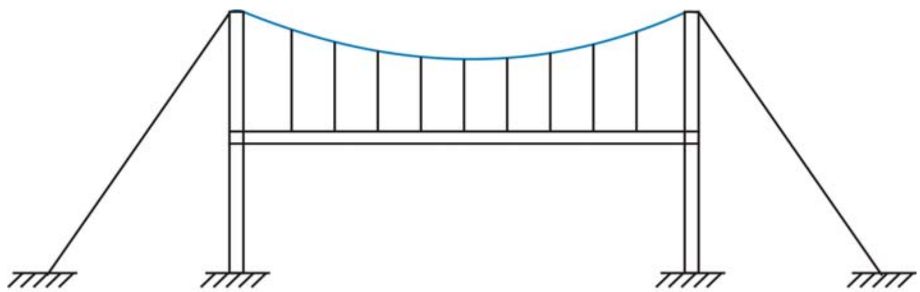


Fig. 31.1 Deformable structure.

Fig 31.2a Unloaded cable (when dead load is neglected)

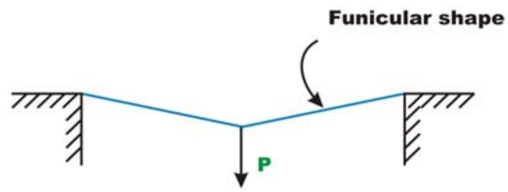


Figure 31.2b Cable in tension.

The shape assumed by a rope or a chain (with no stiffness) under the action of external loads when hung from two supports is known as a funicular shape. Cable is a funicular structure. It is easy to visualize that a cable hung from two supports subjected to external load must be in tension (vide Fig. 31.2a and 31.2b). Now let us modify our definition of cable. A cable may be defined as the structure in pure tension having the funicular shape of the load.

31.2 Cable subjected to Concentrated Loads

As stated earlier, the cables are considered to be perfectly flexible (no flexural stiffness) and inextensible. As they are flexible they do not resist shear force and bending moment. It is subjected to axial tension only and it is always acting tangential to the cable at any point along the length. If the weight of the cable is negligible as compared with the externally applied loads then its self weight is neglected in the analysis. In the present analysis self weight is not considered.

Consider a cable $ACDEB$ as loaded in Fig. 31.2. Let us assume that the cable lengths L_1, L_2, L_3, L_4 and sag at C, D, E (h_c, h_d, h_e) are known. The four reaction components at A and B, cable tensions in each of the four segments and three sag values: a total of eleven unknown quantities are to be determined. From the geometry, one could write two force equilibrium equations ($\sum F_x = 0, \sum F_y = 0$) at each of the point A, B, C, D and E i.e. a total of ten equations and the required one more equation may be written from the geometry of the cable. For example, if one of the sag is given then the problem can be solved easily. Otherwise if the total length of the cable S is given then the required equation may be written as

$$S = \sqrt{L_1^{\ 2} + h_c^{\ 2}} + \sqrt{L_2^{\ 2} + (h_d - h_c)^2} + \sqrt{L_2^{\ 2} + (h_d - h_e)^2} + \sqrt{L_2^{\ 2} + (h \ + h_e)^2} \qquad (31.1)$$

31.3 Cable subjected to uniform load.

Cables are used to support the dead weight and live loads of the bridge decks having long spans. The bridge decks are suspended from the cable using the hangers. The stiffened deck prevents the supporting cable from changing its shape by distributing the live load moving over it, for a longer length of cable. In such cases cable is assumed to be uniformly loaded.

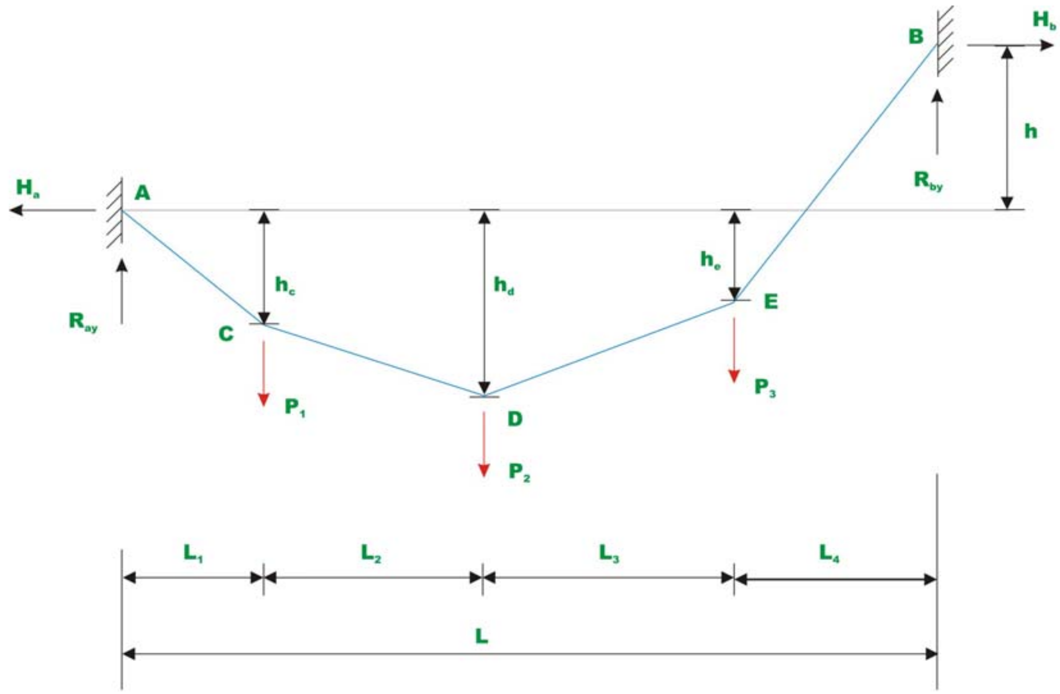


Fig. 31.3a Cable subjected to concentrated load.

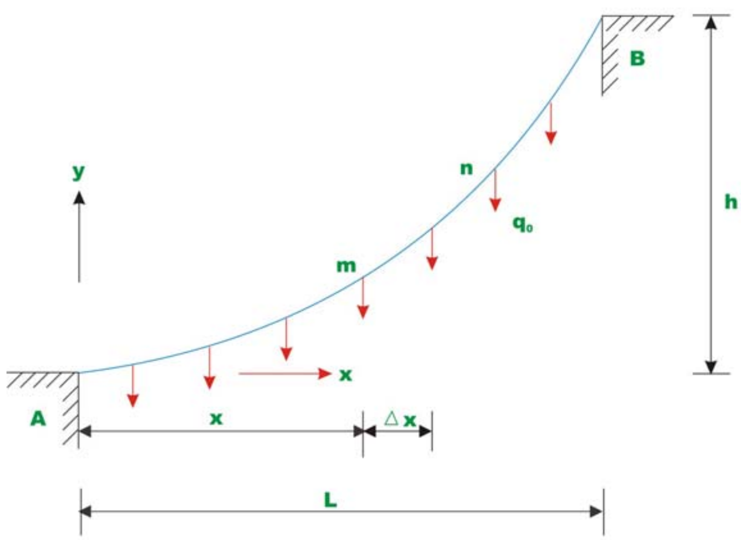


Fig. 31.3b Cable subjected to uniformly distributed load.

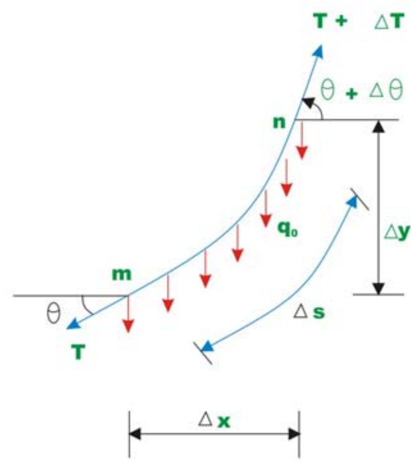


Fig. 31.3c Free-body diagram

Consider a cable which is uniformly loaded as shown in Fig 31.3a. Let the slope of the cable be zero at A. Let us determine the shape of the cable subjected to uniformly distributed load q_0. Consider a free body diagram of the cable as shown in Fig 31.3b. As the cable is uniformly loaded, the tension in the cable changes continuously along the cable length. Let the tension in the cable at m end of the free body diagram be T and tension at the n end of the cable be $T+\Delta T$. The slopes of the cable at m and n are denoted by θ and $\theta+\Delta\theta$ respectively. Applying equations of equilibrium, we get

$$\sum Fy = 0 \qquad -T\sin\theta + (T+\Delta T)\sin(\theta+\Delta\theta) - q_0(\Delta x) = 0 \qquad (31.2a)$$

$$\sum Fx = 0 \qquad -T\cos\theta + (T+\Delta T)\cos(\theta+\Delta\theta) = 0 \qquad (31.2b)$$

$$\sum Mn = 0 \qquad -(T\cos\theta)\Delta y + (T\sin\theta)\Delta x + (q_0\Delta x)\frac{\Delta x}{2} = 0 \qquad (31.2c)$$

Dividing equations 31.2a, b, c by Δx and noting that in the limit as $\Delta x \to 0, \Delta y \to 0 \;\; \Delta\theta \to 0$ and $\Delta T \to 0$.

$$\lim_{\Delta x \to 0} \qquad \frac{\Delta T}{\Delta x}\sin(\theta+\Delta\theta) = q_0$$

$$\frac{d}{dx}(T\sin\theta) = q_0 \qquad (31.3a)$$

$$\frac{d}{dx}(T\cos\theta) = 0 \qquad (31.3b)$$

$$\lim_{\Delta x \to 0} \qquad -T\cos\theta\frac{\Delta y}{\Delta x} + T\sin\theta + q_0\frac{x_0}{2} = 0$$

$$\frac{dy}{dx} = \tan\theta \qquad (31.3c)$$

Integrating equation (31.3b) we get

$$T\cos\theta = \text{constant}$$

At support (i.e., at $x=0$), $\qquad T\cos\theta = H \qquad$ (31.4a)

i.e. horizontal component of the force along the length of the cable is constant.

Integrating equation 31.3a,

$$T\sin\theta = q_0 x + C_1$$

At $x = 0,$ $T\sin\theta = 0,$ $C_1 = 0$ as $\theta = 0$ at that point.

Hence, $T\sin\theta = q_0 x$ (31.4b)

From equations 31.4a and 31.4b, one could write

$$\tan\theta = \frac{q_0 x}{H} \qquad (31.4c)$$

From the figure, $\tan\theta = \frac{dy}{dx} = \frac{q_0 x}{H}$

$$\therefore y = \frac{q_0 x^2}{2H} + C$$

At $x = 0, y = 0 \Rightarrow C = 0$ and $y = \frac{q_0 x^2}{2H}$ (31.5)

Equation 31.5 represents a parabola. Now the tension in the cable may be evaluated from equations 31.4a and 31.4b. i.e,

$$T = \sqrt{q_0{}^2 x^2 + H^2}$$

$T = T_{\max}$, when $x = L$.

$$T_{\max} = \sqrt{q_0{}^2 L^2 + H^2} = q_0 L\sqrt{1 + \left(\frac{H}{q_0 L}\right)^2} \qquad (31.6)$$

Due to uniformly distributed load, the cable takes a parabolic shape. However due to its own dead weight it takes a shape of a catenary. However dead weight of the cable is neglected in the present analysis.

Example 31.1

Determine reaction components at A and B, tension in the cable and the sag y_B, and y_E of the cable shown in Fig. 31.4a. Neglect the self weight of the cable in the analysis.

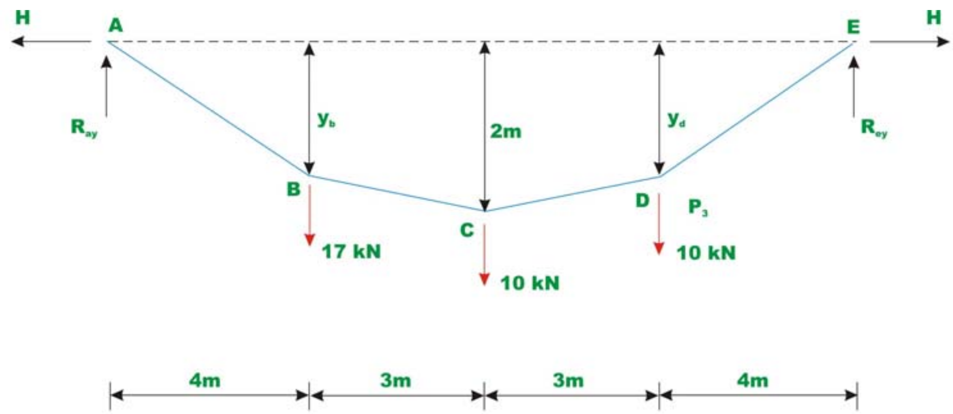
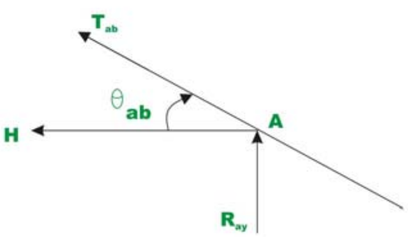
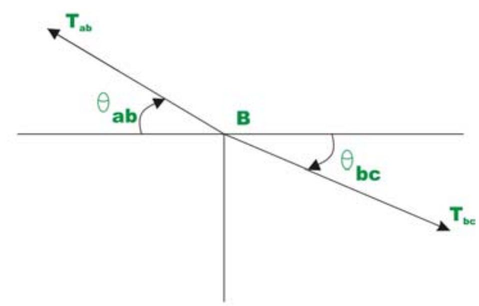


Fig. 31.4 Example 31.1



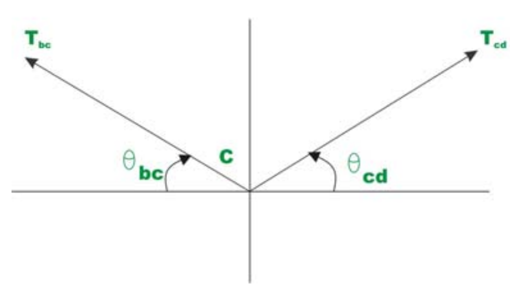
Joint A

Fig. 31.4b



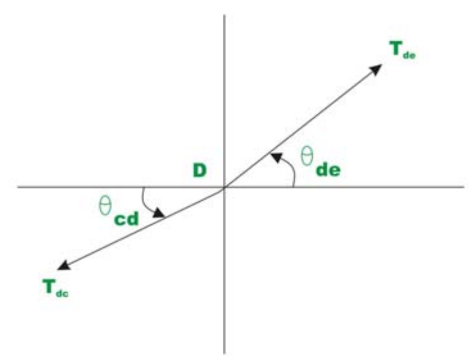
Joint B

Fig. 31.4c



Joint C

Fig. 31.4d



Joint D

Fig. 31.4e

Since there are no horizontal loads, horizontal reactions at A and B should be the same. Taking moment about *E*, yields

$$R_{ay} \times 14 - 17 \times 20 - 10 \times 7 - 10 \times 4 = 0$$

$$R_{ay} = \frac{280}{14} = 20 \quad \text{kN}; \qquad R_{ey} = 37 - 20 = 17\text{kN}.$$

Now horizontal reaction H may be evaluated taking moment about point C of all forces left of C.

$$R_{ay} \times 7 - H \times 2 - 17 \times 3 = 0$$

$$H = 44.5 \text{ kN}$$

Taking moment about B of all the forces left of B and setting $M_B = 0$, we get

$$R_{ay} \times 4 - H \times y_B = 0; \qquad y_B = \frac{80}{44.50} = 1.798\, m$$

Similarly, $y_D = \dfrac{68}{44.50} = 1.528\, m$

To determine the tension in the cable in the segment AB, consider the equilibrium of joint A (vide Fig.31.4b).

$$\sum F_x = 0 \Rightarrow T_{ab} \cos\theta_{ab} = H$$

$$T_{ab} = \frac{44.5}{\left(3 \Big/ \sqrt{3^2 + 0.298^2} \right)} = 48.789 \text{ kN}$$

The tension T_{ab} may also be obtained as

$$T_{ab} = \sqrt{R_{ay}{}^2 + H^2} = \sqrt{20^2 + 44.5^2} = 48.789 \quad \text{kN}$$

Now considering equilibrium of joint $B, C,$ and D one could calculate tension in different segments of the cable.

Segment bc

Applying equations of equilibrium,

$$\sum F_x = 0 \Rightarrow T_{ab} \cos\theta_{ab} = T_{bc} \cos\theta_{bc}$$

$$T_{bc} = \frac{44.5}{\left(3 \Big/ \sqrt{3^2 + 0.298^2}\right)} \cong 44.6 \quad \text{kN}$$

See Fig.31.4c

Segment cd

$$T_{cd} = \frac{T_{bc}\cos\theta_{bc}}{\cos\theta_{cd}} = \frac{44.5}{\left(3 \Big/ \sqrt{3^2 + 0.472^2}\right)} = 45.05 \quad \text{kN}$$

See Fig.31.4d.
See Fig.31.4e.

Segment de

$$T_{de} = \frac{T_{cd}\cos\theta_{cd}}{\cos\theta_{de}} = \frac{44.5}{4 \Big/ \sqrt{4^2 + 1.528^2}} = 47.636 \quad \text{kN}$$

The tension T_{de} may also be obtained as

$$T_{de} = \sqrt{R_{ey}{}^2 + H^2} = \sqrt{17^2 + 44.5^2} = 47.636 \quad \text{kN}$$

Example 31.2

A cable of uniform cross section is used to span a distance of 40m as shown in Fig 31.5. The cable is subjected to uniformly distributed load of 10 kN/m. run. The left support is below the right support by 2 m and the lowest point on the cable C is located below left support by 1 m. Evaluate the reactions and the maximum and minimum values of tension in the cable.

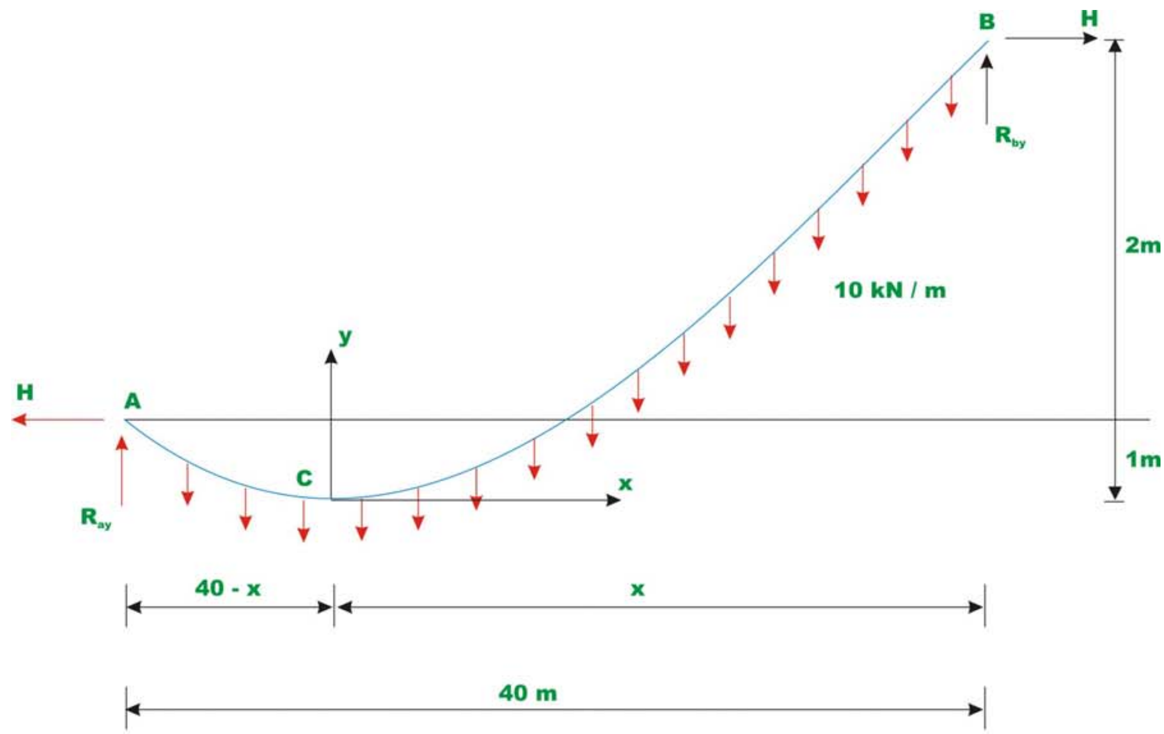


Fig. 31.5 Example 31.2

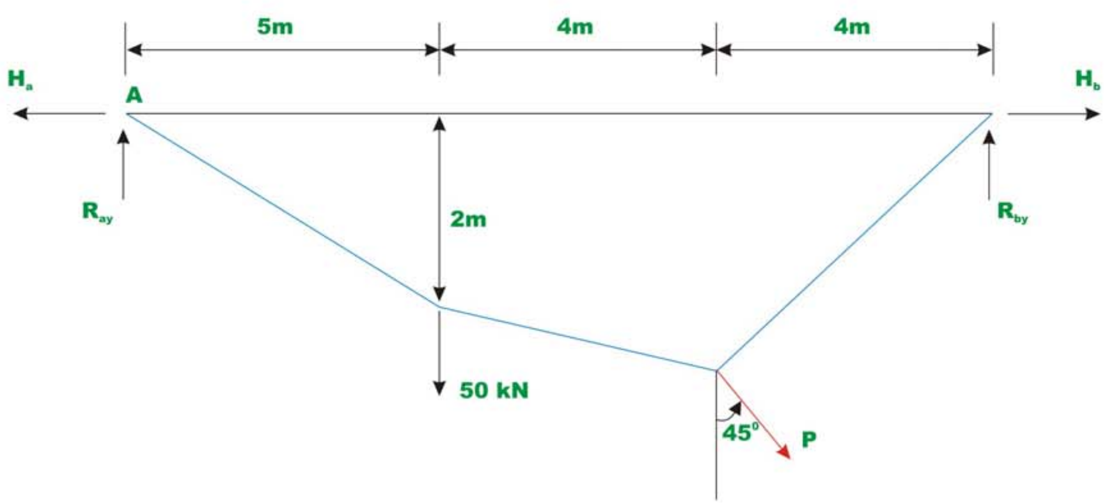


Fig. 31.6 Example 31.3

Assume the lowest point C to be at distance of x m from B. Let us place our origin of the co-ordinate system xy at C. Using equation 31.5, one could write,

$$y_a = 1 = \frac{q_0 (40-x)^2}{2H} = \frac{10(40-x)^2}{2H} \quad (1)$$

$$y_b = 3 = \frac{10x^2}{2H} \quad (2)$$

where y_a and y_b be the y co-ordinates of supports A and B respectively. From equations 1 and 2, one could evaluate the value of x. Thus,

$$10(40-x)^2 = \frac{10x^2}{3} \quad \Rightarrow \quad x = 25.359\,m$$

From equation 2, the horizontal reaction can be determined.

$$H = \frac{10 \times 25.359^2}{6} = 1071.80 \quad \text{kN}$$

Now taking moment about A of all the forces acting on the cable, yields

$$R_{by} = \frac{10 \times 40 \times 20 + 1071.80 \times 2}{40} = 253.59 \quad \text{kN}$$

Writing equation of moment equilibrium at point B, yields

$$R_{ay} = \frac{40 \times 20 \times 10 - 1071.80 \times 2}{40} = 146.41 \quad \text{kN}$$

Tension in the cable at supports A and B are

$$T_A = \sqrt{146.41^2 + 1071.81^2} = 1081.76 \quad \text{kN}$$

$$T_B = \sqrt{253.59^2 + 1071.81^2} = 1101.40 \quad \text{kN}$$

The tension in the cable is maximum where the slope is maximum as $T\cos\theta = H$. The maximum cable tension occurs at B and the minimum cable tension occurs at C where $\frac{dy}{dx} = \theta = 0$ and $T_C = H = 1071.81 \quad \text{kN}$

Example 31.3

A cable of uniform cross section is used to support the loading shown in Fig 31.6. Determine the reactions at two supports and the unknown sag y_C.

Taking moment of all the forces about support B,

$$R_{ay} = \frac{1}{10}\left[350 + 300 + 100 y_c\right] \tag{1}$$

$$R_{ay} = 65 + 10 y_c$$

Taking moment about B of all the forces left of B and setting $M_B = 0$, we get,

$$R_{ay} \times 3 - H_a \times 2 = 0$$
$$\Rightarrow H_a = 1.5 R_{ay} \tag{2}$$

Taking moment about C of all the forces left of C and setting $M_C = 0$, we get

$$\sum M_C = 0 \qquad R_{ay} \times 7 - H_a \times y_C - 50 \times 4 = 0$$

Substituting the value of H_a in terms of R_{ay} in the above equation,

$$7R_{ay} - 1.5 R_{ay} y_C - 200 = 0 \tag{3}$$

Using equation (1), the above equation may be written as,

$$y_C^2 + 1.833 y_C - 17 = 0 \tag{4}$$

Solving the above quadratic equation, y_C can be evaluated. Hence,

$$y_C = 3.307 m.$$

Substituting the value of y_C in equation (1),

$$R_{ay} = 98.07 \text{ kN}$$

From equation (2),

$$H_a = 1.5 R_{ay} = 147.05 \text{ kN}$$

Now the vertical reaction at D, R_{dy} is calculated by taking moment of all the forces about A,

$$R_{dy} \times 10 - 100 \times 7 + 100 \times 3.307 - 50 \times 3 = 0$$

$$R_{dy} = 51.93 \text{ kN.}$$

Taking moment of all the forces right of C about C, and noting that $\sum M_C = 0$,

$$R_{dy} \times 3 = H_d \times y_C \quad \Rightarrow H_d = 47.109 \text{ kN.}$$

Summary

In this lesson, the cable is defined as the structure in pure tension having the funicular shape of the load. The procedures to analyse cables carrying concentrated load and uniformly distributed loads are developed. A few numerical examples are solved to show the application of these methods to actual problems.

Module 5

Cables and Arches

Lesson

32

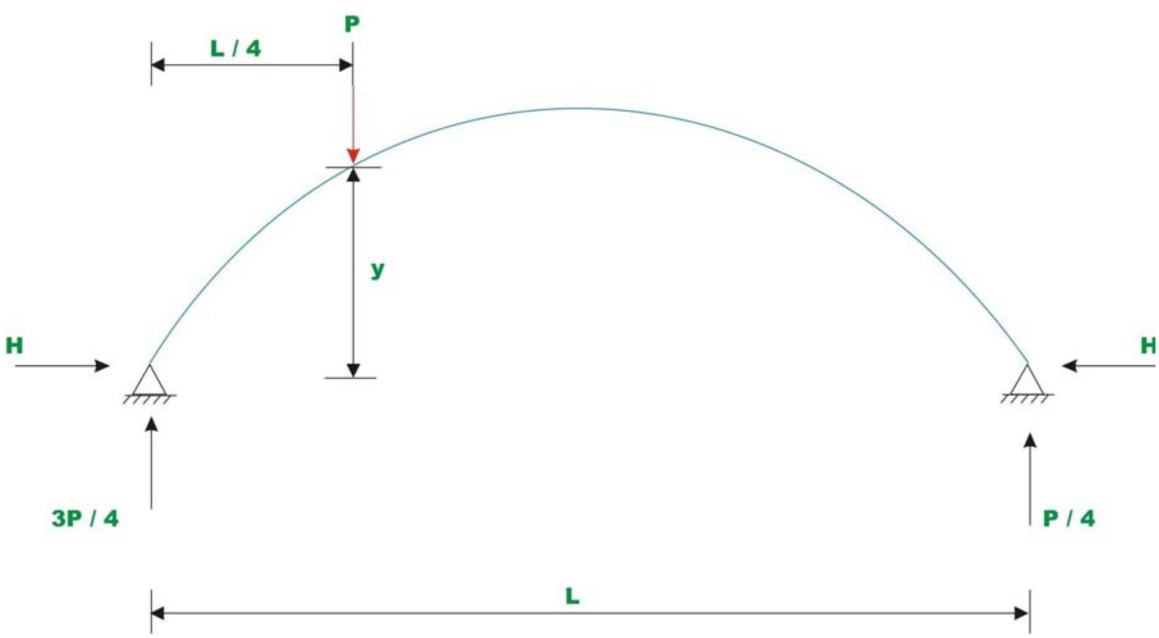
Three Hinged Arch

Instructional Objectives:

After reading this chapter the student will be able to
1. Define an arch.
2. Identify three-hinged, two-hinged and hingeless arches.
3. State advantages of arch construction.
4. Analyse three-hinged arch.
5. Evaluate horizontal thrust in three-hinged arch.

32.1 Introduction

In case of beams supporting uniformly distributed load, the maximum bending moment increases with the square of the span and hence they become uneconomical for long span structures. In such situations arches could be advantageously employed, as they would develop horizontal reactions, which in turn reduce the design bending moment.



(a) Arch

Fig. 32.1 Beam and Arch comparison.

For example, in the case of a simply supported beam shown in Fig. 32.1, the bending moment below the load is $\frac{3PL}{16}$. Now consider a two hinged symmetrical arch of the same span and subjected to similar loading as that of simply supported beam. The vertical reaction could be calculated by equations of statics. The horizontal reaction is determined by the method of least work. Now the bending moment below the load is $\frac{3PL}{16} - Hy$. It is clear that the bending moment below the load is reduced in the case of an arch as compared to a simply supported beam. It is observed in the last lesson that, the cable takes the shape of the loading and this shape is termed as funicular shape. If an arch were constructed in an inverted funicular shape then it would be subjected to only compression for those loadings for which its shape is inverted funicular.

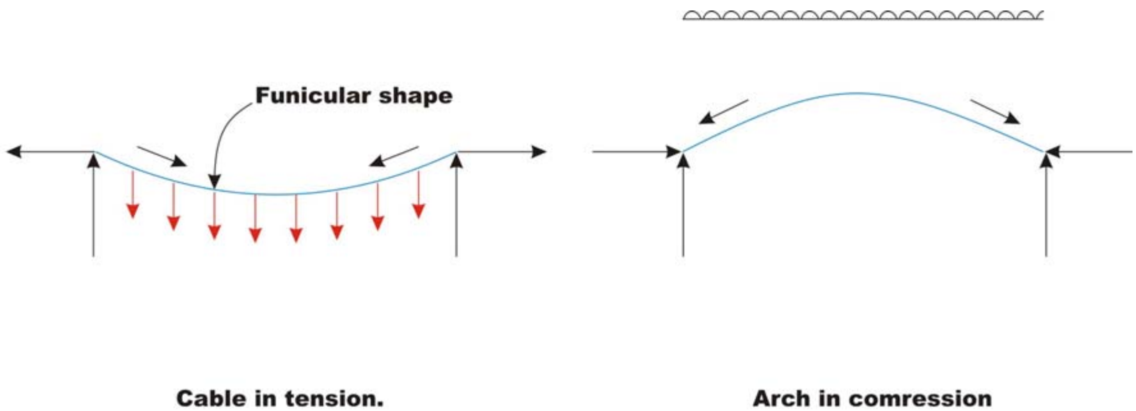


Fig. 32.2 Cable and Arch structure.

Since in practice, the actual shape of the arch differs from the inverted funicular shape or the loading differs from the one for which the arch is an inverted funicular, arches are also subjected to bending moment in addition to compression. As arches are subjected to compression, it must be designed to resist buckling.

Until the beginning of the 20th century, arches and vaults were commonly used to span between walls, piers or other supports. Now, arches are mainly used in bridge construction and doorways. In earlier days arches were constructed using stones and bricks. In modern times they are being constructed of reinforced concrete and steel.

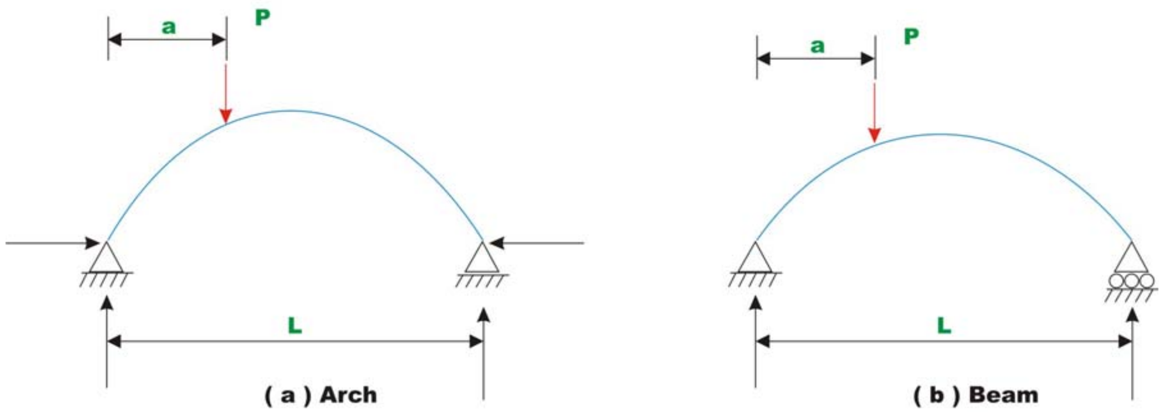
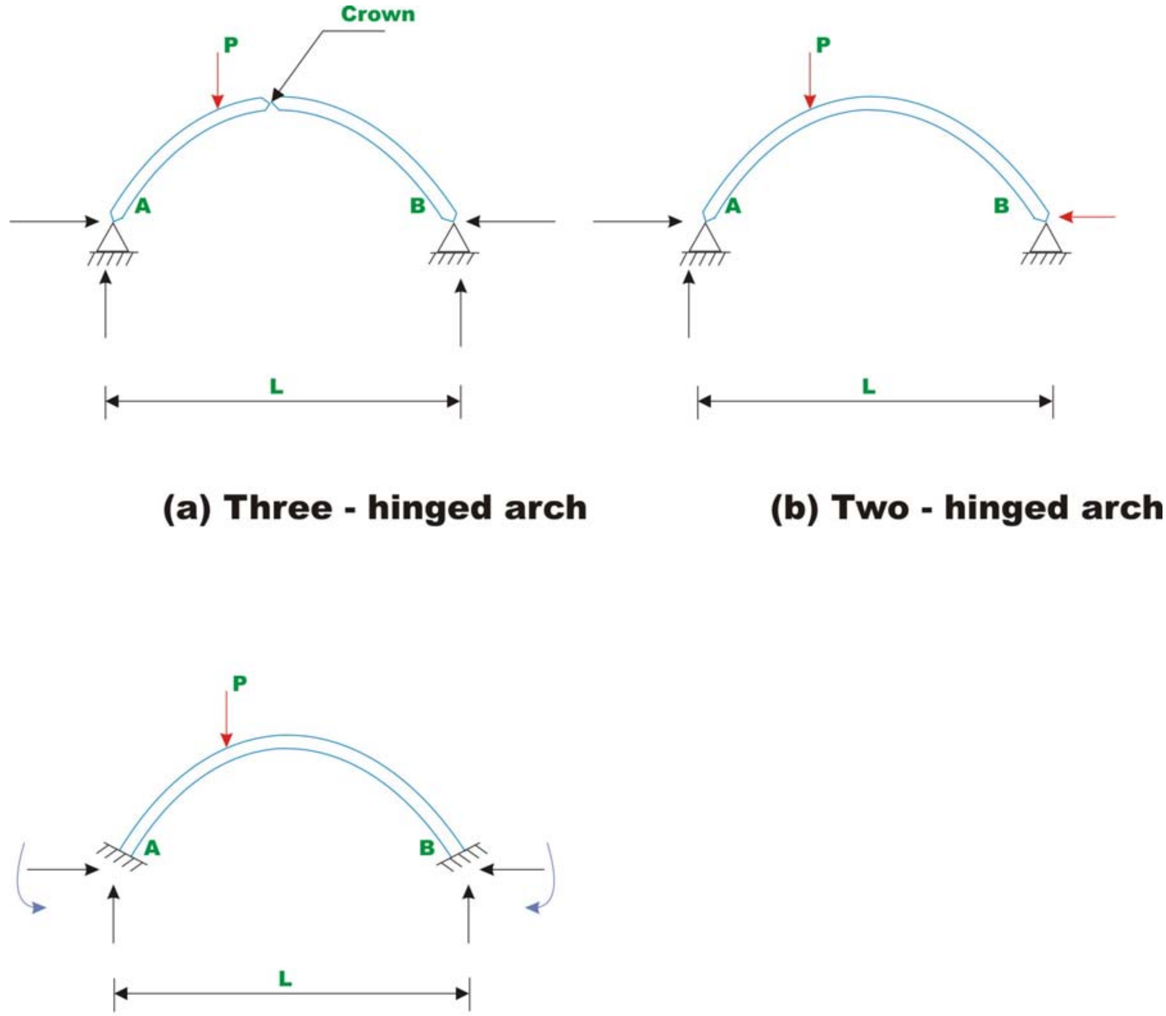


Fig. 32.3

A structure is classified as an arch not based on its shape but the way it supports the lateral load. Arches support load primarily in compression. For example in Fig 32.3b, no horizontal reaction is developed. Consequently bending moment is not reduced. It is important to appreciate the point that the definition of an arch is a structural one, not geometrical.

32.2 Type of arches

There are mainly three types of arches that are commonly used in practice: three hinged arch, two-hinged arch and fixed-fixed arch. Three-hinged arch is statically determinate structure and its reactions / internal forces are evaluated by static equations of equilibrium. Two-hinged arch and fixed-fixed arch are statically indeterminate structures. The indeterminate reactions are determined by the method of least work or by the flexibility matrix method. In this lesson three-hinged arch is discussed.



(a) Three - hinged arch **(b) Two - hinged arch**

(c) Fixed hinged arch

Fig. 32.4 Types of arches.

32.3 Analysis of three-hinged arch

In the case of three-hinged arch, we have three hinges: two at the support and one at the crown thus making it statically determinate structure. Consider a three hinged arch subjected to a concentrated force P as shown in Fig 32.5.

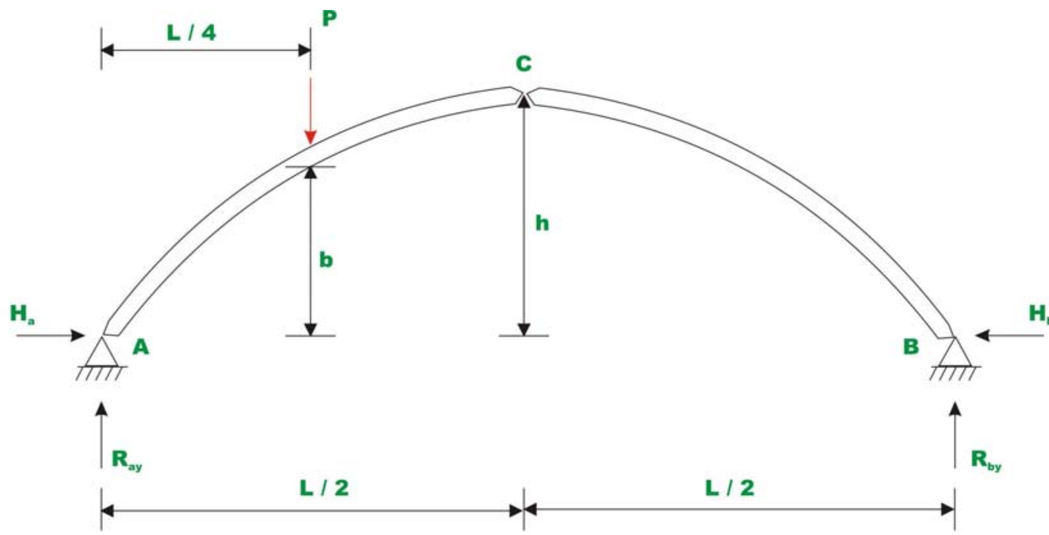


Fig. 32.5 Three hinged arch.

There are four reaction components in the three-hinged arch. One more equation is required in addition to three equations of static equilibrium for evaluating the four reaction components. Taking moment about the hinge of all the forces acting on either side of the hinge can set up the required equation. Taking moment of all the forces about hinge A, yields

$$R_{by} = \frac{PL}{4L} = \frac{P}{4} \tag{32.1}$$

$$\sum Fy = 0 \qquad \Rightarrow \qquad R_{ay} = \frac{3P}{4} \tag{32.2}$$

Taking moment of all forces right of hinge C about hinge C leads to

$$H_b \times h = \frac{R_{by} L}{2}$$

$$\Rightarrow \qquad H_b = \frac{R_{by} L}{2h} = \frac{PL}{8h} \tag{32.3}$$

Applying $\sum Fx = 0$ to the whole structure gives $\qquad H_a = \frac{PL}{8h}$

Now moment below the load is given by ,

$$M_D = \frac{R_{ay} L}{4} - H_a b$$

$$M_D = \frac{3PL}{16} - \frac{PLb}{8h} \tag{32.4}$$

If $\frac{b}{h} = \frac{1}{2}$ then $M_D = \frac{3PL}{16} - \frac{PL}{16} = 0.125PL$ (32.5)

For a simply supported beam of the same span and loading, moment under the loading is given by,

$$M_D = \frac{3PL}{16} = 0.375PL \tag{32.6}$$

For the particular case considered here, the arch construction has reduced the moment by 66.66 %.

Example 32.1

A three-hinged parabolic arch of uniform cross section has a span of 60 m and a rise of 10 m. It is subjected to uniformly distributed load of intensity 10 kN/m as shown in Fig. 32.6 Show that the bending moment is zero at any cross section of the arch.

Solution:

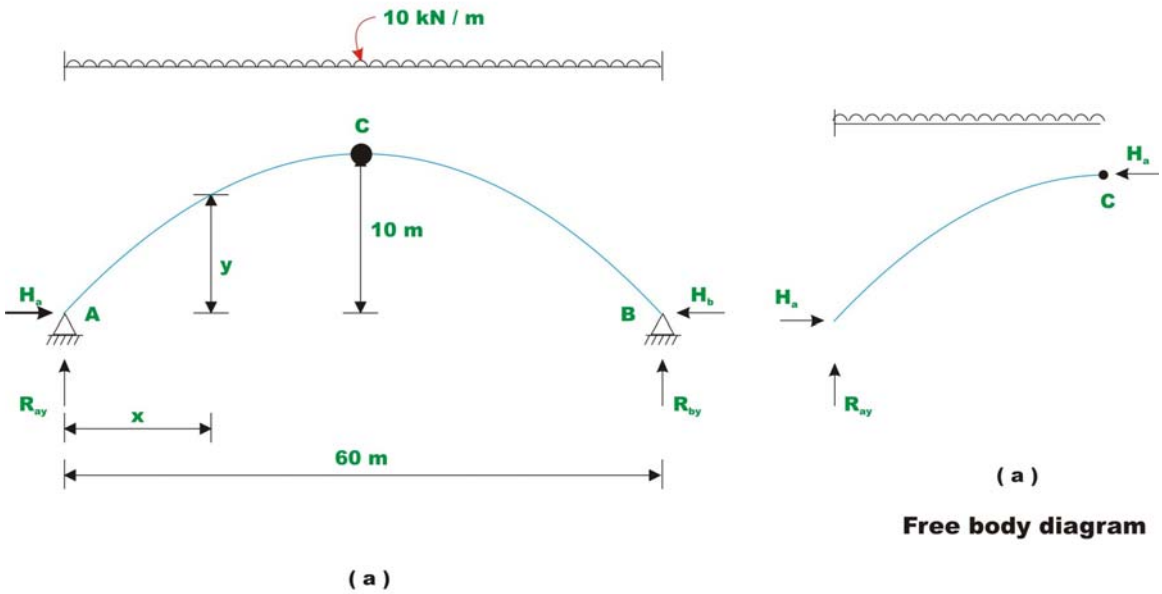


Fig. 32.6 Three hinged arch of Example 32.1

Reactions:

Taking moment of all the forces about hinge A, yields

$$R_{ay} = R_{by} = \frac{10 \times 60}{2} = 300 \quad \text{kN} \tag{1}$$

Taking moment of forces left of hinge C about C, one gets

$$R_{ay} \times 30 - H_a \times 10 - 10 \times 30 \times \frac{30}{2} = 0$$

$$H_a = \frac{300 \times 30 - 10 \times 30 \times \left(\frac{30}{2}\right)}{10} \tag{2}$$
$$= 450 \quad \text{kN}$$

From $\sum Fx = 0$ one could write, $H_b = 450 \quad \text{kN}$.
The shear force at the mid span is zero.

Bending moment

The bending moment at any section x from the left end is,

$$M_x = R_{ay} x - H_a y - 10 \frac{x^2}{2} \tag{3}$$

The equation of the three-hinged parabolic arch is

$$y = \frac{2}{3}x - \frac{10}{30^2}x^2 \tag{4}$$

$$M_x = 300x - \left(\frac{2}{3}x - \frac{10}{30^2}x^2\right)450 - 5x^2$$

$$= 300x - 300x + 5x^2 - 5x^2 = 0$$

In other words a three hinged parabolic arch subjected to uniformly distributed load is not subjected to bending moment at any cross section. It supports the load in pure compression. Can you explain why the moment is zero at all points in a three-hinged parabolic arch?

Example 32.2

A three-hinged semicircular arch of uniform cross section is loaded as shown in Fig 32.7. Calculate the location and magnitude of maximum bending moment in the arch.

Solution:

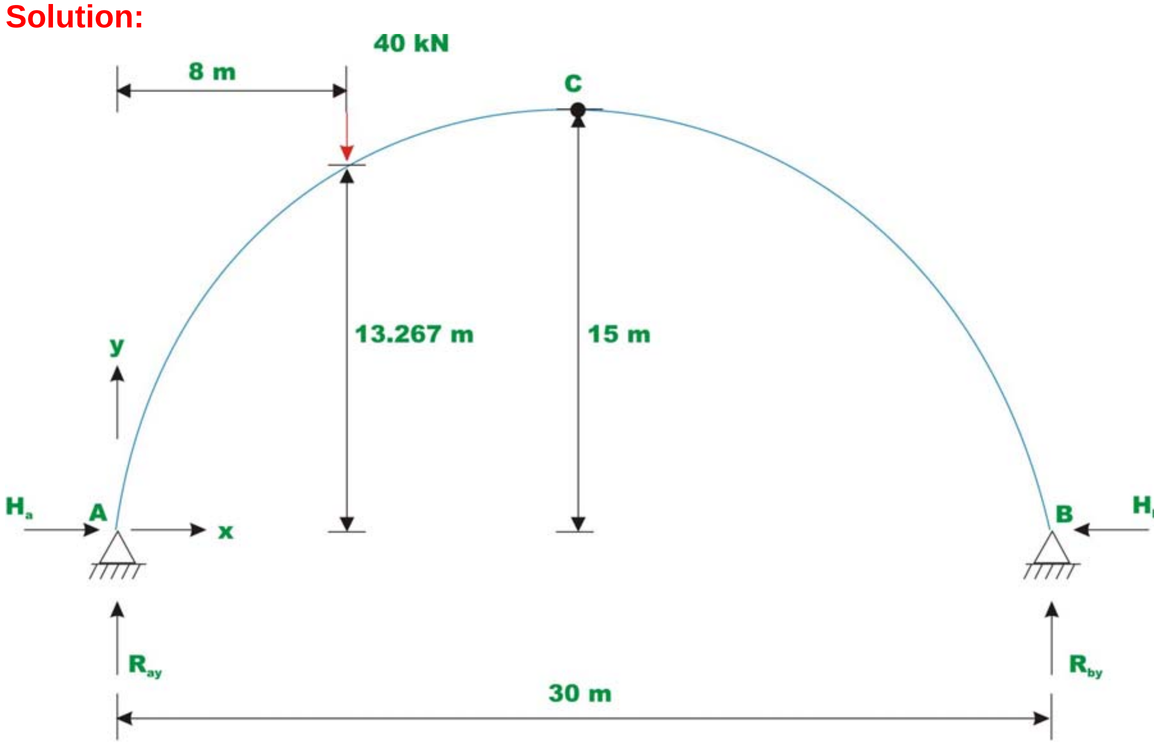


Fig. 32.7 A semi circular arch of Example 32.2

Reactions:

Taking moment of all the forces about hinge B leads to,

$$R_{ay} = \frac{40 \times 22}{30} = 29.33 \quad \text{kN} \quad (\uparrow)$$

$$\sum Fy = 0 \qquad \Rightarrow R_{by} = 10.67 \quad \text{kN} \quad (\uparrow) \tag{1}$$

Bending moment

Now making use of the condition that the moment at hinge C of all the forces left of hinge C is zero gives,

$$M_c = R_{ay} \times 15 - H_a \times 15 - 40 \times 7 = 0 \tag{2}$$

$$H_a = \frac{29.33 \times 15 - 40 \times 7}{15} = 10.66 \quad \text{kN} \quad (\rightarrow)$$

Considering the horizontal equilibrium of the arch gives,

$$H_b = 10.66 \quad \text{kN} \quad (\leftarrow)$$

The maximum positive bending moment occurs below D and it can be calculated by taking moment of all forces left of D about D.

$$M_D = R_{ay} \times 8 - H_a \times 13.267 \tag{3}$$

$$= 29.33 \times 8 - 10.66 \times 13.267 = 93.213 \quad \text{kN}$$

Example 32.3

A three-hinged parabolic arch is loaded as shown in Fig 32.8a. Calculate the location and magnitude of maximum bending moment in the arch. Draw bending moment diagram.

