CHAPTER III

LINEAR ALGEBRAIC EQUATIONS

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3.1 INTRODUCTION

- 3.1.1 Objective
  - How to solve systems that have the form of:

$$f_1(x_1, x_2, \dots, x_n) = 0$$
$$f_2(x_1, x_2, \dots, x_n) = 0$$
$$\dots\dots$$
$$f_n(x_1, x_2, \dots, x_n) = 0$$

Where $f_1, f_2, f_3, \dots, f_n$ are linear functions dependent on $x_1, x_2 \dots$

3.1.2 Contents

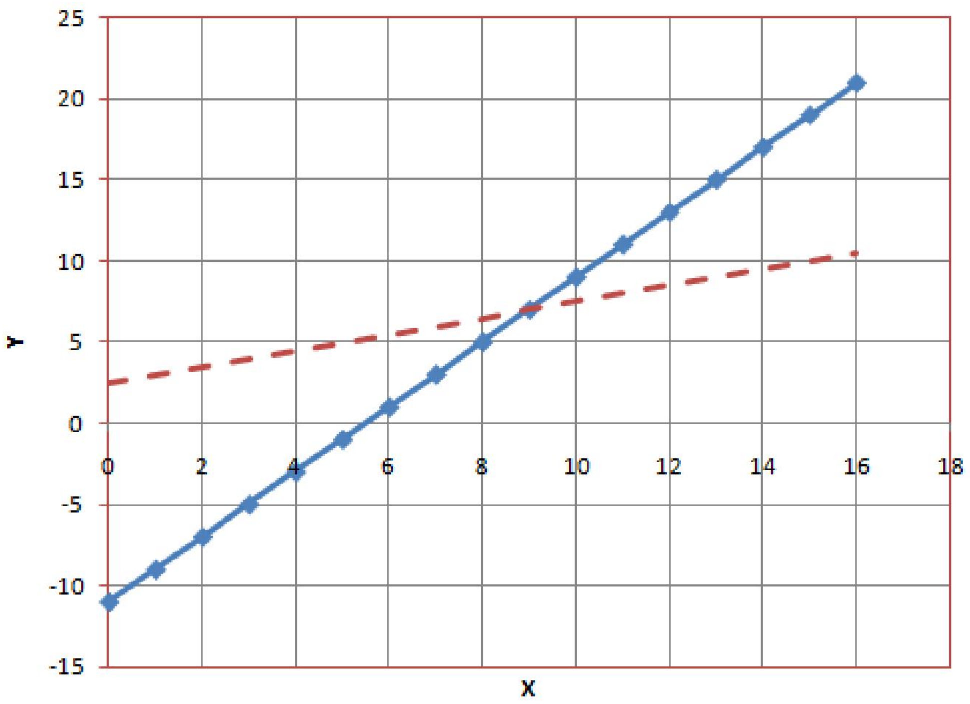
- Graphical Method
- Cramer's rule
- Elimination
- Naïve Gauss Elimination
- Gauss-Jordan Elimination
- LU-Decomposition
- Gauss-Seidel Method

3.2 Graphical Method

- For a system of linear equations, representing every equation graphically i.e.
  - Lines for 2 variables
  - Planes for 3 variables
  - For n variables, holding m variables constants and studying behavior graphically by varying the rest of the variables(n-m <3)

