Chapter **Four**

Roots of Equations

Numerical Methods

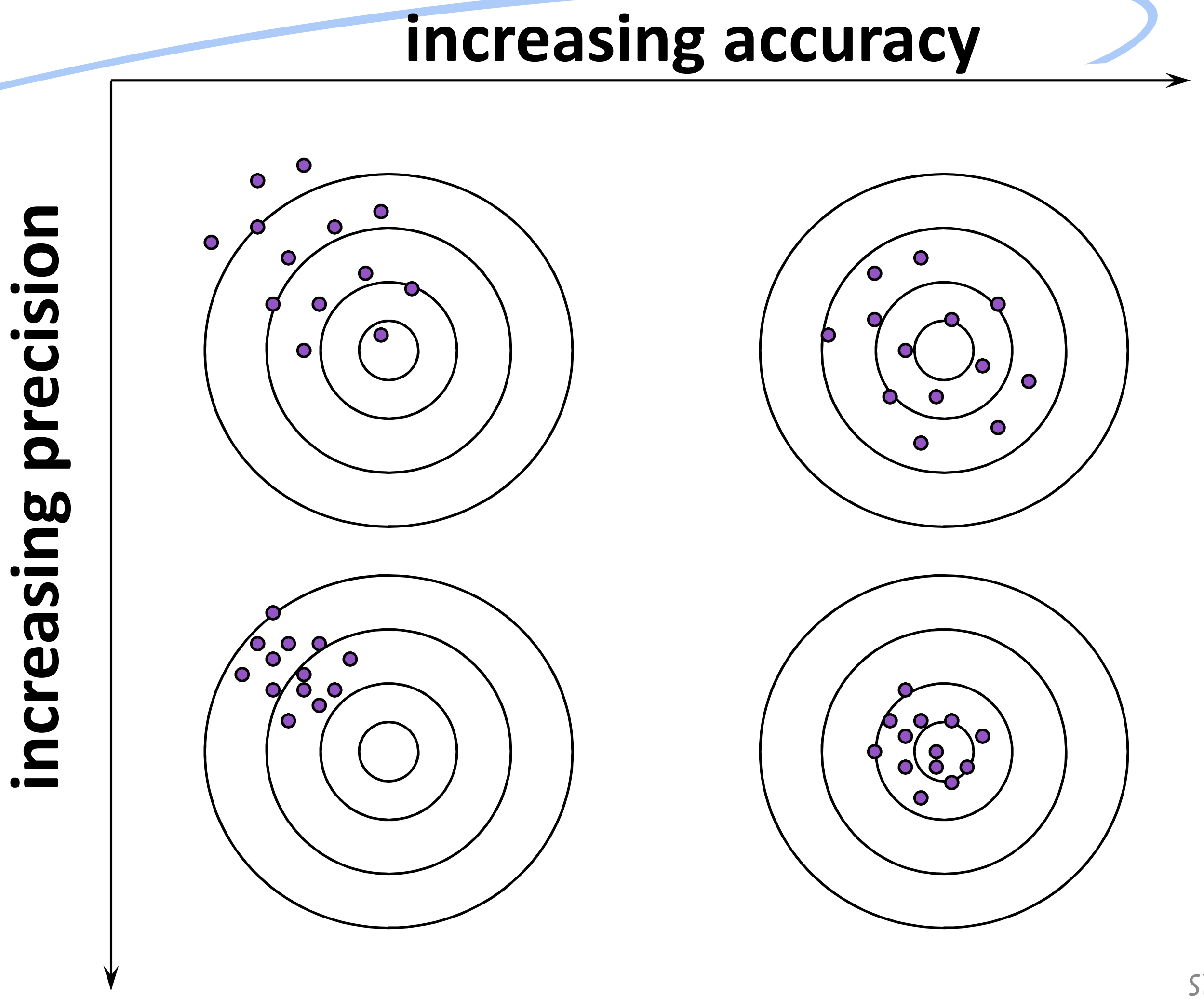
Numerical methods:

- techniques by which mathematical problems are formulated so that they can be solved with arithmetic operations.
- algorithms that are used to obtain numerical solutions of a mathematical problem

Approximation and errors

- **Accuracy** - how closely a computed or measured value agrees with the true value
- **Precision** - how closely individual computed or measured values agree with each other
  - number of significant figures
  - spread in repeated measurements or computations

Accuracy and Precision



Error Definitions

- **Numerical error** - use of approximations to represent exact mathematical operations and quantities
- **true value (x_t)** = **approximation** (x_a) + **error**
  - error, ε_t = true value(x_t) - approximation (x_a)
  - subscript ***t*** represents the true error
  - shortcoming....gives no sense of magnitude

$$\varepsilon_t = true\ \ error = x_t - x_a$$

Error Definitions

- True relative percent error

$$\varepsilon_t = \frac{x_t - x_a}{x_t} \times 100\% = \frac{true\ \ error}{true\, value} \times 100\%$$

- True relative error

$$\varepsilon_t = \frac{true\ \ error}{true\, value} = \frac{x_t - x_a}{x_t}$$

Example

- Consider a problem where the true answer is 7.91712. If you report the value as 7.92, answer the following questions.