Solution:

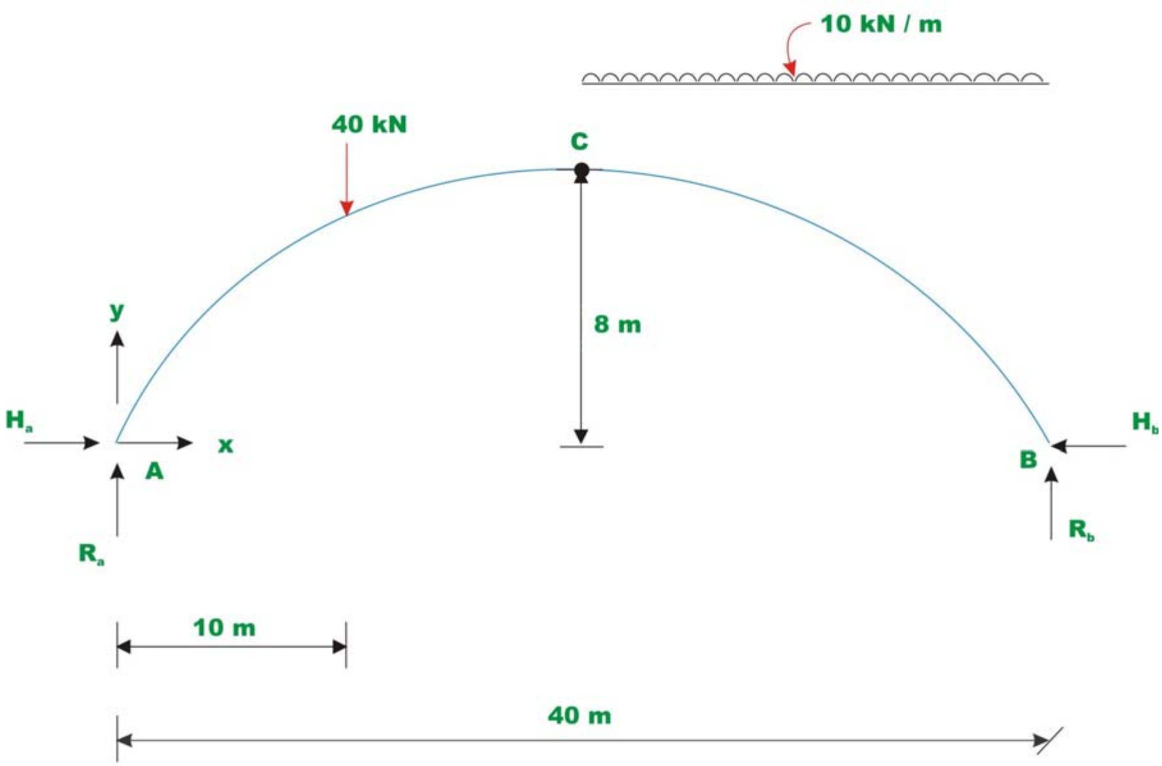


Fig. 32.8a Eaxmple 32.3

Reactions:

Taking A as the origin, the equation of the three-hinged parabolic arch is given by,

$$y = \frac{8}{10}x - \frac{8}{400}x^2 \qquad (1)$$

Taking moment of all the forces about hinge B leads to,

$$R_{ay} = \frac{40\times 30 + 10\times 20\times \left(20/2\right)}{40} = 80 \quad \text{kN} \quad (\uparrow)$$

$$\sum Fy = 0 \qquad \Rightarrow R_{by} = 160 \text{ kN } (\uparrow) \tag{2}$$

Now making use of the condition that, the moment at hinge C of all the forces left of hinge C is zero gives,

$$M_c = R_{ay} \times 20 - H_a \times 8 - 40 \times 10 = 0$$

$$H_a = \frac{80 \times 20 - 40 \times 10}{8} = 150 \text{ kN } (\rightarrow) \tag{3}$$

Considering the horizontal equilibrium of the arch gives,

$$H_b = 150 \text{ kN } (\leftarrow) \tag{4}$$

Location of maximum bending moment

Consider a section x from end B. Moment at section x in part CB of the arch is given by (please note that B has been taken as the origin for this calculation),

$$M_x = 160x - \left(\frac{8}{10}x - \frac{8}{400}x^2\right)150 - \frac{10}{2}x^2 \tag{5}$$

According to calculus, the necessary condition for extremum (maximum or minimum) is that $\frac{\partial M_x}{\partial x} = 0$.

$$\begin{aligned} \frac{\partial M_x}{\partial x} &= 160 - \left(\frac{8}{10} - \frac{8 \times 2}{400}x\right)150 - 10x \\ &= 40 - 4x = 0 \end{aligned} \tag{6}$$

$$x = 10 \text{ m.}$$

Substituting the value of x in equation (5), the maximum bending moment is obtained. Thus,

$$M_{\max} = 160(10) - \left(\frac{8}{10}(10) - \frac{8}{400}(10)^2\right)150 - \frac{10}{2}(10)^2$$

$$M_{\max} = 200 \text{ kN.m.} \tag{7}$$

Shear force at D just left of 40 kN load

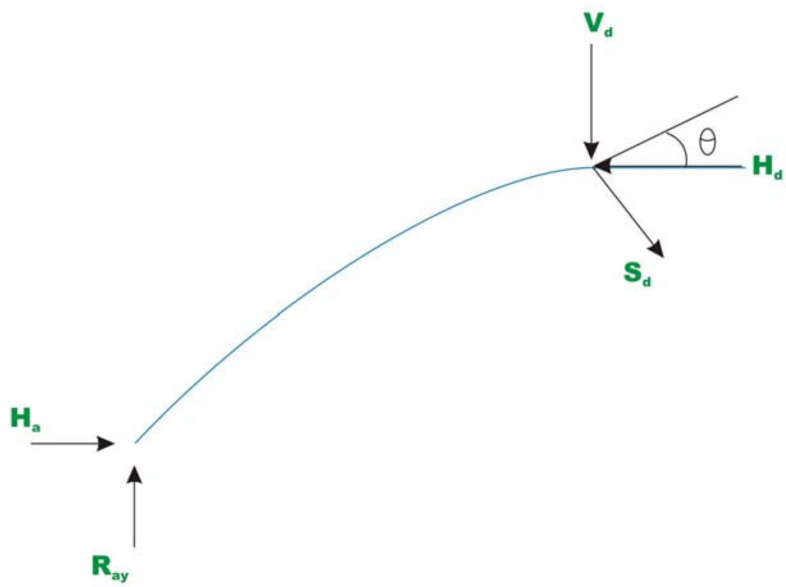


Fig. 32.8b

The slope of the arch at D is evaluated by,

$$\tan\theta = \frac{dy}{dx} = \frac{8}{10} - \frac{16}{400}x \tag{8}$$

Substituting $x = 10$ m. in the above equation, $\theta_D = 21.8^0$

Shear force S_d at left of D is

$$S_d = H_a \sin\theta - R_{ay}\cos\theta \tag{9}$$

$$S_d = 150\sin(21.80) - 80\cos(21.80)$$

$$= -18.57 \text{ kN.}$$

Example 32.4

A three-hinged parabolic arch of constant cross section is subjected to a uniformly distributed load over a part of its span and a concentrated load of 50 kN, as shown in Fig. 32.9. The dimensions of the arch are shown in the figure. Evaluate the horizontal thrust and the maximum bending moment in the arch.

Solution:

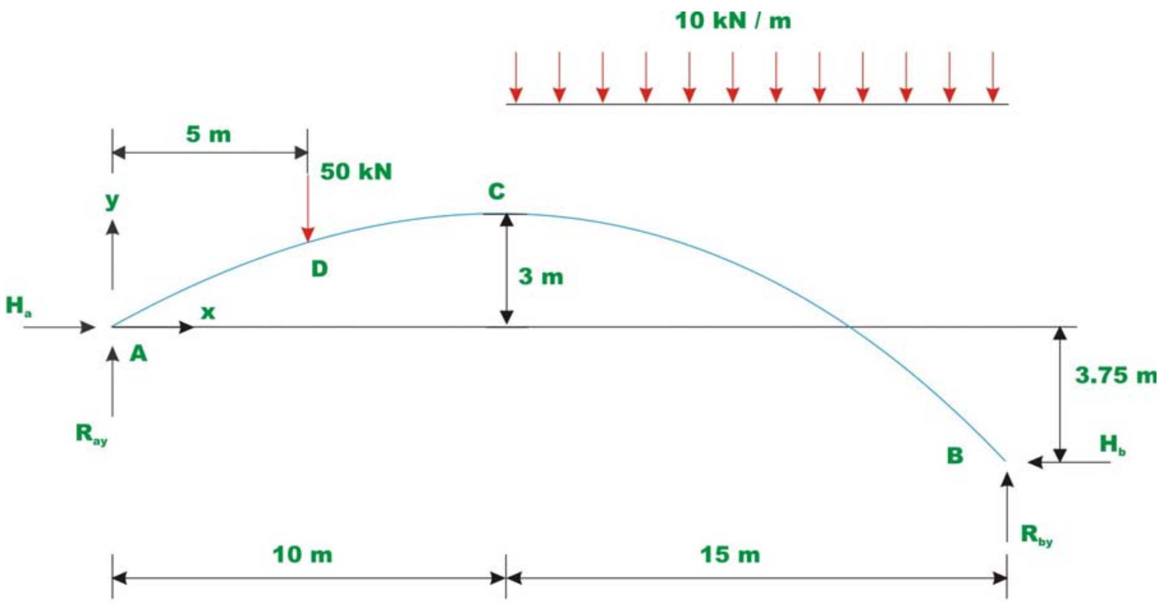


Fig. 32.9

Reactions:

Taking A as the origin, the equation of the parabolic arch may be written as,

$$y = -0.03x^2 + 0.6x \qquad (1)$$

Taking moment of all the loads about B leads to,

$$\begin{aligned} R_{ay} &= \frac{1}{25}\left[50\times 20 + 10\times 15\times \frac{15}{2} - H_a \times 3.75\right] \\ &= \frac{1}{25}\left[2125 - 3.75 H_a\right] \end{aligned} \qquad (2)$$

Taking moment of all the forces right of hinge C about the hinge C and setting $M_c = 0$ leads to,

$$R_{by} \times 15 - 6.75 H_b - 10 \times 15 \times \frac{15}{2} = 0$$

$$R_{by} = \frac{1}{15}\left[1125 + 6.75 H_b\right] \qquad (3)$$

Since there are no horizontal loads acting on the arch,

$$H_a = H_b = H \quad (say)$$

Applying $\sum Fy = 0$ for the whole arch,

$$R_{ay} + R_{by} = 10 \times 15 + 50 = 200$$

$$\frac{1}{25}\left[2125 - 3.75 H\right] + \frac{1}{15}\left[1125 + 6.75 H\right] = 200$$

$$85 - 0.15 H + 75 + 0.45 H = 200$$

$$H = \frac{40}{0.3} = 133.33 \quad \text{kN} \qquad (4)$$

From equation (2),

$$R_{ay} = 65.0 \quad \text{kN}$$

$$R_{by} = 135.0 \quad \text{kN} \qquad (5)$$

Bending moment

From inspection, the maximum negative bending moment occurs in the region AD and the maximum positive bending moment occurs in the region CB.

Span AD

Bending moment at any cross section in the span AD is

$$M = R_{ay} x - H_a(-0.03x^2 + 0.6x) \qquad 0 \le x \le 5 \qquad (6)$$

For, the maximum negative bending moment in this region,

$$\frac{\partial M}{\partial x} = 0 \Rightarrow \qquad R_{ay} - H_a(-0.06x + 0.6) = 0$$

$$x = 1.8748 \quad \text{m}$$

$$M = -14.06 \quad \text{kN.m.}$$

For the maximum positive bending moment in this region occurs at D,

$$M_D = R_{ay} 5 - H_a(-0.03 \times 25 + 0.6 \times 5)$$
$$= +25.0 \quad \text{kN.m}$$

Span CB

Bending moment at any cross section, in this span is calculated by,

$$M = R_{ay} x - H_a(-0.03x^2 + 0.6x) - 50(x-5) - 10(x-10)\frac{(x-10)}{2}$$

For locating the position of maximum bending moment,

$$\frac{\partial M}{\partial x} = 0 = R_{ay} - H_a(-0.06x + 0.6) - 50 - \frac{10}{2} \times 2(x-10) = 0$$

$$x = 17.5 \quad \text{m}$$

$$M = 65 \times 17.5 - 133.33(-0.03(17.5)^2 + 0.6(17.5)) - 50(12.5) - \frac{10}{2}(7.5)^2$$
$$M = 56.25 \quad \text{kN.m}$$

Hence, the maximum positive bending moment occurs in span CB.

Summary

In this lesson, the arch definition is given. The advantages of arch construction are given in the introduction. Arches are classified as three-hinged, two-hinged and hingeless arches. The analysis of three-hinged arch is considered here. Numerical examples are solved in detail to show the general procedure of three-hinged arch analysis.

Module 5

Cables and Arches

Lesson
33
Two-Hinged Arch

Instructional Objectives:

After reading this chapter the student will be able to
1. Compute horizontal reaction in two-hinged arch by the method of least work.
2. Write strain energy stored in two-hinged arch during deformation.
3. Analyse two-hinged arch for external loading.
4. Compute reactions developed in two hinged arch due to temperature loading.

33.1 Introduction

Mainly three types of arches are used in practice: three-hinged, two-hinged and hingeless arches. In the early part of the nineteenth century, three-hinged arches were commonly used for the long span structures as the analysis of such arches could be done with confidence. However, with the development in structural analysis, for long span structures starting from late nineteenth century engineers adopted two-hinged and hingeless arches. Two-hinged arch is the statically indeterminate structure to degree one. Usually, the horizontal reaction is treated as the redundant and is evaluated by the method of least work. In this lesson, the analysis of two-hinged arches is discussed and few problems are solved to illustrate the procedure for calculating the internal forces.

33.2 Analysis of two-hinged arch

A typical two-hinged arch is shown in Fig. 33.1a. In the case of two-hinged arch, we have four unknown reactions, but there are only three equations of equilibrium available. Hence, the degree of statical indeterminacy is one for two-hinged arch.

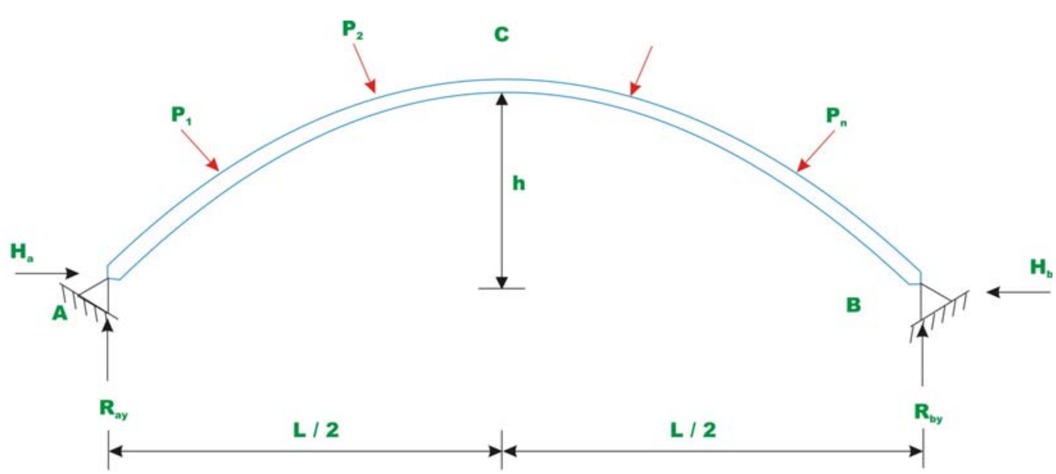


Fig. 33.1a Two - hinged arch.

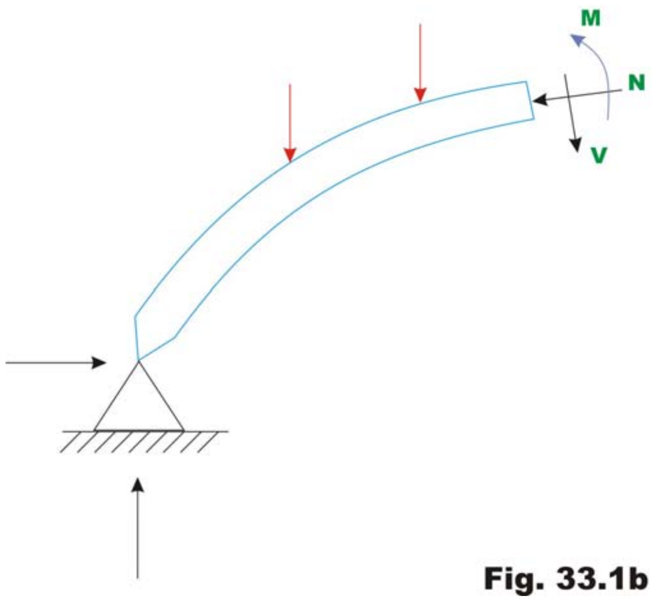


Fig. 33.1b

The fourth equation is written considering deformation of the arch. The unknown redundant reaction H_b is calculated by noting that the horizontal displacement of hinge B is zero. In general the horizontal reaction in the two hinged arch is evaluated by straightforward application of the theorem of least work (see module 1, lesson 4), which states that the partial derivative of the strain energy of a statically indeterminate structure with respect to statically indeterminate action should vanish. Hence to obtain, horizontal reaction, one must develop an expression for strain energy. Typically, any section of the arch (vide Fig 33.1b) is subjected to shear force V, bending moment M and the axial compression N. The strain energy due to bending U_b is calculated from the following expression.

$$U_b = \int_0^s \frac{M^2}{2EI} ds \tag{33.1}$$

The above expression is similar to the one used in the case of straight beams. However, in this case, the integration needs to be evaluated along the curved arch length. In the above equation, s is the length of the centerline of the arch, I is the moment of inertia of the arch cross section, E is the Young's modulus of the arch material. The strain energy due to shear is small as compared to the strain energy due to bending and is usually neglected in the analysis. In the case of flat arches, the strain energy due to axial compression can be appreciable and is given by,

$$U_a = \int_0^s \frac{N^2}{2AE} ds \tag{33.2}$$

The total strain energy of the arch is given by,

$$U = \int_0^s \frac{M^2}{2EI} ds + \int_0^s \frac{N^2}{2AE} ds \tag{33.3}$$

Now, according to the principle of least work

$\frac{\partial U}{\partial H} = 0$, where H is chosen as the redundant reaction.

$$\frac{\partial U}{\partial H} = \int_0^s \frac{M}{EI} \frac{\partial M}{\partial H} ds + \int_0^s \frac{N}{AE} \frac{\partial N}{\partial H} ds = 0 \tag{33.4}$$

Solving equation 33.4, the horizontal reaction H is evaluated.

33.2.1 Symmetrical two hinged arch

Consider a symmetrical two-hinged arch as shown in Fig 33.2a. Let C at crown be the origin of co-ordinate axes. Now, replace hinge at B with a roller support. Then we get a simply supported curved beam as shown in Fig 33.2b. Since the curved beam is free to move horizontally, it will do so as shown by dotted lines in Fig 33.2b. Let M_0 and N_0 be the bending moment and axial force at any cross section of the simply supported curved beam. Since, in the original arch structure, there is no horizontal displacement, now apply a horizontal force H as shown in Fig. 33.2c. The horizontal force H should be of such magnitude, that the displacement at B must vanish.

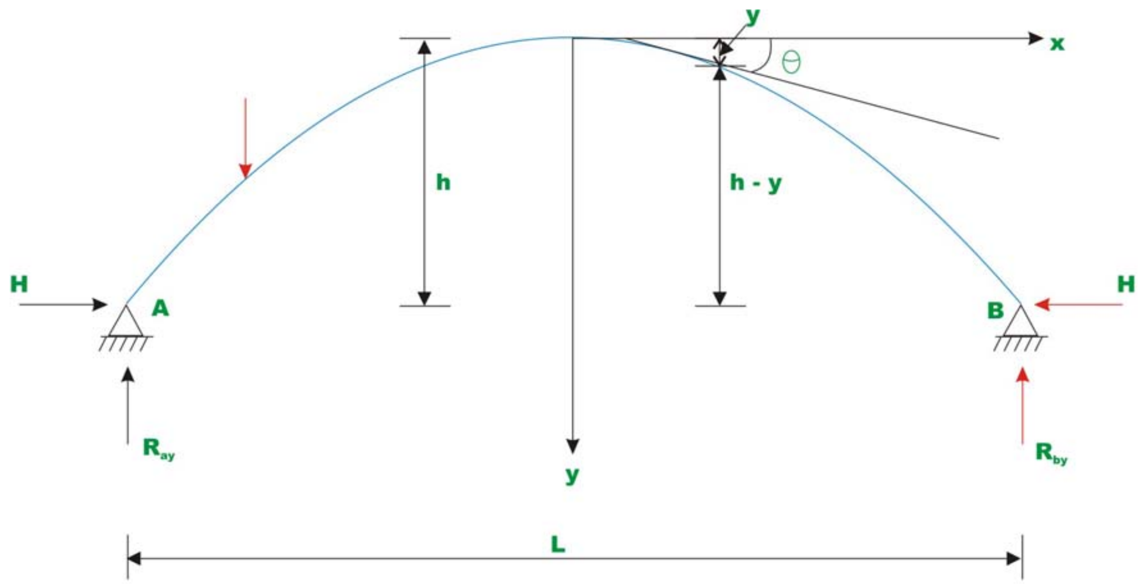


Fig. 33.2a

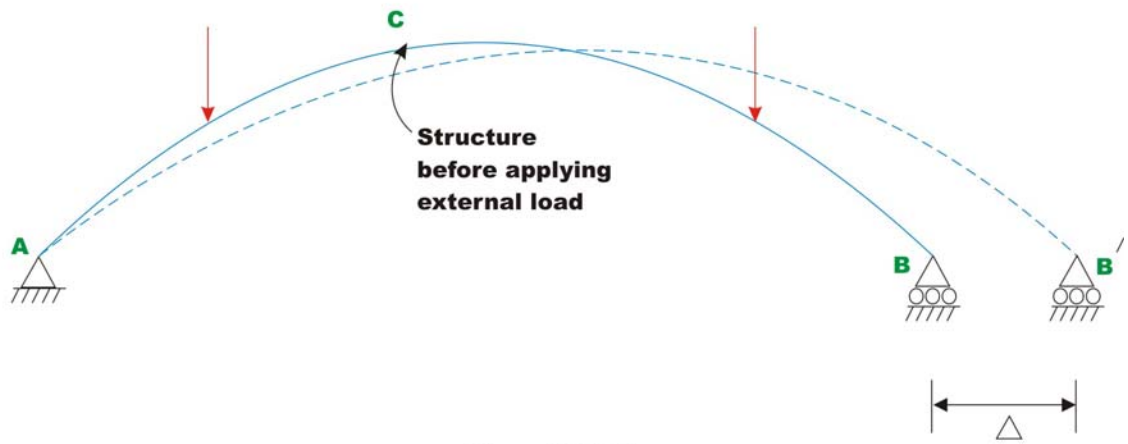


Fig. 33.2b.

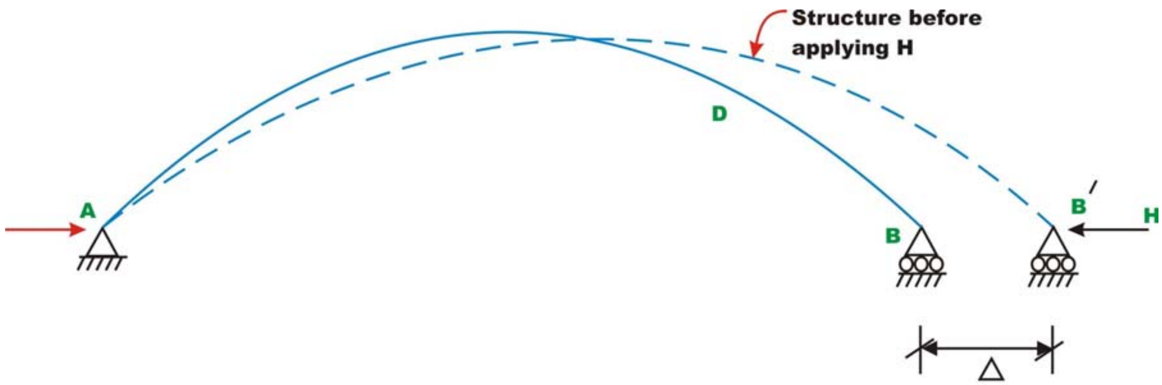


Fig. 33.2c.

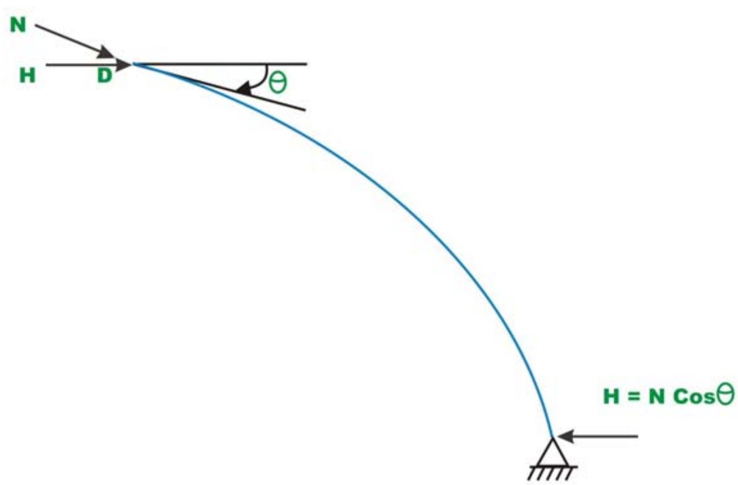


Fig. 33.2d.

From Fig. 33.2b and Fig 33.2c, the bending moment at any cross section of the arch (say D), may be written as

$$M = M_0 - H(h - y) \tag{33.5}$$

The axial compressive force at any cross section (say D) may be written as

$$N = N_0 + H\cos\theta \tag{33.6}$$

Where θ is the angle made by the tangent at D with horizontal (vide Fig 33.2d). Substituting the value of M and N in the equation (33.4),

$$\frac{\partial U}{\partial H} = 0 = -\int_0^s \frac{M_0 - H(h-y)}{EI}(h-y)ds + \int_0^s \frac{N_0 + H\cos\theta}{EA}\cos\theta\, ds \tag{33.7a}$$

Let, $\tilde{y} = h - y$

$$-\int_0^s \frac{M_0 - H\tilde{y}}{EI}\tilde{y}\,ds + \int_0^s \frac{N_0 + H\cos\theta}{EA}\cos\theta\,ds = 0 \tag{33.7b}$$

Solving for H , yields

$$-\int_0^s \frac{M_0}{EI}\tilde{y}\,ds + \int_0^s \frac{H\tilde{y}^2}{EI}ds + \int_0^s \frac{N_0}{EA}\cos\theta\,ds + \int_0^s \frac{H\cos^2\theta}{EA}ds = 0$$

$$H = \frac{\int_0^s \frac{M_0}{EI}\tilde{y}\,ds - \int_0^s \frac{N_0}{EA}\cos\theta\,ds}{\int_0^s \frac{\tilde{y}^2}{EI}ds + \int_0^s \frac{\cos^2\theta}{EA}ds} \tag{33.8}$$

Using the above equation, the horizontal reaction H for any two-hinged symmetrical arch may be calculated. The above equation is valid for any general type of loading. Usually the above equation is further simplified. The second term in the numerator is small compared with the first terms and is neglected in the analysis. Only in case of very accurate analysis second term s considered. Also for flat arched, $\cos\theta \cong 1$ as θ is small. The equation (33.8) is now written as,

$$H = \frac{\int_0^s \frac{M_0}{EI}\tilde{y}\,ds}{\int_0^s \frac{\tilde{y}^2}{EI}ds + \int_0^s \frac{ds}{EA}} \tag{33.9}$$

As axial rigidity is very high, the second term in the denominator may also be neglected. Finally the horizontal reaction is calculated by the equation

$$H = \frac{\int_0^s \frac{M_0}{EI}\tilde{y}\,ds}{\int_0^s \frac{\tilde{y}^2}{EI}ds} \tag{33.10}$$

For an arch with uniform cross section EI is constant and hence,

$$H = \frac{\int_0^s M_0 \tilde{y}\, ds}{\int_0^s \tilde{y}^2\, ds} \qquad (33.11)$$

In the above equation, M_0 is the bending moment at any cross section of the arch when one of the hinges is replaced by a roller support. $\tilde{y}$ is the height of the arch as shown in the figure. If the moment of inertia of the arch rib is not constant, then equation (33.10) must be used to calculate the horizontal reaction H.

33.2.2 Temperature effect

Consider an unloaded two-hinged arch of span L. When the arch undergoes a uniform temperature change of $T°C$, then its span would increase by αLT if it were allowed to expand freely (vide Fig 33.3a). α is the co-efficient of thermal expansion of the arch material. Since the arch is restrained from the horizontal movement, a horizontal force is induced at the support as the temperature is increased.

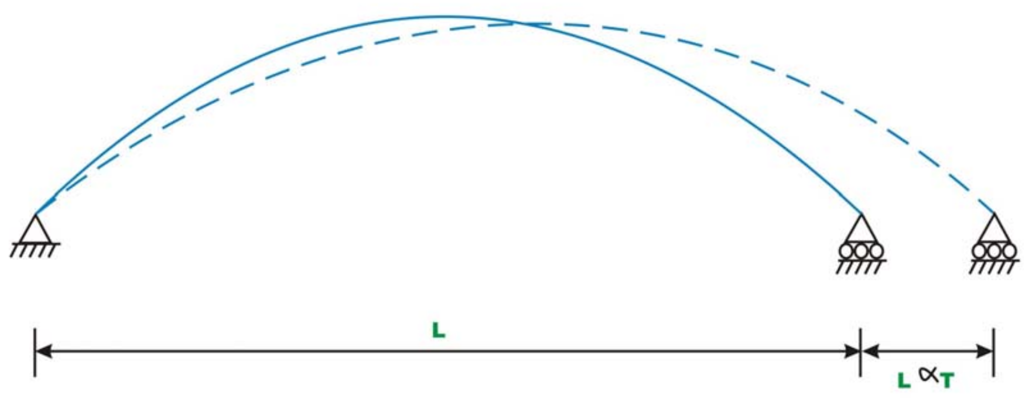


Fig. 33.3a

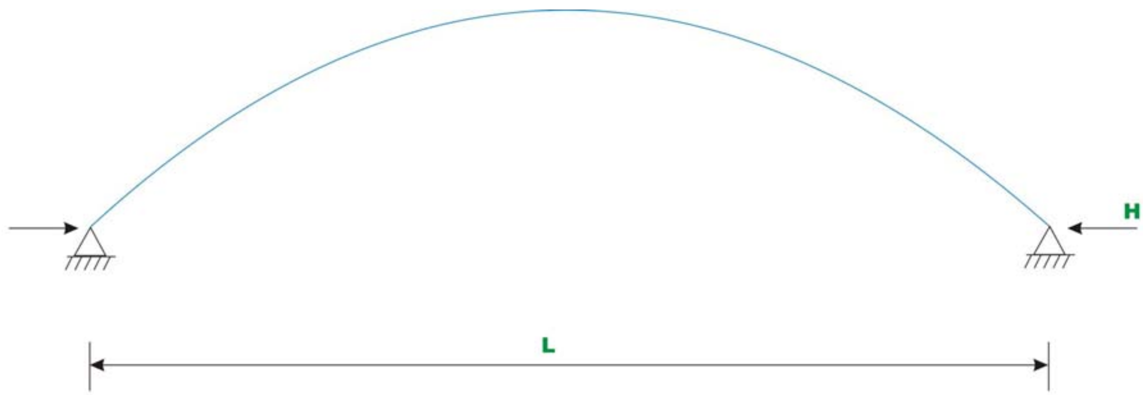


Fig. 33.3b.

Now applying the Castigliano's first theorem,

$$\frac{\partial U}{\partial H} = \alpha LT = \int_0^s \frac{H\tilde{y}^2}{EI} ds + \int_0^s \frac{H\cos^2\theta}{EA} ds \tag{33.12}$$

Solving for H ,

$$H = \frac{\alpha LT}{\int_0^s \frac{\tilde{y}^2}{EI} ds + \int_0^s \frac{\cos^2\theta}{EA} ds} \tag{33.13}$$

The second term in the denominator may be neglected, as the axial rigidity is quite high. Neglecting the axial rigidity, the above equation can be written as

$$H = \frac{\alpha LT}{\int_0^s \frac{\tilde{y}^2}{EI} ds} \tag{33.14}$$

Example 33.1

A semicircular two hinged arch of constant cross section is subjected to a concentrated load as shown in Fig 33.4a. Calculate reactions of the arch and draw bending moment diagram.

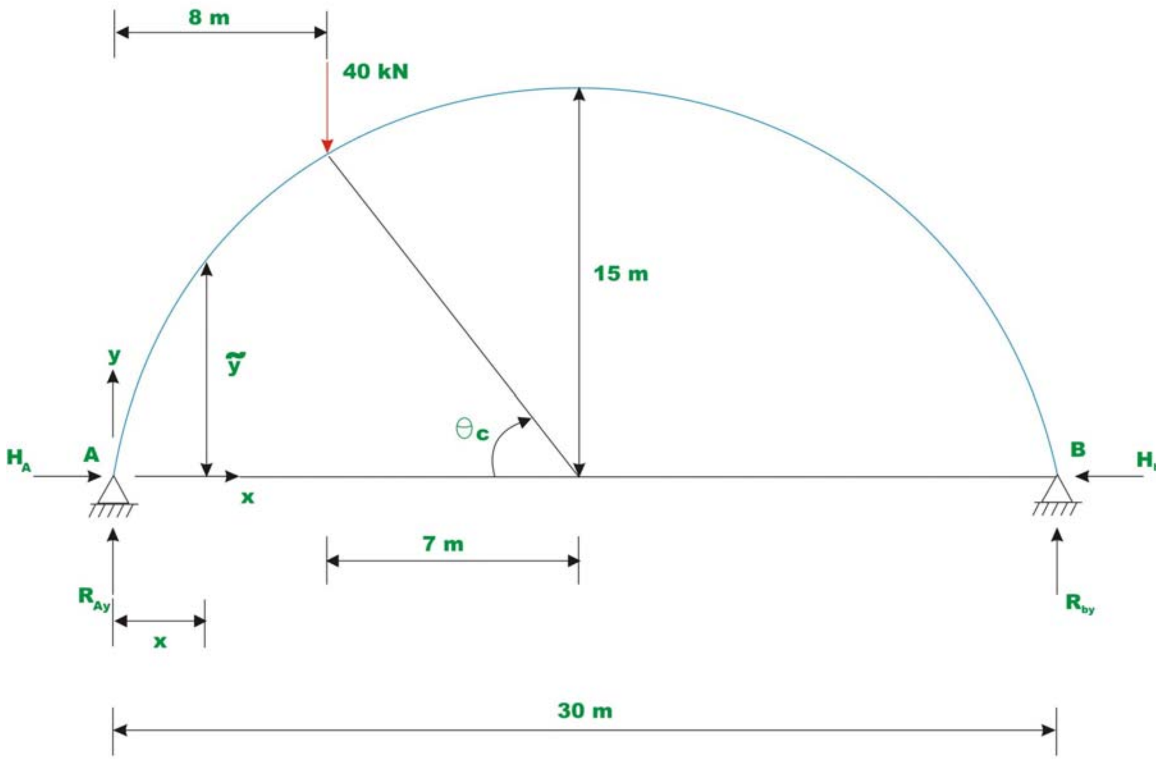


Fig. 33.4a.

Solution:

Taking moment of all forces about hinge B leads to,

$$R_{ay} = \frac{40 \times 22}{30} = 29.33 \text{ kN } (\uparrow)$$

$$\sum Fy = 0 \qquad \Rightarrow R_{by} = 10.67 \text{ kN } (\uparrow) \qquad (1)$$

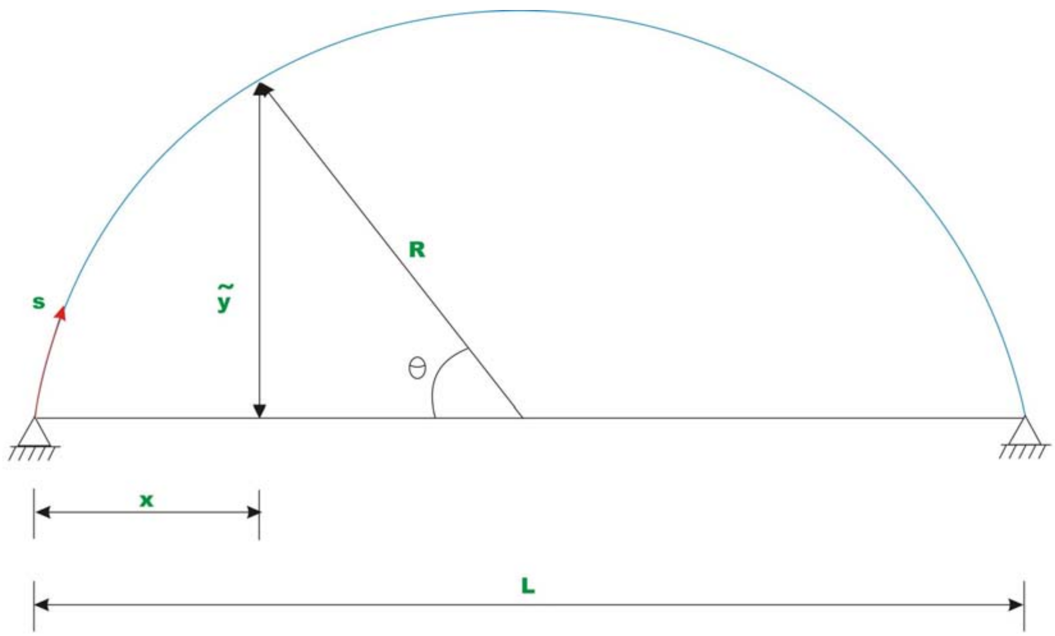


Fig. 33.4b.

From Fig. 33.4b,

$$\tilde{y} = R\sin\theta$$

$$x = R(1 - \cos\theta)$$

$$ds = R\,d\theta \qquad (2)$$

$$\tan\theta_c = \frac{13.267}{7} \qquad \Rightarrow \theta_c = 62.18^\circ = \pi / 2.895\, rad$$

Now, the horizontal reaction H may be calculated by the following expression,

$$H = \frac{\int_0^s M_0 \tilde{y}\, ds}{\int_0^s \tilde{y}^2\, ds} \qquad (3)$$

Now M_0 the bending moment at any cross section of the arch when one of the hinges is replaced by a roller support is given by,

$$M_0 = R_{ay} x = R_{ay}\ R(1-\cos\theta) \qquad\qquad 0 \le \theta \le \theta_c$$

and,

$$\begin{aligned} M_0 &= R_{ay}\ R(1-\cos\theta) - 40(x-8) \\ &= R_{ay}\ R(1-\cos\theta) - 40\{R(1-\cos\theta)-8\} \qquad \theta_c \le \theta \le \pi \end{aligned} \qquad (4)$$

Integrating the numerator in equation (3),

$$\int_0^s M_0 \tilde{y} ds = \int_0^{\theta_c} R_{ay} R^3 (1-\cos\theta)\sin\theta\, d\theta + \int_{\theta_c}^{\pi} [R_{ay}\ R(1-\cos\theta) - 40\{R(1-\cos\theta)-8\}] R\sin\theta\, R d\theta$$

$$= R_{ay} R^3 \int_0^{\pi/2.895} (1-\cos\theta)\sin\theta\, d\theta + R^2 \int_{\pi/2.895}^{\pi} [R_{ay}\ R(1-\cos\theta)\sin\theta - 40\{R(1-\cos\theta)\sin\theta - 8\sin\theta\}] d\theta$$

$$= R_{ay} R^3 [-\cos\theta]_0^{\pi/2.895} + R^2 \left[\left[R_{ay}\ R(-\cos\theta)\right]_{\pi/2.895}^{\pi} - \left[40\, R(-\cos\theta)\right]_{\pi/2.895}^{\pi} + \left[40\times 8(-\cos\theta)\right]_{\pi/2.895}^{\pi} \right]$$

$$= 0.533 R_{ay} R^3 + R^2 \left[[1.4667 R_{ay}\ R] - [40\, R(1.4667)] + [40\times 8(1.4667)] \right]$$

$$= 52761.00 + 225(645.275 - 410.676) = 105545.775 \qquad (5)$$

The value of denominator in equation (3), after integration is,

$$\begin{aligned} \int_0^s \tilde{y}^2 ds &= \int_0^{\pi} (R\sin\theta)^2 R d\theta \\ &= R^3 \int_0^{\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta = R^3 \left(\frac{\pi}{2}\right) = 5301.46 \end{aligned} \qquad (6)$$

Hence, the horizontal thrust at the support is,

$$H = \frac{105545.775}{5301.46} = 19.90 \quad \text{kN} \qquad (7)$$

Bending moment diagram

Bending moment M at any cross section of the arch is given by,

$$M = M_0 - H\tilde{y}$$
$$= R_{ay} R(1-\cos\theta) - HR\sin\theta \qquad 0 \le \theta \le \theta_c \qquad (8)$$
$$= 439.95(1-\cos\theta) - 298.5\sin\theta$$

$$M = 439.95(1-\cos\theta) - 298.5\sin\theta - 40(15(1-\cos\theta)-8) \qquad \theta_c \le \theta \le \pi \qquad (9)$$

Using equations (8) and (9), bending moment at any angle θ can be computed. The bending moment diagram is shown in Fig. 33.4c.

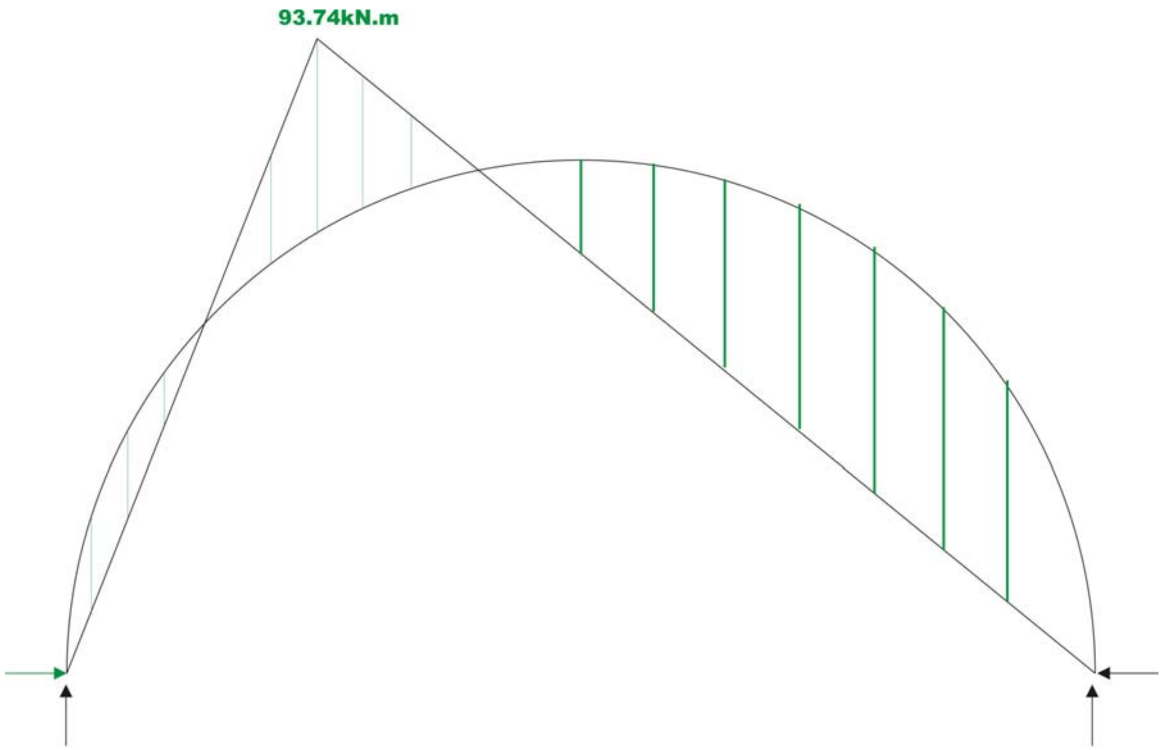


Fig. 33.4c Bending moment diagram

Example 33.2

A two hinged parabolic arch of constant cross section has a span of 60m and a rise of 10m. It is subjected to loading as shown in Fig.33.5a. Calculate reactions of the arch if the temperature of the arch is raised by $40°C$. Assume co-efficient of thermal expansion as $\alpha = 12 \times 10^{-6} / °C$.

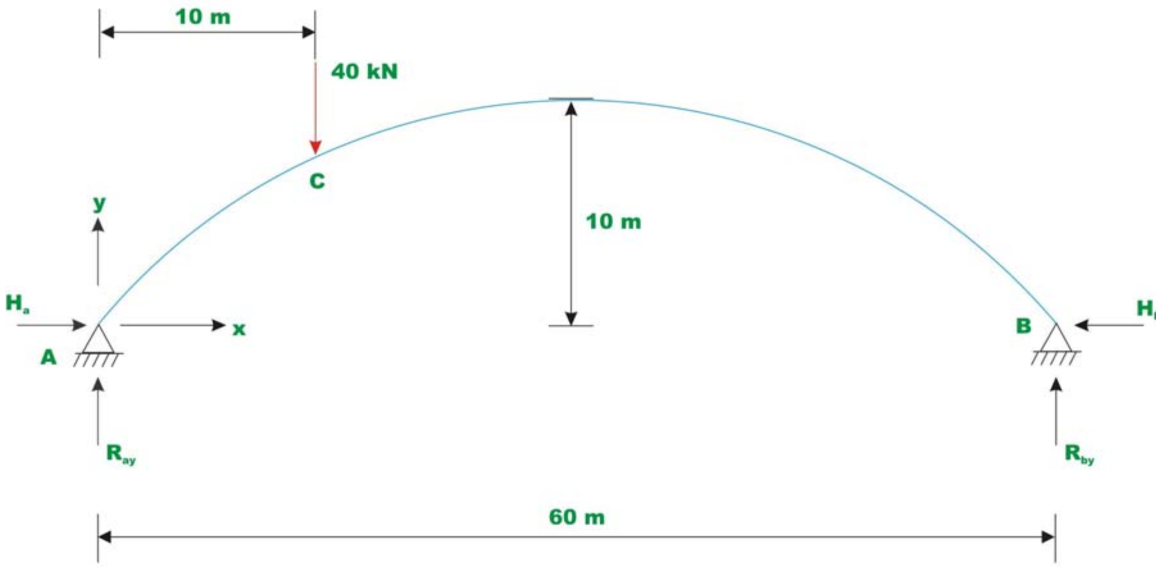


Fig. 33.5

Taking A as the origin, the equation of two hinged parabolic arch may be written as,

$$y = \frac{2}{3}x - \frac{10}{30^2}x^2 \qquad (1)$$

The given problem is solved in two steps. In the first step calculate the horizontal reaction due to 40 kN load applied at C. In the next step calculate the horizontal reaction due to rise in temperature. Adding both, one gets the horizontal reaction at the hinges due to combined external loading and temperature change. The horizontal reaction due to 40 kN load may be calculated by the following equation,

$$H_1 = \frac{\int_0^s M_0 y\, ds}{\int_0^s \tilde{y}^2\, ds} \tag{2a}$$

For temperature loading, horizontal reaction is given by,

$$H_2 = \frac{\alpha L T}{\int_0^s \frac{y^2}{EI} ds} \tag{2b}$$

Where L is the span of the arch.

For 40 kN load,

$$\int_0^s M_0 y\, ds = \int_0^{10} R_{ay}\, x y\, dx + \int_{10}^{60} \left[R_{ay} x - 40(x-10) \right] y\, dx \tag{3}$$

Please note that in the above equation, the integrations are carried out along the *x*-axis instead of the curved arch axis. The error introduced by this change in the variables in the case of flat arches is negligible. Using equation (1), the above equation (3) can be easily evaluated.

The vertical reaction A is calculated by taking moment of all forces about B. Hence,

$$R_{ay} = \frac{1}{60}\left[40 \times 50\right] = 33.33 \quad \text{kN}$$

$$R_{by} = 6.67 \quad \text{kN}.$$

Now consider the equation (3),

$$\int_0^l M_0 y\, dx = \int_0^{10} (33.33)\, x \left(\frac{2}{3} x - \frac{10}{30^2} x^2\right) dx + \int_{10}^{60} \left[(33.33) x - 40(x-10)\right] \left(\frac{2}{3} x - \frac{10}{30^2} x^2\right) dx$$

$$= 6480.76 + 69404.99 = 74885.75 \tag{4}$$

$$\int_0^l y^2\,dx = \int_0^{60}\left[\frac{2}{3}x - \frac{10}{30^2}x^2\right]^2 dx \tag{5}$$

$$= 3200$$

Hence, the horizontal reaction due to applied mechanical loads alone is given by,

$$H_1 = \frac{\int_0^l M_0 y \quad dx}{\int_0^l y^2 \quad dx} = \frac{75885.75}{3200} = 23.71 \quad \text{kN} \tag{6}$$

The horizontal reaction due to rise in temperature is calculated by equation (2b),

$$H_2 = \frac{12\times10^{-6}\times60\times40}{3200/EI} = \frac{EI\times12\times10^{-6}\times60\times40}{3200}$$

Taking $E = 200 \quad \text{kN/mm}^2$ and $I = 0.0333m^4$

$$H_2 = 59.94 \quad \text{kN}. \tag{7}$$

Hence the total horizontal thrust $H = H_1 + H_2 = 83.65$ kN.

