Groundwater Hydraulics

Chapter 1 – Introduction

CENG 6606

Contents

What is groundwater?

- Equivalent terminologies: groundwater, subsurface water, ground water
- **Definitions**
  - Water occupying all the voids within a geologic stratum (Todd, 1980)
  - All the water found beneath the surface of the ground (Bear and Verruijt, 1987)
  - Practically, all the water beneath the water table (i.e., in the saturated zone) and above the water table (i.e., in the unsaturated zone, vadose zone, zone of aeration) are called groundwater.
- Groundwater velocity is very small and depends on local hydrogeologic conditions , 2 m/year to 2 m/day are normal (Todd, 1980).

1. Definitions of aquifers

- Aquifer, Aquitards, Aquicludes, and Aquifuges are a geological formation or a group of formations that can/cannot contain water, and that water can/cannot move within the formation

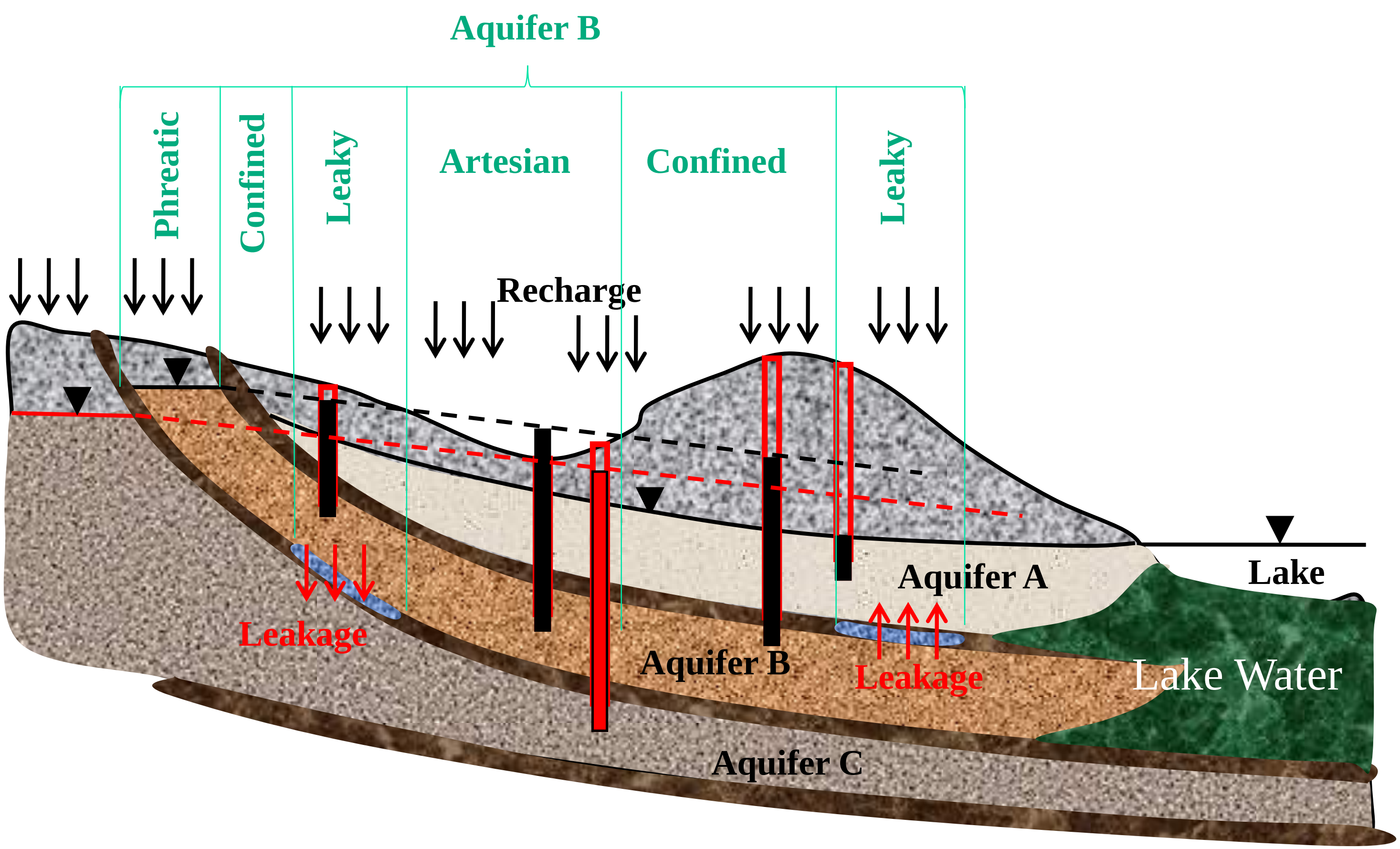
	Formation nature	Store water?	Transmit water?
Aquifer	Pervious	Yes	Yes
Aquitard	Semi pervious	Yes	Yes but slower than that in an aquifer
Aquiclude	Semi pervious	Yes	No
Aquifuge	Impervious	No	No

Latin: Aqui ≡ water; -fer ≡ "to bear", aquifer ≡ "water bearer"
-tard ≡ "slow"; -clude ≡ "to shut or close"; -fuge ≡ "to drive away"
(Todd, 1980)

- **Confined aquifer**: an aquifer bounded from above and from below by impervious formations (aquiclude or aquifuge)
- **Unconfined aquifer** (phreatic aquifer or water table aquifer): an aquifer in which water table serves as its upper boundary
- **Perched aquifer**: An unconfined aquifer which has an impervious layer of limited areal extent located between the ground surface and the water table (of the unconfined aquifer)
- **Confining layer**: a geologic formation that is impervious to water, e.g., unconsolidated soils such as silt and clay; consolidated bed rock such as limestone, sandstone, siltstone, basalt, granite, …, etc. The latter rock formations can also be aquifers when only consolidated formations are considered.

- **Piezometric surface or Potentiometric surface**: an imaginary surface connecting the water levels of a number of observation wells tapping into a confined aquifer.
    - **Note**: use water table instead of piezometric surface for unconfined aquifers
- **Artesian aquifer**: a confined aquifer whose piezometric surface is above the ground surface (i.e., water comes out automatically from a well in an artesian aquifer)
- **Double porosity aquifer**
    - For fractured rocks
        - Matrix blocks: low permeability, high storativity
        - Fractures: high permeability, low sotrativity

Schematic of aquifers



Schematic of perched aquifers

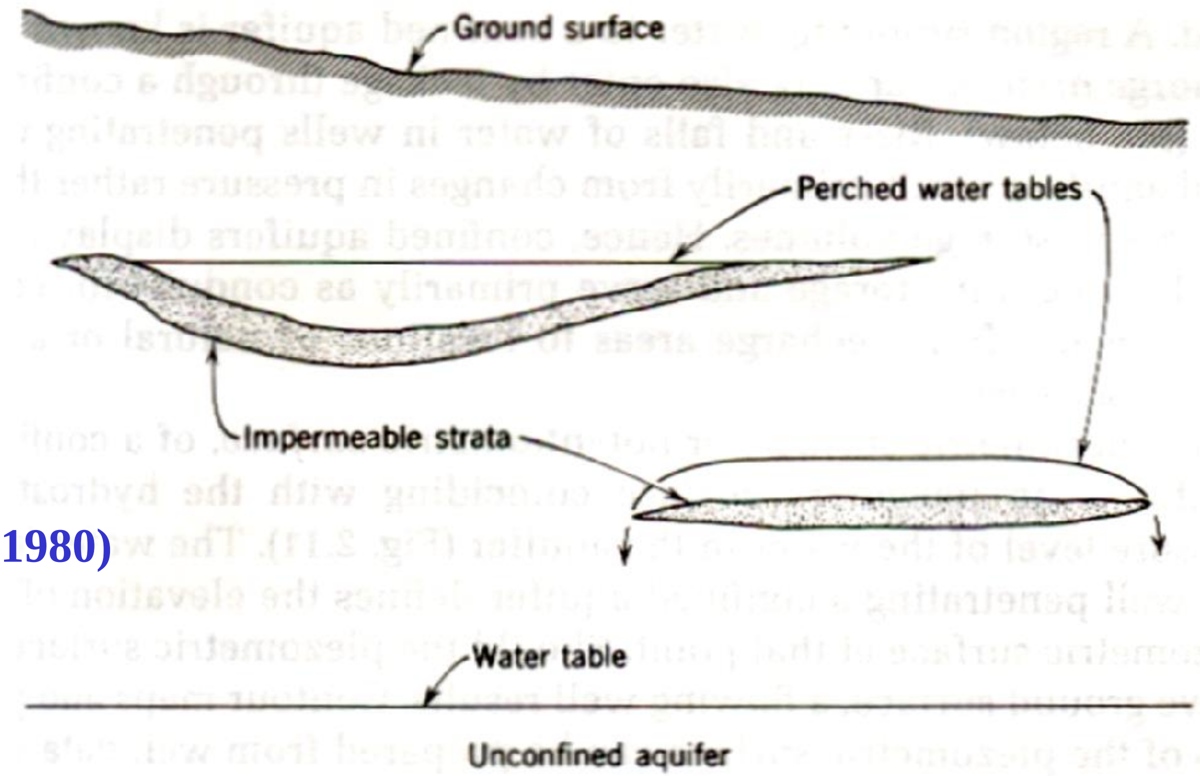


Fig. 2.12 Sketch of perched water tables.

(After Todd, 1980)

2. Fundamental physical properties

- Six fundamental physical properties for describing hydraulic aspects of saturated groundwater flow in aquifers
  - Three fluid properties
    - Density, ρ (M/L^3)
    - Dynamic viscosity, μ ($M/L \cdot T$) (or kinematic viscosity, $\nu = \mu/\rho$, L^2/T)
    - Fluid compressibility, β (LT^2/M) (or 1/Pa)
  - Three medium properties
    - Porosity, n
    - Permeability, k (L^2)
    - Matrix compressibility, α (LT^2/M) (or 1/Pa)

3. Porosity (pore space)

- **Pore space** (voids, pores, or interstices): The portion of a geologic formation that is not occupied by solid matter (e.g., soil grain or rock matrix).
- **Effective or interconnected pore space**: Pores that form a continuous phase through which water or solute can move
- **Isolated or non-inter connected pore space**: Pores that are dispersed (scattered) over the medium. These pores cannot contribute to transport of matter across the porous medium. Also known as **dead-end pores**.
- **Saturated zone**: Aquifers in which pore space is completely filled with water.
- **Unsaturated zone**: Aquifers in which the pore space is filled partially with liquid phase (water), and partially with gas phase.

3. Porosity (pore space)

- Only the connected pores (gray) can transmit water; the unconnected pores (white) are not a part of the effective porosity

Schematic of various pore space

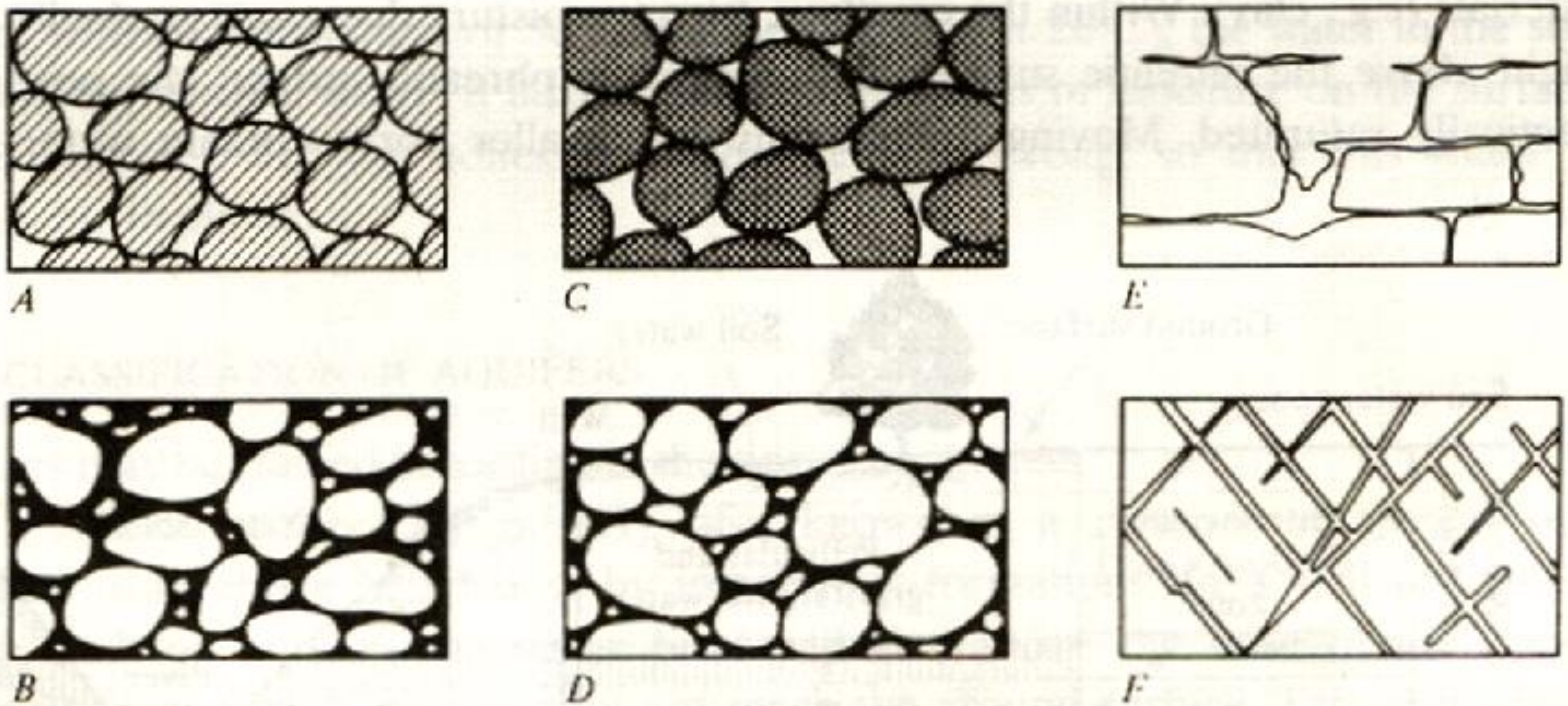


Fig. 1.2. Diagram showing several types of rock interstices. A Well-sorted sedimentary deposit having high porosity; B. Poorly sorted sedimentary deposit having low porosity; C. Well-sorted sedimentary deposit consisting of pebbles that are themselves porous, so that the deposit as a whole has a very high porosity; D. Well-sorted sedimentary deposit whose porosity has been diminished by the deposition of mineral matter in the interstices; E. Rock rendered porous by solution; F. Rock rendered porous by fracturing (*after Meinzer, 1942*).

(After Bear and Verruijt, 1987)

n (d_{50}, shape, arrangement, clay)

- **Porosity is a function of**

(a) **Grain size distribution**: soils with uniformly distributed grains have larger porosities than soils with un-uniformly distributed grains

(b) **Grain shape:** Sphere-shaped grains will pack more tightly and have less porosity than particles of other shapes

(c) **Grain arrangement:** Porosities of well-rounded sediments range from 26% (rhombohedral packing) to 48% (cubic packing)

(d) **Clays**, clay-rich or organic soils have very high porosities because:

  - Grain shapes are highly irregular
  - Dispersive effect of the electrostatic charge on the surfaces of certain book-shaped clay minerals causes clay particles to be repelled by each other, resulting in high porosity.

Unconsolidated deposits (sediments)	Porosity (%)
Gravel	25-40
Sand	25-50
Silt	35-50
Clay	40-70
Rocks	
Fractured basalt	5-50
Karst limestone	5-50
Sandstone	5-30
Limestone, dolomite	0-20
Shale	0-10
Fractured crystalline rock	0-10
Dense crystalline rock	0-5

(After Freeze and Cherry, 1979)

- Porosity of rocks (Two porosities)
  - **Primary:** pore space between grains
  - **Secondary:** pore space caused by fracturing
- **Sedimentary rocks (Fetter, 1994):** Formed by sediments by digenesis. Sediments are products of weathering of rocks or chemically precipitated materials. Changes in sediments due to overlying materials and physiochemical reactions with fluid in the pore space result in pore space variation.
- **Compaction:** porosity is reduced
- **Dissolution:** porosity is increased
- **Precipitation** (e.g., cementing materials such as calcite, dolomite, or silica): porosity is reduced

- **Limestone and dolomites**: Formed respectively by calcium carbonate and calcium-magnesium carbonate, which were originally part of an aqueous solution. These rocks may be dissolved in a zone of circulating groundwater, resulting in huge caverns that have sizes as large as a building. For example, the caverns at Carlsbad, New Mexico.

Bottomless Pit

Hall of Giants

The Big Room

4. Hydraulic head

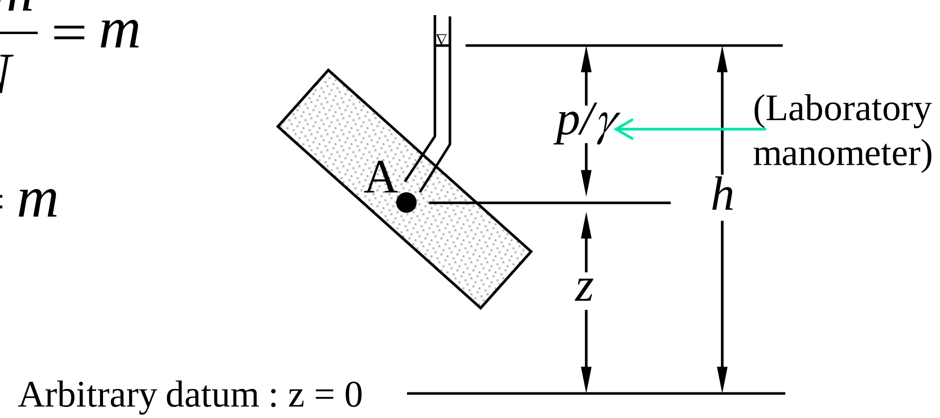
- Hydraulic head, h (L): For incompressible fluids (density is a constant) it is the sum of potential energy and pressure energy **per unit weight** of water. Sum of elevation head (z) and pressure head (p/γ)

$$[h] = \frac{energy}{weight\ of\ \ water} = \frac{Nm}{N} = m$$

$$\frac{p}{\gamma} = \frac{p}{\rho g} = \frac{N/m^2}{(Kg/m^3)(m/s^2)} = \frac{Nm}{N} = m$$

$$[z] = \frac{Potential\ \ Energy}{Weight\ \ of\ \ Water} = \frac{Nm}{N} = m$$

$$h = \frac{p}{\gamma} + z \qquad (2)$$



Schematic of hydraulic head, pressure head, and elevation (potential) head

5. Hydraulic conductivity

- Fluid conductivity, K [L/T]: It is measurement of the ease of a particular fluid passing through the pore space of a porous medium (i.e., conductive properties of a porous medium for a particular fluid) (Hubbert, 1956). The proportionality constant in Darcy's law, which depends on medium and fluid properties i.e. grain size, density and viscosity of fluid.
- If the fluid is water K is hydraulic conductivity

$$K = \frac{Cd^2 \rho g}{\mu}$$

C: shape factor, a medium property
d: representative grain diameter, a medium property
ρ: fluid density, g: gravity

$\gamma = \rho g$: specific weight of fluid, driving force exerted by gravity on a unit volume of the fluid ($[\gamma] = kg/m^3 \cdot m/s^2 = (kg \cdot m/s^2)/m^3 = N/m^3$ = force per unit volume)

μ: Dynamic viscosity (resistance of the fluid to shearing)

Table 2.1. Typical values of hydraulic conductivity and permeability[a]

$-\log_{10} \cdot K$ (cm/sec)	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11

<table>
<tr><td>Permeability</td><td colspan="4">Pervious</td><td colspan="4">Semipervious</td><td colspan="5">Impervious</td></tr>
<tr><td>Aquifer</td><td colspan="5">Good
(Aquifer)</td><td colspan="4">Poor
(Aquitard, aquiclude)</td><td colspan="4">None
(Aquifuge)</td></tr>
<tr><td rowspan="2">Soils</td><td colspan="2">Clean gravel</td><td colspan="3">Clean sand or sand and gravel</td><td colspan="4">Very fine sand, silt, loess, loam, solonetz</td><td colspan="4"></td></tr>
<tr><td colspan="4"></td><td colspan="2">Peat</td><td colspan="3">Stratified clay</td><td colspan="4">Unweathered clay</td></tr>
<tr><td>Rocks</td><td colspan="4"></td><td colspan="3">Oil rocks</td><td colspan="2">Sandstone</td><td colspan="2">Good limestone, dolomite</td><td colspan="2">Breccia, granite</td></tr>
</table>

$-\log_{10} \cdot k$ (cm²)	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\log_{10} k$ (md)	8	7	6	5	4	3	2	1	0	−1	−2	−3	−4	−5

[a] From Bear *et al.* (1968).

(After Bear and Verruijt, 1987)

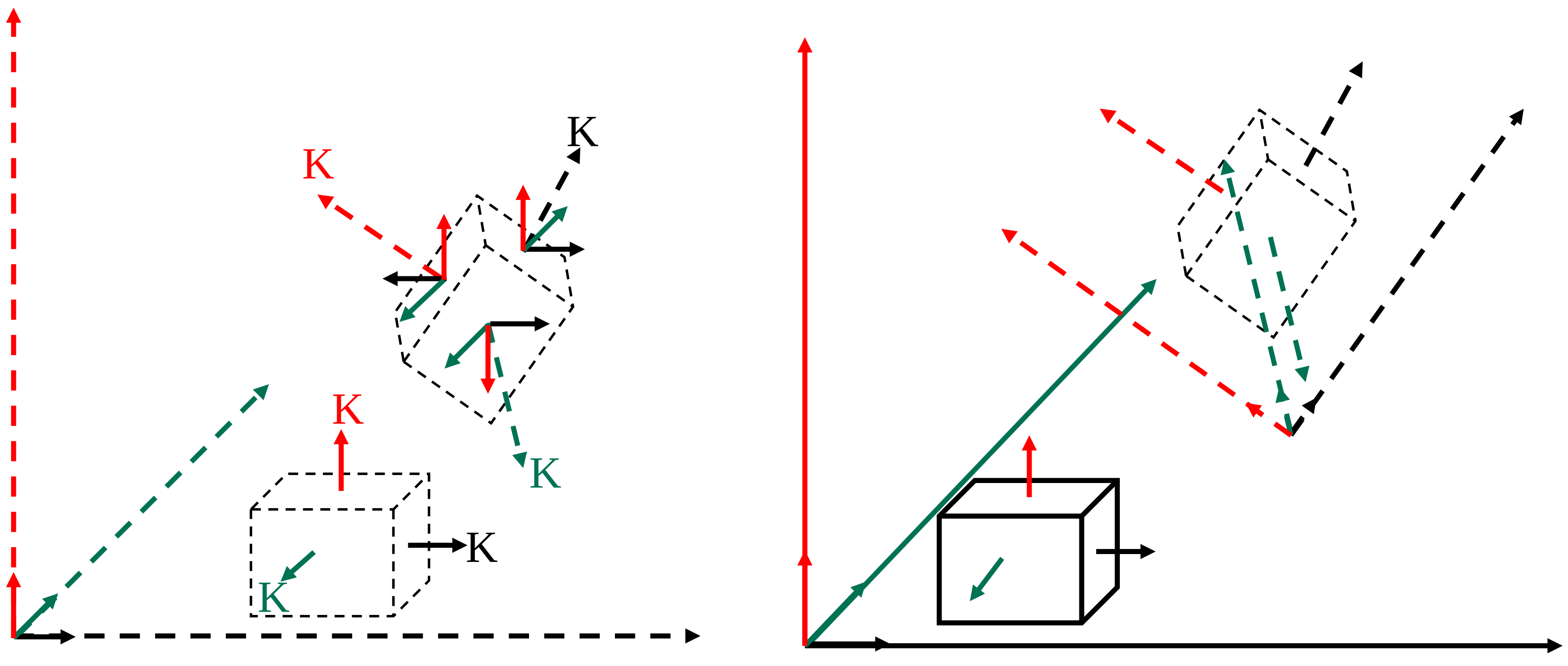
Hydraulic conductivity as a tensor

- Tensor
  - Zero-order: a scalar such as hydraulic head, a single-valued quantity
  - First-order: a vector such as velocity, having 3 components
  - Second-order: a tensor such as hydraulic conductivity, having 9 components

$$\mathbf{K} = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix}$$

- Symmetric hydraulic conductivity : $K_{ij} = K_{ji}$ $(i \neq j)$
- Principal hydraulic conductivity: the coordinate system aligns along the principal axes of **K** such that $K_{ij} = 0$ $(i \neq j)$
- Isotropic hydraulic conductivity: the value of **K** does not depend on direction, i.e., $K_{ij} = 0$ $(i \neq j)$, and $K_{ii} = K$

Hydraulic conductivity Tensor



- Contravariant and covariant tensors

Determination of hydraulic conductivity

- Theoretical calculation
  - $k = Cd^2 \Rightarrow K = k\rho g/\mu$
  - Few estimates are reliable because of the difficulty of including all possible variables in porous media
- Laboratory measurements
  - Permeameter (Fetter, 1994)
    - **Constant head experiment**: For non cohesive sediments, such as sand and rocks because of the required duration for experiment is short
    - **Falling head experiment**: For cohesive sediments with a low permeability, such as clay and silt
- Field measurements
  - Pumping test, slug test
  - Tracer test

Heterogeneity and anisotropy

- Heterogeneity: $K(\mathbf{x}_1) \neq K(\mathbf{x}_2)$, $\mathbf{x}_1 \neq \mathbf{x}_2$
  - Layered sediments, e.g., sedimentary rocks and unconsolidated marine deposits
  - Spatial discontinuity, e.g., the presence of faults or large-scale stratigraphic features
- Evidences from stochastic studies
  - Hydraulic conductivity tends to be log-normally distributed
  - $Y = \log_{10}K$ has standard deviation in the range 0.5 ~1.5, meaning K values in most geological formations show variations of 1 – 2 orders of magnitude (Freeze, 1975)

Freeze, R. A., A stochastic-Conceptual Analysis of One-Dimensional Groundwater Flow in Nonuniform Homogeneous *Media*, *Water Resources Research*, Vol. 11, No. 5, pp.725-741, 1975.

