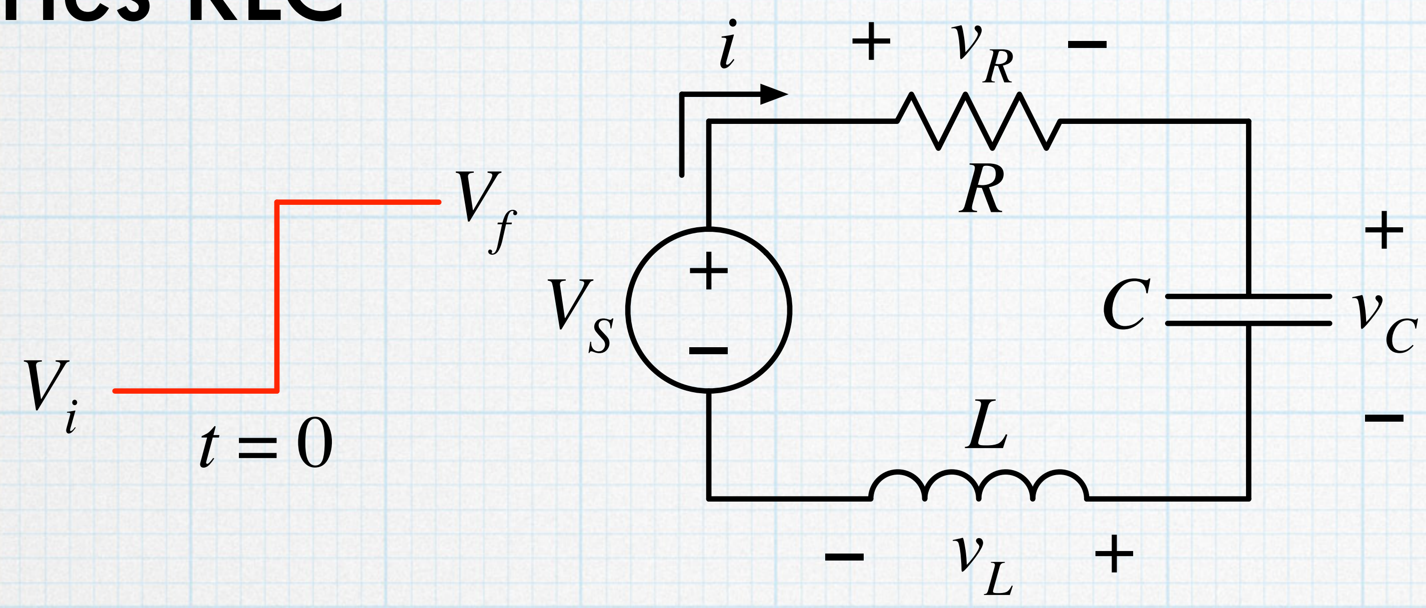
RLC transients

When there is a step change (or switching) in a circuit with capacitors and inductors together, a transient also occurs. With some differences:

- Energy stored in capacitors (electric fields) and inductors (magnetic fields) can trade back and forth during the transient, leading to possible "ringing" effects.
- The transient waveform can be quite different, depending on the exact relationship of the values of C, L, and R.
- The math is more involved.

Series RLC



$$V_S = v_R + v_C + v_L$$

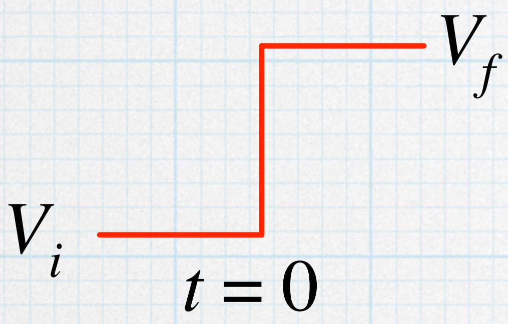
A step voltage source makes an instantaneous change from V_i to V_f at $t = 0$.

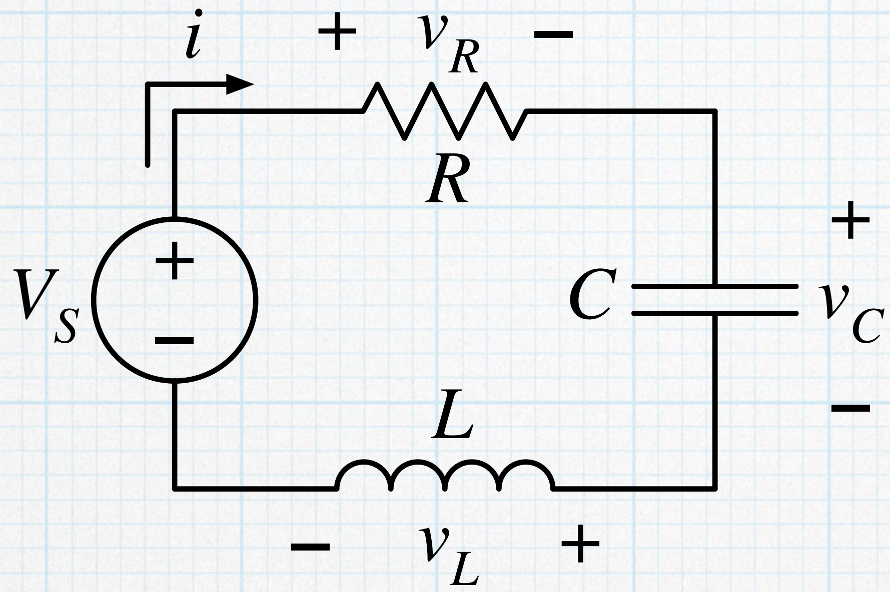
For $t < 0$: $i = 0$, $v_R = 0$, $v_L = 0$, and $v_C = V_i$.

