3. Qualitative modelling

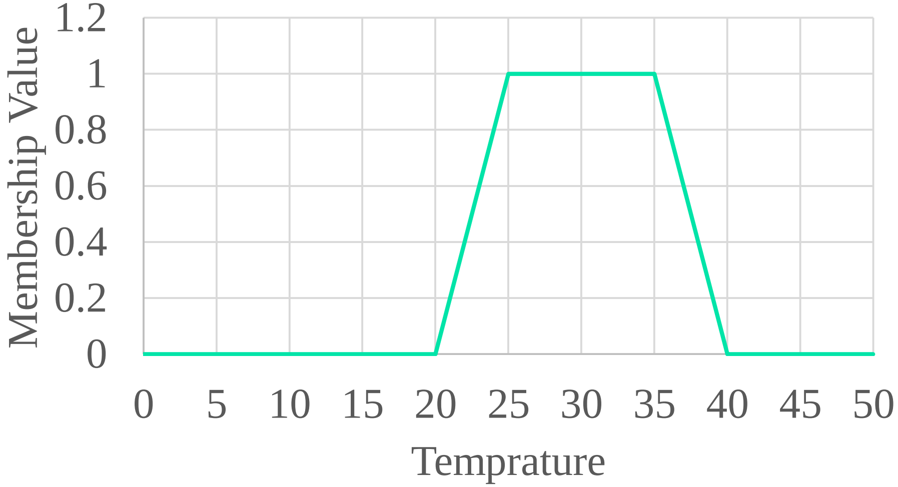
- If values are best described by ***qualitative adjectives***, membership functions indicating the particular quantitative descriptions of parameter or decision variable values can be used to quantify these qualitative descriptions.
- A particular mix of economic and environmental impacts may be more acceptable to some and less acceptable to others
- Qualitative adjectives examples: Large, small, pure, polluted, satisfactory, unsatisfactory, sufficient, insufficient, excellent, good, fair, poor, dry, hot, expensive, clean, high, low, etc.
- Qualitative, or so-called "fuzzy" statements convey information despite the imprecision of the italicized adjectives.
    - The quality of water is good for drinking.
- Membership functions of these uncertain or qualitative variables can be included in quantitative models.

Membership functions (MF)

- Assume a set *A = [18, 25]*.
- Any number *x* is either in or not in the set A.
- The statement "*x belongs to A*" is either *true or false*. Such *set A* is called a **crisp set**.
- If one is not able to say for certain whether or not any number *x* is in the set, then the set A could be referred to as **fuzzy**.
- The degree of truth attached to that statement is defined by a membership function.
- Membership functions range from 0 (completely false) to 1 (completely true)

MFs

- Consider the statement, “The water temperature should be suitable for swimming.”
- Just what temperatures are suitable will depend on the persons asked.
- Thus shall be defined on the basis of the responses of many potential swimmers.
- It would be difficult for anyone to define precisely those temperatures that are suitable, if it is understood that temperatures outside that range are **absolutely not suitable**.
- A function defining the interval or range of water temperatures suitable for swimming can be as shown



MFs

- The form or shape of a function depends on the individual subjective feelings of the “members” or individuals who are asked their opinions.
- To define this particular function, each individual *i* could be asked to define his or her comfortable water temperature interval (T_{min}, T_{max}).
- The degree of belonging value associated with any temperature value T equals the number of individuals who place that *T* within their range (T_{min}, T_{max}), divided by the total number of individual opinions obtained.
- It is the fraction of the total number of individuals that consider the water temperature *T* suitable for swimming. For this reason such functions are often called membership functions

MF Implication for optimization

- Now suppose the water temperature applied to a swimming pool where the temperature could be regulated. The hotter the temperature the more it will cost.
- If we could quantify the economic benefits associated with various temperatures we could perform a benefit–cost analysis by maximizing the net benefits.
- Alternatively, we could maximize the fraction of people who consider the temperature good for swimming subject to a cost constraint using a membership function shown above in place of an economic benefit function.

MF Implication for optimization

- Continuing with this example, assume you are asked to provide the desired temperature at a reasonable cost.
- Just what is reasonable can also be defined by another membership function, but this time the function applies to cost, not temperature.
- In this case one could consider there are in fact two objectives,
  - Suitable temperature and
  - Acceptable/Reasonable cost.
- A model that **maximizes the minimum value of both membership functions** is one approach for finding an acceptable policy for controlling the water temperature at this swimming pool.

