ADDIS ABABA INSTITUTE OF TECHNOLOGY

SCHOOL OF CIVIL AND ENVIRONMENTAL ENGINEERING

**CENG 2102 - Theory of structures I**

# COURSE OUTLINE

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    - 1.2 Stability of Structures
    - 1.3 Determinacy of Structures
2. **Loads on Structures**
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    - 2.2. Live Load
    - 2.3. Environmental Loads (wind loads, Earthquake forces, …)
    - 2.4. Hydrostatic and Soil Pressures
    - 2.5. Load Combinations
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    - 3.2 IL for paneled girders
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5. **Consistent Deformation Method**
    - 5.1 Introduction
    - 5.2 Analysis of Indeterminate beams
    - 5.3 Analysis of Indeterminate Trusses

## References


- A. Kassimali, Structural Analysis, ,4th Edition, SI, Indian Institute of Technology, Kanpur
- Harry H. West and L.F. Geshwindner, Fundamentals of Structural Analysis 2nd ed. 2002
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- Popov, E.P., Mechanics of Materials
- ES EN 1991 1-1:2015 (Actions on structures Part 1:1 General actions -Densities, self-weight, imposed loads for buildings)
- ES EN 1991 1-4:2015 (Actions on structures Part 1:1 General actions –Wind actions)
- ES EN1998 1-1:2015 (Design of structures for Earthquake Resistance - Part 1: General rules, seismic actions and rules for buildings)

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# CHAPTER 1

## 1. Stability & Determinacy of Structures

### 1.1 Introduction

A structure refers to a system of connected parts used to support loads. The fundamental purpose of a structure is to transmit loads from the point of application to the point of support and through the foundations to the ground.

Before going into the analysis of any structure, it is necessary to identify its statical type (classification), i.e., whether it is determinate or indeterminate, stable or unstable. An unstable arrangement of supports and structural members should be avoided.

All structures are subjected to loads from their functions and to other unavoidable loads. Establishment of the loads that act on a structure is one of the most difficult and yet important steps in the design process.

In this chapter; criteria for statically classification will be established and different structures will be checked for stability and determinacy.

### 1.2 Stability of Structures

A stable structure is the one, which remains stable and can support any conceivable (imaginable) system of applied loads. Therefore, we do not consider the types of loads, their number and their points of application for deciding the stability or determinacy of the structure. Normally internal and external stability of a structure should be checked separately and if it's overall stable then total degree of indeterminacy should be checked.

To ensure the equilibrium of a structure or its members, it is not only necessary to satisfy the equations of equilibrium, but the members must also be properly held or, constrained by their supports. In structural analysis, a structure is said to be stable when it can support any possible system of applied loads.

A structure in which there are insufficient numbers of reactions to prevent motion from taking place is called an unstable structure. This is external instability.

What matters is not only the number of support reactions but also their arrangement. Structures for which the numbers of reaction components are greater than or equal to the number of available equilibrium equations but that are unstable due to arrangement of these reaction components are said to be geometrically unstable.

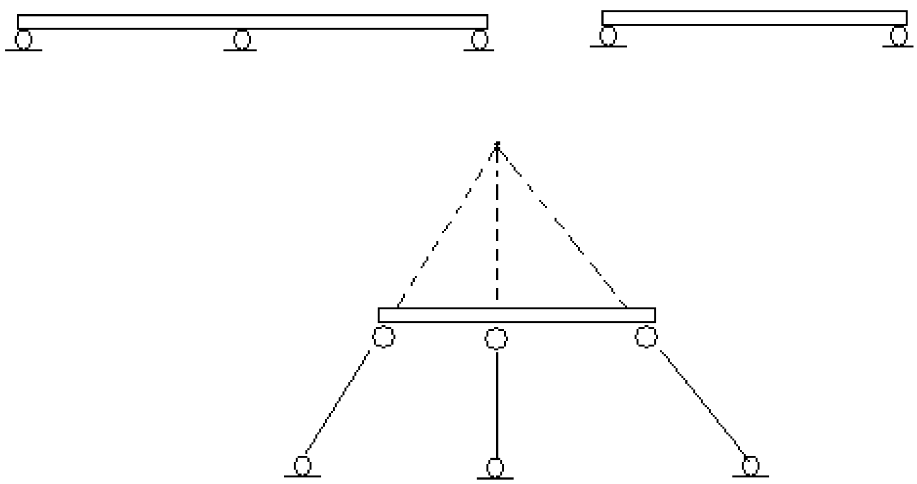
When the reaction elements are three or more like supports that are either parallel or concurrent, they are not sufficient to maintain static equilibrium.

For the case of parallel reactions, they will offer no resistance to horizontal motion, thus making the arrangement unstable. The point of intersection of the concurrent reactions becomes an instantaneous center of rotation and the system is instantaneously unstable.

The stable fundamental element of a plane truss is a triangular arrangement of three members. A truss may have internal instability if four members are used to form an element.

In conclusion, the stability of structures depends on the number and geometric arrangement of reactions and structural members rather than on the strength of individual member or supports. Despite the possibility that an unstable structure could become stable under a particular system of applied loads, the structure is classified as an unstable structure.

A stable structure should have at least three reactive components, (which may not always be sufficient) for external stability of a 2-D structure, which are non-concurrent and non-parallel.



### 1.3 Determinacy of Structures

When all forces in a structure can be determined strictly from equilibrium equations, the structure is referred to as statically determinate. Structures having more unknown forces than available equilibrium equations are called statically indeterminate.

A statically indeterminate structure is one that cannot be analyzed by the equations of static equilibrium alone. Indeterminacy is introduced in structures on account of functional requirements, limitations on types of framing, need for stiffness and often by the nature of inherent continuity introduced by the type of material used like reinforced concrete.

A structure is statically indeterminate when it possesses more members or is supported by more reactive restraints than are strictly necessary for stability (and equilibrium). The excess members or restraints are called redundant. The degree of indeterminacy is the number of unknowns in excess of the available equilibrium equations. In the analysis of indeterminate structures, therefore, ways of establishing additional equations must be sought. These additional equations may be derived from compatibility of deformation or from conditions of symmetry. This additional task would make the analysis of indeterminate structures more difficult than their determinate counterparts.

Indeterminate structures have some advantages and disadvantages over determinate ones. One obvious disadvantage is the computational difficulty involved when establishing the required additional equations. Another disadvantage is that indeterminate structures will be stressed due to differential settlement of supports, temperature changes and errors in fabrication of members. On the other hand, however, indeterminate structures are stiffer and in the case of over loads indeterminate structures can provide an advantage of redistribution of loads within the structure.

The indeterminacy of a structure can be external (with respect to reactions) or internal (with respect to member forces). The question of identifying external or internal indeterminacy is largely of academic interest. What is of primary importance is the total degree of indeterminacy, Nevertheless, determining external and internal indeterminacy is desirable as a method to evaluate the total degree of indeterminacy.

A structure is internally indeterminate when it is not possible to determine all internal forces by using the equations of static equilibrium. For the great majority of structures, the question of whether or not they are indeterminate can be decided by inspection. For certain structures this is not so, and for these types rules have to be established. The internal indeterminacy of trusses will be first considered, and then that of continuous frames.

### 1.4 Criteria for Stability and Determinacy of Structures-Trusses, Beams and Frames

Internal stability of structures and determining which conditions exist in a given case need experience, especially for trusses. In some cases, the structure is different from what our mathematical criteria tell us. Therefore, stability of trusses is most easily settled by inspection.

It is convenient to consider stability and determinacy of structures as follows.

a) With respect to reactions, i.e. external stability and determinacy.

b) With respect to members, i.e. internal stability and determinacy.

c) A combination of external and internal conditions, i.e. overall stability and determinacy.

#### 1.4.1 Beams

A beam is a structural element that is capable of withstanding load primarily by resisting bending. The bending force induced into the material of the beam as a result of the external loads, own weight and external reactions to these loads is called a bending moment. Beams generally carry vertical gravitational forces but can also be used to carry horizontal loads (i.e., loads due to an earthquake or wind).

- Stability depends on external supports
- Determinacy relates on the number of available and conditional equations.
  - $r_a < r$; structure is statically unstable
  - $r_a = r$; structure is statically determinate
  - $r_a > r$; structure is statically indeterminate

where:

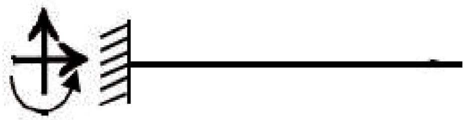
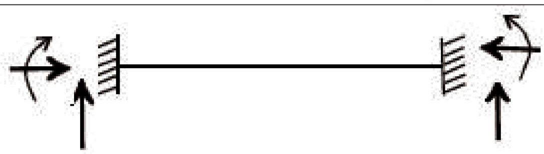
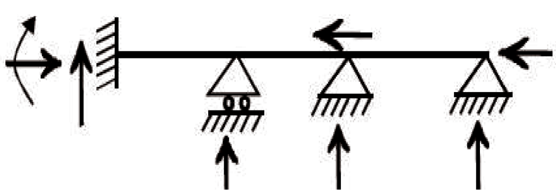
$r_a$ is the available number of reaction components

$r$ is the minimum number of reaction components required for stability, usually 3+n

n is the number of special/ conditional equation

**Remark**: r = 3 is not a sufficient condition for stability

**Example**

| | |
|---|---|
|  | $r_a$ =3 and r=3 → $r_a = r$<br>Stable & determinate |
|  | $r_a$ =5 and r=3 → $r_a > r$<br>Degree of External indeterminacy =2<br>Stable & Indeterminate to 2nd degree. |
|  | $r_a$ =6 and r=3 → $r_a > r$<br>Degree of External indeterminacy =6<br>Stable & Indeterminate to3rd degree. |
|  | $r_a$ =8 and r=3 → $r_a > r$<br>Degree of External indeterminacy =6<br>Stable & Indeterminate to5th degree. |

### 1.4.2 Trusses

A simple truss can be made by combining three bars to form a triangle. Stability depends partly on external supports and partly on the arrangement of members or bars. Three reaction components are required for external stability and determinacy of a plane truss without condition equations.

#### 1.4.2.1 External classification

The external statical classification of the structure depends on the total number of reaction components, $r_a$ and their arrangement. Therefore, the following criteria hold true:

- $r_a < r$; structure is statically unstable externally
- $r_a = r$; structure is statically determinate externally
- $r_a > r$; structure is statically indeterminate externally

where

$r_a$ is the available number of reaction components

$r$ is the minimum number of reaction components required for external stability, usually 3+n

n is the number of special/ conditional equation

The condition for $r_a \geq r$ is necessary but not sufficient conditions for statical classification because the arrangement of the reaction components may render the truss unstable.

### 1.4.2.2 Internal classification

For internal classification, in addition to the above definition for r; let m be the total number of bars and j the total number of joints. Then

$$2j = m + r$$

The above equation can be rewritten as: $m = 2j - r$

In this form, m is the number of members required to form an internally statically determinate truss that connects j joints and has r reaction components required for external stability. If $m_a$ is the actual number of bar forces in the truss, then the following criteria hold true for internal classification

- $m_a < m$; truss is statically unstable internally
- $m_a > m$; truss is statically determinate internally
- $m_a > m$; truss is statically indeterminate internally

Consider the trusses shown below. The truss shown in fig (a) is stable where as the truss shown in fig (b) is unstable since the geometric arrangement of the members is not maintained.

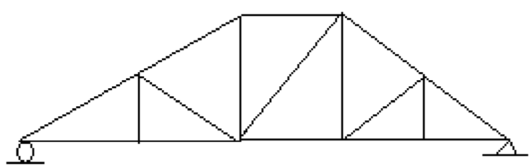


Fig (a)

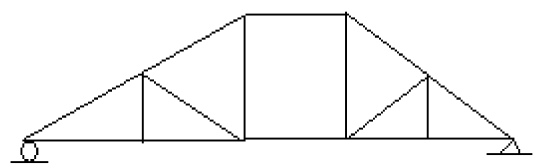
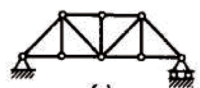
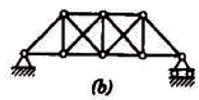
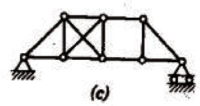
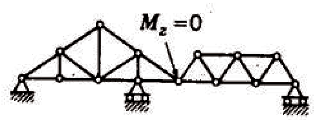
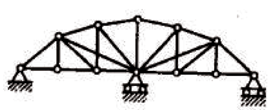
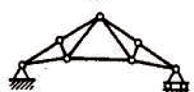
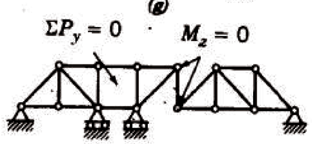


Fig (b)

## Examples

| Structure | Structure Characteristics | | | External Classification | | Internal Classification | |
|---|---|---|---|---|---|---|---|
| | $j$ | $m_a$ | $r_a$ | $r$ | Classification | $m = 2j - r$ | Classification |
|  (a) | 8 | 13 | 3 | 3 | $r_a = r$, deter. stable | $13 = 16 - 3$ | $m_a = m$, deter. stable |
|  (b) | 8 | 15 | 3 | 3 | $r_a = r$, deter. stable | $13 = 16 - 3$ | $m_a > m$, indet. stable |
|  (c) | 8 | 13 | 3 | 3 | $r_a = r$, deter. stable | $13 = 16 - 3$ | $m_a = m$, but unstable |
|  (d) | 7 | 9 | 4 | 3 | $r_a > r$, indet. stable | $11 = 14 - 3$ | $m_a < m$, unstable |
|   (e) | 14 | 24 | 4 | $3 + 1 = 4$ | $r_a = r$, deter. stable | $24 = 28 - 4$ | $m_a = m$, deter. stable |
|  (f) | 12 | 23 | 4 | 3 | $r_a > r$, indet. stable | $21 = 24 - 3$ | $m_a > m$, indet. stable |
|  (g) | 7 | 11 | 3 | 3 | $r_a = r$, deter. stable | $11 = 14 - 3$ | $m_a = m$, deter. stable |
|   (h) | 14 | 22 | 6 | $3 + 3 = 6$ | $r_a = r$, deter. stable | $22 = 28 - 6$ | $m_a = m$, deter. stable |

### 1.4.3 Frames

Frames are composed of continuous members and rigidly connected joints, the degree of indeterminacy (DI) is determined as the difference of the total number of unknown reaction components and the number of static equilibrium equations available. Note that the frame with the hinge has a fourth condition equation, since the bending moment at the hinge must be zero. Stability depends partly on external supports and partly on moment resisting joints.

#### 1.4.3.1 External classification

The external statical classification of the structure depends on the total number of reaction components, $r_a$ and their arrangement. Therefore, the following criteria hold true:

- $r_a < r$; structure is statically unstable externally
- $r_a = r$; structure is statically determinate externally
- $r_a > r$; structure is statically indeterminate externally

where $r_a$ is the available number of reaction components

$r$ is the minimum number of reaction components required for external stability, usually 3+n

n is the number of special/ conditional equation

$r_a \geq r$ is necessary but not sufficient conditions for statical classification because the arrangement of the reaction components may render the frame unstable.

#### 1.4.3.2 Internal classification

$(3\ m_a + r) < (3j + n)$;

Let $m_a$ =the actual number of members

$r$ = the minimum number of independent reaction components required for external stability

j= the total number joints

n= number of special/condition equations

Therefore, 3 $m_a + r$ =the number of unknowns

3j+n=the number of available equations

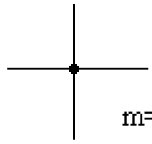
Then the following criteria hold true for internal classification of frames

- $(3m_a + r) < (3j + n)$; structure is statically unstable
- $(3m_a + r) = (3j + n)$; structure is statically determinate
- $(3m_a + r) > (3j + n)$; structure is statically indeterminate

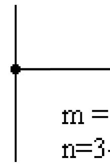
#### Overall classification

The criterion already established for both trusses and frames hold also for investigation of overall effect. To determine the overall classification of a frame, in the above expressions replace $r$ by $r_a$.

