Chapter 4: Optimum Receivers for Additive Gaussian Noise Channel

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Goals

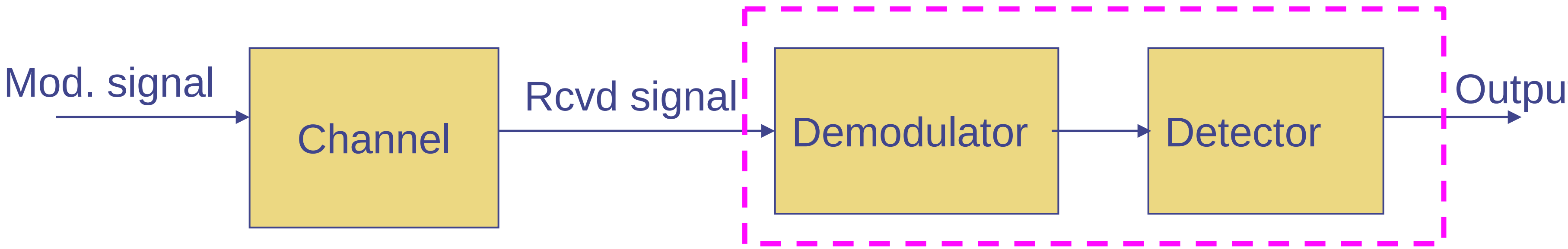
- Design & performance characteristics of optimum receiver
  - Various modulation techniques
  - AWGN channel
- Optimum: Minimize the probability of making errors

Overview

- Optimum receivers
  - Correlation demodulator
  - Matched filter demodulator
  - Optimal detector
- Performance of optimum receiver (memoryless modulation)
- Comparison of digital modulation methods

Optimum Receivers for AWGN Channel

- Consider the following receiver configuration
- Assume the channel does not introduce any changes or disturbances to the modulated signal



Optimum Receivers for AWGN Channel

- The data can be recovered on a component-by-component basis taking ***inner product*** of ***received signal*** and M basis functions, such that

$$x_k = \int_0^T x(t)\, f_k(t)\, dt \qquad \text{where } x(t) \text{ is the modulated waveform}$$

- This is the ***correlative*** demodulation

Optimum Receivers …

- The above integral can also be implemented by noting that

$$\int_0^T x(t)\, f_k(t)\, dt = x(t) * f_k(T-t)\Big|_{t=T}$$

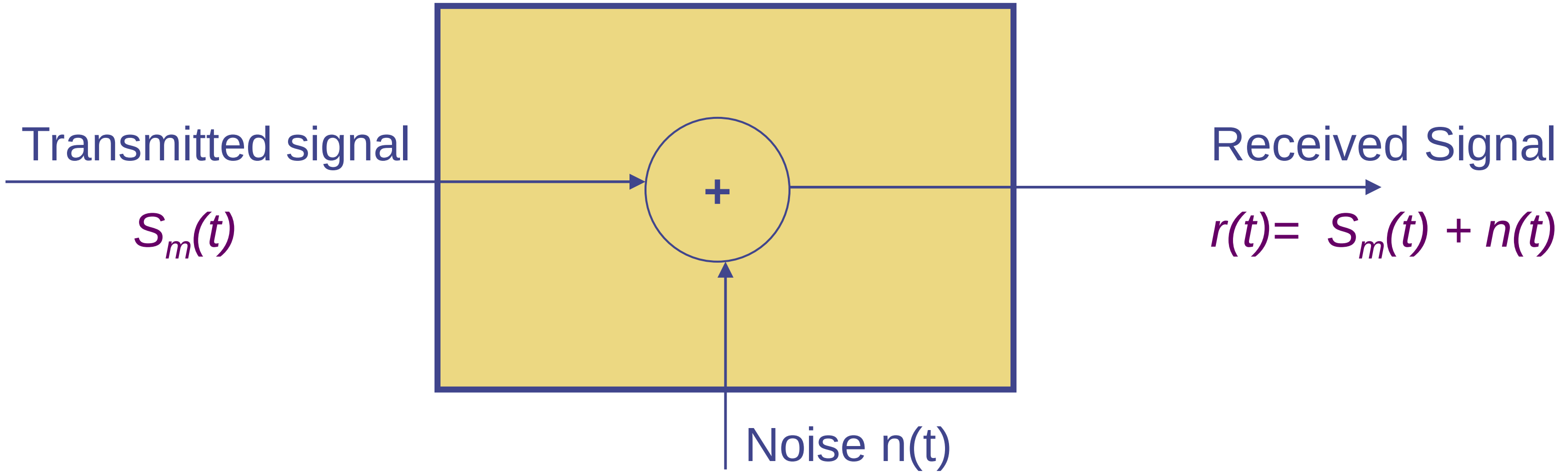
- The component of the modulated waveform $x(t)$ along the ***k^{th} basis function*** is the convolution of the waveform $x(t)$ with a filter whose impulse response is $f_k(T-t)$ at the output sample time T
- This is the ***matched-filter*** demodulation