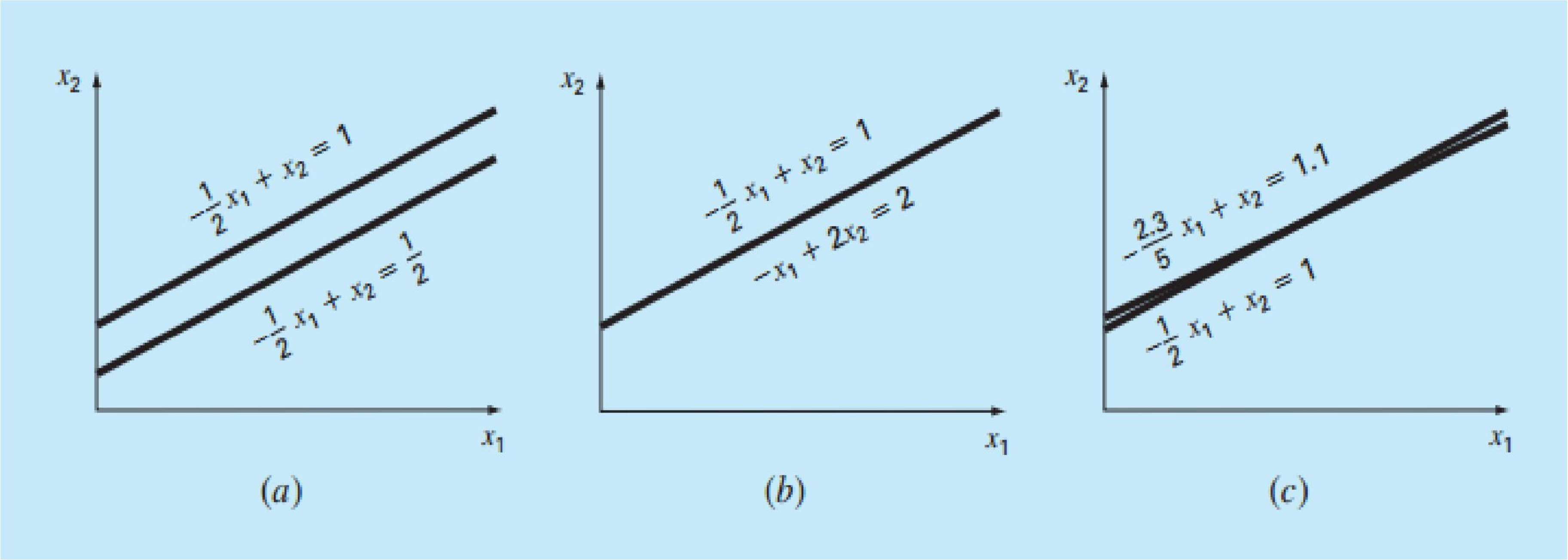
3.2 Graphical Method

- Example
  - {-2x+4y=10; 2x-y=11} solution={x=9.0, y=7.0}



3.2 Graphical Method

- Advantages
  - Help in visualizing the nature of such systems.



Singular | Infinite solutions | Ill-Conditioned

3.2 Graphical Method

- Disadvantages
- Useless for systems with rank>=3.
- 4D and 5D systems aren't what you'd think.

3.3 Cramer's Rule

- Applicable for smaller problems

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}$$

- [EXAMPLE]
- [SCILAB DEMONSTRATION]

3.3 Cramer's Rule

- [EXAMPLE]
  - $3x+5y=10$
  - $x+2y=5$
  - D=1; D1=-5 ; D2=5
  - [solution : x=D1/D=-5; y=D2/D=5]
- [SCILAB]

3.3 Cramer's Rule

- LIMITATIONS
- If system is larger than rank 3, then evaluation of determinants becomes impractical.

3.4 Elimination methods

- Naïve Gauss Elimination
- Gauss-Jordan Elimination
  - Pitfalls of Gauss Elimination
    - Division by Zero
    - Round-off Errors
    - Ill-Conditioned systems
    - Singular systems

3.4.1 Naïve Gaussian Elimination

- Elimination until Upper triangular matrix forms
- [EXAMPLE][MAXIMA demo]

$$\begin{bmatrix} 1 & 2 & 3 \\ -3 & 1 & 5 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 7 & 14 & 7 \\ 0 & 0 & -7 & -7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

3.4.1 Naïve Gaussian Elimination

- [SCILAB] (matrices and the “inv” function)
- >>>a=[1 2 3;-3 1 5;2 4 -1];
- >>>b=[3;-2;-1];
- >>>x=inv(a)*b
- >>>2.

  -1.

  1.

3.4.2 Gauss-Jordan Elimination

- Perform until the IDENTITY matrix forms on the left side.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.4.2 Gauss-Jordan Elimination

$$\begin{bmatrix} 1 & 2 & 3 \\ -3 & 1 & 5 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 7 & 14 & 7 \\ 0 & 0 & -7 & -7 \end{bmatrix}$$