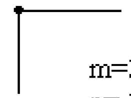
**Note**. The number of conditional equation introduced by a hinge joint is equal to the number of members at the joint minus one.



m=4
n=4-1=3

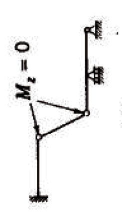
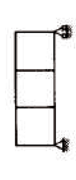
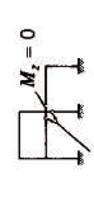
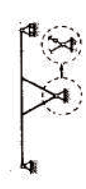


m =3
n=3-1=2



m=2
n= 2-1=1

**Examples**

| Structure | Structure Characteristics | | | | External Classification | | Overall Classification | | |
|---|---|---|---|---|---|---|---|---|---|
| | $j$ | $n$ | $m_a$ | $r_a$ | $r = 3 + n$ | Classification | $(3m_a + r_a)$ | $(3j + n)$ | Classification |
|  (a) | 4 | 1 | 3 | 4 | $3 + 1 = 4$ | $4 = 4$ Determinate, stable | $(9 + 4) = 13$ | $(12 + 1) = 13$ | $13 = 13$ Determinate, stable |
|  (b) | 5 | 2 | 4 | 6 | $3 + 2 = 5$ | $6 > 5$ Indet., 1st degree, stable | $(12 + 6) = 18$ | $(15 + 2) = 17$ | $18 > 17$ Indet., 1st degree, sta-ble |
|  (c) | 4 | 1 | 3 | 3 | $3 + 1 = 4$ | $3 < 4$ Unstable | $(9 + 3) = 12$ | $(12 + 1) = 13$ | $12 < 13$ Unstable |
|  (d) | 8 | 0 | 10 | 3 | 3 | $3 = 3$ Determinate | $(30 + 3) = 33$ | $(24 + 0) = 24$ | $33 > 24$ Indet., 9th degree, stable |
|  (e) | 8 | 3<br>2 (Ext) | 8 | 9 | $3 + 2 = 5$ | $9 > 5$ Indet., 4th degree, stable | $(24 + 9) = 33$ | $(24 + 3) = 27$ | $33 > 27$ Indet., 6th degree, stable |
|  (f) | 5 | 1<br>0 (Ext) | 5 | 5 | 3 | $5 > 3$ Indet., 2nd degree, stable | $(15 + 5) = 20$ | $(15 + 1) = 16$ | $20 > 16$ Indet., 4th degree, stable |

# CHAPTER 2

## 2. Loads on Structures

Determination of loads that act on a structure, evaluation of critical force effects in the member and dimensioning are the most difficult and yet important steps in the overall process of design.

The loads that enter a system are of three different types. Concentrated loads (example a single vehicular wheel load) are those that are applied over a relatively small area. Line loads are distributed along a narrow strip of the structure. The weight of a member itself and the weight of a wall or partition are examples of this type of load. Surface loads are loads that are distributed over an area. The loads on a warehouse floor and the snow load on a roof are examples of surface loads.

The loads that act on a structure can be grouped into three categories: dead loads, live loads, and environmental loads. These categories can be further divided according to the specific nature of the loading. Because the method of analysis is the same for each category of loading, all loads could be combined before the analysis is performed. However, separate analyses for the individual loading cases are usually carried out to facilitate the consideration of various load combinations.

Furthermore, loads can be classified based on:

- **Direction**: - The loads are broadly classified as vertical loads, horizontal loads and longitudinal loads. The vertical loads consist of dead load, live load and impact load. The horizontal loads comprise of wind load and earthquake load. The longitudinal loads i.e. tractive and braking forces are considered in special case of design of bridges, gantry girders etc.
- **Variation with time**: - dead load (permanent) and Live Loads (temporary)
- **Structural Response**: - Static (loads applied gradually) and dynamic (loads applied over a short period of time and vary in magnitude with time)

**Terms relating to actions**

- *permanent actions (G)*, e.g. self-weight of structures, fixed equipment and road surfacing, and indirect actions caused by shrinkage and uneven settlements;
- *variable actions (Q)*, e.g. imposed loads on building floors, beams and roofs, wind actions or snow loads;
- *accidental actions (A),* e.g. *fire,* explosions, or impact from vehicles.
- *seismic action* ($A_E$), action that arises due to earthquake ground motions

### 2.1 Dead loads

Dead loads are those that act on the structure as a result of the weight of the structure itself and of the components of the system that are permanent fixtures. As a result, dead loads are characterized as having fixed magnitudes and positions. Examples of dead loads are the weights of the structural members themselves, such as beams and columns, the weights of roof surfaces, floor slabs, ceilings, or permanent partitions, fixed permanent equipments, weight of different materials and so on.

The dead loads associated with the structure can be determined if the materials and sizes of the various components are known. Nominal density of construction materials and nominal density for stored materials are listed in Tables A-1 to A-12 of ES - EN 1991-1-1:2015.

Some values of density of construction materials are given in Table 2.1 below. [Refer ES - EN 1991-1-1:2015 (Actions on Structures Part 1:1 General Actions-Densities, self-weight, imposed loads for buildings)]

**Table 2.1:** Density of construction materials

| Materials | Density (kN/m$^3$) |
|---|---|
| Concrete (with reinforcement) | 25.0 |
| Metals | |
| Steel | 77.0-78.5 |
| Aluminium | 27.0 |
| Natural stones | |
| Trachyte | 26.0 |
| Granite | 27.0-30.0 |
| Basalt | 27.0-31.0 |
| Mortar | |
| Cement mortar | 19.0-23.0 |
| Lime mortar | 12.0-18.0 |
| Lime Cement mortar | 18.0-20.0 |
| Play wood | |
| Soft wood | 5.0 |
| birch | 7.0 |
| Glass | 22.0 |

## 2.2 Live loads

In a general sense, live loads are considered to include all loads on the structure that are not classified as dead loads, including environmental loads, such as snow loads or wind loads. However, it has become common to narrow the definition of live loads to include only loads that are produced through the construction, use, or occupancy of the structure and not to include environmental loads.

These loads are dynamic in character in that they vary both in magnitude and position. Live loads where the dynamic nature has significance because of the rapidity with which change in position occurs are called moving loads, whereas live loads in which change occurs over an extended period of time, or where there is the potential for change whether exercised or not, are referred to as movable loads. Moving loads include vehicular loads on bridges or crane loads in industrial buildings. Another type of live load is a variable load or a time dependent load-that is, one whose magnitude changes with time, such as a load induced through the operation of machinery.

### 2.2.1 Occupancy live loads

Occupancy live loads for buildings are usually specified in terms of the minimum values that must be used for design purposes. For areas which are intended to be subjected to different categories of loadings the design shall consider the most critical load case.

Areas in residential, social, commercial and administration buildings shall be divided into categories according to their specific uses are listed in Table 6.1 of ES - EN 1991 1-1:2015. The categories of loaded areas shall be designed by using characteristic values $q_k$ (uniformly distributed load) and $Q_k$ (concentrated load). Some representative values for qk and $Q_k$ are given in Table 2.2 below.

**Table 2.2:** Imposed loads on floors

| Categories of loaded areas | Uniformly distributed load (kN/m$^2$) | Concentrated load (kN) |
|---|---|---|
| Rooms in residential buildings and houses; Bedrooms and wards in hospitals; bedrooms in hotels and hostels kitchens and toilets. | 1.5-2.0 | 2.0-3.0 |
| Areas with tables, etc. e.g. areas in schools, cafés, restaurants, dining halls, reading rooms, receptions. | 2.0-3.0 | 3.0-4.0 |
| Areas with fixed seats, e.g. areas in churches, theatres or cinemas, Conference rooms, lecture halls, assembly halls, waiting rooms, railway waiting rooms. | 3.0-4.0 | 2.5-7.0 |
| Areas with possible physical activities, e.g. dance halls, gymnastic rooms, stages. | 4.5-5.0 | 3.5-7.0 |

### 2.2.2 Traffic Loads for Bridges

Bridges must be designed to support the vehicular loads associated with their functional use and minimum loads are mandated for designed purposes. Live loads due to vehicular traffic on highway bridges are specified by the American Association of State Highway and Transportation Officials (AASHTO) in the Standard Specifications for Highway Bridges and Bridge Design Manuals. The approach is to specify the weights and spacing of axles and wheels for a design truck, a design tandem, and the design lane load. These loadings provide for a set of concentrated loads (which represent a truck type loading) and a uniform load (which simulates a line of vehicles).

### 2.2.3 Impact loads

Loads that are applied over a very short period of time have a greater effect on the structure than would occur if the same loads were applied statically. The manner in which a load varies with time and the time over which the full load is placed on the structure will determine the factor by which the static response should be increased to obtain the dynamic response.

For building occupancy loads, the minimum design loads normally include adequate allowance for ordinary impact conditions. However, provisions must be made in the structural design for uses and loads that involve unusual vibrations and impact forces. One situation in which an impact effect (IM is defined as the dynamic load allowance) is applied for moving vehicular loads on a highway bridge.

## 2.3 Environmental loads

Structures experience numerous loading conditions as a result of the environment in which they exist. These are snow and ice loads, roof loads, wind loads and earthquake Loads.

### 2.3.1 Snow and Ice Loads

The procedure for establishing the static snow loads on a building is normally based on ground snow loads and an appropriate ground-to-roof conversion.

The distribution of snow on a roof is complex, and many different approaches are used. Factors considered in calculating snow and ice loads are location, exposure factor, thermal factor, the effects of unloaded portions of roof, unbalanced or non-uniform loads on various roof configurations, drifting, sliding snow, and extra loads induced by rain on snow.

### 2.3.2 Rain Loads

Roof loads that result from the accumulation of rainwater on flat roofs can be a serious problem. This condition is produced by the ponding that occurs when the water accumulates faster than it runs off, either because of the intensity of the rainfall or because of the inadequacy or blockage of the drainage system. The real danger is that as ponding occurs the roof deflects into a dished configuration, which can accommodate more water, and thus greater loads result.

The best way to prevent the problem is to provide a modest slope to the roof (0.25 in. per ft or 2cm. per m or more) and to design an adequate drainage system. In addition to the primary drainage, there should be a secondary system to preclude the accumulation of standing water above a certain level.

### 2.3.3 Wind Loads

The wind loads that act on a structure result from movement of the air against the obstructing surfaces. Wind effects induce forces, vibrations, and in some cases instabilities in the overall structure as well as its non-structural components. The effect of the wind on the structure (i.e. the response of the structure), depends on the wind speed, mass density of the air, location and geometry of the structure, and vibrational characteristics of the system.

**Wind Pressure on Surfaces according to ES - EN 1991 1-4:2015**

The basic wind velocity shall be calculated from Expression (4.1). (ES – EN 1991 1-4:2015 Wind actions)

$$V_b = C_{dir} \cdot C_{season} \cdot V_{b,0}$$

where:

$V_b$ : is the basic wind velocity, defined as a function of wind direction and time of year at 10 m above ground of terrain category II.

$V_{b,0}$ : is the fundamental value of the basic wind velocity, recommended value is 22 m/s

$C_{dir}$ : is the directional factor, the recommended value is 1.0

$C_{season}$ : is the season factor, the recommended value is 1.0

**Peak velocity pressure**

The peak velocity pressure, $q_p(z)$ at height $Z$, includes mean and short-term velocity fluctuations and depends on wind turbulence and mean wind velocity. The recommended rule is given by;

$$q_p(z) = C_e(z).q_b$$

$C_e(z)$ : is the exposure factor and illustrated in Figure 4.2 of ES - EN 1991 1-4:2015 as a function of height above terrain and a function of terrain category.

Terrain categories are given in Table 4.1 of ES - EN 1991 1-4:2015.

The basic velocity pressure that is used to establish the wind load on a structure is given by:

$$q_b = \frac{1}{2}\rho v_b^{\ 2}$$

where: - $\rho$ : is the air density, which depends on the altitude, temperature and barometric pressure to be expected in the region during wind. The recommended value is 1.25 kg/m$^3$.

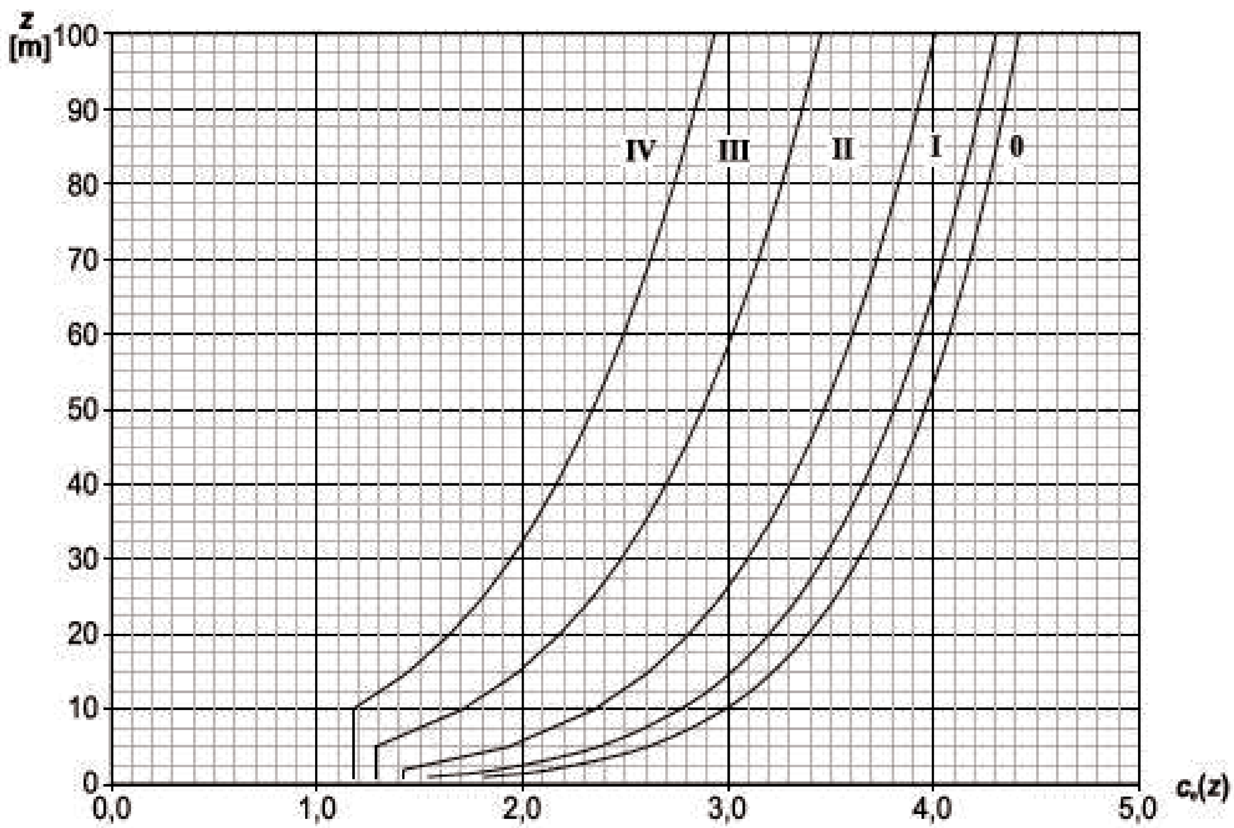



***Figure 2.1:** Illustrations of the exposure factor $C_e(z)$ for $C_o(z)=1$ and a function of terrain category*

**Table 3.3**: Terrain categories

| **Terrain Category** | |
|---|---|
| Category 0 | Sea or coastal area exposed to the open sea |
| Category I | Lakes or flat and horizontal area with negligible vegetation and without obstacles |
| Category II | Area with low vegetation such as grass and isolated obstacles (trees, buildings) with separations of at least 20 obstacle height |
| Category III | Area with regular cover of vegetation or buildings or with isolated obstacles with separations of maximum 20 obstacle heights (such as villages, suburban terrain, permanent forest) |
| Category IV | Area in which at least 15 % of the surface is covered with buildings and their average height exceeds 15 m |

### *Wind Pressure on Surfaces*

The wind pressure acting on the external surfaces $W_e$ is given by:

$$W_e = q_p(z)c_{pe}$$

Where

$q_p(z)$ : is the peak velocity pressure

$z_e$ : is the reference height for the external pressure

$c_{pe}$ : is the pressure coefficient for the external pressure

The wind pressure acting on the internal surfaces $W_i$ is given by:

$$W_i = q_p(z_i)c_{pi}$$

Where

$q_p(z)$ : is the peak velocity pressure

$z_i$ : is the reference height for the internal pressure

$c_{pi}$ : is the pressure coefficient for the internal pressure

### *Pressure coefficient for buildings*

*External pressure coefficient*: The external pressure coefficients $C_{pe}$ for buildings and parts of buildings depend on the size of the loaded area A, which is the area of the structure that produces the wind action in the section to be calculated. The external pressure coefficients are given for loaded areas A of 1 m$^2$ and 10 m$^2$ in the tables for the appropriate building configurations as $C_{pe,1}$, for local coefficients, and $C_{pe,10}$, for overall coefficients, respectively.

$$C_{pe} = \begin{cases} C_{pe,1} & \text{for} \quad A \leq 1\text{m}^2 \\ C_{pe,1} - (C_{pe,1} - C_{pe,10}) \log A & \text{for} \quad 1\text{m}^2 < A < 10\text{m}^2 \\ C_{pe,10} & \text{for} \quad A \geq 10\text{m}^2 \end{cases}$$

Values of external pressure coefficients $C_{pe}$ for different cases are given in Tables 7.1 to 7.5 of ES - EN 1991 1-4:2015.