When source voltage changes, some sort of transient will start. Presumably the transient will last for several time constants, eventually settling into the a final static state, where

For $t > m{\cdot}\tau$: $i = 0$, $v_R = 0$, $v_L = 0$, and $v_C = V_f$., where is t is the time constant (or constants!) of the system and m is the appropriate number of time constants needed for the system to settle.

Questions: What form does the transient take? What is τ? How big should m be?





$$V_S = v_R + v_C + v_L$$

$t > 0$: $v_C = V_I$, $v_R = v_L = 0$.

$t \geq 0$: $$V_S = v_R + v_C + v_L$$

$$V_f = iR + v_C + L\frac{di}{dt}$$

$$i = C\frac{dv_C}{dt}$$

$$V_f = RC\frac{dv_C}{dt} + v_C + LC\frac{d^2v_C}{dt^2}$$

$$\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{V_f}{LC}$$

A second-order differential equation in standard form.

2nd-order differential equations– review (or preview?)

$$\frac{d^2 f(x)}{dx^2} + a\frac{df(x)}{dx} + bf(x) = g(x)$$

$g(x)$ is a "forcing function"

The usual approach to solving second-order diff. eqs. is to split the solution into two functions:

$$f(x) = f_t(s) + f_s(x)$$

$f_t(x) \rightarrow$ transient solution (sometimes called the homogenous solution)

$f_s(x) \rightarrow$ steady-state solution (sometimes called the particular solution)

The steady-state solution, $f_s(x)$ is any function that you can find that is a solution to the full differential equation. Usually, it is found with a trial-and-error approch. The form of $g(x)$ will determine the form of $f_s(x)$.

Then $f_t(x)$ is the solution to the homogenous differential equation.

$$\frac{d^2 f_t(x)}{dx^2} + a\frac{df_t(x)}{dx} + bf_t(x) = 0$$

Usually, there are two functions that satisfy the homogenous equation, f_{t1} and f_{t2} (eg. $\sin x$ and $\cos x$ or e^{+x} and e^{-x}) and the so the complete transient function is a linear combination of the two, $f_t(x) = A{\cdot}f_{t1}(x) + B{\cdot}f_{t2}(x)$. Finally, the two constants A and B are determined using the initial conditions, $f(0)$ and $df(0)/dx$.

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{V_f}{LC}$$

Apply the standard approach the capacitor voltage equation.

$$v_c(t) = v_{ct}(t) + v_{cs}(t)$$

Since the forcing function is a constant, try setting $v_{cs}(t)$ to be a constant. Since we don't know what the constant value should be, we will call it V_1. Insert into the differential equation.

$$\frac{d^2 V_1}{dt^2} + \frac{R}{L}\frac{dV_1}{dt} + \frac{V_1}{LC} = \frac{V_f}{LC}$$

Since V_1 is a constant, the two derivative terms are zero, and we obtain the simple result:

$$V_1 = V_f.$$

Actually, we already knew this — we had used physical arguments to predict that the capacitor voltage would equal V_f at the end of the transient.

So $v_{cs}(t) = V_f$, and we turn our attention to the transient solution.

transient (homogenous) solution

$$\frac{d^2 v_{ct}}{dt^2} + \frac{R}{L}\frac{dv_{ct}}{dt} + \frac{v_{ct}}{LC} = 0$$

Guess: $v_{ct}(t) = Ae^{st}$

We need to determine s. From experience we expect *two* values of s that will give us the two functions that are solutions.

$$s^2\left(Ae^{st}\right) + \frac{R}{L}s\left(Ae^{st}\right) + \frac{1}{LC}\left(Ae^{st}\right) = 0$$

$$\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)\left(Ae^{st}\right) = 0$$

$Ae^{st} \neq 0$ so $\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$

Use the quadratic formula to find the (two!) values of s, giving two separate solutions

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$v_{ct1}(t) = e^{s_1 t}$ and $v_{ct2}(t) = e^{s_2 t}$

The general transient solution is a linear combination: $v_{ct}(t) = Ae^{s_1 t} + Be^{s_2 t}$

Initial conditions

We are getting nearer to obtaining the final form of the capacitor voltage

$$v_C(t) = v_{ct}(t) + v_{cs}(t) = Ae^{s_1 t} + Be^{s_2 t} + V_f$$

We need to determine the values for A and B, and do to that we use the initial conditions. From the expression above, it is obvious that A and B must both have units of volts. The initial conditions are given by evaluating $v_c(t)$ and $dv_c(t)/dt$ at $t = 0$, in the instant just after the source switched to V_f.

At $t = 0$, the capacitor is still at its initial voltage, because we know that the capacitor cannot change instantaneously.

$$v_C(0) = V_i = A + B + V_f$$

The derivative of the capacitor voltage is:

$$\left.\frac{dv_C(t)}{dt}\right|_{t=0} = s_1 A + s_2 B$$

But what is the value at $t = 0$? We need use some secondary arguments to determine the correct value.

Start by recalling that $i_C = C\dfrac{dv_C(t)}{dt}$,

and so $$\left.\frac{dv_C(t)}{dt}\right|_{t=0} = \frac{i_C(0)}{C}.$$

But what is the current at $t = 0$? Just before the voltage source switched, the capacitor current was zero. But the capacitor current can change instantaneously, so we don't know what it may have jumped to after the switch.

However, in the series circuit, $i_C(t) = i_L(t)$ — capacitor current must be identical to the inductor current at all times. We know that the inductor current *cannot* change abruptly, and the inductor was zero before the switch. So just after the switch, $i_L(0) = 0$ — meaning that $i_C(0) = 0$.

$$\left.\frac{dv_C(t)}{dt}\right|_{t=0} = 0 = s_1 A + s_2 B$$

Using the two initial condition expression, we can solve for A and B.

$$A = \frac{V_f - V_i}{\frac{s_1}{s_2} - 1} \qquad B = \frac{V_f - V_i}{\frac{s_2}{s_1} - 1}$$

roots of characteristic equation

Transient behavior depends on the values of s_1 and s_2.