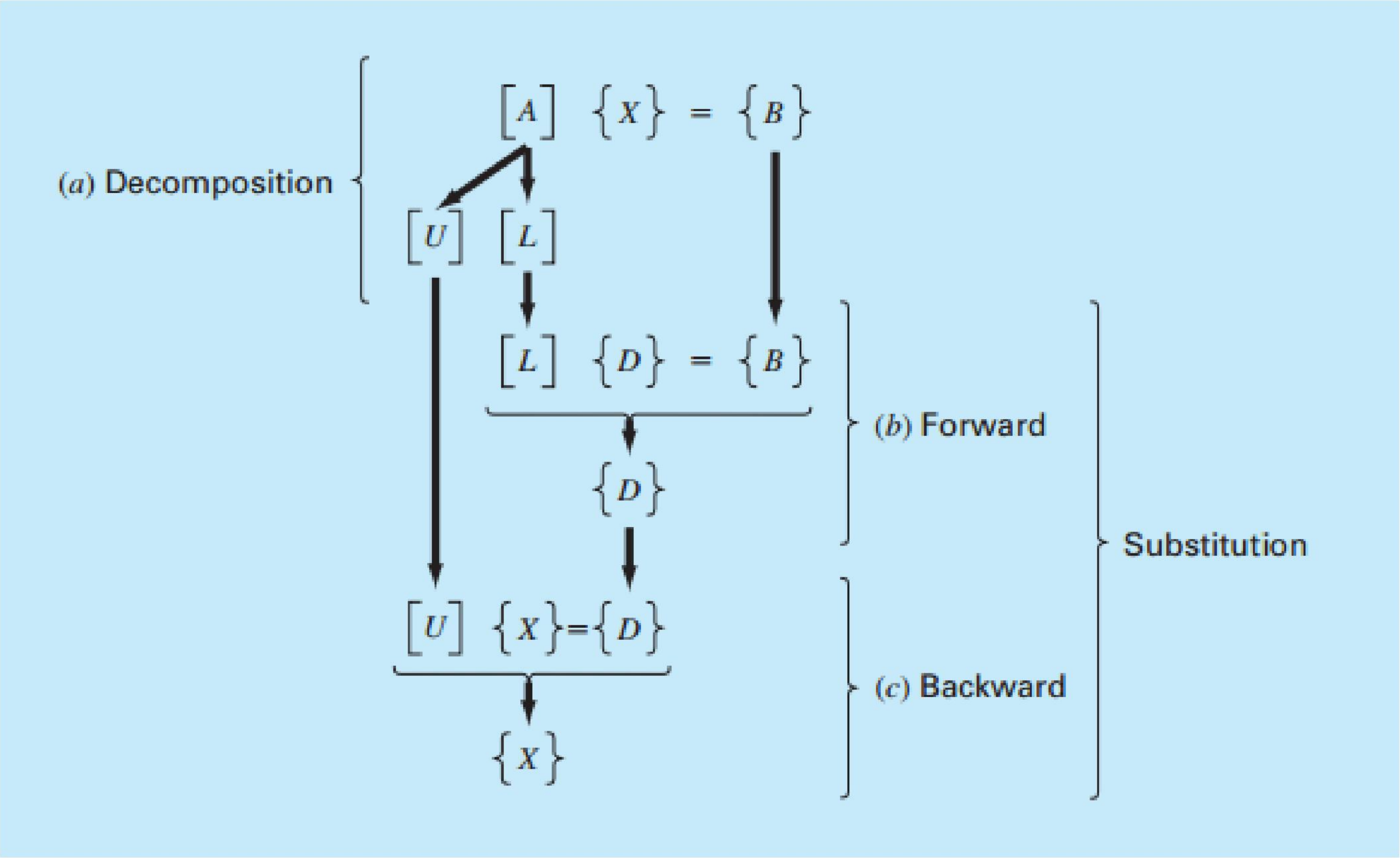
$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

3.5 LU Decomposition

- STEPS:
- 1.Initial : [A]{X}={B}
- 2.Decompose [A] into [U] and [L]
- 3. Construct new sets of systems:
  - [L]{D}={B}…….(1)
  - [U]{x}={D}…….(2)
- 4. Solve (1) and get {D}
- 5. Use {D} from step 4 to solve (2) and get {x}

3.5 LU Decomposition



3.5 LU Decomposition

- [EXAMPLE]

$$\begin{bmatrix} 8 & 4 & -1 \\ -2 & 5 & 1 \\ 2 & -1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \\ 7 \end{bmatrix}$$

- Step 1: Decomposition

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 4 & -1 \\ -2 & 5 & 1 \\ 2 & -1 & 6 \end{bmatrix}$$

3.5 LU Decomposition

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 4 & -1 \\ 0 & 6 & \frac{3}{4} \\ 2 & -1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 4 & -1 \\ 0 & 6 & \frac{3}{4} \\ 0 & -2 & \frac{25}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ \frac{1}{4} & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 8 & 4 & -1 \\ 0 & 6 & \frac{3}{4} \\ 0 & 0 & \frac{26}{4} \end{bmatrix}$$