Example: (Fetter, 1994) Find the geometric mean of the following set of hydraulic conductivity values and compare it to the arithmetic mean : ***K (cm/s)*** *= 2.17* $\times 10^{-2}$ *, 2.58* $\times 10^{-2}$ *, 2.55* $\times 10^{-3}$ *, 1.67* $\times 10^{-1}$ *, 9.50* $\times 10^{-4}$ *;* ***Sum of K (cm/s)*** *= 2.18* $\times 10^{-1}$

Solution :

$$\text{Geometric mean} = K_G = \left(\prod_{i=1}^{n} K_i \right)^{1/n} \Rightarrow \ln K_G = \frac{1}{n} \sum_{i=1}^{n} \ln(K_i)$$

$$\frac{1}{5} \sum_{i=1}^{5} \ln(K_i) = -4.44 \Rightarrow K_G = \exp\left(\sum_{i=1}^{5} \ln(K_i) \right) = \exp(-4.44) = 1.18 \times 10^{-2} \text{ cm/s}$$

$$\text{Arithmetic mean} = \frac{1}{5} \sum_{i=1}^{5} K_i = \frac{2.18 \times 10^{-1}}{5} = 4.36 \times 10^{-2} \text{ cm/s} > 1.18 \times 10^{-2} \text{ cm/s}$$

Which is the best estimate and why? (think)

Answer

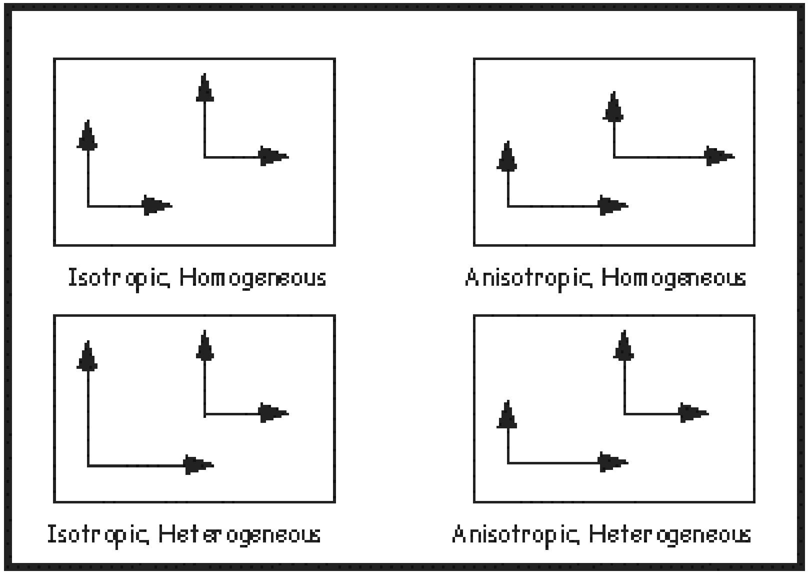
- 1.67×10^{-1} ,
- 2.58×10^{-2} ,
- 2.17×10^{-2} ,
- 2.55×10^{-3} ,
- 9.50×10^{-4}

4.36×10^{-2} Arithmetic mean

1.18×10^{-2} Geometric mean

- Note that arithmetic mean of hydraulic conductivity is dominated by the largest value of K. If observed values of hydraulic conductivity have orders of magnitude differences then the arithmetic mean would be significantly different from the geometric one. The locations with small values of hydraulic conductivity have significant impacts to solute transport than to groundwater flow estimations. Thus, it is more favorable to use the geometric mean instead of an arithmetic one in groundwater hydrology.

- Anisotropy
  - Hydraulic conductivity depends on the direction of measurement, e.g., $K_x \neq K_z$
  - Principal directions of hydraulic conductivity: the directions at which hydraulic conductivity attains its maximum and minimum values, which are always perpendicular to one another.
- Four cases



Transmissivity (T)

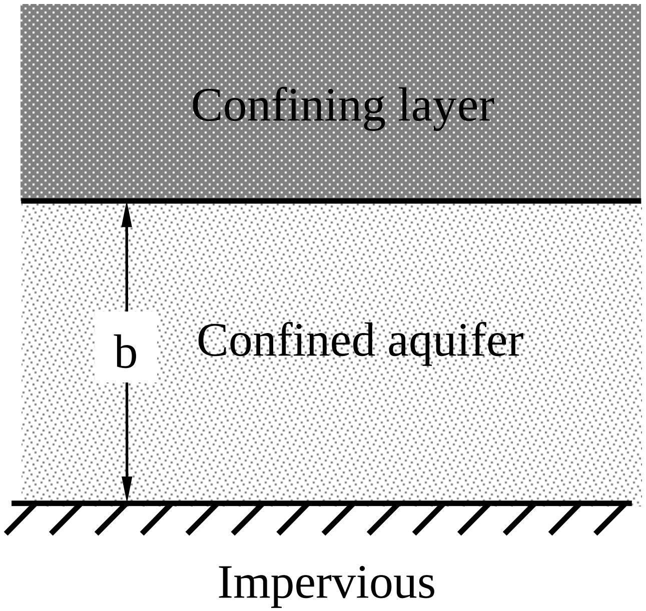
- **Definition:** The rate (Q) at which water is transmitted through a unit width ($\Delta y = 1$) of aquifer under a unit hydraulic gradient ($\nabla h = 1$).

$$Q = vA = (-K\nabla h)A = (-K\nabla h)b\Delta z$$

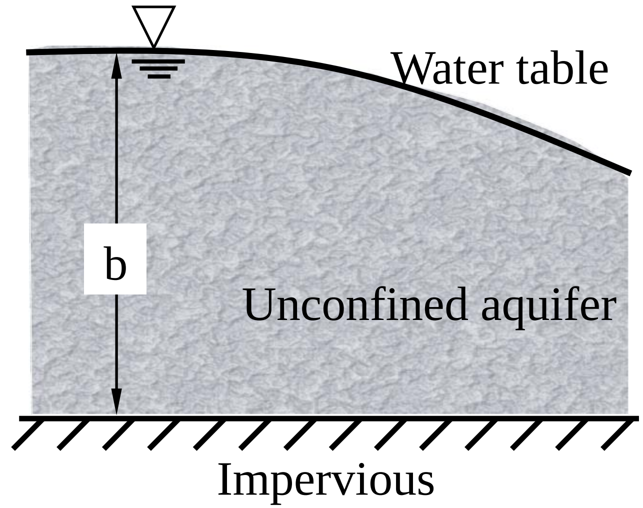
$$\Rightarrow \frac{Q}{\Delta z} = -Kb\nabla h = -T\nabla h$$

  - in which b is the "saturated thickness" of aquifer
- Transmissivity is well defined for the analysis of well hydraulics in a confined aquifer in which the flow field is essentially horizontal and two-dimensional, in which b is the (average) thickness of the aquifer between upper and lower confining layers
- It is, however, not well defined in unconfined aquifer but is still commonly used. In this case, the saturated thickness is the height of the water table above the top of the underlying aquitard (impervious layer) that bounds the aquifer

Schematic representation of the definition of transmissivity



$$T = Kb$$



(Adapted from Freeze and Cherry, 1979)

Note that transmissivities greater than 0.015 m^2/s represent good aquifers for water well exploitation (Freeze and Cherry, 1979)

6. Storage capacity

- Specific storage (specific storativity), S_s (1/L)
- **Definition:** the amount of water released from (or added to) storage per unit decline (or unit rise) in hydraulic head from unit volume of *saturated* aquifer .

$$S_s = \frac{\Delta V_{water}}{V_{aquifer} \times \Delta h}$$

- **Storativity (Storage coefficient), S**: is the amount of water released from (or added to) storage per unit decline (or unit rise) in hydraulic head normal to the unit surface area of saturated aquifer
- Similar to transmissivity, storativity is developed primarily for the analysis of well hydraulics in a confined aquifer

$$S = \frac{\Delta V_{water}}{A_{aquifer} \times \Delta h} = S_s b$$

b is the saturated thickness of the aquifer

- **Storativity of a confined aquifer:** Water is released from a confined aquifer via
  - Expansion of water due to decline of hydraulic head
  - Release of pore water due to compaction of soil skeleton that is again induced by the decline of hydraulic head
- In general, storage coefficients for a confined aquifer are small, in the range of 0.005 to 0.00005 (Freeze and Cherry, 1979)
- **Storativity of an unconfined aquifer:** Water is released in unconfined aquifer via
  - Primary release: storage from the decline of water table, which is generally known as specific yield, S_y
  - Secondary release: the expansion of water and expel of water from aquifer compaction
  - $S = S_y + hS_s$, h is the saturated thickness of the unconfined aquifer . The usual range of S_y is 0.01 ~ 0.30.

- It is customarily to approximate the storativity of an unconfined aquifer by its specific yield.
- **Specific yield, S_y:** The ratio of the volume of water that drains from a saturated aquifer due to the attraction of gravity to the total volume of the aquifer. This is also called gravity drainage.
- **Specific retention (S_r):** the volume of water retained in an aquifer per unit area per unit drop of the water table after drainage has stopped, which is hold between soil particles by surface tension. Hence, the smaller the particle size, the larger the surface tension and the larger the specific retention
  - Specific retention is responsible for the volume of water a soil can retain against gravity drainage.
  - Maximum specific yield occurs in sediments in the medium-to-coarse sand-size range (0.5 to 1.0 mm).

Schematic representation of storativity in confined and unconfined aquifers

Aquifer remains saturated in spite of the decline of piezometric surface

Aquifer above the original water table becomes unsaturated as the water table declines to a new level

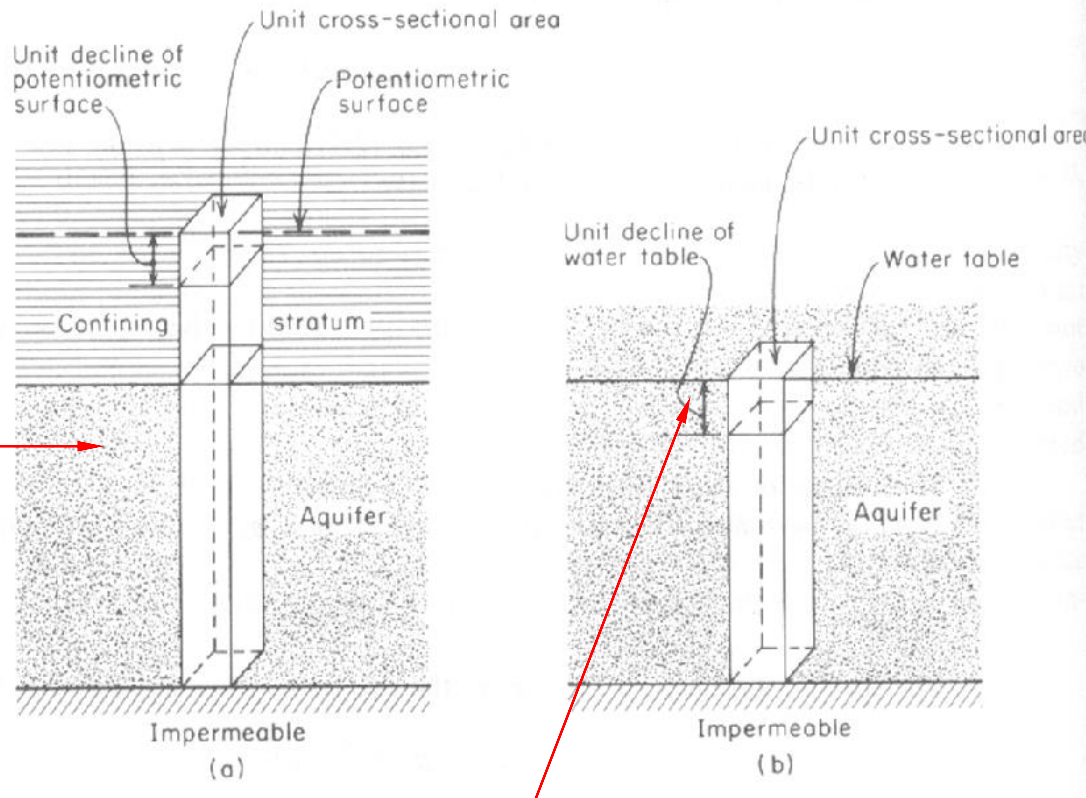


Figure 2.22 Schematic representation of the storativity in (a) confined and (b) unconfined aquifers (after Ferris et al., 1962).

(After Freeze and Cherry, 1979)

Compressibility (Freeze and Cherry):

- The physical phenomena related to the reduction of aquifer volume due to a stress applied to a unit mass of saturated medium
    - Compression of the water in the pores
    - Compression of the individual soil grains (negligible in practice)
    - A rearrangement of soil grains into a more tightly packed configuration
- Definition : ratio of strain to stress
    - **Physical meaning** : the change of volume of a material due to the change of stress applied to that material, with a unit of 1/[stress] (or 1/[pressure])
- Two compressibility in groundwater Hydraulics
    - Water compressibility
    - Aquifer compressibility

Aquifer storativity

- **Storativity:** The amount of water per unit surface area of saturated aquifer released from (or added to) storage per unit decline (or unit rise) in hydraulic head normal to that surface
- Mechanisms
  - Elastic expansion of water
  - Compaction of solid matrix
  - Drainage from pore space between the initial and final water tables (primarily for unconfined aquifers), i.e., specific yield S_y
- Used in two-dimensional, transient groundwater flow equations
- Definition (for both confined and unconfined aquifers)

$$S = \frac{\Delta V_{water}}{A_{aquifer} \times \Delta h} = S_s b \qquad \text{(for constant aquifer thickness)}$$

$$= \int_{b_1(x,y)}^{b_2(x,y)} S_s(x,y,z)\,dz \qquad \text{(for variable aquifer thickness)}$$

(Note again that storativity is dimensionless)

Aquifer storativity/Hydraulic Diffusivity

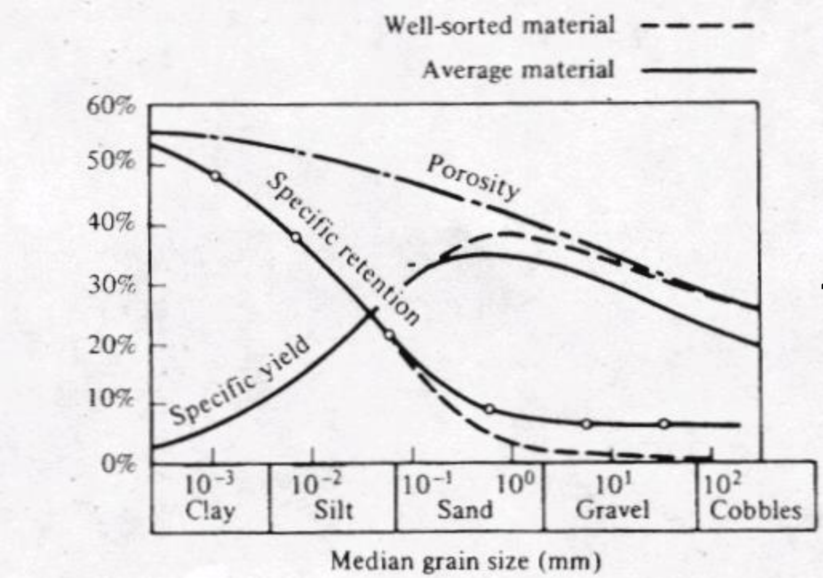


Fig. 4.2. Relationship between specific yield and grain size (*from Conkling et al., 1934, as modified by Davis and De Wiest, 1966*).

(After Bear and Verruijt, 1987)

1. Specific retention increases with decreasing grain size
2. Maximum S_y occurs in medium-to-coarse sand soils.

- Hydraulic diffusivity: D (L^2/T)
  - Represent the diffusive characteristics of groundwater

$$D = \frac{T}{S} = \frac{K}{S_s}$$

7. Porous Vs Fractured media

- Porous media
  - Continuum media
  - A multiphase medium with at least one solid phase and one fluid phase (Greenkorn, 1983; Yeh, 1999)
    - Soils
    - Wood, asphalts
    - Skin, hair, feathers, teeth, and lungs
    - ceramics, contact lenses, membranes
- Fractured media
  - Discrete media
    - Fractured rocks (sedimentary, crystalline, argillaceous)
  - Fractured porous media: discrete media with porous solid matrix

Porous Vs fractured media

- **Soil** (Jury, Gardner, and Gardner, 1991[a])
  - A granular material that is a heterogeneous mixture of solid, liquid, and gaseous material
- **Solid phase:** it has **mineral portion** containing particles of varying sizes, shapes, and chemical composition and **organic portion** containing a diverse population of live, active organisms as well as plant and animal residues in different stages of decomposition
- **Liquid phase:** It consists of **the soil water** held by forces in the soil matrix and varies significantly in mobility depending on its location and **Solute** materials contained in soil water, coming from dissolution of soil mineral phase or from the soil surface.

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[a]Jury, W. A., W. R. Gardner, and W. H. Gardner, Soil Physics, 5th ed., John Wiley & Sons, New York, 1991

Porous Vs Fractured media

Gaseous (vapor) phase

- CO_2 and O_2: mutually complementary gases in soil, depending on plant respiration and biological activity
- Water vapor : formed by the evaporation process in unsaturated soil
- Volatile organic compounds (VOC) : gas phase of VOC's

Examples of porous media: Beach sand; Sand stone; Lime stone; Dry bread; Wood; Lung

Porous Vs Fractured media

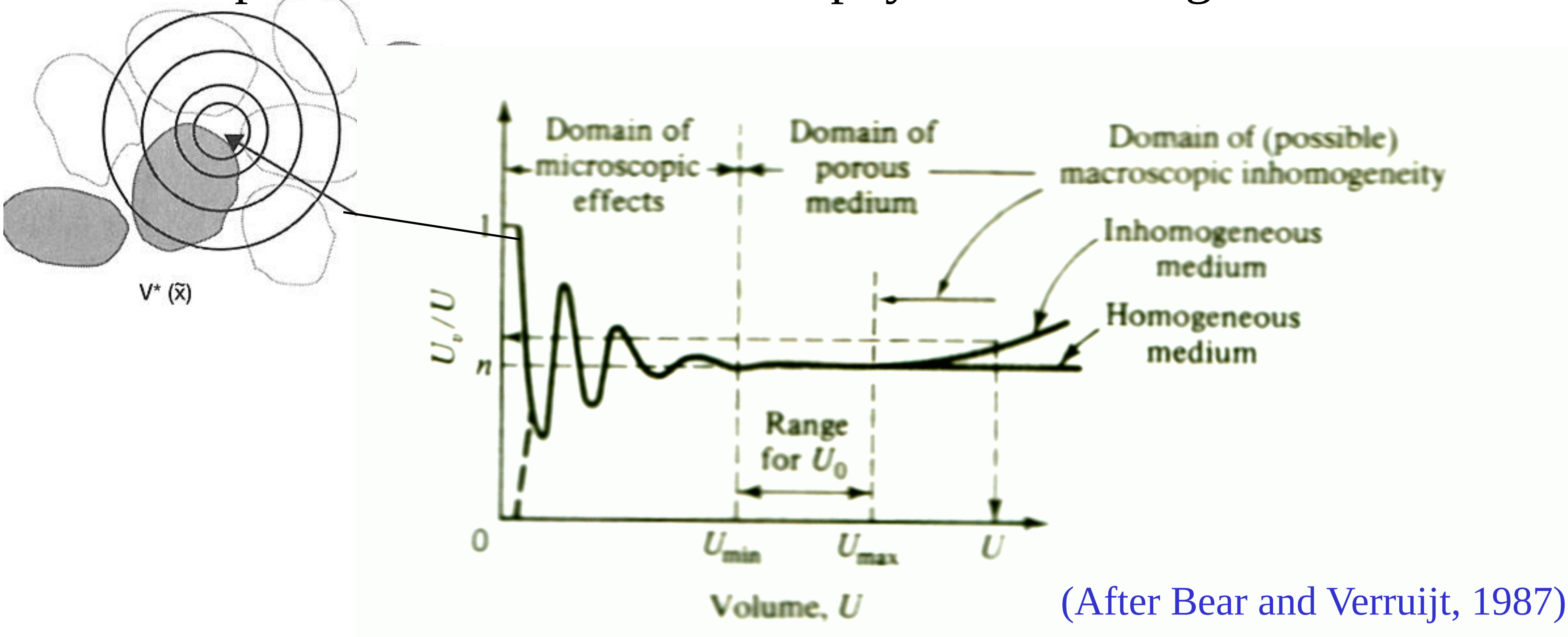
Examples of fractured media

Porous Vs Fractured media

- **Heterogeneity and associated length scales:** A porous system is inherently heterogeneous. Mathematically, we can represent heterogeneity as; $Z(\mathbf{x}_1) \neq Z(\mathbf{x}_2)$ in which Z is a general medium property, e.g., hydraulic conductivity
- **Four length scales of heterogeneity**
  - Microscopic : at the level of pores or grains of the medium
  - Macroscopic: at the level of core plugs
  - Megascopic: at the level of the entire reservoirs which may have large fractures and faults.
  - Gigascopic: heterogeneities at this scale may contain many megascopically heterogeneous reservoirs.
- Note that not all the four heterogeneities are important to all porous media.

Porous Vs Fractured media

- Any property of a medium is an average taken over a suitably selected volume of the medium, which is generally called representative elementary volume (REV)
- Concept of REV: Dimension and physical meaning



(After Bear and Verruijt, 1987)

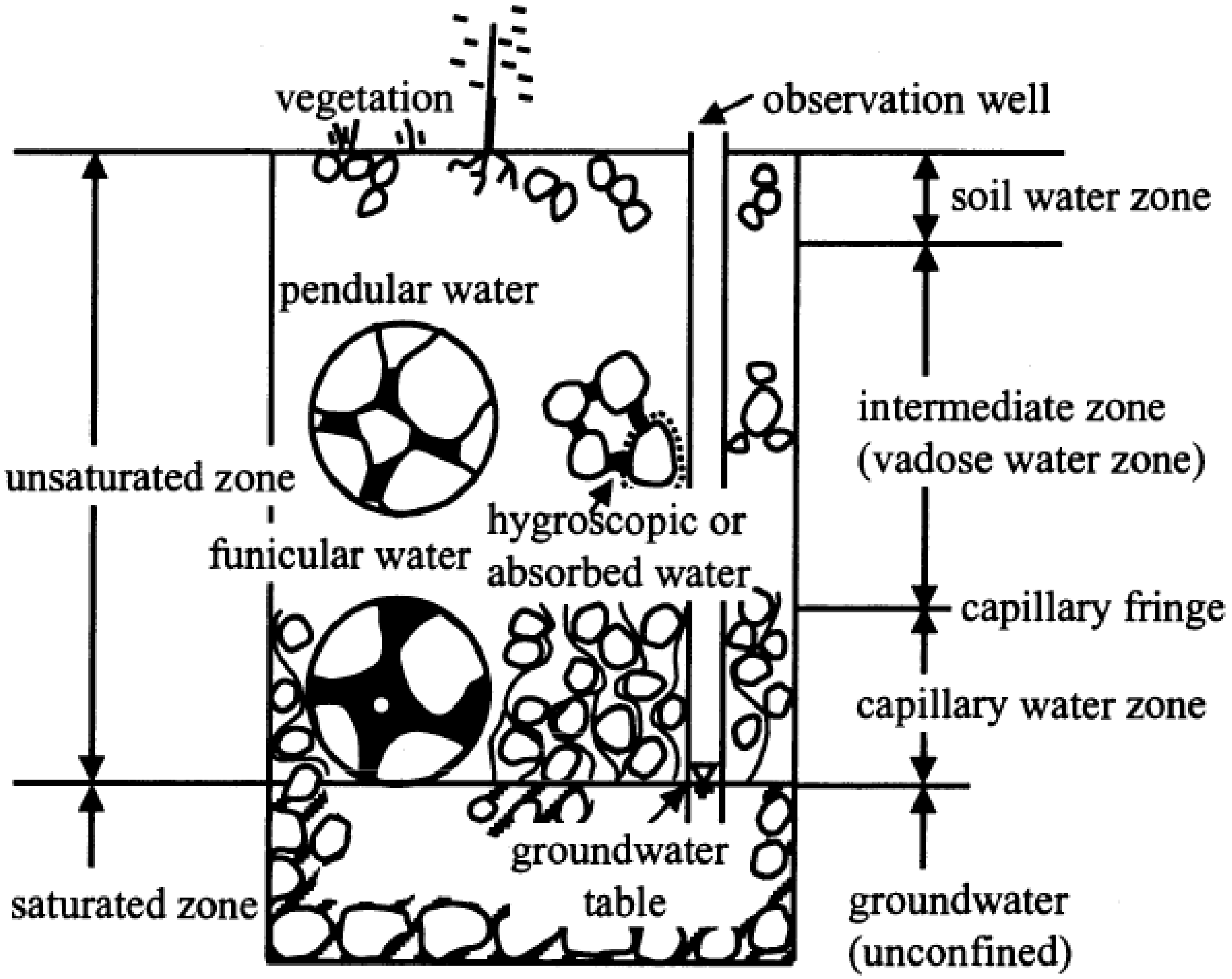
Fig. 1.5. Definition of porosity and Representative Elementary Volume.

Porous Vs Fractured media

- Heterogeneities are spatially correlated at all scales
  - Fractal theories can be used to tell the dependence of property values in various regions of the medium on the length scale of observation
- Long-range correlation of medium properties leads to the spatial variation of interconnectivity of various regions of the medium
  - Percolation theories can be used to tell how the interconnectivity of various regions of a given system affects its overall properties
  - Spatial variation of interconnectivity is more significant in fractured media than in porous media
- Porous media is the focus of this course

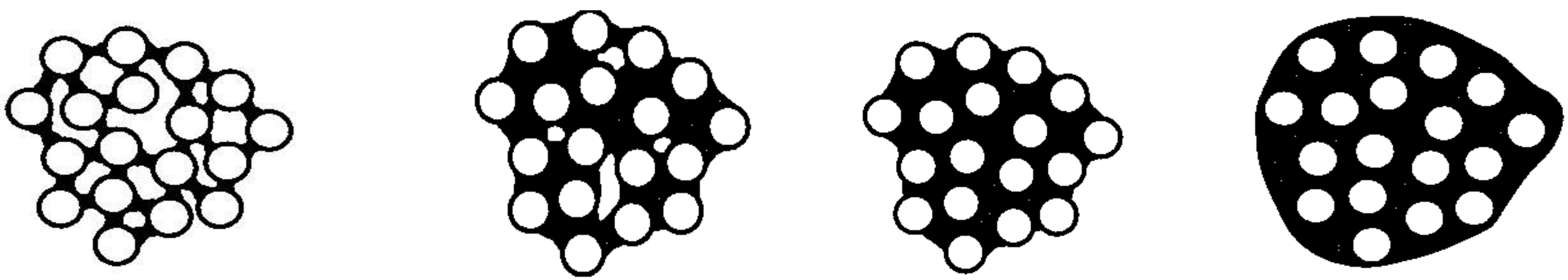
8. Driving forces of groundwater flow

- Moisture distribution



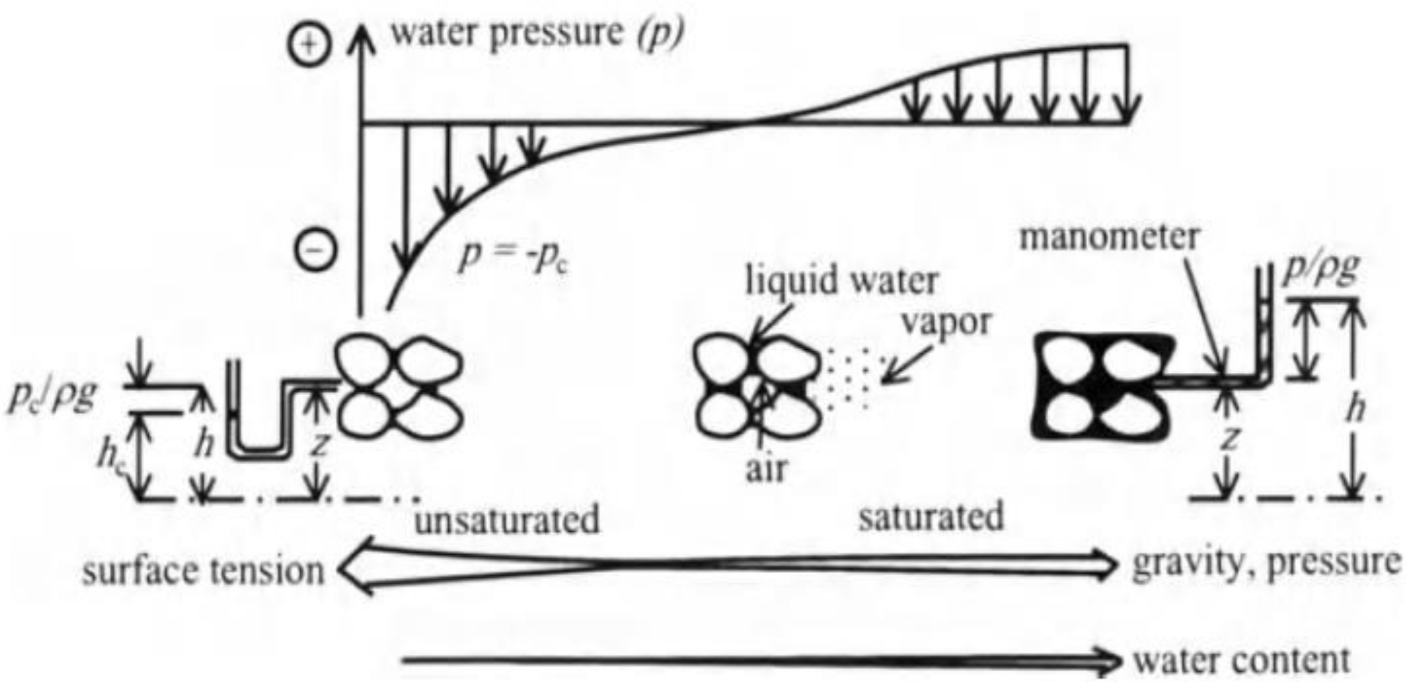
Types of water Bonding

a) Pendular-looks like bridge, but particles not immersed in liquid
b) Funicular-thicker bridges but not completely filled
c) Capillary-particles at edge of cluster not completely wetted by liquid
d) Droplet-all particles completely wet



(a) (b) (c) (d)

- **Figure** shows qualitative relationships between water content (pendular, funicular and saturated cases) and driving forces (surface tension, gravity and pressure head).



