Stress and Equilibrium

Tensor analysis

Stress (Basic assumptions and definitions)

- In continuum mechanics a body is considered **stress free** if the only forces present are those inter-atomic forces required to hold the body together

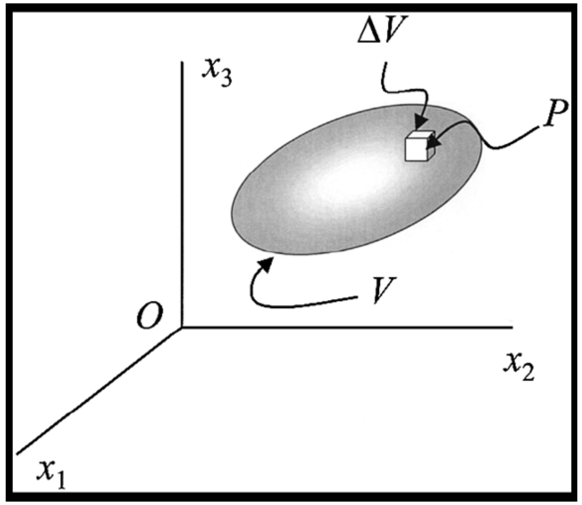
- Types of forces:
  - **Body forces** i.e.: gravity, inertia; designated by vector symbol b_i (force per unit-mass) or p_i (force per unit volume); acting on all volume elements, and distributed throughout the body;

  - **Surface forces** i.e.: pressure; denoted by vector symbol f_i (force per unit area of surface across they which they act); act upon and are distributed in some fashion over a surface element of the body,

Stress (Basic assumptions and definitions)

– **External forces** acting on a body (loads applied to the body);

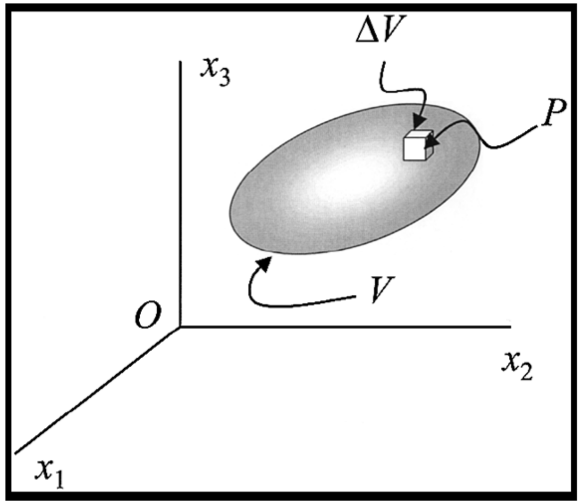
– **Internal forces** acting between two parts of the body (forces which resist the tendency for one part of the member to be pulled away from another part).



Stress (density definition)

- In continuum mechanics we consider a material **body B** having a volume **V** enclosed by a surface **S**, and occupying a regular region $\mathbf{R_0}$ of physical space.

- Let **P** be an interior point of the body located in the small element of volume **ΔV** and mass is **ΔM**. Density



$$\rho_{ave} = \frac{\Delta m}{\Delta V}$$

The density is in general a scalar function of position and time:

$$\rho = (\mathbf{x}, t)$$

Stress (density definition)

The density ρ at point P by the limit of this ratio as the volume shrinks to the point:

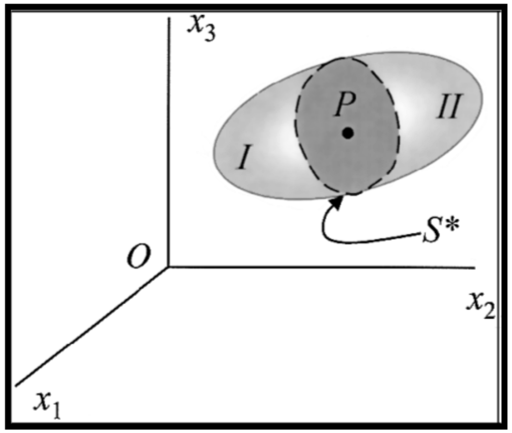
$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

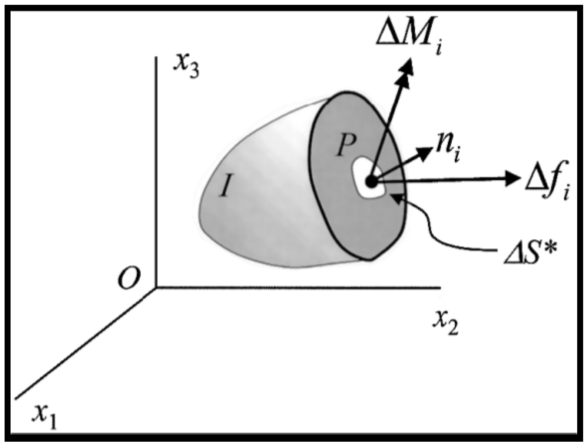
The units of density are kg/m^3. Two measures of body forces, b_i having units of (N/kg), and p_i having units of (N/m^3), are related through the density by the equation:

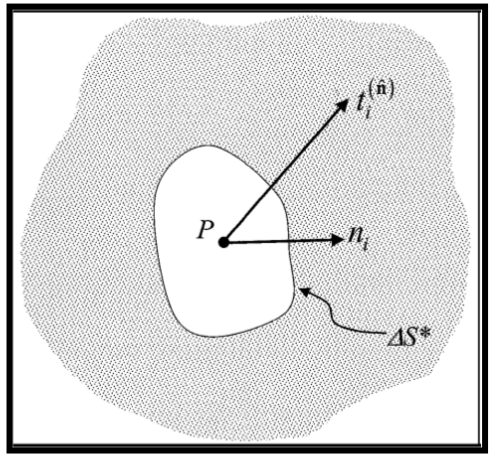
$$\rho b_i = p_i$$

Cauchy Stress Principle

- We consider a homogeneous, isotropic material body ***B*** having a bounding surface ***S***, and a volume ***V***, which is subjected to arbitrary surface forces $\boldsymbol{f_i}$ and body forces $\boldsymbol{b_i}$. Let ***P*** be an interior point of ***B*** and imagine a plane surface ***S**** passing through point ***P***







Cauchy Stress Principle

- Point P is in the small element of area ΔS^* of the cutting plane, which is defined by the unit normal pointing in the direction from Portion I into Portion II .

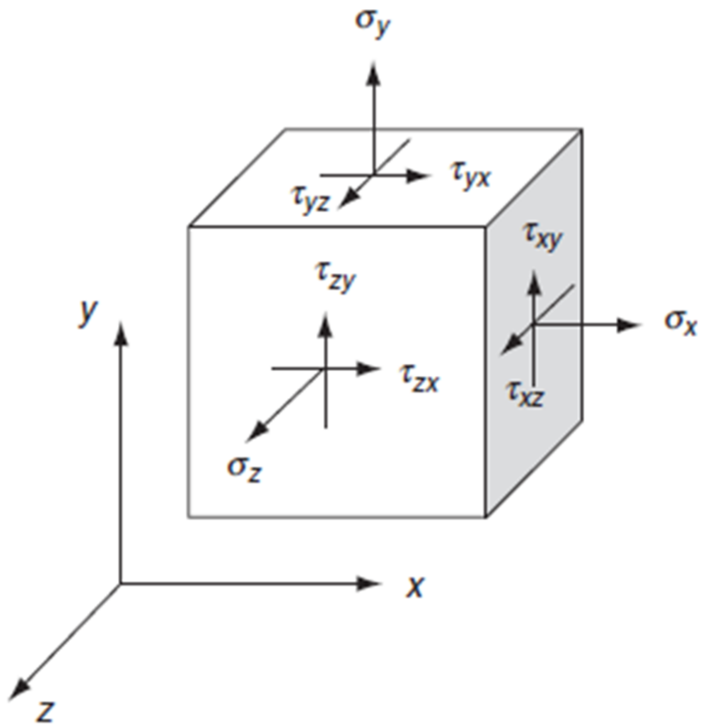
- The internal forces will give rise to a force distribution on ΔS^*equivalent to a resultant force Δf_i and a resultant moment ΔM_i at P.

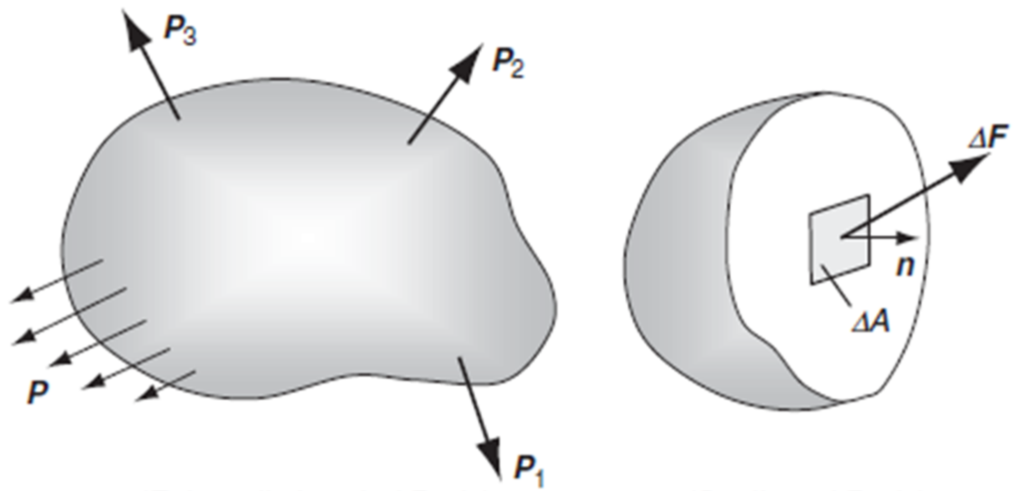
- **The Cauchy stress principle** asserts that in the limit as the area ΔS^* shrinks to zero with P remaining an interior point, we obtain:

$$\lim_{\Delta S^* \to 0} \frac{\Delta f_i}{\Delta S^*} = \frac{df_i}{dS^*} = t_i^{(\hat{n})}$$

$$\lim_{\Delta S^* \to 0} \frac{\Delta M_i}{\Delta S^*} = 0$$

Components of Stress

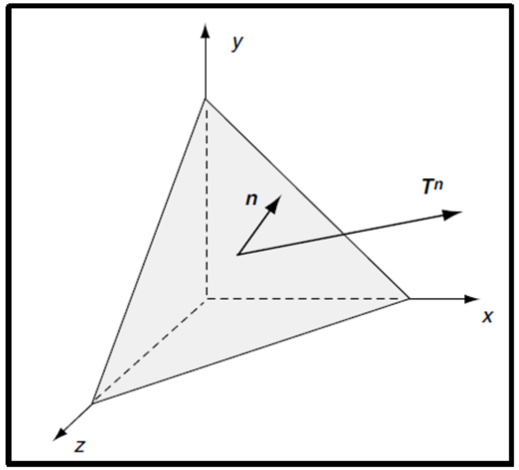




(Externally Loaded Body) (Sectioned Body)

The Stress Tensor (rectangular Cartesian components)

- Consider the traction vector of an oblique plane with arbitrary orientation; with a unit normal to the surface:



$$\boldsymbol{n} = n_x \boldsymbol{e}_1 + n_y \boldsymbol{e}_2 + n_z \boldsymbol{e}_3$$

Where n_x, n_y and n_z are direction cosines of the unit vector ***n*** relative to the given coordinate system.

- Force balance between tractions on the oblique and coordinate faces gives:

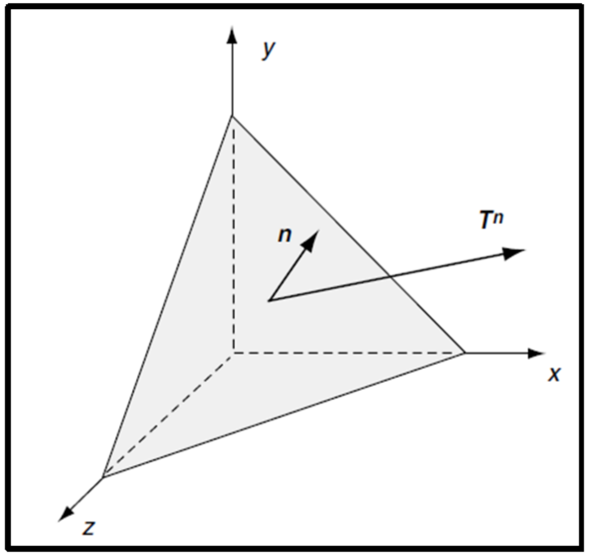
$$\boldsymbol{T}^n = n_x \boldsymbol{T}^n(\boldsymbol{n} = \boldsymbol{e}_1) + n_y \boldsymbol{T}^n(\boldsymbol{n} = \boldsymbol{e}_2) + n_z \boldsymbol{T}^n(\boldsymbol{n} = \boldsymbol{e}_3)$$

The Stress Tensor (rectangular Cartesian components)

This can be written as:

$$T^n = (\sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z) e_1 + (\tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z) e_2 + (\tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z) e_3$$

$$\Rightarrow \quad T_i^n = \sigma_{ji} n_j$$



Stress Transformation

- Stress components at an oblique plane with arbitrary orientation:

$$\sigma'_{ij} = Q_{ip} Q_{jq} \sigma_{pq}$$

Where the rotation matrix $\mathbf{Q}_{ij} = \cos(x'_i, x_j)$

- For the general three-dimensional case, the rotation matrix may be chosen in the form

$$Q_{ij} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

Stress Transformation

- The specific translation then becomes:

$$\sigma_x' = \sigma_x l_1^2 + \sigma_y m_1^2 + \sigma_z n_1^2 + 2(\tau_{xy} l_1 m_1 + \tau_{yz} m_1 n_1 + \tau_{zx} n_1 l_1)$$
$$\sigma_y' = \sigma_x l_2^2 + \sigma_y m_2^2 + \sigma_z n_2^2 + 2(\tau_{xy} l_2 m_2 + \tau_{yz} m_2 n_2 + \tau_{zx} n_2 l_2)$$
$$\sigma_z' = \sigma_x l_3^2 + \sigma_y m_3^2 + \sigma_z n_3^2 + 2(\tau_{xy} l_3 m_3 + \tau_{yz} m_3 n_3 + \tau_{zx} n_3 l_3)$$
$$\tau_{xy}' = \sigma_x l_1 l_2 + \sigma_y m_1 m_2 + \sigma_z n_1 n_2 + \tau_{xy}(l_1 m_2 + m_1 l_2) + \tau_{yz}(m_1 n_2 + n_1 m_2) + \tau_{zx}(n_1 l_2 + l_1 n_2)$$
$$\tau_{yz}' = \sigma_x l_2 l_3 + \sigma_y m_2 m_3 + \sigma_z n_2 n_3 + \tau_{xy}(l_2 m_3 + m_2 l_3) + \tau_{yz}(m_2 n_3 + n_2 m_3) + \tau_{zx}(n_2 l_3 + l_2 n_3)$$
$$\tau_{zx}' = \sigma_x l_3 l_1 + \sigma_y m_3 m_1 + \sigma_z n_3 n_1 + \tau_{xy}(l_3 m_1 + m_3 l_1) + \tau_{yz}(m_3 n_1 + n_3 m_1) + \tau_{zx}(n_3 l_1 + l_3 n_1)$$

