CHAPTER -III
LINEAR ALGEBRAIC EQUATIONS

3.1 Introduction

In this chapter, we will deal with the case of determining the values of $x_1, x_2, ..., x_n$ that simultaneously satisfy the set of equations:

$$\begin{aligned} &f_1(x_1, x_2, ..., x_n) = 0 \\ &f_2(x_1, x_2, ..., x_n) = 0 \\ &...... \\ &f_n(x_1, x_2, ..., x_n) = 0 \end{aligned} \tag{3.1}$$

In particular we will consider *linear algebraic equations* which are of the form:

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ &.... \\ &a_{n1}x_1 + a_{n12}x_2 + ... + a_{n1n}x_n = b_n \end{aligned} \tag{3.2}$$

where the a's are constant coefficients, the b's are constants, and n is the number of equations.

The above system of linear equations may also be written in matrix form as:

$$[A]\{X\} = \{B\} \tag{3.3}$$

where $[A]$ is an n by n square matrix of coefficients (called the *coefficient matrix*),

$$[A] = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{1n} & a_{2n} & ... & a_{nn} \end{bmatrix}$$

$\{B\}$ is an n by 1 column vector of constants,

$$\{B\}^T = \lfloor b_1 \quad b_2 \quad ... \quad b_n \rfloor$$

and $\{X\}$ is an n by 1 column vector of unknowns:

$$\{X\} = \lfloor x_1 \quad x_2 \quad ... \quad x_n \rfloor$$

3.2 Non - Computer methods

There are some non-computer methods which are used to solve small ($n \leq 3$) sets of simultaneous equations that do not require a computer.

3.2.1 The Graphical Method

Plotting the graphs (straight lines) and finding the point of intersection of the graphs.

3.2.2 Cramer's Rule

Cramer's rule states that the solution of a set of linear equations given in Eq. 3.3 can be give as:

$$x_i = \frac{D_i}{D} \tag{3.4}$$

where D is the determinant of the coefficient matrix $[A]$, and D_i is the determinant of the matrix obtained by replacing the coefficients of the unknown x_i in the coefficient matrix by the constants b_1, b_2, ..., b_n. For example, x_1 can be obtained as:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}$$

For more than three equations, Cramer's rule becomes impractical because, as the number of equations increase, the determinants are time-consuming to evaluate by hand (or by computer). Consequently, more efficient alternatives are used.

3.2.3 Elimination of Unknowns

The basic strategy is to multiply the equations by constants so that one of the unknowns will be eliminated when the equations are combined. The result is a single equation that can be solved for the remaining unknown. This can then be substituted into either of the original equations to compute the other variable.

The elimination of unknowns can be extended to systems with more than two or three equations. However, the numerous calculations that are required for larger systems make the method extremely tedious to implement by hand. However, the technique can be formalized and readily programmed for the computer.

3.3 Gauss Elimination

3.3.1 Description of the method

The approach is designed to solve a general set of n equations:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \tag{3.5a}$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \tag{3.5b}$$

$$....$$

$$a_{1n}x_1 + a_{1n}x_2 + ... + a_{nn}x_n = b_n \tag{3.5c}$$

The **naive Gauss elimination** method consists of two phases:

1. *Forward Elimination*: The first step is designed to reduce the set of equations to an upper triangular system. The initial step will be to eliminate the first unknown x_1, from the second through the n^{th} equations. To do this, multiply Eq. (3.5a) by a_{21}/a_{11} to give:

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + ... + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1 \tag{3.6}$$

Now this equation can be subtracted from Eq. 3.5b to give:

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + ... + \left(a_{2n} - \frac{a_{21}}{a_{11}}\right)a_{1n}x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

or

$$a'_{22}x_2 + ... + a'_{2n}x_n = b'_2$$

where the prime indicates that the elements have been changed from their original values.

