Theory of Elasticity

Introduction

CENG-6501

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Introduction to Theory of Elasticity

- Theory of elasticity is concerned with the study of the response of elastic bodies to the action of forces.

- Objective of this study: to analyze the stresses and displacements of structures within the elastic range and thereby to check the sufficiency of their strength, stiffness and stability.

- Elasticity is the branch of solid mechanics which deals with the stresses and deformations in elastic solids produced by direct and indirect actions.

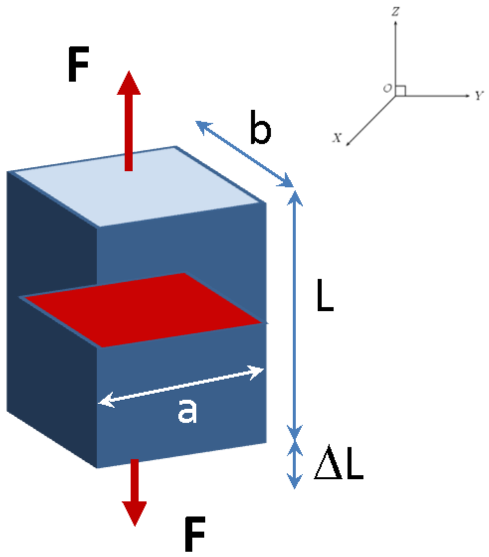
- All structural materials possess to a certain extent the property of elasticity.

- In this course, the matter of an elastic body is assumed to be
    - Homogeneous
    - Isotropic

Introduction to Theory of Elasticity

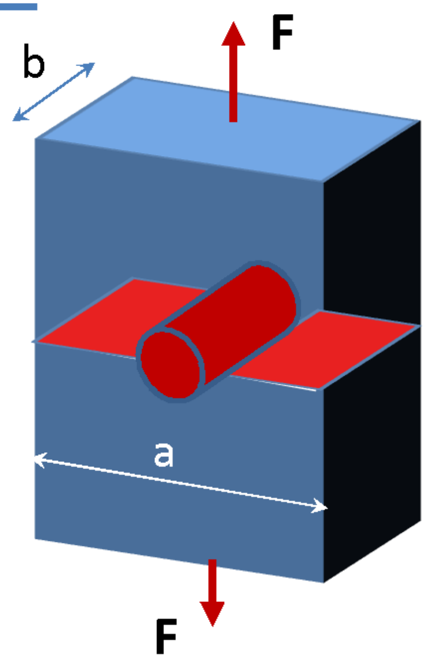
- A body under elastic deformation reverts to its original state on the removal of loads.
- Elasticity theory is also useful for inelastic deformation, such as fracture and plasticity, by studying the microscopic agents of inelasticity, such as crack or a dislocation --- this is called micromechanics.
- In this course we will focus on Linear, infinitesimal elasticity.
  - Stress and displacements are linear with loads
  - Linear superposition can be used to construct solutions (statics only)
- Compared to strength of materials, which makes plausible but unsubstantiated assumptions:
  - Elasticity theory is a more rigorous treatment
  - Only makes mathematical assumptions (usually in the last step, to help solve the equation instead of physical assumptions (hard to justify)
  - Allows us to assess the quality of assumptions made in strength of materials
  - Uses more advanced mathematical tools, tensors, partial differential equations, Fourier transform …

Mechanics of Materials Vs Theory of Elasticity



Axial stress; σ_{zz} = F/A

Axial stress; ε_{zz} = ΔL/L



Rectangular bar with a hole

A' = (a-2r).b

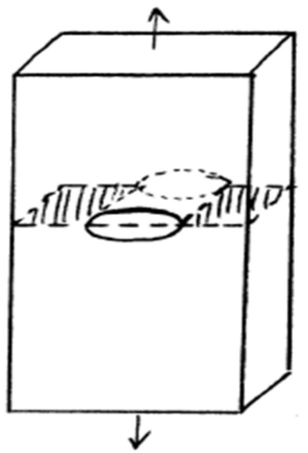
Axial stress; $(\sigma_{zz})_{max}$ = k.F/A'

k – stress concentration factor

Q. Where does it come from?

In **mechanics** of materials there is look-up table
In **Elasticity** we compute the stress distribution around the hole – it can be showed that k = 3 when a >> r .