When the arch shape is more complicated, the integrations $\int_0^s \frac{M_0 y}{EI}ds$ and $\int_0^s \frac{y^2}{EI}ds$ are accomplished numerically. For this purpose, divide the arch span in to n equals divisions. Length of each division is represented by $(\Delta s)_i$ (vide Fig.33.5b). At the midpoint of each division calculate the ordinate y_i by using the equation $y = \frac{2}{3}x - \frac{10}{30^2}x^2$. The above integrals are approximated as,

$$\int_0^s \frac{M_0 y}{EI}ds = \frac{1}{EI}\sum_{i=1}^{n}(M_0)_i\, y_i\,(\Delta s)_i \tag{8}$$

$$\int_0^s \frac{y^2}{EI}ds = \frac{1}{EI}\sum_{i=1}^{n}(y)_i^{\,2}(\Delta s)_i \tag{9}$$

The complete computation for the above problem for the case of external loading is shown in the following table.

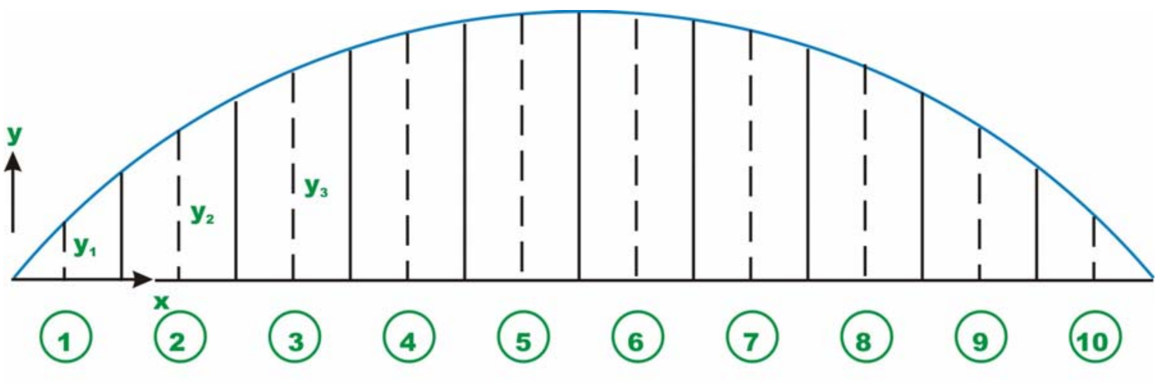


Fig. 33.5(b)

Table 1. Numerical integration of equations (8) and (9)

Segment No	Horizontal distance *x* Measured from A (m)	Corresponding y_i (m)	Moment at that Point $(M_0)_i$ (kNm)	$(M_0)_i y_i (\Delta s)_i$	$(y)_i^2 (\Delta s)_i$
1	3	1.9	99.99	1139.886	21.66
2	9	5.1	299.97	9179.082	156.06
3	15	7.5	299.95	13497.75	337.5
4	21	9.1	259.93	14192.18	496.86
5	27	9.9	219.91	13062.65	588.06
6	33	9.9	179.89	10685.47	588.06
7	39	9.1	139.87	7636.902	496.86
8	45	7.5	99.85	4493.25	337.5
9	51	5.1	59.83	1830.798	156.06
10	57	1.9	19.81	225.834	21.66
			$\sum$	75943.8	3300.3

$$H_1 = \frac{\sum (M_0)_i y_i (\Delta s)}{\sum (y)_i^2 (\Delta s)_i} = \frac{75943.8}{3200.3} = 23.73 \quad \text{kN} \tag{10}$$

This compares well with the horizontal reaction computed from the exact integration.

Summary

Two-hinged arch is the statically indeterminate structure to degree one. Usually, the horizontal reaction is treated as the redundant and is evaluated by the method of least work. Towards this end, the strain energy stored in the two-hinged arch during deformation is given. The reactions developed due to thermal loadings are discussed. Finally, a few numerical examples are solved to illustrate the procedure.

Module
5
Cables and Arches

Lesson 34

Symmetrical Hingeless Arch

Instructional Objectives:

After reading this chapter the student will be able to
1. Analyse hingeless arch by the method of least work.
2. Analyse the fixed-fixed arch by the elastic-centre method.
3. Compute reactions and stresses in hingeless arch due to temperature change.

34.1 Introduction

As stated in the previous lesson, two-hinged and three-hinged arches are commonly used in practice. The deflection and the moment at the center of the hingeless arch are somewhat smaller than that of the two-hinged arch. However, the hingeless arch has to be designed for support moment. A hingeless arch (fixed–fixed arch) is a statically redundant structure having three redundant reactions. In the case of fixed–fixed arch there are six reaction components; three at each fixed end. Apart from three equilibrium equations three more equations are required to calculate bending moment, shear force and horizontal thrust at any cross section of the arch. These three extra equations may be set up from the geometry deformation of the arch.

34.2 Analysis of Symmetrical Hingeless Arch

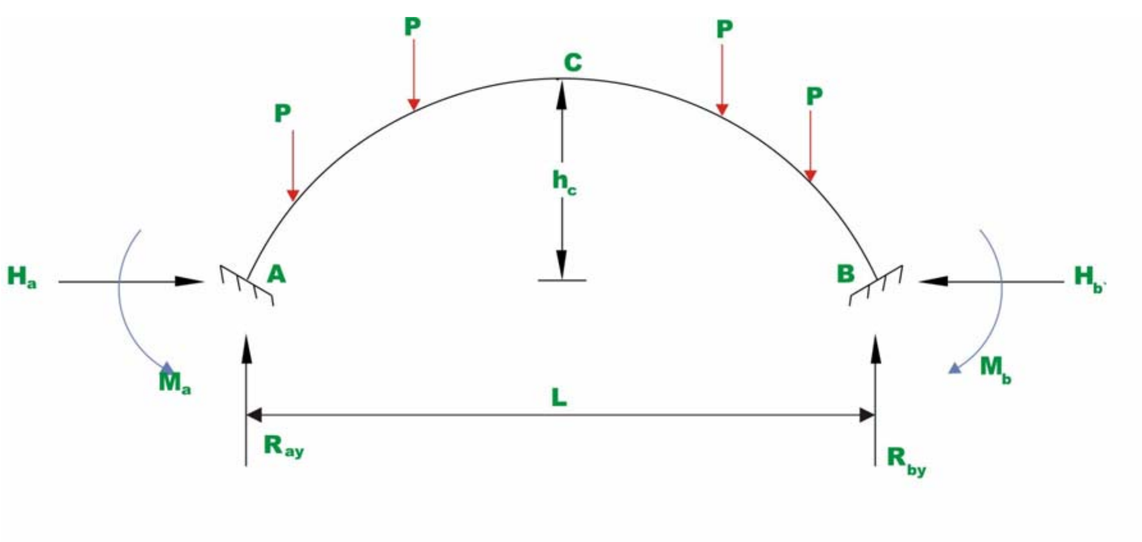


Fig. 34.1 Hingeless Arch

Consider a symmetrical arch of span L and central rise of h_c Let the loading on the arch is also symmetrical as shown in Fig 34.1. Consider reaction components at the left support A i.e., bending moment M_a, vertical reaction R_{ay} and horizontal thrust H_a as redundants.

Considering only the strain energy due to axial compression and bending, the strain energy U of the arch may be written as

$$U = \int_0^s \frac{M^2 ds}{2EI} + \int_0^s \frac{N^2 ds}{2EA} \qquad (34.1)$$

where M and N are respectively the bending moment and axial force of the arch rib. Since the support A is fixed, one could write following three equations at that point.

$$\frac{\partial U}{\partial M_a} = 0 \qquad (34.2a)$$

$$\frac{\partial U}{\partial H_a} = 0 \qquad (34.2b)$$

$$\frac{\partial U}{\partial R_{ay}} = 0 \qquad (34.2c)$$

Knowing dimensions of the arch and loading, using the above three equations, the unknown redundant reactions M_a, H_a and R_{ay} may be evaluated.

Since the arch and the loading are symmetrical, the shear force at the crown is zero. Hence, at the crown we have only two unknowns. Hence, if we take the internal forces at the crown as the redundant, the problem gets simplified.

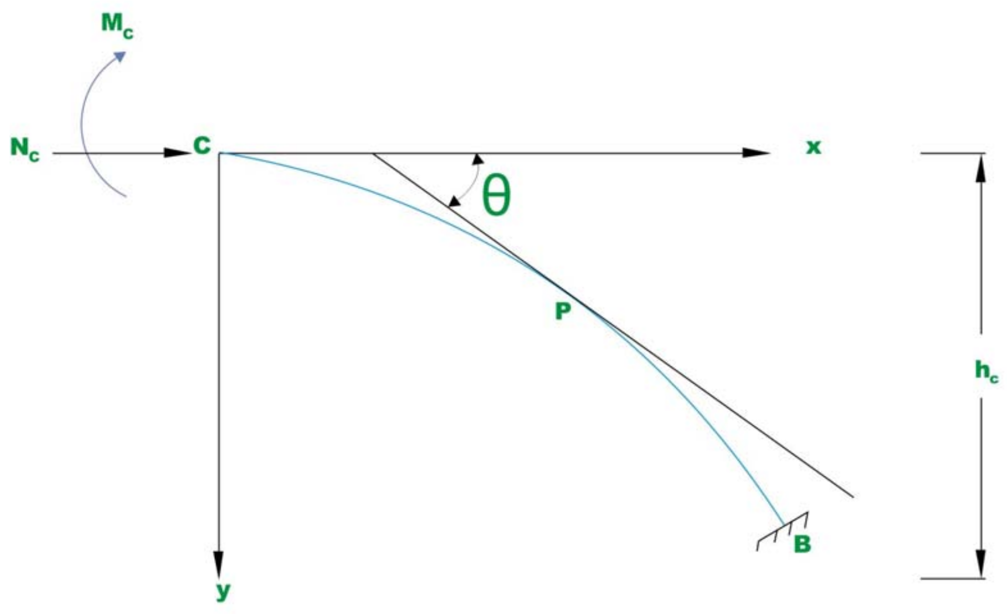


Fig. 34.2

Hence, consider bending moment M_c and the axial force N_c at the crown as the redundant. Since the arch and the loading is symmetrical, we can write from the principle of least work

$$\frac{\partial U}{\partial M_c} = 0 \qquad (34.3a)$$

$$\frac{\partial U}{\partial N_c} = 0 \qquad (34.3b)$$

$$\frac{\partial U}{\partial M_c} = \int_0^s \frac{M}{EI}\frac{\partial M}{\partial M_c} ds + \int_0^s \frac{N}{EA}\frac{\partial N}{\partial M_c} ds = 0 \qquad (34.4a)$$

$$\frac{\partial U}{\partial N_c} = \int_0^s \frac{M}{EI}\frac{\partial M}{\partial N_c} ds + \int_0^s \frac{N}{EA}\frac{\partial N}{\partial N_c} ds = 0 \qquad (34.4b)$$

Where, s is the length of centerline of the arch, I is the moment of inertia of the cross section and A is the area of the cross section of the arch. Let M_0 and N_0 be the bending moment and the axial force at any cross section due to external loading. Now the bending moment and the axial force at any section is given by

$$M = M_c + N_c y + M_0 \tag{34.5a}$$

$$N = N_c \cos\theta + N_0 \tag{34.5b}$$

$$\frac{\partial M}{\partial M_c} = 1; \quad \frac{\partial M}{\partial N_c} = y; \quad \frac{\partial N}{\partial N_c} = \cos\theta; \quad \frac{\partial N}{\partial M_c} = 0. \tag{34.6}$$

Equation (34.4a) and (34.4b) may be simplified as,

$$\int_0^s \frac{M}{EI}(1)ds + \int_0^s \frac{N}{EA}(0)ds = 0$$

$$M_c \int_0^s \frac{ds}{EI} + N_c \int_0^s \frac{y\ ds}{EI} = -\int_0^s \frac{M_0}{EI} ds \tag{34.7a}$$

$$\int_0^s \frac{M}{EI} y ds + \int_0^s \frac{N}{EA} \cos\theta\, ds = 0$$

$$\int_0^s \frac{M_c y}{EI} ds + \int_0^s \frac{N_c y^2}{EI} ds + \int_0^s \frac{N_c}{EA} \cos^2\theta \quad ds = -\int_0^s \frac{M_0 y}{EI} ds - \int_0^s \frac{N_0}{EA} \cos\theta \quad ds \tag{34.7b}$$

From equations 34.7a and 34.7b, the redundant M_c and N_c may be calculated provided arch geometry and loading are defined. If the loading is unsymmetrical or the arch is unsymmetrical, then the problem becomes more complex. For such problems either column analogy or elastic center method must be adopted. However, one could still get the answer from the method of least work with little more effort.

34.3 Temperature stresses

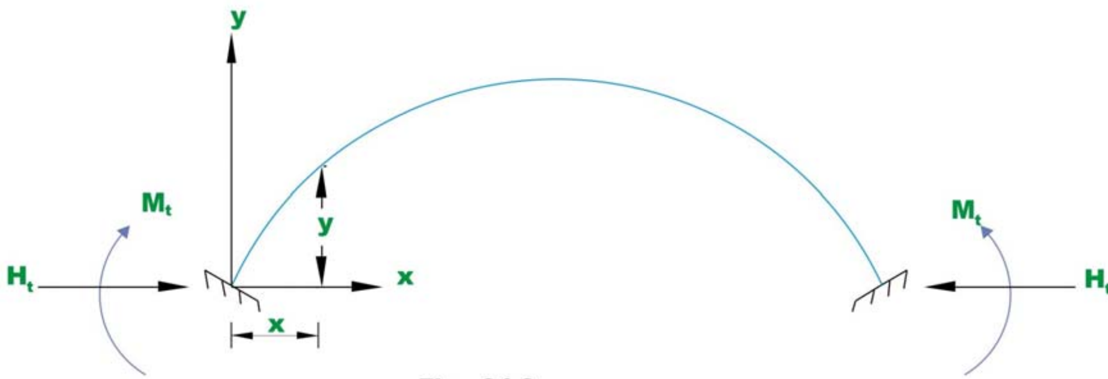


Fig. 34.3

Consider an unloaded fixed-fixed arch of span L. The rise in temperature, would introduce a horizontal thrust H_t and a moment M_t at the supports. Now due to rise in temperature, the moment at any cross-section of the arch

$$M = M_t - H_t t \tag{34.8}$$

Now strain energy stored in the arch

$$U = \int_0^s \frac{M^2 ds}{2EI}$$

Now applying the Castigliano's first theorem,

$$\frac{\partial U}{\partial H_t} = \alpha \quad L \quad T = \int_0^s \frac{M}{EI}\frac{\partial M}{\partial H_t} ds$$

$$\alpha LT == \int_0^s \frac{M_t y}{EI} ds - H_t \int_0^s \frac{y^2}{EI} ds \tag{34.9}$$

Also,

$$\frac{\partial U}{\partial M_t} = 0 = \int_0^s \frac{M}{EI}\frac{\partial M}{\partial M_t} ds$$

$$\int_0^s \frac{(M_t - H_t y)}{EI} ds = 0$$

$$M_t \int_0^s \frac{ds}{EI} - H_t \int_0^s \frac{yds}{EI} = 0 \tag{34.10}$$

Solving equations 34.9 and 34.10, M_t and H_t may be calculated.

Example 34.1

A semicircular fixed-fixed arch of constant cross section is subjected to symmetrical concentrated load as shown in Fig 34.4. Determine the reactions of the arch.

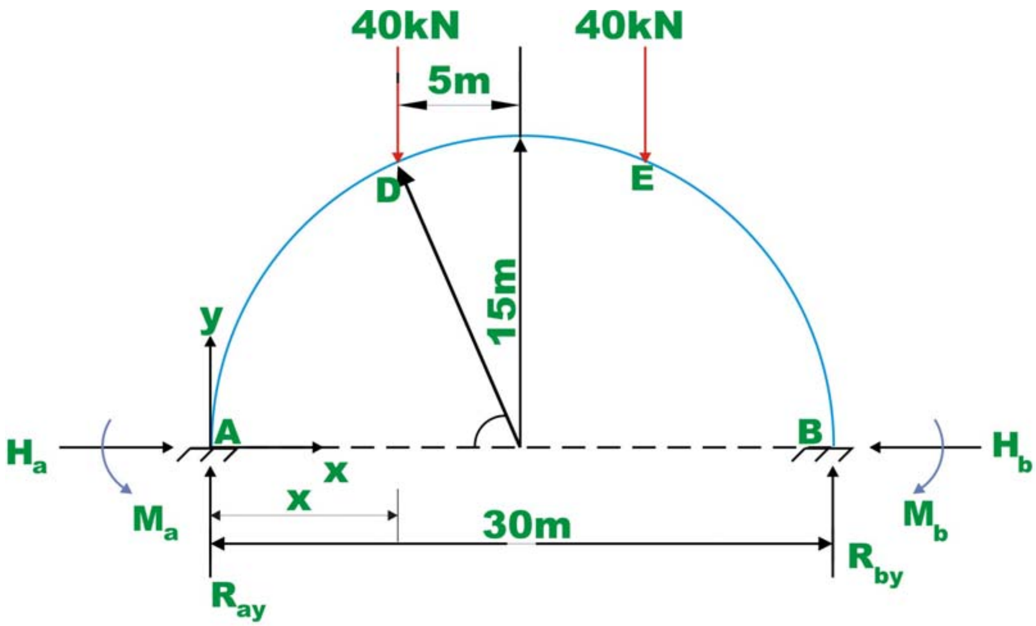


Fig. 34.4 Example 34.1

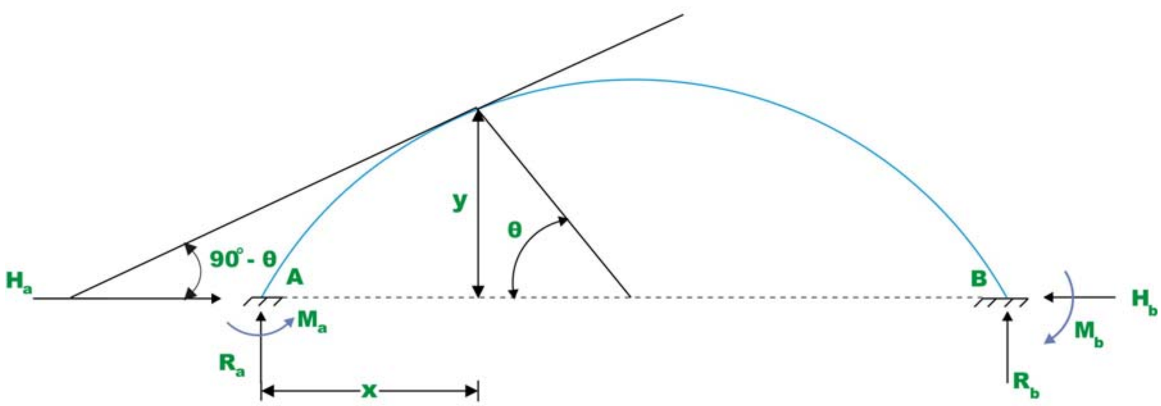


Fig. 34.5

Solution:

Since, the arch is symmetrical and the loading is also symmetrical,

$$R_{ay} = R_{by} = 40\text{kN} \tag{1}$$

Now the strain energy of the arch is given by,

$$U = \int_0^s \frac{M^2 ds}{2EI} + \int_0^s \frac{N^2 ds}{2EA} \qquad (2)$$

Let us choose H_a and M_a as redundants. Then we have,

$$\frac{\partial U}{\partial M_a} = 0 \text{ and } \frac{\partial U}{\partial H_a} = 0 \qquad (3)$$

The bending moment at any cross section is given by,

$$M = R_{ay} x - M_a - H_a y \qquad 0 \le \theta \le \theta_D \qquad (4)$$

$$M = R_{ay} x - M_a - H_a y - 40(x - 10) \qquad \theta_D \le \theta \le \pi / 2$$

$$N = H_a \cos(90 - \theta) + R_a \cos\theta$$

$$N = H_a \sin\theta + R_a \cos\theta \qquad 0 \le \theta \le \theta_D \qquad (5)$$

$$N = H_a \sin\theta + (R_a - 40)\cos\theta \qquad \theta \ \le \theta \le \pi / 2 \qquad (6)$$

$$y = R\sin\theta$$

$$x = R(1 - \cos\theta)$$

And $ds = Rd\theta$

See Fig 34.5.

$$\frac{\partial U}{\partial M_a} = \int_0^s \frac{M}{EI}(-1) ds + \int_0^s \frac{N}{EA}(0) ds = 0$$

$\int_0^s \frac{M}{EI} ds = 0$ Since the arch is symmetrical, integration need to be carried out between limits $0 \text{ to } \pi / 2$ and the result is multiplied by two.

$$2\int_0^{\pi/2} \frac{M}{EI} ds = 0$$

$$\int_0^{\pi/2} 40R(1 - \cos\theta)Rd\theta - \int_0^{\pi/2} M_a Rd\theta - H_a \int_0^{\pi/2} R\sin\theta\, Rd\theta - \int_{\pi/2.552}^{\pi/2} 40[R(1 - \cos\theta) - 10]Rd\theta = 0$$

$$22.8310R^2 - 1.571M_a R - H_a R^2 - 41.304R^2 + 135.92R = 0$$

$$342.477 - 1.571M_a - 15H_a - 169.56 + 135.92 = 0$$

$$1.571M_a + 15H_a - 308.837 = 0$$

$$\frac{\partial U}{\partial H_a} = \int_0^s \frac{M}{EI}(-y)ds + \int_0^s \frac{N}{EA}(\sin\theta)ds = 0$$

$$\frac{1}{EI}\int_0^{\pi/2}(-R\sin\theta)\{[40R(1-\cos\theta)] - M_a - H_a(R\sin\theta)\}Rd\theta - \frac{1}{EI}\int_{\pi/2.552}^{\pi/2}(-R\sin\theta)\{[40[R(1-\cos\theta)-10]]\}Rd\theta +$$
$$\int_0^{\pi/2}\frac{(H_a\sin\theta + R_a\cos\theta)}{EA}(\sin\theta)Rd\theta - \frac{1}{EA}\int_{\pi/2.552}^{\pi/2}(\sin\theta)40\cos\theta\, Rd\theta = 0$$

$$\int_0^{\pi/2}\{-\frac{40R^3}{EI}\sin\theta + \frac{40R^3}{EI}\sin\theta\cos\theta + \frac{M_aR^2}{EI}\sin\theta + \frac{H_aR^3}{EI}\sin^2\theta + \frac{H_aR}{EA}\sin^2\theta - \frac{R(R_{ay})\sin\theta\cos\theta}{EA}\}d\theta +$$
$$\int_{\pi/2.552}^{\pi/2}\{\frac{40R^3}{EI}\sin\theta - \frac{40R^3}{EI}\sin\theta\cos\theta - \frac{400R^2}{EI}\sin\theta - \frac{40R}{EA}\sin\theta\cos\theta\}d\theta = 0$$

$$\frac{-40}{I}(1) + \frac{40}{I}(\frac{1}{2}) + \frac{M_a}{IR}(1) + \frac{H_a}{I}(0.785) + \frac{H_a}{R^2A}(0.785) - \frac{40}{R^2A}(\frac{1}{2}) +$$
$$\frac{40}{I}(0.333) - \frac{40}{I}(0.0554) - \frac{400}{RI}(0.333) - \frac{40}{R^2A}(0.0555) = 0$$

$$-266 + 23.58H_a + 2M_a = 0 \qquad (7)$$

Solving equations (6) and (7), H_a and M_a are evaluated. Thus,

$$H_a = 28.28\,\text{kN}$$
$$M_a = -466.42\,\text{kN} \qquad (8)$$

34.4 Elastic centre method

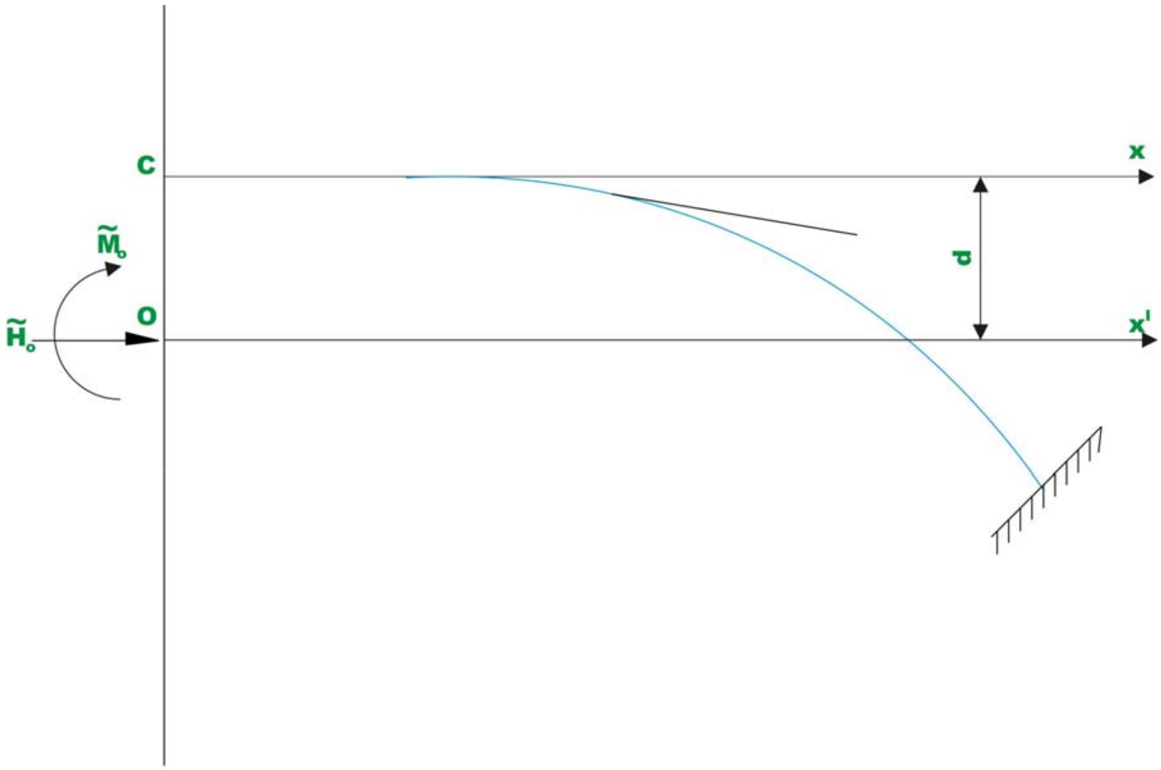


Fig. 34.6 Elastic centre

Equations (34.7a) and (34.7b) are quite difficult to solve. However, they can be further simplified if the origin of co-ordinates is moved from C to O in Fig. 34.3. The distance d is chosen such that $y_1\ (= y - d)$ satisfies the following condition.

$$\int_0^s \frac{y_1}{EI} ds = \int_0^s \frac{(y-d)}{EI} ds = 0 \qquad (34.10a)$$

Solving which, the distance d may be computed as

$$d = \frac{\int_0^s \frac{y}{EI} ds}{\int_0^s \frac{ds}{EI}} \qquad 34.10b)$$

The point O is known as the elastic centre of the arch. Now equation (34.7a) can be written with respect to new origin O. Towards this, substitute $y = y_1 + d$ in equation (34.7a).

$$M_c \int_0^s \frac{ds}{EI} + N_c \int_0^s \frac{(y_1 + d)}{EI} ds = -\int_0^s \frac{M_0}{EI} ds \tag{34.11}$$

In the above equation, $\int_0^s \frac{y_1}{EI} ds$ is zero. Hence the above equation is rewritten as

$$M_c + N_c d = -\frac{\int_0^s \frac{M_0}{EI} ds}{\int_0^s \frac{ds}{EI}} \tag{34.12}$$

Now, $(M_c + N_c d)$ is the moment $\tilde{M}_0$ at O (see Fig. 34.3). Similarly the equation (34.7b) is also simplified. Thus we obtain,

$$\tilde{M}_0 = M_c + N_c d = -\frac{\int_0^s \frac{M_0}{EI} ds}{\int_0^s \frac{ds}{EI}} \tag{34.13}$$

and,

$$\tilde{H}_0 = N_c = -\frac{\int_0^s \frac{M_0 y_1}{EI} ds + \int_0^s \frac{N_0 \cos\theta}{EA} ds}{\int_0^s \frac{y^2{}_1}{EI} ds + \int_0^s \frac{\cos^2\theta}{EA} ds} \tag{34.14}$$

34.4.1Temperature stresses

Consider a symmetrical hinge less arch of span L, subjected to a temperature rise of $T^\circ C$. Let elastic centre O be the origin of co-ordinates and $\tilde{H}_0, \tilde{M}_0$ be the redundants. The magnitude of horizontal force $\tilde{H}_0$ be such as to counteract the increase in the span $\frac{\alpha LT}{2}$ due to rise in temperature T. Also from Symmetry, there must not be any rotation at the crown. Hence,

$$\frac{\partial U}{\partial M_O} = 0 = \int_0^s \frac{M}{EI} \frac{\partial M}{\partial \tilde{M}_O} ds = 0 \tag{34.15}$$

$$\frac{\partial U}{\partial \tilde{H}_O} = \int_0^s \frac{M}{EI} \frac{\partial M}{\partial \tilde{H}_O} ds + \int_0^s \frac{N}{EA} \frac{\partial N}{\partial \tilde{H}_O} ds = \frac{\alpha LT}{2} \tag{34.16}$$

Moment at any section is calculated by,

$$M = \tilde{M}_O + \tilde{H}_O y$$

$$N = \tilde{H}_O \cos\theta$$

$$\int_0^s \frac{\tilde{M}_O}{EI} ds = 0$$

$$\tilde{M}_O = 0 \tag{34.17}$$

and

$$\int_0^s \left(\frac{\tilde{H}_O y_1}{EI}\right) y_1 ds + \int_0^s \left(\frac{\tilde{H}_O \cos\theta}{EA}\right) \cos\theta ds = \frac{\alpha LT}{2}$$

Simplifying the above equation,

$$\tilde{H}_O = \frac{\dfrac{\alpha LT}{2}}{\int_0^s \left(\dfrac{y^2{}_1}{EI}\right) ds + \int_0^s \left(\dfrac{\cos^2\theta}{EA}\right) ds} \tag{34.18}$$

Using equation (34.18), the horizontal thrust $\tilde{H}_O$ due to uniform temperature rise in the arch can be easily calculated provided the dimensions of the arch are known. Usually the area of the cross section and moment of inertia of the arch vary along the arch axis.

Example 30.2

A symmetrical hinge less circular arch of constant cross section is subjected to a uniformly distributed load of $10\ \mathrm{kN/m}$. The arch dimensions are shown in Fig. 34.7a. Calculate the horizontal thrust and moment at A.

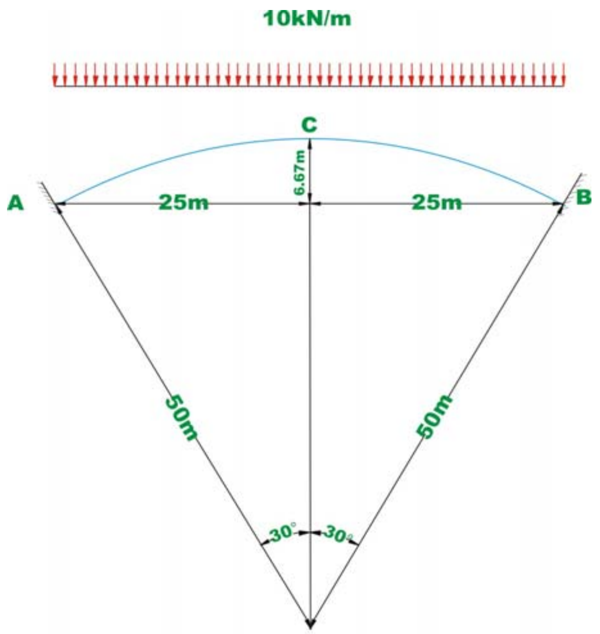


Fig. 34 .7a Example 34.2

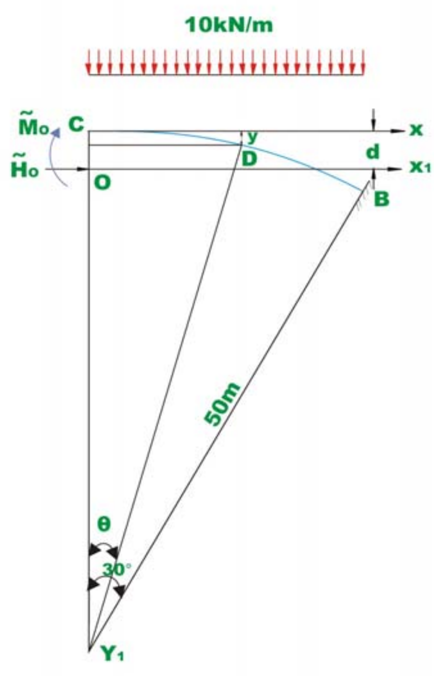


Fig. 34.7b

The distance d of the elastic centre from the crown C is calculated by equation,

$$d = \frac{\int_0^s \frac{y}{EI} ds}{\int_0^s \frac{ds}{EI}} \tag{1}$$

From Fig.34.7b, the ordinate at d , is given by

$$y = 50(1 - \cos\theta)$$

$$d = \frac{\int_0^{\pi/6} \frac{50(1-\cos\theta)}{EI} 50 d\theta}{\int_0^{\pi/6} \frac{50 d\theta}{EI}}$$

$$d = \frac{50\left(\frac{\pi}{6} - \frac{1}{2}\right)}{\frac{\pi}{6}} = 2.2535 \quad \text{m.} \tag{2}$$

The elastic centre O lies at a distance of $2.2535 m$ from the crown. The moment at the elastic centre may be calculated by equation (34.12). Now the bending moment at any section of the arch due to applied loading at a distance x from elastic centre is

$$\tilde{M}_O = \frac{-\int_0^s \frac{5x^2}{EI} ds}{\int_0^s \frac{ds}{EI}} \tag{3}$$

In the present case, $x = 50\sin\theta$ and $ds = 50 d\theta$, $EI = \text{constant}$

$$\tilde{M}_O = \frac{-5 \times 50^3 \int_0^{\pi/6} \sin^2\theta \quad d\theta}{50 \int_0^{\pi/6} d\theta}$$

$$M_C + N_C d = -\frac{5 \times 50^2}{2} \frac{\left(\frac{\pi}{6} - \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right)}{\frac{\pi}{6}} = -1081.29 \quad \text{kN.m} \tag{4}$$

$$N_O = 10\left(\frac{L}{2} - x\right)\cos\theta$$

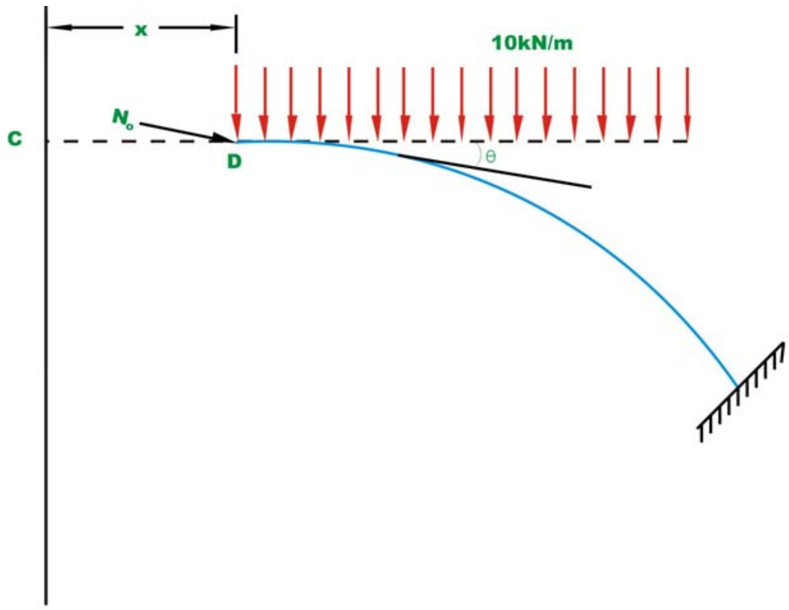


Fig. 34.7c

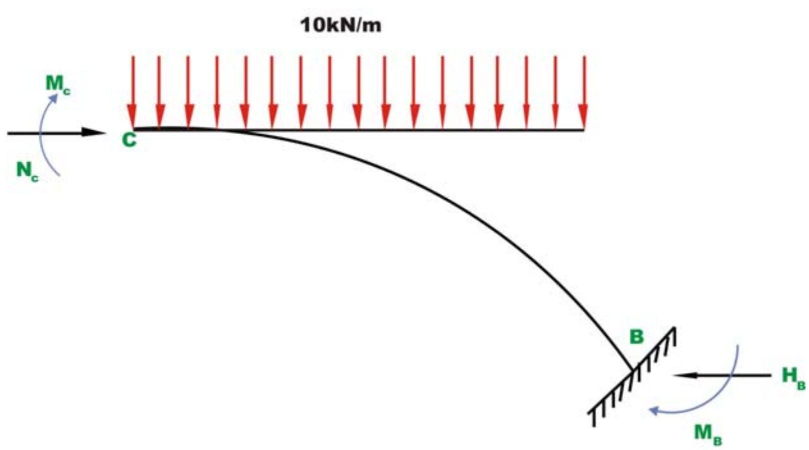


Fig. 34.7d

And.

$$y_1 = y - d$$

$$y_1 = 50(1-\cos\theta) - 2.25$$

$$y_1 = 47.75 - 50\cos\theta$$

Now $\tilde{H}_O$ is given by equation (34.14). Thus

$$\tilde{H}_0 = N_c = -\frac{\int_0^s \frac{M_0 y_1}{EI} ds + \int_0^s \frac{N_0 \cos\theta}{EA} ds}{\int_0^s \frac{y^2{}_1}{EI} ds + \int_0^s \frac{\cos^2\theta}{EA} ds} \qquad (5)$$

$$\int_0^s \frac{M_0 y_1}{EI} ds = \frac{1}{EI}\int_0^{\pi/6} 5x^2 \left(47.75 - 50\cos\theta\right) 50 \quad d\theta$$

$$= \frac{250}{EI}\int_0^{\pi/6} \left(50\sin\theta\right)^2 \left(47.75 - 50\cos\theta\right) \quad d\theta$$

$$= \frac{625000}{EI}\int_0^{\pi/6} \left(23.875(1-\cos 2\theta) - 50\cos\theta\sin^2\theta\right) \quad d\theta$$

$$= \frac{625000}{EI}\int_0^{\pi/6} \left(23.875(1-\cos 2\theta) - 25\left(\cos\theta - \frac{1}{2}(\cos 3\theta + \cos\theta)\right)\right) \quad d\theta$$

$$= \frac{49630.735}{EI} \qquad (6)$$

$$\int_0^s \frac{N_0 \cos\theta}{EA} ds = \frac{1}{EA}\int_0^{\pi/6} 10(25 - x)\cos^2\theta \quad d\theta$$

$$= \frac{10}{EA}\int_0^{\pi/6} \left(25\left(\frac{1+\cos 2\theta}{2}\right) - 50\sin\theta\cos^2\theta\right) \quad d\theta$$

$$= \frac{10}{EA}\int_0^{\pi/6} \left(12.5(1+\cos 2\theta) - 25(\sin\theta + \sin\theta\cos 2\theta)\right) \quad d\theta$$

$$= \frac{10}{EA}\left(12.5(\theta + \sin 2\theta)\right)_0^{\pi/6} - 25\left(-(\cos\theta)_0^{\pi/6} + \frac{1}{2}\left(-\frac{1}{3}\cos 3\theta - \cos\theta\right)_0^{\pi/6}\right)$$

$$= \frac{81.795}{EA} \tag{7}$$

$$\int_0^s \frac{y^2_1}{EI} ds = \frac{1}{EI} \int_0^{\pi/6} \left(47.75 - 50\cos\theta\right)^2 50 \quad d\theta$$

$$= \frac{50}{EI} \int_0^{\pi/6} \left(2280.06 + 2500\cos^2\theta - 4775\cos\theta\right) \quad d\theta$$

$$= \frac{50}{EI} \left(2280.06\left(\frac{\pi}{6}\right) + 1250\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right) - 4775\sin\frac{\pi}{6} \right)$$

$$= \frac{105.046}{EI} \tag{8}$$

$$\int_0^s \frac{\cos^2\theta}{EA} ds = \frac{50}{2EA} \int_0^{\pi/6} (1 + \cos 2\theta) d\theta$$

$$= \frac{25}{EA}\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right) = 23.915 \tag{9}$$

$$\tilde{H}_0 = -\frac{-\left(\frac{49630.735}{EI} + \frac{81.795}{EA}\right)}{\left(\frac{105.046}{EI} + \frac{23.915}{EA}\right)} \tag{10}$$

Consider an arch cross section of $300 \times 500 \quad \text{mm}$; and $I = 3.125 \times 10^{-3} \text{m}^4$ $A = 0.15 \quad \text{m}^2$. Then,

$$\tilde{H}_0 = -\frac{-\left(15881835.2 + 545.3\right)}{\left(33614.72 + 159.43\right)} = -470.25 \quad \text{kN} \tag{11}$$

In equation (5), if the second term in the numerator and the second term in the denominator were neglected then, we get,

$$\tilde{H}_0 = -\frac{-\left(\frac{49630.735}{EI}\right)}{\left(\frac{105.046}{EI}\right)} = -472.67 \quad \text{kN} \tag{12}$$

Thus calculating $\tilde{H}_O$ by neglecting second term in the numerator and denominator induces an error which is less than 0.5%. Hence for all practical purposes one could simplify the expression for $\tilde{H}_O$ as,

$$\tilde{H}_0 = -\frac{\int_0^s \frac{M_0 y_1}{EI} ds}{\int_0^s \frac{y^2{}_1}{EI} ds} \tag{13}$$

Now we have,

$$M_C + N_C d = -1081.29$$

$$N_C = -470.25$$

$$M_C = -23.22 \text{ kN.m} \tag{14}$$

Moment at B, $M_B = M_C + 10 \times 25 \times \frac{25}{2}$

$$= -23.22 + 10 \times 25 \times \frac{25}{2}$$

$$= 3101.78 \quad \text{kN.m} \tag{15}$$

Also $H_B = N_C$.

Since the arch and the loading are symmetrical, $M_A = M_B$ and $H_A = H_B$.

Summary

In this lesson, hingeless arches are considered. The analysis of hingeless arch by the method of least work is given in the beginning. This is followed by the analysis of hingeless arch by the elastic centre method. The procedure to compute stresses developed in the hingeless arch due to temperature change is discussed. A few problems are solved illustrate the various issues involved in the analysis of hingeless arches.

Module 6

Approximate Methods for Indeterminate Structural Analysis

Lesson
35
Indeterminate Trusses and Industrial Frames

Instructional Objectives:

After reading this chapter the student will be able to

1. Make suitable approximations so that an indeterminate structure is reduced to a determinate structure.
2. Analyse indeterminate trusses by approximate methods.
3. Analyse industrial frames and portals by approximate methods.

35.1 Introduction

In module 2, force method of analysis is applied to solve indeterminate beams, trusses and frames. In modules 3 and 4, displacement based methods are discussed for the analysis of indeterminate structures. These methods satisfy both equation of compatibility and equilibrium. Hence they are commonly referred as exact methods. It is observed that prior to analysis of indeterminate structures either by stiffness method or force method; one must have information regarding their relative stiffnesses and member material properties. This information is not available prior to preliminary design of structures. Hence in such cases, one can not perform indeterminate structural analysis by exact methods. Hence, usually in such cases, based on few approximations (which are justified on the structural behaviour under the applied loads) the indeterminate structures are reduced into determinate structures. The determinate structure is then solved by equations of statics. The above procedure of reducing indeterminate structures into determinate and solving them using equations of statics is known as approximate method of analysis as the results obtained from this procedure are approximate when compared to those obtained by exact methods. Also, approximate methods are used by design engineers to detect any gross error in the exact analysis of the complex structures. Depending upon the validity of assumptions, the results of approximate methods compare favourably with exact methods of structural analysis.

In some way, all structural methods of analysis are approximate as the exact loading on the structure, geometry; the material behaviour and joint resistance at beam column joints and soil-structure interaction are never known exactly. However, this is not a good enough reason for using approximate methods of analysis for the final design. After preliminary design, it is important to analyse the indeterminate structure by exact method of analysis. Based on these results, final design must be done. In this module both indeterminate industrial frames and building frames are analysed by approximate methods for both vertical and wind loads.

35.2 Indeterminate Trusses: Parallel-chord trusses with two diagonals in each panel.

Consider an indeterminate truss, which has two diagonals in each panel as shown in Fig. 35.1. This truss is commonly used for lateral bracing of building frames and as top and bottom chords of bridge truss.

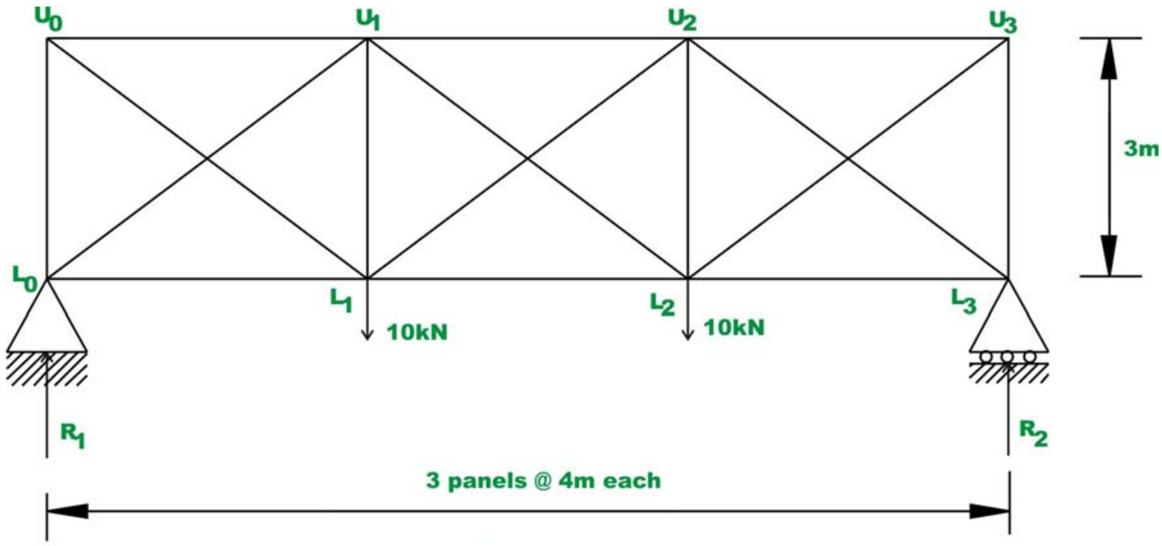


Fig. 35.1

This truss is externally determinate and internally statically indeterminate to 3^{rd} degree. As discussed in lesson 10, module 2, the degree of static indeterminacy of the indeterminate planar truss is evaluated by

$$i = (m + r) - 2j \qquad \text{(reproduced here for convenience)}$$

Where m, j and r respectively are number of members, joints and unknown reaction components. Since the given truss is indeterminate to 3^{rd} degree, it is required to make three assumptions to reduce this frame into a statically determinate truss. For the above type of trusses, two types of analysis are possible.

1. If the diagonals are going to be designed in such a way that they are equally capable of carrying either tensile or compressive forces. In such a situation, it is reasonable to assume, the shear in each panel is equally divided by two diagonals. In the context of above truss, this amounts to 3 independent assumptions (one in each panel) and hence now the structure can be solved by equations of static equilibrium alone.
2. In some cases, both the diagonals are going to be designed as long and slender. In such a case, it is reasonable to assume that panel shear is resisted by only one of its diagonals, as the compressive force

carried/resisted by the other diagonal member is very small or negligible. This may be justified as the compressive diagonal buckles at very small load. Again, this leads to three independent assumptions and the truss may be solved by equations of static alone.

Generalizing the above method, it is observed that one need to make n independent assumptions to solve n^{th} order statically indeterminate structures by equations of statics alone. The above procedure is illustrated by the following examples.

Example 35.1

Evaluate approximately forces in the truss members shown in Fig. 35.2a, assuming that the diagonals are to be designed such that they are equally capable of carrying compressive and tensile forces.

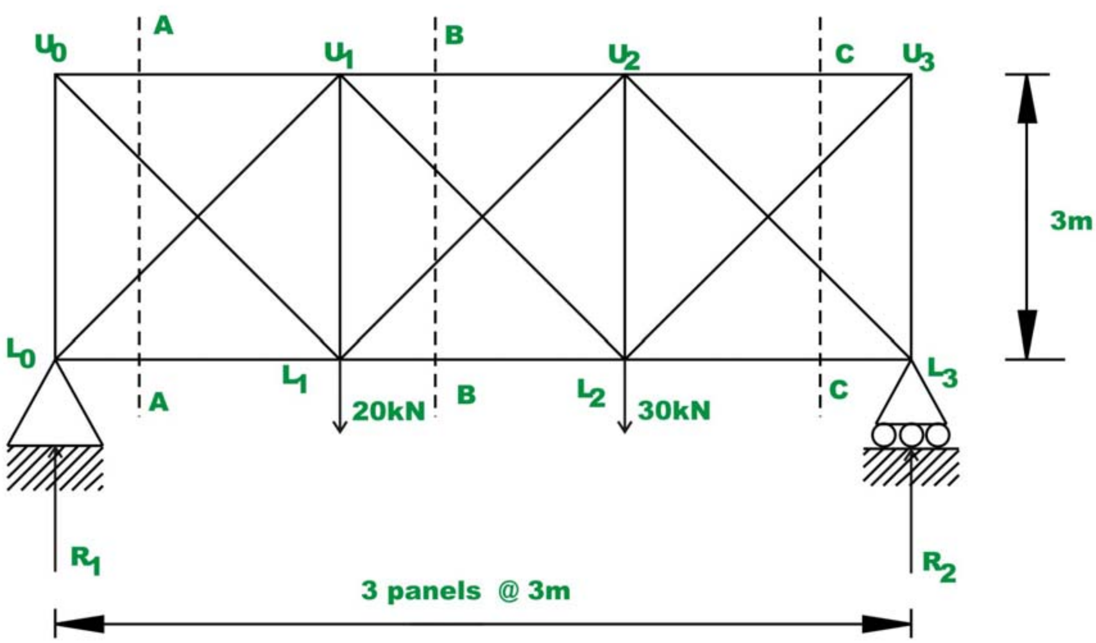


Fig. 35.2a

Solution:

The given frame is externally determinate and internally indeterminate to order 3. Hence reactions can be evaluated by equations of statics only. Thus,

$$R_1 = 23.33 \text{ kN} \quad (\uparrow)$$

$$R_2 = 26.67 \text{ kN} \quad (\uparrow) \qquad (1)$$

Now it is required to make three independent assumptions to evaluate all bar forces. Based on the given information, it is assumed that, panel shear is equally resisted by both the diagonals. Hence, compressive and tensile forces in diagonals of each panel are numerically equal. Now consider the equilibrium of free body diagram of the truss shown left of $A-A$. This is shown in Fig. 35.2b.

