4.3 Yield Criteria

- Cannot perform tests for all combinations of 3D loading so we need yield criterion to generalize from small number of tests.
- What are the ideal properties of a 3D yield criterion?

$f(\sigma_{ij}, Y) < 0$ Elastic Behavior

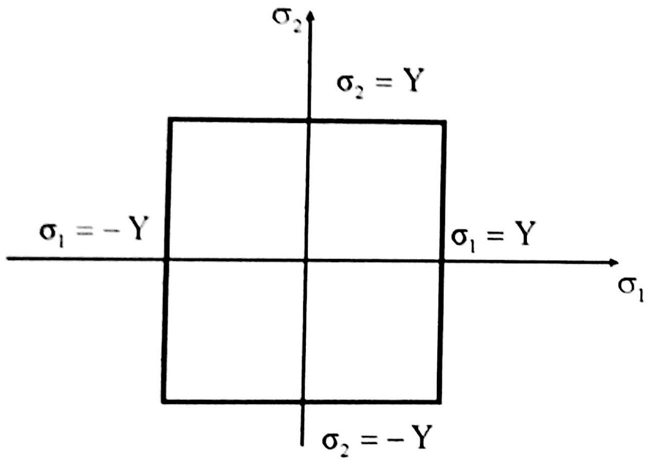
$f(\sigma_{ij}, Y) = 0$ Onset of Inelastic Behavior

- We usually visualize yield criterion by a surface in principal stress space. Why?
- We also calculate "effective" stress to compare with yield stress.

Maximum principal stress criterion
William Rankine (1820-1872)

- Applicable to brittle materials (mostly in tension)

$$f = \max\left(|\sigma_1|, |\sigma_2|, |\sigma_3|\right) = Y$$



Maximum Principal Stress Yield Surface

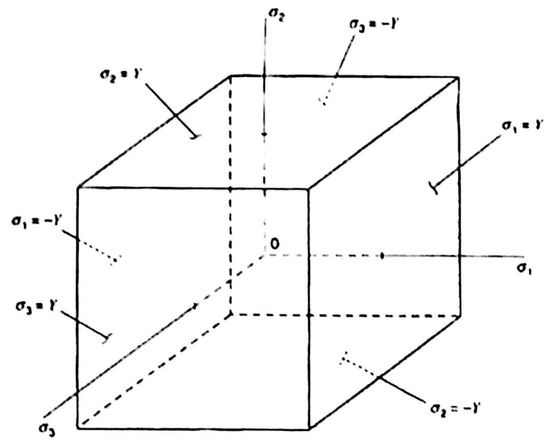


Figure 4.7 Maximum principal stress yield surface.

Maximum principal strain criterion

Adhémar Jean Claude Barré de Saint-Venant

1797 - 1886

- Has the advantage that strains are often easier to measure than stresses
- Assume that epsilon1 is the largest principal strain

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3)$$

$$f_1 = |\sigma_1 - \nu\sigma_2 - \nu\sigma_3| - Y$$

$$\sigma_1 - \nu\sigma_2 - \nu\sigma_3 = \pm Y$$

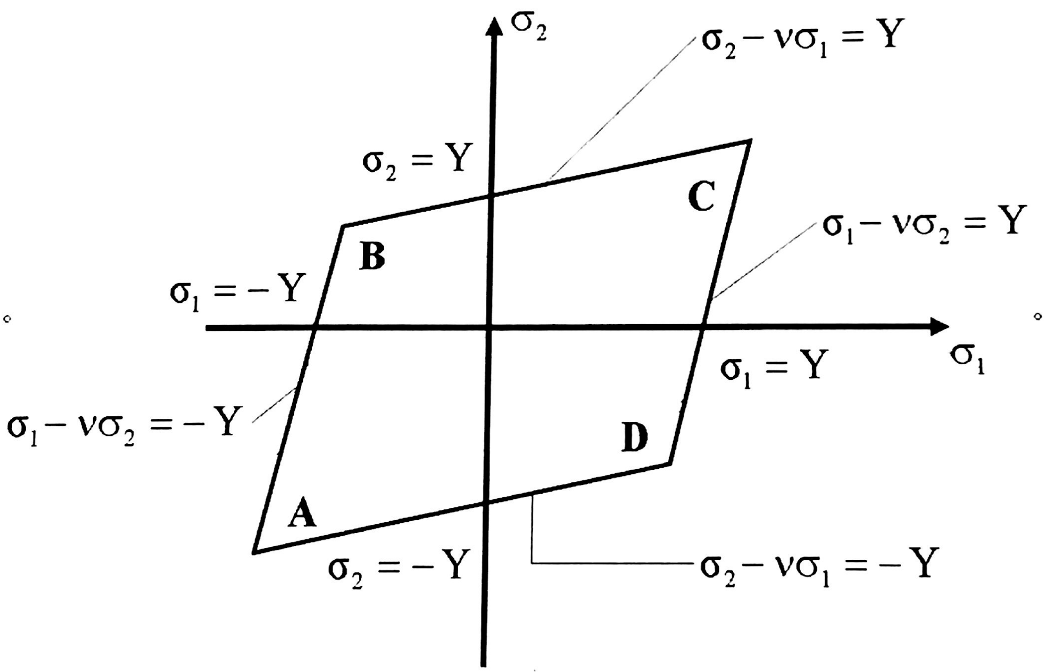
$$\sigma_e = \max_{i \neq j \neq k} |\sigma_i - \nu\sigma_j - \nu\sigma_k| \rightarrow f = \sigma_e - Y$$