Example

- Consider the application of qualitative modeling to the three irrigation farm water allocation problem seen in chapter 2.
- The objective is to find the values of each allocation that maximizes the net benefits, NB(x).
- Maximize Net benefit Subject to constraints:
  - NB= $p_1(12 - p_1) + p_2(20 - 1.5p_2) + p_3(28 - 2.5p_3) - 3(p_1)^{1.3} - 5(p_2)^{1.2} - 6(p_3)^{1.15}$
  - Where $p_1 \leq 0.4(x_1)^{0.9}, p_2 \leq 0.5(x_2)^{0.8}$ *and* $p_3 \leq 0.6(x_3)^{0.7}$
  - Water–allocation restriction: $x_1 + x_2 + x_3 =$ *about 8*
- Remember that the optimal solution using GA was around **x_1 = 0.7217, x_2 = 2.9808, x_3= 4.2985 and NB = 47.3721**

Example

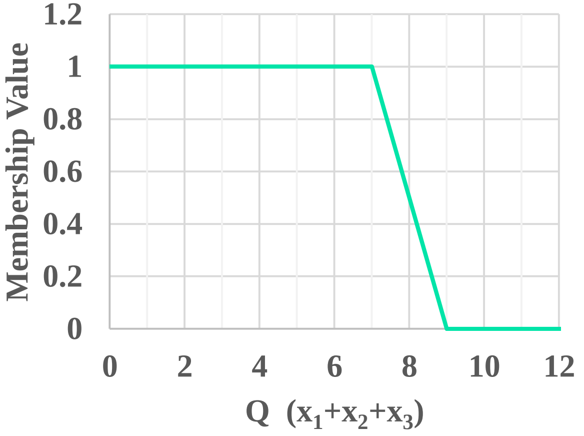
- To create a qualitative equivalent for the model, the objective can be expressed as a membership function of the set of all possible objective values. The higher the objective value the greater the membership function value.
- Since membership functions range from 0 to 1, the objective needs to be scaled so that it also ranges from 0 to 1.
- The highest value of the objective occurs when there is sufficient water to maximize each firm's benefits; i.e. at the value where the derivative of the net benefit function of each firm equal zero.
  - $x_1 = 10.18$ ($NB_1 = 14.552$), $x_2 = 13.56$ ($NB_2 = 29.611$), and $x_3 = 14.54$ ($NB_3 = 42.476$).
- This unconstrained solution would result in a total benefit of 86.639.

Example

- The objective membership function, m_G, can be expressed by
- $m_G = \begin{bmatrix} p_1(12 - p_1) + p_2(20 - 1.5p_2) + p_3(28 - 2.5p_3) \\ -3(p_1)^{1.3} - 5(p_2)^{1.2} - 6(p_3)^{1.15} \end{bmatrix} / 86.639$
- **The goal of maximizing objective function is changed to that of maximizing the degree of reaching the objective target.**
- The optimization problem becomes:
  - $Maximize\ m_G(x)$
  - Subject to: $x_1 + x_2 + x_3 \leq about\ 8$

$$\boldsymbol{m_C(x) = 1, if\ x_1 + x_2 + x_3 \leq 7}$$
$$\boldsymbol{m_C(x) = [9 - (x_1 + x_2 + x_3)]/2, \quad if\ 7 \leq (x_1 + x_2 + x_3) \leq 9}$$
$$\boldsymbol{m_C(x) = 0, if\ x_1 + x_2 + x_3 \geq 9}$$