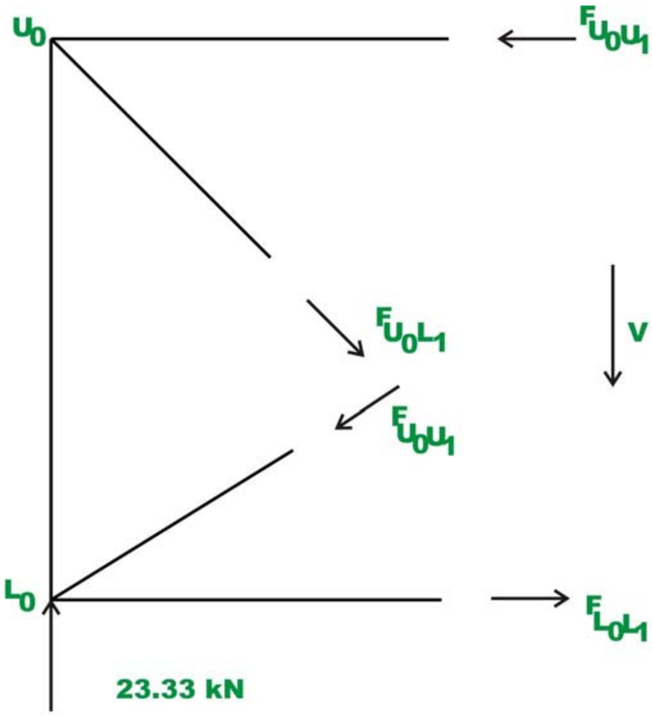


Fig.35.2b

For the first panel, the panel shear is 23.33 kN. Now in this panel, we have

$$F_{U_0L_1} = F_{L_0U_1} = F \tag{2}$$

Considering the vertical equilibrium of forces, yields

$$-F_{L_0U_1}\sin\theta - F_{L_0U_1}\sin\theta + 23.33 = 0 \tag{3}$$

$$2F\sin\theta = 23.33 \qquad\qquad \sin\theta = \frac{1}{\sqrt{2}}$$

$$F = \frac{23.33}{\sqrt{2}} \cong 16.50 \text{ kN} \tag{4}$$

Thus,

$$F_{U_0L_1} = 16.50 \quad \text{kN} \quad (Tension)$$
$$F_{L_0U_1} = 16.50 \quad \text{kN} \quad (Compression)$$

Considering the joint L_0,

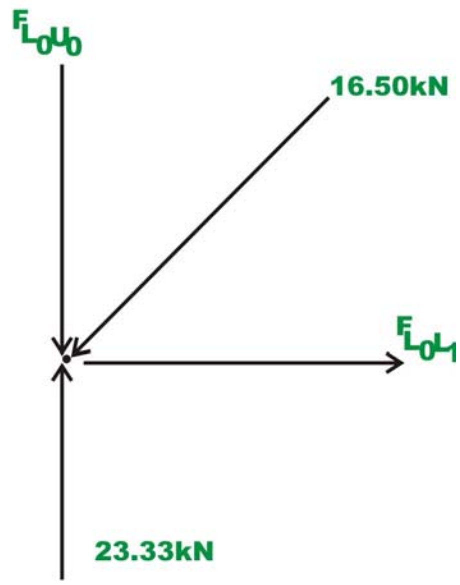


Fig.35.2c

$$\sum F_y = 0 \ \Rightarrow \qquad -F_{L_0U_0} - 16.50\sin 45 + 23.33 = 0$$

$$F_{L_0U_0} = 11.67 \quad \text{kN}(Comp.) \qquad (5)$$

$$\sum F_x = 0 \ \Rightarrow \qquad -16.50\cos 45 + F_{L_0L_1} = 0$$

$$F_{L_0L_1} = 11.67 \quad \text{kN}(Tension) \qquad (6)$$

Similarly, $F_{U_0U_1} = 11.67 \quad \text{kN}(comp.)$

Now consider equilibrium of truss left of section $C-C$ (ref. Fig. 35.2d)

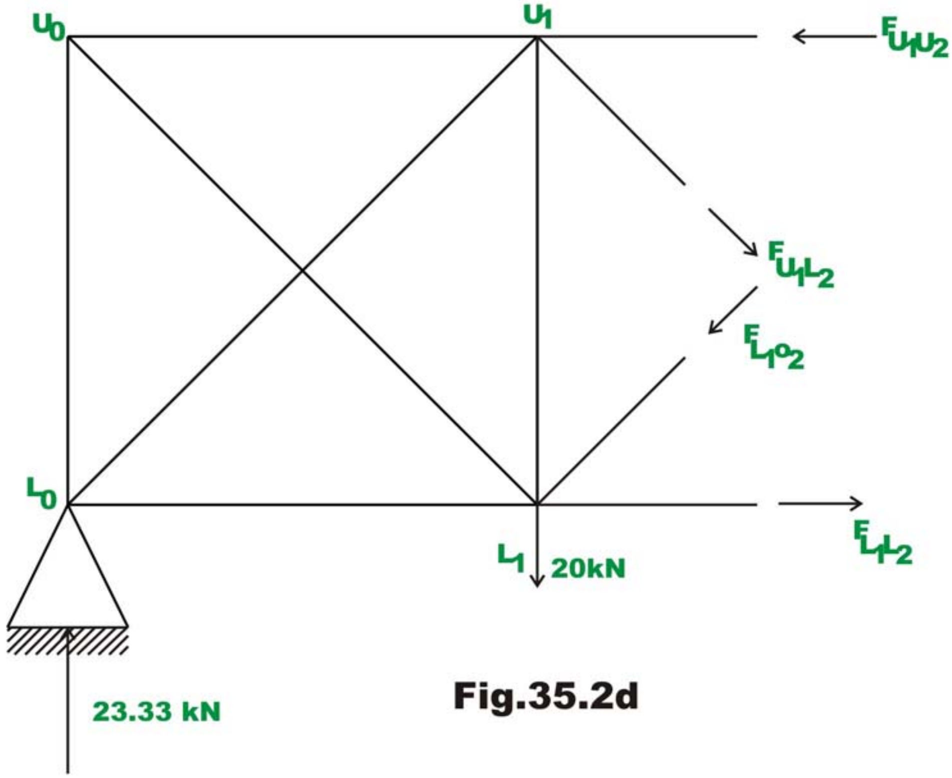


Fig.35.2d

In this panel, the shear is 3.33 kN. Considering the vertical equilibrium of the free body diagram,

$$\sum F_y = 0 \Rightarrow \quad -F_{L_1U_2} \sin 45 - F_{U_1L_2} \sin 45 + 23.33 - 20 = 0 \tag{7}$$

It is given that $F_{L_1U_2} = F_{U_1L_2} = F$

$$2F \sin\theta = 3.33$$

$$F = \frac{3.33}{\sqrt{2}} \cong 2.36 \text{ kN}$$

Thus,

$$F_{U_1L_2} = 2.36 \text{ kN} \quad (Tension)$$
$$F_{L_1U_2} = 2.36 \text{ kN} \quad (Compression)$$

Taking moment about U_1 of all the forces,

$$-F_{L_1L_2} \times 3 + 2.36\left(\frac{1}{\sqrt{2}}\right) \times 3 + 23.33 \times 3 = 0$$

$$F_{L_1L_2} = 25 \quad \text{kN}\left(Tension\right) \tag{8}$$

Taking moment about L_1 of all the forces,

$$F_{U_1U_2} = 25 \quad \text{kN}\left(Comp.\right) \tag{9}$$

Considering the joint equilibrium of L_1 (ref. Fig. 35.2e),

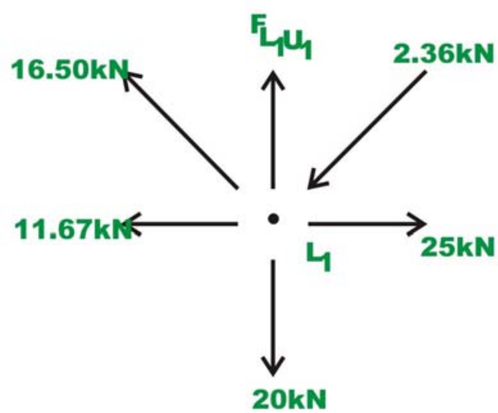


Fig.35.2e

$$\sum F_y = 0 \Rightarrow \qquad F_{L_1U_1} + 16.50 \sin 45 - 2.36 \sin 45 - 20 = 0$$

$$F_{L_1U_1} = 10 \quad \text{kN}\left(Tension\right) \tag{10}$$

Consider the equilibrium of right side of the section $B-B$ (ref. Fig. 35.2f) the forces in the 3^{rd} panel are evaluated.

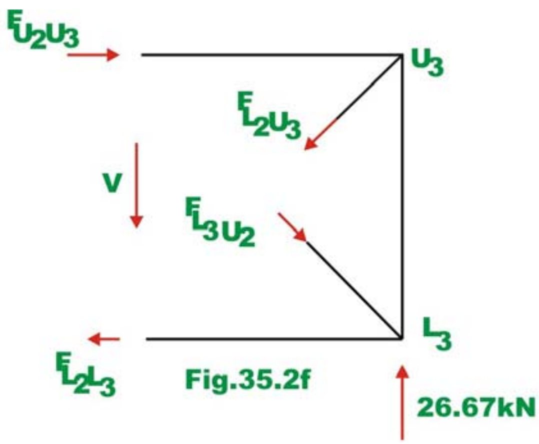


Fig.35.2f

We know that, $F_{L_3U_2} = F_{L_2U_3} = F$

$$\sum F_y = 0 \Rightarrow \qquad -F_{L_3U_2}\sin 45 + F_{L_2U_3}\sin 45 + 26.67 = 0 \tag{11}$$

$$F = \frac{26.67}{\sqrt{2}} \cong 18.86 \quad \text{kN}$$

$$F_{L_3U_2} = 18.86 \quad \text{kN} \quad (Comp.)$$

$$F_{L_2U_3} = 18.86 \quad \text{kN} \quad (Tension) \tag{12}$$

Considering the joint equilibrium of L_3 (ref. Fig. 35.2g), yields

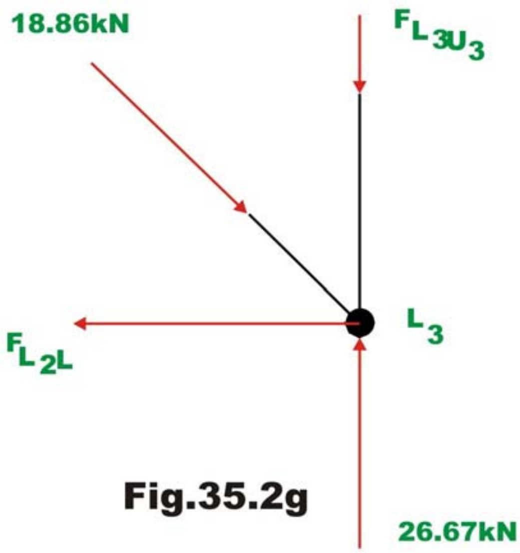


Fig.35.2g

$$\sum F_x = 0 \;\Rightarrow\; \qquad -18.86\cos 45 + F_{L_2L_3} = 0$$

$$F_{L_2L_3} = 13.34 \;\; \text{kN}\,(Tension)$$

$$\sum F_y = 0 \;\Rightarrow\; \qquad F_{L_3U_3} = 13.33 \;\; \text{kN}\,(Comp.)$$

The bar forces in all the members of the truss are shown in Fig. 35.2h. Also in the diagram, bar forces obtained by exact method are shown in brackets.

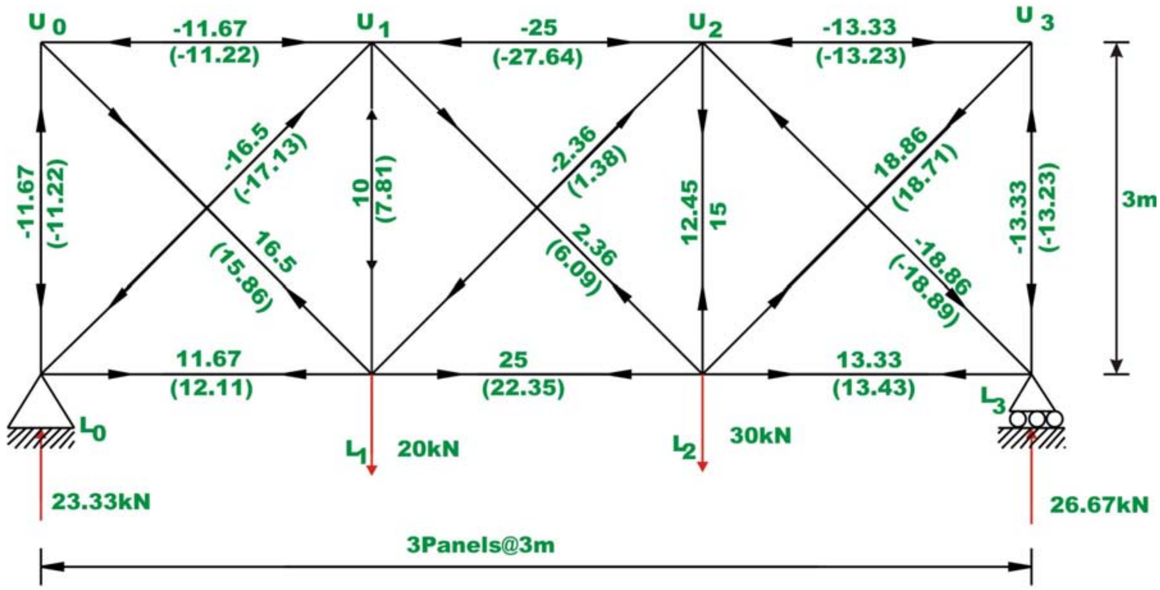


Fig.35.2h

Example 35.2

Determine bar forces in the 3-panel truss of the previous example (shown in Fig. 35.2a) assuming that the diagonals can carry only tensile forces.

Solution:

In this case, the load carried by the compressive diagonal member is zero. Hence the panel shear is completely resisted by the tension diagonal. Reactions of the truss are the same as in the previous example and is given by,

$$R_1 = 23.33 \;\; \text{kN} \qquad (\uparrow)$$
$$R_2 = 26.67 \;\; \text{kN} \qquad (\uparrow) \tag{1}$$

Consider again the equilibrium of free body diagram of the truss shown left of $A - A$. This is shown in Fig. 35.3a.

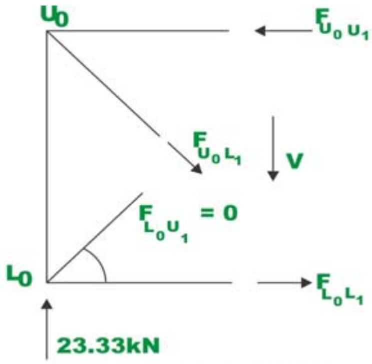


Fig.35.3a

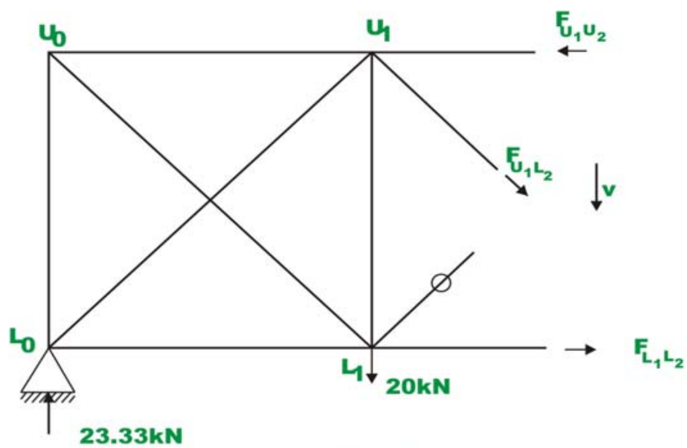


Fig.35.3b

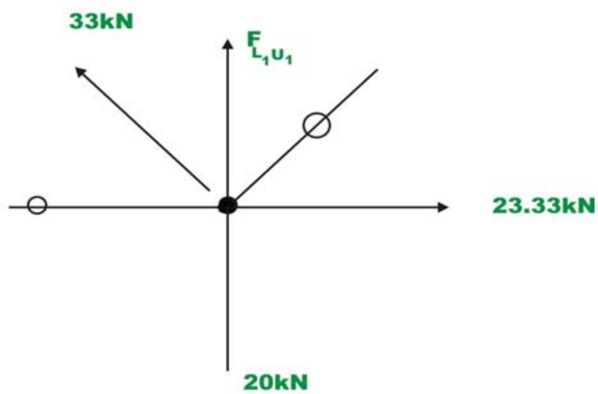


Fig.35.3c

Applying $\sum F_y = 0$,

$$-F_{U_0L_1}\sin 45+23.33=0$$

$$F_{U_0L_1}=23.33\sqrt{2}\cong 33\ \ \text{kN}$$

$$F_{L_0U_1}=0 \tag{2}$$

It is easily seen that, $F_{L_0L_1}=0$ and $F_{U_0U_1}=23.33\ \ \text{kN}$

Considering the vertical equilibrium of joint L_0, we get

$$F_{L_0U_0}=23.33\ \ \text{kN}\ (Comp.) \tag{3}$$

Since diagonals are inclined at $45°$ to the horizontal, the vertical and horizontal components of forces are equal in any panel.

Now consider equilibrium of truss left of section $C-C$ (ref. Fig. 35.3b)

In this panel, the shear is $3.33\ \ \text{kN}$. Considering the vertical equilibrium of the free body diagram,

$$\sum F_y=0 \Rightarrow \qquad -F_{U_1L_2}\sin 45+23.33-20=0 \tag{4}$$

$$F_{U_1L_2}=3.33\sqrt{2}\cong 4.71kN$$

$$F_{L_1U_2}=0 \tag{5}$$

Taking moment of all forces about U_1,

$$-F_{L_1L_2}\times 3+23.33\times 3=0$$
$$F_{L_1L_2}=23.33\ \ \text{kN}(Tension) \tag{6}$$

Taking moment about L_1 of all the forces,

$$-F_{U_1U_2}\times 3+4.71\left(\frac{1}{\sqrt{2}}\right)\times 3+23.33\times 3=0$$
$$F_{U_1U_2}=26.67\ \text{kN}(comp)$$

Considering the joint equilibrium of L_1 (ref. Fig. 35.3c), yields

$$\sum F_y = 0 \;\Rightarrow \qquad F_{L_1U_1} + 33\sin 45 - 20 = 0$$

$$F_{L_1U_1} = 3.33 \;\; \text{kN}\left(comp\right) \tag{7}$$

Considering the equilibrium of right side of the section $B-B$ (ref. Fig. 35.3d) the forces in the 3^{rd} panel are evaluated.

$$\sum F_y = 0 \Rightarrow \qquad -F_{L_2U_3}\sin 45 + 26.67 = 0 \tag{11}$$

$$F_{L_3U_2} = 0$$

$$F_{L_2U_3} = 37.71 \quad \text{kN} \quad \left(Tension\right) \tag{12}$$

Considering the joint equilibrium of L_3 (ref. Fig. 35.3e), yields

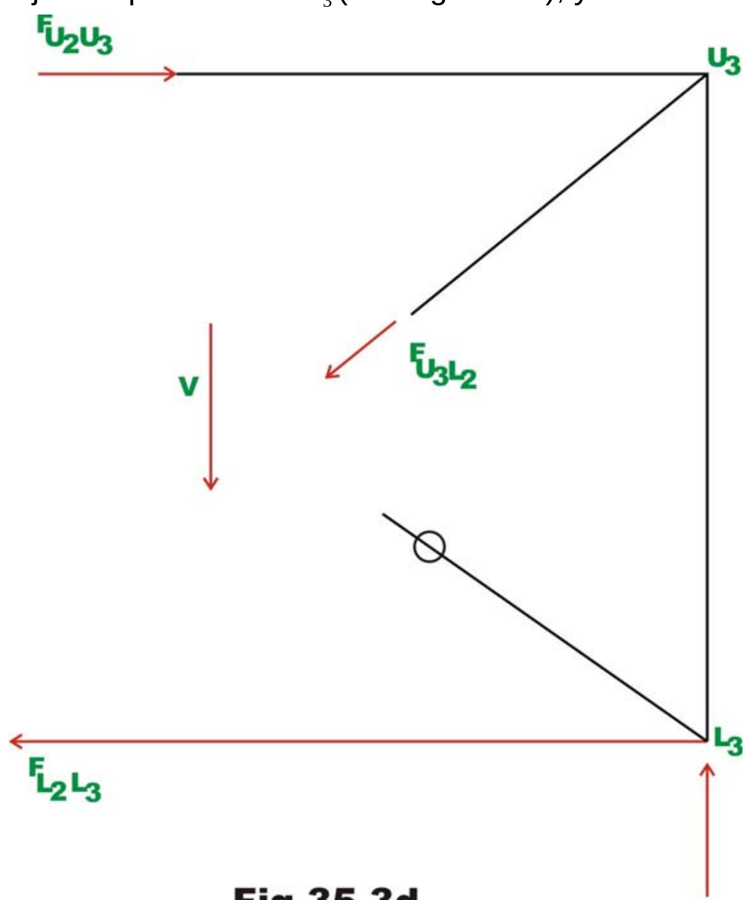


Fig.35.3d

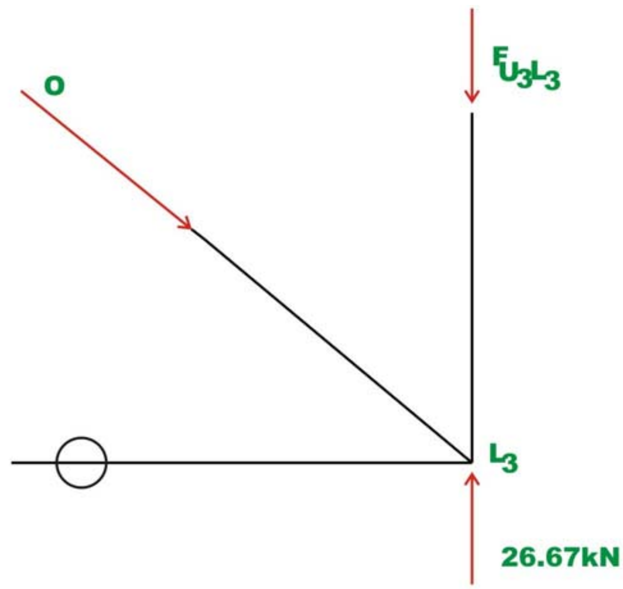


Fig. 35.3e

$$\sum F_x = 0 \Rightarrow \qquad F_{L_2L_3} = 0$$

$$\sum F_y = 0 \Rightarrow \qquad F_{L_3U_3} = 26.66 \ \text{kN}(Comp.)$$

The complete solution is shown in Fig. 35.3f. Also in the diagram, bar forces obtained by exact method are shown in brackets.

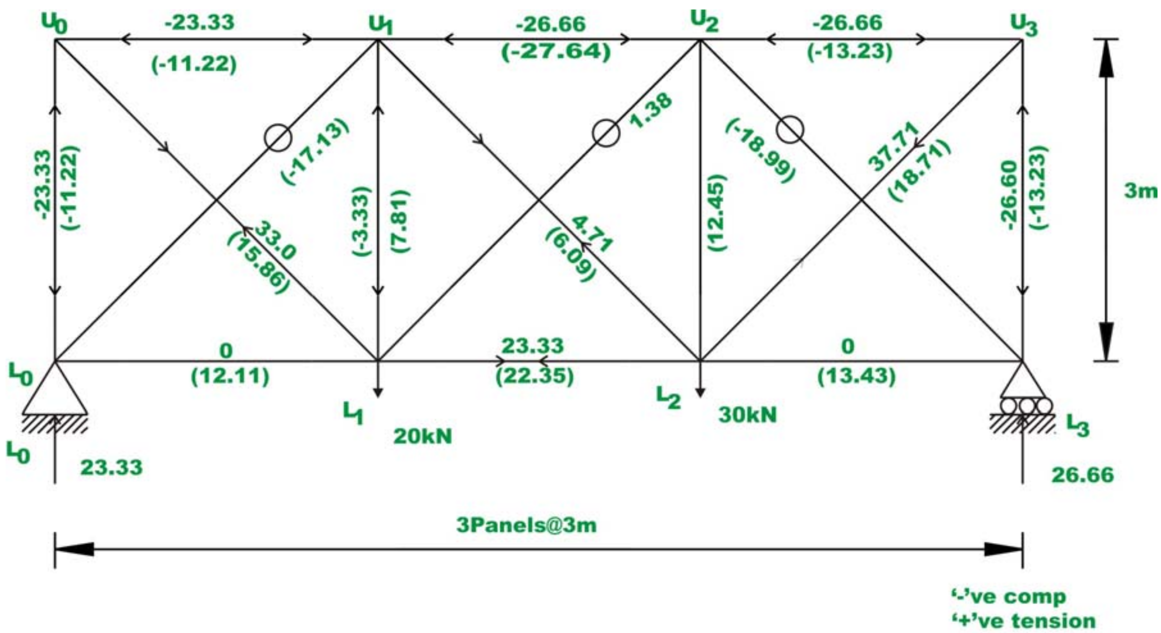
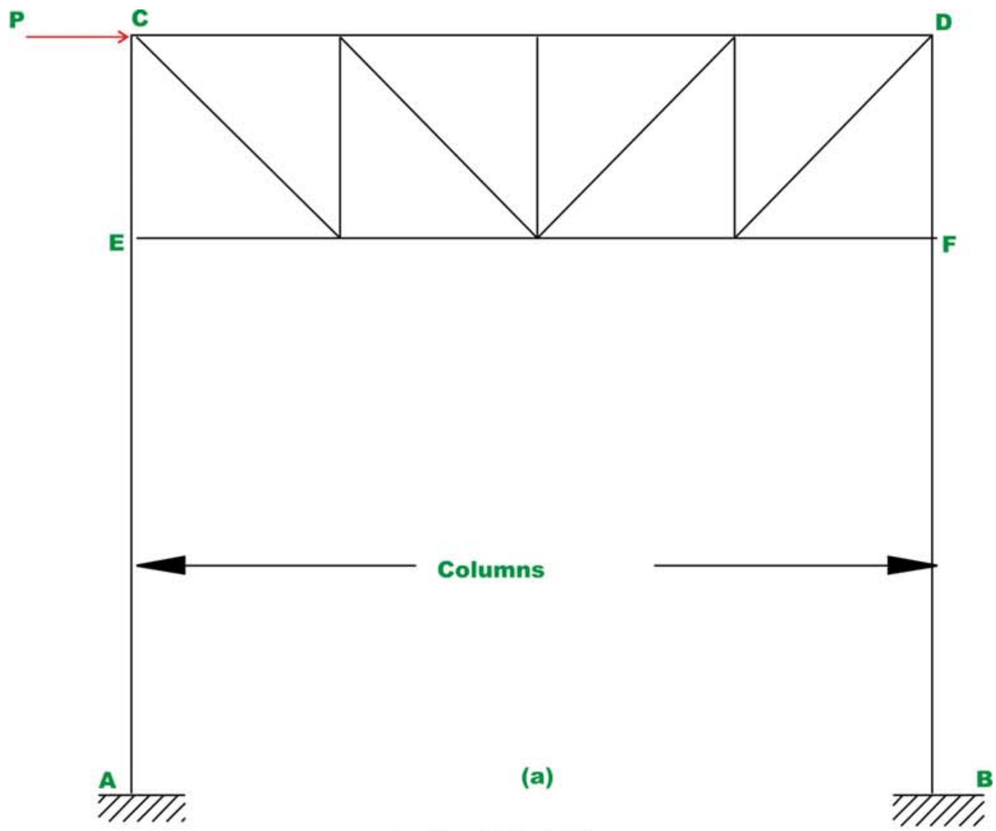


Fig.35.3f Final bar forces

35.3 Industrial frames and portals

Common types of industrial frames are shown in Fig. 35.4a and 35.4b. They consist of two columns and a truss placed over the columns. They may be subjected to vertical loads and wind loads (horizontal loads). While analyzing for the gravity loads, it is assumed that the truss is simply supported on columns. However, while analyzing the frame for horizontal loads it is assumed that, the truss is rigidly connected to columns. The base of the column are either hinged or fixed depending on the column foundation. When the concrete footing at the column base is small, then it is reasonable to assume that the columns are hinged at the base. However if the column are built into massive foundation, then the column ends are considered as fixed for the analysis purposes.



Industrial Frames

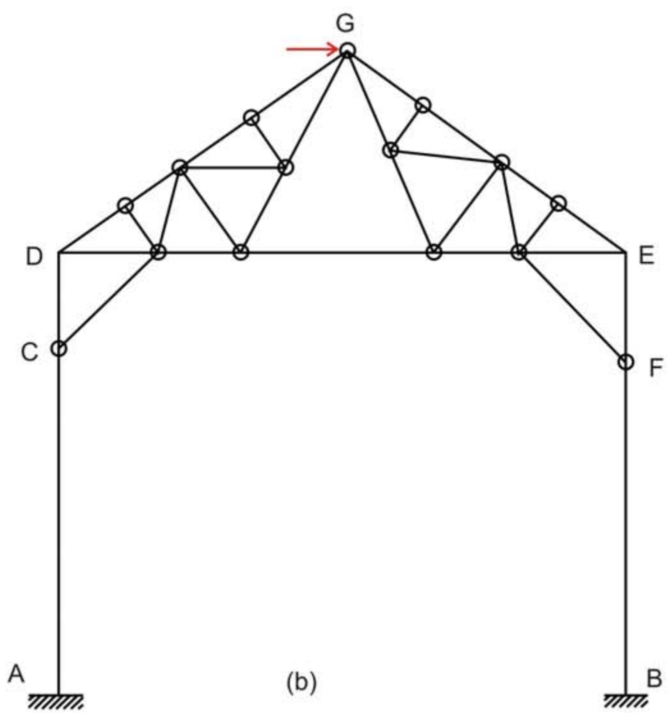


Fig. 35.4

Before considering the analysis of structures to wind load (horizontal load) consider the portals which are also used as the end portals of bridge structure (see Fig. 35.5). Their behaviour is similar to industrial trusses. The portals are also assumed to be fixed or hinged at the base depending on the type of foundation.

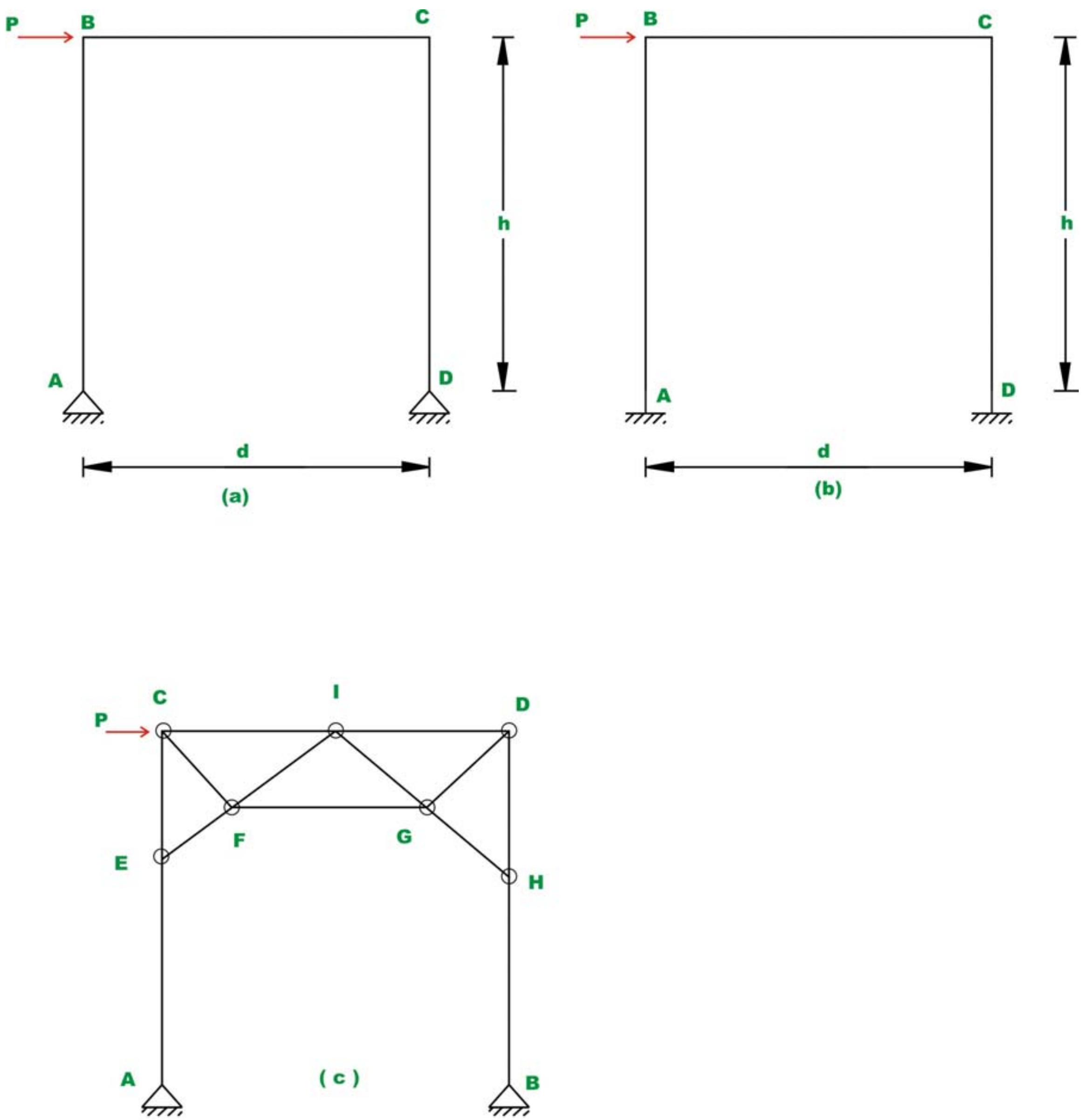


Fig.35.5 Portal Frames

Consider a portal which is hinged at the base, as shown in Fig. 35.5a. This structure is statically indeterminate to degree one. To analyse this frame when subjected to wind loads by only equations of statics, it is required to make one assumption. When stiffness of columns is nearly equal then it is assumed that the shear at the base of each column is equal. If stiffness of columns is unequal then it is assumed that the shear at the base of a column is proportional to its stiffness.

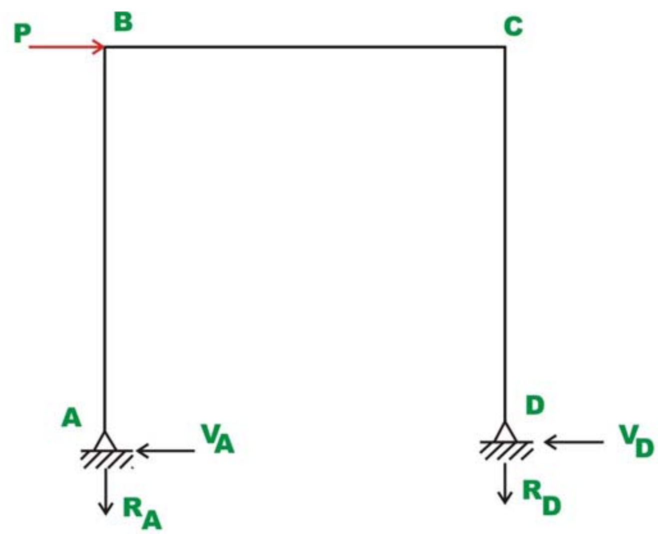


Fig.35.6

Reactions and Bending moments:
As per the assumption, shear at the base of columns is given by (vide Fig. 35.6)

Now $V_A = V_D = \frac{P}{2}$

Taking moment about hinge D,

$$\sum M_D = 0 \qquad \Rightarrow R_A \times d = P \times h$$

$$\Rightarrow R_A = \frac{Ph}{d}(\downarrow)$$

And $$\Rightarrow R_D = \frac{Ph}{d}(\uparrow)$$

The bending moment diagram is shown in Fig. 35.7.

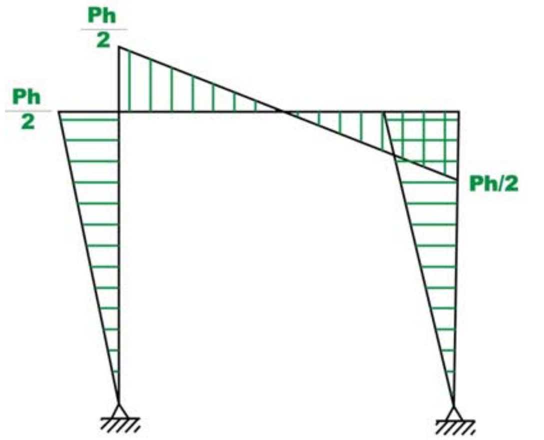


Fig 35.7a Bending Moment Diagram

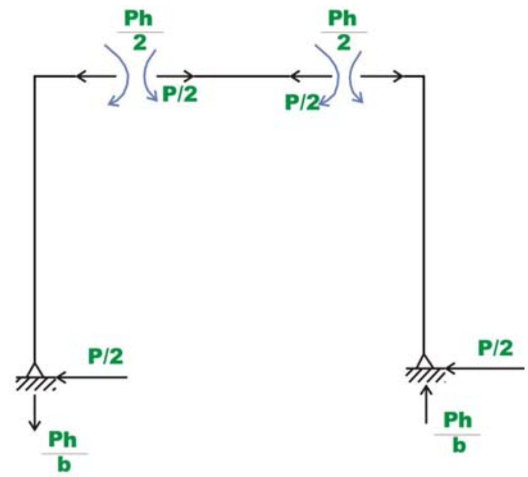


Fig.35.7(b) Reactions

It is clear from the moment diagram, an imaginary hinge forms at the mid point of the girders. Thus instead of making assumption that the shear is equal at the column base, one could say that a hinge forms at the mid point of the girder. Both the assumptions are one and the same.

Now consider a portal frame which is fixed at the base as shown in Fig. 35.5b. This is statically indeterminate to third degree and one needs to make three independent assumptions to solve this problem by equations of static equilibrium alone. Again it is assumed that the shear at the base of each column is equal provided their stiffnesses are equal. The deformed shape of the portal is shown in Fig. 35.8a and the deformed shape of the industrial frame is shown in Fig.35.8b.

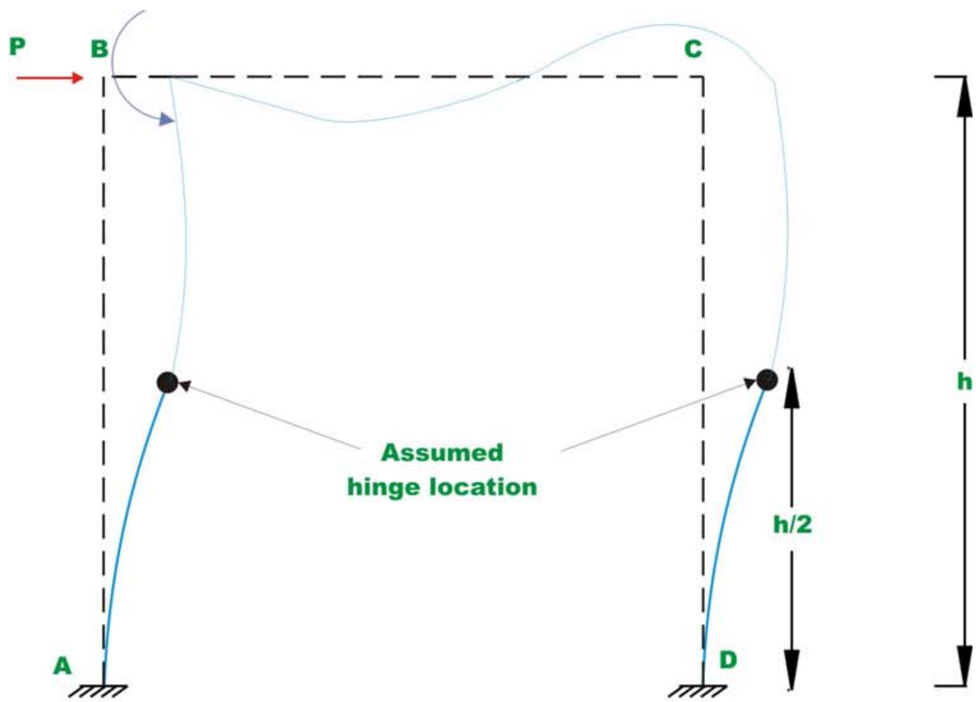


Fig.35.8a

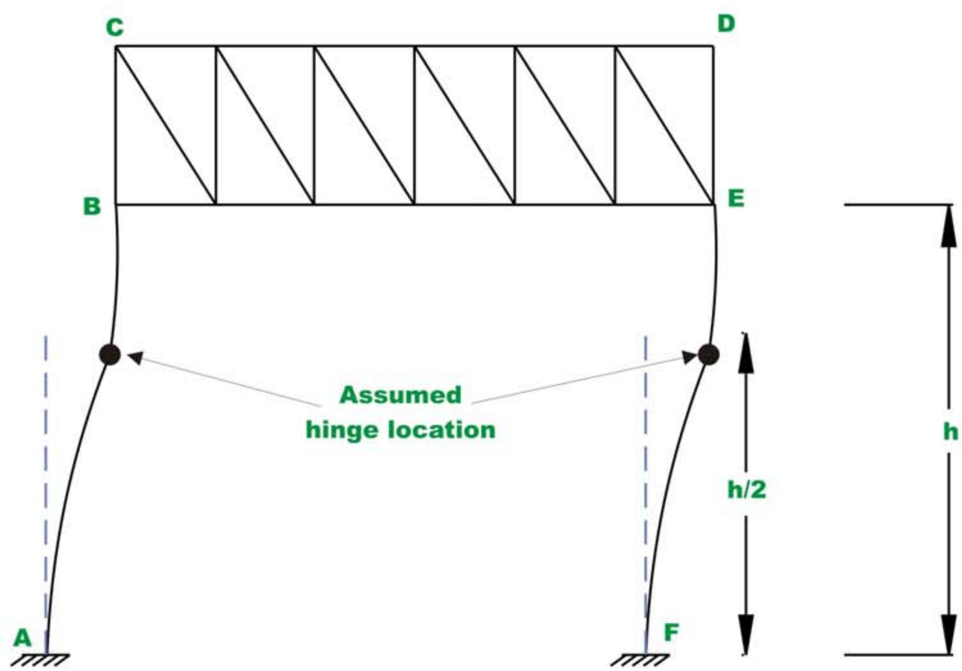


Fig.35.8b

In such a case, the bending moment at the base of the column (at A) produces tension on outside fibres of column cross section. The bending moment at top of column produces tension on inside fibres of column. Hence bending moment changes its sign between column base and top. Thus bending moment must be zero somewhere along the height of the portal. Approximately the inflexion point occurs at the mid height of columns. Now we have three independent assumptions and using them, we could evaluate reactions and moments. In the case of industrial frames, the inflexion points are assumed to occur at mid height between A and B.

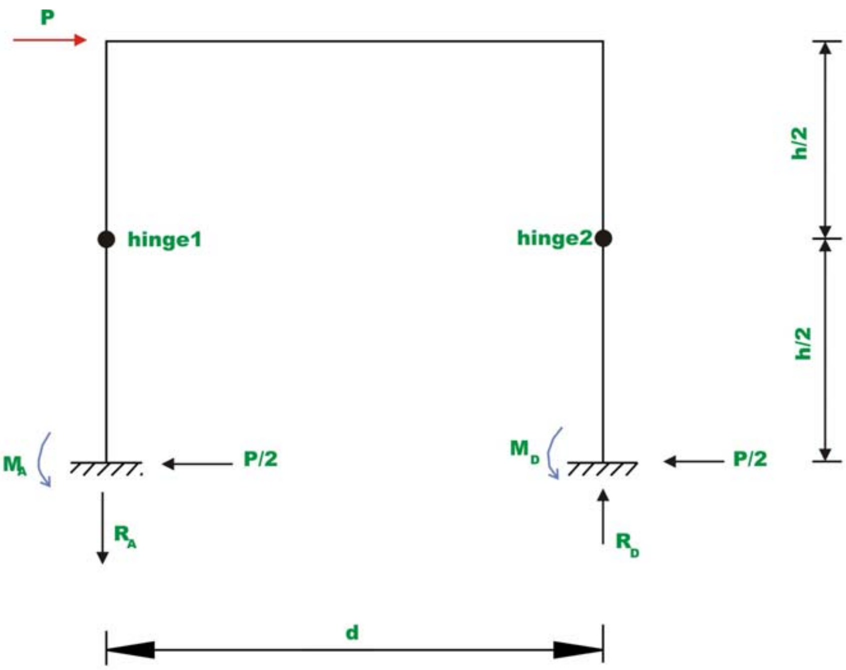


Fig.35.9a

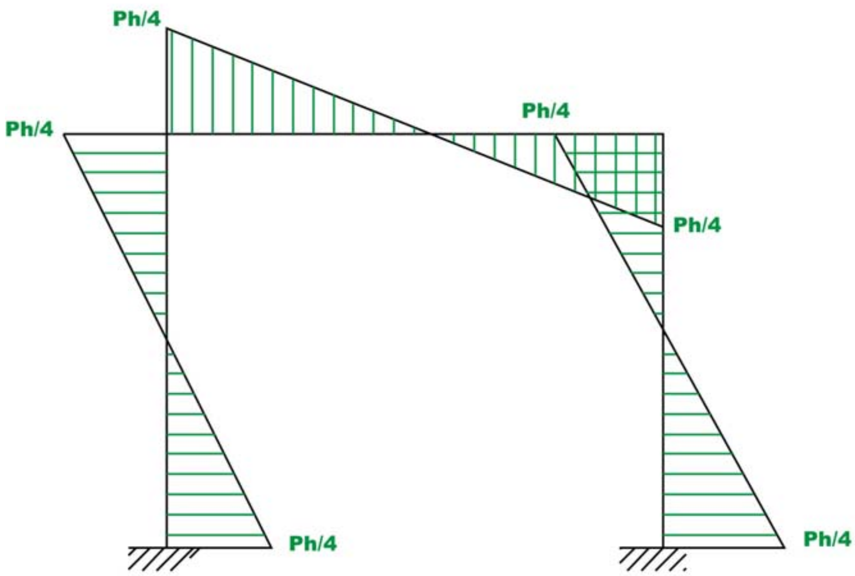


Figure 35.9b

Taking moment of all forces left of hinge 1 about hinge 1 (vide Fig. 35.9a),yields

$$\frac{Ph}{2\times 2} - M_A = 0 \qquad \Rightarrow \qquad M_A = \frac{Ph}{4}$$

Similarly taking moment of all forces left of hinge 2 about hinge 2,

$$\frac{Ph}{2\times 2} - M_D = 0 \qquad \Rightarrow \qquad M_D = \frac{Ph}{4}$$

Taking moment of all forces right of hinge 1 about hinge 1 gives,

$$R_D d + M_D - \frac{P}{2}\frac{h}{2} - \frac{Ph}{2} = 0 \qquad \Rightarrow \qquad R_D = \frac{Ph}{2d}(\uparrow)$$

Similarly

$$R_A = \frac{Ph}{2d}(\downarrow)$$

The bending moment diagram is shown in Fig. 35.9b.

If the base of the column is partially fixed then hinge is assumed at a height of $1/3^{rd}$ from the base. Note that when it is hinged at the base of the column, the inflexion point occurs at the support and when it is fixed, the inflexion point occurs at mid-height.

Example 35.3

Determine approximately forces in the member of a truss portal shown in Fig. 35.10a.