Two dimensional problems

- In-plane stress components transform according to:

$$\sigma_x' = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_y' = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{xy}' = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

- Commonly rewritten in terms of the double angle:

$$\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_y' = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{xy}' = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Principal Stresses, Principal Stress Directions

- The determination of principal stress values and principal stress directions follows precisely the procedure for determining principal values and principal directions of any symmetric second-order tensor.

- The direction determined by the unit vector ***n*** is said to be a principal direction or eigenvector of the symmetric second-order tensor σ_{ij} if there exists a parameter λ such that

$$\sigma_{ij} n_j = \lambda \cdot n_i$$

Principal Stresses, Principal Stress Directions

Where λ is called the principal value or eigenvalue of the tensor, and the substitution property of the Kronecker delta allows to rewrite the above equation as:

$$\left(\sigma_{ij} - \lambda\delta_{ij}\right)n_j = 0$$

The above expression is simply a homogeneous system of three linear algebraic equations in the unknowns n_1, n_2, n_3. The system possesses a nontrivial solution if and only if the determinant of its coefficient matrix vanishes , that is:

$$\det\left[\sigma_{ij} - \lambda\delta_{ij}\right] = 0$$

Principal Stresses, Principal Stress Directions

which upon expansion yields a cubic in σ (called the characteristic equation of the stress tensor)

$$\det\left[\sigma_{ij} - \lambda\delta_{ij}\right] = -\lambda^3 + I_a\lambda^2 - II_a\lambda + III_a = 0$$

Where;

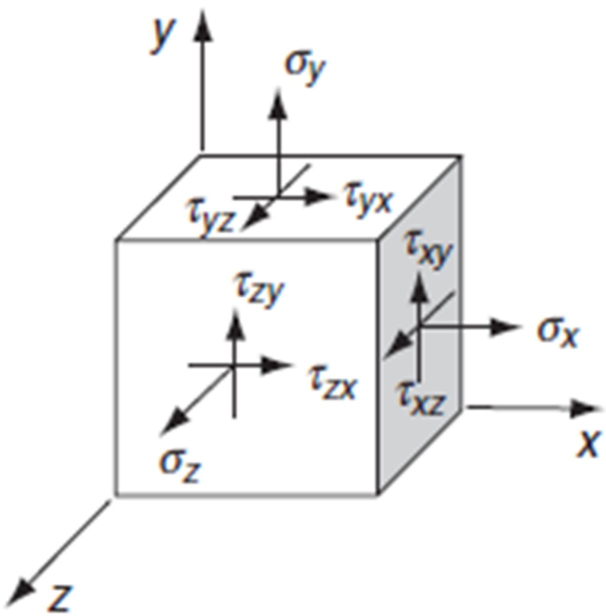
$$I_a = \sigma_{ii}$$

$$II_a = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right)$$

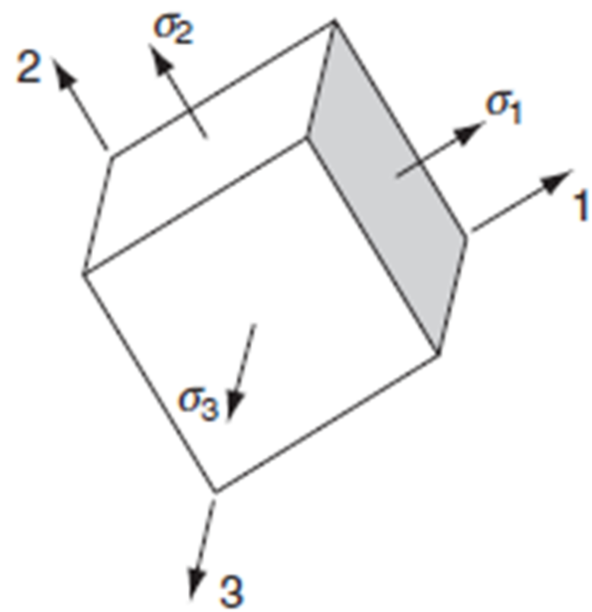
$$III_a = \det\left[\sigma_{ij}\right]$$