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Rename things slightly: $R/2L = \alpha$ and $1/LC = \omega_o$.

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

α is the *damping factor* or *decay constant* [s^{-1}]

ω_o is the *resonant frequency* or *undamped natural frequency* [radian/s].

Overdamped response

When $\dfrac{R}{2L} > \dfrac{1}{\sqrt{LC}}$, s_1 and s_2 will be both be real and negative.

The transient will consist of two decaying exponentials.

$$v_C(t) = Ae^{s_1 t} + Be^{s_2 t} + V_f$$

$$= \left(V_f - V_i\right)\left[\frac{e^{s_1 t}}{\frac{s_1}{s_2} - 1} + \frac{e^{s_2 t}}{\frac{s_2}{s_1} - 1}\right] + V_f$$

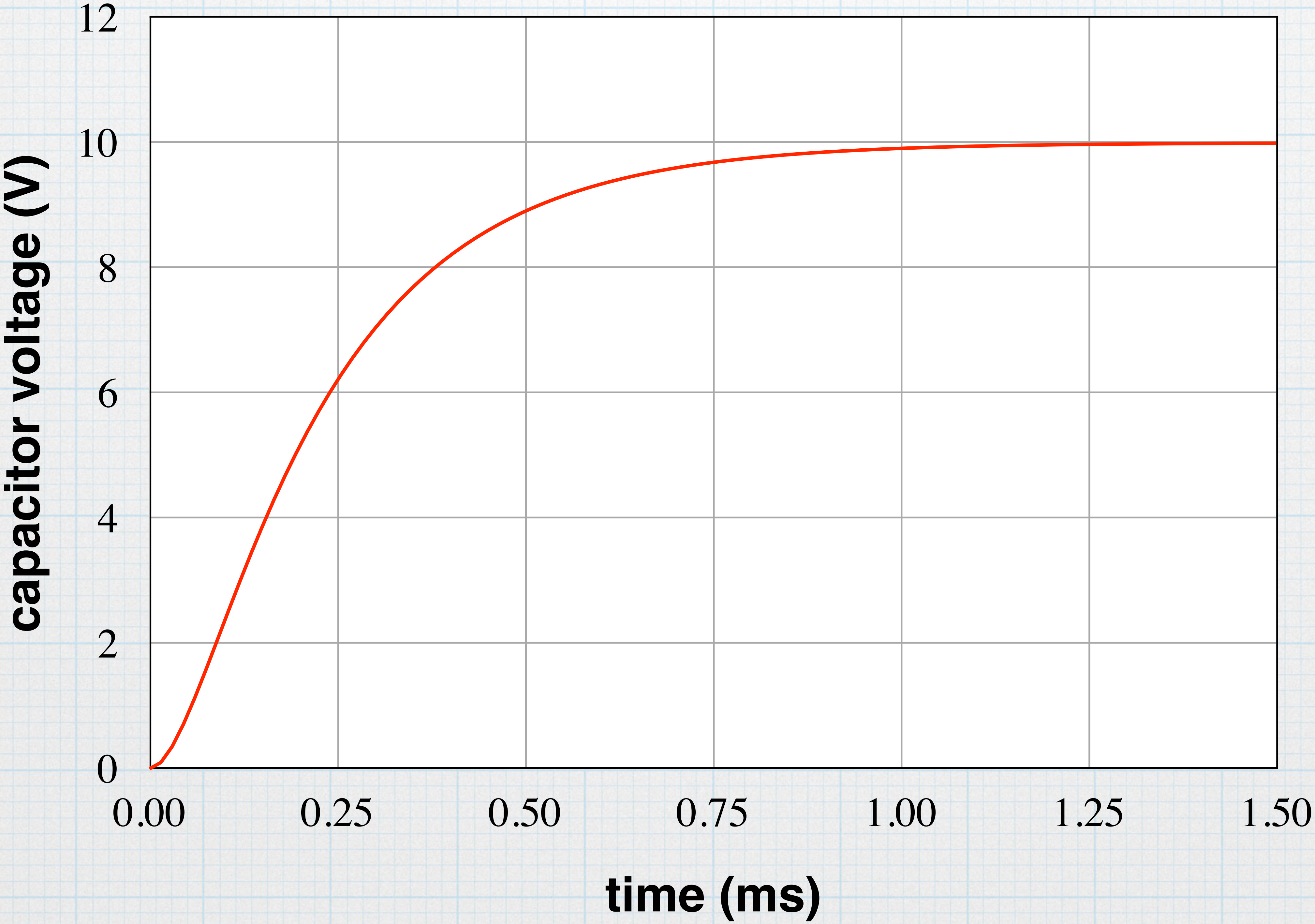
A one-way trip from V_i to V_f.

For $V_i = 0$ V, $V_f = 10$ V, $R = 250\ \Omega$, $L = 10$ mH, and $C = 1\ \mu$F:

$s_1 = 5000\ s^{-1}$ ($\tau_1 = 0.2$ ms) and $s_2 = 20{,}000\ s^{-1}$ ($\tau_2 = 50\ \mu$s), $A = -13.33$ V and $B = 3.33$ V.

$$v_C(t) = -\ 13.33\ \mathrm{V}\exp\left(-\frac{t}{0.2\ \mathrm{ms}}\right) + 3.33\ \mathrm{V}\exp\left(-\frac{t}{50\ \mu\mathrm{s}}\right) + 10\ \mathrm{V}$$

Over-damped response: $V_i = 0$ V, $V_f = 10$ V, $R = 250\ \Omega$, $L = 10$ mH, and $C = 1\ \mu$F.



Underdamped response

When $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$, we crash headfirst into a mathematical difficulty when calculating the roots — the expression inside square root becomes negative, meaning that s_1 and s_2 will be complex numbers.