Groundwater Hydraulics

Chapter 2 – Groundwater motion

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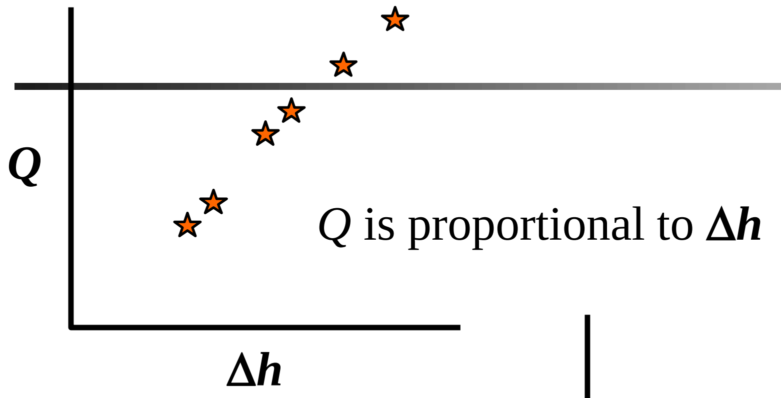
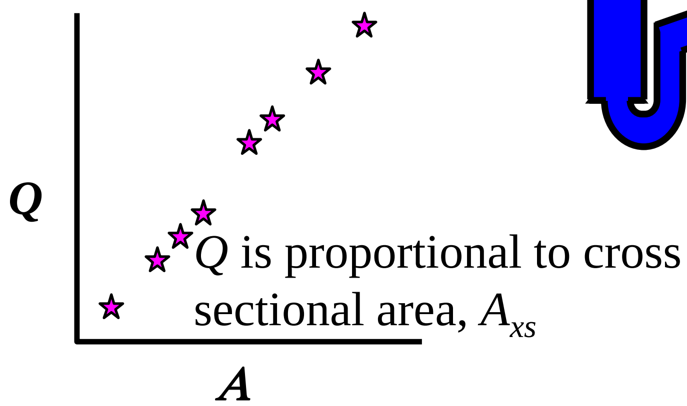
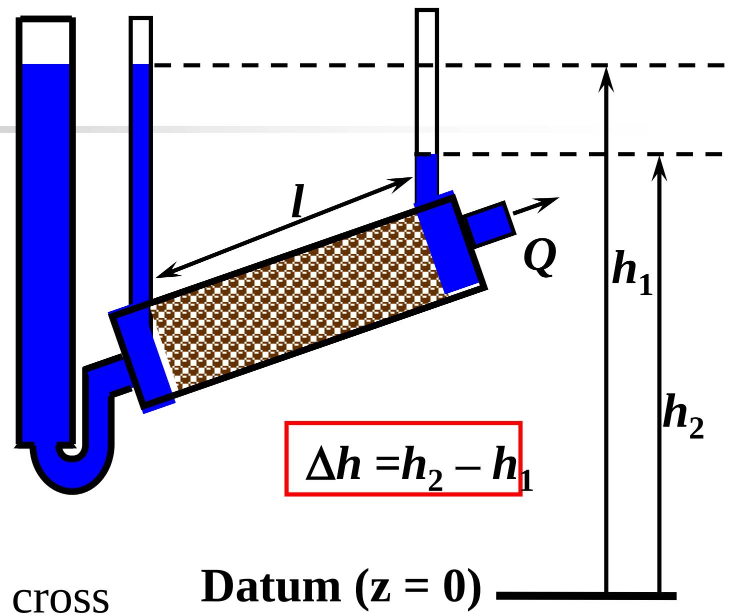
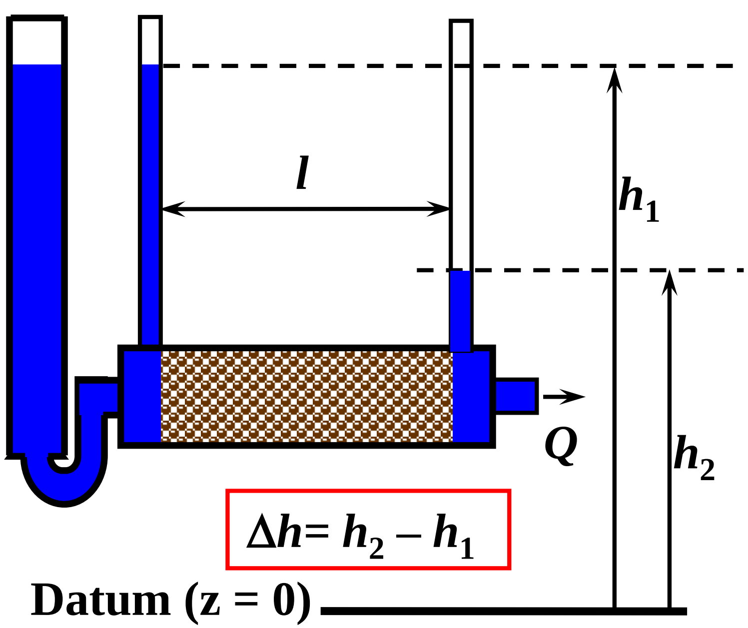
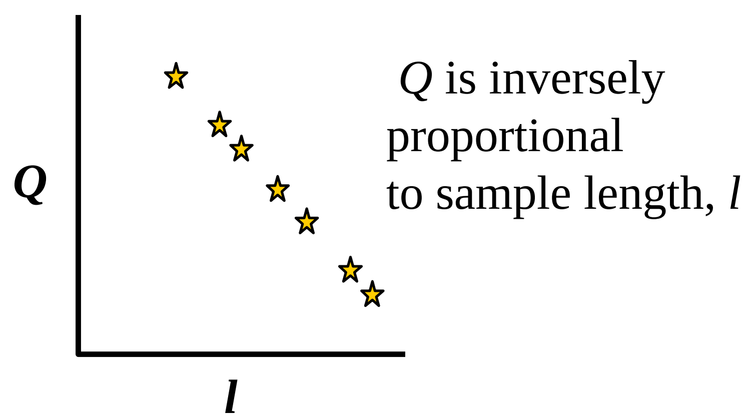
5. Application of Flow net
6. Approaches to groundwater flow analysis in fractured aquifers


Q
Q is proportional to Δh
Δh


Q
Q is proportional to cross sectional area, A_{xs}
A


l
Q
h_1
h_2
$\Delta h = h_2 - h_1$
Datum (z = 0)


l
h_1
h_2
Q
$\Delta h = h_2 - h_1$
Datum (z = 0)


Q
Q is inversely proportional to sample length, l
l

Darcy's Law

- Combine and insert a constant of proportionality

  $$Q = -KA_{xc}\,[\Delta h/l]$$

- A_{xc} = sample cross-sectional area [m^2]
  - *perpendicular* to flow direction
- K = hydraulic conductivity [m/s]
- $\Delta h/l$ = hydraulic gradient [-]
  - l measured along the flow direction
- Sometimes written as $Q/A_{xc} = q = -K[\Delta h/l]$
  - Where q = specific discharge a.k.a. "Darcy velocity"
- Hydraulic gradient often written as a differential, *dh/dl*

Effect of Geologic Material

$$Q = -KA_{xc}[dh/dl]$$

- Re-run experiments with different geologic materials
  - e.g., grain size
- General relationship still holds – but –
- Need a new constant of proportionality (*K*)

***K* is a property of the porous material**

Effect of Fluid Properties

$$Q = -KA_{xc}[dh/dl]$$

- Re-run experiments with a different fluid
  - e.g. petroleum, trichloroethylene, ethanol
- General relationship still holds – but –
- Need a new constant of proportionality (*K*)

***K* is a property of the fluid**

Intrinsic Permeability - K_i

- Separate the effects of the fluid and the porous medium
- K = (porous medium property) x (fluid property)
- Porous medium property: K_i = intrinsic permeability
  - Essentially a function of pore opening size
  - Think of it as a 'friction' term
- $K = K_i$ x fluid driving force/fluid resisting force
- Fluid driving force = specific weight
- Fluid resisting force = dynamic viscosity

$$K = K_i\,[\rho g/\mu]$$

Darcy's Law - Units

- $Q = -KA_{xc}[dh/dl]$
- Solve for K:

$$K = \frac{Qdl}{A_{xc}dh}$$

Where:

- dh/dl is dimensionless
- $Q \equiv [m^3\ s^{-1}]$
- $A_{xc} \equiv [m^2]$
- Therefore,

$$K \equiv \left[\frac{m^3}{s}\right]\left[\frac{1}{m^2}\right]\left[\frac{m}{m}\right] \equiv \left[\frac{m}{s}\right]$$

Since the fluid we consider most frequently (certainly in this course) is water, we almost always be using hydraulic conductivity (K)

Intrinsic Permeability - Units

- Write as: $Q = -K_i(\rho g/\mu)A_{xc}[dh/dl]$
- Solve for K_i

$$K_i \equiv \frac{Q\mu}{A_{xc}\rho g}\frac{dl}{dh}$$

Where:

- $Q \equiv [\mathrm{m}^3\ \mathrm{s}^{-1}]$
- $A_{xc} \equiv [\mathrm{m}^2]$
- $\rho \equiv [\mathrm{kg\ m}^{-3}]$
- $g \equiv [\mathrm{m\ s}^{-2}]$
- $\mu \equiv [\mathrm{kg\ m}^{-1}\ \mathrm{s}^{-1}]$

- Therefore, $K_i \equiv \left[\frac{m^3}{s}\right]\left[\frac{kg}{m\ s}\right]\left[\frac{1}{m^2}\right]\left[\frac{m^3}{kg}\right]\left[\frac{s^2}{m}\right]\left[\frac{m}{m}\right] \equiv \left[m^2\right]$

Magnitude of Intrinsic Permeability

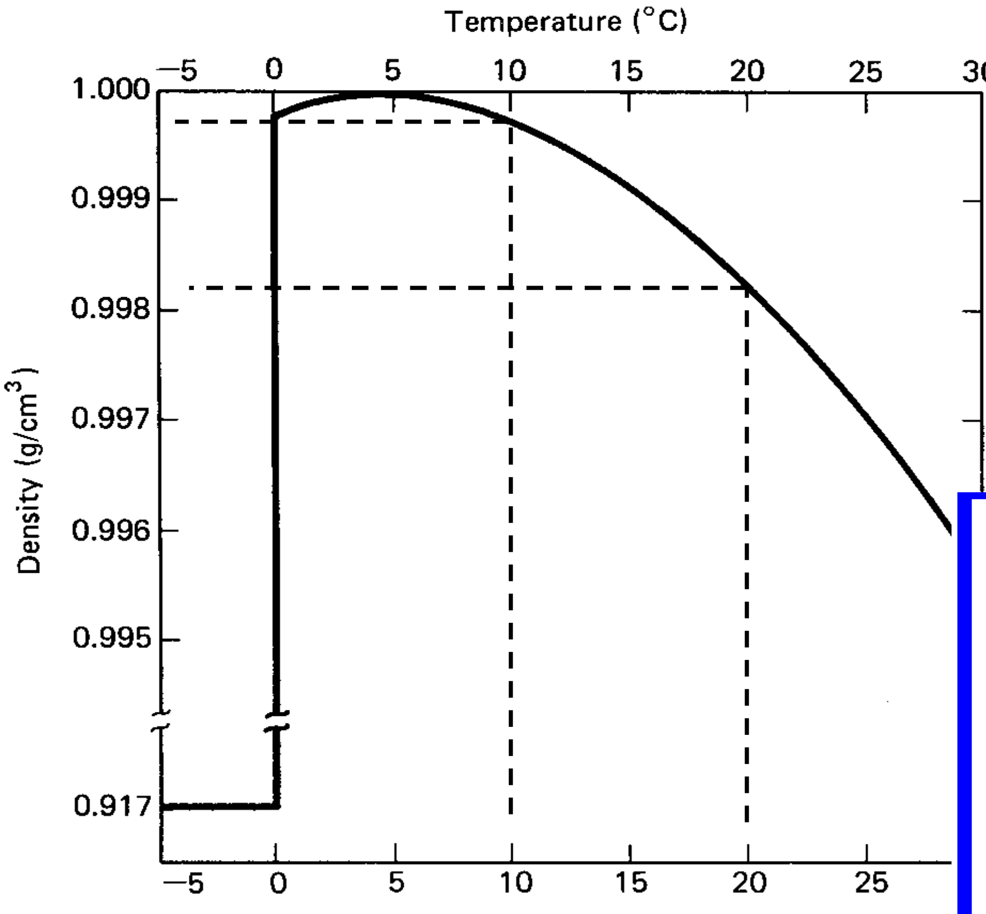
- ($\rho g/\mu$) is a large number

  For water at 15 ^{o}C, $\rho g/\mu$ = 999.1 x 9.81/0.0011 = 8,910,155 [1/(m-s)]
- If K = 1 m/s then, $K_i = K/(\rho g/\mu)$ = 1.12x10^{-7} m^2
- Therefore we usually use a smaller unit –

  1 Darcy = 9.87x10^{-9} cm^2
- This course: typically use *hydraulic* conductivity (K)
- Contaminant transport, petroleum geology: K_i is important

Effect of Temperature

- Density and viscosity (ρ and μ) for water are a function of temperature
- ***K*** is therefore a function of temperature, but
- K_i is NOT a function of temperature
  - More fundamental unit controlling flow
- Lab standards are run at 60 ^oF (15.6 ^oC)
  - For most of the remainder of the course, we will assume that temperature is 15.6 ^oC
- So, how does ***K*** vary as a function of temperature??



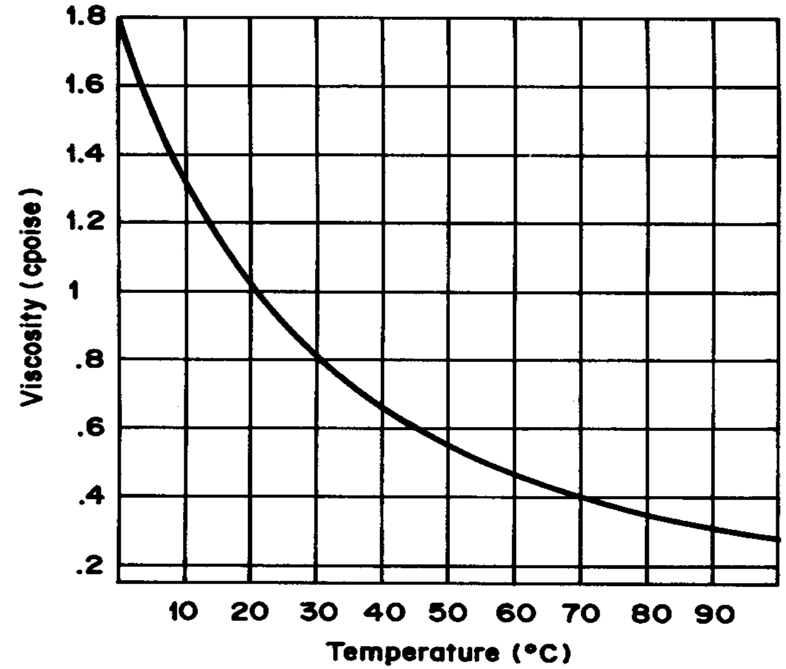
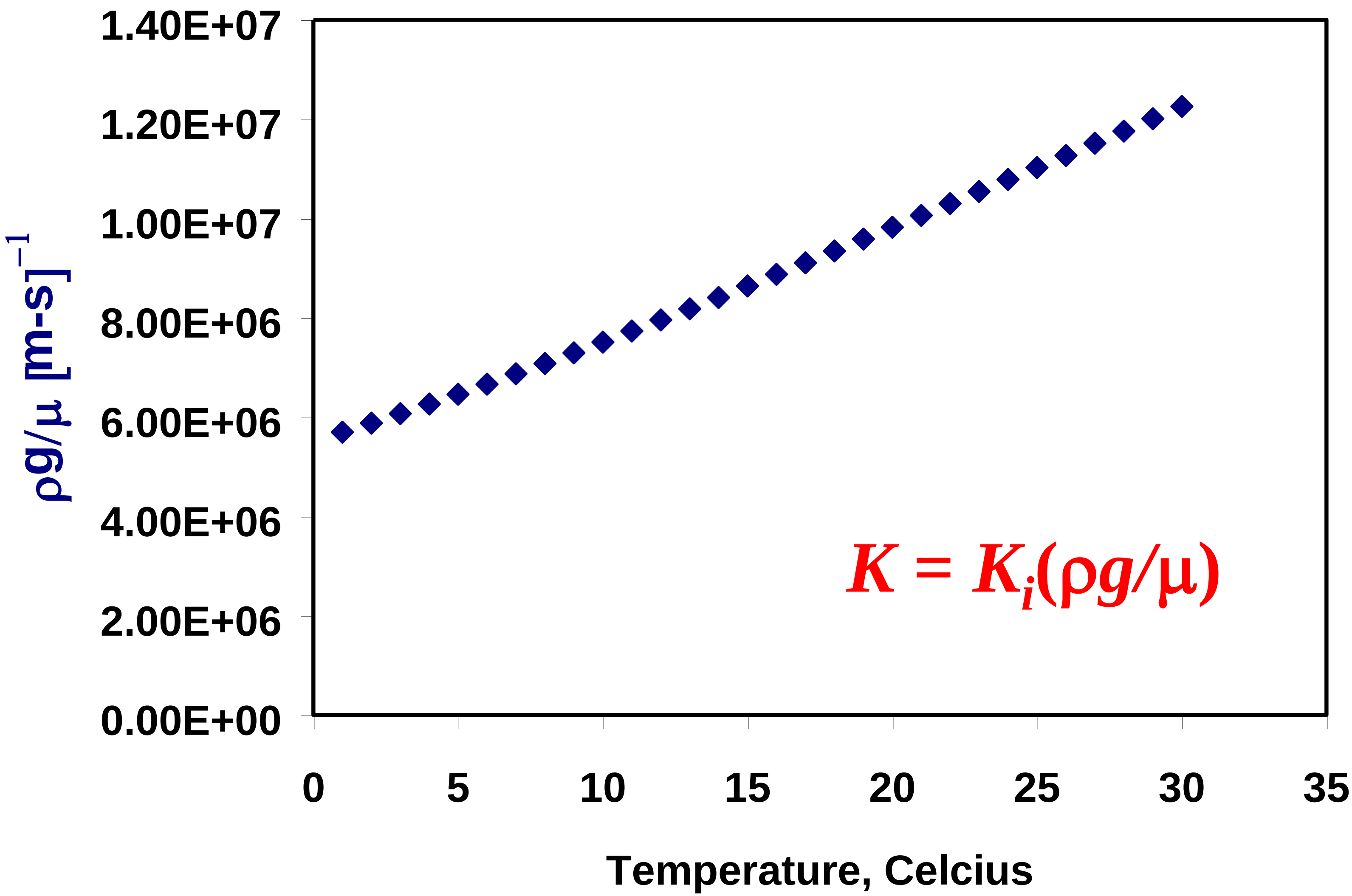


Fig. 3-2. Relationship between viscosity and temperature of water.

$$K = K_i(\rho g/\mu)$$

Effect of *T* on *K*

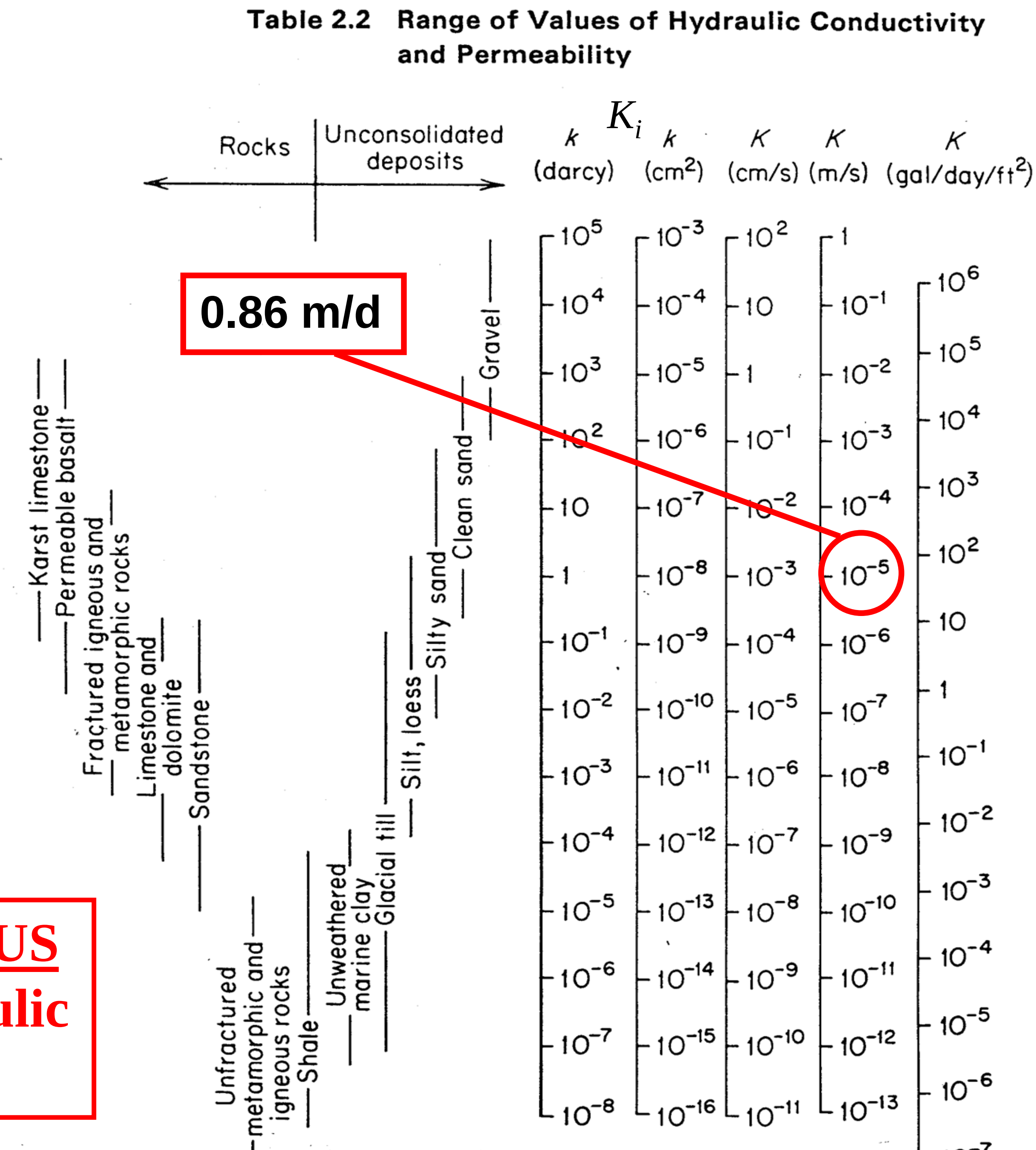


K_i in Rocks

- Primary openings
  - Formed as the rock forms - e.g. the initial porosity of the pre-cementation sediments
  - $\boldsymbol{K_i}$ in sedimentary rocks is the $\boldsymbol{K_i}$ of the sediments from which they form
  - Crystalline rocks (igneous, metamorphic): Low primary permeability (possible exception: some igneous rocks with interconnected pores)
- Secondary openings (after the rock formed)
  - Fractures
  - Dissolution
    - Along fractures, bedding planes
    - Important for chemically precipitated rocks - limestone, dolostone, gypsum, halite
  - Weathering

Estimating K

(1) past experience



Note the <u>ENORMOUS</u> variability in Hydraulic Conductivity!!!

Estimating K, K_i

(2) Empirical Relations to Grain Size

- Where K is hydraulic conductivity in cm/sec
- d_{10} is the effective grain size
    - 10%of the soil by weight is finer in grain size, 90% is coarser
- C is an empirical coefficient

Hazen's Formula

$$K = C\,(d_{10})^2$$

K is proportional to the *square of grain size*

Soil Type	C
Very fine sand, poorly sorted	40-80
Fine sand with appreciable fines	40-80
Medium sand, well sorted	80-120
Coarse sand, poorly sorted	80-120
Coarse sand, well sorted, clean	120-150

- Some other examples of empirical relations….
- Krumbein and Monk (from the oil industry):

$$k_i = C_0 (D_{50})^2 e^{-c\sigma_\Phi}$$

  - Where k_i = intrinsic permeability (in darcies); C_0 = an empirical constant ~760 darcies/mm; c = an empirical constant ~ 1.31; D_{50} = median diameter of sediment (in mm); σ_φ = standard deviation of grain size in φ units

- $K = C(D_{50})^j$

  - Where K = hydraulic conductivity in ft/day; C = a shape factor (40000 for glass spheres, 100 for immature {poorly sorted} sediments); j = 2.0 for glass spheres; j = 1.5 for immature sediments

- Kozeney (1927)

$$K_i = Cn^3 / S^{*2}$$

  - Where K_i = intrinsic permeability (in darcies); C = an empirical constant ~0.5, 0.562, & 0.597 for circular, square, and equilateral triangular pore openings; n = porosity; S^* = specific surface - interstitial surface areas of pores per unit bulk volume of the medium.