Principal Stresses, Principal Stress Directions

- The scalars I_a, II_a, *and* III_a are called the fundamental invariants of the stress tensor σ_{ij} and do not change value under coordinate transformation



(General Coordinate System)



(Principal Coordinate System)

Principal Stresses, Principal Stress Directions

- By denoting the principal directions $\mathbf{n_1}$, $\mathbf{n_2}$, $\mathbf{n_3}$ corresponding to the principal values λ_1, λ_2, λ_3 three possibilities arise:

  I. All three principal values distinct; thus, the three corresponding principal directions are unique (except for sense)

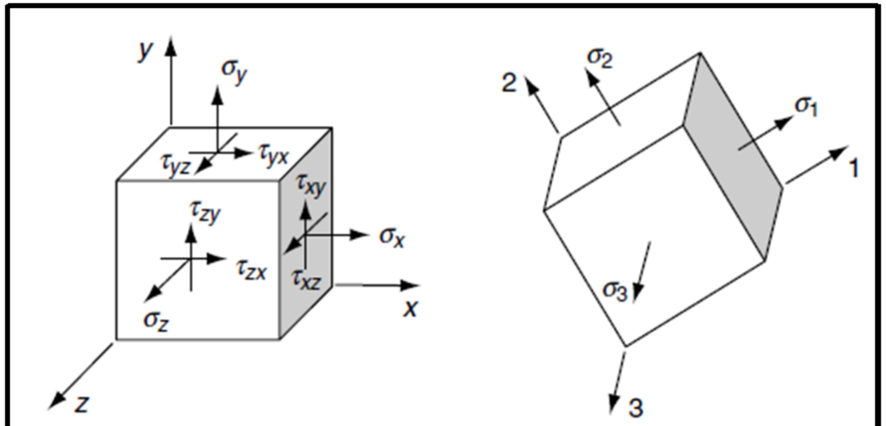
  II. Two principal values equal ($\lambda_1 \neq \lambda_2 = \lambda_3$); the principal direction n_1 is unique (except for sense), and every direction perpendicular to n_1 is a principal direction associated with λ_2, λ_3.

  III. All three principal values equal; every direction is principal, and the tensor is isotropic.

Principal Stresses, Principal Stress Directions

With respect to principal axes the stress tensor reduces to the diagonal form

$$\sigma_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$



(General Coordinate System) (Principal Coordinate System)

and the stress invariants can be expressed as:

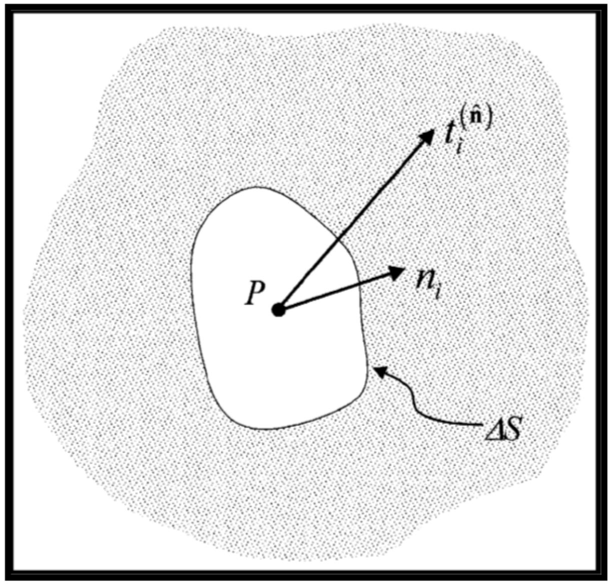
$$I_a = \lambda_1 + \lambda_2 + \lambda_3$$

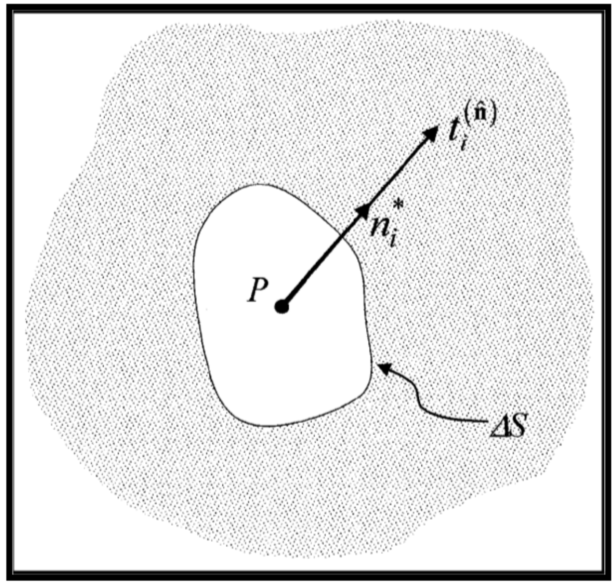
$$II_a = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$

$$III_a = \lambda_1\lambda_2\lambda_3$$

Principal Stresses, Principal Stress Directions

The eigenvalues have important extremal properties. If we arbitrarily rank the principal values such that $\lambda_1 > \lambda_2 > \lambda_3$, then λ_1 will be the largest of all possible diagonal elements, while λ_3 will be the smallest diagonal element possible.