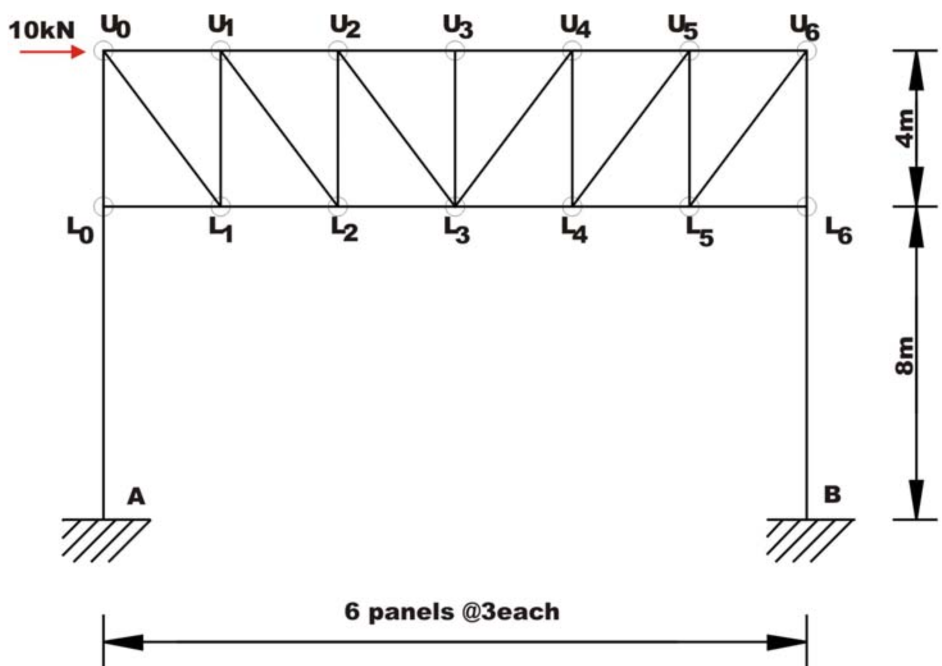


Fig. 35.10a

In this case, as per the first assumption, the shear at the base of each column is the same and is given by (ref. 35.10b)

$$V_A = V_D = \frac{10}{2} = 5\,kN \tag{1}$$

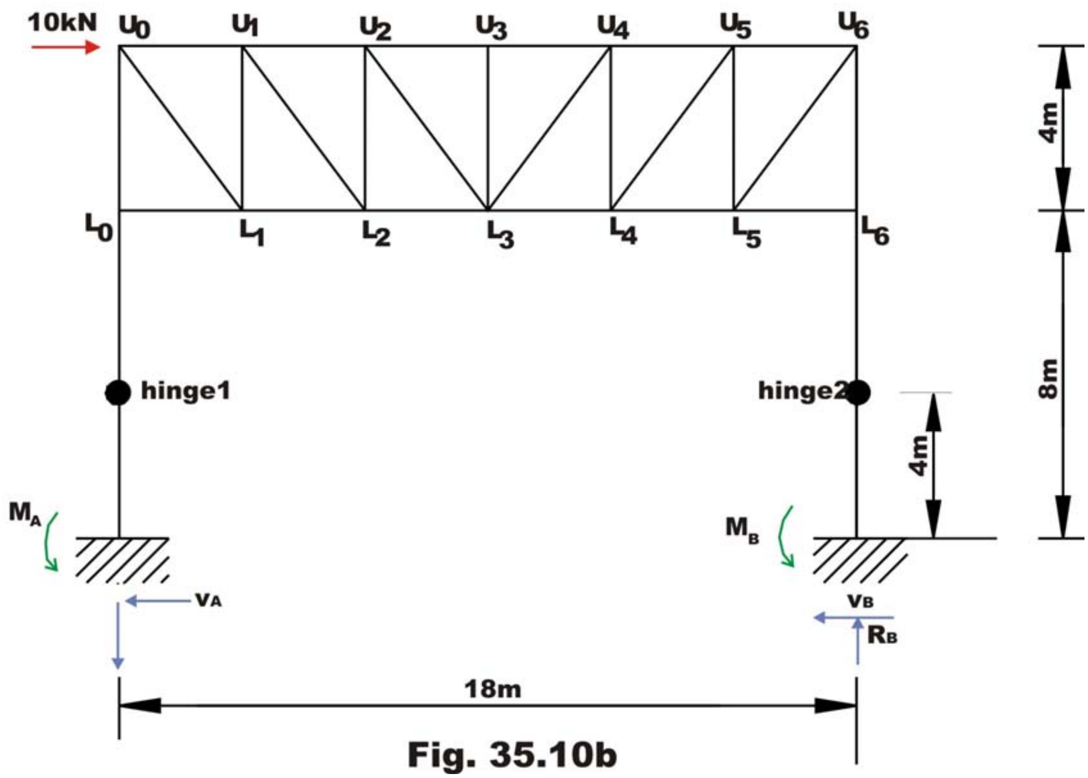


Fig. 35.10b

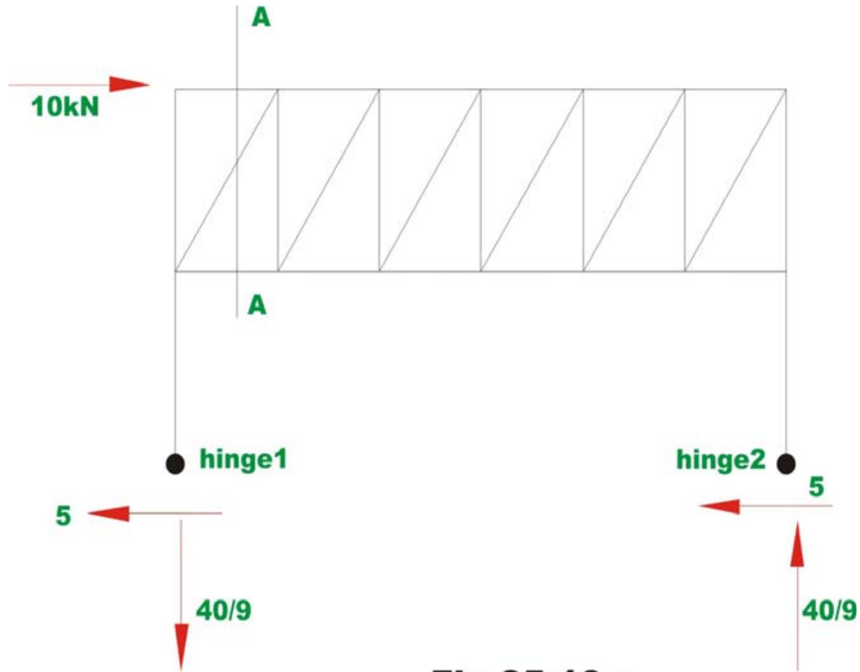


Fig 35.10.c

Taking moment of all forces right of hinge 2 about hinge 2, results

$$M_B = \frac{P}{2} \times 4 \quad \Rightarrow \quad M_B = 20 \text{ kN.m} \tag{2}$$

Similarly $M_A = 20 \text{ kN.m}$ (3)

Taking moment of all forces right of hinge 1 about hinge 1 gives,

$$R_B \times 18 - V_B \times 4 + 20 - 10(4+4) = 0 \quad \Rightarrow \quad R_B = \frac{80}{18} = \frac{40}{9}\, kN(\uparrow)$$

Similarly,

$$R_A = \frac{40}{9} kN(\downarrow) \tag{4}$$

Forces in the truss member can be calculated either by method of sections or by method of joints. For example, consider the equilibrium of truss left of $A - A$ as shown in Fig. 35.10d.

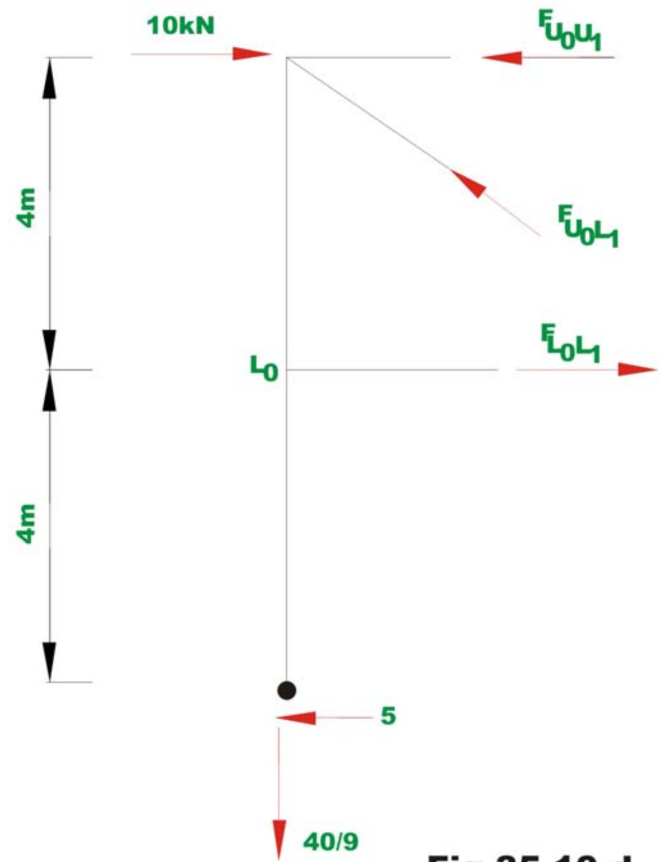


Fig 35.10.d

$$\sum F_y = 0 \Rightarrow \qquad -\frac{40}{9} + F_{U_0L_1} \times \frac{4}{5} = 0$$

$$\Rightarrow F_{U_0L_1} = 5.55\,\text{kN}(Comp.) \qquad (5)$$

Taking moment about U_0,

$$5 \times 8 - F_{L_0L_1} \times 4 = 0$$

$$F_{L_0L_1} = 10\,\text{kN}(Tension) \qquad (6)$$

Taking moment about L_1,

$$10 \times 4 + 5 \times 4 - \frac{40}{9} \times 3 - F_{U_0U_1} \times 4 = 0$$

$$F_{U_0U_1} = 11.66\,\text{kN}(Comp) \qquad (7)$$

Summary

It is observed that prior to analysis of indeterminate structures either by stiffness method or force method; one must have information regarding their relative stiffnesses and member material properties. This information is not available prior to preliminary design of structures. Hence in such cases, one can not perform indeterminate structural analysis by exact methods. Hence, usually in such cases, based on few approximations (which are justified on the structural behaviour under the applied loads) the indeterminate structures are reduced into determinate structures. The determinate structure is then solved by equations of statics. This methodology has been adopted in this lesson to solve indeterminate trusses and industrial frames. Depending upon the validity of assumptions, the results of approximate methods compare favourably with exact methods of structural analysis as seen from the numerical examples.

Module
6
Approximate Methods for Indeterminate Structural Analysis

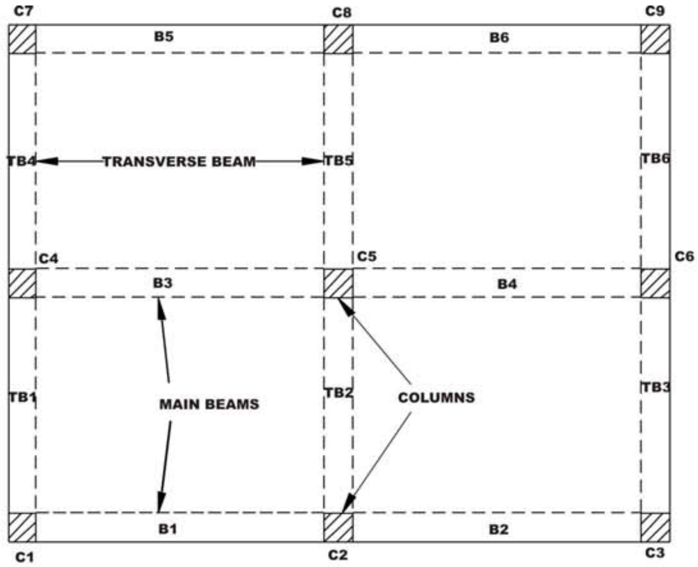
Lesson
36
Building Frames

Instructional Objectives:

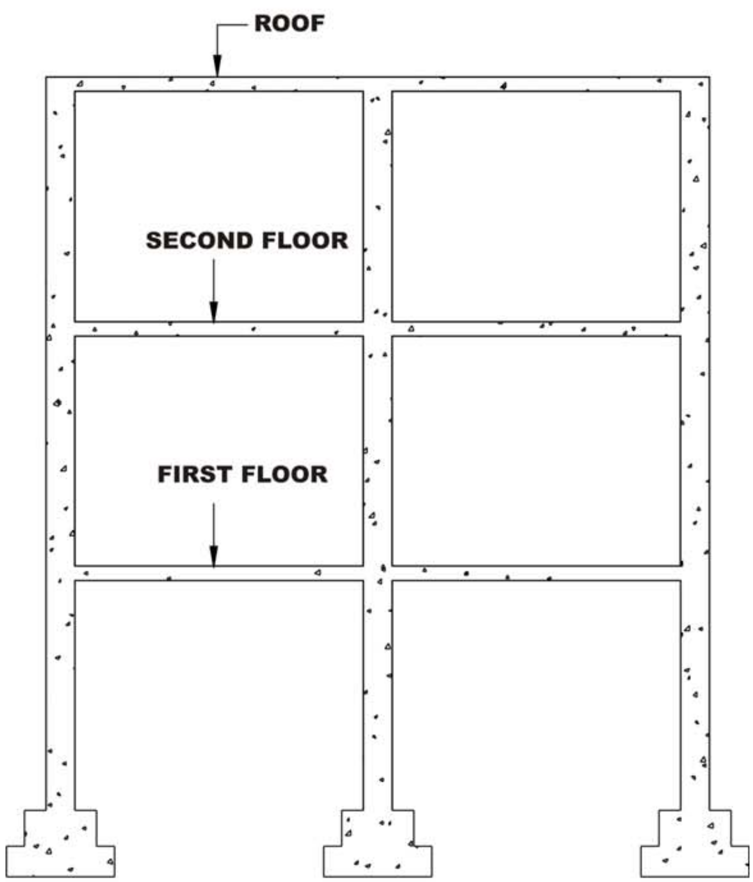
After reading this chapter the student will be able to
1. Analyse building frames by approximate methods for vertical loads.
2. Analyse building frames by the cantilever method for horizontal loads.
3. Analyse building frame by the portal method for horizontal loads.

36.1 Introduction

The building frames are the most common structural form, an analyst/engineer encounters in practice. Usually the building frames are designed such that the beam column joints are rigid. A typical example of building frame is the reinforced concrete multistory frames. A two-bay, three-storey building plan and sectional elevation are shown in Fig. 36.1. In principle this is a three dimensional frame. However, analysis may be carried out by considering planar frame in two perpendicular directions separately for both vertical and horizontal loads as shown in Fig. 36.2 and finally superimposing moments appropriately. In the case of building frames, the beam column joints are monolithic and can resist bending moment, shear force and axial force. The frame has 12 joints(j), 15 beam members(b), and 9 reaction components(r). Thus this frame is statically indeterminate to degree$=((3\times15+9)-12\times3)=18$ (Please see lesson 1, module 1 for more details). Any exact method, such as slope-deflection method, moment distribution method or direct stiffness method may be used to analyse this rigid frame. However, in order to estimate the preliminary size of different members, approximate methods are used to obtain approximate design values of moments, shear and axial forces in various members. Before applying approximate methods, it is necessary to reduce the given indeterminate structure to a determinate structure by suitable assumptions. These will be discussed in this lesson. In lesson 36.2, analysis of building frames to vertical loads is discussed and in section 36.3, analysis of building frame to horizontal loads will be discussed.



Plan



Sectional Elevation Along C_1 - C_3

Fig. 36.1 Building frame

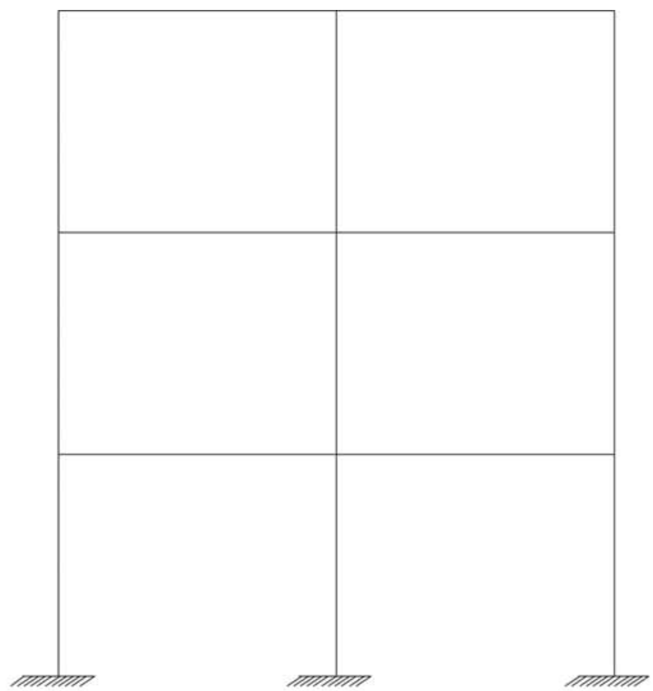


Fig.36.2 Idealized frame for analysis

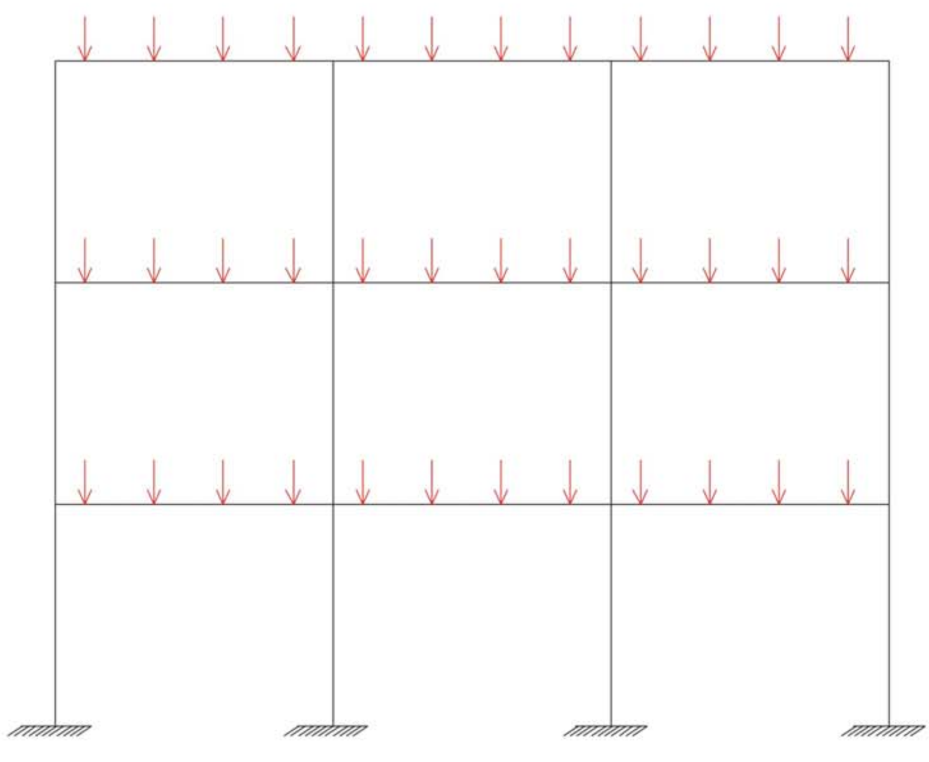


Fig.36.3 Building frame subjected to vertical loads

36. 2 Analysis of Building Frames to Vertical Loads

Consider a building frame subjected to vertical loads as shown in Fig.36.3. Any typical beam, in this building frame is subjected to axial force, bending moment and shear force. Hence each beam is statically indeterminate to third degree and hence 3 assumptions are required to reduce this beam to determinate beam.

Before we discuss the required three assumptions consider a simply supported beam. In this case zero moment (or point of inflexion) occurs at the supports as shown in Fig.36.4a. Next consider a fixed-fixed beam, subjected to vertical loads as shown in Fig. 36.4b. In this case, the point of inflexion or point of zero moment occurs at $0.21L$ from both ends of the support.

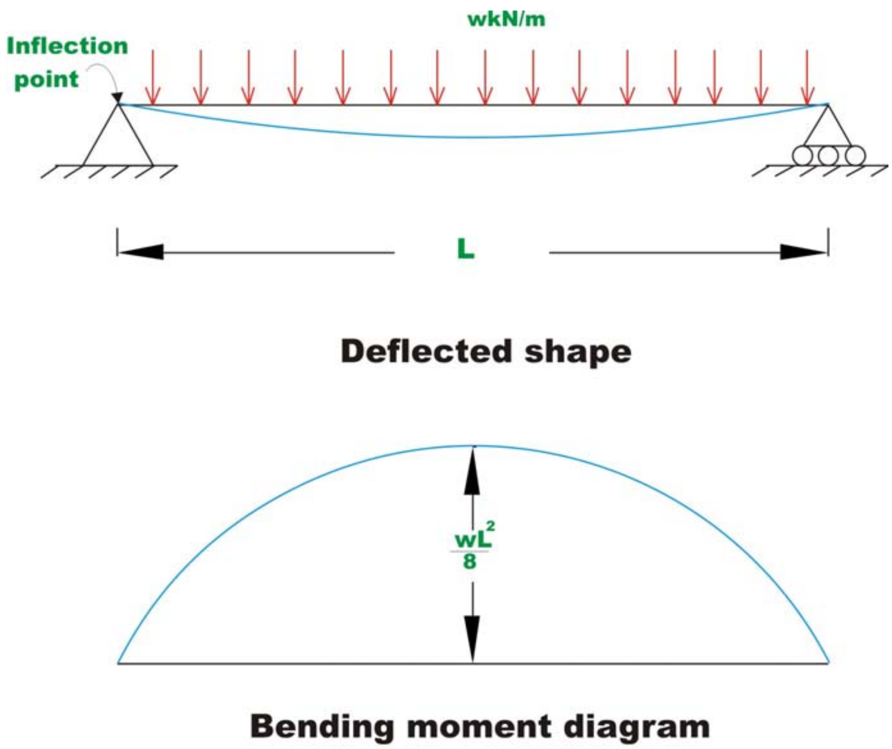
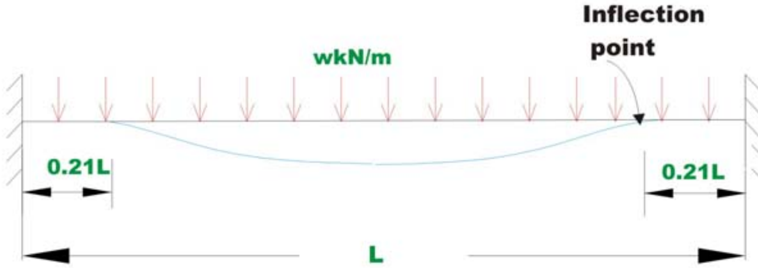
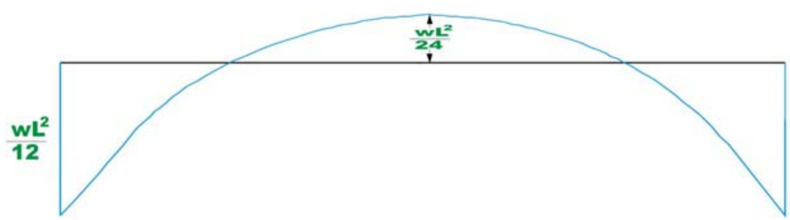


Fig.36. 4a Simply Supported beam



Deflected shape



Bending moment diagram

Fig.36. 4b Fixed - Fixed beam

Now consider a typical beam of a building frame as shown in Fig.36.4c. In this case, the support provided by the columns is neither fixed nor simply supported. For the purpose of approximate analysis the inflexion point or point of zero moment is assumed to occur at $\left(\frac{0+0.21L}{2}\right) \approx 0.1L$ from the supports. In reality the point of zero moment varies depending on the actual rigidity provided by the columns. Thus the beam is approximated for the analysis as shown in Fig.36.4d.

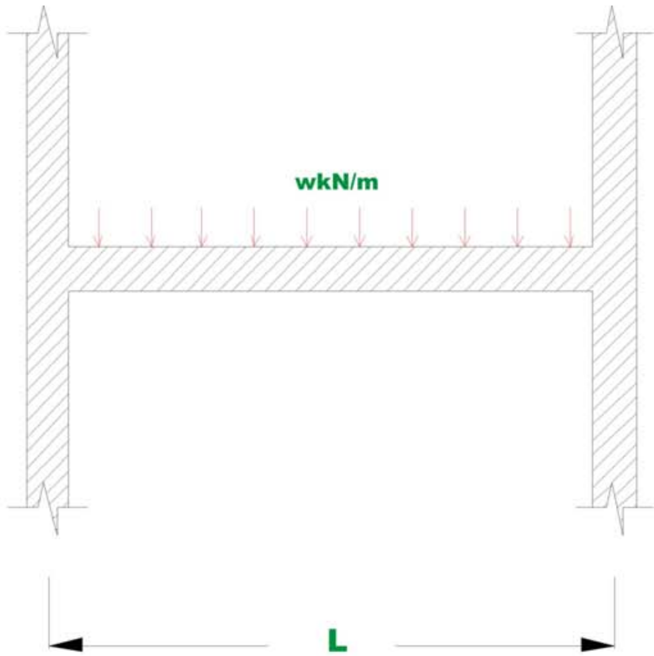


Fig.36.4c

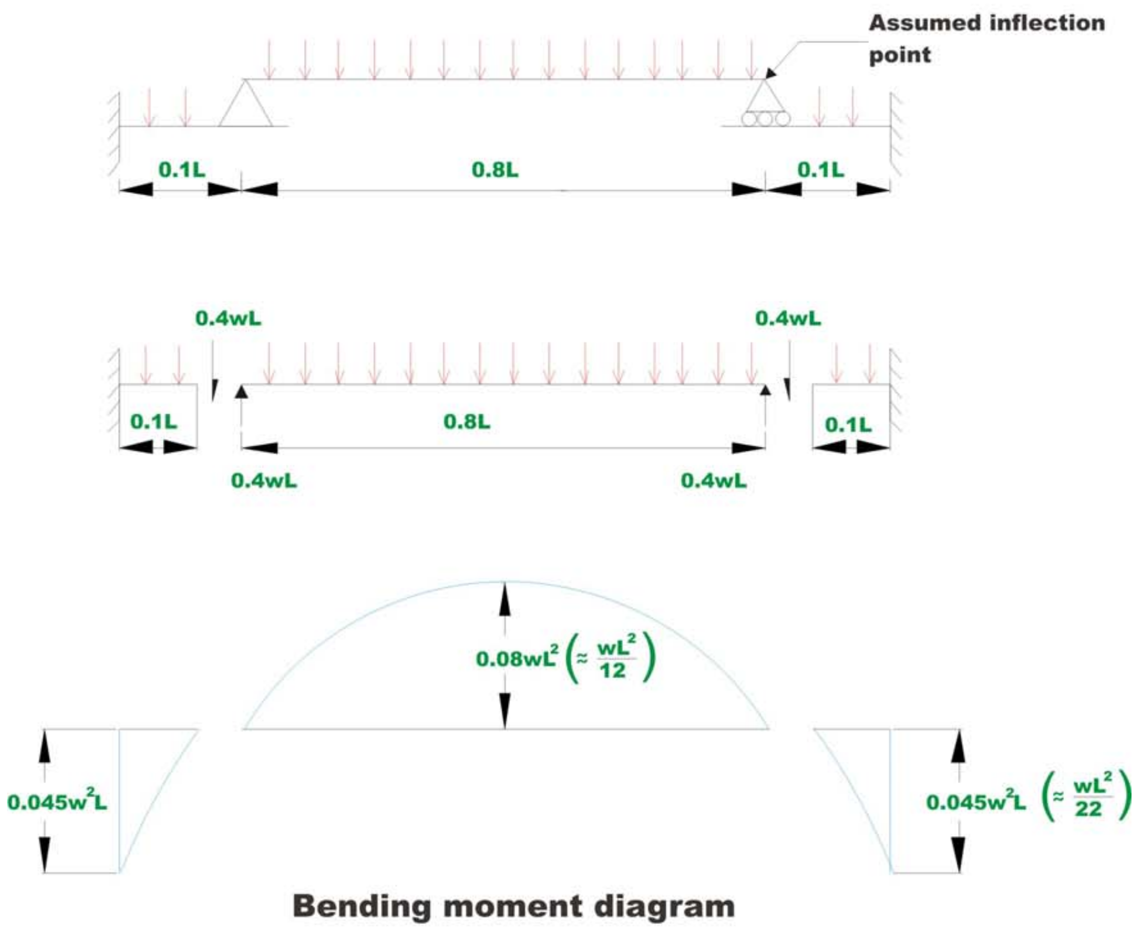


Fig.36.4d

For interior beams, the point of inflexion will be slightly more than $0.1L$. An experienced engineer will use his past experience to place the points of inflexion appropriately. Now redundancy has reduced by two for each beam. The third assumption is that axial force in the beams is zero. With these three assumptions one could analyse this frame for vertical loads.

Example 36.1

Analyse the building frame shown in Fig. 36.5a for vertical loads using approximate methods.

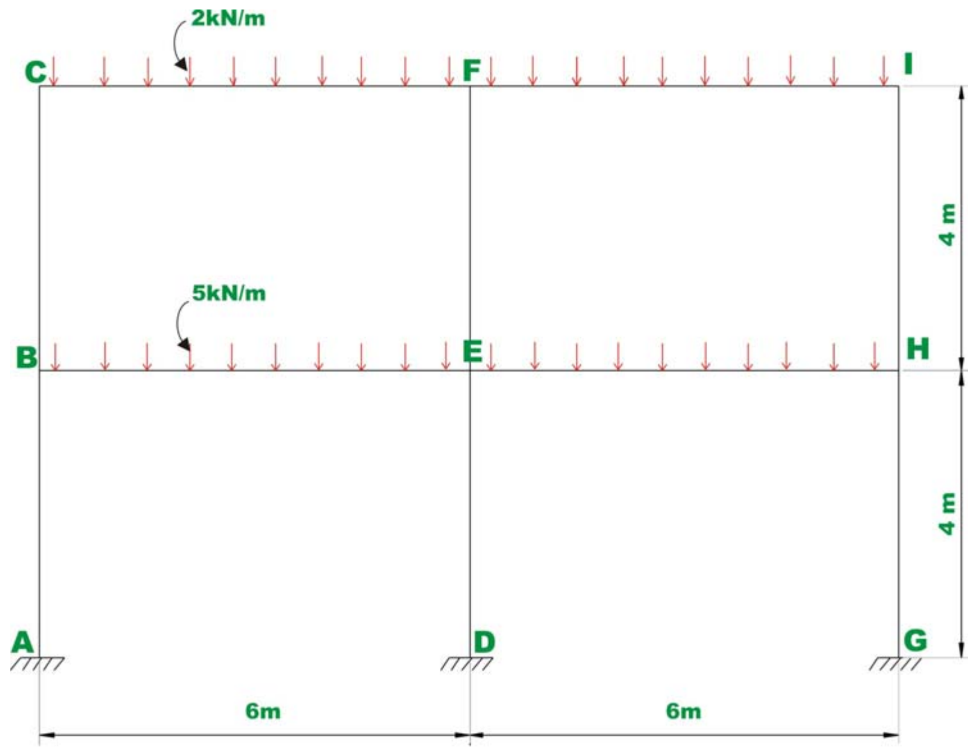


Fig.36.5a

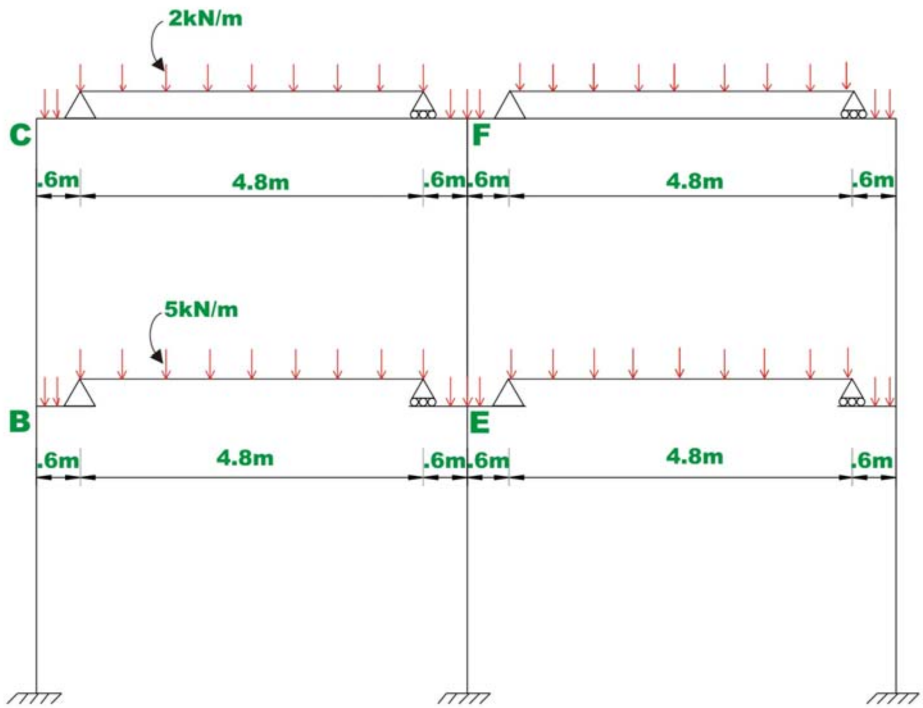
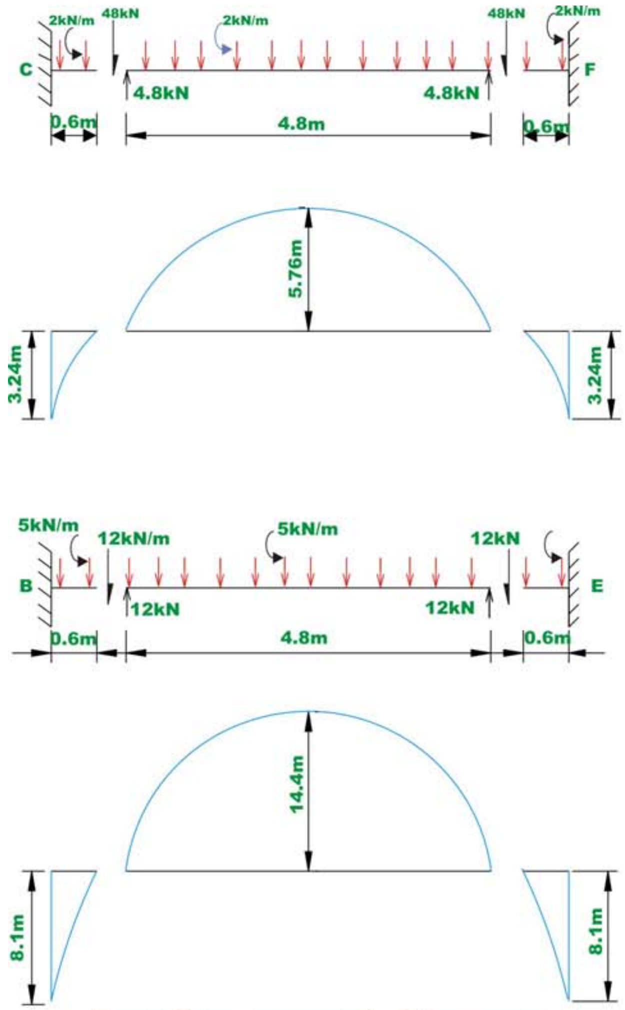


Fig.36.5 b

Solution:

In this case the inflexion points are assumed to occur in the beam at $0.1L(= 0.6m)$ from columns as shown in Fig. 36.5b. The calculation of beam moments is shown in Fig. 36.5c.



Bending moment diagrams

Fig.36.5c

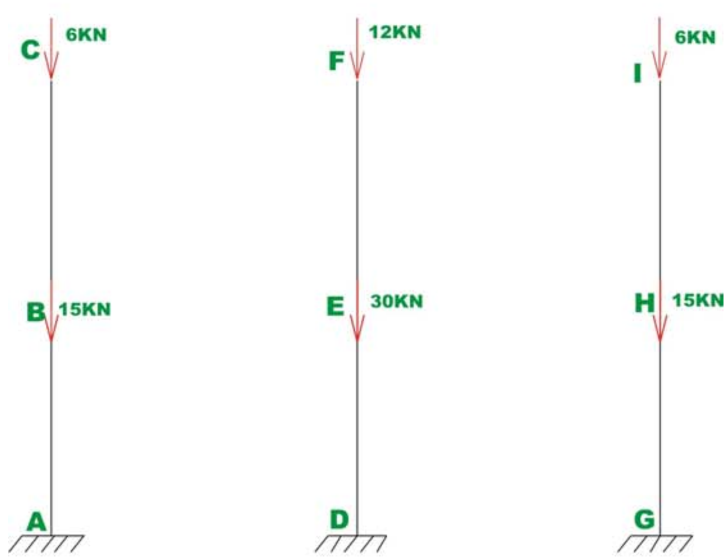


Fig.36.5d Axial force in columns

Now the beam $-ve$ moment is divided equally between lower column and upper column. It is observed that the middle column is not subjected to any moment, as the moment from the right and the moment from the left column balance each other. The $-ve$ moment in the beam BE is $8.1 \mathrm{kN.m}$. Hence this moment is divided between column BC and BA. Hence, $M_{BC} = M_{BA} = \frac{8.1}{2} = 4.05 \mathrm{kN.m}$. The maximum $+ve$ moment in beam BE is $14.4 \mathrm{kN.m}$. The columns do carry axial loads. The axial compressive loads in the columns can be easily computed. This is shown in Fig. 36.5d.

36.3 Analysis of Building Frames to lateral (horizontal) Loads

A building frame may be subjected to wind and earthquake loads during its life time. Thus, the building frames must be designed to withstand lateral loads. A two-storey two-bay multistory frame subjected to lateral loads is shown in Fig. 36.6. The actual deflected shape (as obtained by exact methods) of the frame is also shown in the figure by dotted lines. The given frame is statically indeterminate to degree 12.

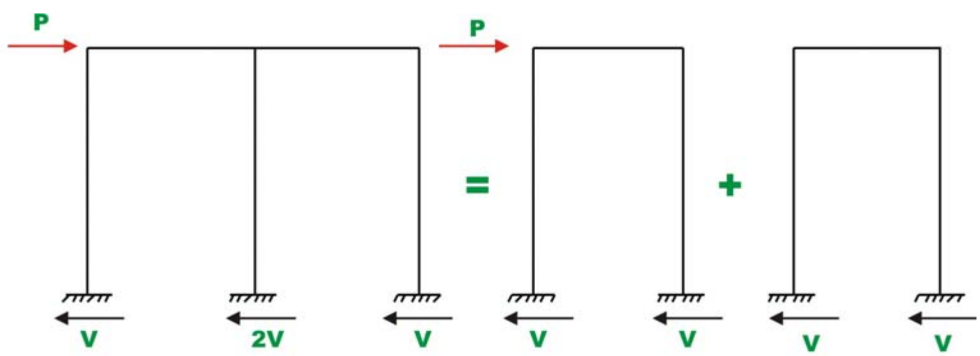


Fig.36.6 Shear in columns

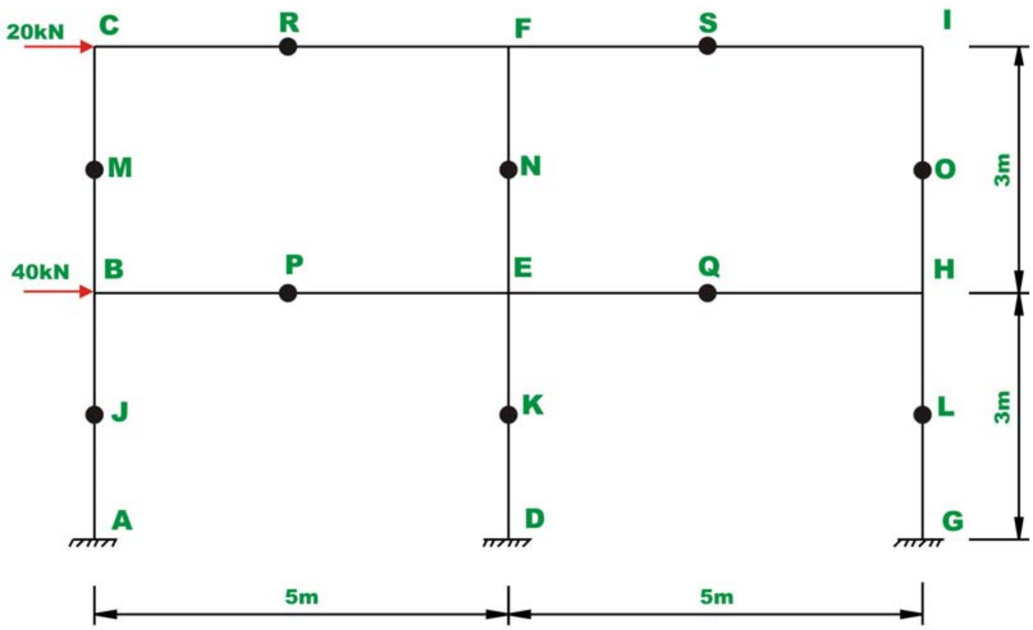


Fig.36.7a Two storey building frame subjected to lateral load of Example 36.2

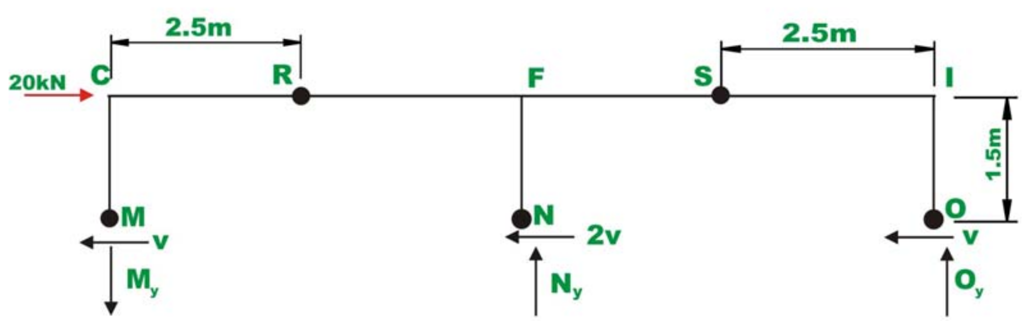


Fig.36.7b

Hence it is required to make 12 assumptions to reduce the frame in to a statically determinate structure. From the deformed shape of the frame, it is observed that inflexion point (point of zero moment) occur at mid height of each column and mid point of each beam. This leads to 10 assumptions. Depending upon how the remaining two assumptions are made, we have two different methods of analysis: $i)$ Portal method and $ii)$ cantilever method. They will be discussed in the subsequent sections.

36.3.1 Portal method

In this method following assumptions are made.

1) An inflexion point occurs at the mid height of each column.
2) An inflexion point occurs at the mid point of each girder.

3) The total horizontal shear at each storey is divided between the columns of that storey such that the interior column carries twice the shear of exterior column.

The last assumption is clear, if we assume that each bay is made up of a portal thus the interior column is composed of two columns (Fig. 36.6). Thus the interior column carries twice the shear of exterior column. This method is illustrated in example 36.2.

Example 36.2

Analyse the frame shown in Fig. 36.7a and evaluate approximately the column end moments, beam end moments and reactions.

Solution:

The problem is solved by equations of statics with the help of assumptions made in the portal method. In this method we have hinges/inflexion points at mid height of columns and beams. Taking the section through column hinges $M.N,O$ we get, (ref. Fig. 36.7b).

$$\sum F_X = 0 \qquad \Rightarrow \qquad V + 2V + V = 20$$

or $V = 5\,\text{kN}$

Taking moment of all forces left of hinge R about R gives,

$$V \times 1.5 - M_y \times 2.5 = 0$$

$$M_y = 3\ \text{kN}(\downarrow)$$

Column and beam moments are calculates as,

$$M_{CB} = 5 \times 1.5 = 7.5\,\text{kN.m} \ ; \ M_{IH} = +7.5\,\text{kN.m}$$

$$M_{CF} = -7.5\,\text{kN.m}$$

Taking moment of all forces left of hinge S about S gives,

$$5 \times 1.5 - O_y \times 2.5 = 0$$

$$O_y = 3\,\text{kN}(\uparrow)$$

$$N_y = 0$$

Taking a section through column hinges J, K, L we get, (ref. Fig. 36.7c).

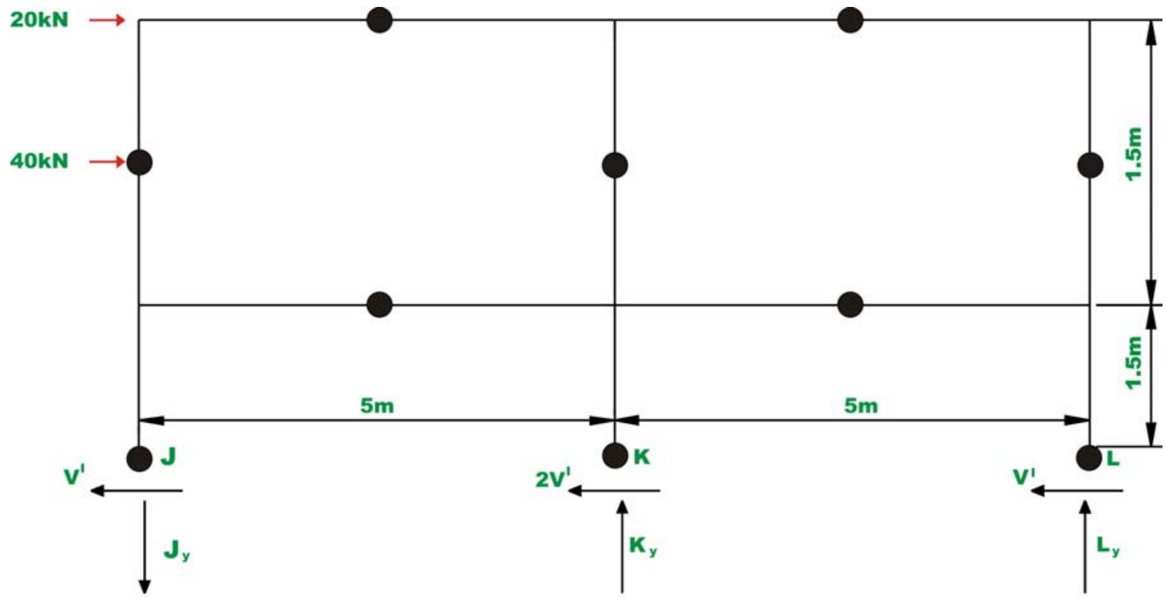


Fig.36.7c

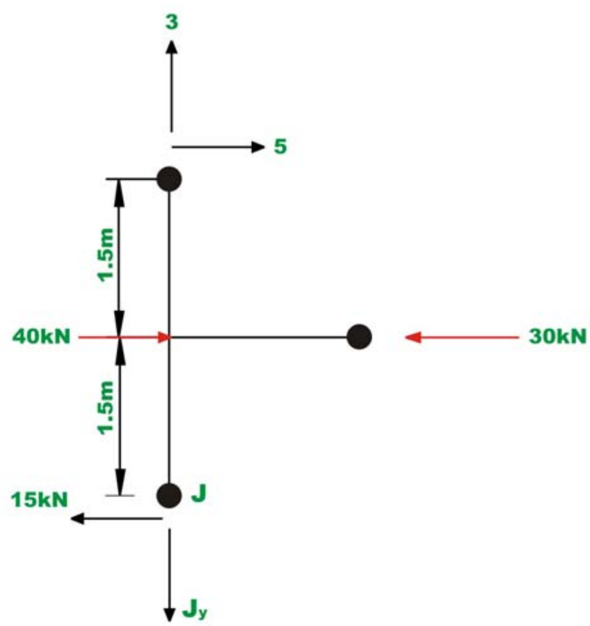


Fig.36.7d

$$\sum F_X = 0 \qquad \Rightarrow \qquad V' + 2V' + V' = 60$$

or $V' = 15$ kN

Taking moment of all forces about P gives (vide Fig. 36.7d)

$$\sum M_p = 015\times1.5+5\times1.5+3\times2.5-J_y\times2.5=0$$

$$J_y = 15 \quad \text{kN}(\downarrow)$$

$$L_y = 15 \quad \text{kN}(\uparrow)$$

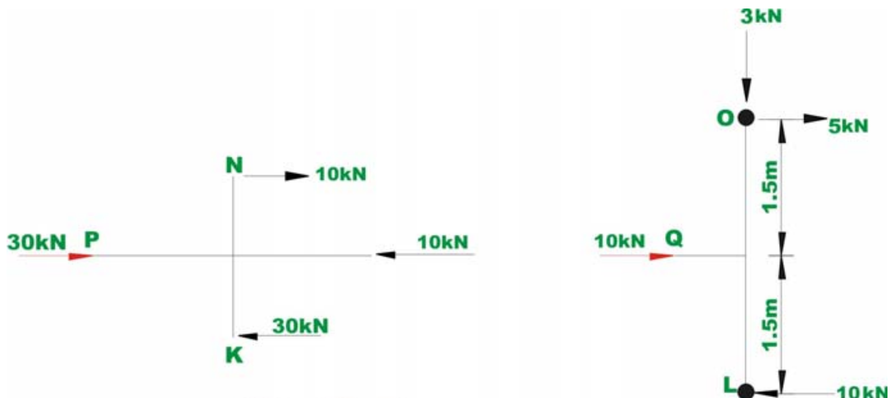


Fig.36.7e

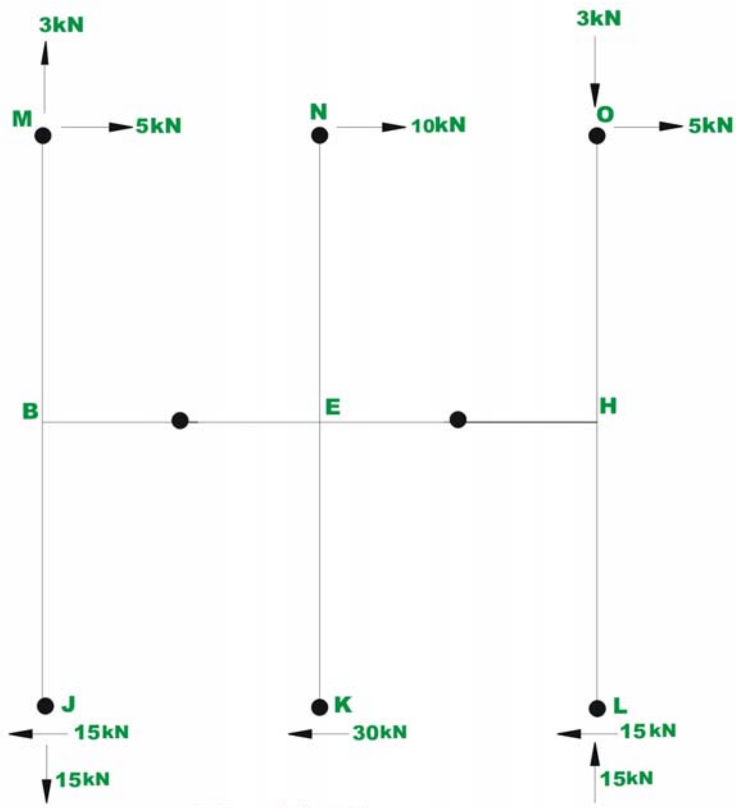


Fig.36.7f

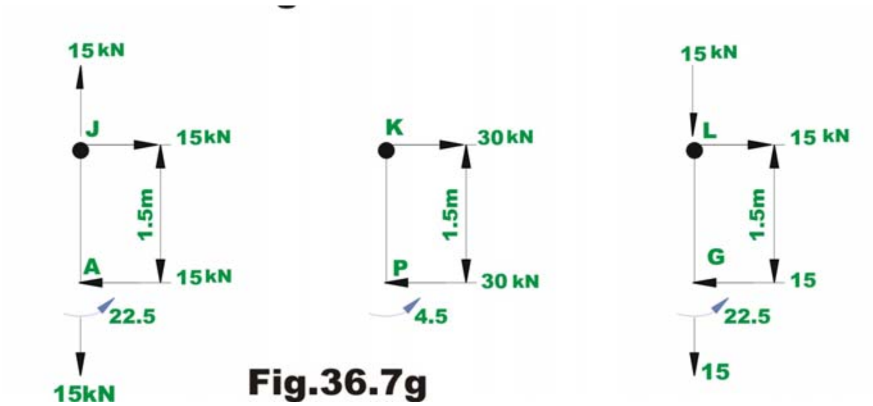


Fig.36.7g

Column and beam moments are calculated as, (ref. Fig. 36.7f)

$$M_{BC} = 5 \times 1.5 = 7.5 kN.m \;;\; M_{BA} = 15 \times 1.5 = 22.5 \text{ kN.m}$$

$$M_{BE} = -30 \text{ kN.m}$$

$$M_{EF} = 10 \times 1.5 = 15 \text{ kN.m} \;;\; M_{ED} = 30 \times 1.5 = 45 \text{ kN.m}$$

$$M_{EB} = -30 \text{ kN.m} \qquad M_{EH} = -30 \text{ kN.m}$$

$$M_{HI} = 5 \times 1.5 = 7.5 \text{ kN.m} \;;\; M_{HG} = 15 \times 1.5 = 22.5 \text{ kN.m}$$

$$M_{HE} = -30 \text{ kN.m}$$

Reactions at the base of the column are shown in Fig. 36.7g**.**

36.3.2 Cantilever method

The cantilever method is suitable if the frame is tall and slender. In the cantilever method following assumptions are made.