- Kozeny-Carmen Bear (1972)

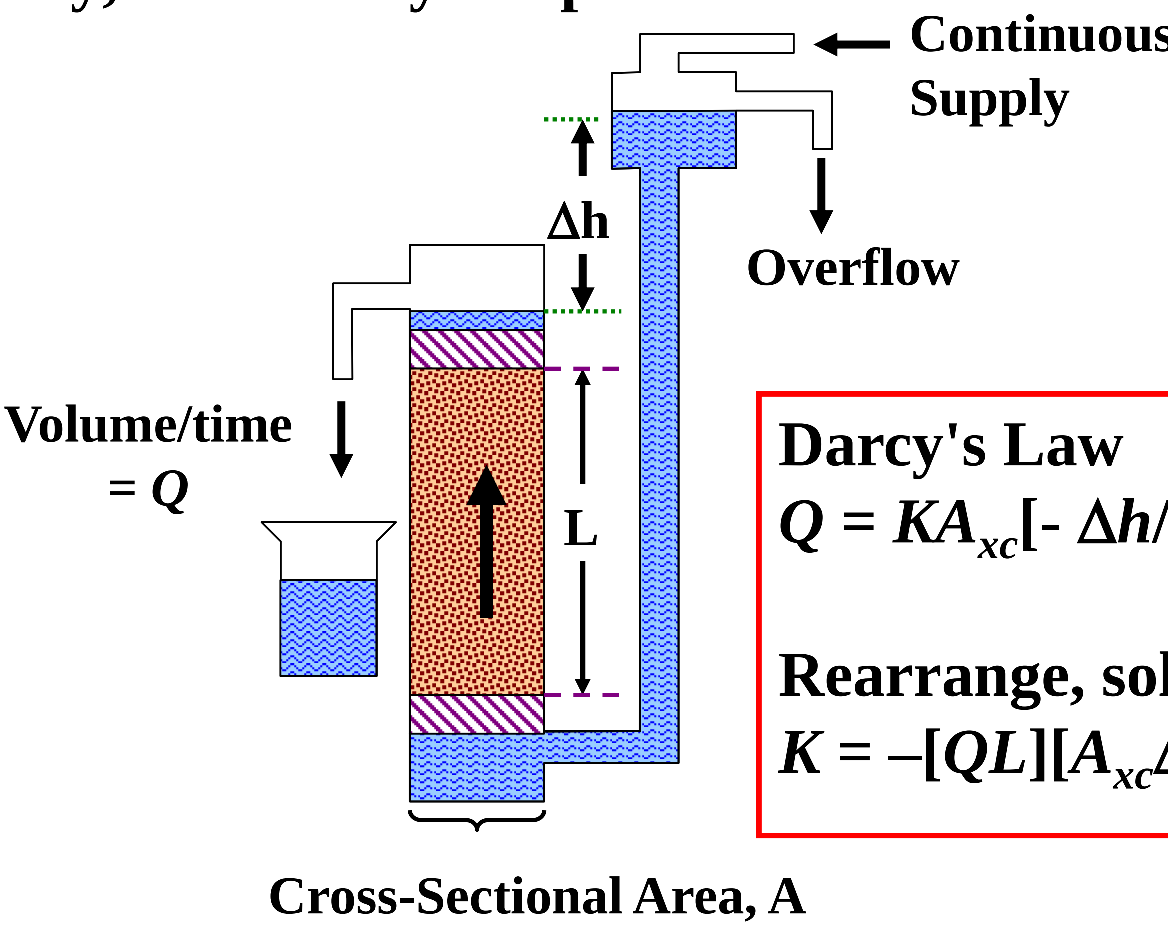
$$K = \frac{\rho_w g}{\mu} * \frac{n^3}{(1-n)^2}\left(\frac{d_m^2}{180}\right)$$

  - Where K = hydraulic conductivity; ρ_w = fluid density; μ = fluid viscosity; g = gravitational constant; d_m = any representative grain size; n = porosity

Measuring K

(3) Constant Head Permeameter

- **Basically, redo Darcy's experiments:**



Darcy's Law

$$Q = KA_{xc}[-\Delta h/L]$$

Rearrange, solve for *K*

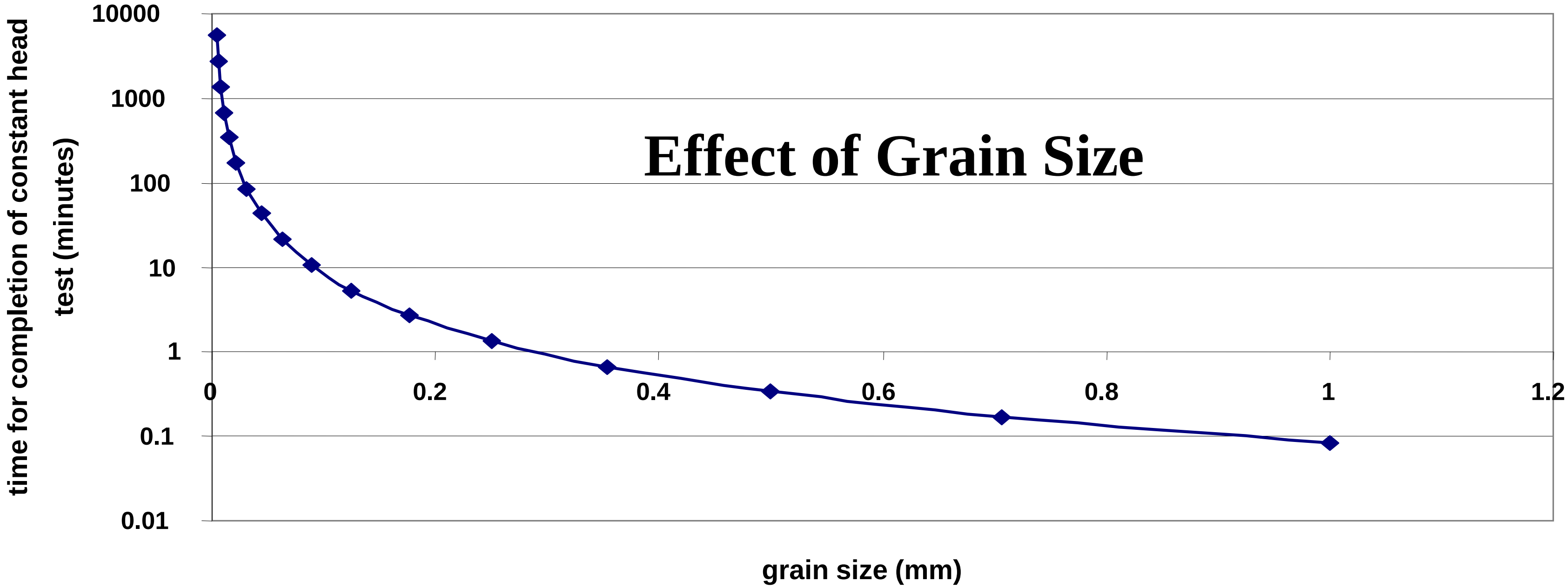
$$K = -[QL][A_{xc}\Delta h]^{-1}$$

Constant Head Permeameter Test Protocols

- Good for relatively coarse grained material with relatively high hydraulic conductivity
- Keep Δh at reasonable field conditions
  - ($< 1/2\ L$)
- Be certain that no air is trapped in the sample
  - Air bubbles will act as impermeable lenses
  - Fill slowly from the bottom to force air upwards
  - De-gas water
- Design test, so all parameters can be measured accurately
- Design test, so it can be conducted in a reasonable amount of time

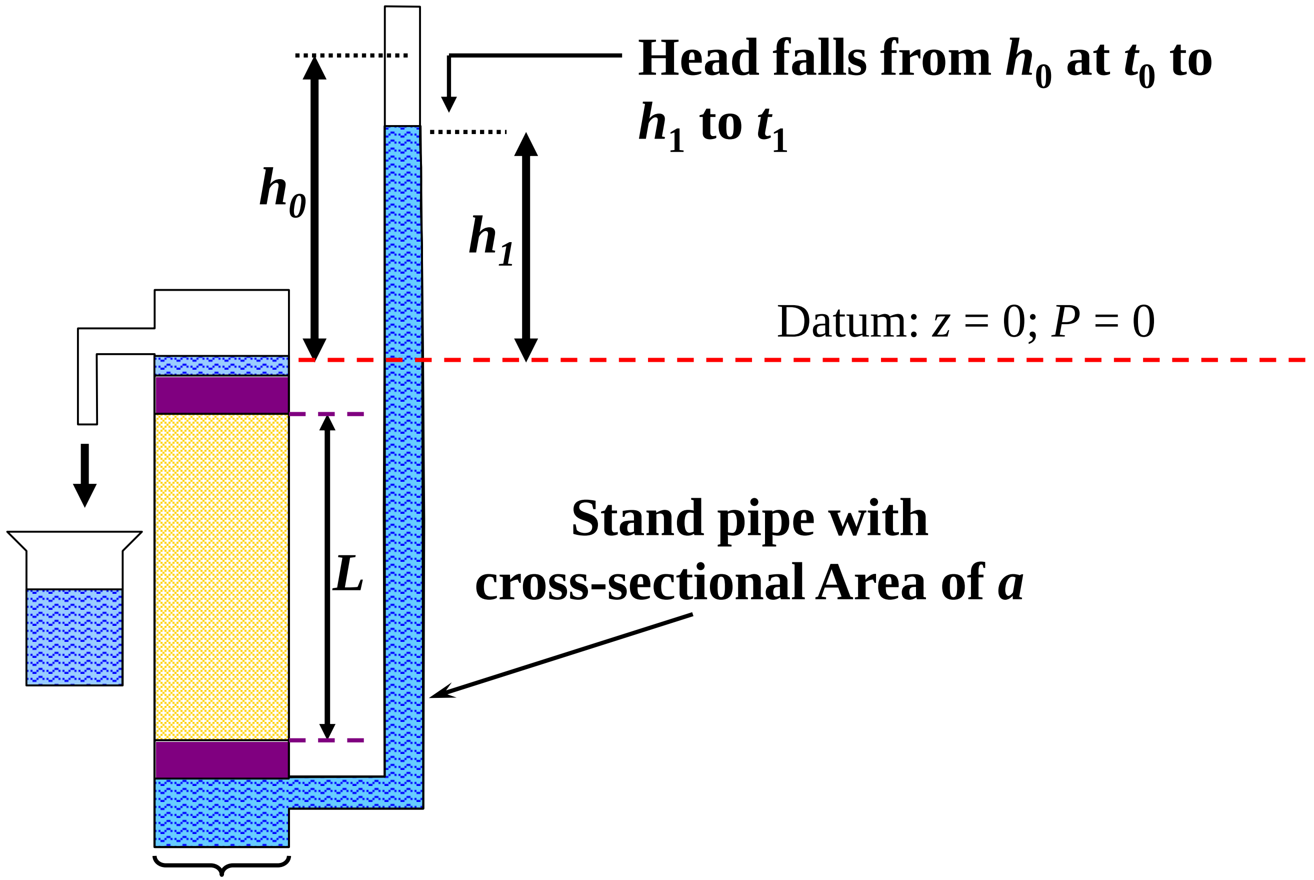
Constant Head Permeameter Test Design

- $K = –QL / [A\Delta h] = –(\text{Vol/time})*L / [A\Delta h]$
- Solve for time = $–(L*\text{Vol}) / [KA\Delta h]$
- Trial Design:
  - L = 10 cm long; A_{xs} = 5 cm^2; Head difference (Δh) = – 5 cm; $K = 10^{-1}$ cm/sec (~ coarse sand); Volume collected = 100 ml;
- Time = 10x100/(0.10x5x5) = 400 s



Measuring K

(3) Falling Head Permeameter



Sample with cross-sectional Area A_{xs}

Falling Head Permeameter Analysis

- Apply to fine grained soils
  - Constant head permeameter test inaccurate, lengthy
- Mass balance – standpipe

  $Q = dV/dt = a\,(dh/dt)$
- Darcy's Law – sample

  $Q = -KA_{xs}(h/L)$
- Set Q equal

  $a\,(dh/dt) = -KA_{xs}(h/L)$

Set datum at outlet

Therefore, $h_{outlet} = 0$ and

$\Delta h = h_{outlet} - h = -h$

At $t = t_o$

$\Delta h = h_{outlet} - h_o = -h_o$

At $t = t_1$

$\Delta h = h_{outlet} - h_1 = -h_1$

Falling Head Permeameter Analysis

- Combine mass balance and Darcy's Law

  $a\,(dh/dt) = -KA_{xs}(h/L)$

- Separate variables and integrate

$$-\int_{h_o}^{h_1} \frac{dh}{h} = \frac{KA_{xs}}{aL}\int_{t_o}^{t_1} dt \qquad \ln\frac{h_o}{h_1} = \frac{KA_{xs}(t_1 - t_o)}{aL}$$

$$K = \frac{aL}{A_{xs}(t_1 - t_o)}\ln\frac{h_o}{h_1}$$

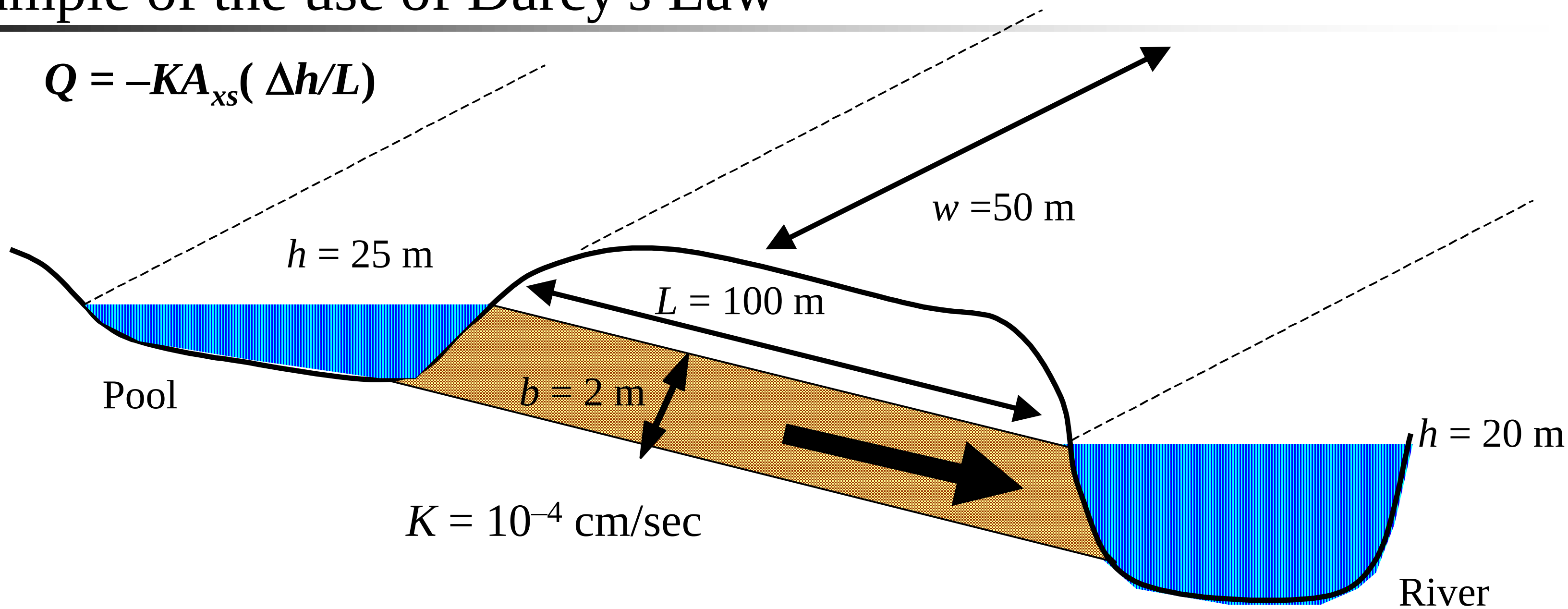
Falling Head Permeameter Test Design

- Solve for time = $$t_1 - t_o = \frac{aL}{KA_{xs}} \ln \frac{h_o}{h_1}$$
- Trial Design:
  - L = 10 cm
  - A_{xs} = 10 cm^2
  - Stand pipe a = 0.5 cm^2
  - h_o = 20 cm; h_1 = 19 cm
  - K = 10^{-3} cm/sec (~ fine sand with silt)

- Time = $$t_1 - t_o = \frac{0.5 \times 10}{0.001 \times 10} \ln \frac{20}{19} = 25.6 \text{ s}$$

Example of the use of Darcy's Law

$$Q = -KA_{xs}(\Delta h/L)$$



- How much water is flowing from the pool into the river per second over a 50 m stretch?
- $\Delta h = -5$ m (head decreases in the direction of flow)
- $l = 100$ m; $\Delta h/l = -0.05$
- $A_{xc} = b \times w = 2 \text{ m} \times 50 \text{ m} = 100 \text{ m}^2$
- $K = 10^{-4}$ cm/sec
- $Q = -10^{-4} \text{ cm/sec} \times 100 \text{ m}^2 \times 100^2 \text{ cm}^2/\text{m}^2 \times (-0.05) = 5 \text{ cm}^3/\text{s}$

PERPENDICULAR to the direction of flow!

Other Ways to Express Flow

Flow per Unit Width

- What is the flow through the aquifer per unit width (per cm)?
  - $Q = -KA_{xs}(\Delta h/l)$ $A_{xs} = b \times w$
  - $Q = -K(b \times w)(\Delta h/l)$ divide both sides by w
  - $Q/w = -Kb(\Delta h/l)$
- $Q/w = -10^{-4}$ cm/sec x 2 m x 100 cm/m x (–.05)=

  $= 0.001\ [\mathrm{cm}^3\ \mathrm{s}^{-1}\ \mathrm{cm}^{-1}]$

Other Ways to Express Flow

Flow per Unit Width per Unit Gradient

- What is the flow through the aquifer per unit width (per cm) per unit hydraulic gradient?
  - This is a measure often used to compare aquifers.
  - $Q = –K(b \times w)(\Delta h/l)$ divide both sides by w
  - $Q/w = –Kb(\Delta h/l)$ divide both sides by $(\Delta h/l)$
  - $(Q/w)/(\Delta h/l) = –Kb$
- $(Q/w)/(\Delta h/l) = –10^{–4}$ cm/sec x 2 m x 100 cm/m =

  = 0.02 [$cm^2\, s^{–1}$]

Transmissivity (*T*)

- The rate at which water is transmitted through a unit width of aquifer under a unit hydraulic gradient
- Our last calculation (flux per unit width per unit hydraulic gradient)
- A common unit in hydrogeology
  - $Q = -KA_{xs}(\Delta h/l)$ $\quad A_{xs} = b \times w$
  - $Q = -K(b \times w)(\Delta h/l)$ divide both sides by w
  - $Q/w = -K(b)(\Delta h/l)$ divide both sides by $-\Delta h/l$
  - $Q/w/(-\Delta h/l) = Kb$

$$T = Kb$$

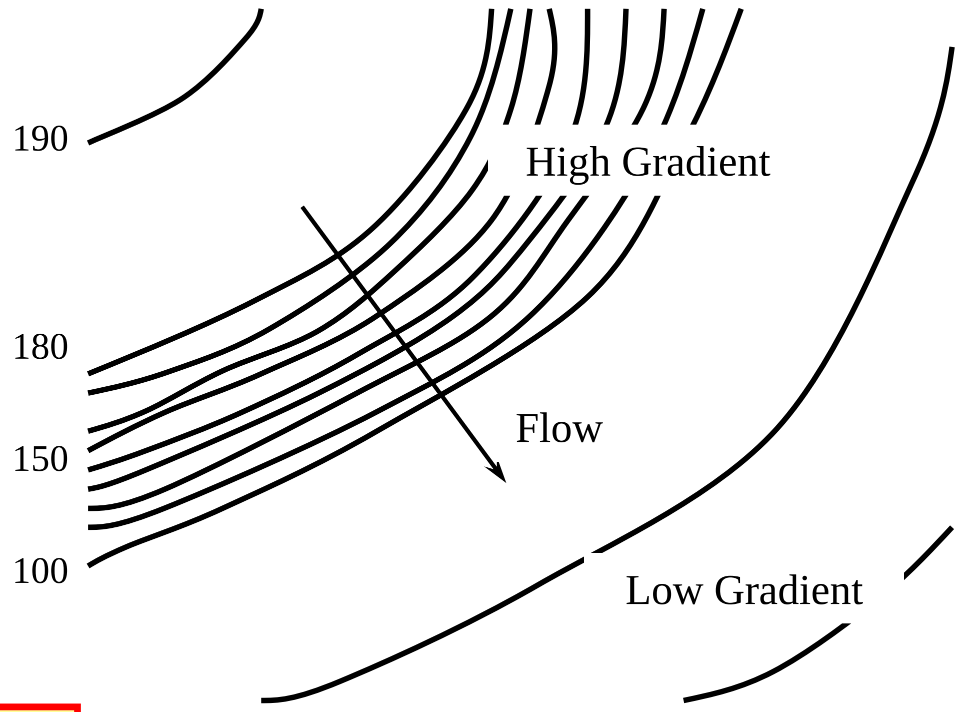
- K is the hydraulic conductivity
- b is the aquifer thickness

Transmissivity (*T*)

- For confined aquifers, *b* is the aquifer thickness (may vary in space)
- For unconfined aquifers, *b* is not as well defined, since it can also change with position and through time
  - use *b* as the saturated thickness
- Alternate way of expressing Darcy's Law
  - $Q = -KA_{xs}(\Delta h/l)$
  - $Q = -K(b \; x \; w)(\Delta h/l)$
  - $Q = -Tw(\Delta h/l)$
    - *w* is the aquifer width (horizontal dimension perpendicular to flow)
- Units: (volume/time)/length (gallons/day/foot) or
- Units: length^2/time [m^2/s]

Gradients in Hydraulic Head

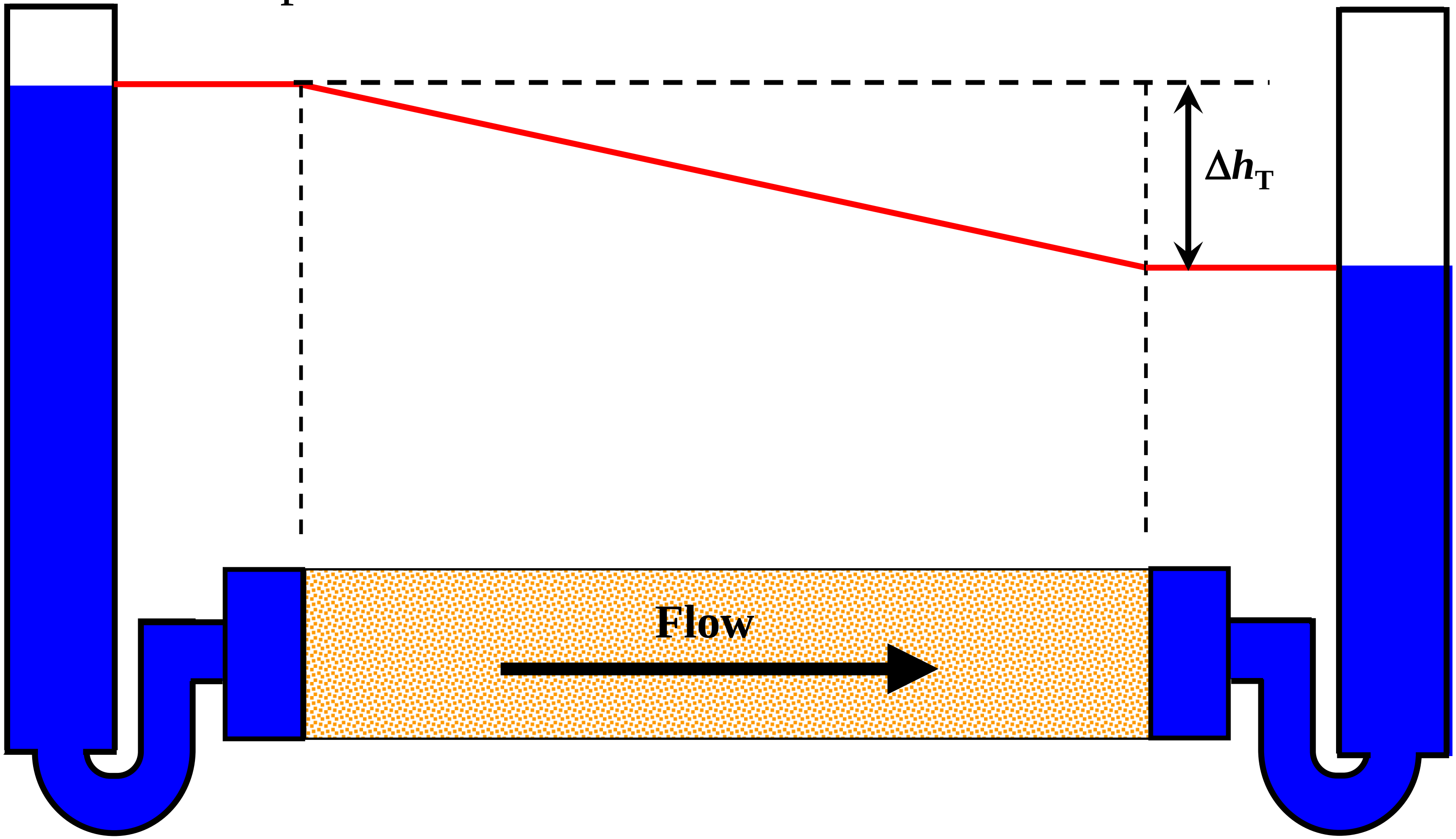
- We measure gradients in head using piezometers
- We can map these as shown
- We often observe changes in head gradient
- What aquifer properties can cause changes in these gradients?



Hydraulic gradient = Δ*h*/*l*

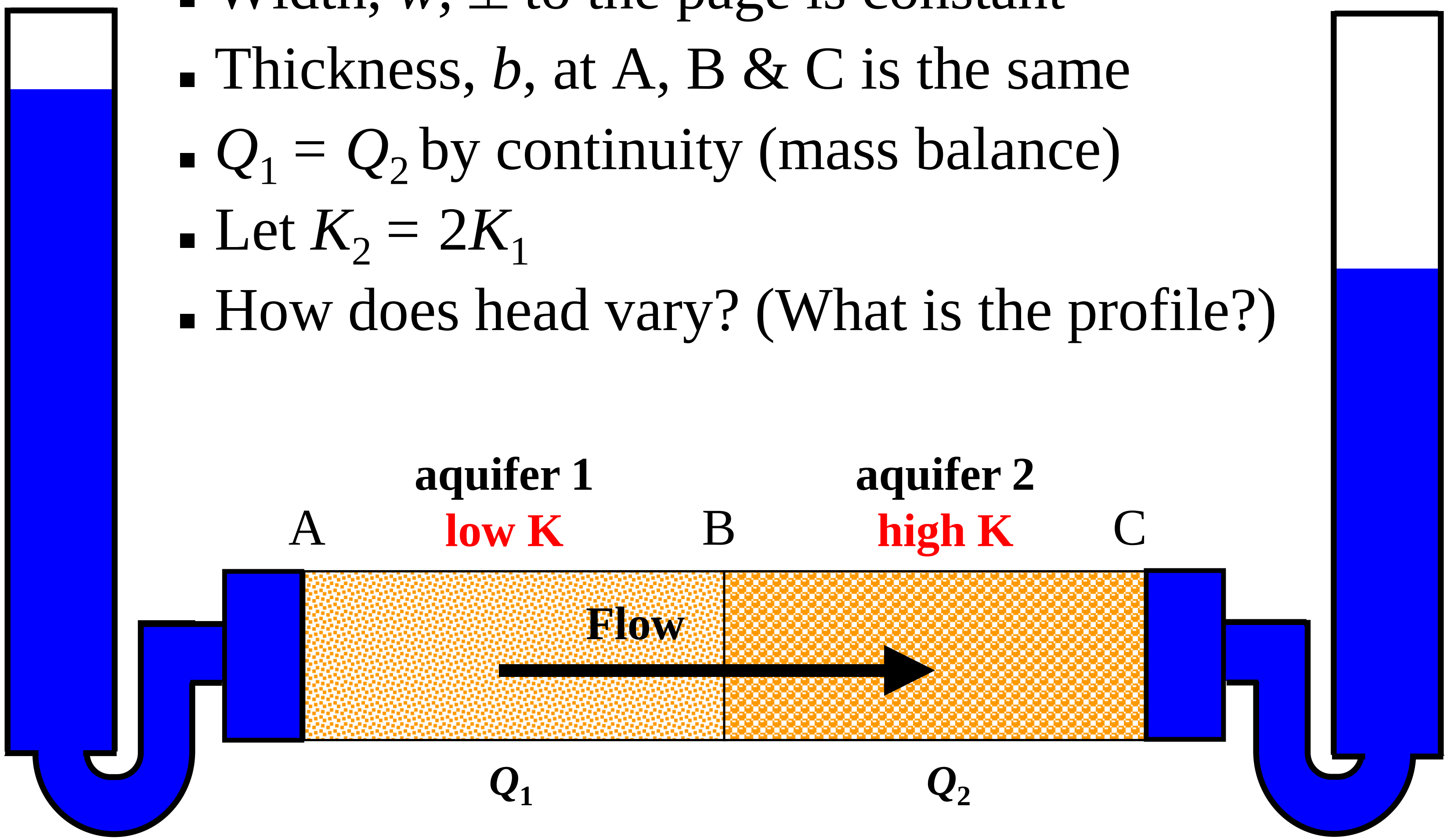
Head Gradient

- Head profile for homogeneous material
- Slope is constant



Effect of K on Change in Head Gradient

- Length, L, from A to B and B to C is same
- Width, w, ⊥ to the page is constant
- Thickness, b, at A, B & C is the same
- $Q_1 = Q_2$ by continuity (mass balance)
- Let $K_2 = 2K_1$
- How does head vary? (What is the profile?)