*Internal pressure coefficient*: The internal pressure coefficient, $c_{pi}$ , depends on the size and distribution of the openings in the building envelope. For a building with a dominant face the internal pressure should be taken as a fraction of the external pressure at the openings of the dominant face. The values given by

- When the area of the openings at the dominant face is twice the area of the openings in the remaining faces, $c_{pi} = 0.75c_{pe}$
- When the area of the openings at the dominant face is at least 3 times the area of the openings in the remaining faces, $c_{pi} = 0.9c_{pe}$

Internal and external pressures shall be considered to act at the same time. The worst combination of external and internal pressures shall be considered for every combination of possible openings and other leakage paths.

### 2.3.4 Earthquake Loads

A common dynamic loading that structures must resist is that associated with earthquake motions. Here, loads are not applied to the structure in the normal fashion. Instead, the base of the structure is subjected to a sudden movement. Since the upper portion of the structure resists motion because of its inertia, a deformation is induced in the structure. This deformation, in turn, induces a horizontal vibration that causes horizontal shear forces throughout the structure.

It results from the acceleration of the supporting earth. Movement of the ground during EQ in the direction parallel to the ground surface has the most damaging effect on structures. The resulting earthquake loads are dependent on the nature of the ground movement and the inertia response characteristics of the structure. The computation of lateral loads due to EQ and load distribution to various levels of a building frame as of ES-EN 1998-1:2015 (Design of structures for Earthquake Resistance) is presented below.

**Methods of analysis:**

Depending on the structural characteristics of the building one of the following methods of analysis may be used:

1. *linear-elastic analysis*
   - "*Equivalent lateral force method of analysis*" for buildings meeting the conditions given in 4.3.3.2 of ES-EN 1998-1:2015;
   - "*modal response spectrum analysis*", which is applicable to all types of buildings
2. *non-linear method*
   - non-linear static (pushover) analysis
   - non-linear time history (dynamic) analysis

Using equivalent lateral force method of analysis, the seismic base shear force $F_b$, for each horizontal direction in which the building is analyzed, shall be determined using the following expression:

$$F_b = S_d(T_1).m.\lambda$$

where

$F_b$ : is the total lateral load on the structure (seismic base shear)

$S_d(T_1)$ : is the ordinate of the design spectrum at period $T_1$;

$T_1$ : is the fundamental period of vibration of the building for lateral motion in the direction considered;

$m$ : is the total mass of the building, above the foundation or above the top of a rigid basement;

$\lambda$ : is the correction factor, the value of which is equal to: $\lambda = 0.85$ if $T_1 < 2T_C$ and the building has more than two storeys, or $\lambda = 1.0$ otherwise.

For buildings with heights of up to 40 m the value of $T_1$(in sec) may be approximated by:

$$T_1 = C_1 H^{3/4}$$

Where: H: is the height of the building, in m, from the foundation or from the top of a rigid basement.

$$C_1 = \begin{cases} 0.085 & \text{for moment resistant space steel frames} \\ 0.075 & \text{for moment resistant space concrete frames and for eccentrically braced steel frames} \\ 0.050 & \text{for other structures} \end{cases}$$

For the horizontal components of the seismic action, the design spectrum, $S_d(T)$ is defined by the following expressions.

$$0 \le T \le T_B: \quad S_d(T) = a_g.S\left[\frac{2}{3} + \frac{T}{T_B}\left(\frac{2.5}{q} - \frac{2}{3}\right)\right]$$

$$T_B \le T \le T_C: \quad S_d(T) = a_g S \frac{2.5}{q}$$

$$T_C \le T \le T_D: \quad S_d(T) = \begin{cases} a_g.S.\frac{2.5}{q}\left[\frac{T_C}{T}\right] \\ \ge \beta.a_g \end{cases}$$

$$T_D \le T: \quad S_d(T) = \begin{cases} a_g.S.\frac{2.5}{q}\left[\frac{T_C.T_D}{T^2}\right] \\ \ge \beta.a_g \end{cases}$$

where

- $T$ : is the vibration period of a linear single-degree-of-freedom system;
- $a_g$ : is the design ground acceleration on type A ground ($a_g = \gamma_I . a_{gR}$);
- $\gamma_I$: :is the importance factor
- $a_{gR}$: : is reference peak ground acceleration on type A ground
- $T_B$ : is the lower limit of the period of the constant spectral acceleration branch;
- $T_C$ : is the upper limit of the period of the constant spectral acceleration branch;
- $T_D$ : is the value defining the beginning of the constant displacement response range of the spectrum;
- $S$ : is the soil factor;
- $q$ : is the behaviour factor;
- $\beta$ :is the lower bound factor for the horizontal design spectrum (the recommended value is 0.2).

The values of the periods $T_B$, $T_C$ and $T_D$ and the soil factor S describing the shape of the elastic response spectrum depend upon the ground type. Ground types (A, B, C, D and E) are described by the stratigraphic profiles and parameters given in Table 3.1 of ES - EN 1998-1:2015.

**Table 2.4: Ground types**

| Ground types | Description of stratigraphic profile |
|---|---|
| A | Rock or other rock-like geological formation, including at most 5 m of weaker material at the surface. |
| B | Deposits of very dense sand, gravel, or very stiff clay, at least several tens of meters in thickness, characterized by a gradual increase of mechanical properties with depth. |
| C | Deep deposits of dense or medium dense sand, gravel or stiff clay with thickness from several tens to many hundreds of meters. |
| D | Deposits of loose-to-medium cohesion less soil (with or without some soft cohesive layers), or of predominantly soft-to-firm cohesive soil. |
| E | A soil profile consisting of a surface alluvium layer with *Vs* values of type C or D and thickness varying between about 5 m and 20 m, underlain by stiffer material with *Vs* > 800 m/s. |

Recommended values of the parameters S, $T_B$, $T_C$ and $T_D$ for the different ground types and type (shape) of spectrum (Type 1 and Type 2 Spectra) are given in Table 3.2 and Table 3.3 of ES-EN 1998-1:2015.

**Table 2.5: Recommended Values of the parameters for Type 1 spectra**

| Ground types | S | $T_B$ (s) | $T_C$ (s) | $T_D$ (s) |
|---|---|---|---|---|
| **A** | 1.0 | 0.05 | 0.25 | 1.2 |
| **B** | 1.35 | 0.05 | 0.25 | 1.2 |
| **C** | 1.5 | 0.1 | 0.25 | 1.2 |
| **D** | 1.8 | 0.1 | 0.30 | 1.2 |
| **E** | 1.6 | 0.05 | 0.25 | 1.2 |

***Importance factors***: Buildings are classified in 4 importance classes, depending on the consequences of collapse for human life, on their importance for public safety and civil protection in the immediate post-earthquake period, and on the social and economic consequences of collapse. Importance classes and importance factors ($\gamma_I$) are described in the following Table 2.6.

**Table 2.6: Importance classes for buildings**

| Importance class | Buildings | Importance factor |
|---|---|---|
| I | Buildings of minor importance for public safety, e.g. agricultural buildings, etc. | 0.8 |
| II | Ordinary buildings, not belonging in the other categories. | 1 |
| III | Buildings whose seismic resistance is of importance in view of the consequences associated with a collapse, e.g. schools, assembly halls, cultural institutions etc. | 1.2 |
| IV | Buildings whose integrity during earthquakes is of vital importance for civil protection, e.g. hospitals, fire stations, power plants, etc. | 1.4 |

The behavior factor $q$ , account for energy dissipation capacity, shall be derived for each design direction as follows:

$$q = q_o k_w \geq 1.5$$

Where

$q_o$: is the basic value of the behavior factor, dependent on the type of the structural system and on its regularity in elevation

$k_w$: is the factor reflecting the prevailing failure mode in structural systems with walls

For buildings that are regular in elevation, the basic values of $q_o$ for the various structural types are given in Table 5.1 of ES - EN 1998-1:2015

**Table 2.7: basic value of the behavior factor, $q_o$, for systems regular in elevation**

| STRUCTURAL TYPE | DCM | DCH |
|---|---|---|
| Frame system, dual system, coupled wall system | $3.0\alpha_u/\alpha_1$ | $4.5\alpha_u/\alpha_1$ |
| Uncoupled wall system | 3.0 | $4.0\alpha_u/\alpha_1$ |
| Torsionally flexible system | 2.0 | 3.0 |
| Inverted pendulum system | 1.5 | 2.0 |

For buildings which are not regular in elevation, the value of $q_o$ should be reduced by 20%

Frames or frame-equivalent dual systems.

- One-storey buildings: $\alpha_u/\alpha_1 = 1.1$;
- multistory, one-bay frames: $\alpha_u/\alpha_1 = 1.2$;
- multistory, multi-bay frames or frame-equivalent dual structures: $\alpha_u/\alpha_1 = 1.3$.

$$k_w = \begin{cases} 1{,}00, \text{ for frame and frame} - \text{equivalent dual systems} \\ (1+\alpha_0)/3 \leq 1, \text{but not less than } 0{,}5, \text{ for wall, wall - equivalent and torsionally} \\ \text{flexible systems} \end{cases}$$

Where: $\alpha_o$ is the prevailing aspect ratio of the walls of the structural system

If the aspect ratios $(h_{wi}/l_{wi})$ of all walls $i$ of a structural system do not significantly differ, the prevailing aspect ratio $\alpha_o$ may be determined from the following expression:

$$\alpha_o = \sum h_{wi} / \sum l_{wi}$$

where: - $h_{wi}$ is the height of wall $i$; and $l_{wi}$ is the length of the section of wall $i$.

Systems of large lightly reinforced walls cannot rely on energy dissipation in plastic hinges and so should be designed as DCM structures.

The seismic hazard map is divided in to 5 zones, where the ratio of the design bedrock acceleration to the acceleration of gravity $g = \alpha_o$ for the respective zones indicated in the following Table 2.8.

**Table 2.8: Bedrock acceleration Ratio $\alpha_o$**

| Zone | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|
| $\alpha_o = a_{gR} / g$ | 0.20 | 0.15 | 0.1 | 0.07 | 0.04 | 0 |

**Table 2.9: Seismic hazard zonnation for some selected Towns**

| **Seismic zone** | **5** | **4** | **3** | **2** | **1** | **0** |
|---|---|---|---|---|---|---|
| **Towns** | Afdera, Dubti,Asaita, Menz – Mama Midir | Adigrat,Alaba, Alamata, Adama, Awassa, Kemise, Jijiga,D/berhan, | Akaki, Arisi-Negele,Dila, A.A,Aleltu Aleta-wondo, Kombolcha | Adwa, Axum, Kebri Beyah, Holeta,Maji | Agaro,Ambo, Jimma,Arek, Merhabete | Adi-Arkay, Ataye, Bahirdar, Assosa, D/markos,Asendabo |

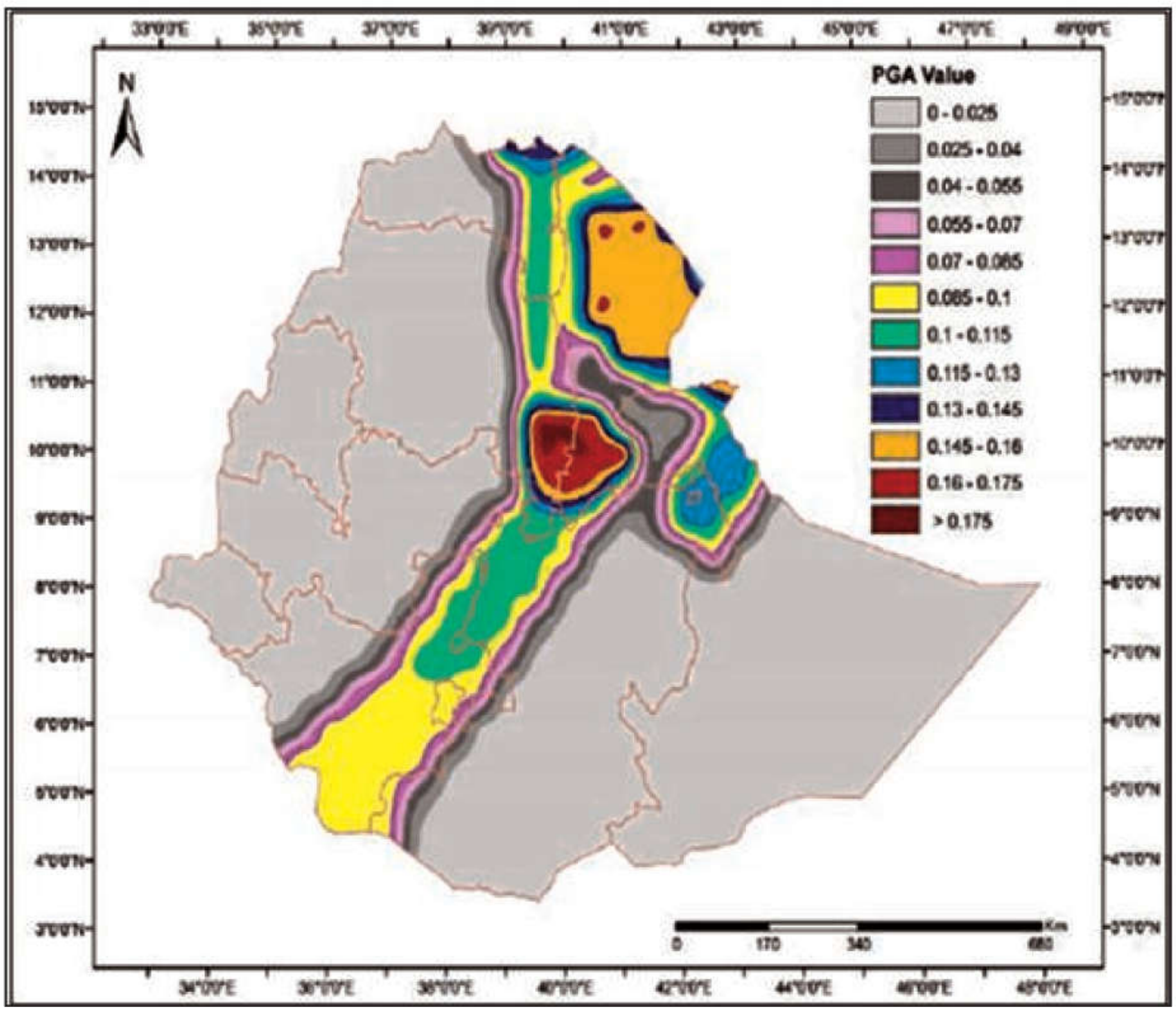



***Figure 2.2**: Ethiopia's seismic hazard map in terms of peak ground acceleration*

**Distribution of the horizontal seismic forces**

The fundamental mode shapes in the horizontal directions of analysis of the building may be calculated using methods of structural dynamics or may be approximated by horizontal displacements increasing linearly along the height of the building.