Optimum Receivers …

- The above two methods can accurately recover the original components of the modulating signal if there is no noise on the channel
- In reality, the received signal at the input of the demodulator is corrupted by noise
- We consider the case where the channel noise is additive White Gaussian **(AWGN)**

Optimum Receivers …

- A channel model for the received signal over an AWGN channel is depicted below



$$r(t) = S_m(t) + n(t) \qquad 0 \le t \le T$$

- $n(t)$ is the sample function of the AWGN process with power spectral density

$$\Phi_{nn}(\omega) = \tfrac{1}{2} N_0 \text{ W/Hz}$$

Optimum Receivers …

- Objective: *Based on the observation of r(t) over the signal interval, design a receiver that is optimum in the sense that it minimizes the probability of error*
- For the receiver configuration shown in the first block diagram, the reception process may be divided into two components; namely, ***signal demodulation*** and ***detection***
- Signal demodulation: Converts the received waveform into an N-dimensional vector
  $\mathbf{r} = [\ r_1,\ r_2,\ r_3,\ \ldots\ldots\ r_N]$
- Detector: Decides which of the M possible signal waveforms are transmitted based on the vector **r**

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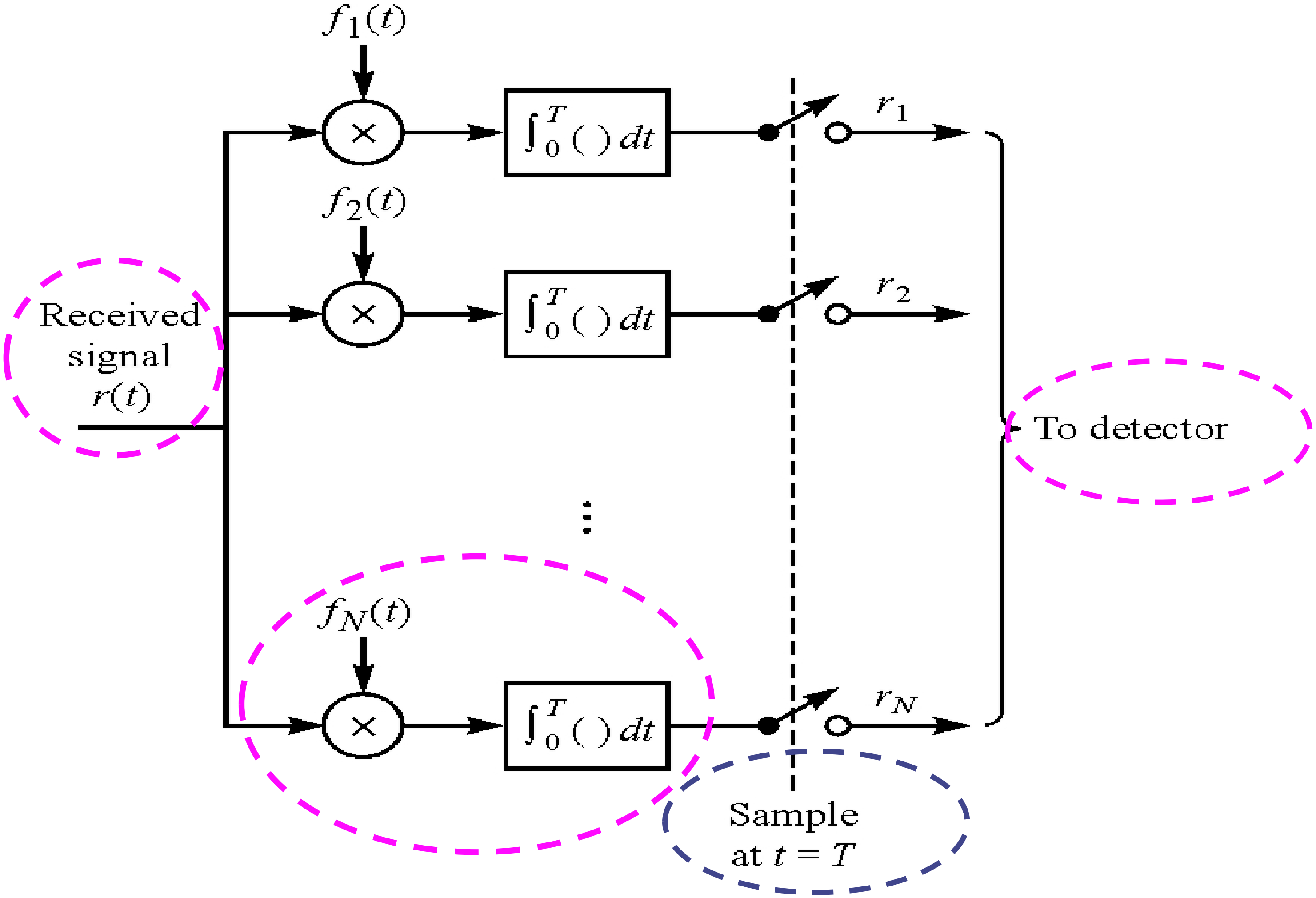
Optimum Receivers - Correlation Demodulator

