CHAPTER-VI

Numerical Solution of Ordinary Differential Equations

6.1 Introduction

Differential equations are equations composed of an unknown function and its derivatives. The following are examples of differential equations:

$$\frac{dv}{dt} = g - \frac{c}{m}v \tag{6.1a}$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{6.1b}$$

Differential equations can be classified as *ordinary differential equations* (ODE) or partial *differential equations* (PDE). ODEs are equations in which the function involves *one* independent variable. PDE's involve functions two or more variables. Both Eq.(6.1a) and Eq.(6.1b) are ordinary differential equations. Differential equations are also classified as to their order. For Eq.(6.1a) is a first-order equation because the highest derivative is a first derivative. A second order equation includes a second derivative, as in Eq. (6.1b) above. Similarly, an n^{th} order equation includes an *n*th derivative.

This chapter is devoted to solving ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y) \tag{6.2}$$

One-step Runge-Kutta (RK) methods can be generally expressed as

$$y_{i+1} = y_i + \phi h \tag{6.3}$$

According to this equation, the slope estimate of ϕ is used to extrapolate from an old value y_i to new value y_{i+1} over a distance h. This formula can be applied step by step to compute values of y. The simplest approach is to use the differential equation to estimate the slope in the form of first derivative at x_i.

6.2 Euler's Method

The first derivative provides a direct estimate of the slope at x_i. That is,

$$\phi = f(x_i, y_i) \tag{6.4}$$

where $f(x_i, y_i)$ is the differential equation evaluated at x_i and y_i. This estimate can be substituted into Eq. (6.3):

$$y_{i+1} = y_i + f(x_i, y_i)h \tag{6.5}$$

This formula is referred to as *Euler's method*.

Error Analysis for Euler's Method

As in any other numerical procedure, the numerical solution of ODEs involves two types of errors (1) Truncation error, and (2) Round-off errors. The truncation errors are composed of two parts. The first is a *local truncation error* that results from the application of the method over a single step. The second is a *propagated truncation error* that results from the approximation produced during the previous steps. The sum of the two is the total or *global truncation error*.

Insight into the magnitude and properties of the truncation error can be gained by deriving Euler's method directly from the Taylor series expansion. Remember that the differential equation to be solved will be of the general form

$$y' = f(x, y) \tag{6.6}$$

where $y' = dy/dx$, and x and y are the independent and the dependent variables, respectively. If the solution has continuous derivatives, it can be represented by a Taylor series expansion about a starting value (x_i, y_i),

$$y_{i+1} = y_i' + y_i' h + \frac{y_i''}{2!} h^2 + ... + \frac{y_i^{(n)}}{n!} h^n + R_n \tag{6.7}$$

where $h = x_{i+1} - x_i$, and R_n = the remainder term, defined by

$$R_n = \frac{y^{(n+1)}(\xi)}{(n+1)!} h^{n+1} \tag{6.8}$$

where ξ lies somewhere in the interval from x_i to x_{i+1}. An alternative form can be developed by substititing Eq.(6.6) into Eq.(6.7) and (6.8) to yield

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{f'(x_i, y_i)}{2!} h^2 + ... + \frac{f^{(n-1)}(x_i, y_i)}{n!} + O(h^{n+1}) \tag{6.9}$$

where $O(h^{n+1})$ specifies that the local truncation error is proportional to the step size raised to the $(n+1)th$ power.

Thus we can see that Euler's method corresponds to the Taylor series up to and including the term $f(x_i, y_i)h$. The true local truncation error, E_t, in the Euler method is thus given as

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + ... + O(h^{n+1}) \tag{6.10}$$

For sufficiently small h, the errors in the terms in Eq.(6.10) usually decrease as the order increases, and the result is often represented as

$$E_a = \frac{f'(x_i, y_i)}{2!} h^2 \tag{6.11}$$

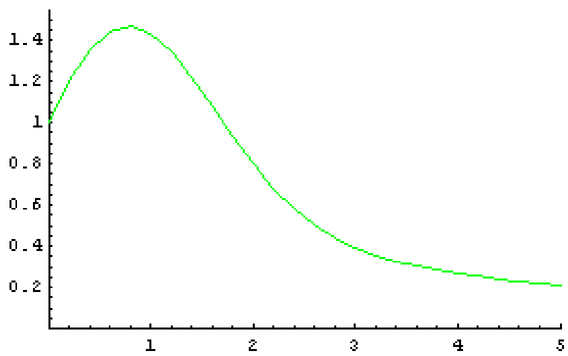
or

$$E_a = O(h^2) \tag{6.12}$$

where E_a = the approximate local truncation error.