Anti-optimization for selecting test conditions

- What test will give us maximum difference between maximum principal stress and maximum principal strain criteria?
- Obviously $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$
- With max principal stress $\sigma_e = \sigma$
- With max principal strain $\sigma_e = \sigma(1-2\nu) = 0.4$
- Alternatively $\sigma_1 = -\sigma_2 = -\sigma_3 = \sigma$
- Max principal strain $\sigma_e = \sigma(1+2\nu) = 1.6$
- What is bad about these test conditions?

In plane stress

- Figure 4.8



Strain energy density criterion (Eugenio Beltrami 1835-1900)

- Strain energy density

$$U_0 = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right)\right]$$

$$\text{Uniaxial test}: \sigma_1 = Y \quad \sigma_2 = \sigma_3 = 0 \qquad U_0 = \frac{1}{2E}\left[\sigma_1^2\right] = \frac{Y^2}{2E}$$

- Criterion $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right) - Y^2 = 0$

- Effective stress $f = \left(\sigma_e\right)^2 - Y^2$ $\sigma_e = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right)}$

- Extreme $\sigma_1 = -\sigma_2 = -\sigma_3 = \sigma$ $\sigma_e = \sigma\sqrt{3 + 2\nu} = 1.90$

Plane stress

- Depends on Poisson's ratio

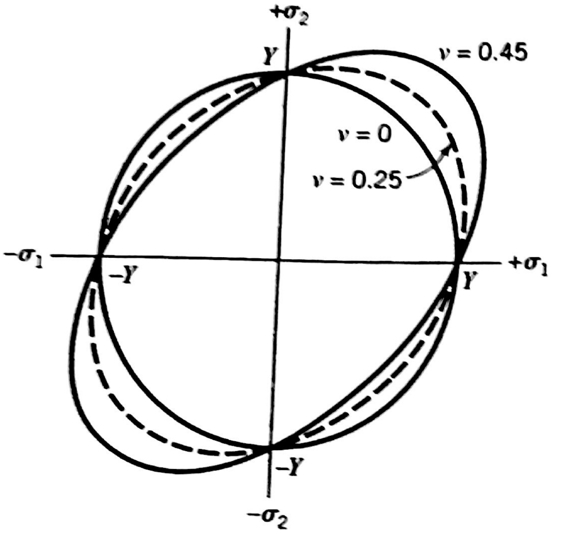


FIGURE 4.9 Strain-energy density yield surface for biaxial stress state ($\sigma_3 = 0$).

- What is common to strain and energy criteria?

4.4. Yielding of ductile metals

- Maximum shear-stress criterion (Henri Eduard Tresca, 1814-1885)
- Uniaxial loading $\sigma_1 = Y; \sigma_2 = 0; \sigma_3 = 0 \rightarrow \tau_{max} = \frac{Y-0}{2} = \frac{Y}{2}$
- Criterion

$$f = \sigma_e - \frac{Y}{2}$$

- Effective stress

$$\sigma_e = \tau_{max} = \max(\tau_1, \tau_2, \tau_3) \qquad \tau_1 = \frac{|\sigma_2 - \sigma_3|}{2} \qquad \tau_2 = \frac{|\sigma_3 - \sigma_1|}{2} \qquad \tau_3 = \frac{|\sigma_1 - \sigma_2|}{2}$$

- Conservative for metals in shear
- Extreme case $\sigma_1 = -\sigma_2 = -\sigma_3 = \sigma \qquad \sigma_e = \sigma$

Distortional energy density criterion (Ludwig von Mises 1881-1973)

- Strain energy density

$$U_0 = U_V + U_D = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right)\right]$$

$$U_V = \frac{\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2}{18K} \qquad U_D = \frac{\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2}{12G}$$

- Uniaxial test

$$\sigma_1 = Y \qquad \sigma_2 = 0 \qquad \sigma_3 = 0$$

$$U_D = \frac{\left(Y-0\right)^2 + \left(0-0\right)^2 + \left(0-Y\right)^2}{12G} = \frac{Y^2}{6G}$$

$$\frac{\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2}{12G} = \frac{Y^2}{6G}$$