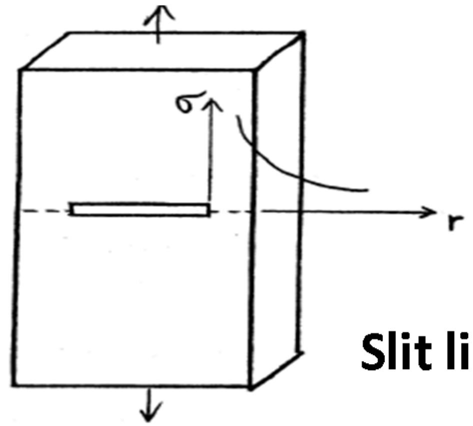
Mechanics of Materials Vs Theory of Elasticity



Elliptical hole

Q. What is the stress concentration factor?

Elasticity Theory can answer.



Slit like crack

Stress field becomes singular at crack tip.

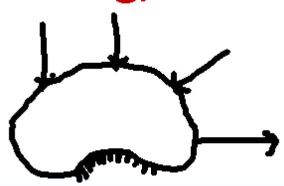
Elasticity theory predicts

- $\sigma \approx \frac{1}{\sqrt{r}}$

- stability criteria for crack advancement.

Given a continuum medium subjected to external loading, we want to find:

- **The displacement field $U_i(x)$ -> vector**
- **The strain field $\varepsilon_{ij}(x)$**
- **The stress field $\sigma_{ij}(x)$**



Mathematical preliminaries

- Scalar, vector, matrix and Tensor
  - Scalar quantities – representing a single magnitude at each point in space
  - Vector quantities – Expressible in terms of components in a 2 or 3 dimensional coordinate system.
  - Formulations with in theory of elasticity also require the need for **matrix variables**.

  Examples:

$$mass\ density\ scalar = \rho$$

$$displacement\ vector = \boldsymbol{U} = u\boldsymbol{e}_1 + v\boldsymbol{e}_2 + w\boldsymbol{e}_3$$

$$stress\ matrix = [\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are unit basis vectors in the coordinate directions

A scalar is a tensor of rank zero, and a vector is a tensor of rank one.

- In n-dimensional space a tensor of rank 3 would have n^3 components.

Mathematical preliminaries

- Index notation

Vector:

$$a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Matrix:

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- **In general a symbol $a_{ij\ldots k}$ with N distinct indices represents 3^N distinct numbers.**

Addition, subtraction, multiplication and equality of index symbols are defined in normal fashion.

$$a_i \pm b_i = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{bmatrix}$$

$$a_{ij} \pm b_{ij} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{bmatrix}$$

$$\lambda a_i = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{bmatrix}, \ \lambda a_{ij} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix}$$

$$a_i b_j = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

Mathematical preliminaries

- Index notation obey usual commutative, associative, distributive laws

$$a_i + b_i = b_i + a_i$$

$$a_{ij} b_k = b_k a_{ij}$$

$$a_i + (b_i + c_i) = (a_i + b_i) + c_i$$

$$a_i (b_{jk} c_l) = (a_i b_{jk}) c_l$$

$$a_{ij} (b_k + c_k) = a_{ij} b_k + a_{ij} c_k$$

- If a subscript appears twice in the same term, then the summation over that subscript from one to three is implied;

$$a_{ii} = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}$$

$$a_{ij} b_j = \sum_{j=1}^{3} a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + a_{i3} b_3$$

Mathematical preliminaries

- Repeated indices – *Dummy subscripts*
- unspecified indices that are not repeated – *free or distinct indices*

- The summation convention may be suspended by underlining one of the repeated indices or by writing no sum.

- The process of setting two free indices equal is called *contraction*.

- A symbol $a_{ij\ldots m\ldots n\ldots k}$ is said to be symmetric with respect to index *mn* if:

$$a_{ij\ldots m\ldots n\ldots k} = a_{ij\ldots n\ldots m\ldots k}$$

- is said to be skewsymmetric with respect to index *mn* if:

$$a_{ij\ldots m\ldots n\ldots k} = -a_{ij\ldots n\ldots m\ldots k}$$

Mathematical preliminaries

- An arbitrary symbol a_{ij} can be expressed as the sum of symmetric and antisymmetric parts

$$a_{ij} = \frac{1}{2}\left(a_{ij} + a_{ji}\right) + \frac{1}{2}\left(a_{ij} - a_{ji}\right)$$

→ *The first term is symmetric while the second is antisymmetric*

Example

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Kronecker Delta

- A special symbol commonly used in index notational schemes → *kronecker delta*

$$\delta_{ij} = \begin{cases} 1, & if \quad i = j(nosum) \\ 0, & if \quad i \neq j \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ *useful properties of the kronecker delta*

$$\delta_{ij} = \delta_{ji}$$
$$\delta_{ii} = 3, \quad \delta_{i\underline{i}} = 1$$
$$\delta_{ij} a_j = a_i, \quad \delta_{ij} a_i = a_j$$

$$\delta_{ij} a_{jk} = a_{ik}, \quad \delta_{jk} a_{ik} = a_{ij}$$
$$\delta_{ij} a_{ij} = a_{ii}, \quad \delta_{ij} \delta_{ij} = 3$$