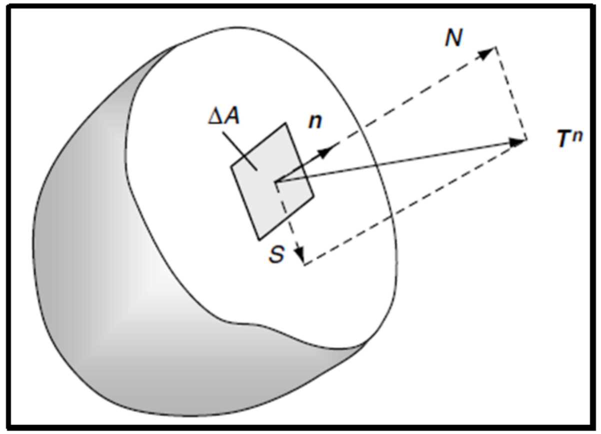




Normal and Shear Stress Components

- Consider the general traction vector $\boldsymbol{T^n}$

Let N and S be the traction vector's normal and shear components



$$N = T^n . n$$

$$S = \left(\left| T^n \right|^2 - N^2 \right)^{1/2}$$

$$N = T^n . n = T_i^n . n_i = \sigma_{ji} n_j n_i$$

$$N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$$

- Using the above relations;

$$\boldsymbol{N = T^n . n}$$

$$\boldsymbol{N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2}$$

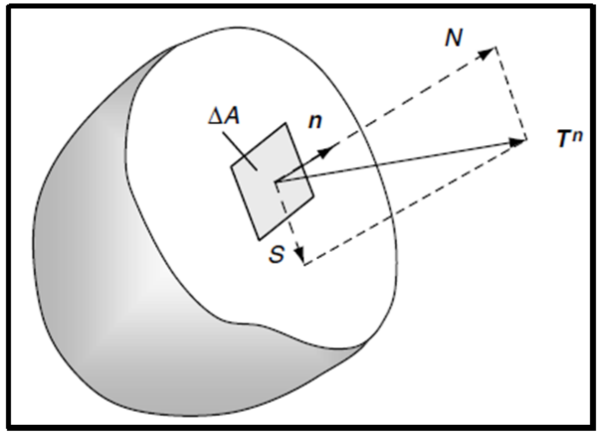
$$\boldsymbol{S^2 + N^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2}$$

In addition;

$$\boldsymbol{1 = n_1^2 + n_2^2 + n_3^2}$$

Mohr's Circles for Stress

Solving for the unknowns n_1^2, n_2^2 and n_3^2



$$n_1^2 = \frac{S^2 + (N - \sigma_2)(N - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}$$

$$n_2^2 = \frac{S^2 + (N - \sigma_3)(N - \sigma_1)}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)}$$

$$n_3^2 = \frac{S^2 + (N - \sigma_1)(N - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}$$

- We can rank the principal stresses as $\sigma_1 > \sigma_2 > \sigma_3$.

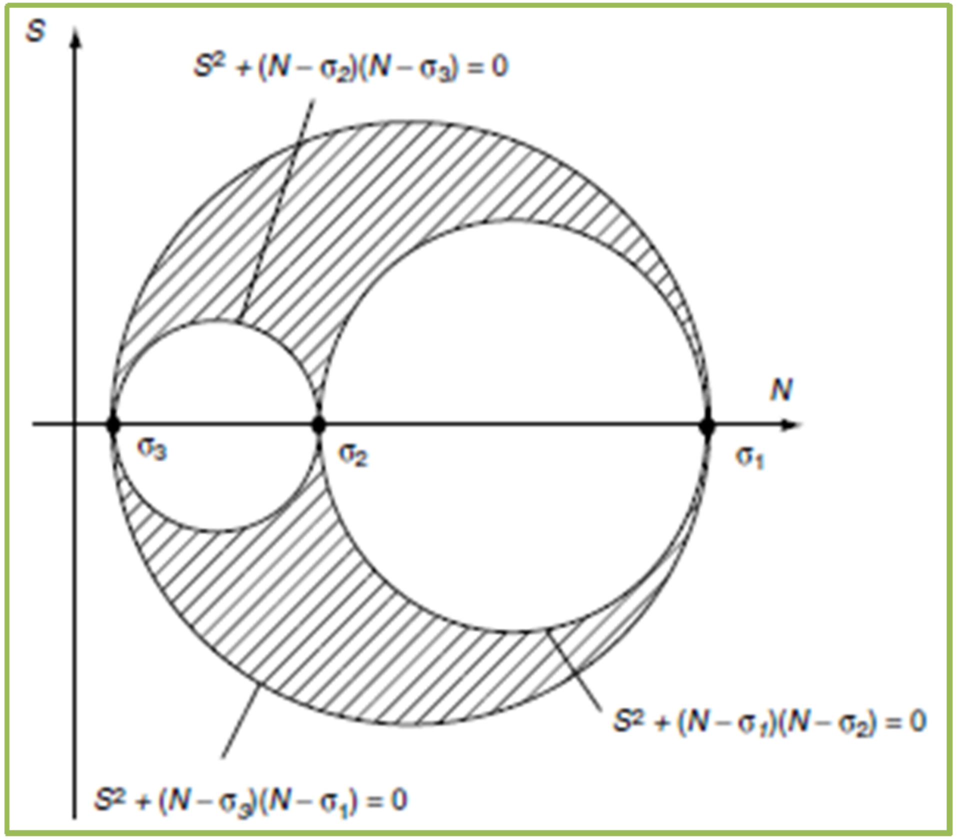
$$S^2 + (N - \sigma_2)(N - \sigma_3) \geq 0$$