Head Profile (Effect of K)

- By continuity, $Q_1 = Q_2$
- Write Darcy's Law

$$-K_1 A_1 (\Delta h/l)_1 = -K_2 A_2 (\Delta h/l)_2$$

- Cancel like terms, A, l
- Substitute $K_2 = 2K_1$

$$K_1 \Delta h_1 = K_2 \Delta h_2 = 2K_1 \Delta h_2$$

- Cancel K_1; therefore,

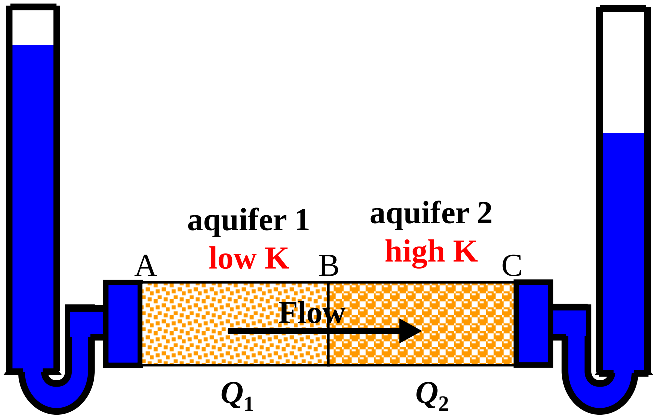
$$\Delta h_1 = 2 \Delta h_2$$

- Determine Δh_1 and Δh_2

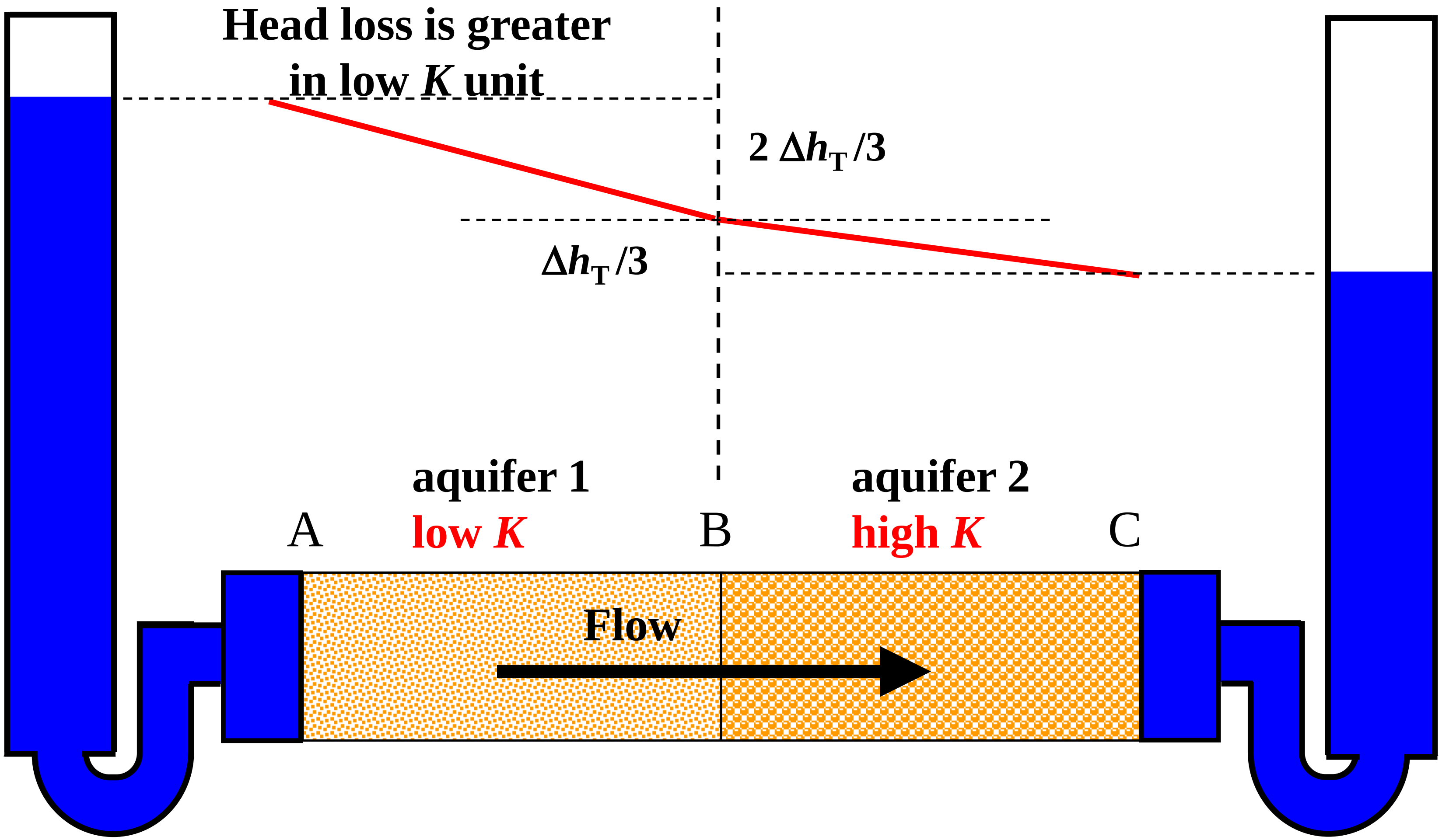
$$\Delta h_T = \Delta h_1 + \Delta h_2 = 2 \Delta h_2 + \Delta h_2 = 3 \Delta h_2$$

$$\Delta h_2 = \Delta h_T /3$$

$$\Delta h_1 = 2 \Delta h_T /3$$

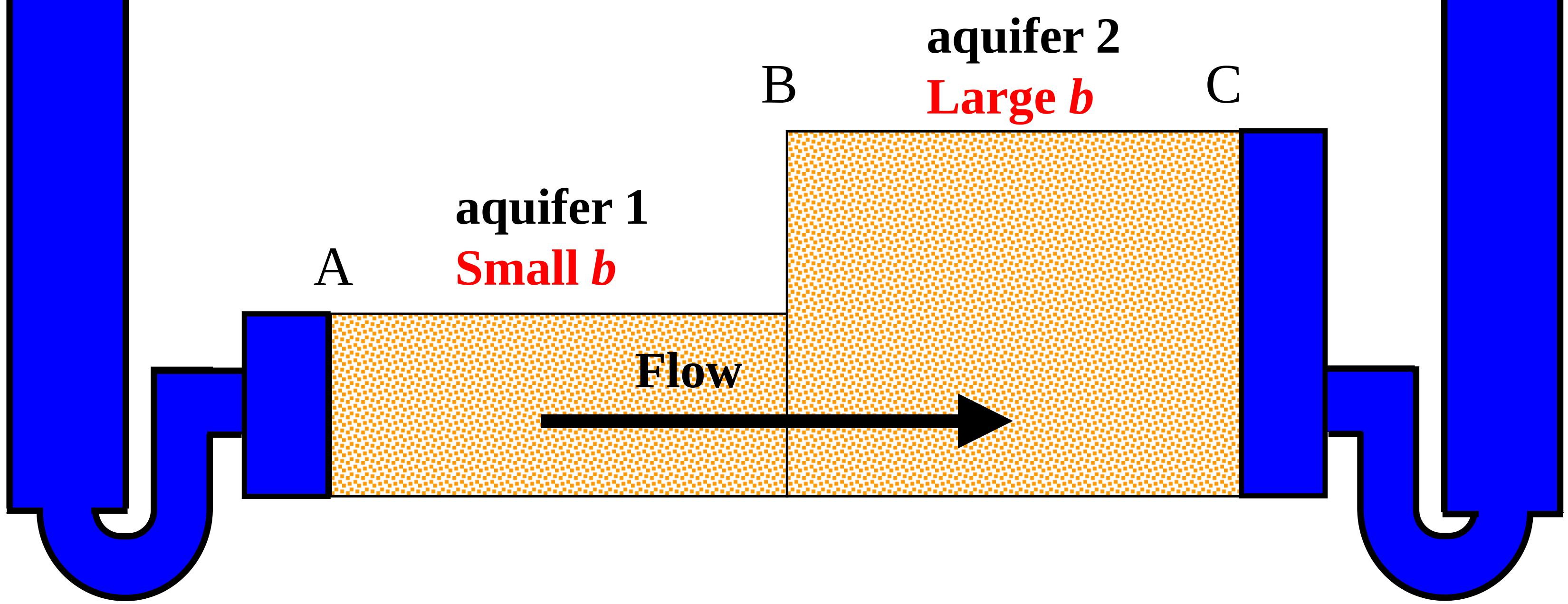


Change in *K* can cause Change in Head Gradient



Effect of *b* on Change in Head Gradient

- Length, L, from A to B and B to C is same
- Width, w, $\perp$ to the page is constant
- Hydraulic conductivity, K, is the same
- $Q_1 = Q_2$ by continuity (mass balance)
- Let $b_2 = 2b_1$; therefore $A_2 = 2A_1$
- How does head vary? (What is the profile?)



Head Profile (Effect of *b*)

- By continuity, $Q_1 = Q_2$
- Write Darcy's Law

$$-K_1 A_1 (\Delta h/l)_1 = -K_2 A_2 (\Delta h/l)_2$$

- Cancel like terms, substitute $A_2 = 2A_1$

$$A_1 \Delta h_1 = A_2 \Delta h_2 = 2A_1 \Delta h_2$$

- Therefore,

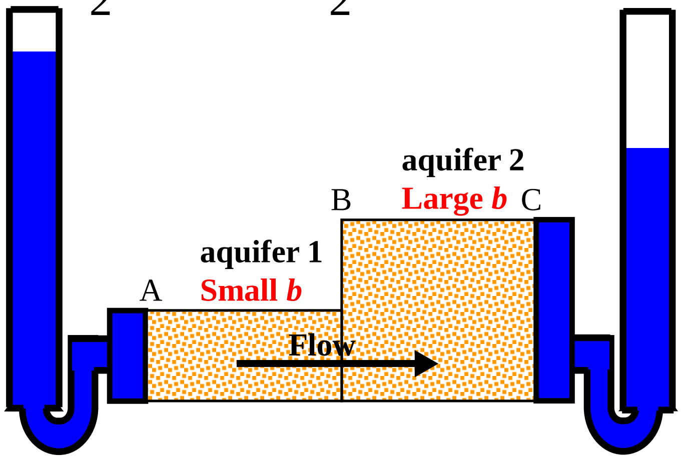
$$\Delta h_1 = 2 \Delta h_2$$

- Determine dh_1 and dh_2

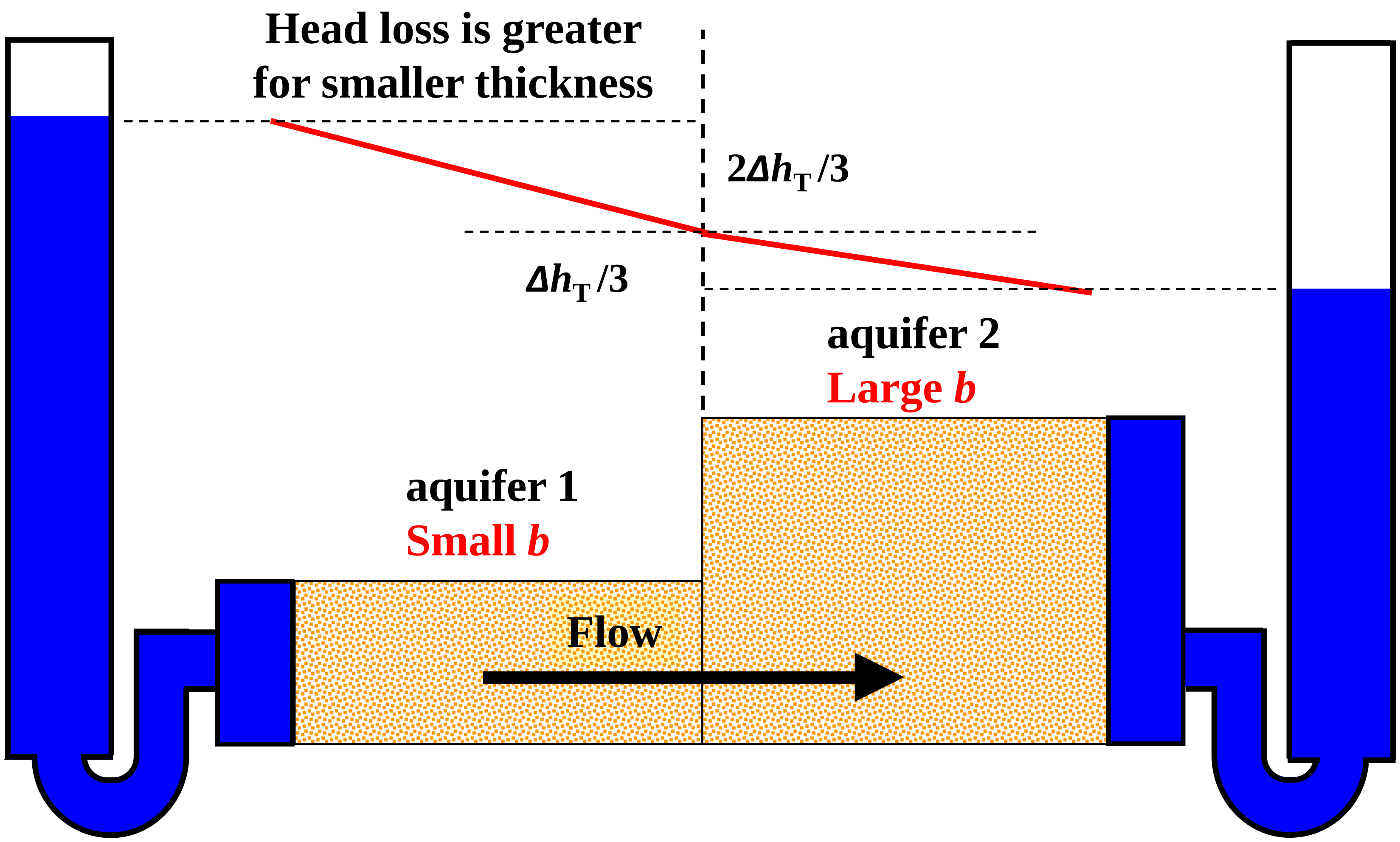
$$\Delta h_T = \Delta h_1 + \Delta h_2 = 2 \Delta h_2 + \Delta h_2 = 3 \Delta h_2$$

$$\Delta h_2 = \Delta h_T /3$$

$$\Delta h_1 = 2 \Delta h_T /3$$

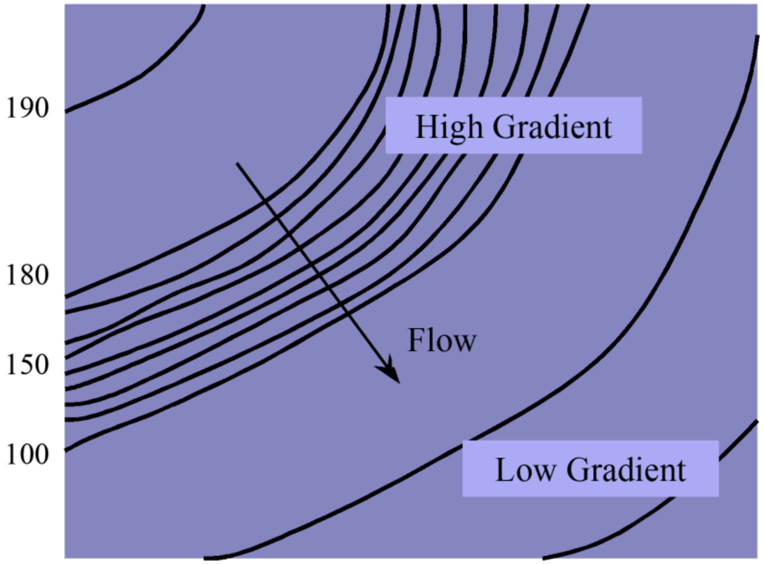


Changes in *b* can cause Changes in Head Gradient

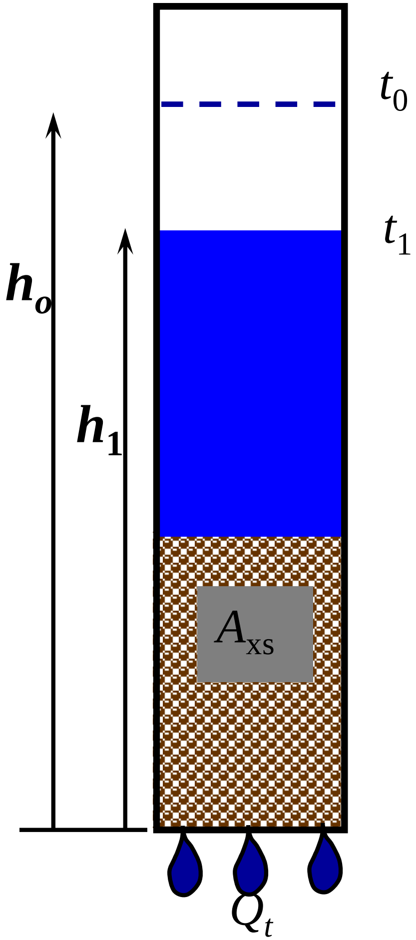


Head Distribution Reflects Transmissivity, *T*, not hydraulic conductivity *K*

- Groundwater computer models calculate the distribution of hydraulic head, try to match measured and calculated head
- Note that the head distribution reflects *T*, not *K*
- You can't determine *K* and *b* separately from head distribution (or hydraulic gradient) measurements. Must know one to calculate the other



How Fast is Groundwater Moving?



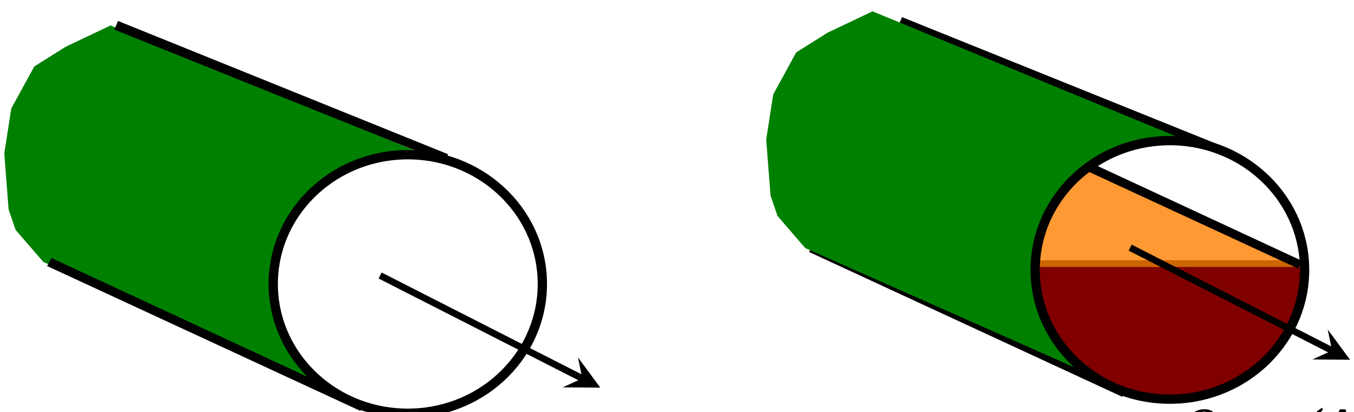
- Consider Darcy's experiment with a vertical sample
  - $Q_t = -KA_{xs}(h_t/L)$ Divide through by A_{xs}:
  - $Q_t/A_{xs} = -K(h_t/L) = q$ [m/s]
  - q = Specific Discharge (Darcy velocity)
  - $Q/A_{xs} = A_{xs}(h_o - h_1)/(t_1 - t_0)/A_{xs}$
  - $q = Q/A_{xs} = (h_o - h_1)/(t_1 - t_0)$

Specific Discharge – Darcy Velocity

- Darcy Velocity is the velocity of water in the standpipe above the sample, not in the sample
- Specific discharge is an *apparent* velocity
  - Does not occur *in* porous media
- Also called an *approach* velocity
- It is the velocity of the water, IF the aquifer had been an open conduit
  - “Empty bed” velocity

How Fast is Groundwater Moving?

- How is groundwater velocity in the porous medium related to specific discharge?
- Consider a pipe carrying water under pressure
- If a pipe became half clogged, but the flow through the pipe was kept constant, the velocity would double.

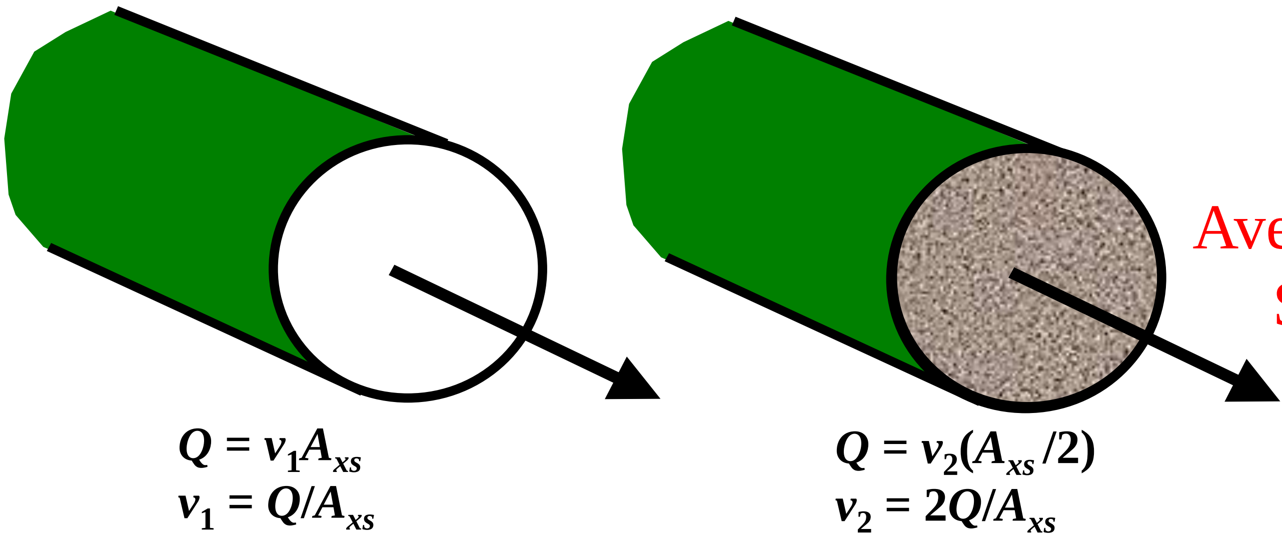


$$Q = v_1 A_{xs}$$
$$v_1 = Q/A_{xs}$$

$$Q = v_2(A_{xs}/2)$$
$$v_2 = 2Q/A_{xs}$$

Effect of Porosity on Velocity

- Similarly, if the pipe was filled with sand having a porosity of 50%, only half the area is available for flow
  - If the flow through the pipe was kept constant, the velocity would double
- The area available for flow is therefore $n_e A_{xs}$
- Groundwater velocity $v = Q/A_{flow} = Q/n_e A_{xs} = q/n_e$



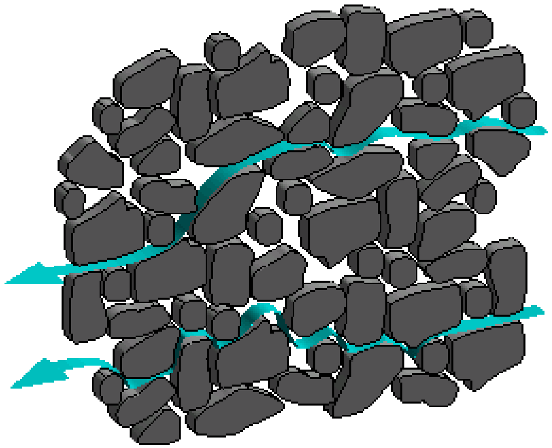
Average linear velocity
Seepage velocity
$v = Q/n_e A_{xs}$
$v = -K/n_e(dh/L)$

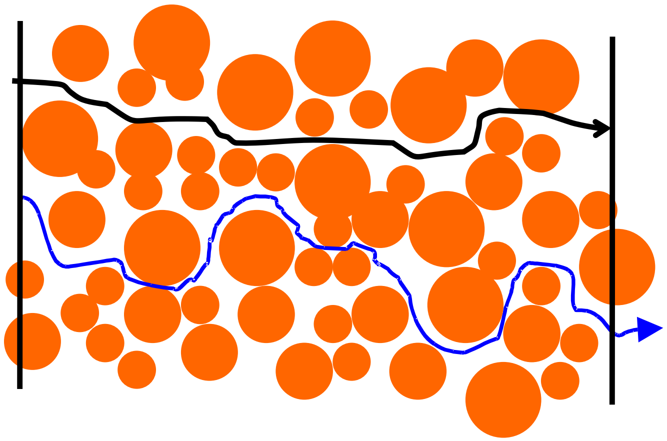
Average Linear Velocity Vs Microscopic Scale