3.5 LU Decomposition

- Step 2: Solve $[\bar{L}][D] = [b]$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ \frac{1}{4} & -\frac{1}{3} & 1 \end{bmatrix} \{D\} = \begin{bmatrix} 11 \\ 4 \\ 7 \end{bmatrix}$$

$$\{D\} = \begin{bmatrix} 11 \\ 6.75 \\ 6.5 \end{bmatrix}$$

3.5 LU Decomposition

- Step 3: Solve $[U][x] = [D]$

$$\begin{bmatrix} 8 & 4 & -1 \\ 0 & 6 & \frac{3}{4} \\ 0 & 0 & \frac{26}{4} \end{bmatrix} \{x\} = \begin{bmatrix} 11 \\ 6.75 \\ 6.5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3.5 LU Decomposition

- Example 2: Alternate Decomposition Method

$$\begin{bmatrix} 8 & 4 & -1 \\ -2 & 5 & 1 \\ 2 & -1 & 6 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} * \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

3.5 LU Decomposition

let l_{11}, l_{22},l_{33} be unity, then we have:

$$\begin{bmatrix} 8 & 4 & -1 \\ -2 & 5 & 1 \\ 2 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} * \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$u_{11} = 8$

$u_{12} = 4$

$u_{13} = -1$

$l_{21} * u_{11} = -2 \Rightarrow l_{21} = -\frac{1}{4}$

$l_{21} * u_{12} + u_{22} = 5 \Rightarrow -\frac{1}{4} * 4 + u_{22} = 5 \Rightarrow u_{22} = 6$

$l_{21} * u_{13} + u_{23} = 1 \Rightarrow -1 * -\frac{1}{4} + u_{23} = 1 \Rightarrow u_{23} = 4$

$l_{31} * u_{11} = 2 \Rightarrow l_{31} = \frac{1}{4}$

$l_{31} * u_{12} + l_{32} * u_{22} = -1 \Rightarrow \frac{1}{4} * 4 + l_{32} * 6 = 5 \Rightarrow l_{32} = -\frac{1}{3}$

$l_{31} * u_{13} + l_{32} * u_{23} + u_{33} = 6 \Rightarrow \frac{1}{4} * -1 + -\frac{1}{3} * \frac{3}{4} + u_{33} = 6 \Rightarrow u_{33} = \frac{26}{4}$

3.6 The Gauss-Seidel Iterative Method

Consider the following system:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

- This system can be transformed into:

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

3.6 The Gauss-Seidel Iterative Method

- Steps:
- 1. Assume initial guesses of x2,x3....xn=selected values(usually zero)
- 2. Compute x1
- 3. Using the result from (2) and initial guesses from step (1),Compute x2,x3,x4....,xn
- 4. Using newly computed values of x2,x3,x4...xn compute x1.
- 5. DO until convergence

3.6 The Gauss-Seidel Iterative Method

- [Example][FORTRAN Demo]

$$5x_1 - x_2 + x_3 = 4$$
$$x_1 + 3x_2 + x_3 = 2$$
$$-x_1 + x_2 + 4x_3 = 3$$

$$x_1 = \frac{4 + x_2 - x_3}{5} \quad x_2 = \frac{2 - x_1 - x_3}{3} \quad x_3 = \frac{3 + x_1 - x_2}{4}$$

3.6 The Gauss-Seidel Iterative Method

- DIAGONAL DOMINANCE
- An NxN matrix is called diagonally dominant, if the diagonal element in every row is greater in magnitude(Absolute Values) than the sum of the elements in that row excluding the diagonal element.
- i.e.

$$|A_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |A_{ij}| \quad (i = 1, 2, \ldots, n)$$

3.6 The Gauss-Seidel Iterative Method

- [Example]
  - The matrix

$$\begin{bmatrix} -2 & 4 & -1 \\ 1 & -1 & 3 \\ 4 & -2 & 1 \end{bmatrix}$$

is not diagonally dominant.

CHECK: row 1: $|-2|<|4|+|-1|$

row 2: $|-1|<|1|+|3|$

row 3: $|1| < |4|+|-2|$

3.6 The Gauss-Seidel Iterative Method

- The matrix can be made diagonally dominant by exchanging rows

$$\begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

- Can be used to facilitate convergence for iterative methods…

3.7 The Conjugate Gradient Method

[READING ASSIGNMENT]

ANY QUESTIONS ?