CHAPTER-II
ROOTS OF EQUATIONS

2.1 Introduction

The roots or zeros of equations can be simply defined as the values of *x* that makes f(x) =0. There are many ways to solve for roots of equations. For some cases, the roots can be found easily by solving the equations directly. However, there are also other cases where solving the equations directly or analytically is not so possible. In such instances, the only alternatives will be approximate solution techniques. There are several techniques of such type and some of them will be discussed in this chapter.

2.2 Graphical method

One alternative to obtain an approximate solution is to plot the function and determine where it crosses the x-axis. This point, which represents the x-value for which f(x)=0 is the root.
Graphical techniques are of limited practical value because they are not precise. However, graphical methods can be utilized to obtain rough estimates of the roots. These estimates can be employed as starting guesses for numerical methods which will be discussed in the next sections.

2.3 The Bisection method

If a function f is real and continuous in an interval from *a* to *b* and *f(a)* and *f(b)* have opposite signs then there exists at least one real root *r* between a and b such that *f(r)* =0.

***Procedure for Bisection method*:**

The procedure in this method consists of continuously halving the interval that contains the root.

i. Choose two initial guesses *a* and *b* which bracket the root. i.e. *f(a)* and *f(b)* have opposite signs [f(a). f(b)<0]. This can be done by evaluating the function at *a* and *b* or by plotting the graph of the function.

ii. Estimate the root *c* from

$$c = \frac{a+b}{2}$$

iii. Make the following evaluations to determine the interval in which the root lies

- If *f(a).f(c)* < 0 ,the root lies in the sub interval (a,c); then b=c and return to step ii
- If *f(a).f(c)* > 0 ,the root lies in the sub interval (c,b); then a=c and return to step ii
- If *f(a.)f(c)* = 0 , then x= c is an exact solution ; terminate the computation

We repeat this procedure until either the exact root has been found or the interval is smaller than some specified tolerance.

Example Find the real solution to the cubic equation $x^3 + 4x^2 - 10 = 0$.

By plotting the graph of the function we can find that there is a real root between 1 and 2 where the graph crosses the x-axis and carrying out the Bisection technique on the interval [1,2] using 10 iterations we have the following:

k	a_k	c_k	b_k	$f[c_k]$
0	1.	1.5	2.	2.375
1	1.	1.25	1.5	-1.796875
2	1.25	1.375	1.5	0.162109375
3	1.25	1.3125	1.375	-0.848388671875
4	1.3125	1.34375	1.375	-0.350982666015625
5	1.34375	1.359375	1.375	-0.0964088439941406
6	1.359375	1.3671875	1.375	0.03235578536987305
7	1.359375	1.36328125	1.3671875	-0.03214997053146362
8	1.36328125	1.365234375	1.3671875	0.00007202476263046265
9	1.36328125	1.3642578125	1.365234375	-0.01604669075459242
10	1.3642578125	1.36474609375	1.365234375	-0.007989262812770903

c = 1.36474609375

Δc = ±0.000488281

f[c] = -0.007989262812770903

After 10 iterations, the interval has been reduced to [a,b] where

a = 1.3642578125
b = 1.365234375

[a, b] = [1.36426, 1.36523]

The root lies somewhere in the interval [a,b] the width of which is

b-a = 0.0009765625

The root is alleged to be
c = 1.36474609375
The accuracy we can guarantee is one half of the interval width.

$$\frac{b-a}{2} = 0.00048828125$$

If the required accuracy is not yet found we can still increase the number of iterations till our satisfaction is met.

2.4 The False position or Regula- Falsi method

An alternative method that exploits a graphical insight over the previous technique is the False position method where by we join *f(a)* and *f(b)* by a straight line and the intersection of this line with the x-axis represents an improved estimate of the root.

Using similarity of triangles, the intersection point of the line with the x-axis can be estimated as

$$\frac{f(a)}{c-a} = \frac{f(b)}{c-b}$$

Rearranging and solving for *c* we have

$$c_n = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}$$

c_n represents a sequence of points generated by the Regula Falsi process and $\{c_n\}$ will converge to zero at x = r.

Example Find the real solution to the cubic equation $x^3 + 4x^2 - 10 = 0$.