Alternating symbol

- Another useful symbol commonly used in index notational schemes
  → *Alternating symbol*

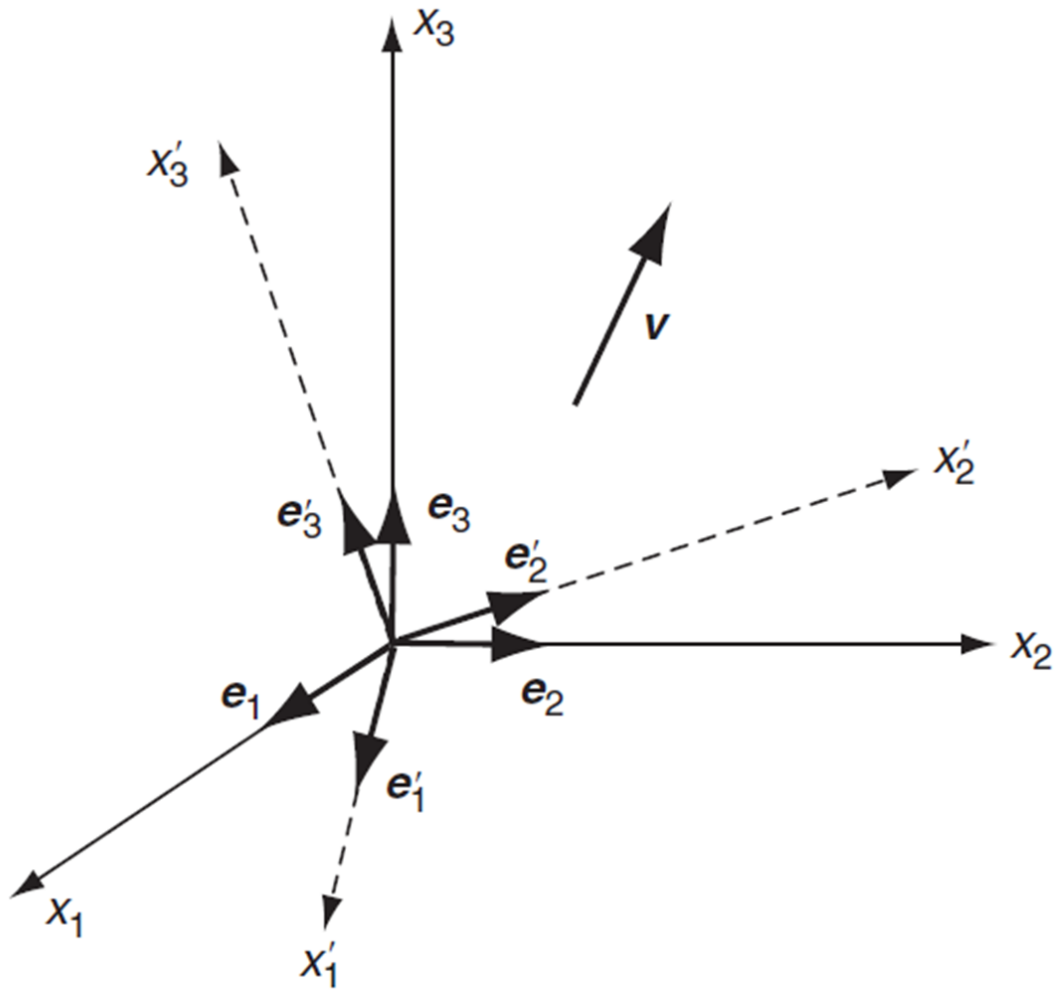
$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if} \quad ijk \text{ is an even permutation of } 1,2,3 \\ -1, & \text{if} \quad ijk \text{ is an odd permutation of } 1,2,3 \\ 0, & \text{otherwise} \end{cases}$$

→ *useful in evaluating determinants*

$$\det[a_{ij}] = |a_{ij}| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$$

Exercise

Coordinate Transformation



2. Stress and Equilibrium

2.1 Body and Surface Forces

2.2 Components of a stress at a point

2.3 State of plane stress at a point-Stress transformation

- a. Uniform Distribution
- b. Non-uniform Distribution

2.4 Principal Stresses

2.5 Equations of Equilibrium

2.6 Analysis of Stress in Three Dimensions