The procedure is then repeated for the remaining equations. For instance, Eq. (3.5a) can be multiplied by a_{31}/a_{11} and the result subtracted from the third equation. Repeating the procedure for the remaining equations results in the following modified system:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1 \tag{3.7a}$$
$$a'_{22}x_2 + a'_{23}x_3 + ... + a'_{2n}x_n = b'_2 \tag{3.7b}$$
$$a'_{32}x_2 + a'_{33}x_3 + ... + a'_{3n}x_n = b'_3 \tag{3.7c}$$
$$.....$$
$$a'_{n2}x_2 + a'_{n3}x_3 + ... + a'_{nn}x_n = b'_n \tag{3.7d}$$

For the foregoing steps, Eq.(3.5a) is called the *pivot equation*, and a_{11} is the *pivot coefficient* or *element*.

Now repeat the above to eliminate the second unknown from Eq. (3.7c) through Eq. (3.7d). To this multiply Eq. (3.7b) by a'_{32}/a'_{22}, and subtract the result from Eq. (3.7c). Perform a similar elimination for the remaining equations to yield

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{13}x_n &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 + ... + a'_{2n}x_n &= b'_2 \\ a''_{33}x_3 + ... + a''_{3n}x_n &= b''_3 \\ ..... \\ a''_{n3}x_3 + ... + a''_{nn}x_n &= b''_n \end{aligned}$$

where the double prime indicates that the elements have been modified twice.

The procedure can be continued using the remaining pivot equations. The final manipulation in the sequence is to use the $(n-1)$th equation to eliminate the x_{n-1} term from the nth equation. At this point, the system will have been transformed to an upper triangular system:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_{1n} \quad (3.8a)$$

$$a'_{22}x_2 + a'_{23}x_3 + ... + a'_{2n}x_n = b'_2 \quad (3.8b)$$

$$a''_{33}x_3 + ... + a''_{3n}x_n = b''_3 \quad (3.8c)$$

$$\vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)} \quad (3.8d)$$

2. *Back Substitution*: Eq. (3.8d) can now be solved for x_n:

$$x_n = \frac{b_n^{(n-1)}}{a_n^{(n-1)}} \quad (3.9)$$

This result can be back substituted into the $(n-1)$th equation to solve for x_{n-1}. The procedure, which is repeated to evaluate the remaining x's, can be represented by the following formula:

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \quad \text{for } i = n-1, n-2, ..., 1 \quad (3.10)$$

3.3.2 Pitfalls of Gauss Elimination

Whereas there are many systems of equations that can be solved with naive Gauss elimination, there are some pitfalls that must be explored before writing general computer program to implement the method.

i, Division by Zero

The primary reason that the foregoing technique is called "naive" is that during both elimination and back-substitution phases, it is possible that a division by zero can occur. Problems also arise when the coefficient is very close to zero. The technique of *pivoting* (to be discussed later) has been developed to partially avoid these problems.

ii, Round-off Errors

The problem of round-off errors can become particularly important when large numbers of equations are to be solved. A rough rule of thumb is that round-off errors may be important when dealing with 100 or more equations. In any event, one should always substitute the answers back into the original equations to check whether a substantial error has occurred.

iii, Ill conditioned Systems

Ill-conditioned systems are those where a small change in coefficients in large changes in the solution. An alternative interpretation of ill-conditioning is that a wide range of answers can approximately satisfy the equations.

An ill-conditioned system is one with a determinant of the coefficient matrix close to zero. It is difficult to specify how close to zero the determinant must be to indicate ill-conditioning. This is complicated by the fact that the determinant can be changed by multiplying one or more of the equations by a scale factor without changing the solution. One way to avoid this difficulty is to scale the equations so that the maximum element in any row is equal to 1 (This process is called *scaling*).

iv, Singular Systems

The system is singular when at least two of the equations are identical. In such cases, we would lose one degree of freedom, and would be dealing with impossible case of $n-1$ equations in n unknowns. Such cases might not be obvious particularly when dealing with large equation sets. Consequently, it would be nice to have some way of automatically detecting singularity. The answer to this problem is neatly offered by the fact that the determinant of a singular system is zero. This idea can, in turn, be connected to Gauss elimination by recognizing that after the elimination step, the determinant can be evaluated as the product of the diagonal elements. Thus, a computer algorithm can test to discern whether a zero diagonal element is created during the elimination stage.