1. How many significant figures did you use?
2. What is the true error?
3. What is the relative error?

Error definitions

- May not know the true answer prior

$$\varepsilon_a = \frac{approximate\ \ error}{approximation} \times 100$$

- This leads us to develop an iterative approach of numerical methods

$$\varepsilon_a = \frac{present\,approx - previous\,approx}{present\,approx} \times 100$$

Error Definitions

- Approximate error

$$\varepsilon_a = x_a(i) - x_a(i-1)$$

- Approximate relative error

$$\varepsilon_a = \frac{x_a(i) - x_a(i-1)}{x_a(i)}$$

- Approximate percentage relative error

$$\varepsilon_a = \frac{x_a(i) - x_a(i-1)}{x_a(i)} \times 100\%$$

Error Definitions

- Round off error - originate from the fact that computers retain only a fixed number of significant figures
- Truncation errors - errors that result from using an approximation in place of an exact mathematical procedure

Introduction to Roots of Equations

- The roots or zeros of equations can be simply defined as the values of *x* that makes f(x) = 0.
- can be found easily by solving the equations directly
- there are also other cases where solving the equations directly or analytically is not so possible
- only alternatives will be approximate solution techniques

Bisection Method

- The Bisection Method is a *successive* approximation method that narrows down an interval that contains a root of the function *f(x)*
- The Bisection Method is *given* an initial interval [*a,b*] that contains a root (We can use the property sign of *f(a)* ≠ sign of *f(b)* to find such an initial interval)
- The Bisection Method will *cut the interval* into 2 halves and check which half interval contains a root of the function
- The Bisection Method will keep *cut the interval* in halves until the resulting interval is extremely small

  The root is then *approximately equal* to *any value* in the final (very small) interval.

Bisection Method

Given a function f(x) continuous on the interval $[a_0,b_0]$ and such that **$f(a_0) \times f(b_0) <= 0$**

For n = 0, 1, 2, …, *until satisfied, do*:

Set $m = (a_n + b_n)/2$

If $f(a_n) \times f(m) <= 0$, set $a_{n+1} = a_n$, $b_{n+1} = m$

Otherwise set $a_{n+1} = m$, $b_{n+1} = b_n$

Then f(x) has a zero in the interval $[a_{n+1}, b_{n+1}]$

Example

Find all the real solutions to the cubic equation

$$x^3 + 4x^2 - 10 = 0$$

i	a_i	$f(a_i)$	c_i	b_i	$f(b_i)$	$f(c_i)$
1	1.0000	**-5.0000**	1.5000	2.0000	**14.0000**	2.3750
2	1.0000	*-5.0000*	1.2500	1.5000	*2.3750*	-1.7969
3	1.2500	*-1.7969*	1.3750	1.5000	*2.3750*	0.1621
4	1.2500	*-1.7969*	1.3125	1.3750	*0.1621*	-0.8484
5	1.3125	*-0.8484*	1.3438	1.3750	*0.1621*	-0.3510
6	1.3438	*-0.3510*	1.3594	1.3750	*0.1621*	-0.0964
7	1.3594	*-0.0964*	1.3672	1.3750	*0.1621*	0.0324
8	1.3594	*-0.0964*	1.3633	1.3672	*0.0324*	-0.0321
9	1.3633	*-0.0321*	1.3652	1.3672	*0.0324*	0.0001
10	1.3633	*-0.0321*	1.3643	1.3652	*0.0001*	-0.0160

Regula Falsi

Given a function f(x) continuous on the interval $[a_0, b_0]$ and such that $f(a_0) \times f(b_0) < 0$

For n = 0, 1, 2, …, until satisfied, do:

Calculate

$$w = [f(b_n) \times a_n - f(a_n) \times b_n]/[f(b_n) - f(a_n)]$$

If $f(a_n) \times f(w) <= 0$,

Set $a_{n+1} = a_n$, $b_{n+1} = w$

Otherwise,

Set $a_{n+1} = w$, $b_{n+1} = b_n$

Example

Find all the real solutions to the cubic equation

$$x^3 + 4x^2 - 10 = 0$$

i	a_i	$f(a_i)$	w_i	b_i	$f(b_i)$	$f(w_i)$
1	1.0000	***-5.0000***	1.2632	2.0000	**14.0000**	-1.6023
2	1.2632	*-1.6023*	1.3388	2.0000	14.0000	-0.4304
3	1.3388	*-0.4304*	1.3585	2.0000	14.0000	-0.1100
4	1.3585	*-0.1100*	1.3635	2.0000	14.0000	-0.0278
5	1.3635	*-0.0278*	1.3648	2.0000	14.0000	-0.0070
6	1.3648	*-0.0070*	1.3651	2.0000	14.0000	-0.0018
7	1.3651	*-0.0018*	1.3652	2.0000	14.0000	-0.0004
8	1.3652	*-0.0004*	1.3652	2.0000	14.0000	-0.0001
9	1.3652	*-0.0001*	1.3652	2.0000	14.0000	0.0000
10	1.3652	*0.0000*	1.3652	2.0000	14.0000	0.0000