1) An inflexion point occurs at the mid point of each girder.
2) An inflexion point occurs at mid height of each column.
3) In a storey, the intensity of axial stress in a column is proportional to its horizontal distance from the center of gravity of all the columns in that storey.

Consider a cantilever beam acted by a horizontal load P as shown in Fig. 36.8. In such a column the bending stress in the column cross section varies linearly from its neutral axis. The last assumption in the cantilever method is based on this fact. The method is illustrated in example 36.3.

Example 36.3

Estimate approximate column reactions, beam and column moments using cantilever method of the frame shown in Fig. 36.8a. The columns are assumed to have equal cross sectional areas.

Solution:

This problem is already solved by portal method. The center of gravity of all column passes through centre column.

$$\bar{x} = \frac{\sum xA}{\sum A} = \frac{(0)A + 5A + 10A}{A + A + A} = 5\,\text{m} \quad \text{(from left column)}$$

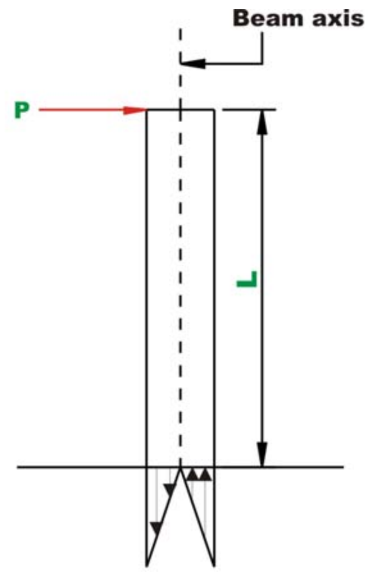


Fig.36.8a Cantilever Column

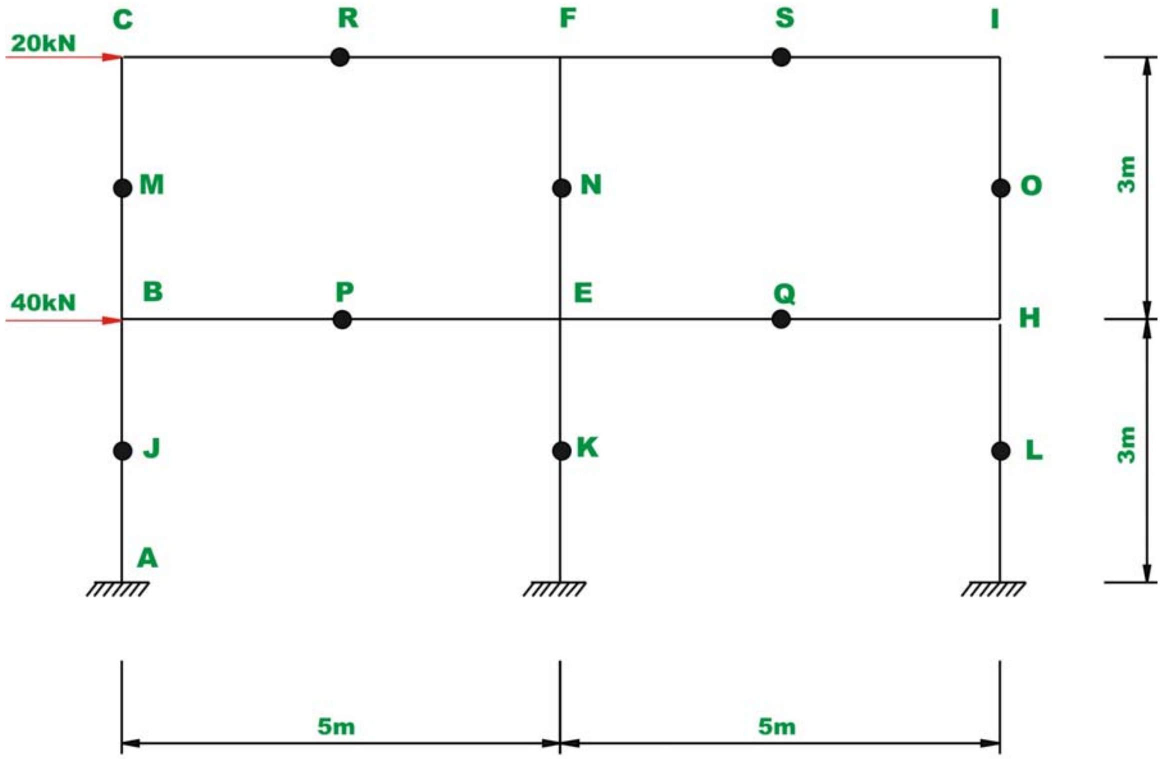


Fig.36.8b

Taking a section through first storey hinges gives us the free body diagram as shown in Fig. 36.8b. Now the column left of C.G. *i.e.* CB must be subjected to tension and one on the right is subjected to compression.

From the third assumption,

$$\frac{M_y}{5 \times A} = -\frac{O_y}{5 \times A} \qquad \Rightarrow \qquad M_y = -O_y$$

Taking moment about O of all forces gives,

$$20 \times 1.5 - M_y \times 10 = 0$$

$$M_y = 3\text{kN}(\downarrow) \quad ; \qquad O_y = 3\text{kN}(\uparrow)$$

Taking moment about R of all forces left of R,

$$V_M \times 1.5 - 3 \times 2.5 = 0$$

$$V_M = 5 \ \text{ kN}(\leftarrow)$$

Taking moment of all forces right of S about S ,

$$V_O \times 1.5 - 3 \times 2.5 = 0 \quad \Rightarrow \qquad V_O = 5\,\text{kN.}$$
$$\sum F_X = 0 \qquad V_M + V_N + V_O - 20 = 0$$

$$V_N = 10\,\text{kN.}$$

Moments

$$M_{CB} = 5 \times 1.5 = 7.5 \text{ kN.m}$$

$$M_{CF} = -7.5 \text{ kN.m}$$

$$M_{FE} = 15 \text{ kN.m}$$

$$M_{FC} = -7.5 \text{ kN.m}$$

$$M_{FI} = -7.5 \text{ kN.m}$$

$$M_{IH} = 7.5 \text{ kN.m}$$

$$M_{IF} = -7.5 \text{ kN.m}$$

Tae a section through hinges J, K, L (ref. Fig. 36.8c). Since the center of gravity passes through centre column the axial force in that column is zero.

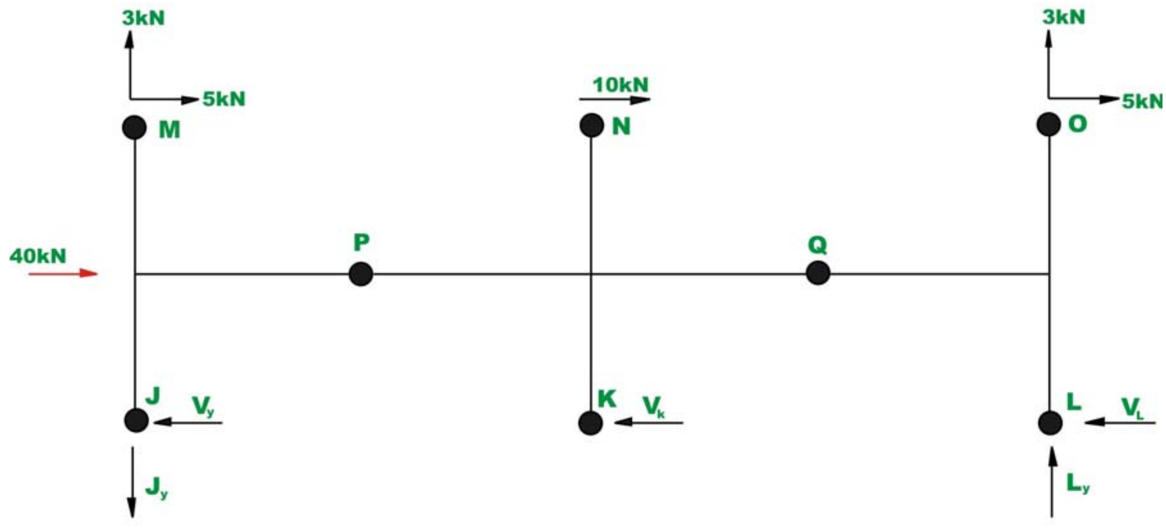


Fig.36.8c

Taking moment about hinge L, J_y can be evaluated. Thus,

$$20 \times 3 + 40 \times 1.5 + 3 \times 10 - J_y \times 10 = 0$$

$$J_y = 15\,\text{kN}(\downarrow) \quad ; \quad L_y = 15\,\text{kN}(\uparrow)$$

Taking moment of all forces left of P about P gives,

$$5 \times 1.5 + 3 \times 2.5 - 15 \times 2.5 + V_j \times 1.5 = 0$$

$$V_J = 15\,\text{kN}(\leftarrow)$$

Similarly taking moment of all forces right of Q about Q gives,

$$5 \times 1.5 + 3 \times 2.5 - 15 \times 2.5 + V_L \times 1.5 = 0$$

$$V_L = 15\,\text{kN}(\leftarrow)$$

$$\sum F_X = 0 \qquad V_J + V_K + V_L - 60 = 0$$

$$V_K = 30 \quad \text{kN.}$$

Moments

$$M_{BC} = 5 \times 1.5 = 7.5 \text{ kN.m} \quad ; \quad M_{BA} = 15 \times 1.5 = 22.5 \text{ kN.m}$$

$$M_{BE} = -30 \text{ kN.m}$$

$$M_{EF} = 10 \times 1.5 = 15 \text{ kN.m} \quad ; \quad M_{ED} = 30 \times 1.5 = 45 \text{ kN.m}$$

$$M_{EB} = -30 \text{ kN.m} \qquad M_{EH} = -30 \text{ kN.m}$$

$$M_{HI} = 5 \times 1.5 = 7.5 \text{ kN.m} \quad ; \quad M_{HG} = 15 \times 1.5 = 22.5 \text{ kN.m}$$

$$M_{HE} = -30 \text{ kN.m}$$

Summary

In this lesson, the building frames are analysed by approximate methods. Towards this end, the given indeterminate building fame is reduced into a determinate structure by suitable assumptions. The analysis of building frames to vertical loads was discussed in section 36.2. In section 36.3, analysis of building frame to horizontal loads is discussed. Two different methods are used to analyse building frames to horizontal loads: portal and cantilever method. Typical numerical problems are solved to illustrate the procedure.

Module 7

Influence Lines

Lesson
37
Moving Load and Its Effects on Structural Members

Instructional Objectives:

The objectives of this lesson are as follows:

- Understand the moving load effect in simpler term
- Study various definitions of influence line
- Introduce to simple procedures for construction of influence lines

37.1 Introduction

In earlier lessons, you were introduced to statically determinate and statically indeterminate structural analysis under non-moving load (dead load or fixed loads). In this lecture, you will be introduced to determination of maximum internal actions at cross-sections of members of statically determinate structured under the effects of moving loads (live loads).

Common sense tells us that when a load moves over a structure, the deflected shape of the structural will vary. In the process, we can arrive at simple conclusion that due to moving load position on the structure, reactions value at the support also will vary.

From the designer's point of view, it is essential to have safe structure, which doesn't exceed the limits of deformations and also the limits of load carrying capacity of the structure.

37.2 Definitions of influence line

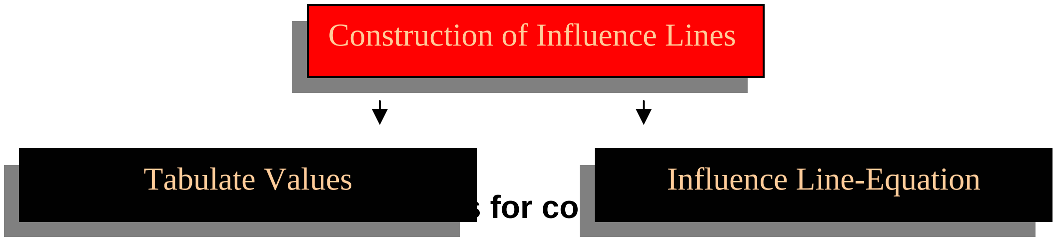
In the literature, researchers have defined influence line in many ways. Some of the definitions of influence line are given below.

- An influence line is a diagram whose ordinates, which are plotted as a function of distance along the span, give the value of an internal force, a reaction, or a displacement at a particular point in a structure as a unit load move across the structure.
- An influence line is a curve the ordinate to which at any point equals the value of some particular function due to unit load acting at that point.
- An influence line represents the variation of either the reaction, shear, moment, or deflection at a specific point in a member as a unit concentrated force moves over the member.

37.3 Construction of Influence Lines

In this section, we will discuss about the construction of influence lines. Using any one of the two approaches (Figure 37.1), one can construct the influence line at a specific point P in a member for any parameter (Reaction, Shear or

Moment). In the present approaches it is assumed that the moving load is having dimensionless magnitude of unity. Classification of the approaches for construction of influence lines is given in Figure 37.1.

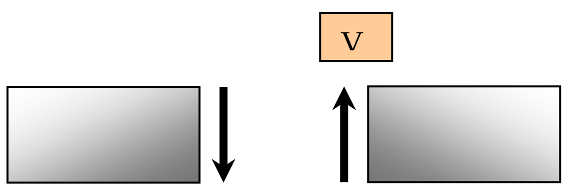
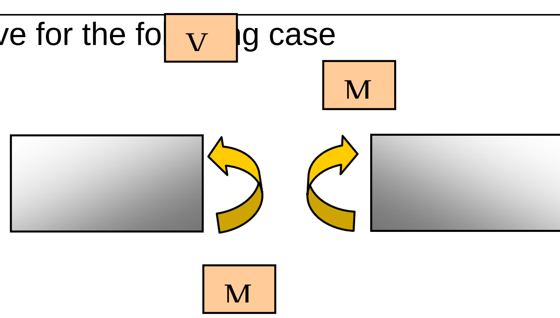


37.3.1 Tabulate Values

Apply a unit load at different locations along the member, say at ***x***. And these locations, apply statics to compute the value of parameter (reaction, shear, or moment) at the specified point. The best way to use this approach is to prepare a table, listing unit load at x versus the corresponding value of the parameter calculated at the specific point (i.e. Reaction R, Shear V or moment M) and plot the tabulated values so that influence line segments can be constructed.

37.3.2 Sign Conventions

Sign convention followed for shear and moment is given below.

Parameter	Sign for influence line
Reaction R	Positive at the point when it acts upward on the beam.
Shear V	Positive for the following case 
Moment M	Positive for the following case 

37.3.3 Influence Line Equations

Influence line can be constructed by deriving a general mathematical equation to compute parameters (e.g. reaction, shear or moment) at a specific point under the effect of moving load at a variable position ***x***.

The above discussed both approaches are demonstrated with the help of simple numerical examples in the following paragraphs.

37.4 Numerical Examples

Example 1:

Construct the influence line for the reaction at support B for the beam of span 10 m. The beam structure is shown in Figure 37.2.

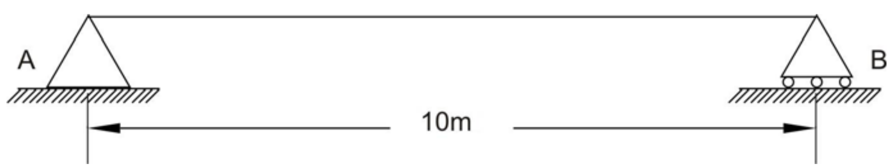


Figure 37.2: The beam structure

Solution:

As discussed earlier, there are two ways this problem can be solved. Both the approaches will be demonstrated here.

Tabulate values:

As shown in the figure, a unit load is places at distance x from support A and the reaction value R_B is calculated by taking moment with reference to support A. Let us say, if the load is placed at 2.5 m. from support A then the reaction R_B can be calculated as follows (Figure 37.3).

$$\Sigma M_A = 0 : \; R_B \times 10 - 1 \times 2.5 = 0 \Rightarrow R_B = 0.25$$

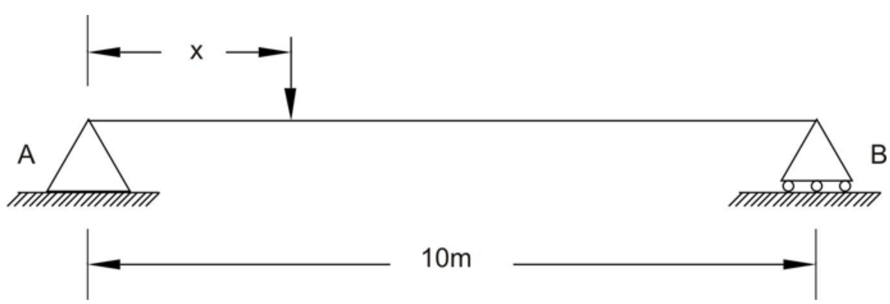


Figure 37.3: The beam structure with unit load

Similarly, the load can be placed at 5.0, 7.5 and 10 m. away from support A and reaction R_B can be computed and tabulated as given below.

x	R_B
0	0.0
2.5	0.25
5.0	0.5
7.5	0.75
10	1

Graphical representation of influence line for R_B is shown in Figure 37.4.

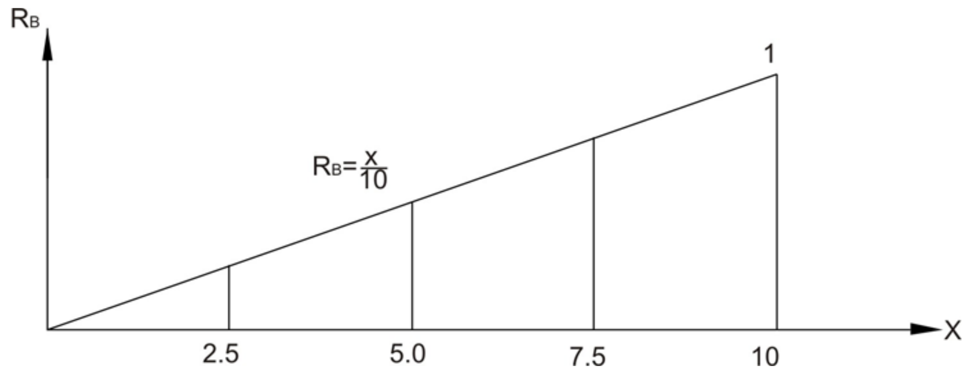


Figure 37.4: Influence line for reaction R_B.

Influence Line Equation:

When the unit load is placed at any location between two supports from support A at distance x then the equation for reaction R_B can be written as

$$\Sigma M_A = 0 : \; R_B \times 10 - x = 0 \;\Rightarrow\; R_B = x/10$$

The influence line using this equation is shown in Figure 37.4.

Example 2:

Construct the influence line for support reaction at B for the given beam as shown in Fig 37.5.

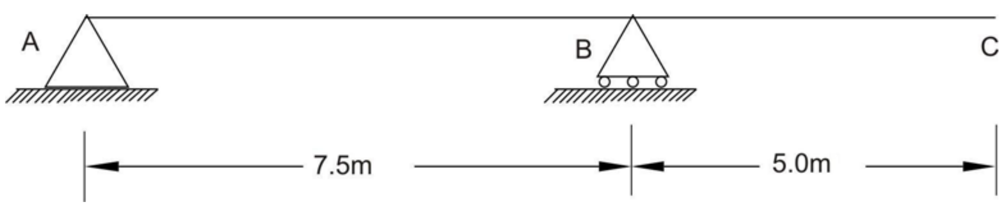


Figure 37.5: The overhang beam structure

Solution:

As explained earlier in example 1, here we will use tabulated values and influence line equation approach.

Tabulate Values:

As shown in the figure, a unit load is places at distance x from support A and the reaction value R_B is calculated by taking moment with reference to support A. Let us say, if the load is placed at 2.5 m. from support A then the reaction R_B can be calculated as follows.

$$\Sigma M_A = 0 : \; R_B \times 7.5 - 1 \times 2.5 = 0 \;\Rightarrow\; R_B = 0.33$$

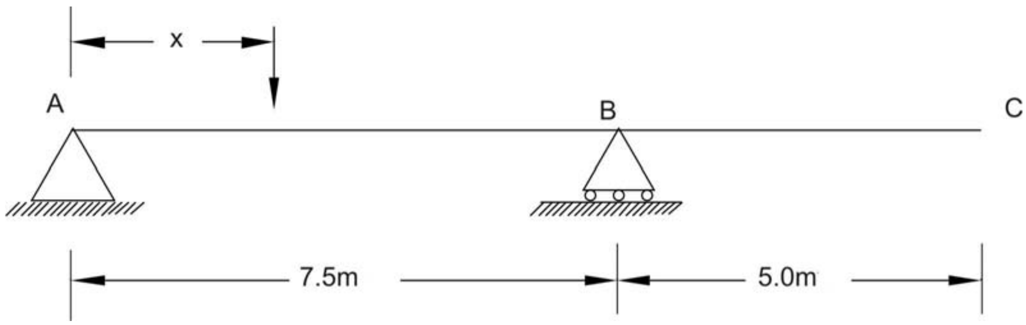


Figure 37.6: The beam structure with unit load

Similarly one can place a unit load at distances 5.0 m and 7.5 m from support A and compute reaction at B. When the load is placed at 10.0 m from support A, then reaction at B can be computed using following equation.

$$\Sigma M_A = 0 : \; R_B \times 7.5 - 1 \times 10.0 = 0 \;\Rightarrow\; R_B = 1.33$$

Similarly a unit load can be placed at 12.5 and the reaction at B can be computed. The values of reaction at B are tabulated as follows.

x	R_B
0	0.0
2.5	0.33
5.0	0.67
7.5	1.00
10	1.33
12.5	1.67

Graphical representation of influence line for R_B is shown in Figure 37.7.

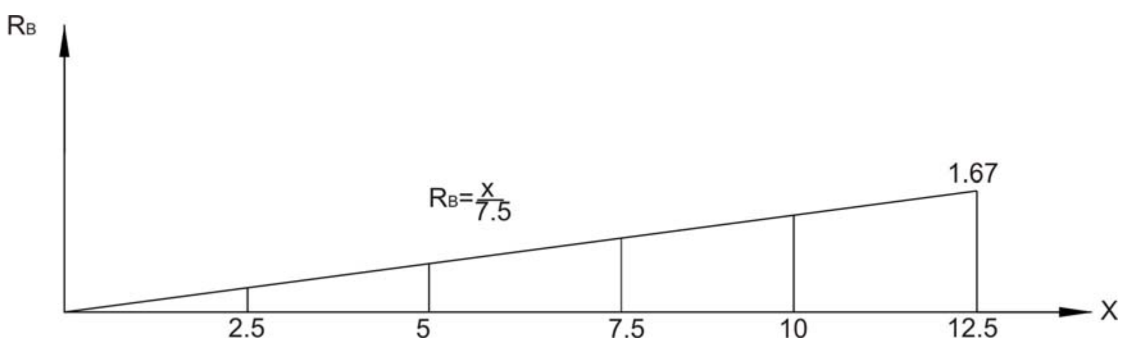


Figure 37.7: Influence for reaction R_B.

Influence line Equation:

Applying the moment equation at A (Figure 37.6),

$$\Sigma M_A = 0: \ R_B \times 7.5 - 1 \times x = 0 \ \Rightarrow \ R_B = x/7.5$$

The influence line using this equation is shown in Figure 37.7.

Example 3:

Construct the influence line for shearing point C of the beam (Figure 37.8)

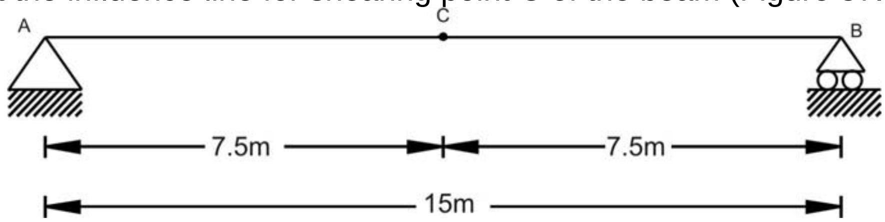


Figure 37.8: Beam Structure

Solution:

Tabulated Values:

As discussed earlier, place a unit load at different location at distance x from support A and find the reactions at A and finally computer shear force taking section at C. The shear force at C should be carefully computed when unit load is placed before point C (Figure 37.9) and after point C (Figure 37.10). The resultant values of shear force at C are tabulated as follows.

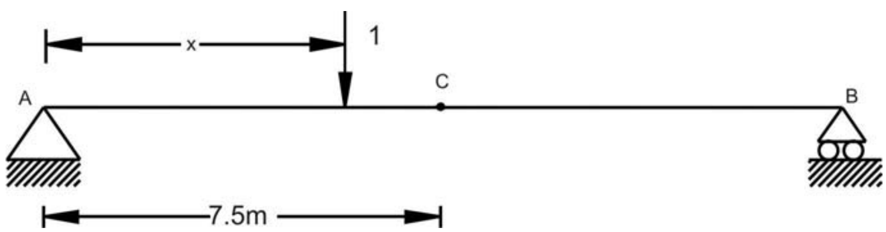


Figure 37.9: The beam structure – a unit load before section

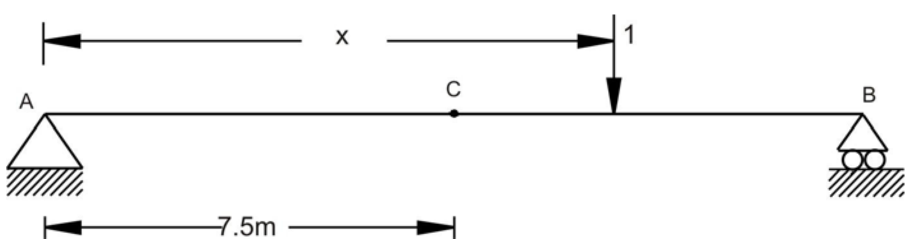


Figure 37.10: The beam structure - a unit load before section

X	V_c
0	0.0
2.5	-0.16
5.0	-0.33
7.5(-)	-0.5
7.5(+)	0.5
10	0.33
12.5	0.16
15.0	0

Graphical representation of influence line for V_c is shown in Figure 37.11.

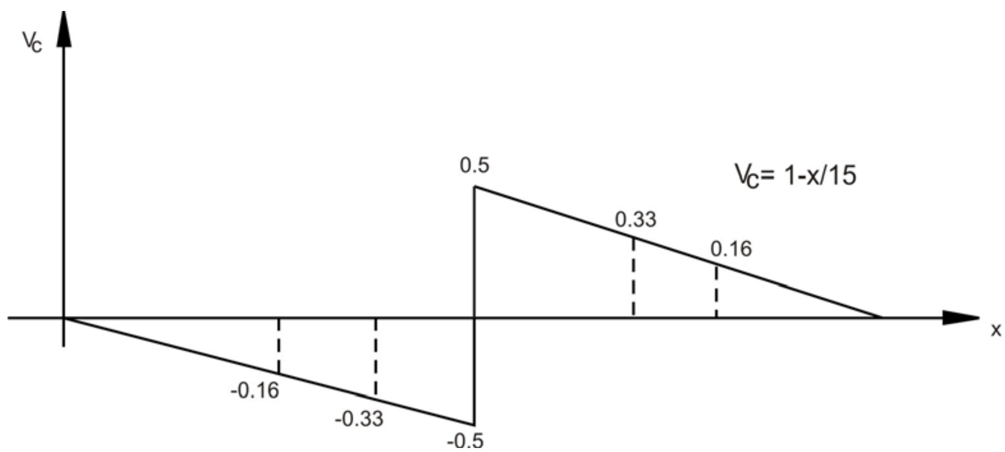


Figure 37.11: Influence line for shear point C

Influence line equation:

In this case, we need to determine two equations as the unit load position before point C (Figure 37.12) and after point C (Figure 37.13) will show different shear force sign due to discontinuity. The equations are plotted in Figure 37.11.

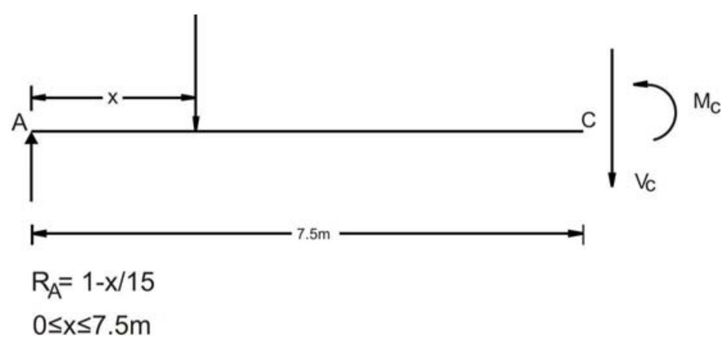


Figure 37.12: Free body diagram – a unit load before section

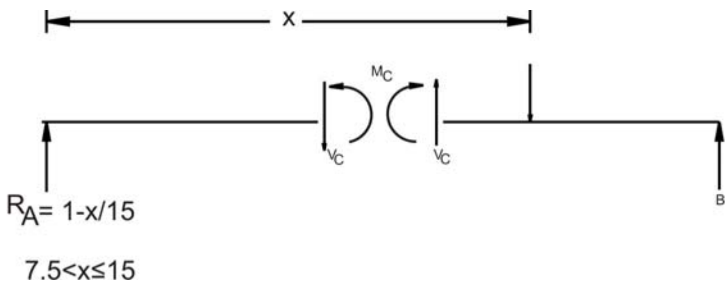


Figure 37.13: Free body diagram – a unit load after section

Influence Line for Moment:

Like shear force, we can also construct influence line for moment.

Example 4:

Construct the influence line for the moment at point C of the beam shown in Figure 37.14

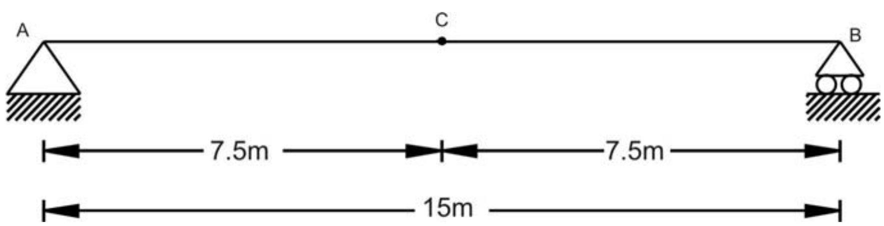


Figure 37.14: Beam structure

Solution:

Tabulated values:

Place a unit load at different location between two supports and find the support reactions. Once the support reactions are computed, take a section at C and compute the moment. For example, we place the unit load at x=2.5 m from support A (Figure 37.15), then the support reaction at A will be 0.833 and support reaction B will be 0.167. Taking section at C and computation of moment at C can be given by

$$\Sigma M_c = 0 : \; -M_c + R_B \times 7.5 - \; = 0 \Rightarrow \; -M_c + 0.167 \times 7.5 - \; = 0 \Rightarrow \; M_c = 1.25$$

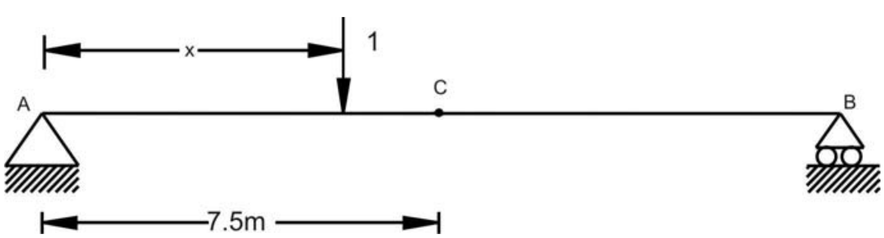


Figure 37.15: A unit load before section

Similarly, compute the moment M_c for difference unit load position in the span. The values of Mc are tabulated as follows.

X	M_c
0	0.0
2.5	1.25
5.0	2.5
7.5	3.75
10	2.5
12.5	1.25
15.0	0

Graphical representation of influence line for M_c is shown in Figure 37.16.

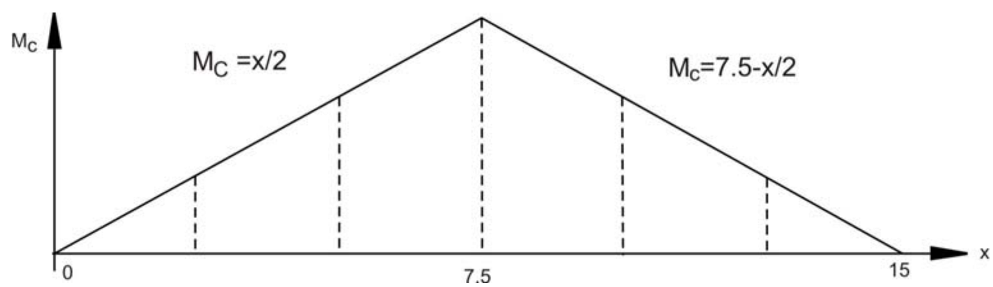


Figure 37.16: Influence line for moment at section C

Influence Line Equations:

There will be two influence line equations for the section before point C and after point C.

When the unit load is placed before point C then the moment equation for given Figure 37.17 can be given by

$\Sigma M_c = 0: \; M_c + 1(7.5 - x) - (1 - x/15)x7.5 = 0 \Rightarrow M_c = x/2$, where $0 \le x \le 7.5$

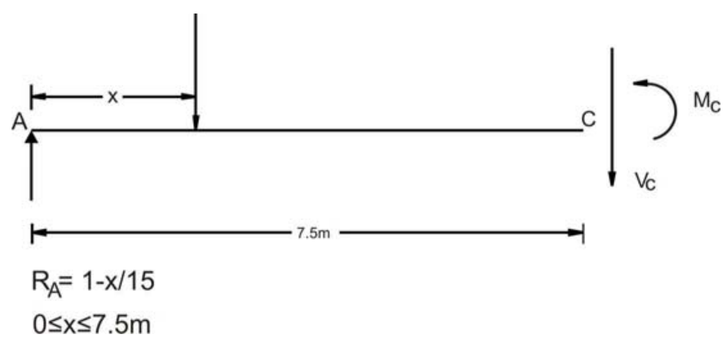


Figure 37.17: Free body diagram - a unit load before section

When the unit load is placed after point C then the moment equation for given Figure 37.18 can be given by

$\Sigma M_c = 0: \; M_c - (1 - x/15) \times 7.5 = 0 \Rightarrow M_c = 7.5 - x/2$, where $7.5 < x \le 15.0$

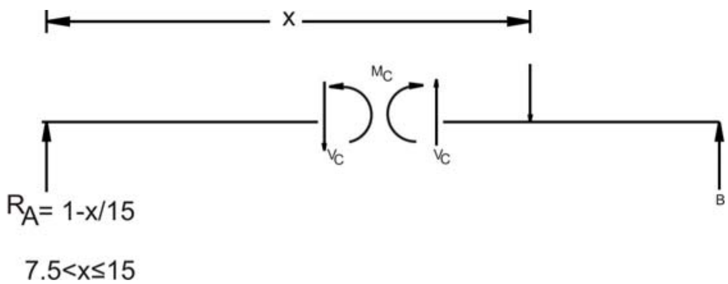


Figure 37.18: Free body diagram - a unit load before section

The equations are plotted in Figure 37.16.

Example 5:
Construct the influence line for the moment at point C of the beam shown in Figure 37.19.

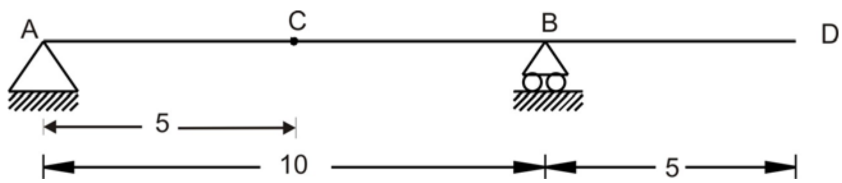


Figure 37.19: Overhang beam structure

Solution:
Tabulated values:
Place a unit load at different location between two supports and find the support reactions. Once the support reactions are computed, take a section at C and compute the moment. For example as shown in Figure 37.20, we place a unit load at 2.5 m from support A, then the support reaction at A will be 0.75 and support reaction B will be 0.25.

Figure 37.20: A unit load before section C

Taking section at C and computation of moment at C can be given by

$$\Sigma M_c = 0 : \; -M_c + R_B \times 5.0 - \; = 0 \;\Rightarrow\; -M_c + 0.25 \times 5.0 = 0 \;\Rightarrow\; M_c = 1.25$$

Similarly, compute the moment M_c for difference unit load position in the span. The values of Mc are tabulated as follows.

x	M_c
0	0
2.5	1.25
5.0	2.5
7.5	1.25
10	0
12.5	-1.25
15.0	-2.5

Graphical representation of influence line for M_c is shown in Figure 37.21.

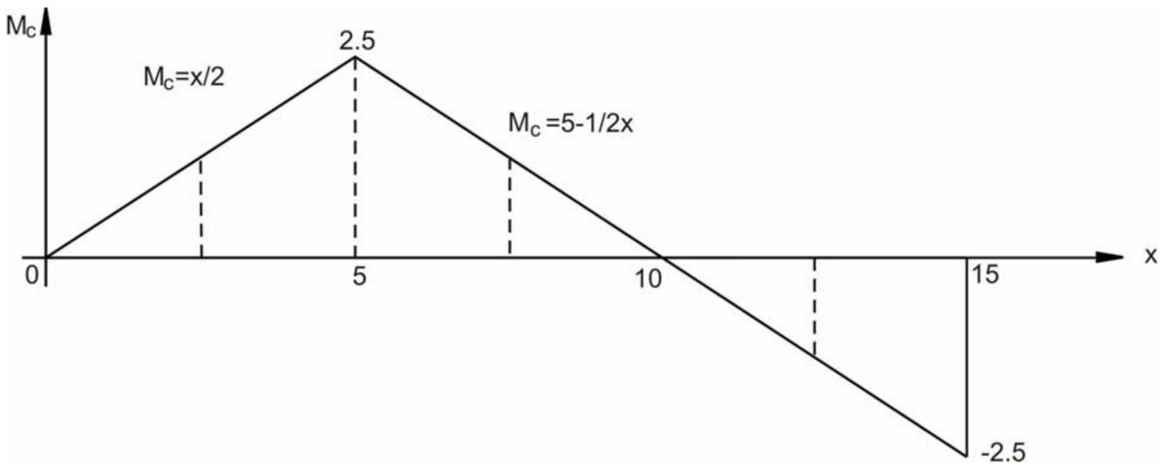


Figure 37.21: Influence line of moment at section C

Influence Line Equations:

There will be two influence line equations for the section before point C and after point C.

When a unit load is placed before point C then the moment equation for given Figure 37.22 can be given by

$\Sigma M_c = 0: \; M_c + 1(5.0 - x) - (1 - x/10)x5.0 = 0 \Rightarrow M_c = x/2$, where $0 \leq x \leq 5.0$

Figure 37.22: A unit load before section C

When a unit load is placed after point C then the moment equation for given Figure 37.23 can be given by

$\Sigma M_c = 0: \; M_c - (1 - x/10) \times 5.0 = 0 \Rightarrow M_c = 5 - x/2$, where $5 < x \leq 15$

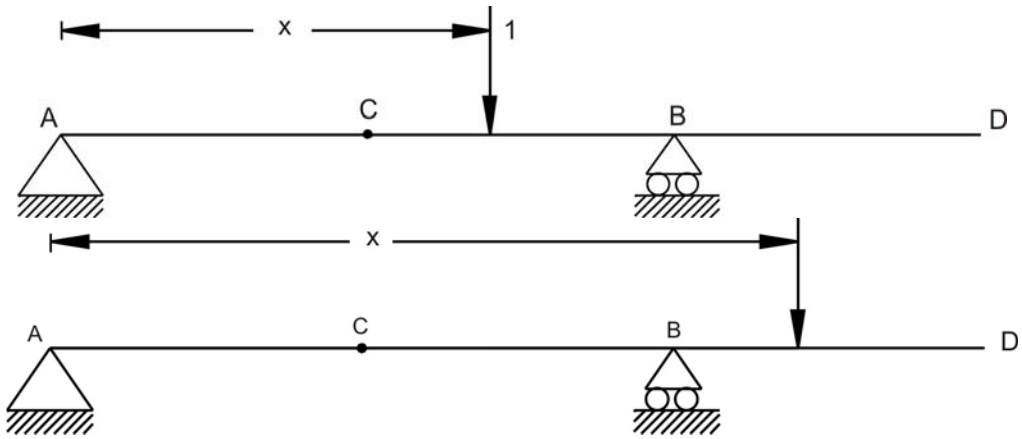


Figure 37.23: A unit load after section C

The equations are plotted in Figure 37.21.

37.5 Influence line for beam having point load and uniformly distributed load acting at the same time

Generally in beams/girders are main load carrying components in structural systems. Hence it is necessary to construct the influence line for the reaction, shear or moment at any specified point in beam to check for criticality. Let us assume that there are two kinds of load acting on the beam. They are concentrated load and uniformly distributed load (UDL).

37.5.1 Concentrated load

As shown in the Figure 37.24, let us say, point load P is moving on beam from A to B. Looking at the position, we need to find out what will be the influence line for reaction B for this load. Hence, to generalize our approach, like earlier examples, let us assume that unit load is moving from A to B and influence line for reaction A can be plotted as shown in Figure 37.25. Now we want to know, if load P is at the center of span then what will be the value of reaction A? From Figure 37.24, we can find that for the load position of P, influence line of unit load gives value of 0.5. Hence, reaction A will be 0.5xP. Similarly, for various load positions and load value, reactions A can be computed.

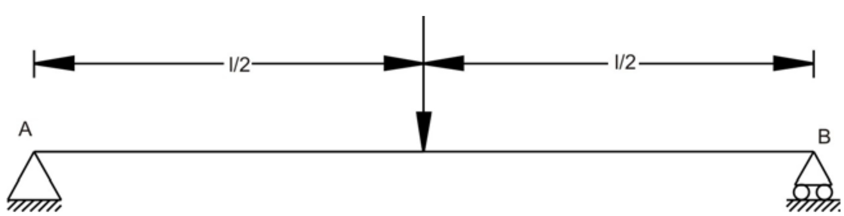


Figure 37.24: Beam structure

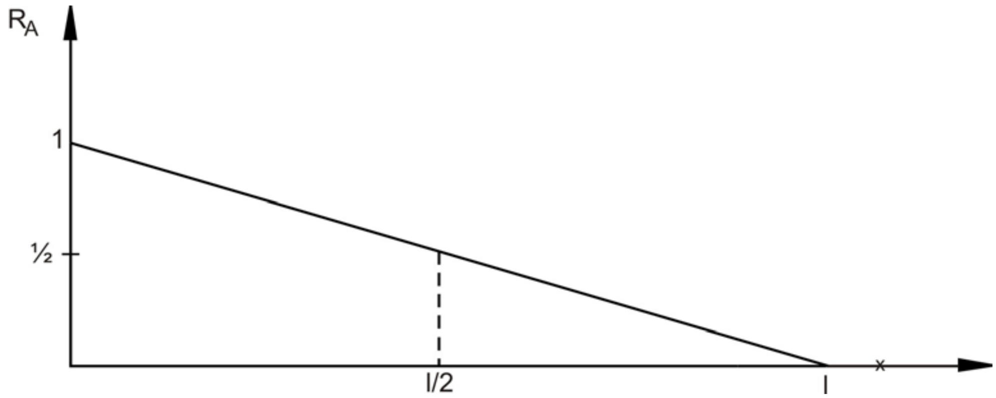


Figure 37.25: Influence line for support reaction at A

37.5.2 Uniformly Distributed Load

Beam is loaded with uniformly distributed load (UDL) and our objective is to find influence line for reaction A so that we can generalize the approach. For UDL of w on span, considering for segment of dx (Figure 37.26), the concentrated load dP can be given by w.dx acting at x. Let us assume that beam's influence line ordinate for some function (reaction, shear, moment) is y as shown in Figure 37.27. In that case, the value of function is given by (dP)(y) = (w.dx).y. For computation of the effect of all these concentrated loads, we have to integrate over the entire length of the beam. Hence, we can say that it will be $\int w.y.dx = w \int y.dx$. The term $\int y.dx$ is equivalent to area under the influence line.

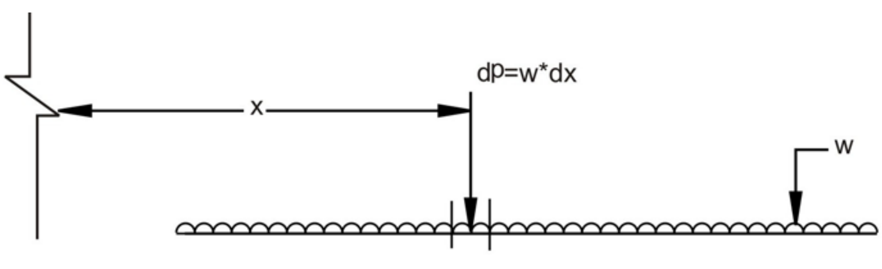


Figure 37.26: Uniformly distributed load on beam

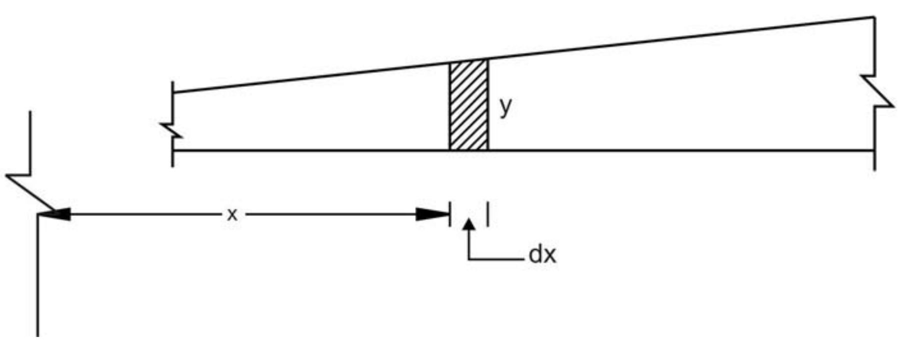


Figure 37.27: Segment of influence line diagram

For a given example of UDL on beam as shown in Figure 37.28, the influence line (Figure 37.29) for reaction A can be given by area covered by the influence line for unit load into UDL value. i.e. [0.5x (1)xl] w = 0.5 w.l.

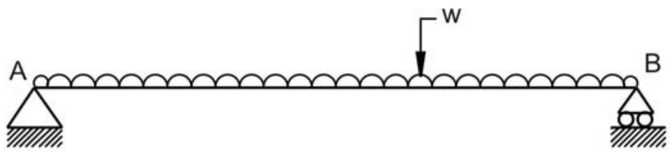


Figure 37.28: UDL on simply supported beam

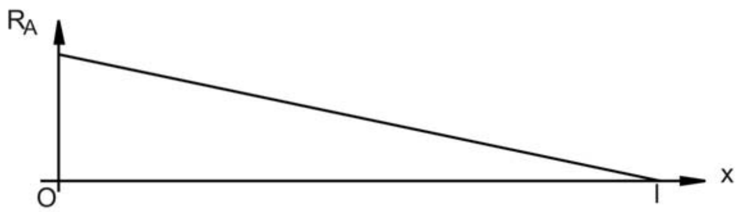


Figure 37.29: Influence line for support reaction at A.