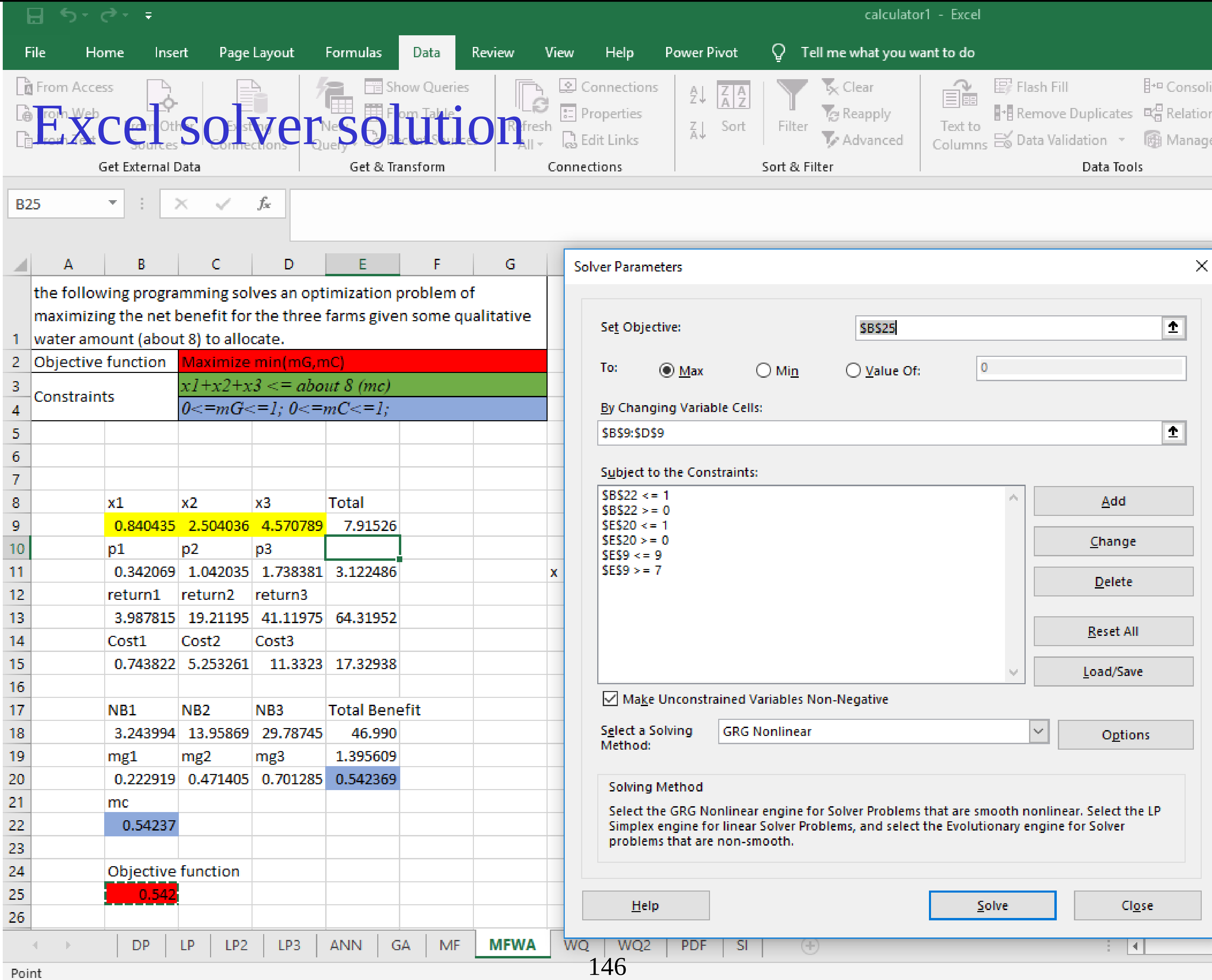
- The total amount of resources available to be allocated is limited to "*about 8*" which is a qualitative constraint.
- The membership function describing the above constraint can be defined by



Example

- The qualitative optimization problem becomes: **maximize the minimum *($m_G(x)$, $m_C(x)$)***
- subject to
  - $m_G = \begin{bmatrix} p_1(12 - p_1) + p_2(20 - 1.5p_2) + p_3(28 - 2.5p_3) \\ -3(p_1)^{1.3} - 5(p_2)^{1.2} - 6(p_3)^{1.15} \end{bmatrix} / 86.639$
  - $m_C(x) = [9 - (x_1 + x_2 + x_3)]/2$
- This yields (using excel solver)
  - $x_1 = 0.84044$, $x_2 = 2.50404$, $x_3 = 4.57079$ and Q = 7.91526
  - $m_G(x) = m_C(x) = 0.542$, and
- The total net benefit, is TB(X) = 46.990.
- Compare this with the GA solution of **x_1 = 0.7217, x_2 = 2.9808, x_3= 4.2985 and NB = 47.3721**.