3.3.3 Techniques for Improving Solutions

1. Use of more significant figures.
2. ***Pivoting***: can be partial or complete.

Partial Pivoting: Determine the largest available coefficient in the column below the pivot element. The rows are then switched so that the largest element is the pivot element.

Complete Pivoting: When columns as well as rows are switched.

3. ***Scaling***: Scaling is the process by which the maximum element in a row is made to be 1 by dividing the equation by the largest coefficient.

Gauss-Jordan Elimination

Gauss-Jordan is a variation of the Gauss elimination. The major difference is that when an unknown is eliminated in the Gauss-Jordan method, it is eliminated from all other equations rather than just the subsequent ones. In addition, all rows are normalized by dividing them by their pivot elements. Thus, the elimination step results in an identity matrix rather than a triangular matrix. Thus, back-substitution is not necessary.
The method is attributed to Johann Carl Friedrich Gauss (1777-1855) and Wilhelm Jordan (1842 to 1899).

Example Use the Gauss-Jordan elimination method to solve the linear system

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & 1 & 5 \\ 2 & 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

First form the augmented matrix **M = [A , B]**

$$\mathbf{M} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{3} \\ \mathbf{-3} & \mathbf{1} & \mathbf{5} & \mathbf{-2} \\ \mathbf{2} & \mathbf{4} & \mathbf{-1} & \mathbf{-1} \end{pmatrix};$$

Then perform Gauss-Jordan elimination.

$$\begin{pmatrix} 1 & 2 & 3 & 3 \\ -3 & 1 & 5 & -2 \\ 2 & 4 & -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 7 & 14 & 7 \\ 0 & 0 & -7 & -7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -7 & -7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Hence, the solution is $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

LU-Decomposition

Gauss elimination is a sound way to solve systems of algebraic equations of the form

$$[A]\{X\} = \{B\} \tag{3.11}$$

However, it becomes inefficient when solving equations with the same coefficients $[A]$, but with different right-hand side constants.

LU decomposition methods separate the time-consuming elimination of the matrix $[A]$ from the manipulations of the right-hand side $\{B\}$. Thus, once $[A]$ has been "decomposed", multiple right-hand side vectors can be evaluated in an efficient manner.

3.5.1 Derivation of LU Decomposition Method

Eq. (2.11) can be rearranged to give:

$$[A]\{X\}-\{B\}=0 \tag{3.12}$$

Suppose that Eq. (3.12) could be expressed as an upper triangular system:

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \tag{3.13}$$

Eq. (3.13) can also be expressed in matrix notation and rearranged to give:

$$[U]\{X\}-\{D\}=0 \tag{3.14}$$

Now, assume that there is a lower diagonal matrix with 1's on the diagonal,

$$[L]=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \tag{3.15}$$

that has the property that when Eq. (3.14) is premultiplied by it, Eq. (3.12) is the result. That is,

$$[L]\{[U]\{X\}-\{D\}\}=[A]\{X\}-[B] \tag{3.16}$$

If this equation holds, it follows that

$$[L][U]=[A] \tag{3.17}$$

and

$$[L]\{D\}=\{B\} \tag{3.18}$$

A two-step strategy (see Fig. 3.1) for obtaining solutions can be based on Eqs. (3.14), (3.17) and (3.18):

1. *LU decomposition step*. $[A]$ is factored or decomposed into the lower $[L]$ and upper $[U]$ triangular matrices.
2. *Substitution step*. $[L]$ and $[U]$ are used to determine a solution $\{X\}$ for a right-hand side $\{B\}$. This step itself consists of two steps. First, Eq. (3.18) is used to generate an intermediate vector by forward substitution. Then, the result is substituted into Eq. (3.14) which can be solved by back substitution for $\{X\}$.