When the fundamental mode shape is approximated by horizontal displacements increasing linearly along the height, the horizontal forces $F_i$ should be taken as being given by:

$$F_i = F_b \frac{z_i m_i}{\sum z_j m_j}$$

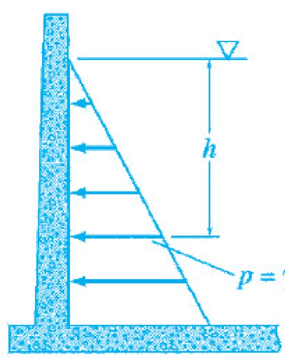
where

$z_i, z_j$ : are the heights of the masses $m_i, m_j$ above the level of application of the seismic action (foundation or top of a rigid basement).

The horizontal forces $F_i$ determined in accordance with this clause shall be distributed to the lateral load resisting system assuming the floors are rigid in their plane.

**2.4 Hydrostatic and Soil Pressures**

Structures used to retain water, such as dams and tanks, as well as coastal structures partially or fully submerged in water must be designed to resist hydrostatic pressure. Hydrostatic pressure acts normal to the submerged surface of the structure, with its magnitude varying linearly with height. The pressure at a point located at a distance h below the surface of the liquid is given as: $P = \gamma\, h$, in which $\gamma$ unit weight of the liquid.




Underground structures, basement walls and floors, and retaining walls must be designed to resist soil pressure. Similarly, the vertical soil pressure is given by $P = \gamma\, h$, with $\gamma$ now representing the unit weight of the soil.

## 2.5 Load Combinations

To estimate the magnitudes of the design loads, it is necessary to consider the possibility that some of these loads might act simultaneously on the structure. The structure is finally designed so that it will be able to withstand the most unfavorable combination of loads that is likely to occur in its lifetime. For each critical load case, the design values of the effects of actions ($E_d$) shall be determined by combining the values of actions that are considered to occur simultaneously. The minimum design loads and the load combinations for which the structures must be designed are usually specified in building codes. [Refer ES - EN 1990:2015 Basis of Structural Design]

**Basic requirements**

A structure shall be designed and executed in such a way that it will, during its intended life, with appropriate degrees of reliability and in an economical way.

- sustain all actions and influences likely to occur during execution and use, and
- remain fit for the use for which it is required.

A structure shall be designed to have adequate

- structural resistance,
- serviceability, and
- durability

**Ultimate Design Load**

The ultimate design load acting on a member will be the summation of the relevant characteristic load combinations multiplied by their respective partial safety factors. Thus, the ultimate design load for the combination of dead and imposed loads will be expressed as follows.

**Partial Safety Factors for Load**

In practice the applied load may be greater than the characteristic load for any of the following reasons:

a. Calculation errors
b. Constructional inaccuracies
c. Unforeseen increases in load (the unfavorable deviation of loads from their nominal values)

To allow for these the respective characteristic loads are multiplied by a partial safety factor $\gamma_f$ to give the ultimate design load appropriate to the limit state being considered. That is,

Ultimate design load = $\gamma_f$ x characteristic load

Load combinations depend on the design philosophy adopted.

**Ultimate limit states**

The limit states that concern:

- the safety of people, and/or
- the safety of the structure

- The general combination of effects of actions should be expressed as:

$$E_d = \sum_{j\geq 1} \gamma_{G,j} G_{k,j} "+" \gamma_{Q,1} Q_{k,1} "+" \sum_{i>1} \gamma_{Q,i} \psi_{0,i} Q_{k,ii}$$

  - Permanent action ($G_k$) and only one variable action ($Q_{ki}$)

$$E_d = 1.35G_k + 1.5Q_{ki}$$

- The combination of actions for accidental design situations can be expressed as:

$$E_d = \sum_{j\geq 1} G_{k,j} "+" A_d "+" (\psi_{1,1} \text{ or } \psi_{2,1}) Q_{k,1} "+" \sum_{i>1} \psi_{2,i} Q_{k,i}$$

- The combination of actions for seismic design situations can be expressed as:

$$E_d = \sum_{j\geq 1} G_{k,j} "+" A_{Ed} "+" \sum_{i\geq 1} \psi_{2,i} Q_{k,i}$$

**Serviceability limit states**

The limit states that concern:

- the functioning of the structure or structural members under normal use;
- the comfort of people;
- the appearance of the construction works,

**Load Combinations for Serviceability Limit States (SLS)**

The combinations of actions to be taken into account in the relevant design situations should be appropriate for the serviceability requirements and performance criteria being verified. The combinations of actions for serviceability limit states are defined symbolically by the following expressions. [See section 6.5.4 of ES - EN 1990:2015: Basis of Structural Design]

Characteristic combination:

$$E_d = \sum_{j\geq 1} G_{k,j} "+" Q_{k,1} "+" \sum_{i>1} \psi_{0,i} Q_{k,i}$$

Frequent combination:

$$E_d = \sum_{j\geq 1} G_{k,j} "+" \psi_{1,1} Q_{k,1} "+" \sum_{i>1} \psi_{2,i} Q_{k,i}$$

**Remark**: - Recommended values of $\gamma$ and $\psi$ factors for actions should be obtained from Annex A of ES - EN 1990:2015: Basis of Structural Design.

The final design of a structure must be consistent with the most critical combination of loads that the structure is to support. However, some judgment is necessary in selecting loading conditions that can reasonably be combined. Obviously, the maximum effects of all loading conditions should not be combined because it is unlikely that they will all occur simultaneously.

# CHAPTER 3

## 3. Influence Lines for Determinate Structures

### Introduction

An influence line for a given function, such as a reaction, axial force, shear force, or bending moment, is a graph that shows the variation of that function at any given point on a structure due to the application of a unit load at any point on the structure.

An influence line for a function differs from a shear, axial or bending moment diagram. Influence lines can be generated by independently applying a unit load at several points on a structure and determining the value of the function due to this load, i.e. shear, axial, and moment at the desired location. The calculated values for each function are then plotted where the load was applied and then connected together to generate the influence line for the function.

### Influence Lines for a Simple Beam

For illustration consider the simply supported beam shown in figure 3.1 and draw the influence lines for the reactions $R_A$, $R_C$, and the shear and bending moment at point $B$, of the simply supported beam shown.

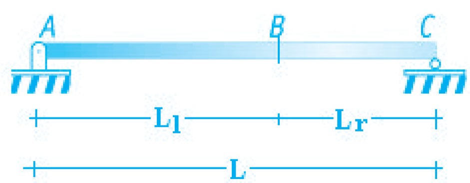



Figure 3.1

i. **Influence Line for Reaction at $A$, $R_A$**

The influence line for a reaction at a support is found by independently applying a unit load at several points on the structure and determining, through statics, what the resulting reaction at the support will be for each case.

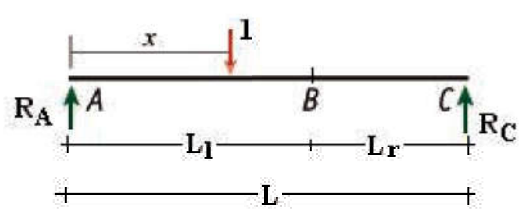



Figure 3.1a

$\Sigma M_C = 0$

$R_A(L)=1(L-x) \quad \rightarrow R_A = (L-x)/L$

If the unit load is applied at $A$, the reaction at $A$ will be equal to unity. Similarly, if the unit load is applied at $B$ (at $x=L_l$), the reaction at $A$ will be equal to $(L- L_l)/L = L_r/L$, and if the unit load is applied at $C$ (at $x=L$), the reaction at $A$ will be equal to 0.

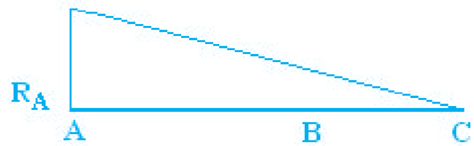



Figure 3.1b - Influence line for the support reaction at *A*

**ii. Influence Line for Reaction at C, $R_C$**

From figure 3.1a above,

$\Sigma M_A = 0$

$R_C(L)=1(x) \rightarrow R_c = X/L$

At $x$=0, $R_C$=0 and at $x=L$, $R_C$=1


Figure 3.1c - Influence line for the reaction at support C

**iii. Influence Line for Shear at B**

The influence line for the shear at point *B* can be found by developing equations for the shear at the section using statics. This can be accomplished as follows:

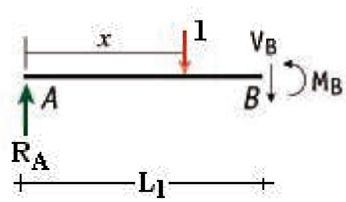



Figure 3.1d

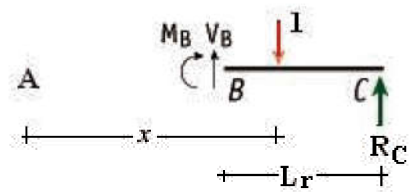



Figure 3.1e

a) if the load moves from A to B, consider figure 3.1d

$\Sigma F_y = 0 \rightarrow V_B=R_A-1$

But, $R_A = (L-x)/L$

Therefore, the shear force at B, $V_B$ becomes

$V_B=-1 +(L-x)/L = -x/L$

At $x$=0, $V_B$=0 and at $x=L_1$, $V_B= -L_l/L$

b) if the load moves from B to C, figure 3.1e

$\Sigma F_y = 0 \rightarrow V_B= 1- R_c$

But, $R_c = x/L$

Up on substitution, the shear force at B, $V_B$ becomes: $V_B=( L-x)/L$

At $x=L_l$, $V_B=( L-L_1)/L= L_r/L$ and at $x=L$, $V_B$= 0

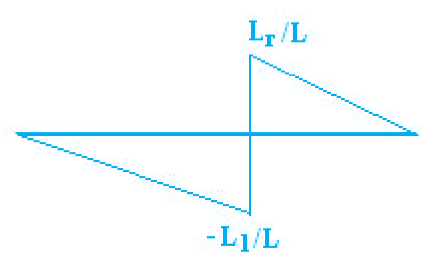



Fig 3.1f Influence line for the Shear at point B.

**iv. Influence line for Moment at B**

The influence line for the moment at point B can be found by using statics to develop equations for the moment at the point of interest, due to a unit load acting at any location on the structure.

a) if the load moves from A to B, consider figure 3.1d

$\Sigma M_B = 0 \rightarrow M_B + 1(L_l - x) - R_A(L_l) = 0$

$BM_B = R_r L_r = \frac{X}{L} L_r$

at $x = 0 \quad BM_B = 0$

at $x = L_\ell \quad BM_B = \frac{L_\ell L_r}{L}$

b) if the load moves from B to C, figure 3.1e

$\Sigma M_B = 0 \rightarrow R_C(L_r) - 1(x - L_l) - M_B = 0$

$BM_B = R_r \ell_t = \frac{L-x}{L} L_\ell$

at $x = l_\ell$ , $BM_B = \frac{L_r L_\ell}{L}$

at $x = L \quad BM_B = 0$

$\overline{m} = \frac{L_\ell L_r}{L}$

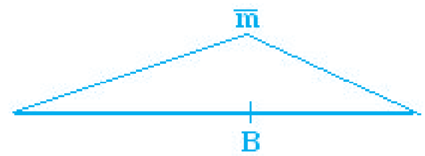



Fig 3.1g Influence line for Bending moment at point B.

✓ **Determination of a stress at a point due to different loads using influence curve**

i) Series of concentrated loads say $P_1$ and $P_2$ as shown.

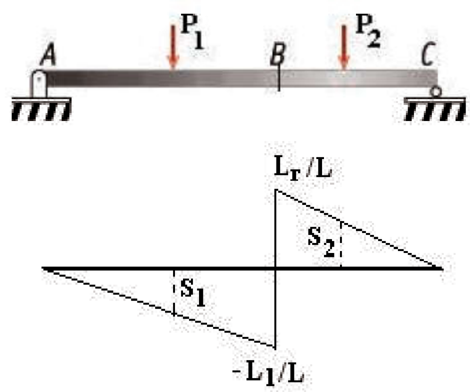



Fig 3.1g IL for $SF_B$

The product of the load $P_1$ and the ordinate of the influence line at the position of the respective load shall give the magnitude of the stress at the section due to the induced load

In the particular case

$SFa = S_1 P_1 + S_2 P_2$

Where $S_1$ *and* $S_2$ represent the ordinates of the IL with appropriate sign for $SFa$, at the respective positions of the loads $P_1$ and $P_2$.

ii) Uniformly distributed load w, say for a length d, position as shown

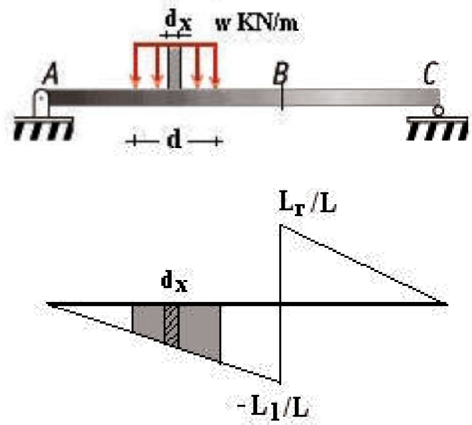



Fig 3.1h IL for $SF_B$

Consider a length $d_x$ and force acting $F = wdx$.

$SF_a$ due to total load over length $d = \sum_{i=1}^{d/dx} wd_x . Si = \sum_{i=1}^{d} Si\, d_x$

Where: $\sum Si\, d_x$ is the area under w of the IL

Hence, the product of $W$ and the area under the influence curve give the stress under consideration due to a uniformly distributed load:

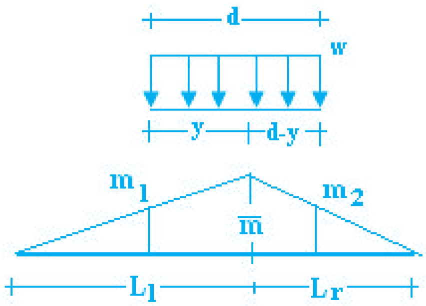
✓ **Determination of position of moving load system for maximum value of a particular function:**

To establish criteria for position of LL (living load) the different type of load system are considered.

**A.** Single moving load.
  - Place the load at the point of the ordinate of the IL for that function is a maximum

**B.** Uniform load longer than the span of the structure for which the ordinates to IL for that function have the same sign.
  - Place the load over all those portion of the structure for which the ordinates to IL for that function have the same sign.

**C.** Uniform load less than the span: -
  - In this particular case, let's consider the shape of influence curve
  - IL for SF type of simple beam: -
    - Place the head of the load at the section and let the load covers the left portion for maximum negative and the right portion for maximum value.
  - IL for BM type of simple beam
    - Critical position may be obtained if the load is placed in such a way that the section divides the load in the same ratio as the section divides the span.