Previously we have found out that the graph will cross the x-axis between the interval 1 and 2

and carrying out the Regula -Falsi technique in 10 iterations we have:

k	a_k	c_k	b_k	$f[c_k]$
0	1.	1.263157894736842	2.	-1.602274384020995
1	1.263157894736842	1.338827838827839	2.	-0.4303647480045276
2	1.338827838827839	1.358546341824779	2.	-0.1100087884743455
3	1.358546341824779	1.36354744004209	2.	-0.02776209100106009
4	1.36354744004209	1.36480703182678	2.	-0.006983415401172977
5	1.36480703182678	1.365123717884378	2.	-0.001755209032341387
6	1.365123717884378	1.3652033036626	2.	-0.000441063010153453
7	1.3652033036626	1.365223301985543	2.	-0.0001108281334247785
8	1.365223301985543	1.365228327025519	2.	-0.0000278479845592372
9	1.365228327025519	1.365229589673847	2.	$-6.99739040177505 \times 10^{-6}$
10	1.365229589673847	1.365229906940572	2.	$-1.758239715154986 \times 10^{-6}$

c = 1.365229906940572

$f[c] = -1.758239715154986 \times 10^{-6}$

After 10 iterations, the interval has been reduced to [a,b] where

a = 1.365229589673847 b = 2.

The root lies somewhere in the interval [a,b] width of which is 0.6347704103261533

The root we have found above is 1.365229906940572

2.5 Fixed Point Iteration method

Iterative techniques are used to find roots of equations, solutions of linear and nonlinear systems of equations and solutions of differential equations. A rule or function g(x) for computing successive terms is needed and it can be found by rearranging the function f(x) = 0 so that x is on the left side of the equation.

$$x = g(x)$$

Moreover a starting value P_0 is also required and the sequence of values $\{P_k\}$ is obtained using the iterative rule $P_{k+1} = g(P_k)$. The sequence has the pattern

$P_1 = g(P_0)$
$P_2 = g(P_1)$
. .
. .
. .
$P_k = g(P_{k-1})$
$P_{k+1}= g(P_k)$

Note: - A fixed point of a function g(x) is a number P such that P = g(P) but not a root of the equation g(x) = 0.

- Geometrically, the fixed points of a function g(x) are the points of intersection of the curve y = g(x) and the line y = x.

Example Use fixed point iteration to find the fixed point(s) for the function $g(x) = 1 + x - \frac{x^2}{3}$

By plotting the graph of the function we can find that there is a real root between 3 and 7 where the graph crosses the x-axis and performing fixed point iteration between 3 and 7 we have:

```
p0 = 3.0
3.
p1 = g[p0]
1.
p2 = g[p1]
1.66667
p3 = g[p2]
1.74074
p4 = g[p3]
1.73068
p5 = g[p4]
1.73226
```

Hence as we can see the series is converging.

```
p0 =  3.000000000000000
p1 =  1.000000000000000
p2 =  1.666666666666670
p3 =  1.740740740740740
p4 =  1.730681298582530
p5 =  1.732262046161430
p6 =  1.732018113970970
p7 =  1.732055864929790
```

The function is $g[x] = 1 + x - \frac{x^2}{3}$

p = 1.732055864929790

g[p] = 1.732050025183900

If seven iterations are not enough to locate the fixed point we can continue the iteration.

p_0 = 3.000000000000000
p_1 = 1.000000000000000
p_2 = 1.666666666666670
p_3 = 1.740740740740740
p_4 = 1.730681298582530
p_5 = 1.732262046161430
p_6 = 1.732018113970970
p_7 = 1.732055864929790
p_8 = 1.732050025183900
p_9 = 1.732050928604050
p_{10} = 1.732050788844670

Example Use a fixed point iteration method find the real root of $f(x) = x^3 + 4x^2 - 10$.

Rearranging the function f(x) in to x = g(x)

i.e. $X = [(10-x^3)^{1/2}]/2$

$\rightarrow$ $g(x) = [(10-x^3)^{1/2}]/2$

Previously we have found out that the root lies on the interval (1,2)

Starting point $P_o = 1$

		$g(P_i)$
P_o =	1.000000000	1.500000000
P_1 =	1.500000000	1.286953768
P_2 =	1.286953768	1.402540804
P_3 =	1.402540804	1.345458374
P_4 =	1.345458374	1.375170253
P_5 =	1.375170253	1.360094193
P_6 =	1.360094193	1.367846968
P_7 =	1.367846968	1.363887004
P_8 =	1.363887004	1.365916733
P_9 =	1.365916733	1.364878217
P_{10} =	1.364878217	1.365410061
P_{11} =	1.365410061	1.365137821

After 11 iterations, the root has been found that x= 1.365137821 with error of 0.02%.

Exercise Use a fixed point iteration method, find the real root of $f(x) = x^4+2x^2-x-10$ .Use an error tolerance of 0.01%.