Alert! Alert! It is time to deal with complex numbers. Unless you are already very comfortable in working with complex number math, stop reading here and go read the notes on complex numbers. In particular, focus on Euler's relation and the connection between complex numbers and sinusoids. After reading through the notes, work complex math practice problems until you are proficient with doing calculations. We could probably fake our way through this one section on underdamped response without having a good understanding complex numbers, but soon we will move to AC analysis where complex numbers will the central feature in how we handle problems. Now is the time to learn — or relearn — complex numbers.

With $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$, the roots can be written in complex form:

$$s_1 = -\frac{R}{2L} + j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \qquad s_2 = -\frac{R}{2L} - j\sqrt{\left(\frac{1}{LC} - \frac{R}{2L}\right)^2}$$

We see that the two roots are complex conjugates, a result that has important implications in the math to come. We can introduce some short-hand to simplify using the expressions.

$$s_1 = \sigma + j\omega_d \qquad s_2 = \sigma - j\omega_d$$

where $\sigma = -\frac{R}{2L}$ is the decay constant or damping factor. It will determine the rate at which the transient response attenuates away and

$\omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$ is the *damped oscillation frequency*. It give the angular of the oscillations that occur.

Inserting the complex conjugate roots into the capacitor voltage expression:

$$v_C(t) = Ae^{s_1 t} + Be^{s_2 t} + V_f$$

$$= A \exp\left[\left(\sigma + j\omega_d\right)t\right] + B \exp\left[\left(\sigma - j\omega_d\right)t\right] + V_f$$

Re-arranging a bit:

$$v_C(t) = A \exp(\sigma t) \exp\left(j\omega_d t\right) + B \exp(\sigma t) \exp\left(-j\omega_d t\right) + V_f$$

$$= \exp(\sigma t)\left[A \exp\left(j\omega_d t\right) + B \exp\left(-j\omega_d t\right)\right] + V_f$$

Using Euler's relation, we can re-write the complex exponentials:

$$\exp\left(j\omega_d t\right) = \cos\omega_d t + j\sin\omega_d t \qquad \exp\left(-j\omega_d t\right) = \cos\omega_d t - j\sin\omega_d t$$

$$v_C(t) = \exp(\sigma t)\left[A\left(\cos\omega_d t + j\sin\omega_d t\right) + B\left(\cos\omega_d t - j\sin\omega_d t\right)\right] + V_f$$

$$= \exp(\sigma t)\left[(A + B)\cos\omega_d t + j(A - B)\sin\omega_d t\right] + V_f$$

We finish by expressing the A and B coefficients in terms of σ and ω_d. The coefficient situation looks to be a bit of a mess, but the end result is surprisingly simple.

$$A + B = \frac{V_f - V_i}{\frac{s_1}{s_2} - 1} + \frac{V_f - V_i}{\frac{s_2}{s_1} - 1}$$

$$= \left(V_f - V_i\right)\left[\frac{s_2}{s_1 - s_2} + \frac{s_1}{s_2 - s_1}\right]$$

$$= \left(V_f - V_i\right)\left[\frac{s_2}{s_1 - s_2} - \frac{s_1}{s_1 - s_2}\right]$$

$$= -\left(V_f - V_i\right)$$

$$j(A - B) = j\frac{V_f - V_i}{\frac{s_1}{s_2} - 1} - \frac{V_f - V_i}{\frac{s_2}{s_1} - 1}$$

$$= j\left(V_f - V_i\right)\left[\frac{s_2}{s_1 - s_2} - \frac{s_1}{s_2 - s_1}\right]$$

$$= j\left(V_f - V_i\right)\left[\frac{s_1 + s_2}{s_1 - s_2}\right]$$

$$= j\left(V_f - V_i\right)\left[\frac{(\sigma + j\omega_d) + (\sigma - j\omega_d)}{(\sigma + j\omega_d) - (\sigma - j\omega_d)}\right]$$

$$= j\left(V_f - V_i\right)\left[\frac{2\sigma}{j2\omega_d}\right]$$

$$= \left(V_f - V_i\right)\left[\frac{\sigma}{\omega_d}\right]$$

Inserting the coefficients and putting it all together:

$$\boxed{v_C(t) = V_f - \left(V_f - V_i\right)\left[\exp(\sigma t)\left(\cos\omega_d t - \frac{\sigma}{\omega_d}\sin\omega_d t\right)\right]}$$