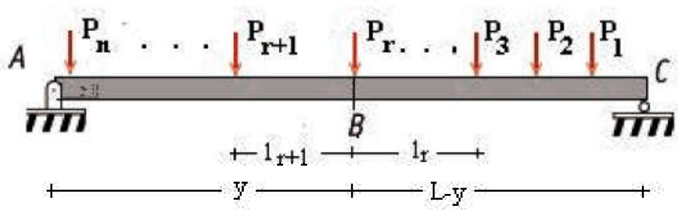



In this case,

$$\frac{wy}{w(d-y)} = \frac{l_l}{l_r}$$

**D.** Series of concentrated leads at fixed distance a part: -
Here also we need to consider the shape of IL

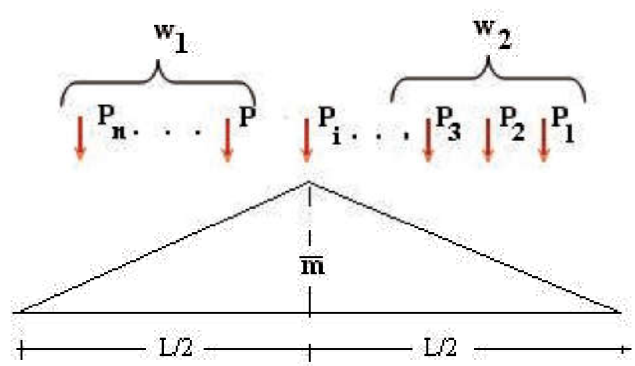
- **Shear force type**




The maximum *SF* occurs the first load of the system which give an intensity of leading equal to or greater than the average intensity of loading for the loads the span moving from left for negative *SF* and from right for positive *SF*

$$i.e\ \frac{\Pr}{\ell r} \geq \frac{\sum P}{\ell}$$

- **BM at a section of simple beam: -**




$w_1$ − resultant of the force at the right

For the load to the left of the section $\frac{w_1}{\ell_1} > \frac{w_2}{\ell_2}$, while for the load to the right of the section $\frac{w_1}{\ell_1} < \frac{w_2}{\ell_2}$,

**E.** Absolute maximum stress for simple beam:

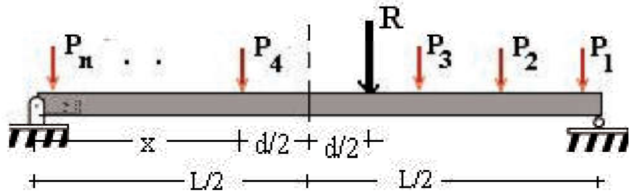
i) Absolute maximum *SF*

- Occurs at a section immediately adjacent to one of the supports.

ii) Absolute max *BM*

a) For simple or uniform load. Absolute maximum *BM* occurs at mid span

b) for simple or concentrated load.




The absolute maximum *BM* occurs under any particular load when the center of the span is mid way between that load and the resultant (R) of all the loads on the span.

$$\Rightarrow x = (l - d)/2$$

### 3.2 IL for paneled girders

In bridge construction, the live loads are usually transmitted to the main beams or girders through floor beams. Consider the following figure shown in Fig. 3.2a to construct IL for shear force at point P.

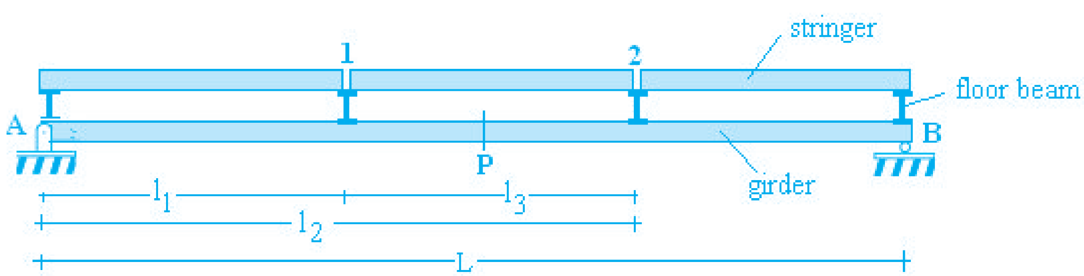
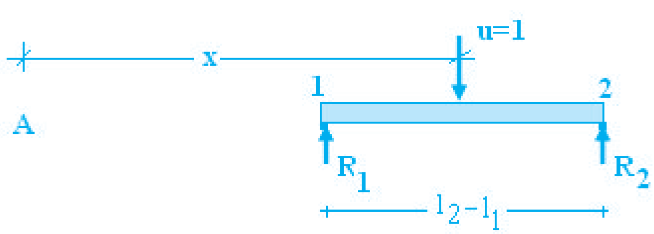



Fig 3.2a Paneled Girders

For u=1 between joint 1 and 2,




$$R_1 = \frac{l_2 - x}{l_2 - l_1} \quad \text{and} \quad R_2 = \frac{x - l_1}{l_2 - l_1}$$

Consider the lower beam shown in Fig. 3.2b

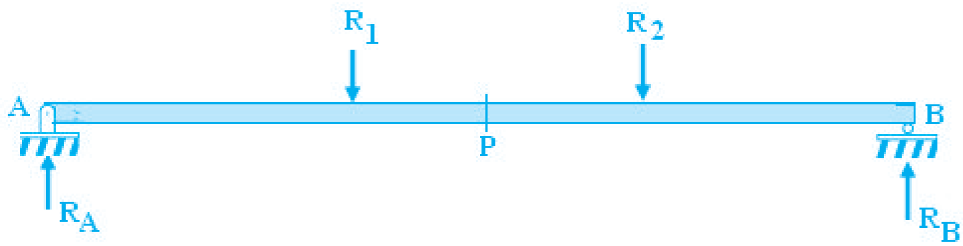



Fig 3.2b Lower beam

From statics,

$$R_A = \frac{l - x}{L} \text{ and } R_B = \frac{x}{L}$$

$SF_P = R_A - R_1$

$$SF_A = \left(\frac{l - x}{L}\right) - \left(\frac{l_2 - x}{l_2 - l_1}\right)$$

At $x = l_1$, $SF_A = -l_1 / L$ at $x = l_2 = l_1 + l_3$, $SF_A = L - l_2 / L$

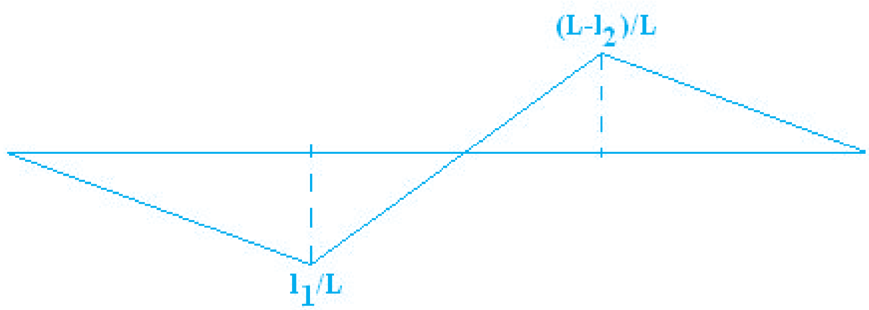



Fig 3.2c IL for shear force at point P, $SF_P$

### 3.3 IL for trusses

As the loads are gradually applied to a truss at panel points (joints), the same procedure as that used for constructing ILs for paneled girders is applicable.

The basic assumption to construct ILs for trusses and girders is the stringers act as simple beams between the adjacent floor beams so that the IL will be a straight line between any two adjacent panel points/joints.

The following procedure may be considering for the construction of influence lines for axial forces in member's trusses:

- Draw the influence lines for the reactions of the given truss.
- Determine the expression(s) of the member force whose influence line is desired. /by using the method of sections or the method of joints/
- If using the method of sections,
  - Apply a unit load to the left of the left end of the panel through which the section passes, and determine the expression for the member force by applying the equilibrium equation to the free body of the truss to the right of the section.
  - Next, apply the unit load to the right of the right end of the section, and determine the member force expression by applying the equilibrium equation to the free body to the left of the section.
  - Construct the influence line by plotting the member force expressions and by connecting the ordinates at the ends of the sectioned panel by a straight line.
- When using the method of joints,
  - If the joint being considered is not located on the loaded chord of the truss, then determine the expression of the desired member force directly by applying the equation of equilibrium to the free body of the joint. Otherwise, apply a unit load at the joint under consideration, and determine the magnitude of the member force by considering the equilibrium of the joint.
  - Next, determine the expression for the member force for a position of the unit load outside the panels adjacent to the joint under consideration.
  - Finally, connect the influence-line segments and ordinates thus obtained by straight lines to complete the influence.

For illustration, consider the truss shown in Fig. 3.3a below. Suppose it is desired to construct IL for forces in members' *cj* and *ck*; for bottom boom loading.

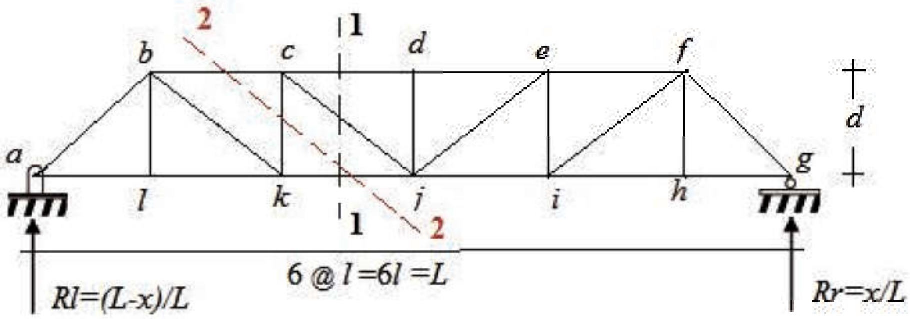



Fig. 3.3a

### i) IL for diagonal member cj

For u=1 between *a* to *k* consider the right portion of section 1-1.

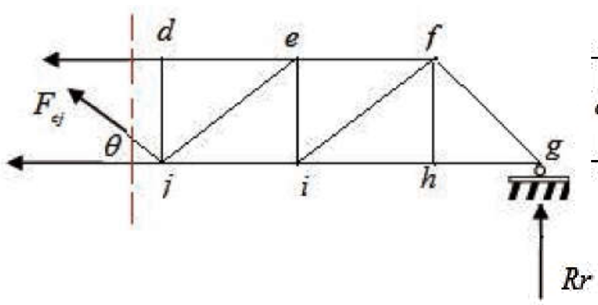



Fig 3.3b

$$\Sigma F_y = 0 \Rightarrow F_{cj} = \frac{R_r}{\sin\theta} = \frac{-x}{L\sin\theta}$$

$$@x = 0,\ F_{cj} = 0 \text{ and } @x = 2l,\ F_{cj} = \frac{-2l}{L\sin\theta}$$

For u=1 between *j* to *g* consider the left portion of section 1-1.

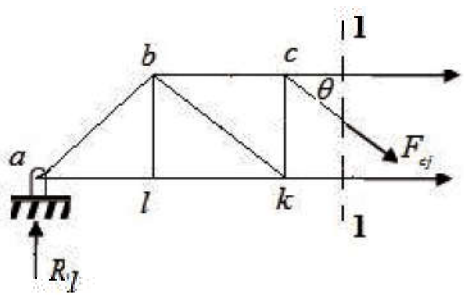



Fig 3.3c

$$\Sigma F_y = 0 \Rightarrow R_l - F_{cj}\sin\theta = 0 \Rightarrow F_{cj} = \frac{L-x}{L\sin\theta}$$

$$@x = 3l,\ F_{cj} = \frac{L-3l}{L\sin\theta} = \frac{3l}{L\sin\theta};\ \text{and}$$

$$@x = L = 6l,\ F_{cj} = 0$$

For u=1 between *k* to *j*, the straight line connects the two points will complete the IL for forces in member *cj*.

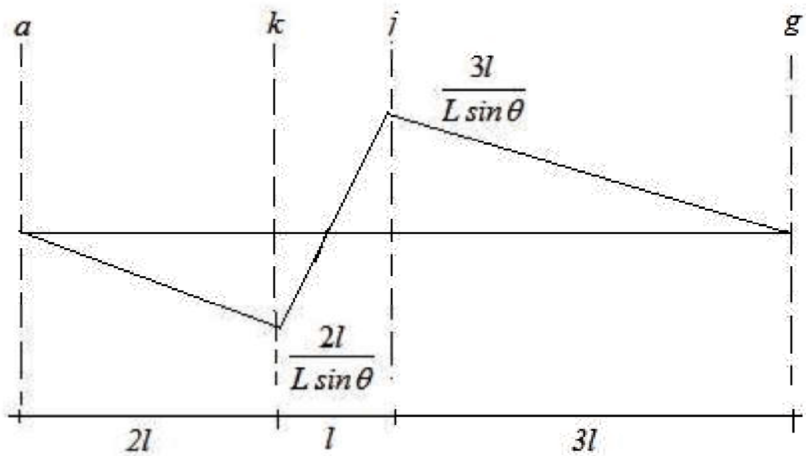



Fig 3.3d IL for $F_{cj}$

### *ii) IL for $F_{ck}$ (Vertical member, for bottom boom loading)*

For u=1 between *a* to *k* consider the right portion of section 2-2.

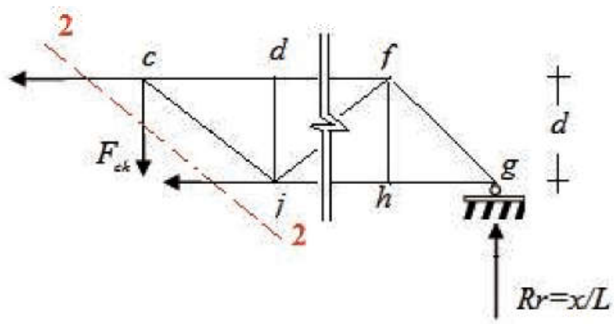



Fig 3.3e

$$\Sigma F_y = 0 \Rightarrow F_{ck} = \frac{x}{L}$$

$$@x = 0,\ F_{ck} = 0 \text{ and } @x = 2l,\ F_{ck} = \frac{2l}{L}$$

For u=1 between *j* to *g* consider the left portion of section 2-2.

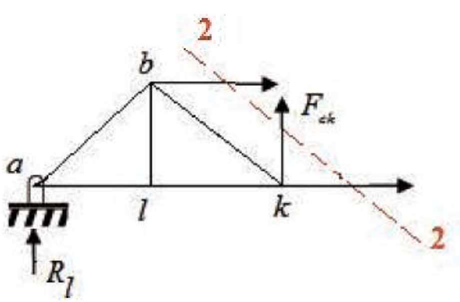



Fig 3.3f

$$F_{ck} + R_l = 0 \Rightarrow F_{ck} = -R_l = -\frac{L-x}{L}$$

$$@x = 3l,\ F_{ck} = \frac{-3l}{L}; \text{ and}$$

$$@x = L = 6l,\ F_{ck} = 0$$

Thus, the IL for force in member ck, $F_{ck}$ for bottom boom loading is drawn in the following figure.

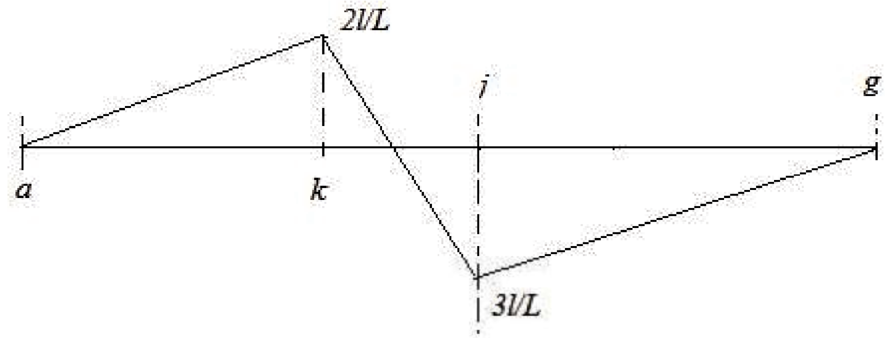



Fig 3.3g IL for $F_{ck}$ (bottom boom loading)

Similarly, for top boom loading the IL for forces in member *ck* is shown in the following figure.

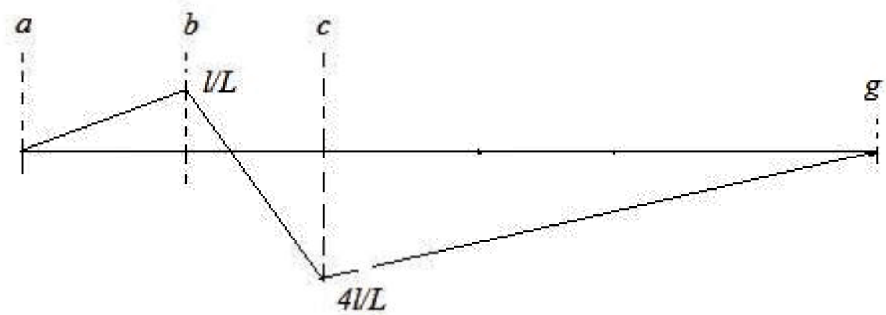



Fig 3.3h IL for $F_{ck}$ (top boom loading)

# CHAPTER 4

## 4. Deflection of Determinate Structures

Structures, like all other physical bodies, deform and change shape when subjected to forces. Other common causes of deformations of structures include temperature changes and support settlements. If the deformations disappear and the structure regains its original shape when the actions causing the deformations are removed, the deformations are termed elastic deformations. The permanent deformations of structures are referred to as inelastic, or plastic, deformations.