Modified Regula Falsi

Given a function f(x) continuous on the interval $[a_0, b_0]$ and such that $\mathbf{f(a_0) \times f(b_0) < 0}$

Set $F = f(a_0)$, $G = f(b_0)$, $w_0 = a_0$

For n = 0, 1, 2, …, until satisfied, do:

Calculate

$w_{n+1} = [G \times a_n - F \times b_n]/[G - F]$

If $f(a_n) \times f(w_{n+1}) <= 0$, set $a_{n+1} = a_n$, $b_{n+1} = w_{n+1}$, $G = f(w_{n+1})$

If also $f(w_n) \times f(w_{n+1}) > 0$, set $F = F/2$

Otherwise, set $a_{n+1} = w_{n+1}$, $F = f(w_{n+1})$, $b_{n+1} = b_n$

If also $f(w_n) \times f(w_{n+1}) > 0$, set $G = G/2$

Then f(x) has a zero in the interval $[a_{n+1}, b_{n+1}]$

Fixed Point Iteration

Given an iteration function g(x) and a starting point x_0

For n = 0, 1, 2, ..., until satisfied , do:

Calculate $x_{n+1} = g(x_n)$

For this algorithm to be useful, we must prove:

i. For the given starting point x0, we can calculate successively x1, x2, …

ii. The sequence x1, x2, … converges to some point ξ

iii. The limit ξ is a fixed point of g(x), that is, $\xi = g(\xi)$

Fixed Point Iteration

- Iterative techniques are used to find roots of equations, solutions of linear and nonlinear systems of equations and solutions of differential equations. ***A rule or function g(x) for computing successive terms is needed and it can be found by rearranging the function f(x) = 0 so that x is on the left side of the equation***.

  $x = g(x)$

- Moreover a starting value P_0 is also required and the sequence of values $\{P_k\}$ is obtained using the iterative rule $P_{k+1} = g(P_k)$. The sequence has the pattern

$$P_1 = g(P_0)$$
$$P_2 = g(P_1)$$
$$\vdots \qquad \vdots$$
$$P_k = g(P_{k-1})$$
$$P_{k+1} = g(P_k)$$

Example

Use fixed point iteration to find the fixed point(s) for the function $g(x) = 1 + x - (x^2/3)$

By plotting the graph of the function we can find that there is a real root between **3** and **7** where the graph crosses the x – axis and performing fixed point iteration between **3** and **7** we have:

solution

P_0		3
P_1	$g(P_0)$	1.000000000
P_2	$g(P_1)$	1.666666667
P_3	$g(P_2)$	1.740740741
P_4	$g(P_3)$	1.730681299
P_5	$g(P_4)$	1.732262046
P_6	$g(P_5)$	1.732018114
P_7	$g(P_6)$	1.732055865
P_8	$g(P_7)$	1.732050025
P_9	$g(P_8)$	1.732050929
P_{10}	$g(P_9)$	1.732050789
P_{11}	$g(P_{10})$	1.732050810
P_{12}	$g(P_{11})$	1.732050807
P_{13}	$g(P_{12})$	1.732050808

Newton's Method

Given f(x) continuous differentiable and a point x_0

For n = 0, 1, 2, … , until satisfied, do:

Calculate

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

Example

Find all the real solutions to the cubic equation

$$x^3 + 4x^2 - 10 = 0$$

i	x_i	$f(x_i)$	$f'(x_i)$
1	1.0000	-5.0000	11.0000
2	1.4545	1.5402	17.9835
3	1.3689	0.0607	16.5729
4	1.3652	0.0001	16.5135
5	1.3652	0.0000	16.5134
6	1.3652	0.0000	16.5134
7	1.3652	0.0000	16.5134
8	1.3652	0.0000	16.5134
9	1.3652	0.0000	16.5134
10	1.3652	0.0000	16.5134

Secant Method

- The secant method is a recursive method used to find the solution to an equation
- it requires two initial guesses for the root.
- The big advantage of the secant method over Newton's Method is that it does not require the given function $f(x)$ to be a differential function or for the algorithm to have to compute a derivative.
- The recursive function $h(x,y)$ depends on two parameters x and y the x-coordinates of two points on the function.

Secant Method

Given a function f(x) and two points x_{-1}, x_0

For n = 0, 1, 2, … , until satisfied, do:

Calculate

$X_{n+1} = [f(x_n)\ (x_{n-1}) - f(x_{n-1})\ (x_n)\ ]/[f(x_n) - f(x_{n-1})]$

Example

Find all the real solutions to the cubic equation

$$x^3 + 4x^2 - 10 = 0$$

i	x_i	$f(x_i)$
0	1.000000	-5.000000000
1	2.000000	14.000000000
2	1.263158	-1.602274384
3	1.338828	-0.430364748
4	1.366616	0.022909431
5	1.365212	-0.000299068
6	1.365230	-0.000000203
7	1.365230	0.000000000
8	1.365230	0.000000000
9	1.365230	0.000000000