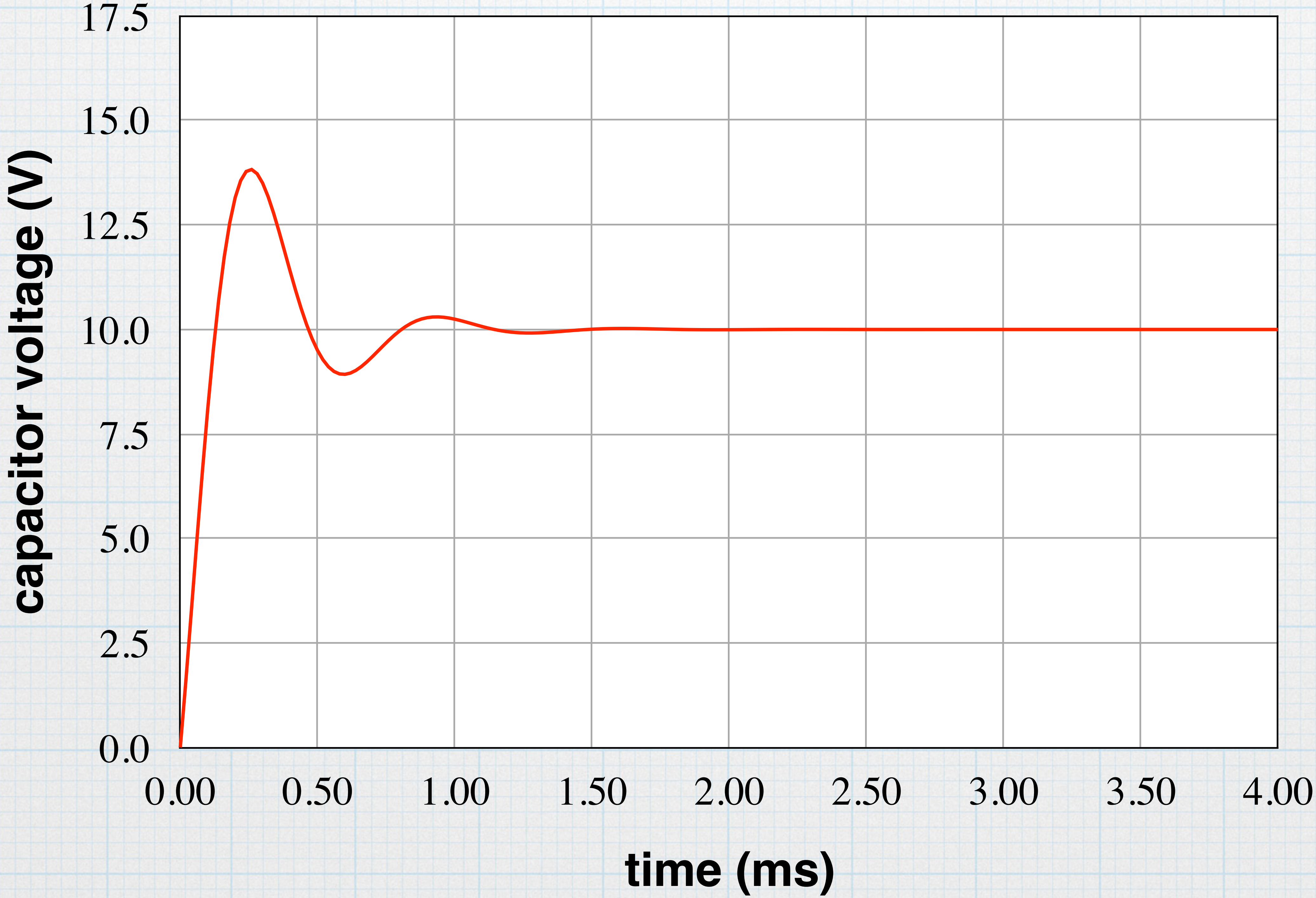
We see oscillating behavior due to the sinusoidal terms. Note $\sigma < 0$, and so the exponential is decaying. This is then a *damped* sinusoid, meaning that the voltage will oscillate, but the amplitude of the oscillation will decrease exponentially with time — disappearing completely after about 5 time constants, where the time constant is $\tau = |\sigma|^{-1}$.

However, ignoring the surprising oscillations, the equation still describes a basic transient. At $t = 0$, the factor inside the bracket reduces to 1, and the starting voltage is $v_C(0) = V_f - \left(V_f - V_i\right) = V_i$, as expected. After a sufficiently long time the exponential will decay away, with the bracketed term heading to zero, leaving $v_C(t \to \infty) = V_f$, also as expected.

The oscillations during the transient are often referred as "ringing". The same phenomenon occurs in mechanical systems (or any second-order system), with percussion instruments being audible examples. Striking a bell or cymbal or drumhead causes an under-damped ringing transient that lasts for a time determined by the mechanical properties of the system.

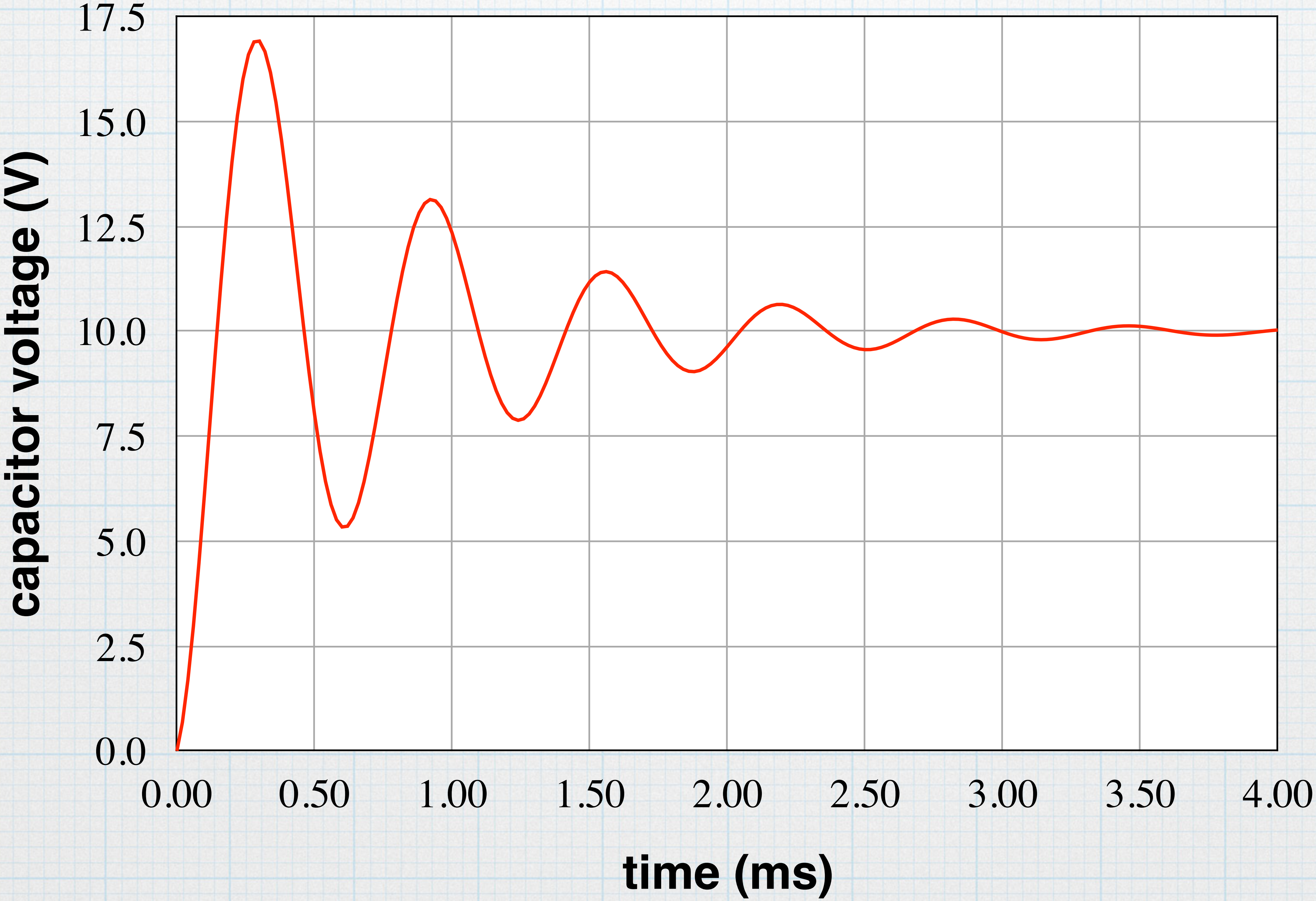
Under-damped response: $V_i = 0$ V, $V_f = 10$ V, $R = 75\ \Omega$, $L = 10$ mH, and $C = 1\ \mu$F.