In this section, we study geometric methods commonly used for determining the slopes and deflections of statically determinate beams and also develop methods for the analysis of deflections of statically determinate structures such as trusses, beams and frames by using basic principles of work and energy.

### Deflections of Beams

As a flexural structure responds to loading, it assumes an equilibrium configuration under the combined action of the loads and reactions. Corresponding to this external state of equilibrium, there is a distribution of internal shears and bending moments throughout the structure. These internal actions would normally be shown in the form of shear and moment diagrams for each member. At any given point within the structure, there is a curvature that is consistent with the moment at that point. These curvatures accumulate as angle changes along the member lengths, causing each member to deflect into a flexed or bent configuration. The individual members of the deformed structure must fit together in a compatible fashion, and all the displacement boundary conditions must be satisfied.

The design of beams is not complete until the amount of deflection has been determined for the specified loads. Failure to control beam deflection with in proper limit in building results in cracks in walls and ceilings. If these deflections are excessive, they may result in psychological frustration of occupants and stacking of opening beams in machine. For most structures, excessive deformations are undesirable, as they may impair the structure's ability to serve its intended purpose.

Deflection of structures can be determined by the following methods:

**Geometric Methods**

- Direct Integration Method
- Moment-Area Method
- Conjugate-Beam Method

**Work–Energy Methods**

- Virtual Work Method
- Graphical Multiplication
- Castigliano's Theorem
- Betti's law and Maxwell's 2nd theorem of reciprocal deflections

## 4.1 Direct Integration Method

### Differential equation of the elastic curve

For a flexural member, the force-deformation relationships must relate member-end moments to the corresponding member-end rotations. In a more general formulation, member-end shears and transverse deflections would be included, but they are omitted here.

Consider a member under flexural action as shown in Fig. 4.1a, and from it isolate the beam element ab as shown in Fig. 4.1b, which is subjected to the positive moment *M*. As the element bends, the top fibers are compressed while the bottom fibers are elongated. In between, there is a longitudinal fiber whose length remains unchanged: this fiber is the so-called neutral fiber of the member.

It is assumed that as the beam deflects, plane cross sections before and after bending remain plane. For the element ab, extensions of lines through cross sections at a and b intersect at point o, the center of curvature, forming the angle de. If tangents to the deflected neutral fiber are constructed at points a and b, it is evident that de also measures the angular deformation over the length of the beam element. The line eb is constructed parallel to the deflected cross section at a creating triangle bde. Then, for small angles, comparing triangles bde and oab, we have

$$d\theta = \frac{dx}{\rho} = \frac{dl}{c} \tag{1}$$

Where ρ is the radius of curvature of the element, c is the distance from the neutral fiber to the topmost fiber, and dl is the shortening of that top fiber. Equation (1) can be rewritten in the form

$$\frac{dl}{dx} = \frac{c}{\rho} = \varepsilon \tag{2}$$

From stress strain relationship, $\sigma = E\varepsilon$

Thus, equation 2 becomes

$$\sigma = -E\frac{c}{\rho} \tag{3}$$

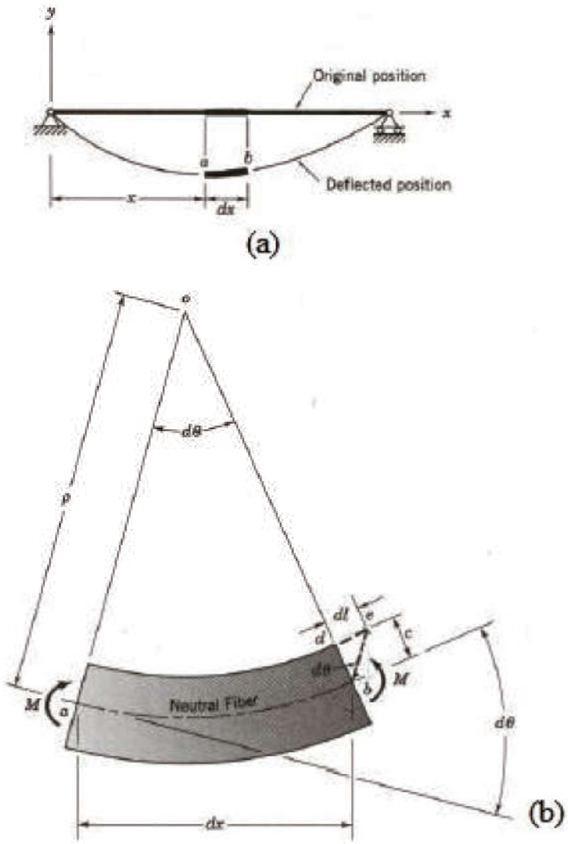



Fig. **4.1** Flexural deformations of beam element

(a) Deflected beam (b) Beam element subjected to moment

Where $\sigma$ is the stress in the top fiber and E is Young's modulus. The minus sign is introduced to indicate that the element is being compressed (negative stress). This fiber stress could also be expressed by the familiar expression from basic mechanics that

$$\sigma = \frac{Mc}{I} \tag{4}$$

where M is the moment acting on the element and I is the moment of inertia. Here the negative sign indicates that a positive moment produces a negative (compressive) stress on the top fiber.
Equating equations 3 and 4, yields

$$\frac{1}{\rho} = \frac{M}{EI} \tag{5}$$

But from elementary calculus, the curvature of a plane curve is given by:

$$k = \frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \tag{6}$$

where: k is defined as the curvature.

Practically the elastic curves of beams are very flat (slope of the deflected structure) thus their slope dy/dx is negligible as compared to unity. Thus, $k = \frac{1}{\rho} = \frac{d^2y}{dx^2}$.

Therefore, $\frac{d^2y}{dx^2} = \frac{M}{EI}$ (This is the basic differential equation of the elastic curve).

From the relationships were developed between static functions load, shear, and bending moment, these static functions can be written as follows:

$$V(x) = \frac{dM}{dx} = EI\frac{d^3y}{dx^3} \quad \text{and} \quad q(x) = \frac{dV}{dx} = EI\frac{d^4y}{dx^4}$$

The family of relationships is extended to include the deformation quantities of slope and deflection.

$$d\theta = \frac{M}{EI}dx$$

The systematic solution of beam deflection problems conditions is called boundary conditions where the constants of integration are determined from the boundary and continuity conditions on $V$, $M$, $\theta$, and $y$.

Fig. 4.2 below serves to summarize the family of relationships spanning from the load intensity $p$ through the displacement $y$. In this figure, a simply supported beam is subjected to a general loading and the member responses are shown through plotted functions for load, shear, moment, slope, and deflection.

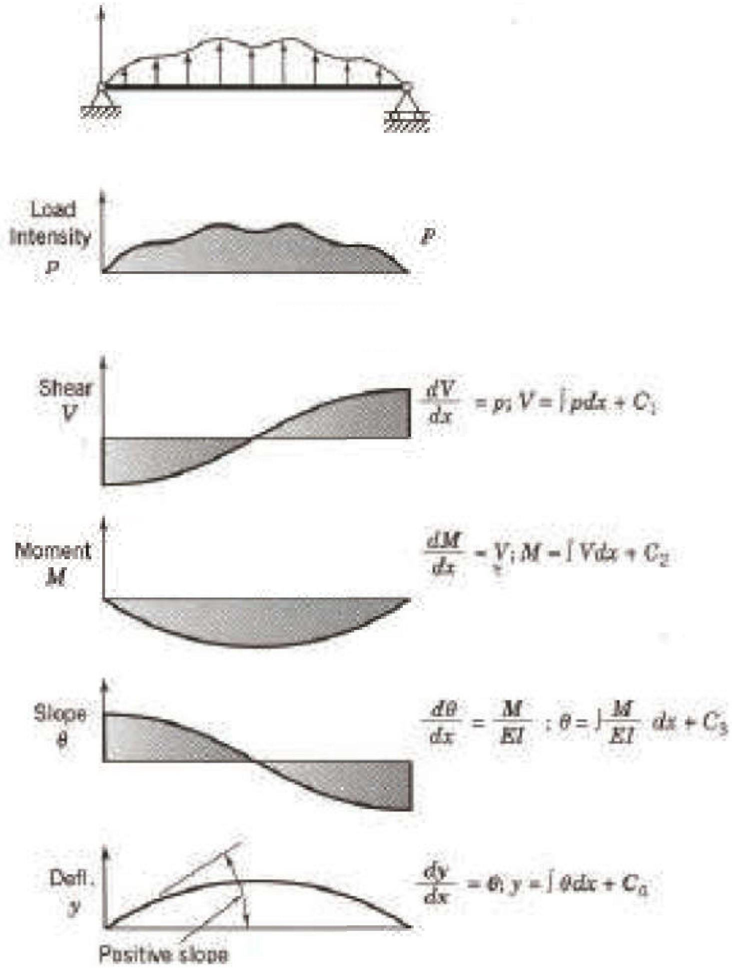



Fig 4.2 Relationships of $p$, $V$, $M$, $\theta$ and $y$

## 4.2 Moment-Area Method

The integration method is of greatest value when the loading is such as to produce a moment diagram that is a continuous function over the entire length of the beam.

When concentrated loads occur along the span, or internal reaction points exist, then the moment diagram has discontinuities. This leads to additional constants of integration that are evaluated by applying continuity conditions at the points of moment discontinuity.

If deflection of certain selected points only is to be determined, the moment area method is preferable and also more efficient for beams with several discontinuities due to change in loading and variation in the rigidity of the beams.

The moment-area method for computing slopes and deflections of beams was developed by Charles E. Greene in 1873. The method is based on two theorems, called the moment-area theorems, relating the geometry of the elastic curve of a beam to its M/EI diagram, which is constructed by dividing the ordinates of the bending moment diagram by the flexural rigidity EI. The method utilizes graphical interpretations of integrals involved in the solution of the deflection differential equation in terms of the areas and the moments of areas of the M/EI diagram.

Consider the beam structure of Fig. 4.3a, which is shown in a deflected configuration under the action of the applied loads. An enlarged view of a portion of the deflected structure between points A and B is isolated in Fig. 4.3b.

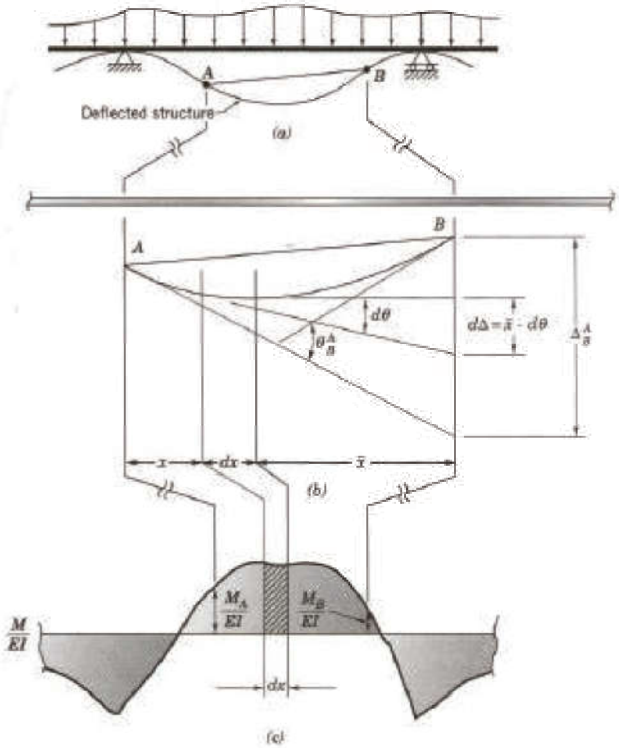



Fig 4.3

Within region AB, an element of length dx with tangents with the deflected member constructed at each end of the element. The angle between these end tangents, which represents the angle change that occurs over the length dx, is denoted dθ. This angle change is given by

$$d\theta = \frac{M}{EI} dx$$

Where M and I are the bending moment and moment of inertia at point x, respectively, and E is the modulus of elasticity of the material. If the M/EI values are plotted as shown in Fig. 4.3c, it is clear that $d\theta$ is given by the shaded area.

The total angle change that occurs between tangents constructed at points A and B is labeled $\theta_B^A$ in Fig. 4.3b. This angle is the slope at B relative to the slope at A; it results from the summation of the incremental angle changes between A and B and is given by

$$\theta_B^A = \int_A^B d\theta$$

Upon substitution it becomes

$$\theta_B^A = \int_A^B \frac{M}{EI} dx$$

This equation is the basis for the first moment-area theorem, which can be stated as follows:

*The angle change between points A and B on the deflected structure, or the slope at point B relative to the slope at point A, is given by the area under the M/EI diagram between these two points.*

Examination of Fig. 4.3b shows that if the tangents to the element of length *dx* are extended, they embrace an intercept of *dΔ* on a vertical line through point *B*. For small angles, this intercept is given by

$$d\Delta = \bar{x} d\theta$$

Up on substitution, we obtain $d\Delta = \bar{x}\frac{M}{EI}dx$ which shows that the intercept *dΔ* is given by the static moment of the shaded area of the *M/EI* diagram taken about an axis through point *B*. The accumulation of these intercepts for all increments between points *A* and *B* gives:

$$\Delta_B^A = \int_A^B d\Delta$$

Where $\Delta_B^A$ is the vertical displacement of point *B* on the deflected structure with, respect to a line drawn tangent to the structure at point *A*.

Upon substitution the above equation yields:

$$\Delta_B^A = \int_A^B \frac{M}{EI} \bar{x} dx$$

This equation is the basis for the second moment-area theorem, which can be stated as follows:

*The deflection of point B on the deflected structure with respect to a line drawn tangent to point A on the structure is given by the static moment of the area under the M/EI diagram between points A and B taken about an axis through point B.*

It is emphasized that the deflection quantities that are determined by using the second moment-area theorem are normal to the original orientation of the member.

Application of the moment-area method involves computation of the areas and moments of areas of various portions of the M/EI diagram. When a beam is subjected to a combination of distributed and concentrated loads/different types of loads /, determination of the properties of the resultant M/EI diagram, due to the combined effect of all the loads, can become a difficult task.

This difficulty can be avoided by constructing the bending moment diagram in parts that is, constructing a separate bending moment diagram for each of the loads. The ordinates of the bending moment diagrams thus obtained are then divided by EI to obtain the M/EI diagrams. These diagrams usually consist of simple geometric shapes, so their areas and moments of areas can be easily computed.

**Example**: - Figure 4.4 shows the M/EI diagram by parts about point B for a beam subjected to a combination of a uniformly distributed load and a concentrated load.

- From statics, $R_A = 116.667kN$ and $R_B = 208.333kN$

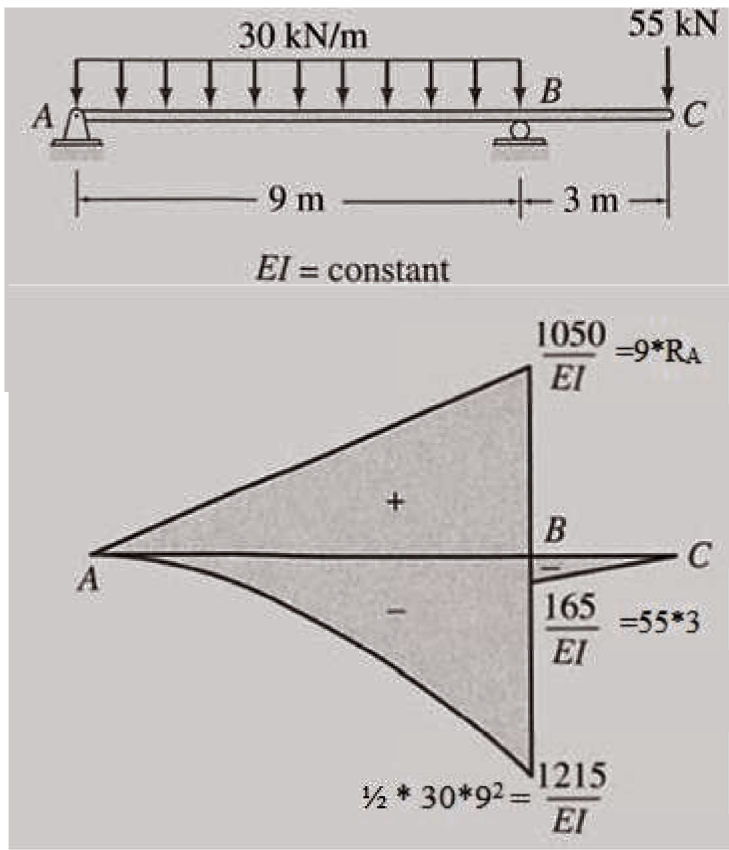



Fig. 4.4 M/EI diagram by parts about point B

### 4.3 Conjugate-Beam Method

Another method, which is derived from the moment area principle, is the conjugate-beam method, developed by Otto Mohr in 1868. Essentially, it requires the same amount of computation as the moment-area theorems to determine a beam's slope or deflection; however, this method relies only on the principles of statics, and hence its application will be more familiar.