2.6 Newton-Raphson Method

If f(x) and f'(x) are continuous near a root P, then this extra information regarding the nature of f(x) can be used to develop algorithms that will produce sequences $\{P_k\}$that converge faster to P than either the bisection or false position method. The Newton-Raphson (or simply Newton's) method is one of the most useful and best known algorithms that relies on the continuity of f(x) and f'(x). The method is attributed to Sir Isaac Newton (1643-1727) and Joseph Raphson (1648-1715).
If the initial guess at the root is P_0, a tangent can be extended from the point $[P_0, f(P_0)]$. The point where this tangent crosses the x-axis usually reperesents an improved estimate of the root and the Newton-Raphson mehtod can be derived based on the basis of this geometrical interpretation and is expressed by the iteration

$$P_{k+1} = g(P_k) = P_k - \frac{f(P_k)}{f'(P_k)} \quad \text{for } k=0, 1, 2,...$$

Example Use Newton- Raphson method to find the three roots of the cubic polynomial.

$$f[x] = 4x^3 - 15x^2 + 17x - 6$$

Determine the Newton-Raphson iteration formula that is used

$$g[x] = x - \frac{f[x]}{f'[x]}$$

Use the starting value $P_0 = 3$.

$$f[x] = -6 + 17x - 15x^2 + 4x^3$$

The Newton-Raphson iteration formula g[x] is

$$g[x] = x - \frac{-6 + 17x - 15x^2 + 4x^3}{17 - 30x + 12x^2}$$

$$g[x] = \frac{6 - 15x^2 + 8x^3}{17 - 30x + 12x^2}$$

Starting with $P_0 = 3$

p0 = 3.0
3.

p1 = g[p0]
2.48571

p2 = g[p1]
2.18342

p3 = g[p2]
2.04045

p4 = g[p3]
2.00265

p5 = g[p4]
2.00001

p6 = g[p5]
2.

Hence,

$p_0 = 3.0000000000000000,\quad f[p_0] = 18.$
$p_1 = 2.4857142857142860,\quad f[p_1] = 5.010192419825074$
$p_2 = 2.1834197620337600,\quad f[p_2] = 1.244567116269891$
$p_3 = 2.0404526629830990,\quad f[p_3] = 0.2172558662514135$
$p_4 = 2.0026544732145300,\quad f[p_4] = 0.01333585694116124$
$p_5 = 2.0000125925878950,\quad f[p_5] = 0.00006296436663433269$
$p_6 = 2.0000000002854240,\quad f[p_6] = 1.427117979346804\times 10^{-9}$
$p_7 = 2.0000000000000000,\quad f[p_7] = 0.$
$p = 2.$

$\Delta p = \pm 2.85424\times 10^{-10}$
$f[p] = 0.$

If we plot the graph we can see that there are two more real roots, using the starting values 0.0 and 1.4 we can find them.

Starting with $P_0 = 0.0$

$p_0 = 0.0000000000000000,\quad f[p_0] = -6.$
$p_1 = 0.3529411764705882,\quad f[p_1] = -1.692652147364137$
$p_2 = 0.5670227828549363,\quad f[p_2] = -0.4541102868356983$
$p_3 = 0.6850503150510711,\quad f[p_3] = -0.1075938266370042$
$p_4 = 0.7367776746893981,\quad f[p_4] = -0.01758613258850938$
$p_5 = 0.7492433382396966,\quad f[p_5] = -0.000949264155361673$
$p_6 = 0.7499972689032857,\quad f[p_6] = -3.413915646177657\times 10^{-6}$
$p_7 = 0.7499999999641980,\quad f[p_7] = -4.475264603343021\times 10^{-11}$
$p_8 = 0.7500000000000001,\quad f[p_8] = -1.110223024625157\times 10^{-15}$
$p = 0.7500000000000001$
$\Delta p = \pm 3.58021\times 10^{-11}$
$f[p] = -1.110223024625157\times 10^{-15}$

Starting with $P_0 = 1.4$

$p_0 = 1.4000000000000000,\quad f[p_0] = -0.6240000000000006$
$p_1 = 0.9783783783783790,\quad f[p_1] = 0.02017870609835537$
$p_2 = 1.0017155262247010,\quad f[p_2] = -0.001724335120004028$
$p_3 = 1.0000086994622160,\quad f[p_3] = -8.69968925432119\times 10^{-6}$
$p_4 = 1.0000000002270260,\quad f[p_4] = -2.270263976811293\times 10^{-10}$
$p_5 = 1.0000000000000000,\quad f[p_5] = 8.88178419700125\times 10^{-16}$

$p = 1.$
$\Delta p = \pm 2.27026\times 10^{-10}$
$f[p] = 8.88178419700125\times 10^{-16}$

2.7 Secant Method

The Newton-Raphson algorithm requires two functions evaluations per iteration, $f(P_k)$ and $f'(P_k)$.However, many functions have derivatives which may be extremely difficult or inconvenient to evaluate. Hence, it is desirable to have a method for finding a root that does not depend on the computation of a derivative. The secant method does not need a formula for the derivative and it can be coded so that only one new function evaluation is required per iteration.