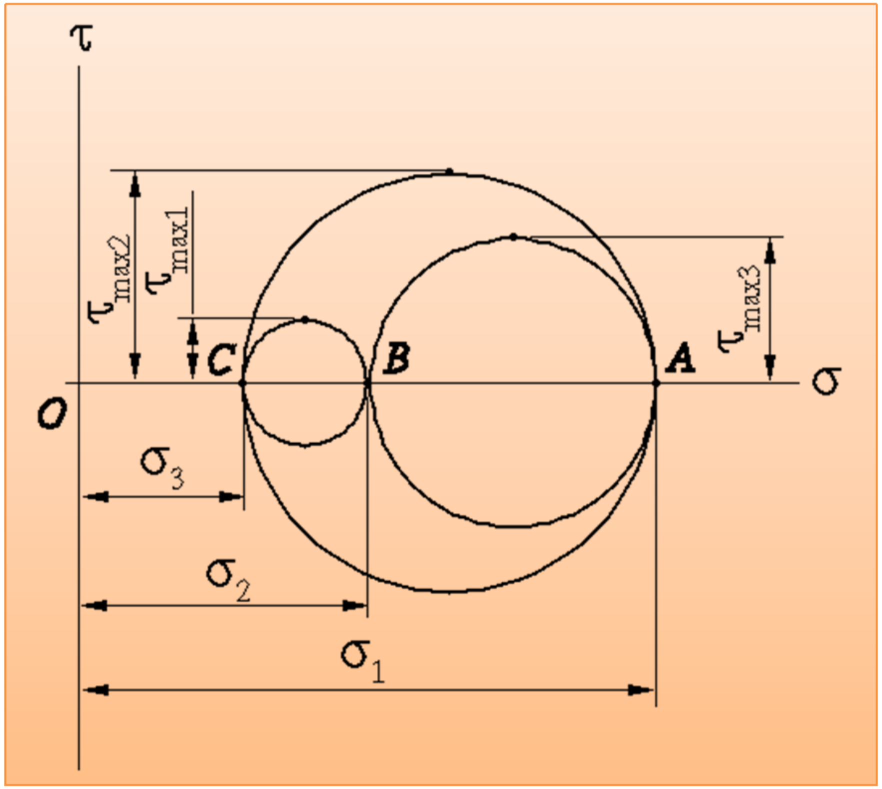
$$S^2 + (N - \sigma_3)(N - \sigma_1) \leq 0$$

$$S^2 + (N - \sigma_1)(N - \sigma_2) \geq 0$$

- For the equality case, the above equations represent three circles in S-N coordinate system

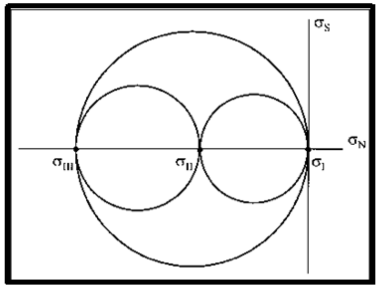
Mohr’s Circles for Stress

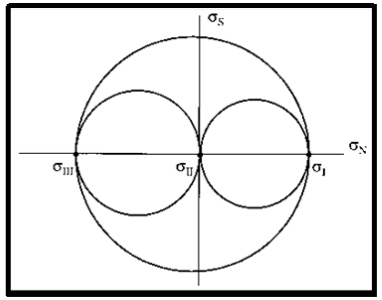


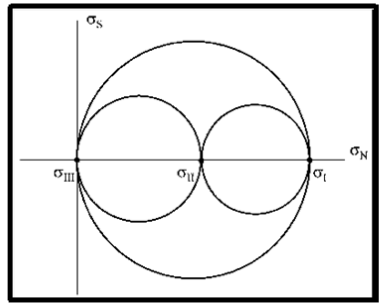


Plane Stress

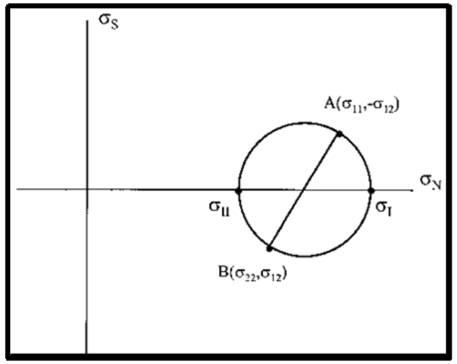
$$[\sigma_{ij}^*] = \begin{bmatrix} \sigma_{(1)} & 0 & 0 \\ 0 & \sigma_{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