37.6 Numerical Example

Find the maximum positive live shear at point C when the beam (Figure 37.30) is loaded with a concentrated moving load of 10 kN and UDL of 5 kN/m.

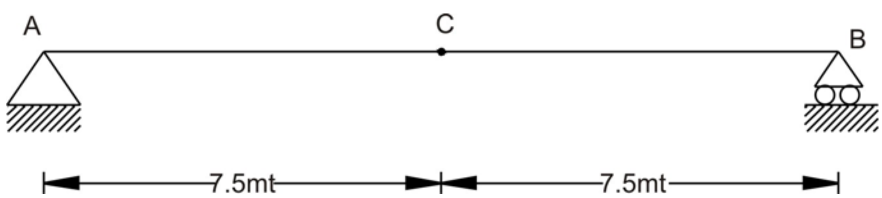


Figure 37.30: Simply supported beam

Solution:

As discussed earlier for unit load moving on beam from A to B, the influence line for the shear at C can be given by following Figure 37.31.

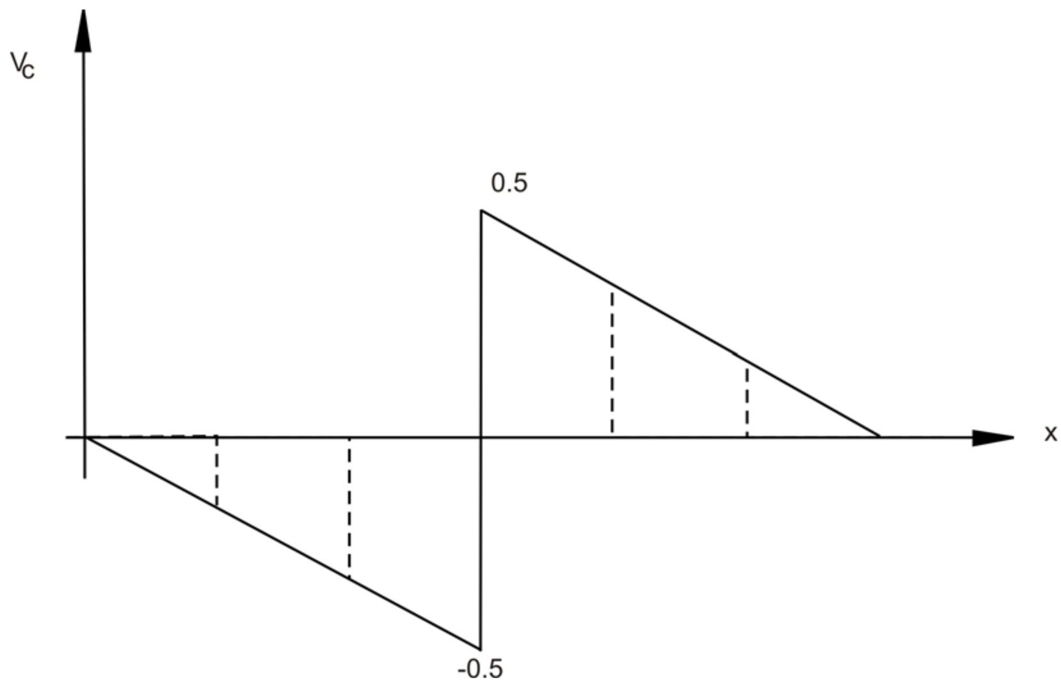


Figure 37.31: Influence line for shear at section C.

Concentrated load: As shown in Figure 37.31, the maximum live shear force at C will be when the concentrated load 10 kN is located just before C or just after C. Our aim is to find positive live shear and hence, we will put 10 kN just after C. In that case,

$$V_c = 0.5 \times 10 = 5 \text{ kN.}$$

UDL: As shown in Figure 37.31, the maximum positive live shear force at C will be when the UDL 5 kN/m is acting between x = 7.5 and x = 15.

$$V_c = [\,0.5 \times (15 - 7.5)\,(0.5)] \times 5 = 9.375$$

Total maximum Shear at C:

$$(V_c)\ \max = 5 + 9.375 = 14.375.$$

Finally the loading positions for maximum shear at C will be as shown in Figure 37.32. For this beam one can easily compute shear at C using statics.

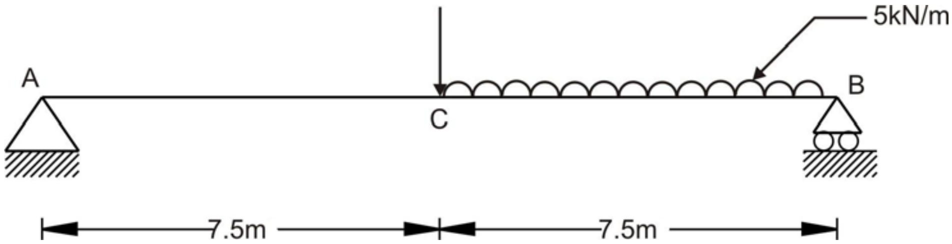


Figure 37.32: Simply supported beam

37.7 Closing Remarks

In this lesson we have studied the need for influence line and their importance. Further we studied the available various influence line definitions. Finally we studied the influence line construction using tabulated values and influence line equation. The understanding about the simple approach was studied with the help of many numerical examples.

Suggested Text Books for Further Reading

- Armenakas, A. E. (1988). *Classical Structural Analysis – A Modern Approach*, McGraw-Hill Book Company, NY, ISBN 0-07-100120-4

- Hibbeler, R. C. (2002). *Structural Analysis*, Pearson Education (Singapore) Pte. Ltd., Delhi, ISBN 81-7808-750-2

- Junarkar, S. B. and Shah, H. J. (1999). *Mechanics of Structures – Vol. II*, Charotar Publishing House, Anand.

- Leet, K. M. and Uang, C-M. (2003). *Fundamentals of Structural Analysis*, Tata McGraw-Hill Publishing Company Limited, New Delhi, ISBN 0-07-058208-4

- Negi, L. S. and Jangid, R.S. (2003). *Structural Analysis*, Tata McGraw-Hill Publishing Company Limited, New Delhi, ISBN 0-07-462304-4

- Norris, C. H., Wilbur, J. B. and Utku, S. (1991). *Elementary Structural Analysis*, Tata McGraw-Hill Publishing Company Limited, New Delhi, ISBN 0-07-058116-9

Module 7

Influence Lines

Lesson
38
Influence Lines for Beams

Instructional Objectives:

The objectives of this lesson are as follows:
- How to draw qualitative influence lines?
- Understand the behaviour of the beam under rolling loads
- Construction of influence line when the beam is loaded with uniformly distributed load having shorter or longer length than the span of the beam.

38.1 Müller Breslau Principle for Qualitative Influence Lines

In 1886, Heinrich Müller Breslau proposed a technique to draw influence lines quickly. The Müller Breslau Principle states that the ordinate value of an influence line for any function on any structure is proportional to the ordinates of the deflected shape that is obtained by removing the restraint corresponding to the function from the structure and introducing a force that causes a unit displacement in the positive direction.

Let us say, our objective is to obtain the influence line for the support reaction at A for the beam shown in Figure 38.1.

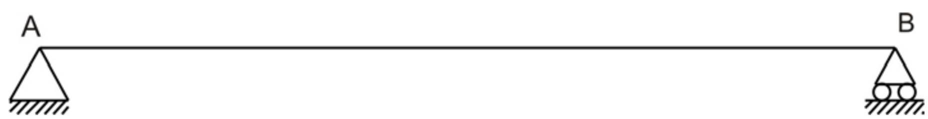


Figure 38.1: Simply supported beam

First of all remove the support corresponding to the reaction and apply a force (Figure 38.2) in the positive direction that will cause a unit displacement in the direction of R_A. The resulting deflected shape will be proportional to the true influence line (Figure 38.3) for the support reaction at A.

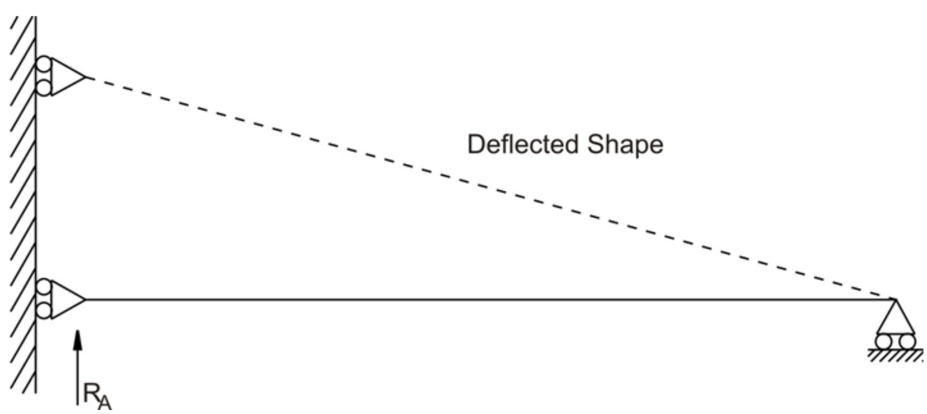


Figure 38.2: Deflected shape of beam

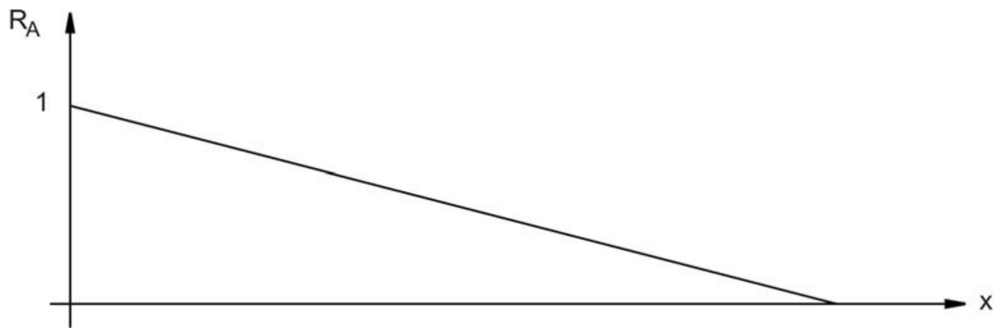


Figure 38.3: Influence line for support reaction A

The deflected shape due to a unit displacement at A is shown in Figure 38.2 and matches with the actual influence line shape as shown in Figure 38.3. Note that the deflected shape is linear, i.e., the beam rotates as a rigid body without any curvature. This is true only for statically determinate systems.

Similarly some other examples are given below.

Here we are interested to draw the qualitative influence line for shear at section C of overhang beam as shown in Figure 38.4.

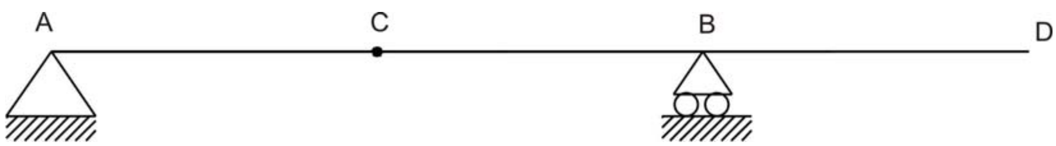


Figure 38.4: Overhang beam

As discussed earlier, introduce a roller at section C so that it gives freedom to the beam in vertical direction as shown in Figure 38.5.

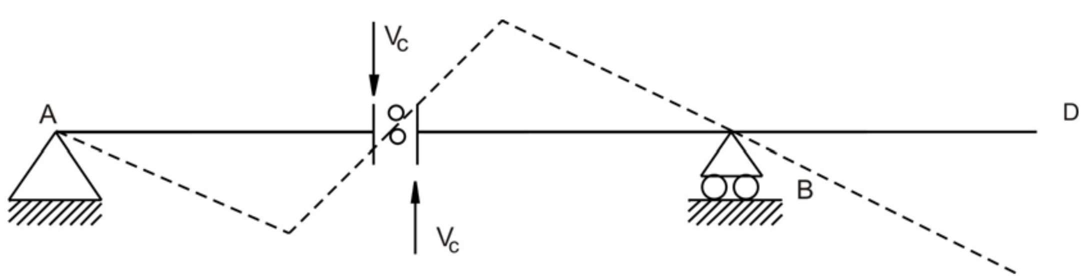


Figure 38.5: Deflected shape of beam

Now apply a force in the positive direction that will cause a unit displacement in the direction of V_C. The resultant deflected shape is shown in Figure 38.5. Again, note that the deflected shape is linear. Figure 38.6 shows the actual influence, which matches with the qualitative influence.

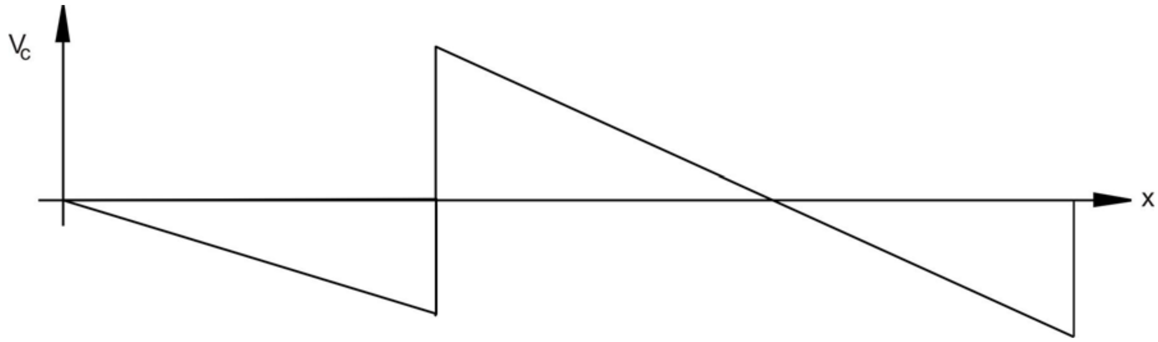


Figure 38.6: Influence line for shear at section C

In this second example, we are interested to draw a qualitative influence line for moment at C for the beam as shown in Figure 38.7.

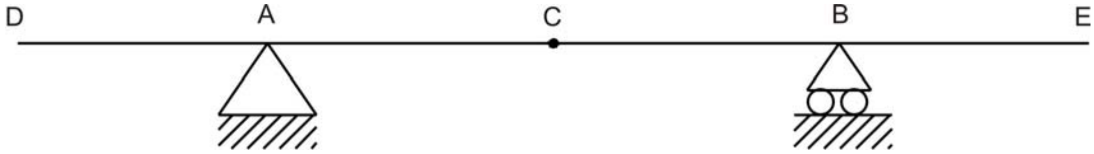


Figure 38.7: Beam structure

In this example, being our objective to construct influence line for moment, we will introduce hinge at C and that will only permit rotation at C. Now apply moment in the positive direction that will cause a unit rotation in the direction of M_c. The deflected shape due to a unit rotation at C is shown in Figure 38.8 and matches with the actual shape of the influence line as shown in Figure 38.9.

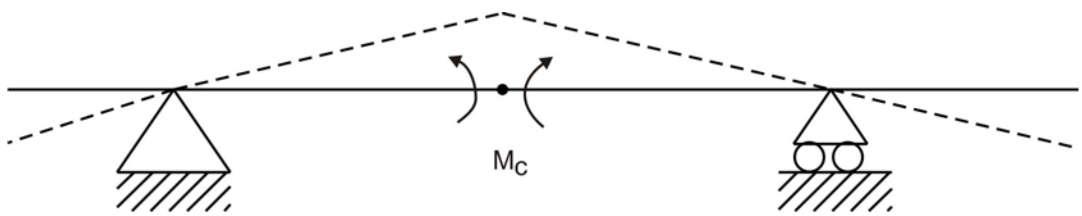


Figure 38.8: Deflected shape of beam

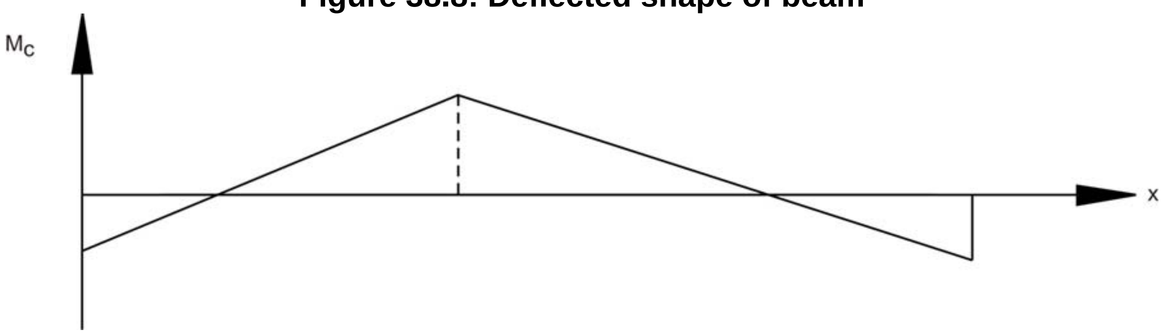


Figure 38.9: Influence line for moment at section C

38.2. Maximum shear in beam supporting UDLs

If UDL is rolling on the beam from one end to other end then there are two possibilities. Either Uniformly distributed load is longer than the span or uniformly distributed load is shorter than the span. Depending upon the length of the load and span, the maximum shear in beam supporting UDL will change. Following section will discuss about these two cases. It should be noted that for maximum values of shear, maximum areas should be loaded.

38.2.1 UDL longer than the span

Let us assume that the simply supported beam as shown in Figure 38.10 is loaded with UDL of w moving from left to right where the length of the load is longer than the span. The influence lines for reactions R_A, R_B and shear at section C located at x from support A will be as shown in Figure 38.11, 38.12 and 38.13 respectively. UDL of intensity w per unit for the shear at supports A and B will be given by

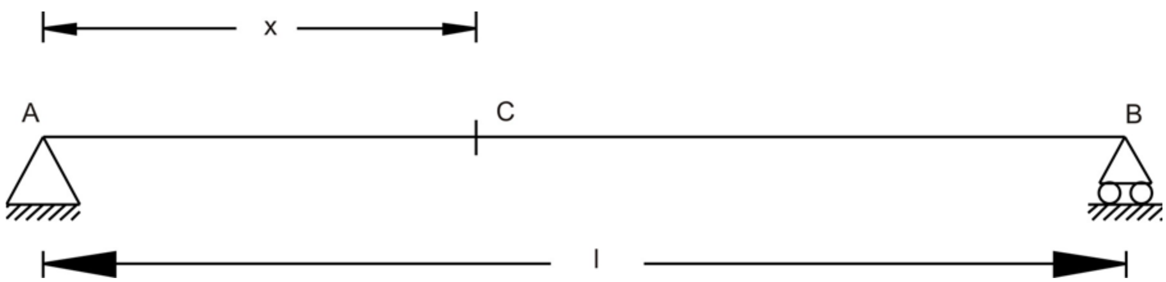


Figure 38.10: Beam Structure

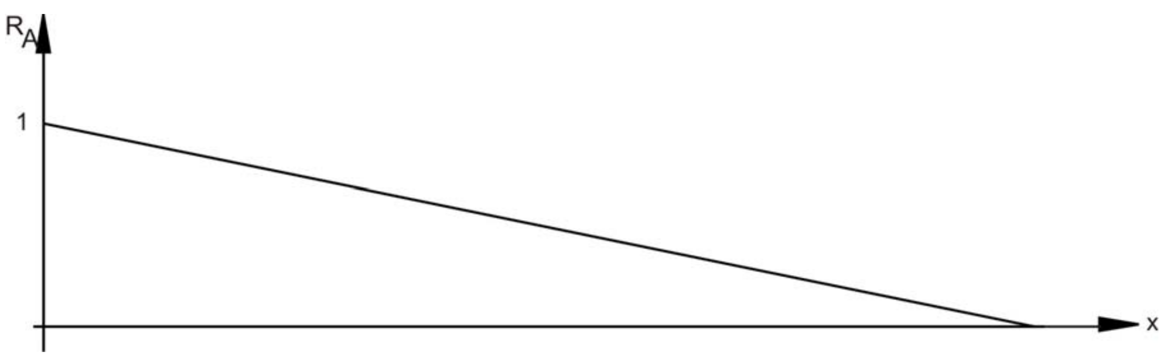


Figure 38.11: Influence line for support reaction at A

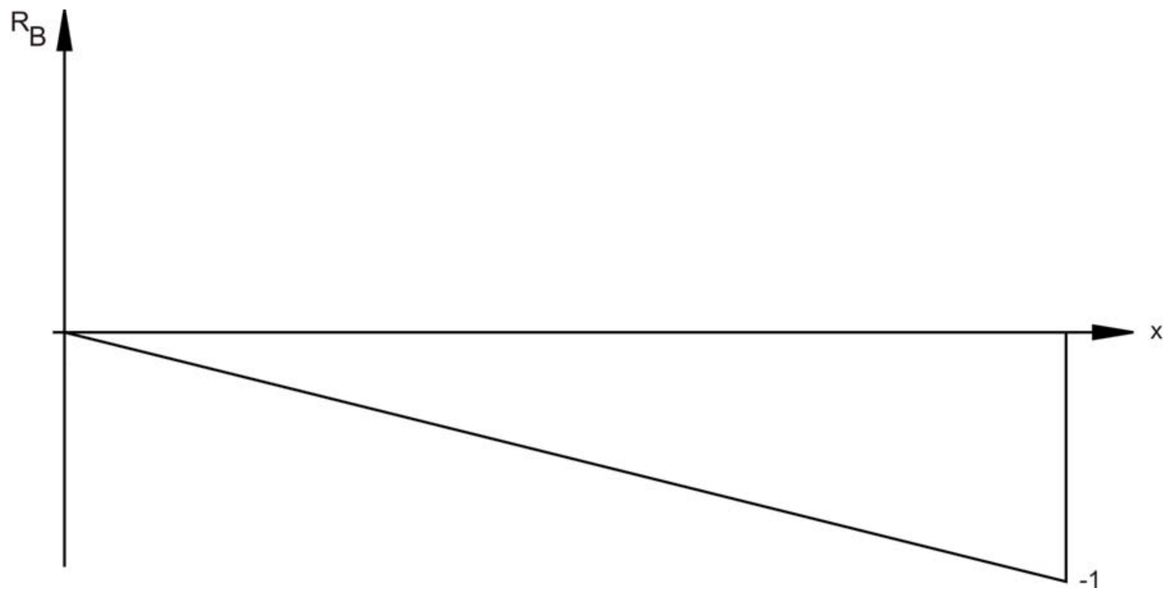


Figure 38.12: Influence line for support reaction at B

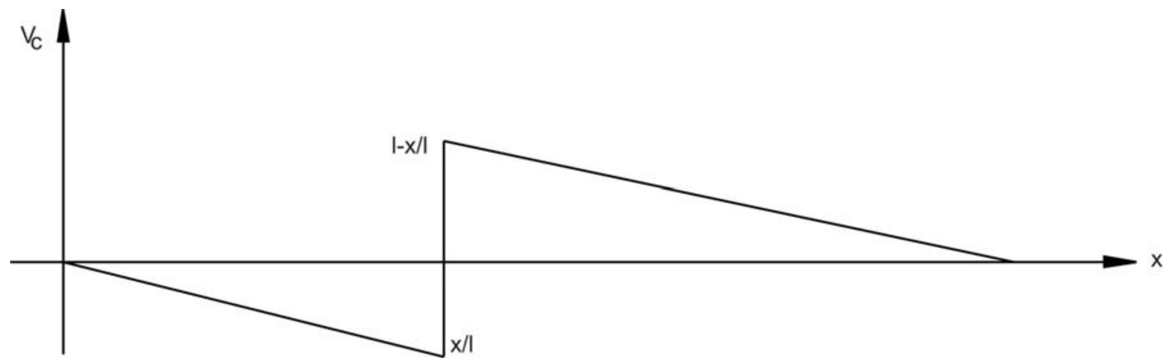


Figure 38.13: Influence line for shear at section C

$$R_A = w \times \frac{1}{2} \times l \times 1 = \frac{wl}{2}$$

$$R_B = -w \times \frac{1}{2} \times l \times 1 = \frac{-wl}{2}$$

Suppose we are interested to know shear at given section at C. As shown in Figure 38.13, maximum negative shear can be achieved when the head of the load is at the section C. And maximum positive shear can be obtained when the tail of the load is at the section C. As discussed earlier the shear force is computed by intensity of the load multiplied by the area of influence line diagram covered by load. Hence, maximum negative shear is given by

$$= -\frac{1}{2} \times x \times \frac{x}{l} \times w = -\frac{wx^2}{2l}$$

and maximum positive shear is given by

$$=\frac{1}{2}\times\left(\frac{l-x}{l}\right)\times(l-x)\times w=-\frac{w(l-x)^2}{2l}$$

38.2.2 UDL shorter than the span

When the length of UDL is shorter than the span, then as discussed earlier, maximum negative shear can be achieved when the head of the load is at the section. And maximum positive shear can be obtained when the tail of the load is at the section. As discussed earlier the shear force is computed by the load intensity multiplied by the area of influence line diagram covered by load. The example is demonstrated in previous lesson.

38.3 Maximum bending moment at sections in beams supporting UDLs.

Like the previous section discussion, the maximum moment at sections in beam supporting UDLs can either be due to UDL longer than the span or due to ULD shorter than the span. Following paragraph will explain about computation of moment in these two cases.

38.3.1 UDL longer than the span

Let us assume the UDL longer than the span is traveling from left end to right hand for the beam as shown in Figure 38.14. We are interested to know maximum moment at C located at x from the support A. As discussed earlier, the maximum bending moment is given by the load intensity multiplied by the area of influence line (Figure 38.15) covered. In the present case the load will cover the completed span and hence the moment at section C can be given by

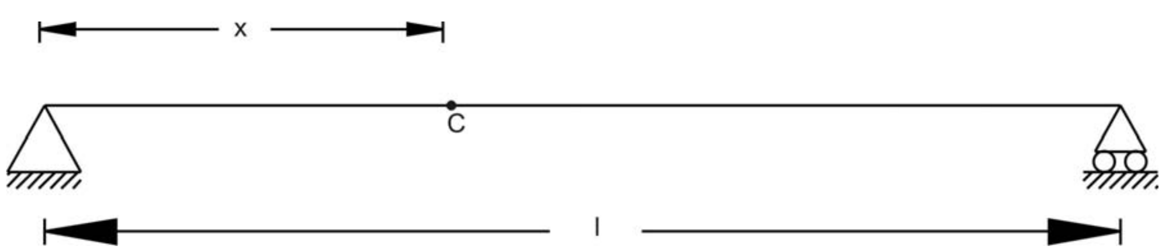


Figure 38.14: Beam structure

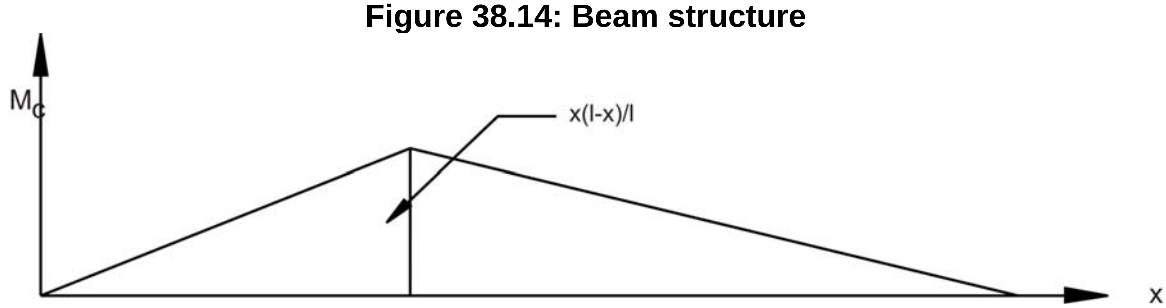


Figure 38.15: Influence line for moment at section C

$$w \times \frac{1}{2} \times l \times \frac{x(l-x)}{l} = -\frac{wx(l-x)}{2}$$

Suppose the section C is at mid span, then maximum moment is given by

$$\frac{w \times \frac{l}{2} \times \frac{l}{2}}{2} = \frac{wl^2}{8}$$

38.3.2 UDL shorter than the span

As shown in Figure 38.16, let us assume that the UDL length y is smaller than the span of the beam AB. We are interested to find maximum bending moment at section C located at x from support A. Let say that the mid point of UDL is located at D as shown in Figure 38.16 at distance of z from support A. Take moment with reference to A and it will be zero.

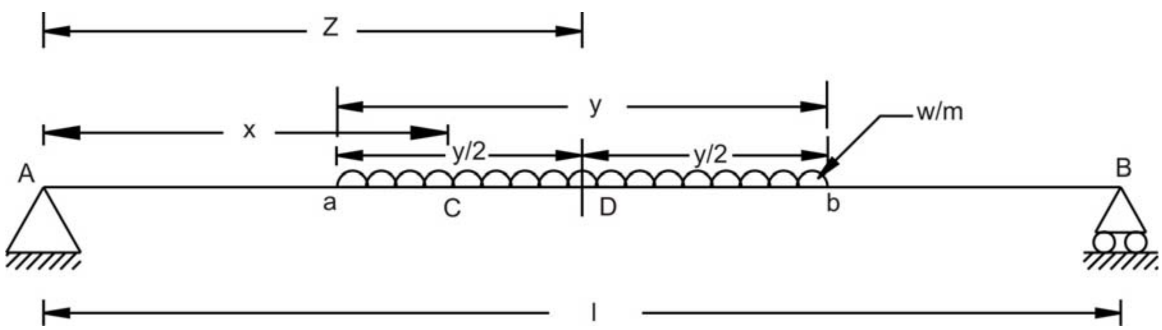


Figure 38.16: Beam loaded with UDL shorter in length than span

Hence, the reaction at B is given by

$$R_B = w \times y \times \frac{z}{l} = -\frac{wx(l-x)}{2}$$

And moment at C will be

$$M_C = R_B(l-x) - \frac{w}{2}(z + \frac{y}{2} - x)^2$$

Substituting value of reaction B in above equation, we can obtain

$$M_C = \frac{wyz}{l}(l-x) - \frac{w}{2}(z + \frac{y}{2} - x)^2$$

To compute maximum value of moment at C, we need to differentiate above given equation with reference to z and equal to zero.

$$\frac{dM_c}{dz} = \frac{wy}{l}(l - x) - w(z + \frac{y}{2} - x) = 0$$

Therefore,

$$\frac{y}{l}(l - x) = (z + \frac{y}{2} - x)$$

Using geometric expression, we can state that

$$\frac{ab}{AB} = \frac{Cb}{CB}$$

$$\therefore \frac{CB}{Cb} = \frac{AB}{ab} = \frac{AB - CB}{ab - Cb} = \frac{AC}{aC}$$

$$\therefore \frac{aC}{Cb} = \frac{AC}{CB}$$

The expression states that for the UDL shorter than span, the load should be placed in a way so that the section divides it in the same proportion as it divides the span. In that case, the moment calculated at the section will give maximum moment value.

38.4 Closing Remarks

In this lesson we studied how to draw qualitative influence line for shear and moment using Müller Breslau Principle. Further we studied how to draw the influence lines for shear and moment when the beam is loaded with UDL. Here, we studied the two cases where the UDL length is shorter or longer than span. In the next lesson we will study about two or more than two concentrated loads moving on the beam.

Suggested Text Books for Further Reading

- Armenakas, A. E. (1988). *Classical Structural Analysis – A Modern Approach*, McGraw-Hill Book Company, NY, ISBN 0-07-100120-4

- Hibbeler, R. C. (2002). *Structural Analysis*, Pearson Education (Singapore) Pte. Ltd., Delhi, ISBN 81-7808-750-2

- Junarkar, S. B. and Shah, H. J. (1999). *Mechanics of Structures – Vol. II*, Charotar Publishing House, Anand.

- Leet, K. M. and Uang, C-M. (2003). *Fundamentals of Structural Analysis*, Tata McGraw-Hill Publishing Company Limited, New Delhi, ISBN 0-07-058208-4

- Negi, L. S. and Jangid, R.S. (2003). *Structural Analysis*, Tata McGraw-Hill Publishing Company Limited, New Delhi, ISBN 0-07-462304-4

- Norris, C. H., Wilbur, J. B. and Utku, S. (1991). *Elementary Structural Analysis*, Tata McGraw-Hill Publishing Company Limited, New Delhi, ISBN 0-07-058116-9

Module
7
Influence Lines

Lesson
39
Influence Lines for Beams (Contd.)

Instructional Objectives:

The objectives of the present lesson are as follows.

- Construction of influence line for maximum shear at sections in a beam supporting two concentrated loads
- Construction of influence line for maximum moment at sections in a beam supporting two concentrated loads
- Construction of influence line for maximum end shear in a beam supporting a series of moving concentrated loads
- Construction of influence line for maximum shear at a section in a beam supporting a series of moving concentrated loads
- Construction of influence line for maximum moment at a section in a beam supporting a series of moving concentrated loads
- Construction of influence line for absolute maximum moment in s beam supporting a series of moving concentrated loads
- Understanding about the envelopes of maximum influence line values

39.1 Introduction

In the previous lessons, we have studied about construction of influence line for the either single concentrated load or uniformly distributed loads. In the present lesson, we will study in depth about the beams, which are loaded with a series of two or more then two concentrated loads.

39.2 Maximum shear at sections in a beam supporting two concentrated loads

Let us assume that instead of one single point load, there are two point loads P_1 and P_2 spaced at y moving from left to right on the beam as shown in Figure 39.1. We are interested to find maximum shear force in the beam at given section C. In the present case, we assume that $P_2<P_1$.

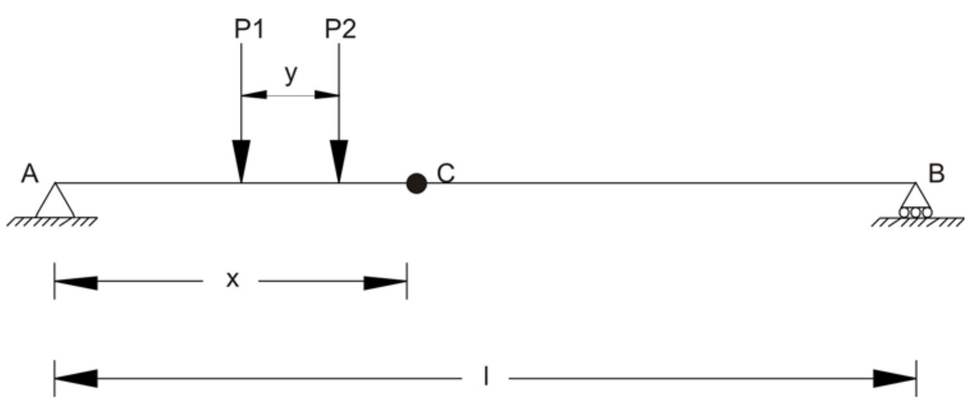


Figure 39.1: Beam loaded with two concentrated point loads

Now there are three possibilities due to load spacing. They are: $x<y$, $x=y$ and $x>y$.

Case 1: x<y

This case indicates that when load P_2 will be between A and C then load P_1 will not be on the beam. In that case, maximum negative shear at section C can be given by

$$V_C = -P_2 \frac{x}{l}$$

and maximum positive shear at section C will be

$$V_C = P_2 \frac{(l-x)}{l}$$

Case 2: x=y

In this case, load P_1 will be on support A and P_2 will be on section C. Maximum negative shear can be given by

$$V_C = -P_2 \frac{x}{l}$$

and maximum positive shear at section C will be

$$V_C = P_2 \frac{(l-x)}{l}$$

Case 3: x>y

With reference to Figure 39.2, maximum negative shear force can be obtained when load P_2 will be on section C. The maximum negative shear force is expressed as:

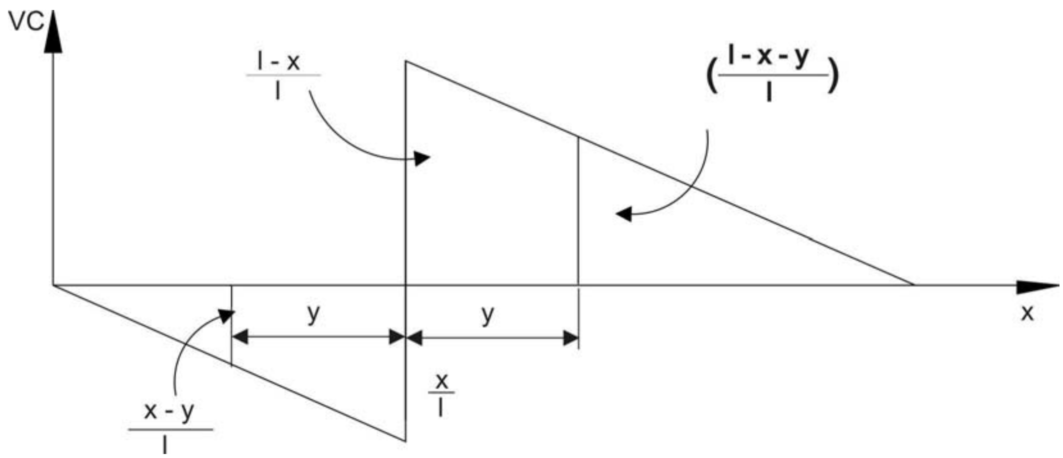


Figure 39.2: Influence line for shear at section C

$$V_C{}^1 = -P_2 \frac{x}{l} - P_1 \left(\frac{x-y}{l} \right)$$

And with reference to Figure 39.2, maximum positive shear force can be obtained when load P_1 will be on section C. The maximum positive shear force is expressed as:

$$V_C{}^2 = -P_1 \frac{x}{l} + P_2 \left(\frac{l-x-y}{l} \right)$$

From above discussed two values of shear force at section, select the maximum negative shear value.

39.3 Maximum moment at sections in a beam supporting two concentrated loads

Let us assume that instead of one single point load, there are two point loads P_1 and P_2 spaced at y moving left to right on the beam as shown in Figure 39.3. We are interested to find maximum moment in the beam at given section C.

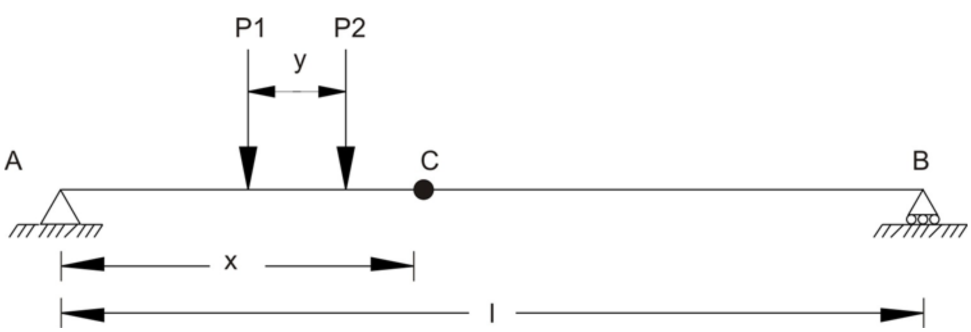


Figure 39.3: Beam loaded with two concentrated loads

With reference to Figure 39.4, moment can be obtained when load P_2 will be on section C. The moment for this case is expressed as:

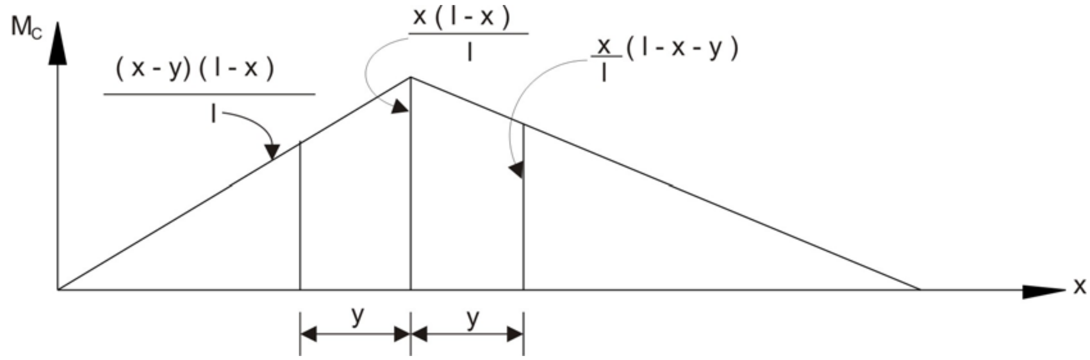


Figure 39.4: Influence line for moment at section C

$$M_C^{\ 1} = P_1(x-y)\left(\frac{l-x}{l}\right) + P_2 x\left(\frac{l-x}{l}\right)$$

With reference to Figure 39.4, moment can be obtained when load P_1 will be on section C. The moment for this case is expressed as:

$$M_C^{\ 2} = P_1 x\left(\frac{l-x}{l}\right) + P_2 x\left(\frac{l-x-y}{l}\right)$$

From above two cases, maximum value of moment should be considered for maximum moment at section C when two point loads are moving from left end to right end of the beam.

39.4 Maximum end shear in a beam supporting a series of moving concentrated loads

In real life situation, usually there are more than two point loads, which will be moving on bridges. Hence, in this case, our aim is to learn, how to find end shear in beam supporting a series of moving concentrated loads. Let us assume that as shown in Figure 39.5, four concentrated loads are moving from right end to left end on beam AB. The spacing of the concentrated load is given in Figure 39.5.

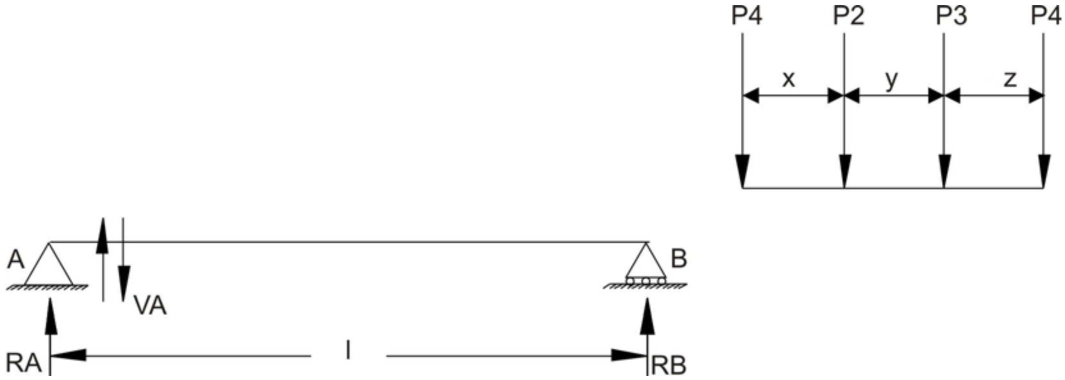


Figure 39.5: Beam loaded with a series of loads

As shown in figure, we are interested in end shear at A. We need to draw influence line for the support reaction A and a point away from the support at infinitesimal distance on the span for the shear V_A. The influence lines for these cases are shown in Figure 39.6 and 39.7.

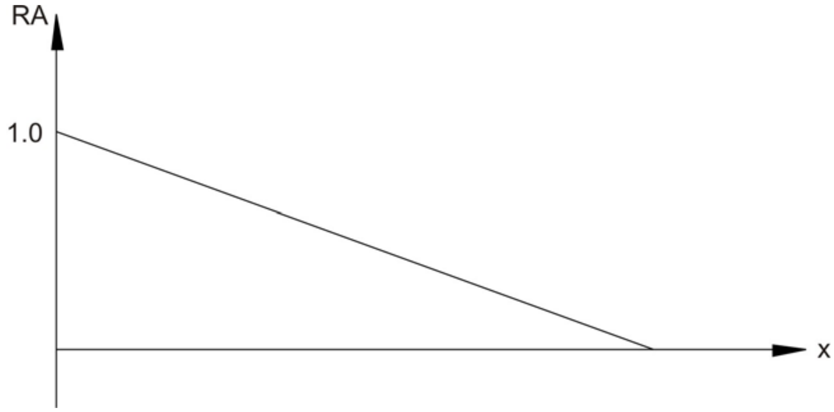


Figure 39.6: Influence line for reaction at support A

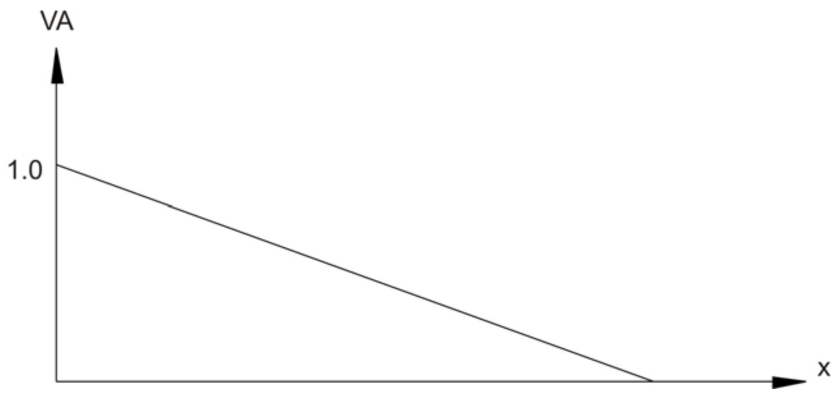


Figure 39.7: Influence line for shear near to support A.

When loads are moving from B to A then as they move closer to A, the shear value will increase. When load passes the support, there could be increase or decrease in shear value depending upon the next point load approaching support A. Using this simple logical approach, we will find out the change in shear value near support and monitor this change from positive value to negative value. Here for the present case let us assume that ΣP is summation of the loads remaining on the beam. When load P_1 crosses support A, then P_2 will approach A. In that case, change in shear will be expressed as

$$dV = \frac{\sum Px}{l} - P_1$$

When load P_2 crosses support A, then P_3 will approach A. In that case change in shear will be expressed as

$$dV = \frac{\sum Py}{l} - P_2$$

In case if *dV* is positive then shear at A has increased and if *dV* is negative, then shear at A has decreased. Therefore, first load, which crosses and induces negative changes in shear, should be placed on support A.

39.4.1 Numerical Example

Compute maximum end shear for the given beam loaded with moving loads as shown in Figure 39.8.

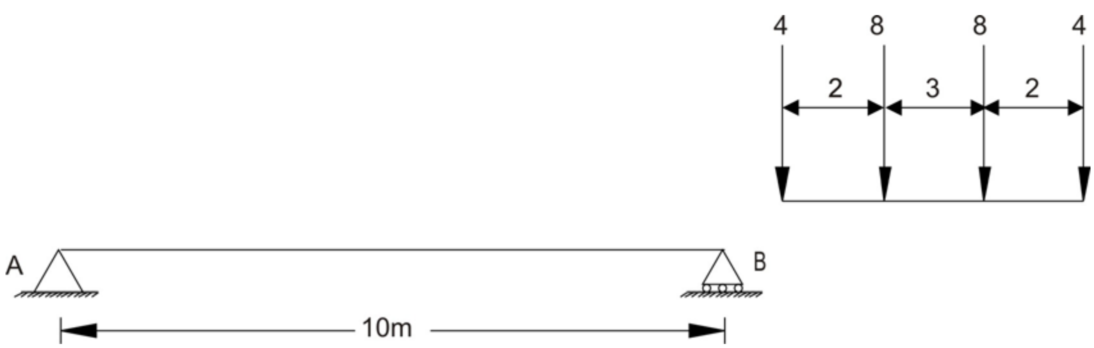


Figure 39.8: Beam loaded with a series of four concentrated loads

When first load of 4 kN crosses support A and second load 8 kN is approaching support A, then change in shear can be given by

$$dV = \frac{\sum (8+8+4)2}{10} - 4 = 0$$

When second load of 8 kN crosses support A and third load 8 kN is approaching support A, then change in shear can be given by

$$dV = \frac{\sum (8+4)3}{10} - 8 = -3.8$$

Hence, as discussed earlier, the second load 8 kN has to be placed on support A to find out maximum end shear (refer Figure 39.9).

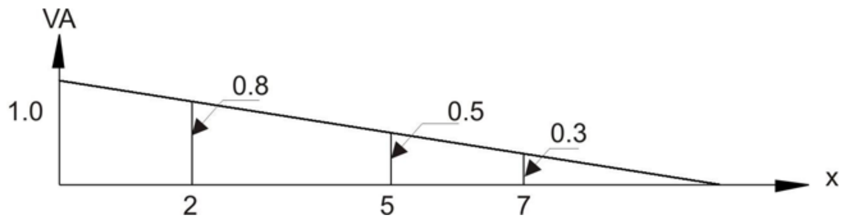


Figure 39.9: Influence line for shear at A.

$$V_A = 4\times 1 + 8\times 0.8 + 8\times 0.5 + 4\times 0.3 = 15.6 kN$$

39.5 Maximum shear at a section in a beam supporting a series of moving concentrated loads

In this section we will discuss about the case, when a series of concentrated loads are moving on beam and we are interested to find maximum shear at a section. Let us assume that series of loads are moving from right end to left end as shown in Figure. 39.10.