The formula for the secant method is the same one that was used in the regula falsi method, except that the logical decisions regarding how to define each succeeding term are different and is expressed as:

$$p_{k+1} = g(p_{k-1}, p_k) = p_k - \frac{f(p_k)(p_k - p_{k-1})}{f(p_k) - f(p_{k-1})} \quad \text{for} \quad k = 0, 1, \ldots$$

Example Use the secant method to find the three roots of the cubic polynomial.

$$f[x] = 4x^3 - 16x^2 + 17x - 4$$

Determine the secant iteration formula that is used.

- Use the starting values $P_0=3$ and $P_1=2.8$

The secant iteration formula $g[x_0, x_1]$ is

$$p_2 = g[p_0, p_1] = p_1 - \frac{(-p_0 + p_1)(-4 + 17p_1 - 16p_1^2 + 4p_1^3)}{-17p_0 + 16p_0^2 - 4p_0^3 + 17p_1 - 16p_1^2 + 4p_1^3}$$

$$p_2 = g[p_0, p_1] = \frac{4(1 + p_0^2 p_1 + p_0(-4 + p_1)p_1)}{17 + 4p_0^2 + 4p_0(-4 + p_1) - 16p_1 + 4p_1^2}$$

First, do the iteration one step at a time.

p0 = 3.0
3.

p1 = 2.8
2.8

p2 = g[p0, p1]
2.5628

p3 = g[p1, p2]
2.45956

p4 = g[p2, p3]
2.41596

p5 = g[p3, p4]
2.40742

p6 = g[p4, p5] 2.40681

$p_0 =$ 3.0000000000000000, $f[p_0] =$ 11.
$p_1 =$ 2.8000000000000000, $f[p_1] =$ 5.967999999999989
$p_2 =$ 2.5627980922098570, $f[p_2] =$ 1.809778114233566
$p_3 =$ 2.4595609795289590, $f[p_3] =$ 0.5373616335744913
$p_4 =$ 2.4159623116721990, $f[p_4] =$ 0.0880463943210401
$p_5 =$ 2.4074188545030930, $f[p_5] =$ 0.00584986163346457
$p_6 =$ 2.4068108234745750, $f[p_6] =$ 0.00007189617626579548
$p_7 =$ 2.4068032576442970, $f[p_7] = 6.000787777793448 \times 10^{-8}$
$p_8 =$ 2.4068032513242300, $f[p_8] = 6.110667527536862 \times 10^{-13}$
$p_9 =$ 2.4068032513241660, $f[p_9] =$ 0.
$p_{10} =$ 2.4068032513241660, $f[p_{10}] =$ 0.

p = 2.406803251324166
Δp = ±0.
f[p] = 0.

- Using the starting values P_0=0.6 and P_1=0.5

$p_0 =$ 0.6000000000000000, $f[p_0] =$ 1.303999999999999
$p_1 =$ 0.5000000000000000, $f[p_1] =$ 1.
$p_2 =$ 0.1710526315789467, $f[p_2] =$ -1.540229989794438
$p_3 =$ 0.3705048874540401, $f[p_3] =$ 0.3056437021120457
$p_4 =$ 0.3374791576072164, $f[p_4] =$ 0.06861572052792555
$p_5 =$ 0.3279187494467301, $f[p_5] =$ -0.004827220357223688
$p_6 =$ 0.3285471311038295, $f[p_6] =$ 0.00006748859864649792
$p_7 =$ 0.3285384669324079, $f[p_7] = 6.475412978046435 \times 10^{-8}$
$p_8 =$ 0.3285384586113031, $f[p_8] = -8.70692407062279 \times 10^{-13}$
$p_9 =$ 0.3285384586114150, $f[p_9] = 1.387778780781446 \times 10^{-16}$
$p_{10} =$ 0.3285384586114150, $f[p_{10}] = 1.387778780781446 \times 10^{-16}$

p = 0.328538458611415
Δp = ±0.
f[p] = $1.387778780781446 \times 10^{-16}$

- Using the starting values P_0=1.0 and P_1=1.1

$p_0 =$ 1.0000000000000000, $f[p_0] =$ 1.
$p_1 =$ 1.1000000000000000, $f[p_1] =$ 0.6640000000000015
$p_2 =$ 1.2976190476190490, $f[p_2] =$ -0.1417166072778375
$p_3 =$ 1.2628600508339780, $f[p_3] =$ 0.007687910275651078
$p_4 =$ 1.2646486450005470, $f[p_4] =$ 0.00004124933638571804
$p_5 =$ 1.2646582934370370, $f[p_5] = -1.442380259675247 \times 10^{-8}$
$p_6 =$ 1.2646582900644130, $f[p_6] = 2.664535259100376 \times 10^{-14}$
$p_7 =$ 1.2646582900644200, $f[p_7] =$ 0.
$p_8 =$ 1.2646582900644200, $f[p_8] =$ 0.

p = 1.26465829006442
Δp = ±0.
f[p] = 0.