$$\sigma = -\frac{R}{2L} = -3750\,\mathrm{s}^{-1}\ (\tau = 0.267\,\mathrm{ms}) \qquad \omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = 9270\,\mathrm{s}^{-1}$$



Under-damped response: $V_i = 0$ V, $V_f = 10$ V, $R = 25\ \Omega$, $L = 10$ mH, and $C = 1\ \mu$F.

$$\sigma = -\frac{R}{2L} = -1250\ \mathrm{s}^{-1}\ (\tau = 0.8\ \mathrm{ms}) \qquad \omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = 9920\ \mathrm{s}^{-1}$$



Observations of underdamped response

Critically damped response

If $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$ ($\alpha = \omega_o$) then $s_1 = s_2$ (double root).

Note: This will almost *never* happen. It will be the wildest fluke if the components have *exactly* the correct ratios to meet the above requirement. For the most part, critical damping is only of academic interest.

$$v_C(t) = Ae^{s_1 t} + Be^{s_1 t} + V_f = Ce^{s_1 t} + V_f$$

$$= C \exp\left(-\frac{t}{2L/R}\right) + V_f$$

This causes a bit of a problem, because we are left with only one term in the general solution, and hence only one coefficient – not enough to satisfy the initial conditions.

This suggests that there must be another solution lurking around in the math. In the special circumstances for the critically damped case, the homogeneous equation can be written:

$$\frac{d^2 v_{tr}}{dt^2} + 2\alpha \frac{dv_{tr}}{dt} + \alpha^2 v_{tr} = 0$$

$$\frac{d^2 v_{tr}}{dt^2} + 2\alpha \frac{dv_{tr}}{dt} + \alpha^2 v_{tr} = 0$$

$$\frac{d}{dt}\left[\frac{dv_{tr}}{dt} + \alpha v_{tr}\right] + \alpha\left[\frac{dv_{tr}}{dt} + \alpha v_{tr}\right] = 0$$

$$\frac{dy}{dt} + \alpha y = 0 \qquad \text{where} \qquad y = \frac{dv_{tr}}{dt} + \alpha v_{tr}$$

$$\downarrow$$

$$y = Ae^{-\alpha t} \quad \longrightarrow \quad Ae^{-\alpha t} = \frac{dv_{tr}}{dt} + \alpha v_{tr}$$

$$A = e^{\alpha t}\frac{dv_{tr}}{dt} + \alpha e^{\alpha t} v_{tr}$$

$$A = \frac{d}{dt}\left[e^{\alpha t} v_{tr}\right]$$

$$At + B = e^{\alpha t} v_{tr}$$

$$\boxed{v_{tr}(t) = (At + B)\, e^{-\alpha t}}$$

Now there are two constants.

$$v_C(t) = (At + B)\, e^{-\alpha t} + V_f$$

Also a fast trip from V_i to V_f.

Use initial conditions to find A and B.

$$v_c(0) = V_i = B + V_f$$

$$\left.\frac{dv_C}{dt}\right|_{t=0} = 0 = A - \alpha B$$

solve to give:

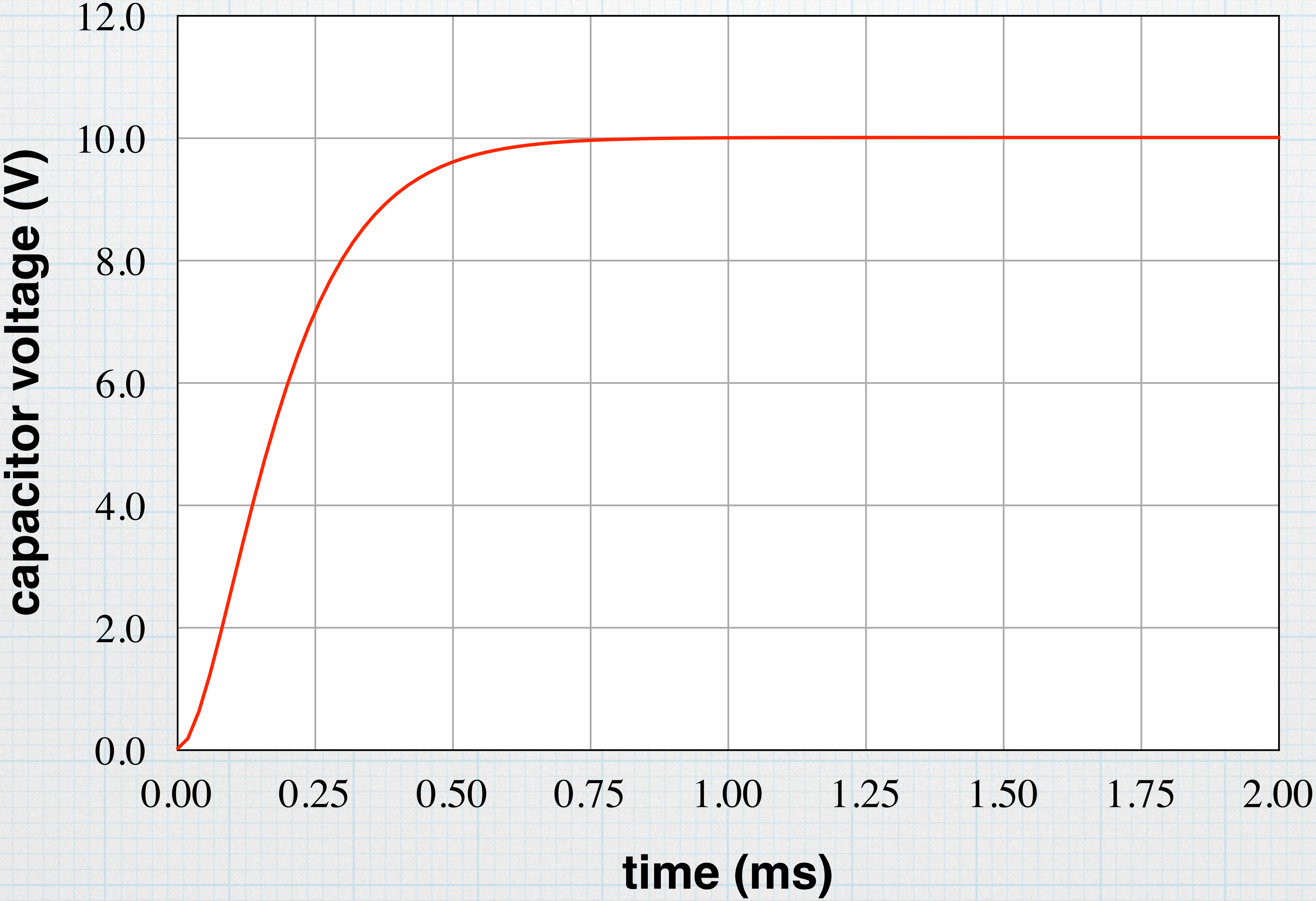
$$A = \alpha\left(V_i - V_f\right)$$

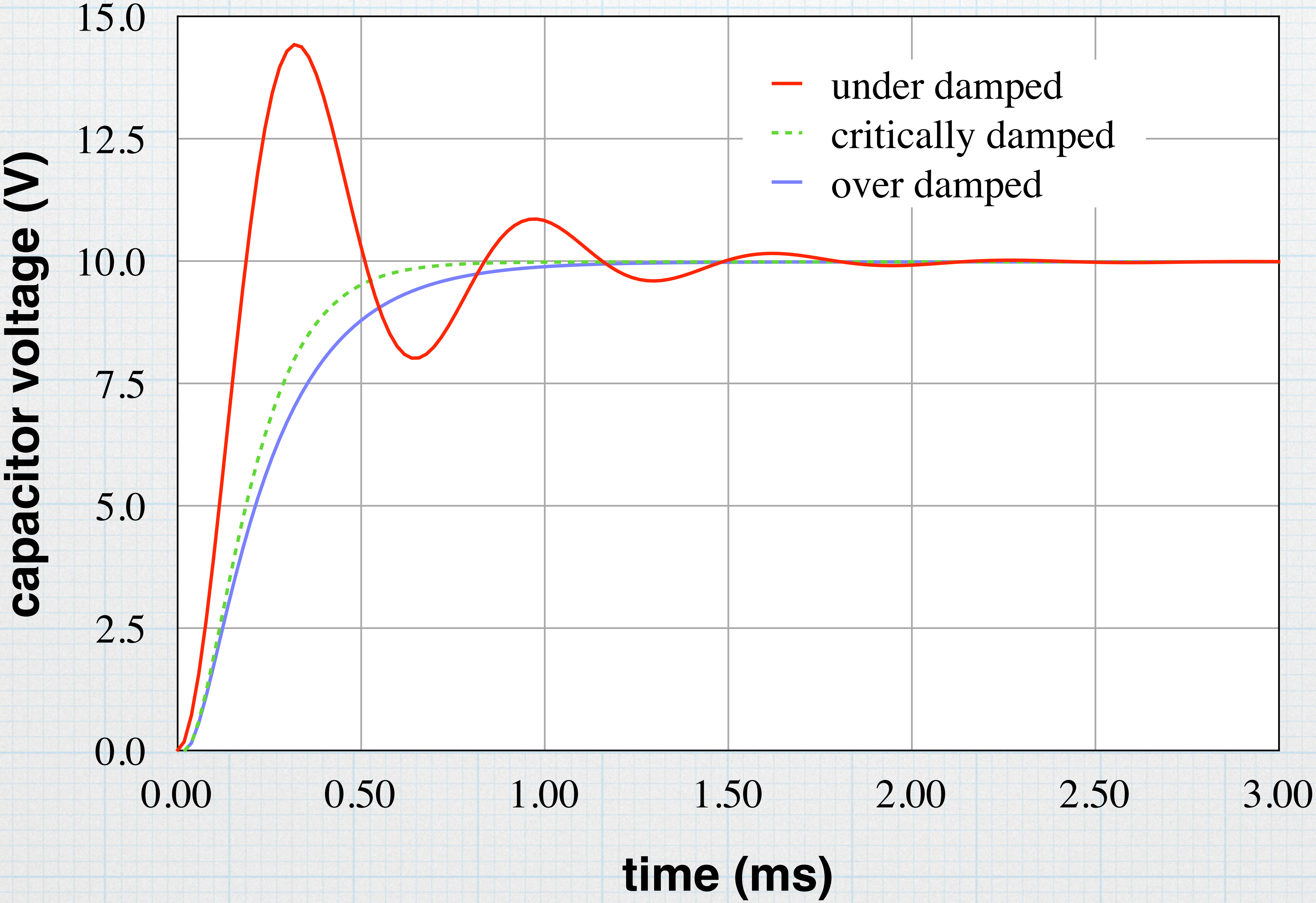
$$B = \left(V_i - V_f\right)$$

$$v_C(t) = \left(V_i - V_f\right)(1 + \alpha t)\, e^{-\alpha t} + V_f$$

Under-damped response: $V_i = 0$ V, $V_f = 10$ V, $R = 200\ \Omega$, $L = 10$ mH, and $C = 1\ \mu$F.

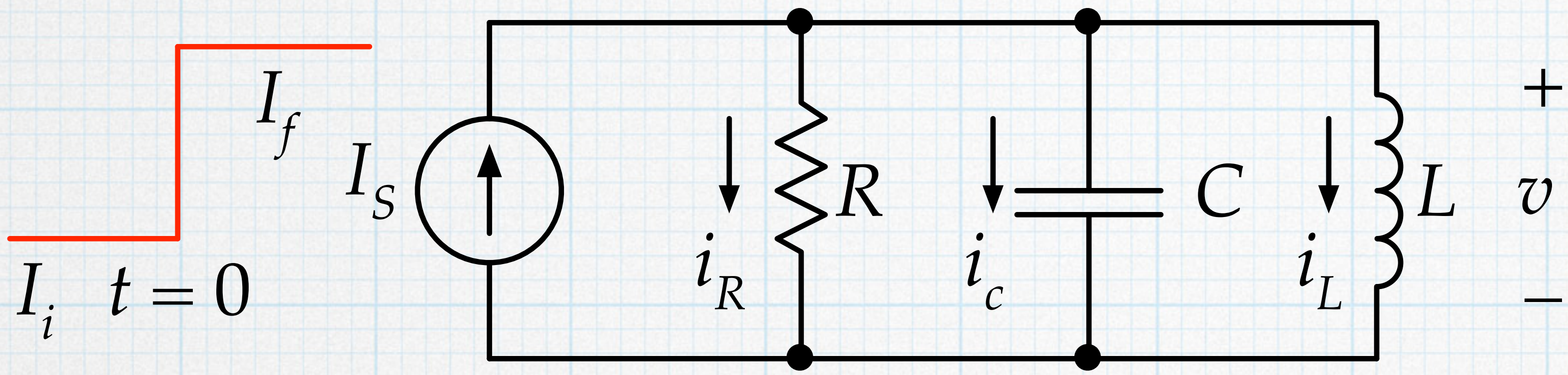
$$\frac{R}{2L} = 10{,}000\ \mathrm{s}^{-1}$$





under damped
critically damped
over damped
capacitor voltage (V)
time (ms)
15.0
12.5
10.0
7.5
5.0
2.5
0.0
0.00
0.50
1.00
1.50
2.00
2.50
3.00

parallel RLC



$t < 0$:	$t >> \tau$:
$v = 0.$	$v = 0.$
$i_R = 0.$	$i_R = 0.$
$i_C = 0.$	$i_C = 0.$
$i_L = I_i.$	$i_L = I_f.$

Current source makes an abrupt change from I_i to I_f at t = 0.

$$I_S = i_R + i_C + i_L$$

$$= \frac{v}{R} + C\frac{dv}{dt} + i_L \qquad\qquad v = L\frac{di_L}{dt}$$

$t > 0$:

$$I_f = \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} + i_L$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I_f}{LC}$$

initial conditions ($t = 0$)

$$i_L(0) = I_i$$

$$\left.\frac{di_L(t)}{dt}\right|_{t=0} = \frac{v_L(0)}{L} = \frac{v_C(0)}{L} = 0$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I_f}{LC}$$

This has exactly the same form as the series case, with inductor current replacing capacitor voltage. The steady-state function will be $i_{ss} = I_f$, and the transient function will have the general form:

$$i_{tr}(t) = Ae^{s_1 t} + Be^{s_2 t}$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \quad = -\alpha + \sqrt{\alpha^2 - \omega_o^2}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \quad = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

$\alpha = \frac{1}{2RC}$ damping factor - note the difference from series case

$\omega_o = \frac{1}{\sqrt{LC}}$ resonant frequency

We will obtain the exact same set of results for the parallel case:

$$\frac{1}{2RC} > \frac{1}{\sqrt{LC}} \qquad (\alpha > \omega_o)$$

overdamped – both roots are real and negative.

$$i_L(t) = \left(I_i - I_f\right)\left(\frac{e^{s_1 t}}{1 - \frac{s_1}{s_1}} + \frac{e^{s_2 t}}{1 - \frac{s_2}{s_1}}\right) + I_f$$

$$\frac{1}{2RC} < \frac{1}{\sqrt{LC}} \qquad (\alpha < \omega_o)$$

underdamped – roots are complex conjugates.

$$i_L(t) = \left(I_i - I_f\right) e^{-\alpha t}\left[\cos\omega_d t + \frac{\alpha}{\omega_d}\sin\omega_d t\right] + I_f \qquad \omega_d = \sqrt{\omega_o^2 - \alpha^2}$$

$$\frac{1}{2RC} = \frac{1}{\sqrt{LC}} \qquad (\alpha = \omega_o)$$

critically damped – repeated root.

$$i_L(t) = \left(I_i - I_f\right)\left(1 + \alpha t\right) e^{-\alpha t} + I_f$$

(Virtually impossible to have critical damping.)