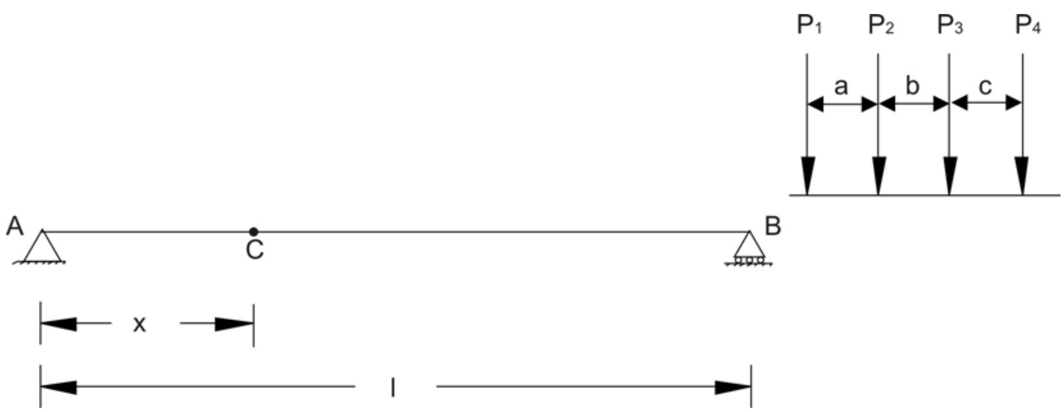


Figure 39.10: Beam loaded with a series of loads

The influence line for shear at the section is shown in Figure 39.11.

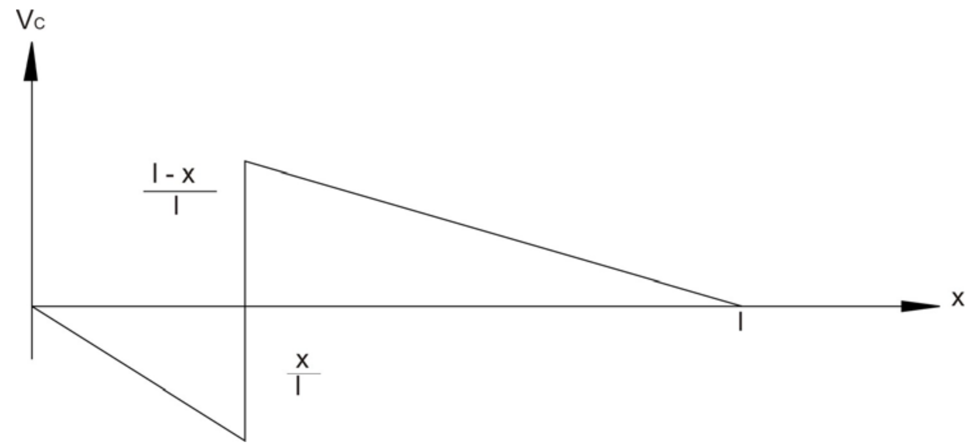


Figure 39.11: Influence line for shear at section C

Monitor the sign of change in shear at the section from positive to negative and apply the concept discussed in earlier section. Following numerical example explains the same.

39.5.1 Numerical Example

The beam is loaded with concentrated loads, which are moving from right to left as shown in Figure 39.12. Compute the maximum shear at the section C.

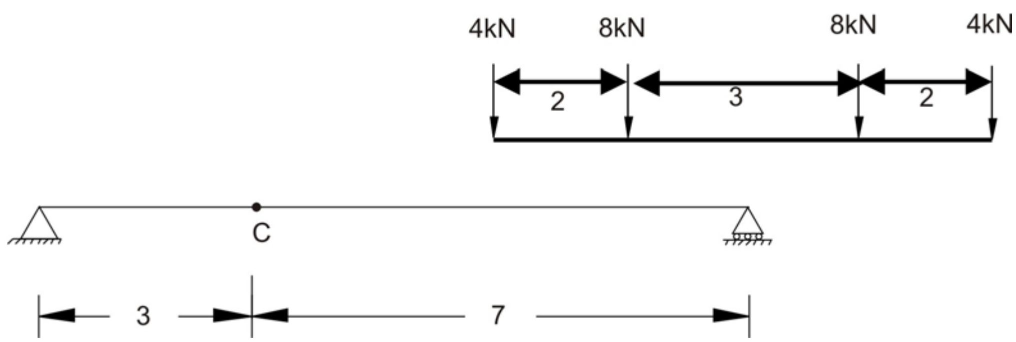


Figure 39.12: Beam loaded with a series of loads

The influence line at section C is shown in following Figure 39.13.

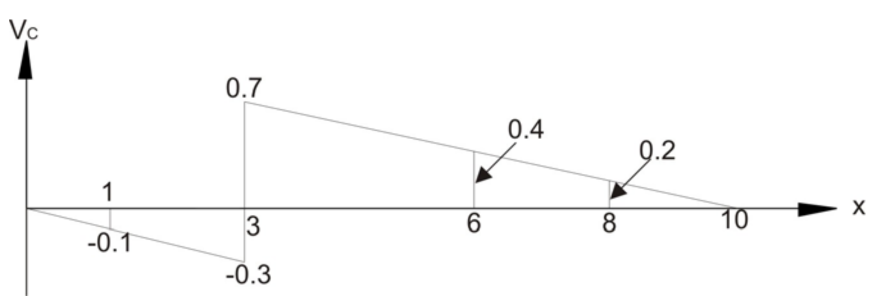


Figure 39.13: Influence line for shear at section C

When first load 4kN crosses section C and second load approaches section C then change in shear at a section can be given by

$$dV = \frac{20 \times 2}{10} - 4 = 0$$

When second load 8 kN crosses section C and third load approaches section C then change in shear at section can be given by

$$dV = \frac{12 \times 3}{10} - 8 = -4.4$$

Hence place the second concentrated load at the section and computed shear at a section is given by

$$V_C == 0.1 \times 4 + 0.7 \times 8 + 0.4 \times 8 + 0.2 \times 4 = 9.2kN$$

39.6 Maximum Moment at a section in a beam supporting a series of moving concentrated loads

The approach that we discussed earlier can be applied in the present context also to determine the maximum positive moment for the beam supporting a series of moving concentrated loads. The change in moment for a load P_1 that moves from position x_1 to x_2 over a beam can be obtained by multiplying P_1 by the change in ordinate of the influence line i.e. $(y_2 - y_1)$. Let us assume the slope of the influence line (Figure 39.14) is S, then $(y_2 - y_1) = S\,(x_2 - x_1)$.

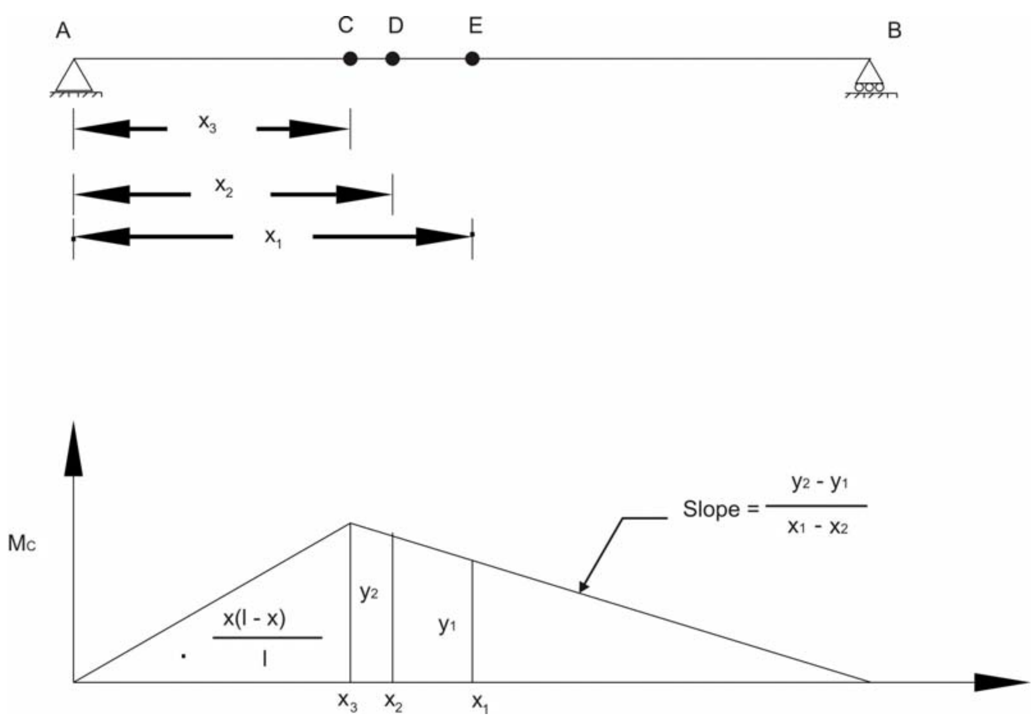


Figure 39.14: Beam and Influence line for moment at section C

Hence change in moment can be given by

$$dM = P_1 S(x_2 - x_1)$$

Let us consider the numerical example for better understanding of the developed concept.

39.6.1 Numerical Example

The beam is loaded with concentrated loads, which are moving from right to left as shown in Figure 39.15. Compute the maximum moment at the section C.

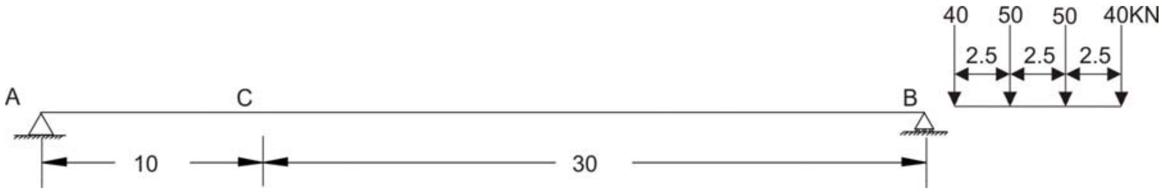


Figure 39.15: Beam loaded with a series of loads

The influence line for moment at C is shown in Figure 39.16.

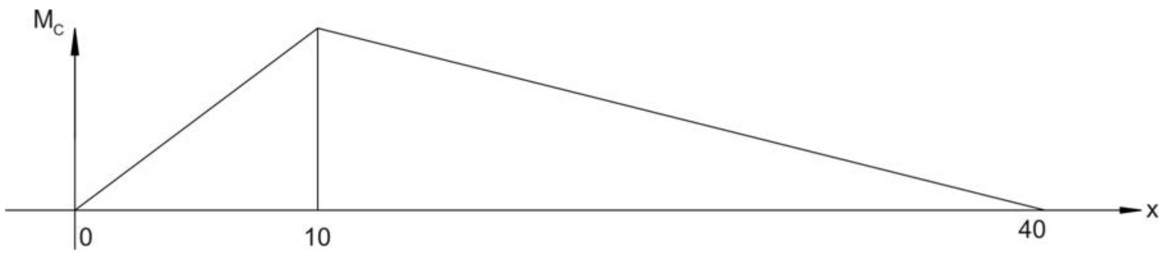


Figure 39.16: Beam loaded with a series of loads

If we place each of the four-concentrated loads at the peak of influence line, then we can get the largest influence from each force. All the four cases are shown in Figures 39.17-20.

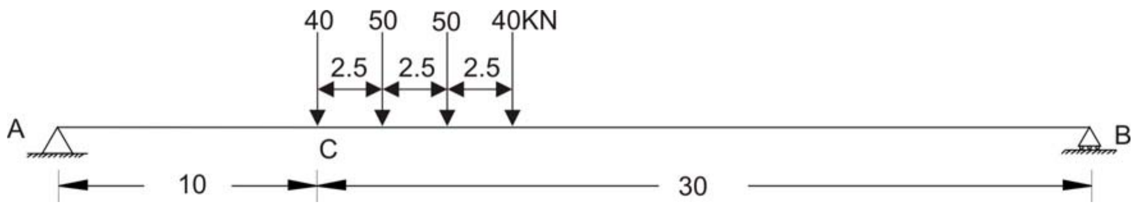


Figure 39.17: Beam loaded with a series of loads – First load at section C

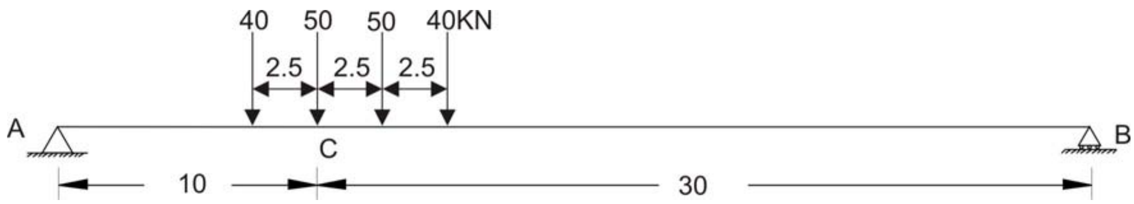


Figure 39.18: Beam loaded with a series of loads – Second load at section C

Figure 39.19: Beam loaded with a series of loads - – Third load at section C

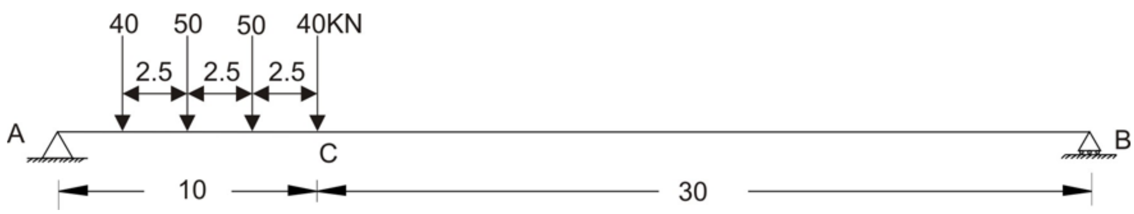


Figure 39.20: Beam loaded with a series of loads - – Third load at section C

As shown in Figure 39.17, when the first load crosses the section C, it is observed that the slope is downward (7.5/10). For other loads, the slope is upward (7.5/(40-10)). When the first load 40 kN crosses the section and second load 50 kN is approaching section (Figure 39.17) then change in moment is given by

$$dM = -40\left(\frac{7.5}{10}\right)2.5 + (50+50+40)\left(\frac{7.5}{(40-10)}\right)2.5 = 12.5 kN.m$$

When the second load 50 kN crosses the section and third load 50 kN is approaching section (Figure 39.18) then change in moment is given by

$$dM = -(40+50)\left(\frac{7.5}{10}\right)2.5 + (50+40)\left(\frac{7.5}{(40-10)}\right)2.5 = -112.5 kN.m$$

At this stage, we find negative change in moment; hence place second load at the section and maximum moment (refer Figure 39.21) will be given by

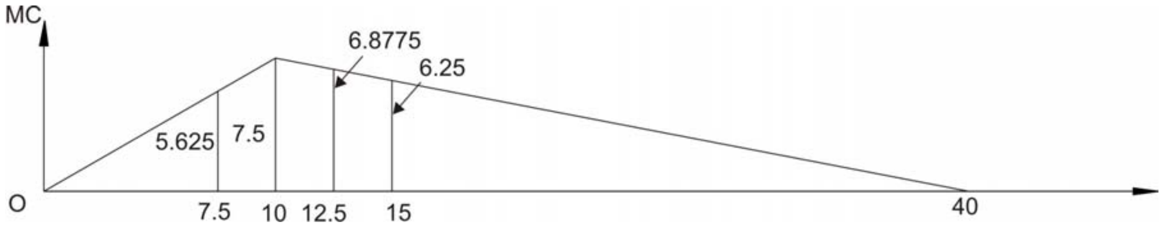


Figure 39.21: Influence line for moment at C

$$M_c = 40(5.625) + 50(7.5) + 50(6.8775) + 40(6.25) = 1193.87 kNm$$

39.7 Absolute maximum moment in s beam supporting a series of moving concentrated loads.

In earlier sections, we have learned to compute the maximum shear and moment for single load, UDL and series of concentrated loads at specified locations. However, from design point of view it is necessary to know the critical location of the point in the beam and the position of the loading on the beam to find maximum shear and moment induced by the loads. Following paragraph explains briefly for the cantilever beam or simply supported beam so that quickly maximum shear and moment can be obtained.

Maximum Shear: As shown in the Figure 39.22, for the cantilever beam, absolute maximum shear will occur at a point located very near to fixed end of the beam. After placing the load as close as to fixed support, find the shear at the section close to fixed end.

Figure 39.22: Absolute maximum shear case – cantilever beam

Similarly for the simply supported beam, as shown in Figure 39.23, the absolute maximum shear will occur when one of the loads is placed very close to support.

Figure 39.23: Absolute maximum shear – simply supported beam

Moment:

The absolute maximum bending moment in case of cantilever beam will occur where the maximum shear has occurred, but the loading position will be at the free end as shown in Figure 39.24.

Figure 39.24: Absolute maximum moment – cantilever beam

The absolute maximum bending moment in the case of simply supported beam, one cannot obtain by direct inspection. However, we can identify position analytically. In this regard, we need to prove an important proposition.

Proposition:

When a series of wheel loads crosses a beam, simply supported ends, the maximum bending moment under any given wheel occurs when its axis and the center of gravity of the load system on span are equidistant from the center of the span.

Let us assume that load P_1, $P_{2,}$ P_3 etc. are spaced shown in Figure 39.25 and traveling from left to right. Assume P_R to be resultant of the loads, which are on the beam, located in such way that it nearer to P_3 at a distance of d_1 as shown in Figure 39.25.

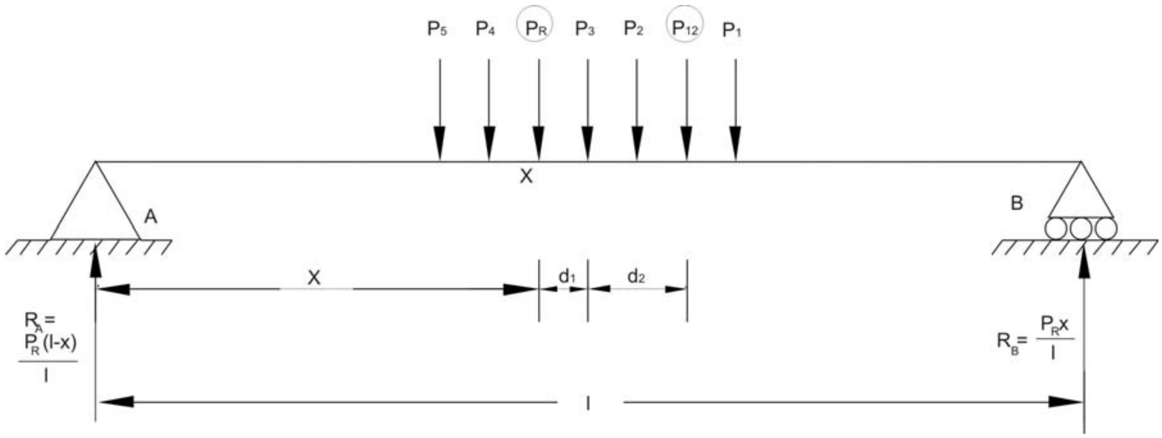


Figure 39.25: Absolute maximum moment case – simply supported beam

If P_{12} is resultant of P_1 and P_2, and distance from P_3 is d_2. Our objective is to find the maximum bending moment under load P_3. The bending moment under P_3 is expressed as

$$M = \frac{P_R x}{l}(l - x - d_1) - P_{12}(d_2)$$

Differentiate the above expression with respect to x for finding out maximum moment.

$$\frac{dM}{dx} = \frac{P_R}{l}(l - 2x - d_1) = 0 \Rightarrow l - 2x + d_1 = 0 \Rightarrow x = \frac{l}{2} - \frac{d_1}{2}$$

Above expression proves the proposition.

Let us have a look to some examples for better understanding of the above-derived proposition.

39.7.1 Numerical Examples

Example 1:

The beam is loaded with two loads 25 kN each spaced at 2.5 m is traveling on the beam having span of 10 m. Find the absolute maximum moment

Solution:

When the a load of 25kN and center of gravity of loads are equidistant from the center of span then absolute bending moment will occur. Hence, place the load on the beam as shown in Figure 39.26.

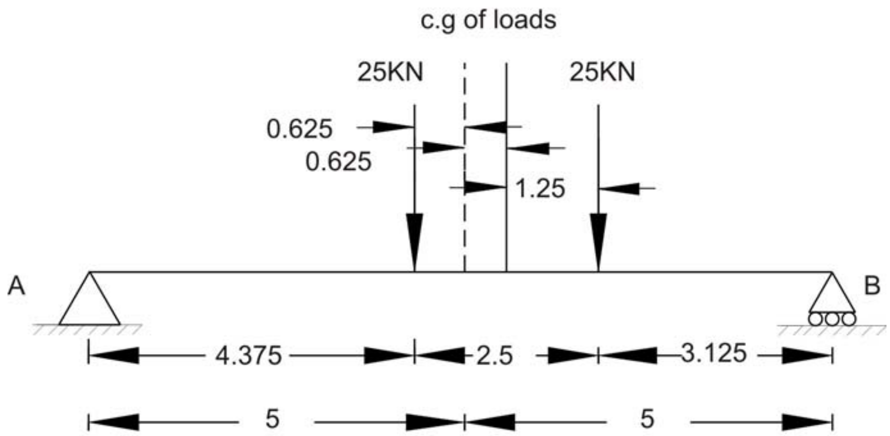


Figure 39.26: Simply supported beam (Example 1)

The influence line for M_x is shown in Figure 39.27

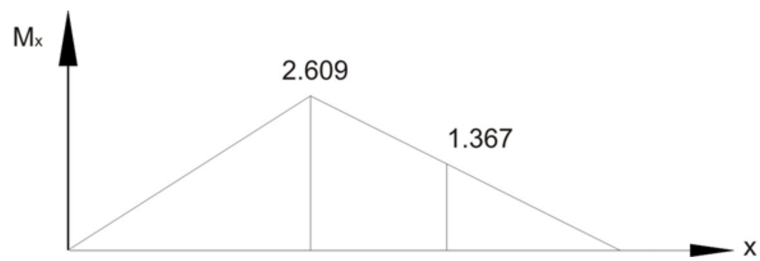


Figure 39.27: Influence line for moment at X (Example 1)

Computation of absolute maximum moment is given below.

$$M_x = 25(2.461) + 25(1.367) = 95.70 kN.m$$

Example 2:

Compute the absolute maximum bending moment for the beam having span of 30 m and loaded with a series of concentrated loads moving across the span as shown in Figure 39.28.

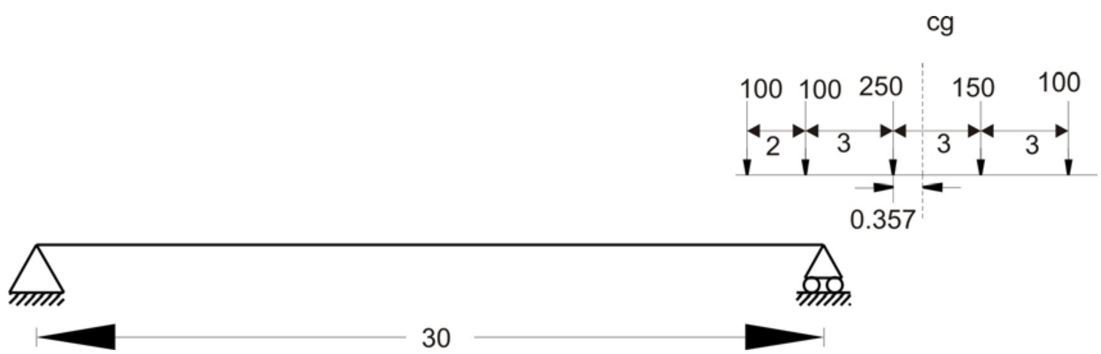


Figure 39.28: Simply supported beam (Example 2)

First of all compute the center of gravity of loads from first point load of 100 kN

$$= \frac{100(2)+250(5)+150(8)+100(11)}{100+100+250+150+100} = \frac{3750}{700} = 5.357m$$

Now place the loads as shown in Figure 39.29.

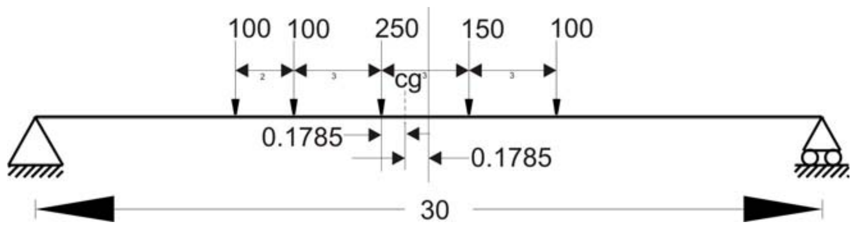


Figure 39.29: Simply supported beam with load positions (Example 2)

Also, draw the influence line as shown in Figure 39.30 for the section X.

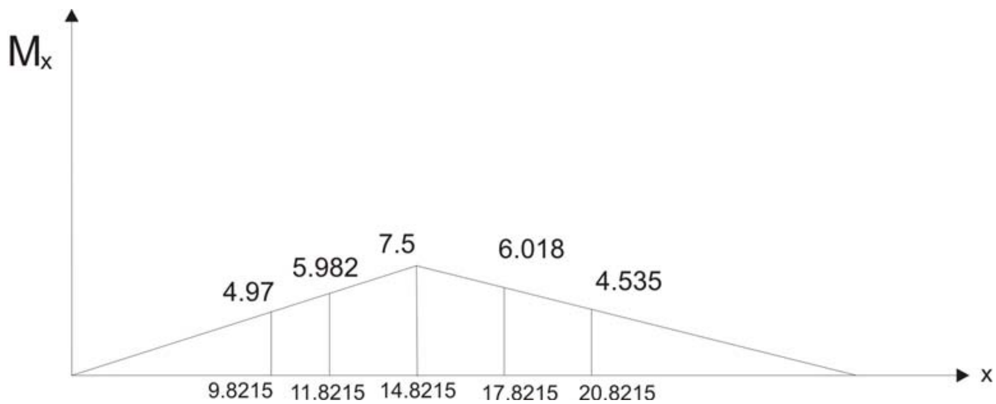


Figure 39.30: Influence Line for moment at section X (Example 2)

$$M_x = 100(4.97)+100(5.982)+250(7.5)+150(6.018)+100(4.535) = 4326.4kN.m$$

39.8 Envelopes of maximum influence line values

For easy calculations steps of absolute maximum shear and moment rules for cantilever beam and simply supported beam were discussed in previous section. Nevertheless, it is difficult to formulate such rules for other situations. In such situations, the simple approach is that develop the influence lines for shear and moment at different points along the entire length of the beam. The values easily can be obtained using the concepts developed in earlier sections. After obtaining the values, plot the influence lines for each point under consideration in one plot and the outcome will be envelop of maximums. From this diagram, both the absolute maximum value of shear and moment and location can be obtained. However, the approach is simple but demands tedious calculations for each point. In that case, these calculations easily can be done using computers.

39.9 Closing Remarks

In this lesson, we have learned various aspects of constructing influence lines for the cases when the moving concentrated loads are two or more than two. Also, we developed simple concept of finding out absolute maximum shear and moment values in cases of cantilever beam and simply supported beam. Finally, we discussed about the need of envelopes of maximum influence line values for design purpose.

Suggested Text Books for Further Reading

- Armenakas, A. E. (1988). *Classical Structural Analysis – A Modern Approach*, McGraw-Hill Book Company, NY, ISBN 0-07-100120-4

- Hibbeler, R. C. (2002). *Structural Analysis*, Pearson Education (Singapore) Pte. Ltd., Delhi, ISBN 81-7808-750-2

- Junarkar, S. B. and Shah, H. J. (1999). *Mechanics of Structures – Vol. II*, Charotar Publishing House, Anand.

- Leet, K. M. and Uang, C-M. (2003). *Fundamentals of Structural Analysis*, Tata McGraw-Hill Publishing Company Limited, New Delhi, ISBN 0-07-058208-4

- Negi, L. S. and Jangid, R.S. (2003). *Structural Analysis*, Tata McGraw-Hill Publishing Company Limited, New Delhi, ISBN 0-07-462304-4

- Norris, C. H., Wilbur, J. B. and Utku, S. (1991). *Elementary Structural Analysis*, Tata McGraw-Hill Publishing Company Limited, New Delhi, ISBN 0-07-058116-9

Module 7

Influence Lines

Lesson 40

Influence Lines for Simple Trusses

Instructional Objectives:

The objectives of this lesson are as follows.

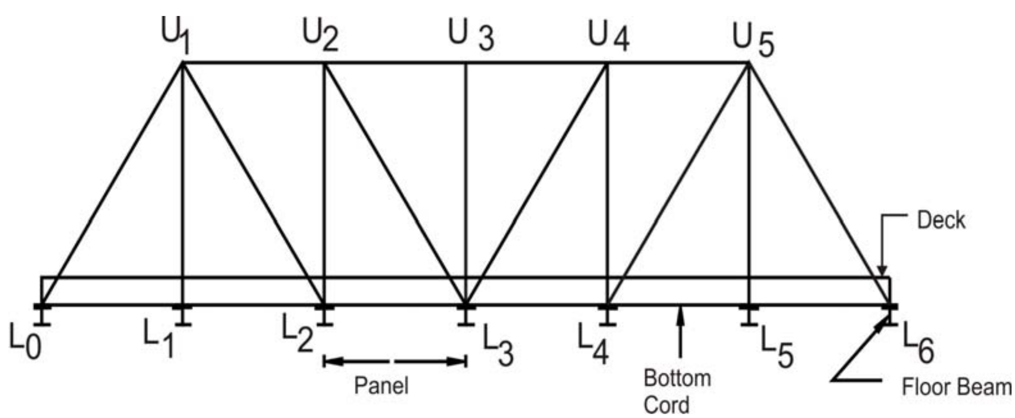
- Understand the bridge truss floor system and load transfer mechanism
- Draw the influence line for the truss reactions
- Draw the influence line for the truss member forces

40.1 Introduction

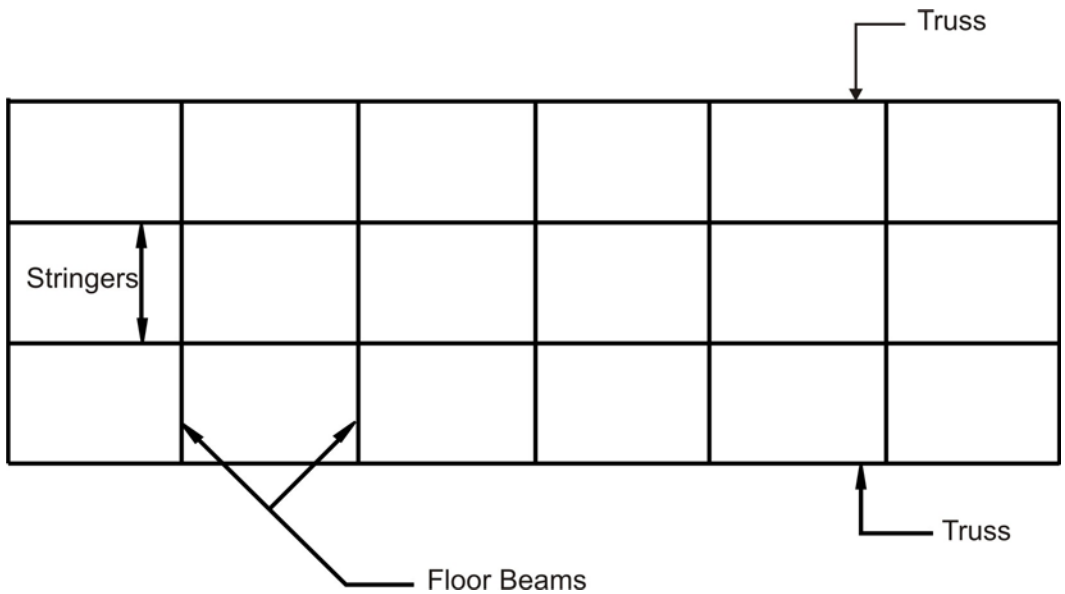
In previous lessons, we have studied the development of influence lines for beams loaded with single point load, UDL and a series of loads. In similar fashion, one can construct the influence lines for the trusses. The moving loads are never carried directly on the main girder but are transmitted across cross girders to the joints of bottom chord. Following section will explain load transmission to the trusses followed by the influence lines for the truss reactions and influence lines for truss member forces.

40.2 Bridge Truss Floor System

A typical bridge floor system is shown in Figure 40.1. As shown in Figure, the loading on bridge deck is transferred to stringers. These stringers in turn transfer the load to floor beams and then to the joints along the bottom chord of the truss.



Front view



Floor plan

Figure 40.1 Bridge floor system

It should be noted that for any load position; the truss is always loaded at the joint.

40.3 Influence lines for truss support reaction

Influence line for truss reactions are of similar to that a simply supported beam. Let us assume that there is truss with overhang on both ends as shown in Figure 40.2. In this case, the loads to truss joints are applied through floor beams as discussed earlier. These influence lines are useful to find out the support, which will be critical in terms of maximum loading.

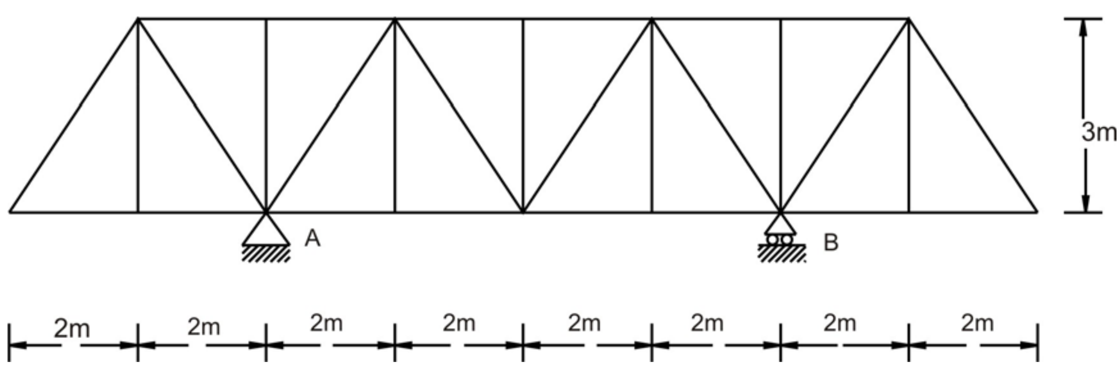
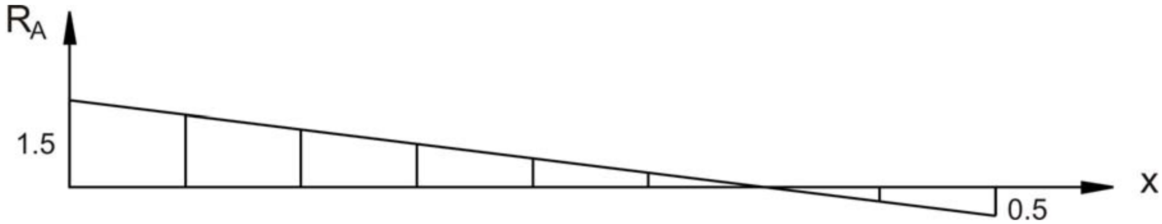
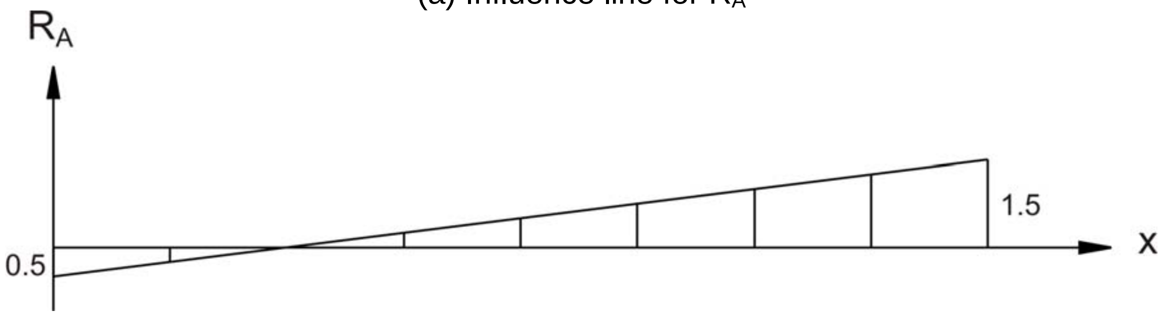


Figure 40.2 Bridge truss

The influence lines for truss reactions at A and B are shown in Figure 40.3.



(a) Influence line for R_A



(b) Influence line for R_B

Figure 40.3: Influence lines for support reactions

40.4 Influence lines for truss member forces

Influence lines for truss member force can be obtained very easily. Obtain the ordinate values of influence line for a member by loading each joint along the deck with a unit load and find member force. The member force can be found out using the method of joints or method of sections. The data is prepared in tabular form and plotted for a specific truss member force. The truss member carries axial loads. In the present discussion, tensile force nature is considered as positive and compressive force nature is considered as negative.

40.4.1 Numerical Examples

Example 1:

Construct the influence line for the force in member GB of the bridge truss shown in Figure 40.4.

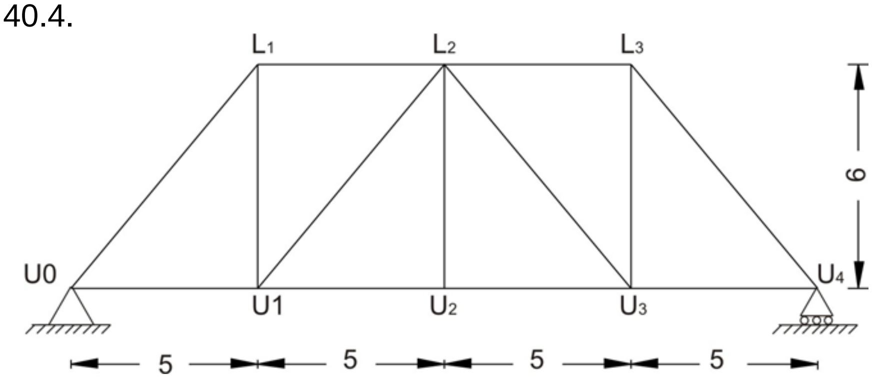


Figure 40.4: Bridge Truss (Example 1)

Solution:

Tabulated Values:

In this case, successive joints L_0, L_1, L_2, L_3, and L_4 are loaded with a unit load and the force F_{L2U3} in the member L_2U_3 are using the method of sections. Figure 40.5 shows a case where the joint load is applied at L_1 and force F_{L2U3} is calculated.

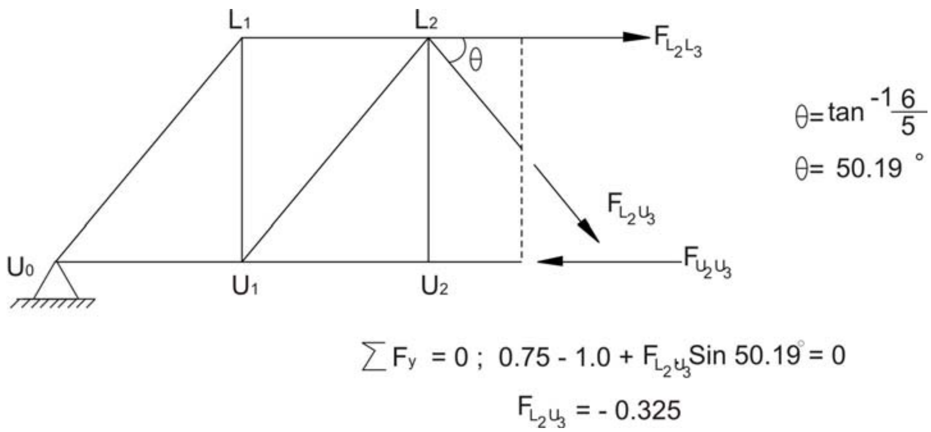


Figure 40.5: Member Force F_{L2U3} Calculation using method of sections.

The computed values are given below.

x	F_{L2U3}
0	0
5	-0.325
10	-0.650
15	0.325
20	0

Influence line: Let us plot the tabular data and connected points will give the influence line for member L_2U_3. The influence line is shown in Figure 40.6. The figure shows the behaviour of the member under moving load. Similarly other influence line diagrams can be generated for the other members to find the critical axial forces in the member.

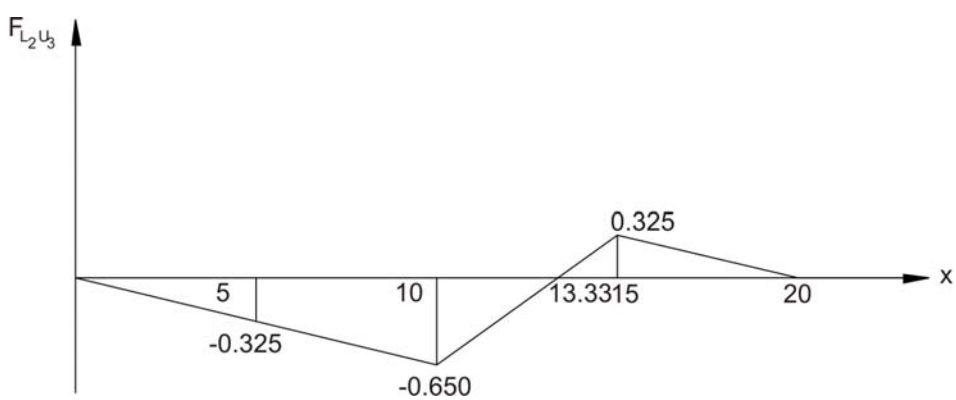


Figure 40.6: Influence line for member force F_{L2U3}

Example 2:

Tabulate the influence line values for all the members of the bridge truss shown in Figure 40.7.

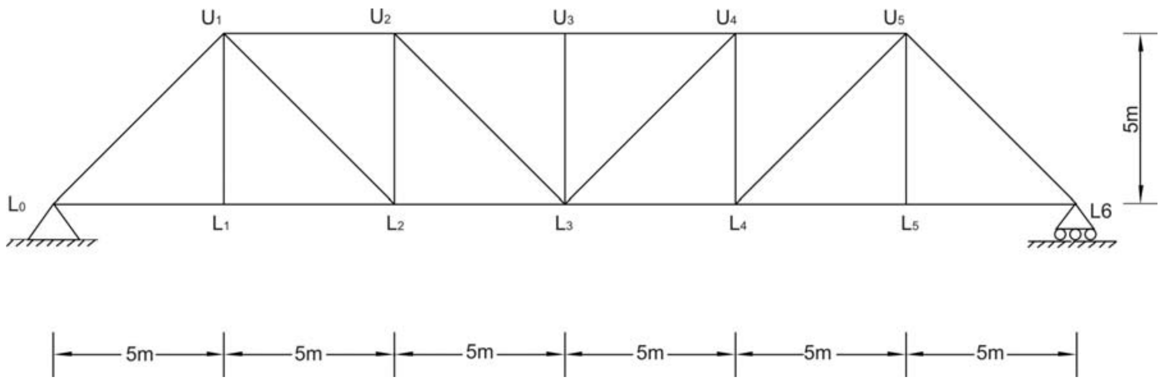


Figure 40.7: Bridge Truss (Example 2)

Solution:

Tabulate Values:

Here objective is to construct the influence line for all the members of the bridge truss, hence it is necessary to place a unit load at each lower joints and find the forces in the members. Typical cases where the unit load is applied at L_1, L_2 and L_3 are shown in Figures 40.8-10 and forces in the members are computed using method of joints and are tabulated below.

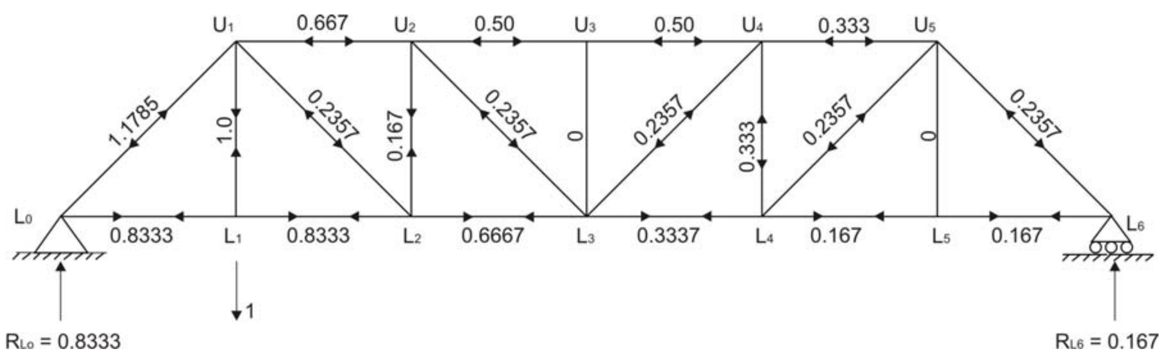


Figure 40.8: Member forces calculation when unit load is applied at L_1

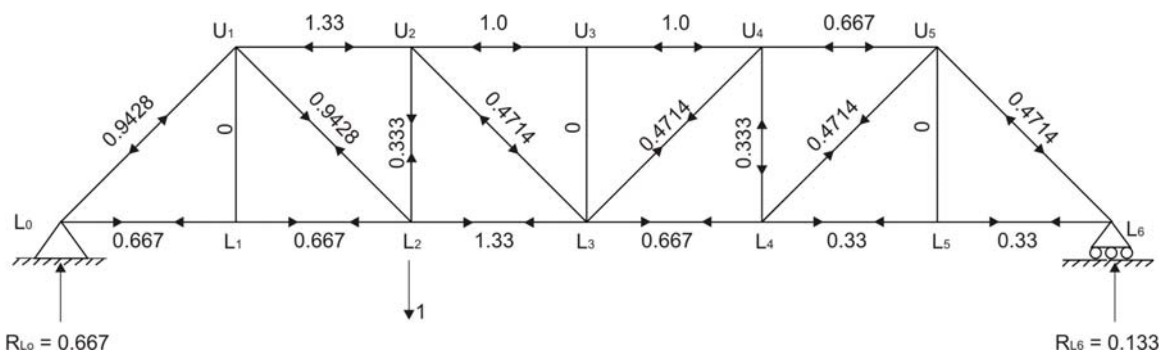


Figure 40.9: Member forces calculation when unit load is applied at L_2

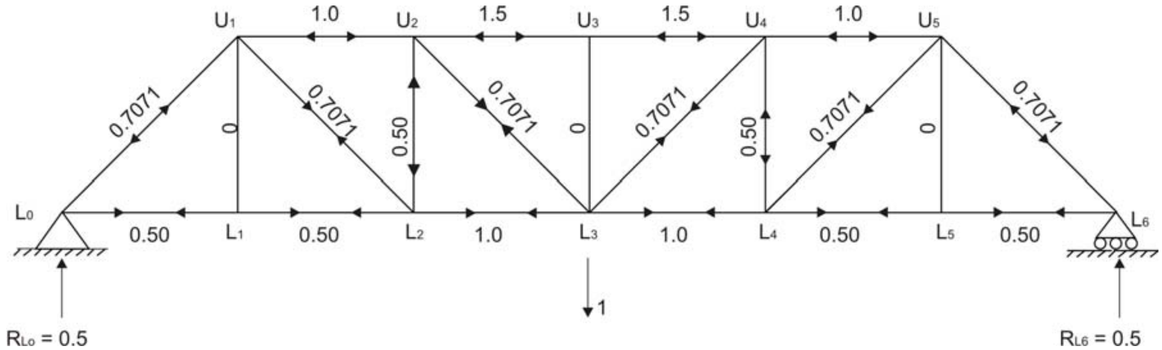


Figure 40.10: Member forces calculation when unit load is applied at L_3

Member	Member force due to unit load at:						
	L_0	L_1	L_2	L_3	L_4	L_5	L_6
L_0L_1	0	0.8333	0.6667	0.5	0.3333	0.1678	0
L_1L_2	0	0.8333	0.6667	0.5	0.3333	0.1678	0
L_2L_3	0	0.6667	1.3333	1.0	0.6667	0.3336	0
L_3L_4	0	0.3336	0.6667	1.0	1.3333	0.6667	0
L_4L_5	0	0.1678	0.3333	0.5	0.6667	0.8333	0
L_5L_6	0	0.1678	0.3333	0.5	0.6667	0.8333	0
U_1U_2	0	-0.6667	-1.333	-1.0	-0.6667	-0.333	0
U_2U_3	0	-0.50	-1.000	-1.5	-1.0	-0.50	0
U_3U_4	0	-0.50	-1.000	-1,5	-1.0	-0.50	0
U_4U_5	0	-0.333	-0.6667	-1.0	-1.333	-0.6667	0
L_0U_1	0	-1.1785	-0.9428	-0.7071	-0.4714	-0.2357	0
L_1U_1	0	1	0	0	0	0	0
L_2U_1	0	-0.2357	0.9428	0.7071	0.4714	0.2357	0
L_2U_2	0	0.167	0.3333	-0.50	-0.3333	-0.3333	0
L_3U_2	0	-0.2357	-0.4714	0.7071	0.4714	0.2357	0
L_3U_3	0	0	0	0	0	0	0
L_3U_4	0	0.2357	0.4714	0.7071	-0.4714	-0.2357	0
L_4U_4	0	-03333	-0.3333	-0.50	0.3333	0.167	0
L_4U_5	0	0.2357	0.4714	0.7071	0.9428	-0.2357	0
L_5U_5	0	0	0	0	0	1	0
L_6U_5	0	-0.2357	-0.4714	-0.7071	-0.9428	-1.1785	0

Influence lines:

Using the values obtained in the above given table, the influence line can be plotted very easily for truss members.

40.5 Closing Remarks

In this lesson we have studied how the loads are transferred in bridge truss floor system. Further, we found that there is similarity between the influence line of support reactions for simply supported beam and truss structures. Finally we studied the influence line for truss member forces. It was essential to know the method of sections and method of joints for the analysis of trusses while drawing influence lines.

Suggested Text Books for Further Reading

- Armenakas, A. E. (1988). *Classical Structural Analysis – A Modern Approach*, McGraw-Hill Book Company, NY, ISBN 0-07-100120-4

- Hibbeler, R. C. (2002). *Structural Analysis*, Pearson Education (Singapore) Pte. Ltd., Delhi, ISBN 81-7808-750-2

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