The method may be used to obtain an expression for the entire deflection curve over the whole of a structure. The problem of beam statics is governed by the following equation.

$$q(x) = \frac{dV}{dx} = \frac{d^2M}{dx^2}$$

This is a second-order linear differential equation, and the solution is the familiar shear and moment diagram problem: starting with the load, the first integration gives the shear and the second integration gives the moment. Similarly, the beam deflection problem is governed by:

$$\frac{M}{EI} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

This is also a second-order linear differential equation. Here, we start with the curvature, *M/EI*; the first integration yields the slope, and the second integration gives the deflection. This observation would enable us utilize the semi graphical method to obtain $y'(x)$ and $y(x)$.

As shown in Table below, a pin or roller support at the end of the real beam provides *zero displacement,* but the beam has a nonzero slope. Consequently, the conjugate beam must be supported by a pin or roller, since this support has *zero moment* but has a shear or end reaction. When the real beam is fixed supported, both the slope and displacement at the support are zero. Here the conjugate beam has a free end, since at this end there is zero shear and zero moment.

To make this comparison, consider a beam having the same length as the real beam, but referred to here as the "conjugate beam" and "loaded" with the *M/EI* diagram derived from the load *w* on the real beam.

The conjugate beam has a set of boundary conditions and internal continuity conditions on shear and moment that match the corresponding real beam boundary conditions and internal continuity conditions on slope and deflection.

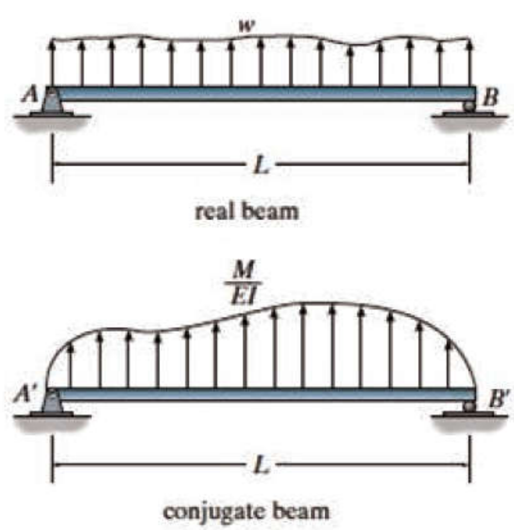



Fig. 4.5, real and conjugate beams

Unlike the moment-area method, an orderly sign convention can be employed with the conjugate beam method. If positive curvature (*M/EI*) is applied as positive (upward) load intensity on the conjugate beam will correspond to the correct signs of the resulting shears and moments on the conjugate beam correspond to the correct signs of the slope and deflection, respectively, on the real beam.

The following table shows that the geometric conditions of the real beam and the corresponding force conditions of the conjugate beam.

| | Real Beam | | Conjugate Beam | |
|---|---|---|---|---|
| 1) | $\theta$ <br> $\Delta = 0$ |  pin | $V$ <br> $M = 0$ |  pin |
| 2) | $\theta$ <br> $\Delta = 0$ |  roller | $V$ <br> $M = 0$ |  roller |
| 3) | $\theta = 0$ <br> $\Delta = 0$ |  fixed | $V = 0$ <br> $M = 0$ |  free |
| 4) | $\theta$ <br> $\Delta$ |  free | $V$ <br> $M$ |  fixed |
| 5) | $\theta$ <br> $\Delta = 0$ |  internal pin | $V$ <br> $M = 0$ |  hinge |
| 6) | $\theta$ <br> $\Delta = 0$ |  internal roller | $V$ <br> $M = 0$ |  hinge |
| 7) | $\theta$ <br> $\Delta$ |  hinge | $V$ <br> $M$ |  internal roller |

Therefore, from the above comparisons, the two theorems related to the conjugate beam and the real beam can be stated as follows:

**Theorem 1**: *The slope at a point in the real beam is numerically equal to the shear at the corresponding point in the conjugate beam.*

**Theorem 2**: *The displacement of a point in the real beam is numerically equal to the moment at the corresponding point in the conjugate beam.*

**Sign Convention**

*If the positive ordinates of the M/EI diagram are applied to the conjugate beam as upward loads (in the positive y direction) and vice versa, then a positive shear in the conjugate beam denotes a positive (counterclockwise) slope of the real beam with respect to the undeformed axis of the real beam; also, a positive bending moment in the conjugate beam denotes a positive (upward or in the positive y direction) deflection of the real beam with respect to the undeformed axis of the real beam and vice versa.*

**Procedure for Analysis**

The following procedures can be used for determining the slopes and deflections of beams by the conjugate-beam method.

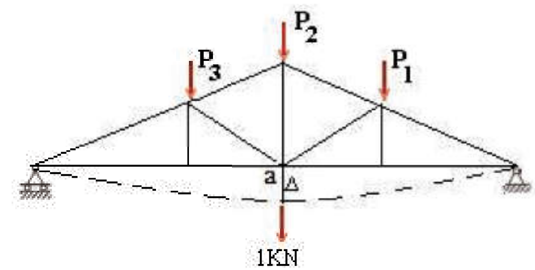
- Construct the M/EI diagram for the given (real) beam subjected to the specified (real) loading. If the beam is subjected to a combination of different types of loads, the analysis can be considerably expedited by constructing the M/EI diagram by parts.
- Determine the conjugate beam corresponding to the given real beam. The external supports and internal connections for the conjugate beam must be selected so that the shear and bending moment at any point on the conjugate beam are consistent with the slope and deflection, respectively, at that point on the real beam.
- Apply the M/EI diagram as the load on the conjugate beam. The positive ordinates of the M/EI diagram are applied as upward loads on the conjugate beam and vice versa.
- Calculate the reactions at the supports of the conjugate beam by applying the equations of equilibrium and condition (if any).
- Determine the shears at those points on the conjugate beam where slopes are desired on the real beam. Determine the bending moments at those points on the conjugate beam where deflections are desired on the real beam.

## 4.4 The Method of Virtual Work

If a deformable structure, in equilibrium and sustaining a given system loads is subjected to a virtual deformation as result of some additional action the external virtual work of the given system of loads is equal to the internal virtual work of the stress caused by the given system of load.

i.e External virtual work = Internal virtual work

**Deflections of Trusses by the Virtual Work Method**




Consider statically determinate truss as shown in figure above. Suppose $\Delta$ , the vertical displacement of the truss at point a, is required.

- First remove the real loads and apply unit fictitious load at point in $\Delta$ direction.

- Superpose the real loads next. Thus, the unit fictitious force will move through a distance $\Delta$.

→ Therefore, the external virtual work is $W_{ve} = 1.\Delta$

On the other hand, let ***u*** be the fictitious bar force resulting from the action of unit force (the virtual axial force)

- The strain (axial deformation, $\delta$ ) in each member is given by

$$\delta = \frac{SL}{EA} \tag{1}$$

- The virtual internal work done on each member by the virtual axial force ***u***, acting through the real axial deformation, $\delta$ is equal to $u.\delta$ . Therefore, the total virtual internal work done on all the members of the truss can be written as $W_{vi} = \sum u.\delta$

By equating the virtual external work to the virtual internal work in accordance with the principle of virtual forces for deformable bodies, we obtain the following expression for the method of virtual work for truss deflections:

$$1 \times \Delta = \sum u.\delta \tag{2}$$

When the deformations are caused by external loads, Eq. (1) can be substituted into Eq. (2) to obtain $1 \times \Delta = \sum u \frac{SL}{EA}$ (3)

**Note: -** If the sign of $\sum u\frac{SL}{EA}$ is positive, then the actual deflection has the same sense as the unit force otherwise opposite to it. However, it is important that the proper sign for tension (+) and compression (-) be consistent throughout due to the computation for $u$ and $S$.

**Temperature Changes and Fabrication Errors**

The expression of the virtual work method as given by Eq. (2) is general in the sense that it can be used to determine truss deflections due to temperature changes, fabrication errors, and any other effect for which the member axial deformations, $\sigma$ , are either known or can be evaluated beforehand.

In some cases, truss members may change their length due to temperature. If $\alpha$ is the coefficient of thermal expansion for a member and DT is the change in its temperature, the change in length of a member is $\delta = \alpha.L.\Delta T$ . Up on substituting the expression $\delta = \alpha.L.\Delta T$ in Eq. (2), we obtain the following expression: $1\times\Delta = \sum u\alpha L(\Delta T)$ which can be used to compute truss deflections due to the changes in temperature.

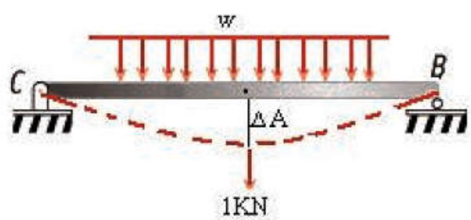
Errors in fabricating the lengths of the members of a truss may occur. If a truss member is shorter or longer than intended, the displacement of a truss joint from its expected position can be determined from $1.\Delta = \sum u.\Delta L$

Where: $\Delta L$ : difference in length of the member from its intended size as caused by a fabrication error.

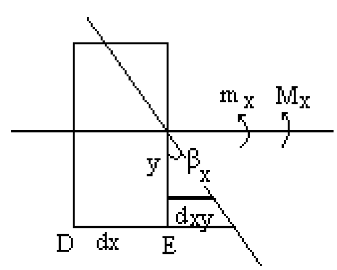
**Deflections of Beams by the Virtual Work Method**

**i) Deflection resulting from flexural strains**

$1\times\Delta$ and $1\times\theta$ expression for external virtual work.




For the expression of internal virtual work consider the simple beam shown. Suppose $\Delta_A$ is required. A unit fictitious load is applied at $A$ as shown above.




- Let $m_x$ be internal moment at $x$ due to the fictitious unit load.

Due to the flexural strains resulting from the application of real loads, the internal fictitious moment on one face of the *DE* is caused to rotate through some virtual angle $\beta_x$ relative to the other face.

Internal virtual work for a length $dx = m_x B_x$

- Since $\delta = \dfrac{M_x y}{I}$ due to the real load, $\Delta dxy = \dfrac{\delta d_x}{E} = \dfrac{M_x y d_x}{EI}$
- For small angle $\beta_x$,

$$\beta x = \tan \beta x = \frac{\Delta dxy}{y} = \frac{M_x d_x}{EI}$$

Substitution yields $m_x \beta x = m_x \dfrac{M_x d_x}{EI}$

Then the expression for internal work resulting from flexural strains in the total length becomes $\int_O^L m_x \dfrac{M_x d_x}{EI}$

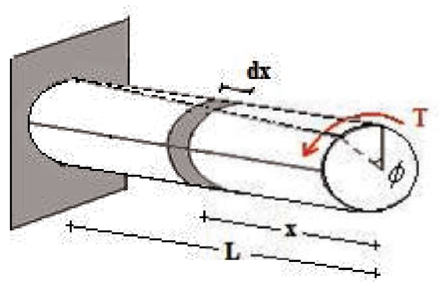
Hence, for linear deflection $1 \times \Delta = \int_O^L M_x \dfrac{m_x d_x}{EI}$,

For rotational deflection $1 \times \theta = \int_O^L m_\alpha \dfrac{M d_x}{EI}$,

In which $M$ is the internal moment due to real loads, $m_x$ and $m_\alpha$ are the internal moments caused due to unit fictitious force and fictitious couple, respectively.

**ii) Deflection resulting from torsional strains**

Consider the cantilever shaft under real torque $T$.



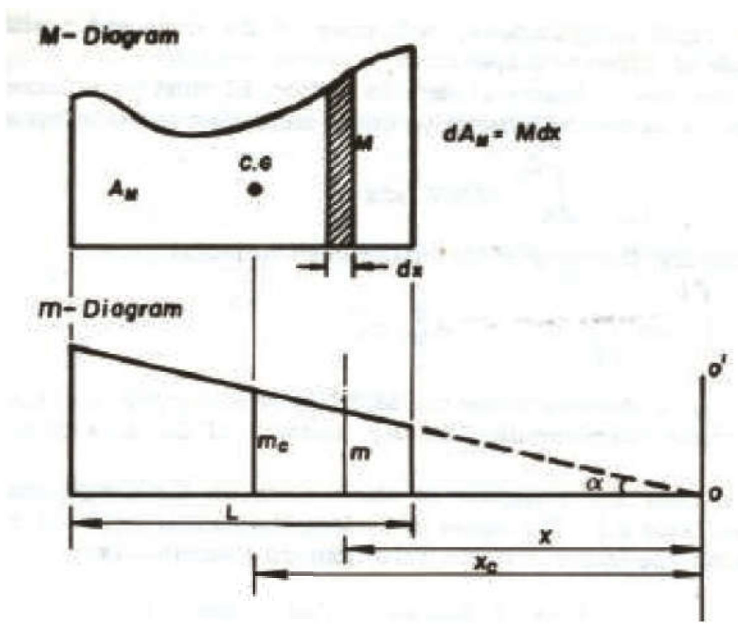

Due to the applied torque, $T$, the free end rotates a $\phi$ rad. The fictitious torque $t$ will move through $\phi$ radians $t\phi = t\dfrac{td_x}{JG}$, for a differential length $dx$. For the entire length $L$, the internal virtual work is given as: $\int_O^L t \dfrac{Td_x}{JG}$.

Thus, $1 \times \Delta = \int_O^L t \dfrac{Td_x}{JG}$ and $1 \times \theta = \int_O^L t \dfrac{Td_x}{JG}$

### 4.5 Graphical Multiplication

Displacement computations may be simplified considerably by the introduction of a special technique known as the graphical multiplication method for the calculation of product integrals belonging to the type $\int_0^L mMdx$.

Note that the integral contains the products of two ordinates to the *m* and *M* curves. For the technique to apply, at least one of the curves must be a straight line while the other may be bounded by any curve.




Since $m = x \tan\alpha$

$$\int_0^L mMdx = \tan\alpha \int_0^L xMdx = \tan\alpha \int_0^L xdA_M$$

Where:

$Mdx = dA_M$ represents the differential of the area $A_M$ bounded by the $M$ - curve.

Consequently, $\int_0^L xdA_M = A_M x_c$

where: $Q$ = statical moment of the area $A_M$ about o -o' axis

$Xc$ = abscissa of the centroid of $A_M$

Therefore, $\int_0^L mMdx = x_c(\tan\alpha)A_M = A_M x_c$

Hence, the product of the multiplication of two graphs, one of which at least is bounded by a straight line, equals the area bounded by the area of the graph of arbitrary outline multiplied by the ordinate to the first graph measured along the vertical passing through the centroid of the second one. It should be noted that the ordinate $m_c$ must be measured on the graph bounded by a straight line.

When a graph of a trapezoid shape or of a quadratic parabola has to be multiplied by another trapezoid graph, it is convenient to subdivide into triangular and parabolic segments.