Average linear velocity

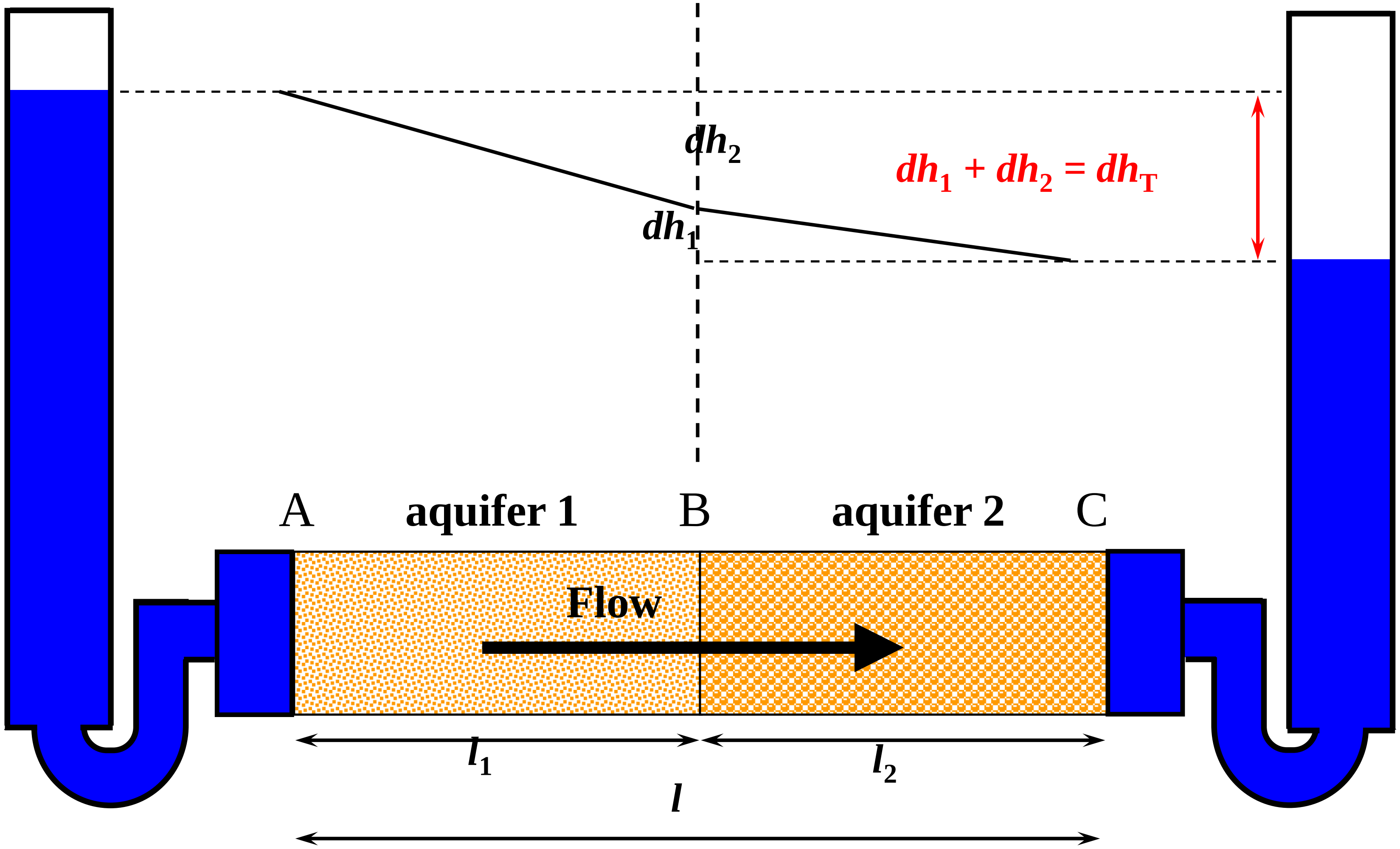
$$v = q/n_e = -K\,(dh/L)/n_e$$

- Pores have different sizes – velocity will differ in different size pores
- Water flowing near the pore walls will be slowed by viscosity, flow near the center of the pore throat will move fastest
- Flow paths are of different lengths, and some must split and branch around grains
- Actual *v* will vary about the mean

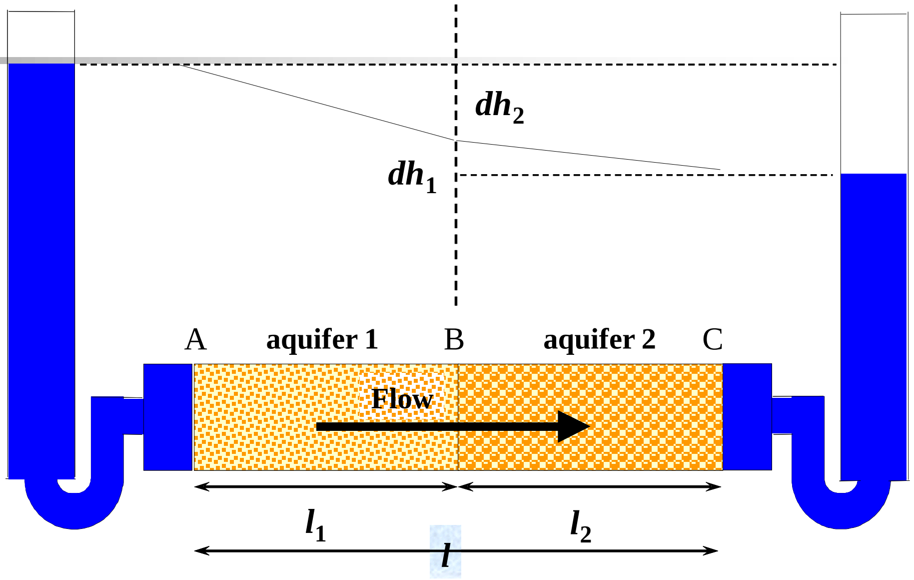




Flow Across Layers – Effective *K*



- Continuity: $Q_1 = Q_2$
- Head: $dh_1 + dh_2 = dh_T$
- Flow path: $l_1 + l_2 = l$
- Darcy's Law – solve for K_{eff}



$$Q = K_{eff} A \frac{dh_T}{l} = K_{eff} A \frac{dh_1 + dh_2}{l_1 + l_2}$$

$$K_{eff} = \frac{Q(l_1 + l_2)}{A(dh_1 + dh_2)}$$

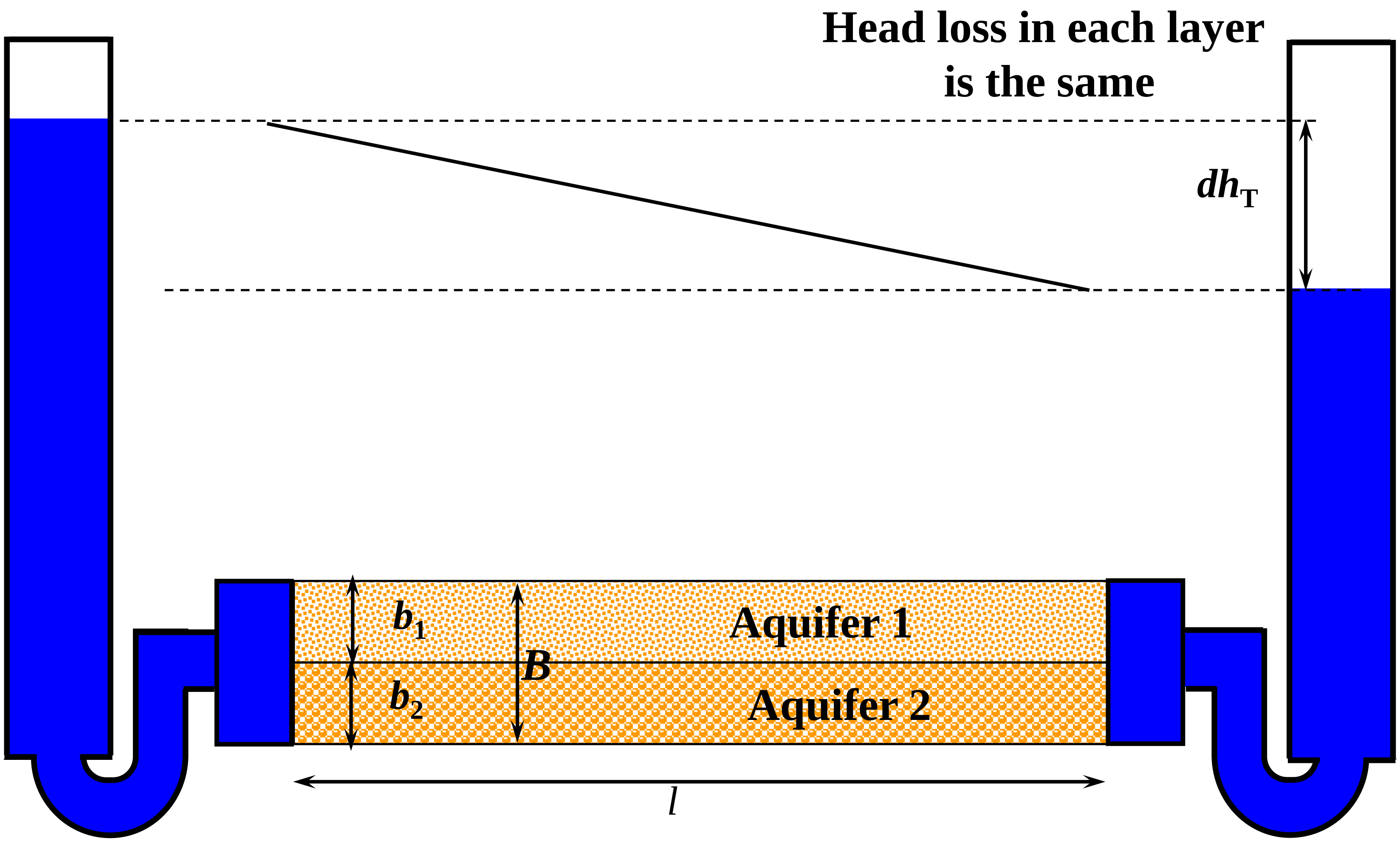
- **Darcy's Law – solve for dh_1 and dh_2**

$$Q = K_1 A \frac{dh_1}{l_1} \qquad dh_1 = \frac{Q l_1}{A K_1}$$

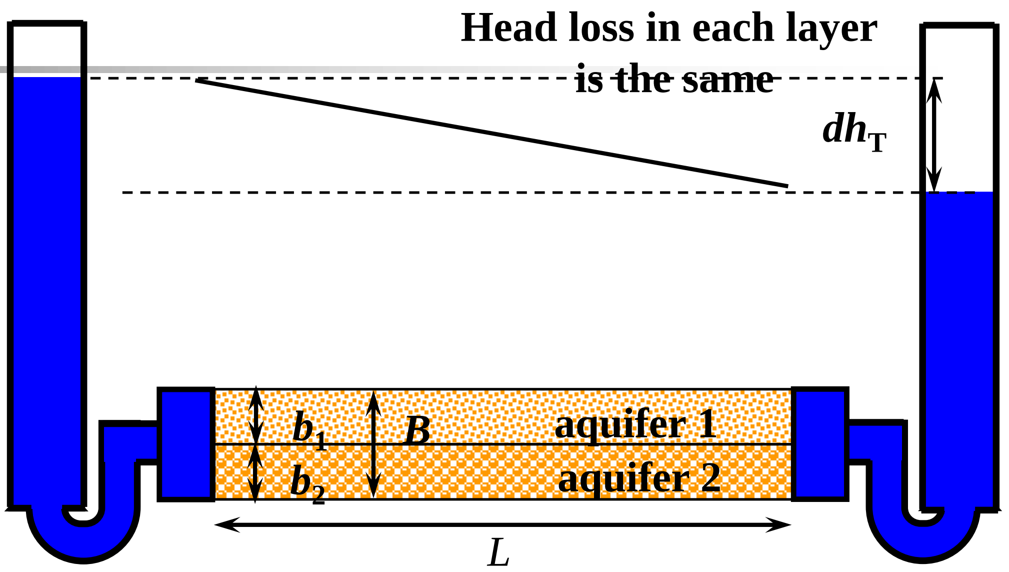
- **Substitute**

$$K_{eff} = \frac{(l_1 + l_2)}{\left[\frac{l_1}{K_1} + \frac{l_2}{K_2}\right]} = \frac{l}{\left[\frac{l_1}{K_1} + \frac{l_2}{K_2}\right]}$$

Flow Along Layers – Effective K



- Continuity: $Q_1 + Q_2 = Q_T$
- Head: $dh_1 = dh_2 = dh_T$
- Flow area: $b_1 w + b_2 w = A$
- Darcy's Law – solve for K_{eff}



$$Q_T = K_{eff}(b_1 + b_2)w\frac{dh_T}{L} \qquad K_{eff} = \frac{Q_T L}{(b_1 + b_2)w\,dh_T}$$

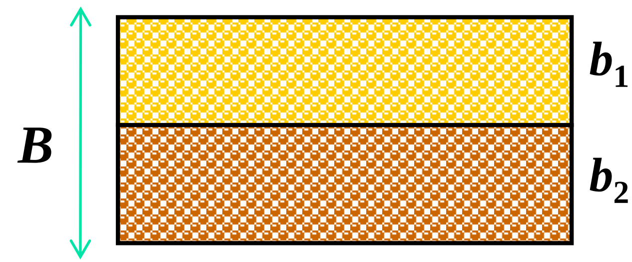
- **Darcy's Law – solve for Q_1**

$$Q_1 = K_1 b_1 w \frac{\Delta h_T}{l}$$

- **Substitute $Q_1 + Q_2 = Q_T$**

$$K_{eff} = \frac{K_1 b_1 + K_1 b_1}{(b_1 + b_2)} = \frac{K_1 b_1 + K_2 b_2}{B}$$

Vertical vs Horizontal K



- Vertical flow – across layers
- Horizontal flow – along layers
- Example
  - $K_1 = 1$ and $K_2 = 100$ m/d
  - $b_1 = 2$ and $b_2 = 2$ m
- Find K_{eff} for horizontal and vertical flow
- For vertical K (Flow across layers)

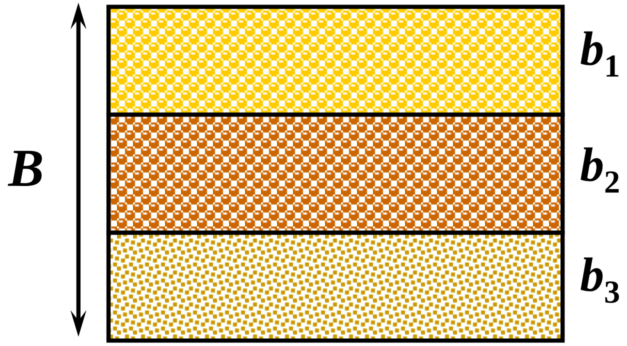
$$K_{eff} = \frac{(b_1 + b_2)}{\left[\frac{b_1}{K_1} + \frac{b_2}{K_2}\right]} = \frac{4}{\left[\frac{2}{1} + \frac{2}{100}\right]} = 1.98\,[\text{m/d}]$$

- For horizontal K (Flow along Layers)

$$K_{eff} = \frac{K_1 b_1 + K_1 b_1}{(b_1 + b_2)} = \frac{1\times 2 + 100\times 2}{4} = 50.5\,[\text{m/d}]$$

Vertical vs Horizontal *K*

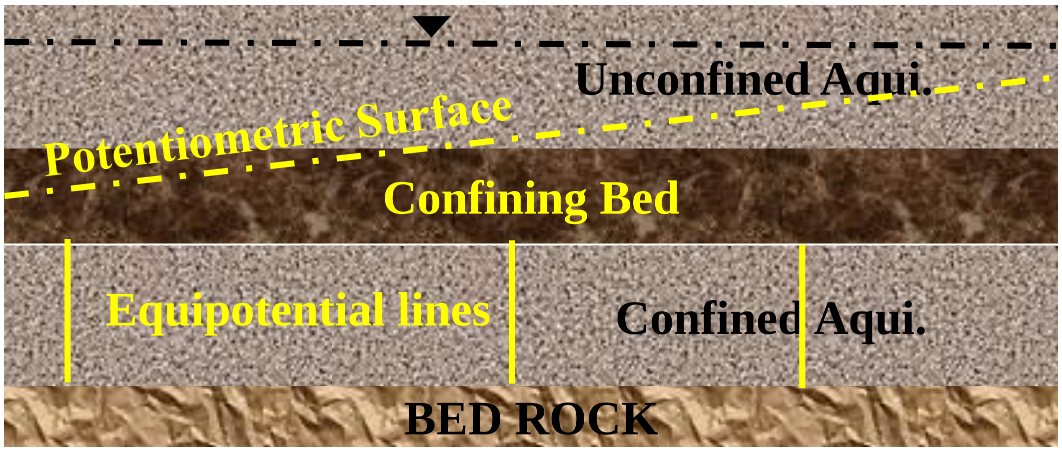
- Vertical effective conductivity is dominated by the layer having the lowest *K*
- Horizontal effective conductivity is dominated by the high *K* layer
- Horizontal effective *K* is much larger than the vertical effective *K*



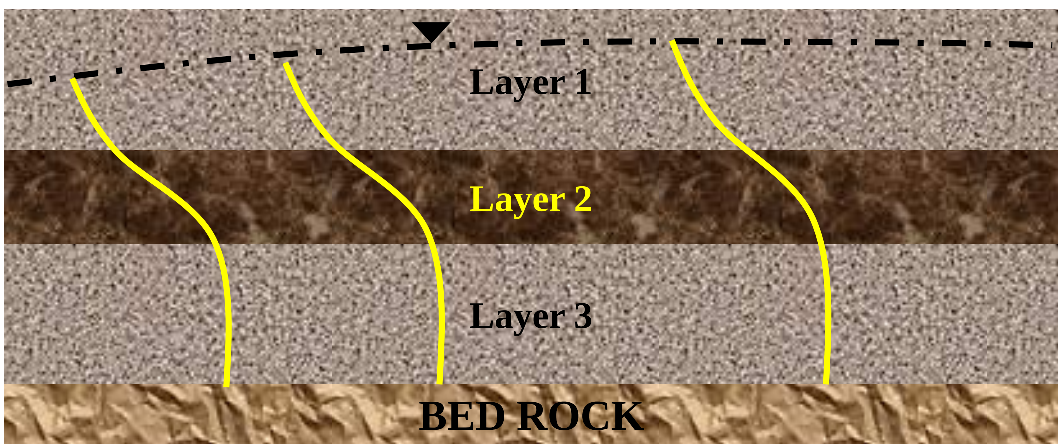
2. Governing equation

- Two conceptual views of groundwater:
  - Aquifer system view point
  - Flow system view point
- The aquifer view point:
  - Is based on the concept of confined and unconfined aquifers.
  - Is especially suited to analysis of flow to pumping wells
  - Is the basis for many analytical solutions including those of Theim, Theis and Jacob.
  - The groundwater flow assumed to be strictly horizontal through aquifers and strictly vertical through confining beds.
  - Is used to simulate two dimensional horizontal flow.
- In the flow system view point equipotential lines pass through all geologic units, both aquifers and confining beds.

The geologic system



The aquifer
System view point



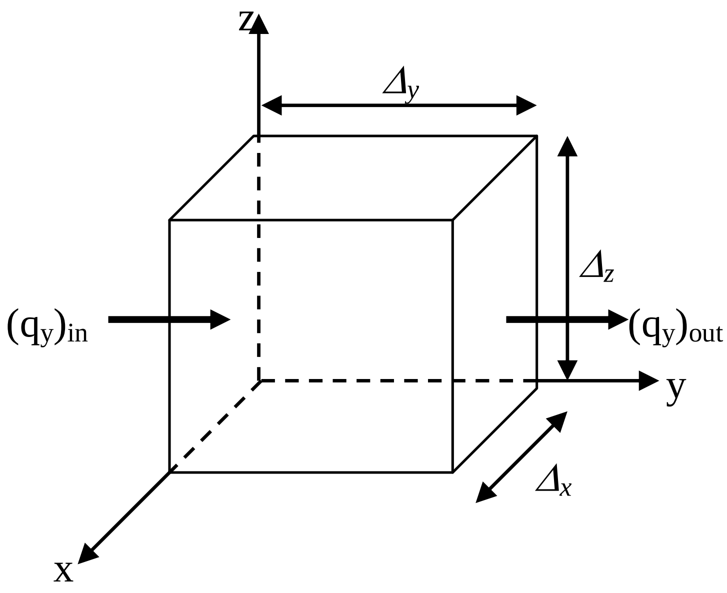
The flow system
view point

Modified from Anderson and Woessner, 1992

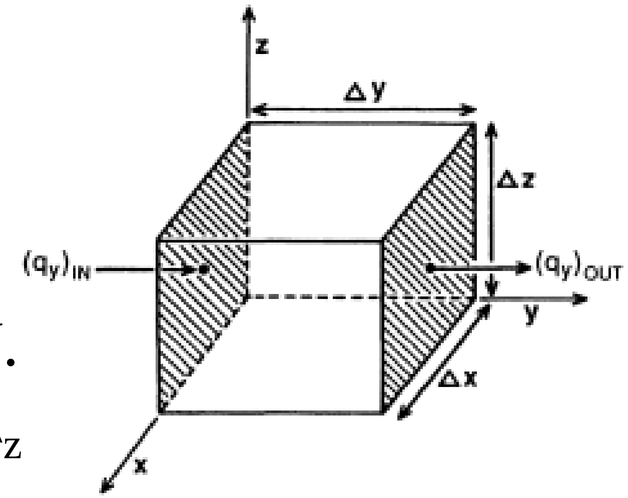
- The governing equation for water flow in saturated medium can be obtained by combining a special form of Darcy’s law (derived from the water phase momentum balance) and the continuity equation written for the water phase.
- The derivation is traditionally done by referring to a cube of porous material (Figure 1) that is large enough to be representative of the properties of the porous medium and yet small enough so that the change of head within the volume is relatively small (Anderson and Woessner, 1992).

Groundwater Flow Equation

- **Figure 1** Representative elementary volume used in the derivation
- The cube in Figure 1 is called the representative elementary volume (REV). Its volume is equal to $\Delta_x\Delta_y\Delta_z$. The flow of water through the REV is expressed in terms of the discharge rate (q), whose magnitude in the three coordinates will be q_x, q_y, and q_z.



- The water balance equation (conservation of mass) states that:
  - Mass Out – Mass In = Change of the Mass in storage
- Consider flow along the y-axis of the REV. Influx to REV occurs through the face $\Delta_x\Delta_z$ and is equal to $(q_y)_{in}$. Flux out is $(q_y)_{out}$.



The volumetric flow rate along y-axis is: $\left(q_{y,out} - q_{y,in}\right)\Delta_x\Delta_z$

This can also be written as: $\frac{\left(q_{y,out} - q_{y,in}\right)}{\Delta_y}\Delta_x\Delta_y\Delta_z$

Dropping the *'in'* and *'out'* subscripts, the change in flow rate through the REV along the y-axis is: $\frac{\partial q_y}{\partial y}\Delta_x\Delta_y\Delta_z$

- Similar expression can be written for the change in flow rate along the *x*- and *z*- axes. The total change in flow rate is equal to the rate of change in storage:

$$\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)\Delta_x\Delta_y\Delta_z = Rate\ of\ \ Change\ in\ storage \qquad 1$$

- The existence of sink (e.g. a pumping well) or source of water (e.g. injection well or some other source of recharge) within the REV is undeniable. The **volumetric inflow rate** of such sources is represented by $R^*\Delta_x\Delta_y\Delta_z$. Here the R^* is defined to be intrinsically positive when it is a source of water; therefore it is added to the right hand side of Eq. 1. Therefore Eq. 1 becomes:

$$\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}-R^*\right)\Delta_x\Delta_y\Delta_z = Rate\ of\ \ Change\ in\ storage \qquad 2$$

- The change in storage is represented by specific storage (S_S). It is defined as the volume of water released from storage per unit change in head (*h*) per unit volume of aquifer (Anderson and Woessner, 1992) i.e.

$$S_S = -\frac{\Delta V}{\Delta h \Delta_x \Delta_y \Delta_z}$$

- The sign convention is that the *ΔV* is intrinsically positive when the *Δh* is negative, in other words, water is released from the REV when head decreases.
- The rate of change in storage in REV will be:

$$\frac{\Delta V}{\Delta t} = -S_S \frac{\Delta h}{\Delta t} \Delta_x \Delta_y \Delta_z \qquad 3$$

- Combining Eq. 2 and Eq. 3:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = -S_S \frac{\partial h}{\partial t} + R^* \qquad 4$$

- Darcy law is used to set the relationship between q and h. Darcy law in three dimension is written as (Anderson and Woessner, 1992):

$$q_x = -K_x \frac{\partial h}{\partial x} \qquad q_y = -K_y \frac{\partial h}{\partial y} \qquad q_z = -K_z \frac{\partial h}{\partial z}$$

- Substituting these expressions in Eq. A.4 the desired groundwater flow equation is formulated:

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = S_S \frac{\partial h}{\partial t} - R^*$$

- Where K_x, K_y, and K_z are components of the hydraulic conductivity.

- In the above derivation it is assumed that K_x, K_y, and K_z are collinear to the *x*, *y*- and *z*- axes.
- If the geology is such that it is not possible to align the principal direction of the hydraulic conductivity tensor with the rectilinear coordinate system, a modified form of equation that utilizes the hydraulic conductivity tensor is required.

$$K = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix}$$

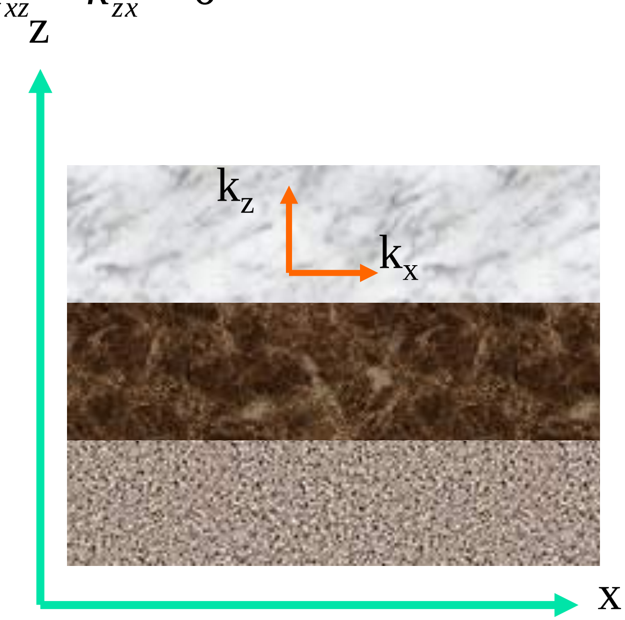
- By using a global coordinate system for the entire problem domain and a local coordinate system for each REV in the grid, the off diagonal terms in the hydraulic conductivity tensor could have zero value (Anderson and Woessner, 1992).

$$K = \begin{bmatrix} K_x & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & K_z \end{bmatrix}$$

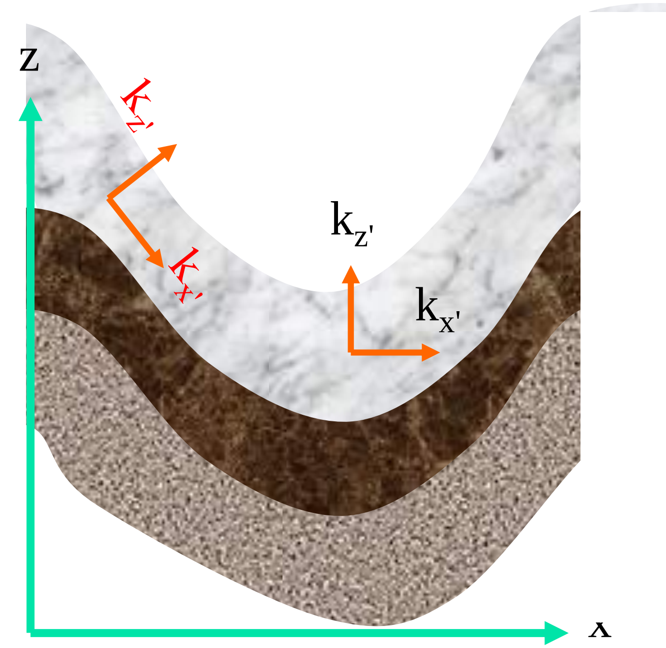
$$\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial z}\left(k_{zz}\frac{\partial h}{\partial z}\right)=S_S\frac{\partial h}{\partial t}-R^*$$

$$k_{xz}=k_{zx}=0$$

$$\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial x}\left(k_{xz}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial z}\left(k_{zz}\frac{\partial h}{\partial z}\right)+\frac{\partial}{\partial z}\left(k_{zx}\frac{\partial h}{\partial z}\right)=S_S\frac{\partial h}{\partial t}-R^*;\quad k_{xz}\neq 0; k_{zx}\neq 0$$



- The x-z coordinate system is aligned with the principal directions of the hydraulic conductivity tensor.