Excel solver solution



Example: Qualitative Water Quality

- Consider the stream pollution problem where the stream receives waste, W_i from sources located at sites *i = 1* and *2*.
- Without some waste treatment at these sites, the pollutant concentrations at *sites 2* and *3* will exceed the maximum desired concentration. The problem is to find the fraction of wastewater removal, x_i, at *sites i = 1* and *2* required to meet the quality standards at *sites 2* and *3* at a minimum total cost.
- The data used for the problem shown in next slide
- The crisp model for this problem is: **Minimize $C_1x_1 + C_2x_2$,** Subject to
- Water quality constraint at site 2:
  - $[P_1Q_1 + W_1(1 - x_1)]\sigma_{12}/Q_2 \leq P_2^{max}$
  - $[32 \times 20 + 250{,}000(1 - x_1)/86.4]\, 0.25/12 \leq 20 \rightarrow x_1 \geq 0.78$

Example

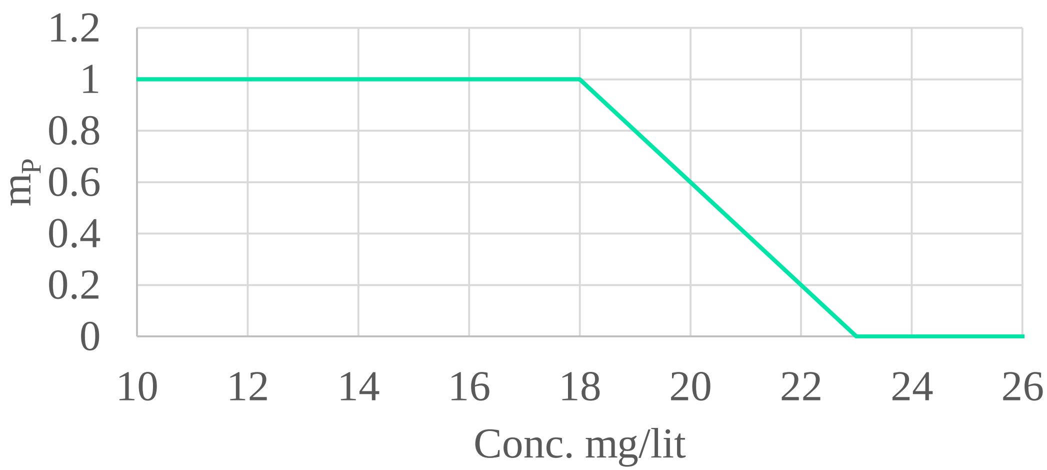
	Parameter	Unit	Value	Remark
Flow	Q_1	m^3/s	10	Flow just u/s of site 1
	Q_2	m^3/s	12	Flow just u/s of site 2
	Q_3	m^3/s	13	Flow at park
Waste	W_1	Kg/day	250,000	Pollutant mass produced at site 1
	W_2	Kg/day	80,000	Pollutant mass produced at site 2
Pollutant Concentration	P_1	mg/l	32	Concentration Just upstream of site 1
	P_2	mg/l	20	Maximum Allowable concentration upstream of site 2
	P_3	mg/l	20	Maximum Allowable concentration at site 3
Decay Fraction	σ_{12}	-	0.25	Fraction of site 1 pollutant mass at site 2
	σ_{13}	-	0.15	Fraction of site 1 pollutant mass at site 3
	σ_{23}	-	0.6	Fraction of site 2 pollutant mass at site 3

Example

- Water quality constraint at site 3:
  - $[[P_1 Q_1 + W_1(1 - x_1)]\sigma_{13} + W_2(1 - x_2)\sigma_{23}]/Q_3 \leq P_3^{max}$
  - $\begin{bmatrix} [32 \times 20 + 250{,}000 \times (1 - x_1)/86.4] \times 0.15 \\ + 80{,}000(1 - x_2)/86.4 \times 0.6 \end{bmatrix} /13 \leq 20$
  - $\rightarrow x_1 + 1.28x_2 \geq 1.79$
- Restrictions on fractions of waste removal:
  - $0 \leq x_i \leq 1$, for *sites i = 1* and *2*
- Excel solver for this problem using linear programming will result in:
  - $x_1 = 0.779$ and $x_2 = 0.791$
  - Essentially 80% removal efficiency is expected.
- Compare this solution with that of the next qualitative model.

Example

- The maximum allowable pollutant concentrations in the stream at *sites 2* and *3* were expressed as "*about 20 mg/l*"
- Obtaining opinions of individuals of what they consider to be "*about 20 mg/l*" a membership function (m_P) can be defined as shown below:

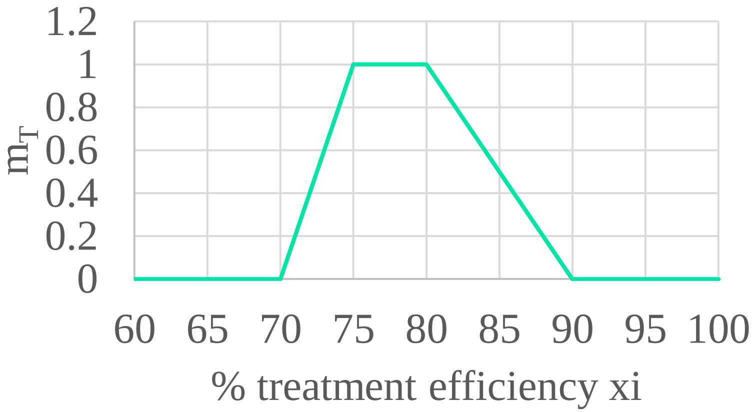


$$\boldsymbol{m_P(x) = 1, if\ conc. \le 18}$$
$$\boldsymbol{m_P(x) = [23 - Conc.]/5,}$$
$$\boldsymbol{if\ 18 \le conc \le 23}$$
$$\boldsymbol{m_P(x) = 0, if\ conc \ge 23}$$

Example

- Regardless of whether or not this is required to meet stream quality standards, the government environmental agency expects each polluter to install:
  - Best available technology (BAT) or To carry out best management practices (BMP)
- Asking experts just what BAT or BMP means with respect to treatment efficiencies could result in a variety of answers.
- These responses were used to define membership function (m_T) for each of the two firms in this example.

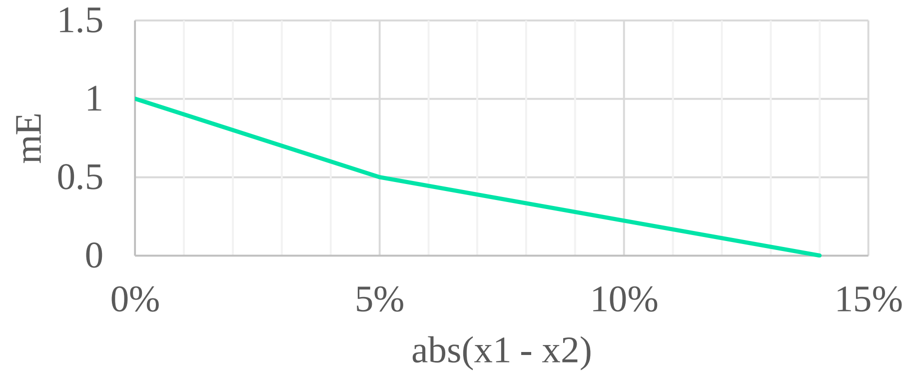
Membership function for best available treatment technology



$$\boldsymbol{m_T(x) = 0, if\ x_i \le 0.7}$$
$$\boldsymbol{m_T(x) = 20x_i - 14,}$$
$$\boldsymbol{if\ 0.7 \le x_i \le 0.75}$$
$$\boldsymbol{m_T(x) = 1, if\ 0.75 \le x_i \le 0.8}$$
$$\boldsymbol{m_T(x) = 9 - 10x_i,}$$
$$\boldsymbol{if\ 0.8 \le x_i \le 0.9}$$
$$\boldsymbol{m_T(x) = 0, if\, x_i \ge 0.9}$$

Example

- There is a third concern that has to do with equity. It is expected that no polluter should be required to treat at a much higher efficiency than any other polluter.
- A membership function (m_E) defining just what differences are acceptable or equitable could quantify this concern.



$$m_E(y) = 1 - 10y, \text{ if } y \leq 0.05$$
$$m_E(y) = 0.7778 - 5.5556y, \text{ if } 0.05 \leq y \leq 0.14$$
$$m_E(y) = 0, \text{ if } y \geq 0.14$$
$$\text{where } y = |x_1 - x_2|$$

Example

- Considering each of these membership functions as objectives, a number of fuzzy optimization models can be defined.
- One is to find the treatment efficiencies that maximize the minimum value of each of these membership functions.
  - $Maximize\ m = max\ \{min\ (m_P, m_T, m_E)\}$