For rapid computations, evaluation of the areas and positions of centroids of different shapes must be readily available. In the case of beams of variable section, *EI* must be included under the sign. A numerical integration could be applied to the integral

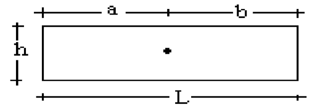
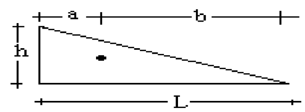
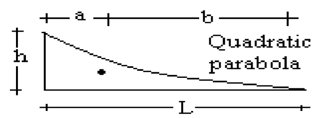
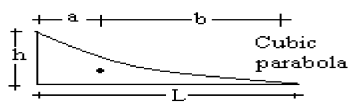
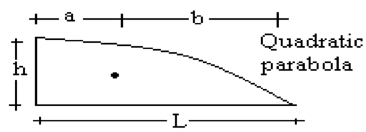
$$\int_0^L m(\frac{M}{EI})dx$$

Alternatively, the value of the integral can be found from

$$\int_0^L m(\frac{M}{EI})dx = A_M^* m_c^*$$

where $A_M^*$ is the area under the *M/EI* curve and $m_c^*$ is the ordinate of the m-curve corresponding to the centroid of the area of the *M/EI* curve.

The areas and positions of their centroids for simple curves are given in the following Table.

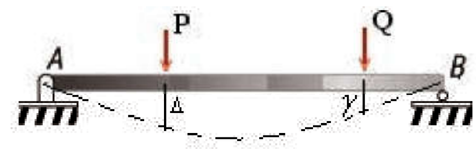
| Shape of graph | Area | Position of centroid | |
|---|---|---|---|
| | | a | b |
|   | $hL$ | $L/2$ | $L/2$ |
|   | $hL/2$ | $L/3$ | $2L/3$ |
|   | $hL/3$ | $L/4$ | $3L/4$ |
|   | $hL/4$ | $L/5$ | $4L/5$ |
|   | $2hL/3$ | $3L/8$ | $5L/8$ |

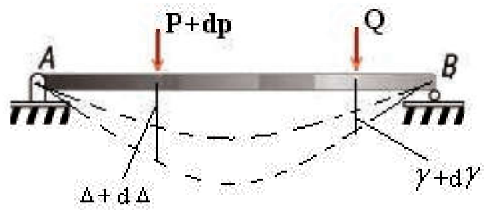
### 4.6. Castiglione's Theorem

The deflection component of the point of application of an action on a structure in the direction of that action will be obtained by evaluating the 1st partial derivative of the total internal strain energy of the structure wrt the applied action.

$$i.e\ \Delta = \partial w / \partial p$$

Consider the simple beam loaded as shown below




Internal strain energy = external work for gradual application (a)

$$\Rightarrow w = \frac{p\Delta}{2} + \frac{Q\gamma}{2} \quad --- (1)$$

When $dp$ is added $lP \Rightarrow$ the resulting increment of internal energy

$$dw = \left(p + \frac{dp}{2}\right) d\Delta \ + Qd\gamma \quad \text{neglecting differential of higher order}$$

$$dw = pd\Delta \ + Qd\gamma \quad --- (2)$$

If $p$ and $dp$ and $Q$ were gradually and simultaneously placed,

Internal strain energy

$$w^1 = \frac{p\Delta}{2} + \frac{dp\Delta}{2} + \frac{pd\Delta}{2} + \frac{Q\gamma}{2} + \frac{Qd\gamma}{2} \quad --- (3)$$

But $dw = w^1 - w$

$$dw = \frac{dp\Delta}{2} + \frac{pd\Delta}{2} + \frac{Qd\gamma}{2} \quad --- (4) \ \{subtract\ 1\ from\ 3\}$$

Upon simplifying using 2,

$$\Delta = \frac{dw}{dp}, \ thus\ the\ thm\ hold$$

Since more than one action will usually be applied to the structure, the general expression for deflection by this theorem should be written as a partial derivative.

$$i.e\ \Delta = \frac{\partial w}{\partial p}$$

Moreover, this may be simplified as, say for internal work resulting from bending as.

$$\Delta = \frac{\partial}{\partial p}\int \frac{M^2 dx}{2EI} = \int M \frac{\partial M}{\partial p}\frac{dx}{EI}$$

Likewise, for axial strain

$$\Delta = \sum S \frac{\partial s}{\partial p}\frac{L}{AE}$$

For rotational deflection, the partial derivates will be taken with respect to a moment and is written as:

$$\theta = \int M \frac{\partial M}{\partial M_o}\frac{dx}{EI}$$

Similar expression for deflection resulting from torsional strains could be applied as:

$$\Delta = \int T \frac{\partial T}{\partial p}\frac{dx}{JG} \quad and\ \theta = \int T \frac{\partial T}{\partial M_o}\frac{dx}{JG}$$

During computation, if the sign of the answer is negative, the actual deflection is opposite to the sense of the action with respect to which the derivative taken. If a deflection component is required for a pt where no action is applied or if an action exist at the point but not in the direction of the desired deflection component, and then an imaginary action is applied at the point and in the desired direction until the derivative of the total internal strain energy has been found. The imaginary action is then reduced to zero.

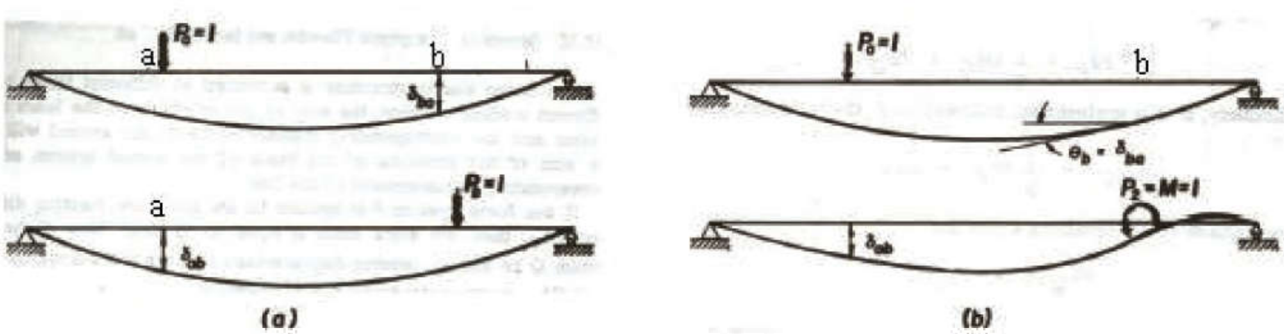
### 4.7 Betti's law and Maxwell's 2nd Theorem of reciprocal deflections

The principle of reciprocal deflections is one of the most important in the theory of structures and has a wide application in the analysis of statically indeterminate structures. Maxwell's Reciprocal Theorem may be stated as follows:

*In any elastic system, the displacement caused by a unit load along the line of action of another unit load is equal to the displacement due to the second unit load along the line of action of the first load. The theorem for the beam shown below can be expressed as:*

$$\delta_{ab} = \int m_a (\frac{m_b dx}{EI}) = \int m_b (\frac{m_a dx}{EI}) = \delta_{ba}$$

In a similar manner for the beam shown below, Maxwell's Theorem states that the slope (rotational displacement) at point ***b*** due to a unit force at ***a*** is equal to the linear displacement at ***a***, due to a unit couple at ***b***.




That is; $\delta_{ab} = \int m_6 (\frac{m_a dx}{EI}) = \int m_a (\frac{m_6 dx}{EI}) = \delta_{ba}$

It will be found that Maxwell's Reciprocal Theorem is perfectly general. Using the symbol δ to indicate any type of displacement, the theorem can be written as

$$\delta_{ij} = \delta_{ji}$$

This equation expresses Maxwell's law. That is, for a linearly elastic structure, the displacement at point ***i*** due to a unit load at point ***j*** is equal to the displacement at point ***j*** due to a unit load at point ***i***.

# CHAPTER 5

## 5. The Consistent Deformation Method

The method of consistent deformations, or sometimes referred to as the force or flexibility method, is one of the several techniques available to analyze indeterminate structures. The following is the procedure that describes the concept of this method for analyzing externally indeterminate structures with single or double degrees of indeterminacy.

**Principle**: - Given a set of forces on a structure, the reactions must assume such a value as are not only in static equilibrium with the applied forces but also satisfy the conditions of geometry at the supports as well as the indeterminate points of the structure.

This method involves with the replacement of redundant supports or restrains by unknown actions in such a way that one obtains a basic determinate structure under the action of the applied loading and these unknown reactions or redundant. Then, the derived basic determinate structure must still satisfy the physical requirements at the location of the excess supports now replace by redundant reactions.

### 5.1 Analysis of Indeterminate Beams by Consistent Deformation Method

The basic procedures to solve intermediate beams by the method of consistent deformation method are as follows:

- determine the degree of indeterminacy
- select redundant and remove restraint
- determine reactions and draw moment diagram for the primary structure
- calculate deformations at redundant due to the applied and unit loads
- write consistent deformation equation
- solve consistent deformation equation
- determine support reactions
- draw moment, shear, and axial load diagrams

For illustration consider the beam loaded as shown

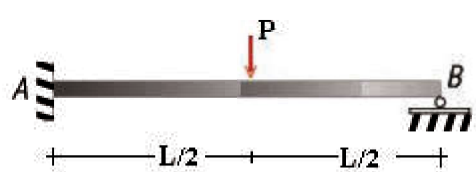



Fig 5.1 (a)

Basic determinate beam under applied loading is shown below.

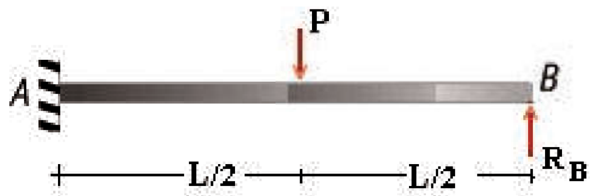



Fig 5.1 (b)

Basic beam under the reaction of $R_B = 1$

The condition of geometry dictates that

$$\Delta_B - R_B\delta_B = 0 \quad \Rightarrow R_B = \frac{\Delta_B}{\delta B} = 0$$

Using either of the methods discussed in chapter four, compute the deflection of end *B* due to the applied load P and the redundant reaction $R_B$.

$$\Delta_B = \frac{5}{48}\frac{PL^3}{EI} \quad \text{and} \quad \delta_B = \frac{R_B L^3}{3EI}$$

From statics, $R_B = \frac{11}{16}P$ and $M_A = \frac{3}{16}PL$

**5.2 Analysis of Indeterminate Trusses by Consistent Deformation Method**

The method essentially consists of choosing a basic determinate truss (structure) on which the applied loading and the redundant force act and the applying conditions of geometry requiring the deflection in the direction of the redundant force must be zero or specified value. Once the redundant are determined, the member forces and other desired reaction components can be determined by the principle of super position.

For illustration consider the followings.

***i) Indeterminate to the 1st degree***

The truss shown in Fig. 5.2 (a) below is indeterminate to the 1st degree internally. A basic determinate structure shown in Fig. 5.2 (b) is selected with external redundant $H_D$ (the horizontal reaction).

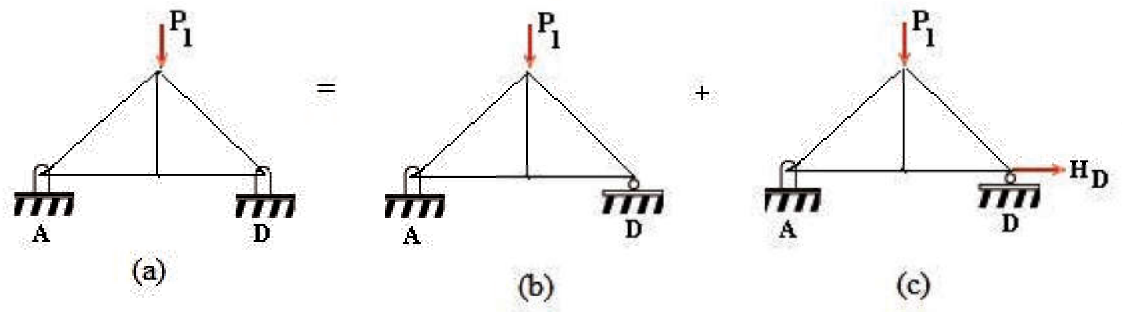



Fig. 5.2

From the geometry of the original structure shown in Fig.5.2(b), the horizontal displacement of support D due to the applied load becomes:

$$\Delta_0 = \Sigma u_1 \frac{SL}{AE}$$

Similarly, for Fig. 5.2 (c), the horizontal displacement of support D due to the fictitious load $u_1$=1kN, becomes:

$$\delta_1 = \Sigma \frac{u_1^{\,2} L}{AE}$$

The horizontal displacement of support D of the actual structure is zero. Thus the following equation holds true.

$$\Delta_0 + H_D \delta_1 = 0 \;,\; \Rightarrow H_D = -\frac{\Delta_0}{\delta_1}$$

***ii) Indeterminate to the 2<sup>nd</sup> degree***

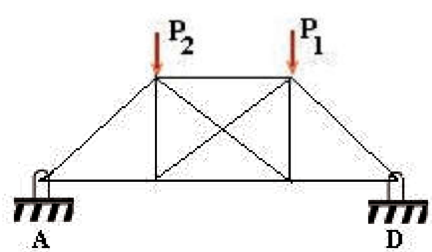



Fig 5.3

The given truss is indeterminate to the $1^{st}$ degree internally and to the $1^{st}$ degree externally. A basic determinate structure shown is selected with external redundant $H_D$ (the horizontal reaction) and the internal redundant (diagonal member *f*).

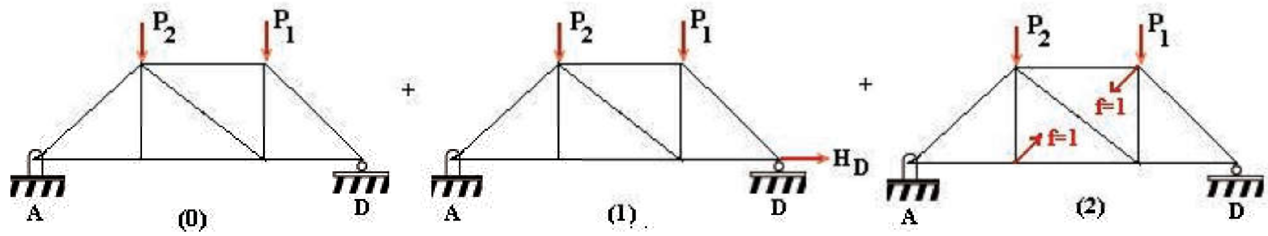



Fig. 5.4

From the geometry of the original structure (Fig. 5.3):

$$\Delta_{10} = \Sigma u_1 \frac{SL}{AE}, \Delta_{20} = \Sigma u_2 \frac{SL}{AE}$$

$$\delta_{11} = \Sigma u_1 \frac{u_1 L}{AE}, \delta_{22} = \Sigma u_2 \frac{u_2 L}{AE}$$

$$\delta_{12} = \delta_{21} = \sum u_1 \frac{u_2 L}{AE}$$

Then, the consistent deformation equation is

$$\begin{cases} \Delta_{10} + \delta_{11} H_D + \delta_{12} * f = 0 \\ \Delta_{20} + \delta_{21} H_D + \delta_{22} * f = 0 \end{cases}$$

For this the following format is of great value

| Member | Length | Area | $S$ | $u_1$ | $u_2$ | $E\Delta_{10}$ $\frac{u_1 SL}{A}$ | $E\Delta_{20}$ $\frac{u_2 SL}{A}$ | $E\delta_{11}$ $\frac{u_1 u_1 L}{A}$ | $E\delta_{21}$ $\frac{u_1 u_2 L}{A}$ | $E\delta_{22}$ $\frac{u_2 u_2 L}{A}$ | $Fi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *ab* | | | | | | | | | | | |
| . | | | | | | | | | | | |
| . | | | | | | | | | | | |
| *yz* | | | | | | | | | | | |

The final member force is obtained as $F_i = S_i + H_D u_{1i} + f u_{2i}$

Where $S$ – internal member forces of basic determinate structures under the action of the applied load

$u_i$ – internal member forces under the action of unit load at the redundant, (u=1)