A global coordinate system (x-z) is defined. Local coordinates (x'-z') are aligned with the principal directions of the local hydraulic conductivity tensor.

3. Initial and boundary conditions

- For a well posed boundary value problem: (i) A solution must exist, (ii) The solution must be unique and (iii) The solution must be stable, in the sense that sufficiently small variations in the given data should lead to arbitrary small changes in the solution
- Initial and boundary conditions are needed for a unique solution of the groundwater flow equations (second-order partial differential equations) for a specific flow domain of interest
- **Initial conditions:** specification of the distribution of the state variable (hydraulic head for the groundwater flow equation) at some initial time, usually at t = 0.
- For example $$h = h(x, y, z, 0) = f(x, y, z) \quad \text{in D}$$
- in which $f(x,y,z)$ is a known function, D is the flow domain.

- **Boundary conditions:** specification of the interaction between the flow domain and its surrounding environment, which is a mathematical representation of the physical reality
  - Known water fluxes
  - Known values of state variables, such as hydraulic head, that the external domain imposes on the flow regime
- Different initial and boundary conditions result in different solutions
- Three mathematical boundary conditions:
  1. Dirichlet
  2. Neumann
  3. Cauchy

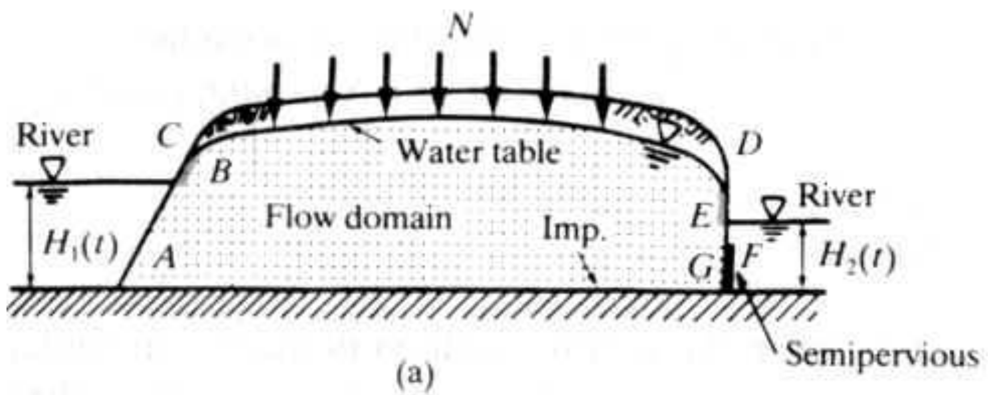
- Three mathematical boundary conditions
  - **Dirichlet condition** (boundary condition of the first kind): the fluid pressure (or hydraulic head) is specified as a known function of space and time.
    - This occurs whenever the porous medium flow domain is in contact with a body of open water (AB, EG surfaces)

$$p(\mathbf{x},\mathrm{t}) = f(\mathbf{x},t) \quad \text{on B}$$

$$h(\mathbf{x},t) = g(\mathbf{x},t) \quad \text{on B}$$

***f* and *g* are two known functions**

  - Special case : Equipotential boundary

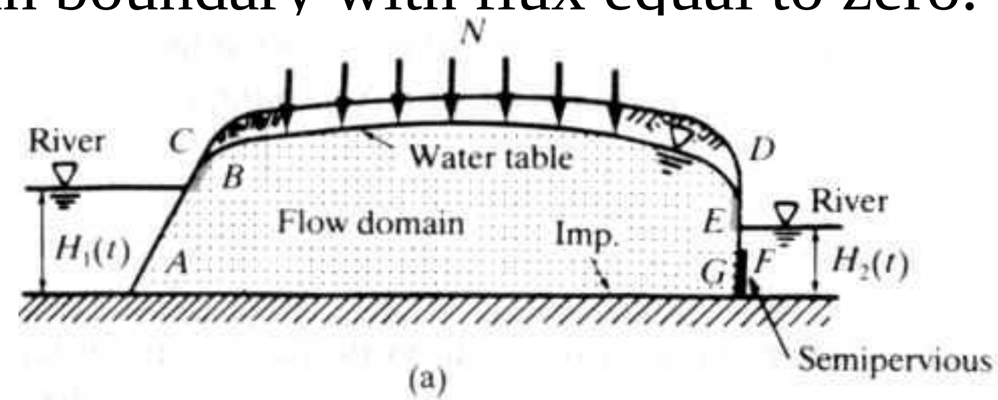


- **Neumann condition** (boundary condition of the second kind): the pressure gradient (or hydraulic gradient), or a linear combination of their components, is specified as a known function of space and time on the boundary.

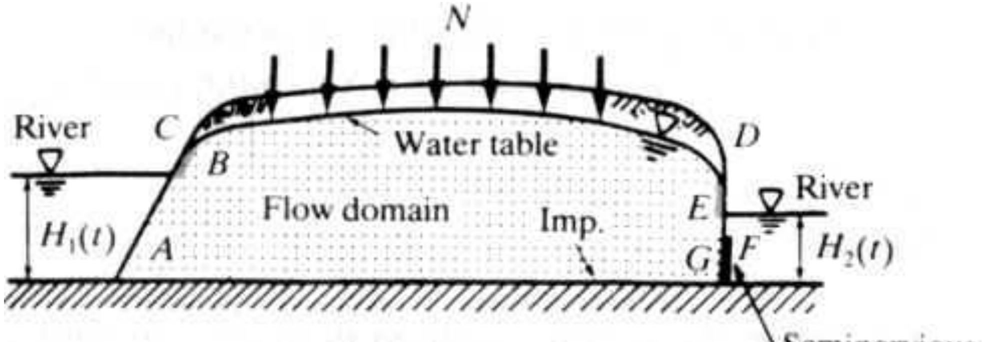
$$q_r \cdot \upsilon = m(x,t) \text{ on } B$$

m(x, t) is a known function

- This occurs when constant flux (discharge) is seen across a certain portion of the boundary (BE).
- Thus an impervious boundary (Boundary along AG) is Neumann boundary with flux equal to zero.



- **Cauchy,** Mixed boundary condition, boundary condition of the third kind) : the condition which specifies the information on the relationship between the state variable and its derivatives
- This occurs when the porous medium domain is in contact with a body of water continuum (or another porous medium domain) through a relatively thin semi pervious layer separating the two domains (e.g., FG in the bottom figure)



(a)

Analytical Method Example:

- **The ends A and B of a soil column, 200 cm long, have head at 0 cm and 40 cm until steady state prevails. If the head of the ends are changed to 0 cm. Find the head distribution in the soil column at any time t. Take Ss as 10^{-3} and K as 10^{-5} cm/s.**

$$Ss\frac{\partial h}{\partial t} = K\frac{\partial^2 h}{\partial x^2} \Rightarrow \frac{\partial h}{\partial t} = \frac{10^{-5}}{10^{-3}}\frac{\partial^2 h}{\partial x^2}$$

$$\Rightarrow \frac{\partial h}{\partial t} = 0.01\frac{\partial^2 h}{\partial x^2}$$

For steady state:

$$\frac{\partial h}{\partial t} = 0.01\frac{\partial^2 h}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 h}{\partial x^2} = 0$$

Upon Integration: $h(x) = c_1 x + c_2$

$$h(0) = c_1 \times 0 + c_2 = 0 \Rightarrow c_2 = 0cm$$

$$h(x) = c_1 x$$

$$h(200) = c_1 \times 200 = 40 \Rightarrow c_1 = 0.2$$

$$h(x) = 0.2x$$

Since the head at A and B are suddenly changed we gain transient state whose initial condition could be described by the above equation.

$$\frac{\partial h}{\partial t} = 0.01\frac{\partial^2 h}{\partial x^2}$$

Let $h = TX$ *is the solution*

$$\frac{\partial h}{\partial t} = \frac{dT}{dt}X; \frac{\partial h}{\partial x} = T\frac{dX}{dx}$$

$$\Rightarrow \frac{\partial^2 h}{\partial x^2} = T\frac{d^2 X}{dx^2}$$

Substituting :

$$\frac{dT}{dt}X = 0.01T\frac{d^2 X}{dx^2}$$

$$\frac{dT}{Tdt} = 0.01\frac{d^2 X}{Xdx^2}$$

$$\frac{dT}{Tdt} = -c^2 \Rightarrow \frac{dT}{T} = -c^2 dt$$

$$\ln\frac{T}{c_1} = -c^2 t \Rightarrow T = c_1 e^{-c^2 t}$$

$$0.01\frac{d^2 X}{Xdx^2} = -c^2 \Rightarrow \frac{d^2 X}{dx^2} + 100c^2 X = 0$$

Solving this ODE

$$X = c_2 \sin(10cx) + c_3 \cos(10cx)$$

thus $h = TX$

$$\Rightarrow h = c_1 e^{-c^2 t}\left(c_2 \sin(10cx) + c_3 \cos(10cx)\right)$$

$$h = e^{-c^2 t}\left(C_1 \sin(10cx) + C_2 \cos(10cx)\right)$$

$$h(0,t) = e^{-c^2 t}\left(C_1 \times 0 + C_2 \times 1\right) = 0 \Rightarrow C_2 = 0$$

$$\Rightarrow h(x,t) = C_1 e^{-c^2 t}\sin(10cx)$$

$$h(200,t) = C_1 e^{-c^2 t}\sin(10c \times 200) = 0$$

$$\sin(2000c) = 0 = \sin(n\pi) \Rightarrow c = \frac{n\pi}{2000}$$

The general solution would be

$$h(x,t)=\sum_{n=0}^{\infty} b_n e^{-\left(\frac{n\pi}{2000}\right)^2 t}\sin\left(\frac{n\pi x}{200}\right);\ at\ t=0\ h=0.2x \Rightarrow x=\sum_{n=0}^{\infty} 5b_n \sin\left(\frac{n\pi x}{200}\right)$$

$$\Rightarrow 5b_n=\frac{2}{200}\int_0^{200} x\sin\left(\frac{n\pi x}{200}\right)dx\ \ Note: Fourier\ \sin e\ series$$

$$5b_n=\frac{2}{200}\left[x\frac{200}{n\pi}\left(-\cos\left(\frac{n\pi x}{200}\right)\right)-\left(\frac{200}{n\pi}\right)^2\sin\left(\frac{n\pi x}{200}\right)\right]_0^{200}$$

$$5b_n=(-1)^{n+1}\frac{400}{n\pi}\Rightarrow b_n=(-1)^{n+1}\frac{80}{n\pi}$$

$$h(x,t)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{80}{n\pi}e^{-\left(\frac{n\pi}{2000}\right)^2 t}\sin\left(\frac{n\pi x}{200}\right)$$

$$h(x,t)=\frac{80}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\left(\frac{n\pi}{2000}\right)^2 t}\sin\left(\frac{n\pi x}{200}\right)$$

4. Dupuit assumption

- The boundary of an unconfined aquifer (z) is indeed the solution (h) that needs to be determined.
- Dupuit assumptions: First developed by Dupuit (1863) and then advanced by Forchheimer (1930), or called Dupuit-Forchheimer theory
  - From observations, the slope of phreatic surface (water table) is very small (commonly 1/1000)
  - Two assumptions
    - The hydraulic gradient is equal to the slope of the free surface and is invariant with depth
    - The equipotential lines are vertical, i.e., the flow lines are horizontal, i.e.,

$$\frac{\partial p}{\partial z} = -\rho g$$

$$q_s = -K\frac{dh}{ds}$$

$$= -K\frac{dz}{ds}$$

$$= -K\sin\theta$$

$$\approx -K\tan\theta$$

$$= -K\frac{dh}{dx}$$

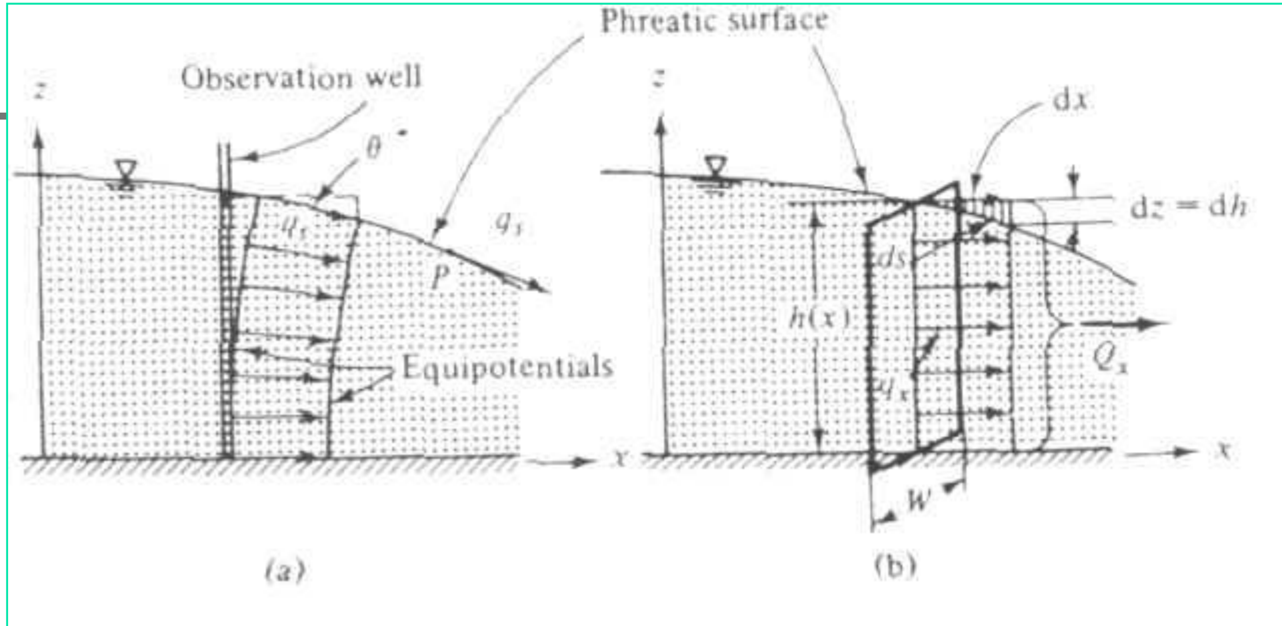


Fig. 2.6. The Dupuit assumption (Bear, 1979).

a. The real flow field with non-vertical equipotential lines near the water table

b. The flow field obtained by the Dupuit assumption, i.e., vertical equipotential lines

For small θ, sin θ can be replaced by tan θ, then

$$q_s \equiv q_x = -K\tan\theta = -K\frac{dh}{dx} \qquad (\text{for } h = h(x))$$

(A) For horizontal bottom and 3-D steady-state, free surface flows

$h(x,y,z) \to h(x,y)$ because the assumption of vertical equipotential lines (or horizontal flows)

$$q_x = -K\frac{\partial h}{\partial x}, \quad q_y = -K\frac{\partial h}{\partial y} \Rightarrow \underline{\mathbf{q} = -K\nabla' h} \quad or \quad Q_x = -KWh\frac{\partial h}{\partial x}, \quad Q_y = -KWh\frac{\partial h}{\partial y} \Rightarrow \mathbf{Q} = -KWh\nabla' h \text{ or } \underline{\mathbf{Q}/W = -Kh\nabla' h}$$

in which $\nabla' = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}$ (W is the width of the unconfined aquifer)

(Note that for 1-D flows, $h(x,y) \to h(x)$ and ∇' is replaced by $\frac{d}{dx}$)

(Note that $\left[\frac{\mathbf{Q}}{W}\right] = \left[\frac{Q_x}{W}\right] = \left[\frac{Q_y}{W}\right] = \frac{L^2}{T} =$ discharge per unit width)

Example : two-dimensional steady-state flow without accretion

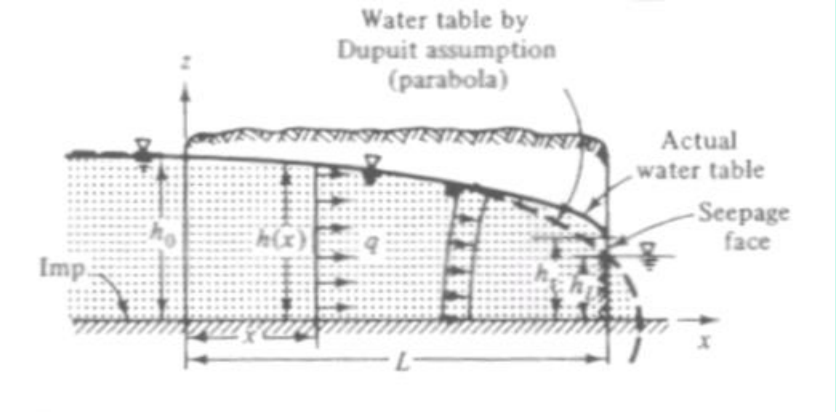


Fig. 2.7. Steady unconfined flow between two reservoirs (Bear, 1979).

(After Bear and Verruijt, 1987)

$$Q_x = -Kh\frac{dh}{dx} = \text{constant}$$

$$\Rightarrow Q_x dx = -Khdh$$

$$\Rightarrow Q_x \int_0^L dx = -K\int_{h_0}^{h_L} hdh = K\frac{\left(h_0^2 - h_L^2\right)}{2}$$

$$\Rightarrow Q_x = \frac{K\left(h_0^2 - h_L^2\right)}{2L} \qquad (1)$$

(Dupuit equation)

(Q_x = flow per unit width)

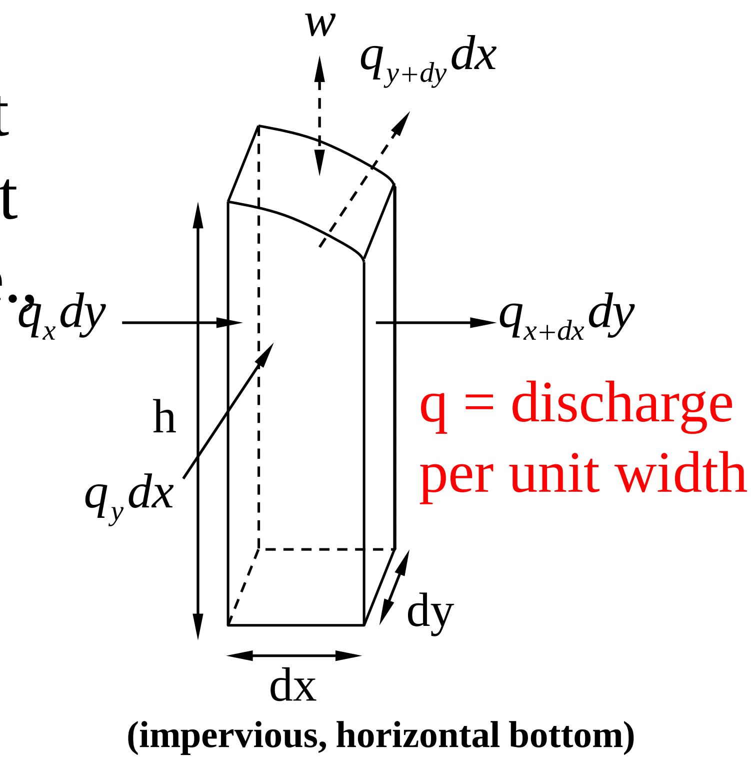
Example : three-dimensional steady-state flow with accretion

Mass in – mass out = Δ M = 0 (steady state)

Under Dupuit assumptions : $h(x, y, z) \rightarrow h(x, y)$

w [L/T] : rate of water into or out of the unconfined aquifer per unit area of the unconfined aquifer i.e., $w > 0$ for infiltration, $w < 0$ for evaporation

(Adapted from Fetter, 1994)



$$\rho q_x dy\Delta t - \rho q_{x+dx} dy\Delta t + \rho q_y dx\Delta t - \rho q_{y+dy}\Delta t + \underbrace{\rho w dx dy \Delta t}_{\text{sink/source}} = 0$$

$$\Rightarrow \rho\Delta t dy\left[\underbrace{-Kh\left(\frac{\partial h}{\partial x}\right)_x}_{q_x} + \underbrace{K\left(h\frac{\partial h}{\partial x}\right)_{x+dx}}_{q_{x+\Delta x}}\right] + \rho\Delta t dx\left[\underbrace{-Kh\left(\frac{\partial h}{\partial y}\right)_y}_{q_y} + \underbrace{K\left(h\frac{\partial h}{\partial y}\right)_{y+dy}}_{q_{y+\Delta y}}\right]$$

$$+\rho w dx dy \Delta t = 0$$

$$\Rightarrow \rho\Delta t dy\left[K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)dx\right] + \rho\Delta t dx\left[K\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)dy\right] + \rho w dx dy \Delta t = 0$$

$$\Rightarrow K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)dxdy + K\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)dxdy + w dxdy = 0$$

$$\Rightarrow K\left(\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2}\right) = -2w \qquad (2)$$

Furthermore, for one-dimensional flows, Eq(2) reduces to

$$K\frac{d^2}{dx^2}\left(h^2\right)=-2w$$

BC's : $h(x=0)=h_1$, $h(x=L)=h_2$

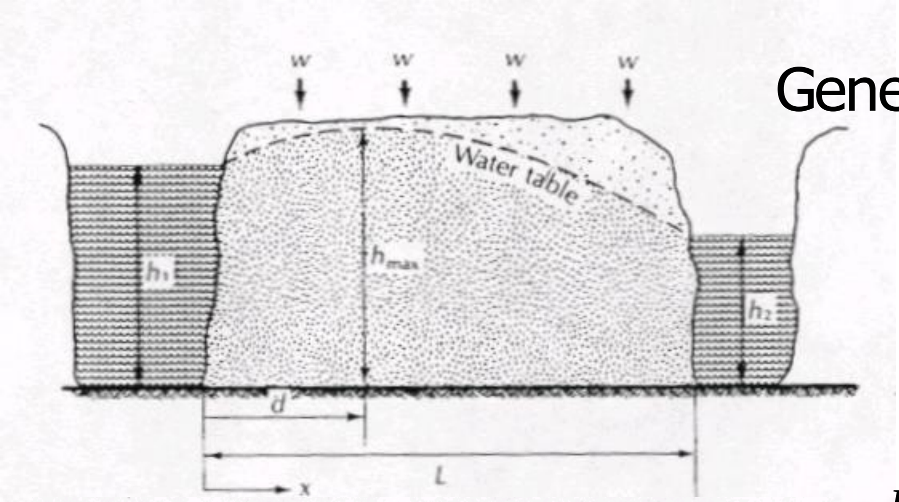


FIGURE 5.19 Unconfined flow, which is subject to infiltration or evaporation.

General solution : $h^2(x)=-\frac{w}{K}x^2+c_1x+c_2$

$$h(x=0)=h_1 \Rightarrow c_2=h_1^2$$

$$h(x=L)=h_2 \Rightarrow c_1=\left(\frac{h_2^2-h_1^2}{L}\right)+\frac{wL}{K}$$

$$h^2(x)=h_1^2-\frac{\left(h_1^2-h_2^2\right)x}{L}+\frac{w}{K}(L-x)x$$

(After Fetter, 1994)

Hence

$$h(x)=\sqrt{h_1^2-\frac{\left(h_1^2-h_2^2\right)x}{L}+\frac{w}{K}(L-x)x} \quad (3)$$

From Eq. (3)

$$h(x)=\sqrt{h_1^2-\frac{\left(h_1^2-h_2^2\right)x}{L}+\frac{w}{K}(L-x)x}=\sqrt{-ax^2+bx+c}=\sqrt{T}$$

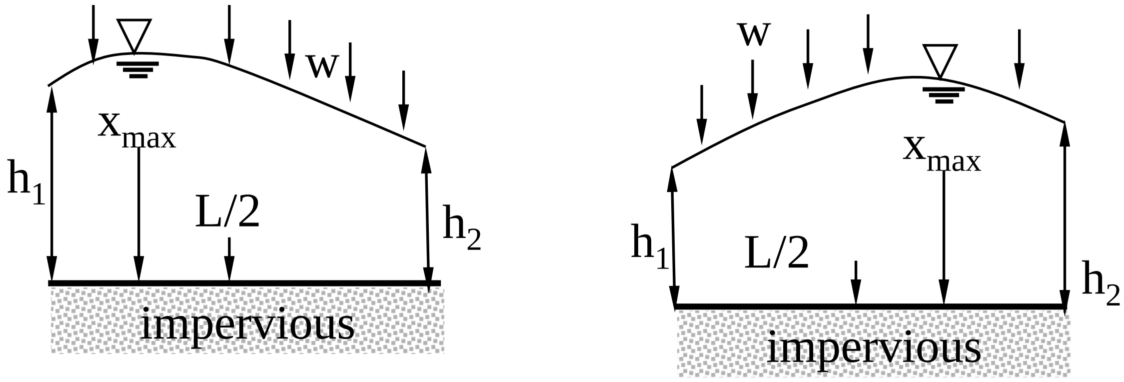
$$a=\frac{w}{K}>0\ ,\ b=\frac{\left(h_2^2-h_1^2\right)}{L}+\frac{wL}{K},\ c=h_1^2>0$$

$$\Rightarrow \begin{aligned} &\frac{dh}{dx}=\frac{-2ax+b}{2\sqrt{T}},\ x=\frac{b}{2a}=\frac{L}{2}+\frac{K\left(h_2^2-h_1^2\right)}{2wL}\ \text{ where }\ \frac{dh}{dx}=0 \\ &\frac{d^2h}{dx^2}=\frac{-4ac-b^2}{4\sqrt{T^3}}<0 \end{aligned}$$

- Hence, the water table surface is a hyperbola with maximum elevation occurs at

$$x_{max} = \frac{L}{2} + \frac{K\left(h_2^2 - h_1^2\right)}{2wL}, \quad h_{max} = \sqrt{\frac{\left(h_1^2 + h_2^2\right)}{2} + \frac{wL^2}{4K} + \frac{K\left(h_2^2 - h_1^2\right)^2}{4wL^2}}$$

- The location of maximum h occurs to the left of the midpoint if $h_2 < h_1$, or to the right if $h_2 > h_1$.