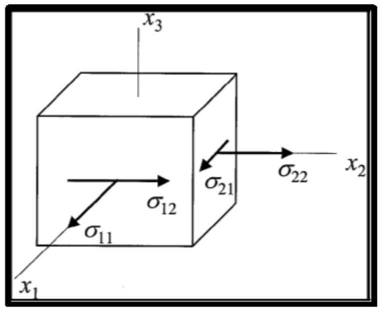




$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$$\left.\begin{matrix} \sigma_{(1)} \\ \sigma_{(2)} \end{matrix}\right\} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\frac{\sigma_{11} - \sigma_{22}}{2} + \sigma_{12}{}^2}$$



Example

EXAMPLE ·1: Stress Transformation

For the following state of stress, determine the principal stresses and directions and find the traction vector on a plane with unit normal $\boldsymbol{n} = (0,1,1)/\sqrt{2}$.

$$\sigma_{ij} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

The principal stress problem is started by calculating the three invariants, giving the result $I_1 = 3$, $I_2 = -6$, $I_3 = -8$. This yields the following characteristic equation:

$$-\sigma^3 + 3\sigma^2 + 6\sigma - 8 = 0$$

The roots of this equation are found to be $\sigma = 4,\ 1,\ -2$. Back-substituting the first root into the fundamental system (see 1.6.1) gives

$$\begin{aligned} -n_1^{(1)} + n_2^{(1)} + n_3^{(1)} &= 0 \\ n_1^{(1)} - 4n_2^{(1)} + 2n_3^{(1)} &= 0 \\ n_1^{(1)} + 2n_2^{(1)} - 4n_3^{(1)} &= 0 \end{aligned}$$

Solving this system, the normalized principal direction is found to be $\boldsymbol{n}^{(1)} = (2,\ 1,\ 1)/\sqrt{6}$. In similar fashion the other two principal directions are $\boldsymbol{n}^{(2)} = (-1,\ 1,\ 1)/\sqrt{3}$, $\boldsymbol{n}^{(3)} = (0,\ -1,\ 1)/\sqrt{2}$.

The traction vector on the specified plane is calculated by using the relation

$$T_i^n = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix}$$

Deviator and Spherical Stress

It is often convenient to decompose the stress in to two parts, called the spherical and deviatoric stresses

$$\bar{\sigma}_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} \qquad \sigma_M = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{kk}$$

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_M & 0 & 0 \\ 0 & \sigma_M & 0 \\ 0 & 0 & \sigma_M \end{bmatrix}$$

While the deviatoric stress becomes:

$$\hat{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

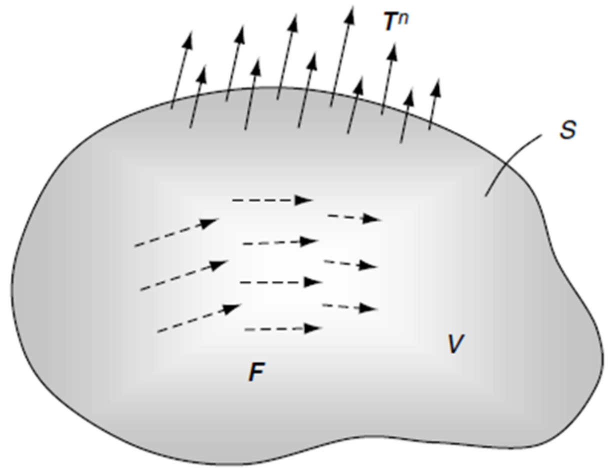
- The spherical stress is an isotropic tensor.
- The principal directions of the deviatoric stress are the same as those of the stress tensor.

Equilibrium equations

- Consider a closed sub-domain with volume V and surface S within a body in equilibrium.
- For static equilibrium the forces acting on this region are balanced and thus the resultant force mush vanish.

$$\iint_S T_i^n dS + \iiint_V F_i dV = 0$$

$$\iint_S \sigma_{ji} n_j dS + \iiint_V F_i dV = 0$$



Applying the divergence theorem;

$$\iiint_V (\sigma_{ji,j} + F_i) dV = 0$$

Because the region V is arbitrary and the integrand is continuous, then by the zero-value theorem, the integrand must vanish:

Equilibrium equations

$$(\sigma_{ji,j} + F_i) = \mathbf{0}$$

The above equation represents the three scalar equations of equilibrium;

All elasticity stress fields must satisfy these relations in order to be in static equilibrium.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = \mathbf{0}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = \mathbf{0}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = \mathbf{0}$$