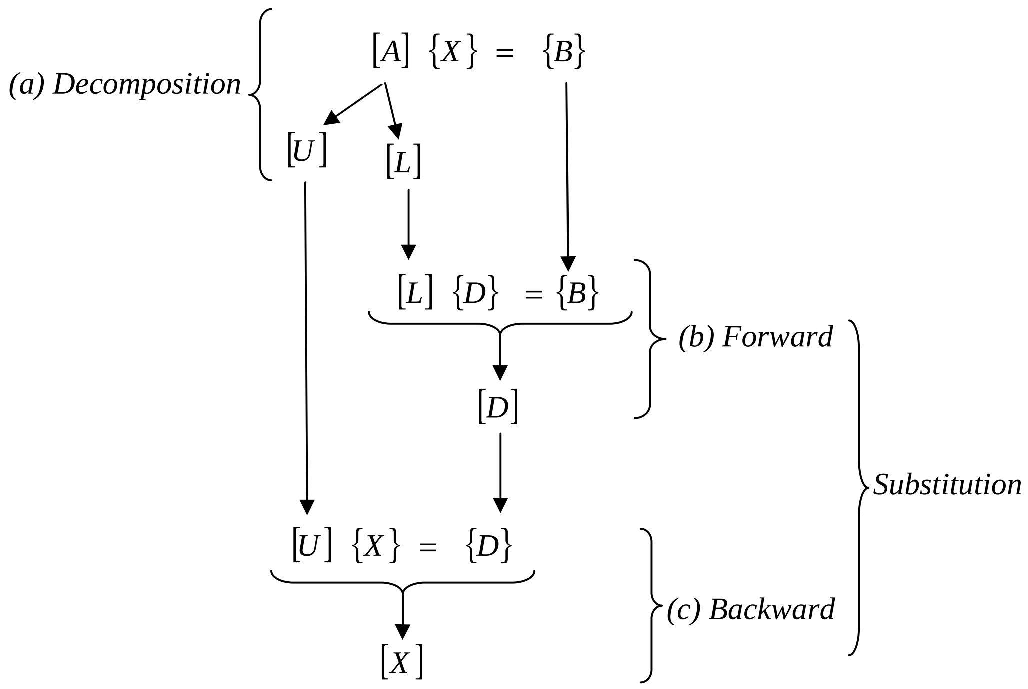

Fig. 3.1 Steps in LU Decomposition

Example Given $A = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$

Find matrices **L** and **U** so that **LU** = **A**.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 3 & \frac{7}{2} \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{2} & \frac{6}{5} & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

Hence,

$$A = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{2} & \frac{6}{5} & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 0 & -4 \end{pmatrix}$$

Gauss-Seidel Method

Iteration is a popular technique finding roots of equations. Generalization of fixed point iteration can be applied to systems of linear equations to produce accurate results. The Gauss-Seidel mehtod is the most common iterative method and is attributed to Philipp Ludwig von Seidel (1821-1896).

Consider that the n×n square matrix **A** is split into three parts, the main diagonal **D**, below diagonal **L** and above diagonal **U**. We have **A = D - L - U**.

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} & a_{n-2,n-1} & a_{n-2,n} \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-2} & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} a_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_{2,2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_{3,3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2,n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1,n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_{n,n} \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} 0 & -a_{1,2} & -a_{1,3} & \cdots & -a_{1,n-2} & -a_{1,n-1} & -a_{1,n} \\ 0 & 0 & -a_{2,3} & \cdots & -a_{2,n-2} & -a_{2,n-1} & -a_{2,n} \\ 0 & 0 & 0 & \cdots & -a_{3,n-2} & -a_{3,n-1} & -a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-2,n-1} & -a_{n-2,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -a_{2,1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -a_{3,1} & -a_{3,2} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -a_{n-2,1} & -a_{n-2,2} & -a_{n-2,3} & \cdots & 0 & 0 & 0 \\ -a_{n-1,1} & -a_{n-1,2} & -a_{n-1,3} & \cdots & -a_{n-1,n-2} & 0 & 0 \\ -a_{n,1} & -a_{n,2} & -a_{n,3} & \cdots & -a_{n,n-2} & -a_{n,n-1} & 0 \end{pmatrix}$$

The solution to the linear system **AX=B** can be obtained starting with $\mathbf{P}_0$, and using iteration scheme

$$\mathbf{P}_{k+1} = \mathbf{M}_S \mathbf{P}_k + \mathbf{C}_S$$

where

$$\mathbf{M}_S = (\mathbf{D} - \mathbf{L})^{-1} \mathbf{U}$$

and

$$\mathbf{C}_S = (\mathbf{D} - \mathbf{L})^{-1} \mathbf{B}.$$

A sufficient condition for the method to be applicable is that **A** is strictly *diagonally dominant*.