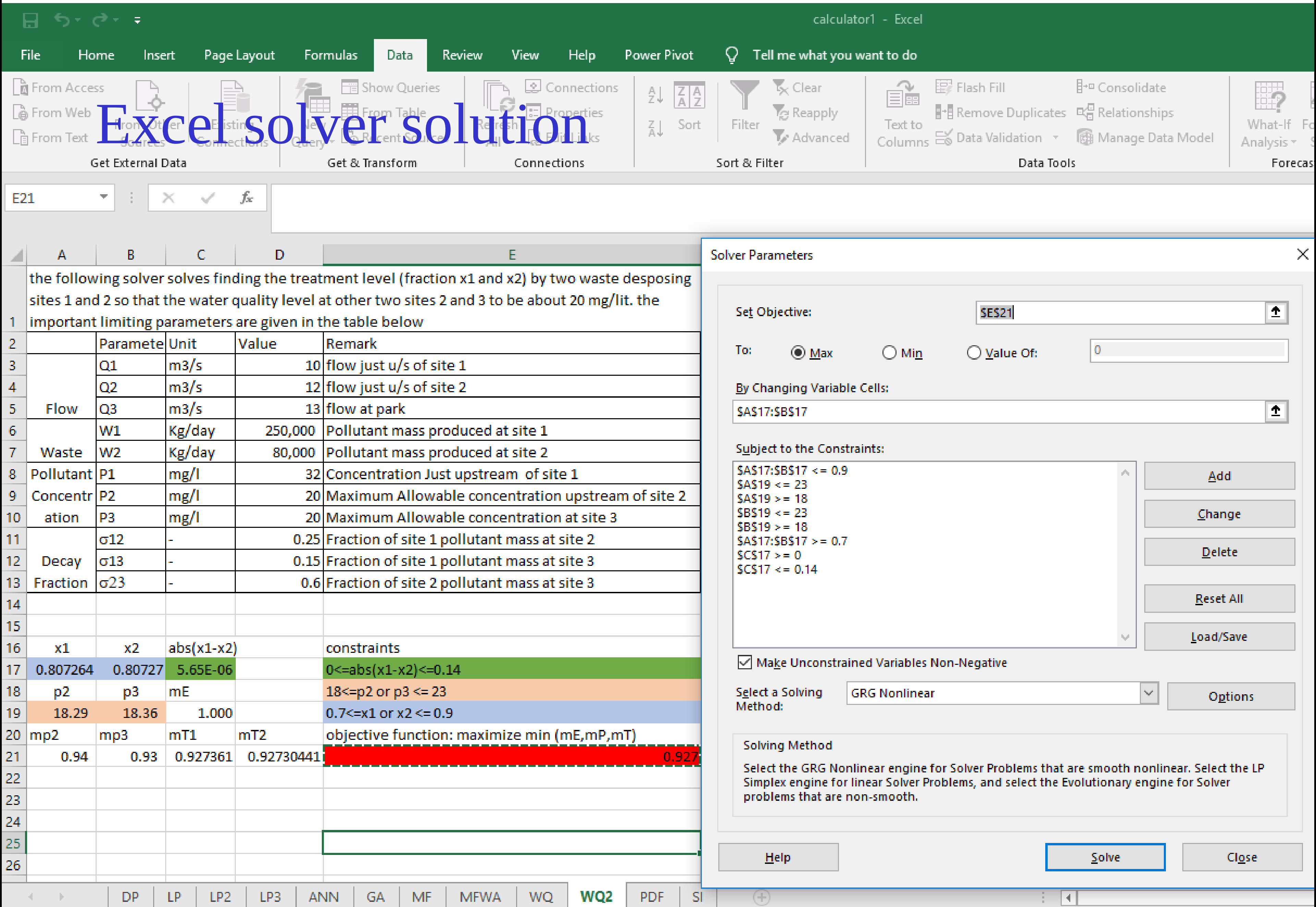
Example

- Considering each of these membership functions as objectives, a number of fuzzy optimization models can be defined.
- One is to find the treatment efficiencies that maximize the minimum value of each of these membership functions.
  - *Maximize* $m = max \{min (m_P, m_T, m_E)\}$
- Solving this model yields the results shown below.

Variable	Value	Remark	Variable	Value	Remark
m	0.93		m_{p2}	0.94	
x_1	0.81		m_{p3}	0.93	
x_2	0.81		m_{T1}	0.93	
P_2	18.28		m_{T2}	0.93	
P_3	18.36		m_E	1	

- This solution confirms the assumptions made when constructing the representations of the membership functions in the model.

Excel solver solution



Exercise 3

- Consider the application of qualitative modeling to the three irrigation farm water allocation problem seen in GA section.
- The problem is to find the allocations of water to each farm that maximize the total benefits TB(X):
  - $Maximize\ TB(x) = (6x_1 - x_1^2) + (7x_2 - 1.5x_2^2) + (8x_3 - 0.5x_3^2)$
- These allocations cannot exceed the amount of water available, Q, less any that must remain in the river, R.
- Assuming the available flow for allocations is about 6 units
- The maximize equation is subject to the resource constraint:
  - $x_1 + x_2 + x_3 \leq about\ 6\ unit$