- Criterion

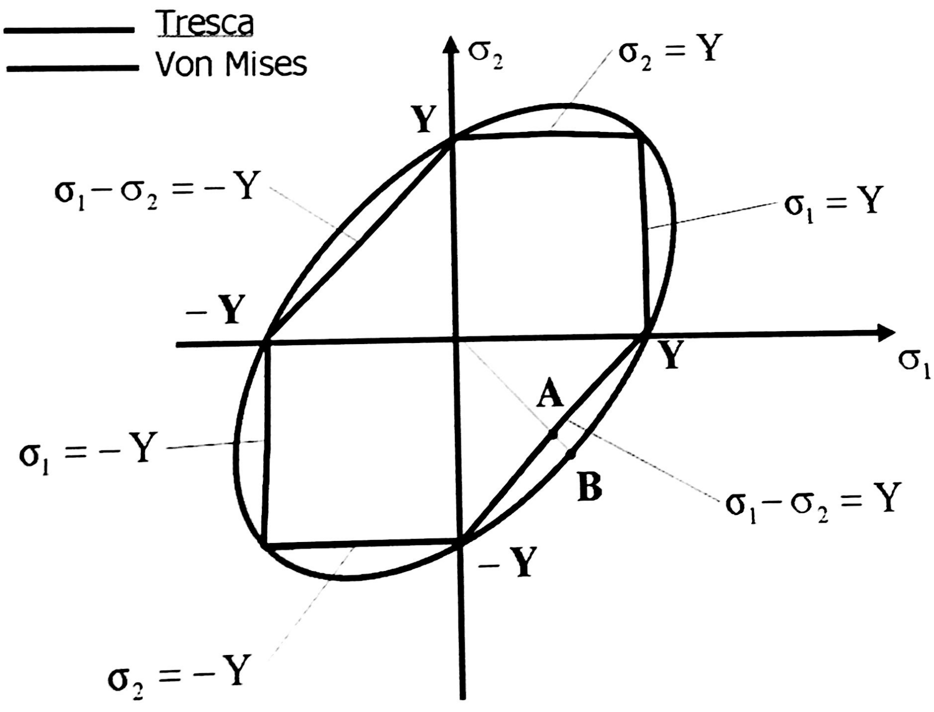
Max difference?

$$\sigma_e = \sqrt{0.5\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right]}$$

$$f = \sigma_e - Y$$

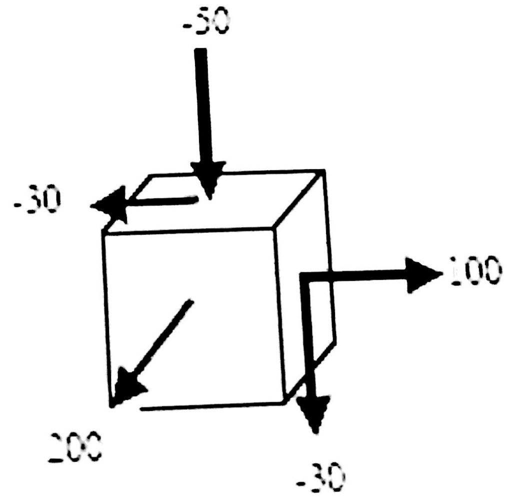
In plane stress

- Where is maximum difference?



Example comapring failure criteria

Stress analysis of a spacecraft structural member gives the state of stress as shown below. If the part is made from an alloy with Y = 500 MPa, check yielding according to Rankine, Tresca and von Mises criteria. What is its safety factor for each criterion?



Maximum principal stress Criterion (Rankine)

$$\sigma = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 100 & -30 \\ 0 & -30 & -50 \end{bmatrix} \textbf{ MPa}$$

- The principal stresses are:

$$\sigma_1 = 200; \quad \sigma_2 = 105.77; \quad \sigma_3 = -55.77;$$

- Maximum principal stress is 200MPa
- Factor of safety FS=500/200=2.5

Maximum shear stress Criterion (Tresca)

- Yield function $f = \sigma_c - \frac{Y}{2}$

- Maximum shear stress $\sigma_c = \frac{|\sigma_1 - \sigma_3|}{2} = \frac{200 + 55.78}{2} = 127.89 \text{ MPa}$

- Shear stress for uniaxial tension

$$\frac{Y}{2} = 250 \text{ MPa} \qquad f = 127.89 - 250 < 0$$

- Factor of safety FS=250/127.89=1.95

Distortional Energy Density Criterion (von Mises)

- Yield function

$$f = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} - Y = 224 - 500$$

- Factor of safety FS=500/224=2.2
- Why are all the results so close?

$$\sigma_1 = 200; \quad \sigma_2 = 105.77; \quad \sigma_3 = -55.77;$$

Other yield criteria

- For isotropic materials there is usually substantial difference between yield in tension and compression. Why?
- For orthotropic materials there is also differences between the behavior along the three principal directions and between shear and normal stresses
- Fortunately, for most applications we have at least transverse isotropy

Reading assignment

Sections 4.4-5: Question: Under what conditions will we have maximum differences between Tresca and Von-Mises criteria? (consider both the case that Tresca is more conservative and the case that Von Mises is)

Source: www.library.veryhelpful.co.uk/ Page11.htm