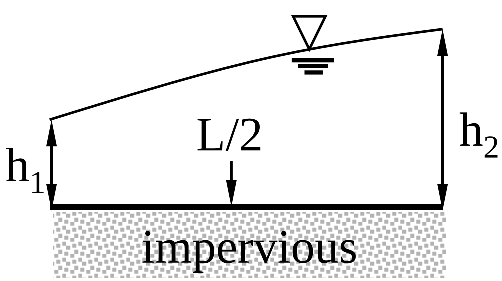


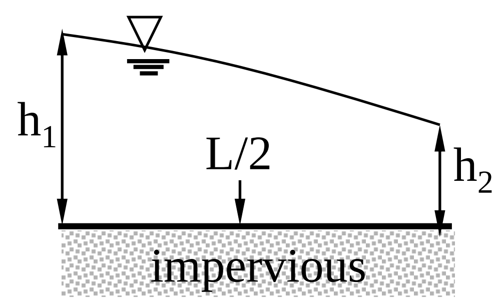
From Eq(3) and if w = 0 then

$$h(x) = \sqrt{h_1^2 - \frac{(h_1^2 - h_2^2)x}{L}} = \sqrt{ax+b} = \sqrt{T}, \quad a = \frac{h_2^2 - h_1^2}{L}, \quad b = h_1^2 > 0$$

$$\Rightarrow \begin{array}{l} \dfrac{dh}{dx} = \dfrac{a}{2\sqrt{ax+b}} \\ \dfrac{d^2h}{dx^2} = -\dfrac{a^2}{4(ax+b)^{3/2}} < 0 \end{array}$$

Hence, the water table surface is a parabola with a positive slope when $h_2 > h_1$, or a negative slope when $h_2 < h_1$



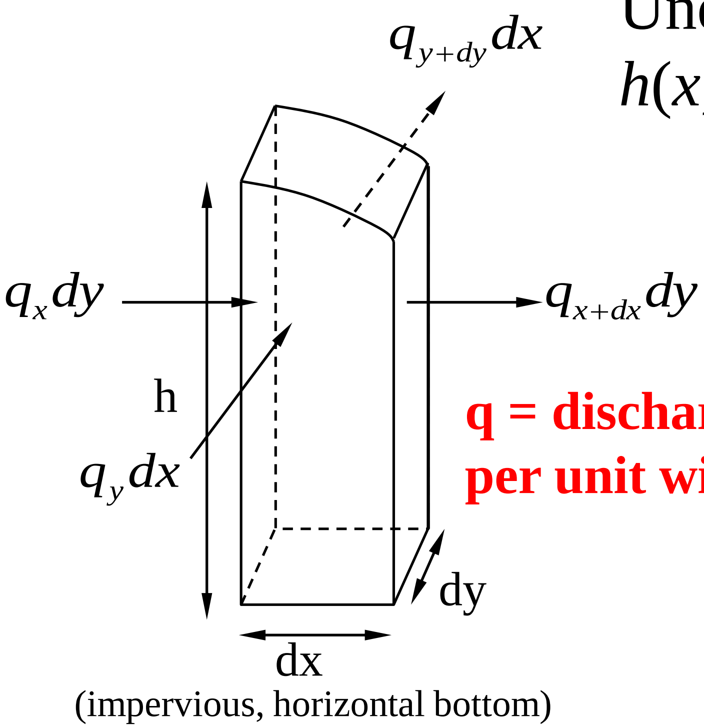


- Transient 2-D unconfined flows

For incompressible fluids and homogeneous and isotropic aquifers

Mass in – mass out = Δ M

Under Dupuit assumptions: $h(x,y,z) \rightarrow h(x,y)$



q = discharge per unit width

$$\rho q_x dy\Delta t - \rho q_{x+dx} dy\Delta t + \rho q_y dx\Delta t - \rho q_{y+dy}\Delta t = \rho S_y dxdy\Delta h$$

$$\Rightarrow \rho\Delta t dy\left[\underbrace{-Kh\left(\frac{\partial h}{\partial x}\right)_x}_{q_x} + \underbrace{K\left(h\frac{\partial h}{\partial x}\right)_{x+dx}}_{q_{x+\Delta x}}\right]$$

$$S_y = \text{specific yield} \equiv \frac{\Delta V_w}{A\Delta h}$$

$$+\rho\Delta t dx\left[\underbrace{-Kh\left(\frac{\partial h}{\partial y}\right)_y}_{q_y} + \underbrace{K\left(h\frac{\partial h}{\partial y}\right)_{y+dy}}_{q_{y+\Delta y}}\right] = \rho S_y dxdy\Delta h$$

$$\Rightarrow K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)dxdy + K\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)dxdy = S_y dxdy\frac{\partial h}{\partial t}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right) = \frac{S_y}{K}\frac{\partial h}{\partial t} \quad \text{(Boussinesq equation)} \qquad (4)$$

- Boussinesq equation is a non-linear PDE, which can not be solved analytically except under some idealized conditions
- Approximations: Drawdown in the aquifer is small, i.e., h ≈ b (averaged thickness assumed to be constant over the aquifer)

From (4):

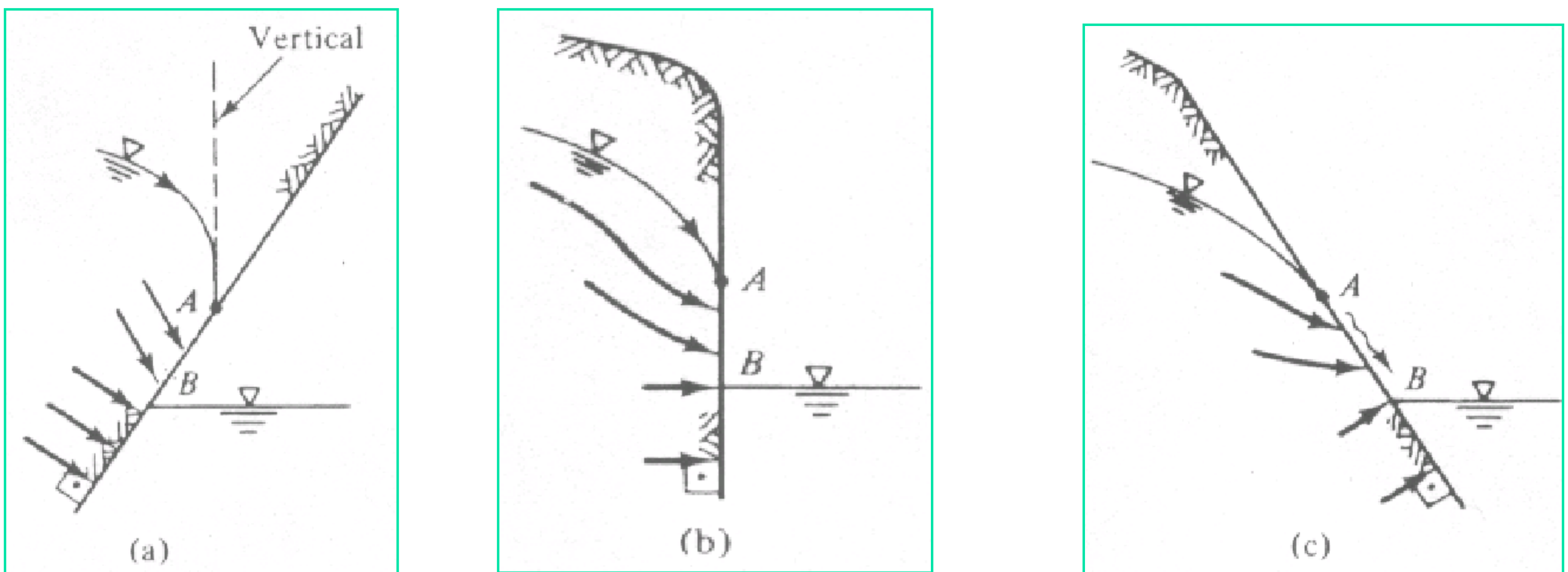
$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)\approx\frac{\partial}{\partial x}\left(b\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(b\frac{\partial h}{\partial y}\right)=\frac{S_y}{K}\frac{\partial h}{\partial t}$$

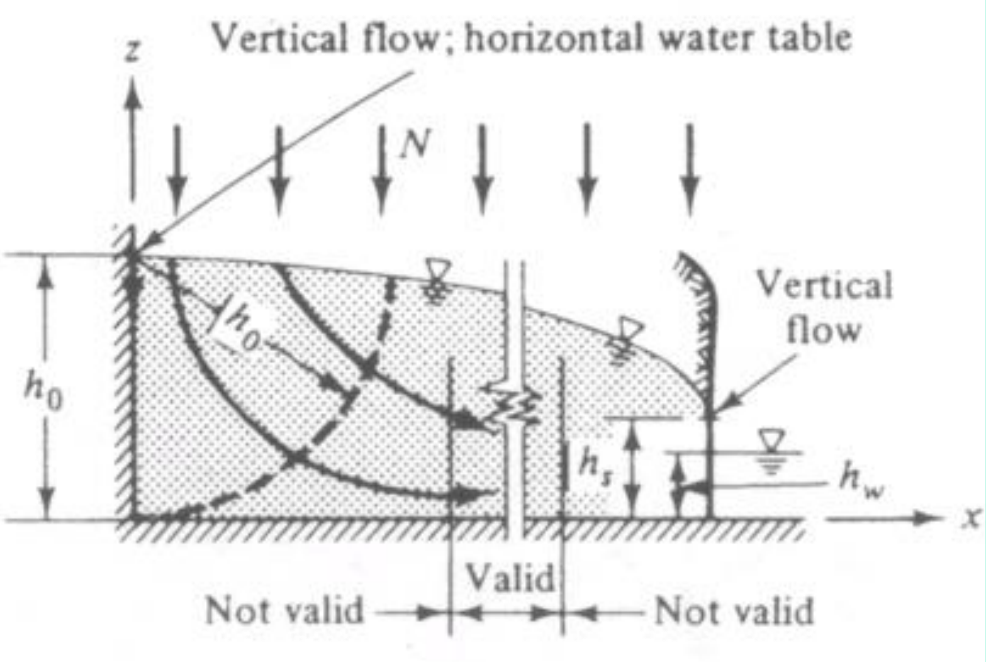
$$\Rightarrow \frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=\frac{S_y}{Kb}\frac{\partial h}{\partial t} \qquad (5)$$

(Note that (5) is similar to the 2-D flow in a confined aquifer, except that S, Storativity of a confined aquifer, is used instead of S_y)

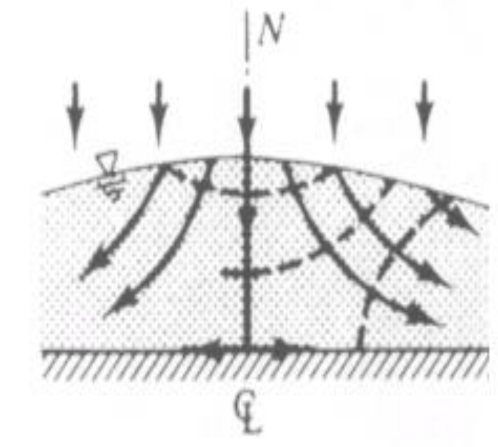
- Conditions when Dupuit assumption does not work
- Vertical flow is not negligible (Vertical impervious boundary; Crest of water table (or water divide); Seepage face
- Rule of thumb (Bear and Verruijt, 1987): Dupuit assumption is valid at distances from the downstream end larger than twice the average height of the flow domain. However, discharge calculated from Dupuit assumption is a satisfactory estimation for most cases

Examples of seepage face

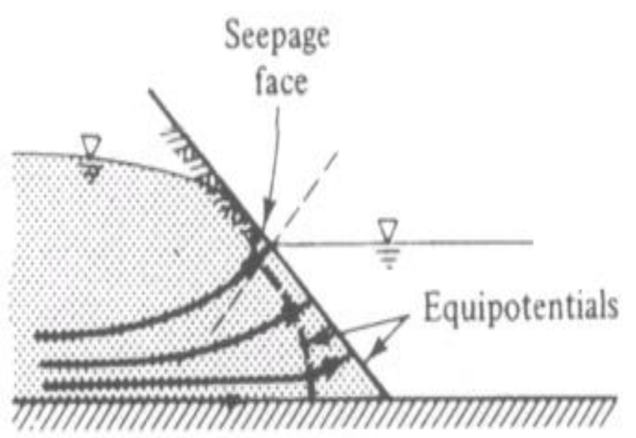




(a)



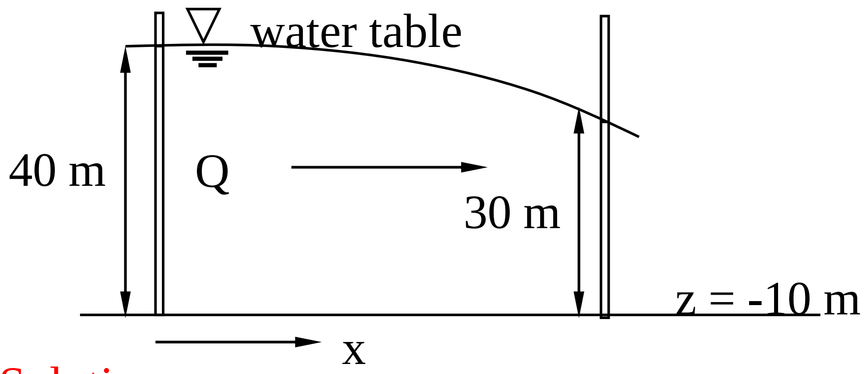
(b)



(c)

- Examples where Dupuit assumption is not valid

Example : (Problem 2.14, Bear and Verruijt, 1987)



(a) Determine Q if K = 18 m/d

(b) Repeat (a) if K = 30 m/d from x = 0 to x = 800 m, and K = 10 m/d for the remaining 400 m.

Solution :

(a) Because the flow field is steady-state, Q is a constant. Hence

$$Q = -Kh\frac{dh}{dx} = -\frac{K}{2}\frac{dh^2}{dx} = -\frac{18}{2}\frac{40^2 - 30^2}{1200} = 5.25\, m^2/d$$

(b) The hydraulic head at x = 800 must be continuous. Furthermore, Q is a constant because the flow field is steady-state. Hence

$$Q = \left(-Kh\frac{dh}{dx}\right)_1 = \left(-Kh\frac{dh}{dx}\right)_2 = -\frac{30}{2}\frac{h^2 - 40^2}{800} = -\frac{10}{2}\frac{30^2 - h^2}{400}$$

$$\Rightarrow h = 36.33\, m$$

$$\Rightarrow Q = 5.25\, m^2/d$$

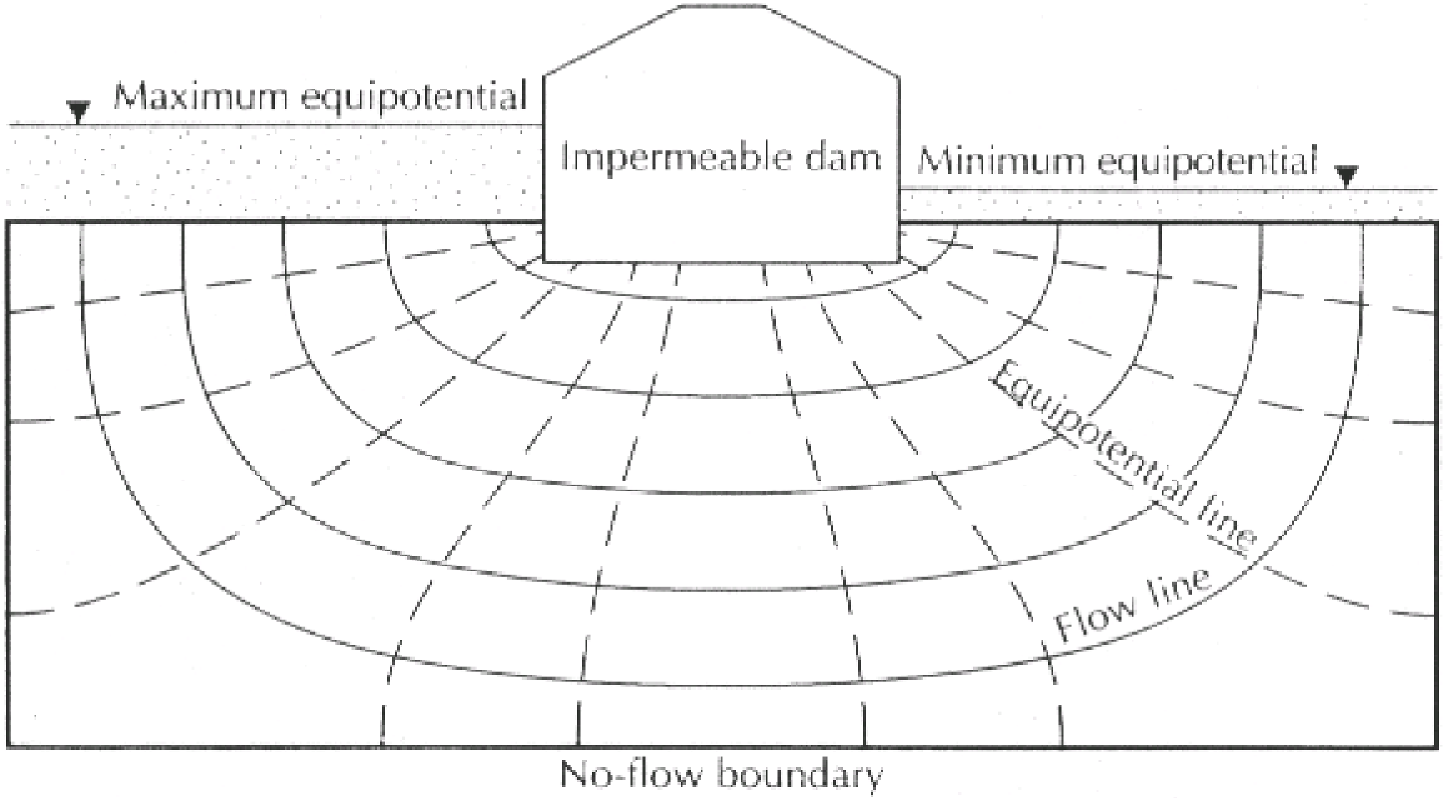
5. Flow net

- The 2D steady state Groundwater flow equation in isotropic and homogeneous porous medium can be expressed by Laplace's Equation:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

- Graphically, the equation can be represented by two sets of curves known as '*Equipotential line*' and '*flow lines*', that intersect at right angles. The combined representation of two sets of lines is called a flow net. With the help of a flow net, the groundwater flow problems can be analyzed.

- Equipotential line: A line on which values of hydraulic head are the same.
  - Potential of groundwater $\phi = \nabla h$ = mechanical energy (pressure energy + elevation energy) per unit mass of groundwater. Equipotential lines are always perpendicular to the direction of ∇h, no matter the isotropy of the medium
- Flow line (Fetter, 1994): An imaginary line that traces the path that a particle of groundwater would follow as it flows through an aquifer.
  - Flow lines will cross equipotential lines at right angles in an isotropic aquifer
  - Flow lines will cross the equipotential lines at an angle dictated by the degree of anisotropy and the orientation of ∇h to the hydraulic conductivity tensor ellipsoid
  - Flow lines are parallel to ∇h in isotropic media but not in anisotropic media



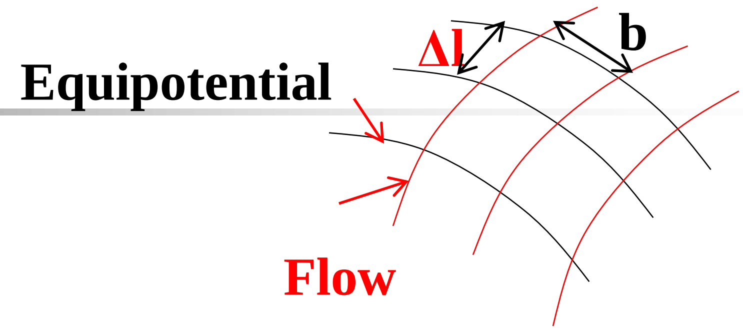
$$Q = \frac{KmH}{n}$$

Q : flow per unit width $[L^2/T]$

K : homogeneous/isotropic hydraulic conductivity [L/T]

m : # of stream tubes (flow tubes, i.e., area between two adjacent flow lines)

n : # of divisions of head in the flow net



- Darcy's equation: $v = ki$; where k is hydraulic conductivity (m/s) and i $(= \Delta h/\Delta l)$ is hydraulic gradient. The seepage flow q, through a cross sectional area A is computed as; $q = vA = kiA$.
- In the flow net case: for a single net $A = bX1 = b$; $q = kb\Delta h/\Delta l$, but $\Delta h = H/N_d$ where N_d is the number of equipotential drops; and H is the head difference between the initial and end section along the groundwater flow direction.
- The total discharge per unit width $Q = N_f(q) = N_f kbH/(N_d \Delta l)$; however if the flow net is drawn so that $b \approx \Delta l$, $Q = kHN_f/N_d$
    - Where N_f is the number of flow tubes.

Boundary conditions Vs flow lines

- Boundary conditions vs flow lines and equipotential lines
- **No-flow boundary** (Neumann): Adjacent flow lines are parallel to this boundary, and equipotential lines are perpendicular to this boundary
- **Constant-head boundary** (Dirichlet): This boundary represents an equipotential line and adjacent equipotential lines are parallel to this boundary. Flow lines will intersect the constant-head boundary at right angles
- **Water-table boundary**: the water table, in general, is neither a flow line nor an equipotential line. It is a line where head is known. If Dupuit assumption is valid, equipotential lines are vertical and flow lines are horizontal. If there is recharge or discharge across the water table, flow lines will be at an oblique angle to the water table.

Three BC's vs flow lines and equipotential lines

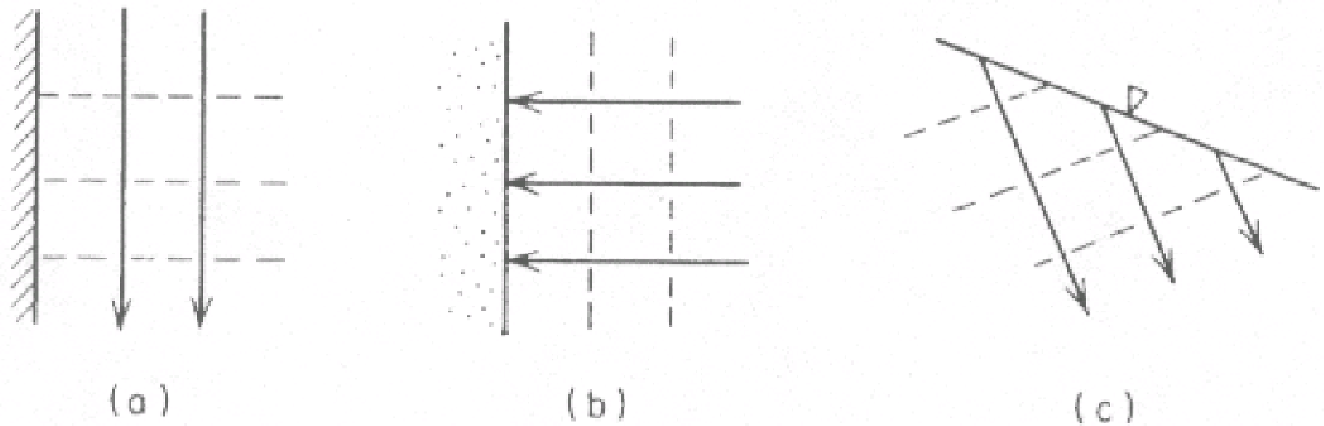


Figure 5.1 Groundwater flow in the vicinity of (a) an impermeable boundary, (b) a constant-head boundary, and (c) a water-table boundary.

(After Freeze and Cherry, 1979)

Flow nets for anisotropic media

- For isotropic soil the flow net is orthogonal; however the flow net in case of anisotropic soil is not orthogonal. Thus the two dimensional seepage flow equation is not a Laplace equation.
- As the permeability is different in the two directions. For example in horizontally stratified aquifers, the horizontal permeability is usually greater than the vertical. Thus the seepage flow equation in an isotropic soils will be:

$$k_x \frac{\partial^2 h}{\partial x^2} + k_y \frac{\partial^2 h}{\partial y^2} = 0$$

- However this equation can be modified to work as Laplace equation as:

$$\frac{k_x}{k_y}\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=0 \; let \; x_t = x\sqrt{k_y/k_x};$$

$$\frac{\partial^2 h}{\partial x_t{}^2}+\frac{\partial^2 h}{\partial y^2}=0$$

- For example if $k_x = 4k_y$; $x_t = x/2$; The section of the medium is transformed by halving the horizontal dimension. Draw the flow net for the transformed section then transfer the flow net back to the original section.

Steps:

1. Transform the coordinates according to a specific scaling
2. Construct a flow net for the transformed, isotropic medium
3. Invert the scaling ratio

The total discharge per unit width:

$Q = N_f kbH/(N_d \Delta l)$;

Where $k = (k_x k_y)^{1/2}$

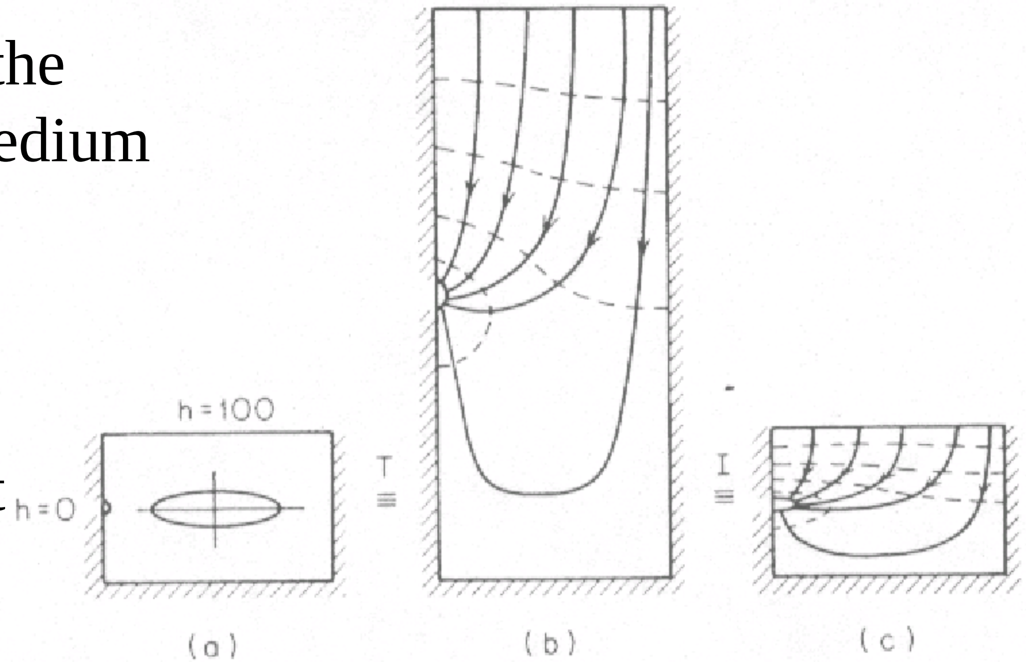


Figure 5.8 (a) Flow problem in a homogeneous anisotropic region with $\sqrt{K_x}/\sqrt{K_z} = 4$. (b) Flow net in the transformed isotropic section. (c) Flow net in the actual anisotropic section. *T*, transformation; *I*, inversion.

6. Approaches to groundwater flow analysis in fractured aquifers