Example Using Euler's method solve the following initial value problem

$y' = 1 - t y$ with $y(0) = 1$ over $0 \le t \le 5$



The Euler solution for $y' = 1 - t y$
Using $n = 26$ points.

{{0., 1.}, {0.2, 1.2}, {0.4, 1.352}, {0.6, 1.44384}, {0.8, 1.47058}, {1., 1.43529}, {1.2, 1.34823}, {1.4, 1.22465}, {1.6, 1.08175}, {1.8, 0.935591}, {2., 0.798778}, {2.2, 0.679267}, {2.4, 0.580389}, {2.6, 0.501803}, {2.8, 0.440865}, {3., 0.393981}, {3.2, 0.357592}, {3.4, 0.328733}, {3.6, 0.305195}, {3.8, 0.285454}, {4., 0.268509}, {4.2, 0.253702}, {4.4, 0.240592}, {4.6, 0.228871}, {4.8, 0.21831}, {5., 0.208732}}

The final value is $y(5) = y_{26} = 0.208732$

6.3 Runge-Kutta Methods

Runge-Kutta (RK) methods achieve the accuracy of a Taylor series approach without having requiring the calculation of higher derivatives. Many variations exist but all can be cast in the generalized form:

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h \tag{6.13}$$

where $\phi(x_i, y_i, h)$ is called an increment function, which can be interpreted as a representative slope over the interval. The increment function can be written in the general form as

$$\phi = a_1 k_1 + a_2 k_2 + ... + a_n k_n \tag{6.14}$$

where the a's are constants and the k's are

$$k_1 = f(x_i, y_i) \tag{6.15a}$$

$$k_2 = f(x_i + p_1 h, .y_i + q_{11} k_1 h) \tag{6.15b}$$

$$k_3 = f(x_i + p_2 h, y_i + q_{22} k_2 h) \tag{6.15c}$$

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$$k_n = f(x_i + p_{n-1} h, y_i + q_{n-1,2} k_2 h + ... + q_{n-1,n-1} k_{n-1} h) \tag{6.15d}$$

Notice that the k's are recurrence relationships. Various types of Runge-Kutta methods can be devised by employing different numbers of terms in the increment function as specified by n. Note that the first-order RK method with n = 1 is, in fact, Euler's method. Once n is chosen, values for the *a's, p's* and *q's* are evaluated by setting eqn. 6.14 equal to terms in a Taylor series expansion.

Fourth-order Runge-Kutta Methods

The most popular and relatively accurate RK methods are the fourth order. There are an infinite number of versions. The following, known as the *classical fourth-order RK method*, is the most commonly used form

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{6.16}$$

where

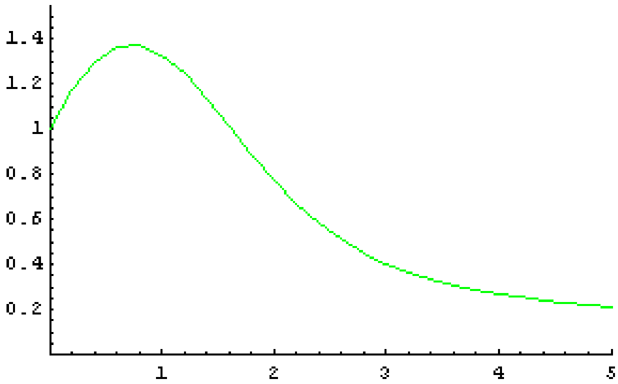
$$k_1 = f(x_i, y_i)h \qquad k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2\right)h$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1\right)h \qquad k_4 = f(x_i + h, y_i + k_3)h$$

NB. The nth order RK methods have local errors of $o(h^{n+1})$ and global error of $o(h^n)$

Example Using 4^{th} order Runge-Kutta methods solve the following initial value problem

$$y' = 1 - t\,y \text{ with } y(0) = 1 \text{ over } 0 \le t \le 5.$$



The Runge-Kutta solution for $y' = 1 - t\,y$

Using $n = 26$ points.

{{0., 1.}, {0.2, 1.17755}, {0.4, 1.30245}, {0.6, 1.36819}, {0.8, 1.37546}, {1., 1.3313}, {1.2, 1.24732}, {1.4, 1.13727}, {1.6, 1.01473}, {1.8, 0.89124}, {2., 0.77533}, {2.2, 0.672211}, {2.4, 0.584154}, {2.6, 0.511195}, {2.8, 0.451948}, {3., 0.404325}, {3.2, 0.366085}, {3.4, 0.335157}, {3.6, 0.309811}, {3.8, 0.288687}, {4., 0.270764}, {4.2, 0.255297}, {4.4, 0.241752}, {4.6, 0.229741}, {4.8, 0.218982}, {5., 0.209267}}

The final value is $y(5) = y_{26} = 0.209267$