Example1 Use Gauss-Seidel iteration to solve the linear system

$$\begin{pmatrix} 7 & -2 & 1 & 2 \\ 2 & 8 & 3 & 1 \\ -1 & 0 & 5 & 2 \\ 0 & 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \\ 4 \end{pmatrix}.$$

Try 10 iterations.

The system can be expressed as

$$\begin{pmatrix} 7 & -2 & 1 & 2 \\ 2 & 8 & 3 & 1 \\ -1 & 0 & 5 & 2 \\ 0 & 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \\ 4 \end{pmatrix}$$

Using 10 iterations we have:

$P_0 = \{0, -1, 1, 1\}$
$P_1 = \{-0.285714, -0.678571, 0.542857, 1.475\}$
$P_2 = \{-0.264286, -0.571875, 0.357143, 1.37522\}$
$P_3 = \{-0.178763, -0.511141, 0.414158, 1.35911\}$
$P_4 = \{-0.164951, -0.53396, 0.423366, 1.37282\}$
$P_5 = \{-0.176704, -0.536189, 0.415531, 1.37198\}$
$P_6 = \{-0.17598, -0.533326, 0.416013, 1.37067\}$
$P_7 = \{-0.174857, -0.533624, 0.416762, 1.371\}$
$P_8 = \{-0.175145, -0.533875, 0.41657, 1.37108\}$
$P_9 = \{-0.175211, -0.533796, 0.416526, 1.37103\}$
$P_{10} = \{-0.175168, -0.533784, 0.416555, 1.37103\}$

Hence,

$$A\,X = \begin{pmatrix} 3.00001 \\ -1.99991 \\ 5. \\ 4. \end{pmatrix} \approx \begin{pmatrix} 3 \\ -2 \\ 5 \\ 4 \end{pmatrix} = B$$

For the purpose of hand calculation let's see 3 set of linear equations containing 3 unknowns.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

If the diagonal elements are all nonzero, the first equation can be solved for x_1, the second for x_2 and the third for x_3 to yield:

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}} \qquad \text{(a)}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}} \qquad \text{(b)}$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}} \qquad \text{(c)}$$

Steps to be followed

i. Using the initial guess $x_2 = x_3 = 0$ solve for x_1 from (a)
ii. Using the values of x_1 from step *i* and $x_3 = 0$ solve for x_2 from (b)
iii. Using the value of x_1 from step *i* and that of x_2 from step *ii* solve for x_3 from (c)
iv. Using the value of x_2 from step *ii* and that of x_3 from step *iii* solve for x_1 from (a)
v. Using the value of x_1 from step *iv* and that of x_3 from step *iii* solve for x_2 from(b)
vi. Using the value of x_1 from step *iv* and that of x_2 from step *v* solve for x_3 from(c)
vii. Repeat the process until the required accuracy is achieved.

Example 2 Use the Gauss-Seidel method to obtain the solution of the following system of linear equations.

$$5x_1 - x_2 + x_3 = 4$$
$$x_1 + 3x_2 + x_3 = 2$$
$$-x_1 + x_2 + 4x_3 = 3$$

Solving for: x_1 from eq1 $\quad x_1 = \dfrac{4 + x_2 - x_3}{5}$

x_2 from eq2 $\quad x_2 = \dfrac{2 - x_1 - x_3}{3}$

x_3 from eq3 $\quad x_3 = \dfrac{3 + x_1 - x_2}{4}$

Executing the above steps repetitively we will have the following result.

x_1	x_2	x_3
0.8	0.4	0.85
0.71	0.146667	0.890833
0.651167	0.152667	0.874625
0.655608	0.156589	0.874755
0.656367	0.156293	0.875019
0.656255	0.156242	0.875003
0.656248	0.15625	0.875
0.65625	0.15625	0.875
0.65625	0.15625	0.875

As we can see the values start to repeat after the 8th iteration hence we can stop the calculation and take the final values as the solution of the linear system of equations.

Hence, x_1 = 0.65625

x_2 = 0.15625

x_3 = 0.875