- Correlation demodulator: Decomposes *r(t)* into N-dimensional vectors
- The signal and noise are expanded into a series of linearly weighted *orthonormal basis functions* $\{f_n(t)\}$ which spans the signal space so that all possible members of the signal set $\{S_m(t), 1 \leq m \leq M\}$ can be represented
- The basis functions ***do not span*** the noise space
- However, the noise terms that fall outside the signal space are irrelevant to the detection of the required signal *(see text for proof)*

Correlation Demodulator …



Correlation-type demodulator

Correlation Demodulator …

- The N correlators essentially compute the projection of $r(t)$ onto the N basis functions $\{f_n(t)\}$ such that

$$\int_0^T r(t)f_k(t)dt = \int_0^T [S_m(t)+n(t)]\, f_k(t)dt$$

$$r_k = s_{mk} + n_k \qquad k = 1,2,\ldots\ldots.N \quad \textbf{Where}$$

$$s_{mk} = \int_0^T S_m(t)f_k(t)dt \qquad k = 1,2,\ldots.N \; and$$

$$n_k = \int_0^T n(t)f_k(t)dt \qquad k = 1,2,\ldots\ldots.N$$

- The signal is now a vector $\mathbf{s}_{mk} = [\, s_{m1},\ s_{m2},\ s_{m3},\ \ldots\ldots\ s_{mN}]$ whose values depend on which of the M possible signals was transmitted

Correlation Demodulator …

- The components $\{n_k\}$ are random variables that arise from the additive white Gaussian noise
- In the interval 0 ≤ t ≤ T

$$r(t) = \sum_{k=1}^{N} s_{mk} f_k(t) + \sum_{k=1}^{N} n_k(t) f_k(t) + n'(t) = \sum_{k=1}^{N} r_k f_k(t) + n'(t)$$

$$\text{Where} \quad n'(t) = n(t) - \sum_{k=1}^{N} n_k(t) f_k(t)$$

- n'(t) is a zero-mean Gaussian noise process that represents the difference between original noise process $n(t)$ and part corresponding to projection of $n(t)$ onto $\{f_n(t)\}$

Correlation Demodulator …

- Recall that *n’(t)* is irrelevant to the decision as to which signal is transmitted and thus the decision can be based on the correlator output signal and noise only
  - I.e $r_k = s_{mk} + n_k$
- Note that the signal components are deterministic and the noise components are Gaussian with zero-mean such that

$$E\{n_k\} = \int_0^t E[n(t)]\, f_k(t)\, dt = 0 \qquad \textit{for all } n$$

- And the covariances are

$$E\{n_k n_m\} = \int_0^T \int_0^T E[n(t)n(\tau)]\, f_k(t)\, f_m(t)\, dt d\tau$$

$$= \frac{1}{2} N_0 \int_0^T \int_0^T \delta(t-\tau) f_k(t)\, f_m(t)\, dt d\tau = \frac{1}{2} N_0 \delta_{mk}$$

Correlation Demodulator …

- Thus, $\{n_k\}$ are zero-mean, uncorrelated Gaussian random variables (also independent) with common variance

  $\sigma_n^2 = ½ N_0$

- Further, $\{r_k\}$ is also Gaussian with mean s_{mk} and the same variance $\sigma_r^2 = \sigma_n^2 = ½ N_0$

- The output $\{r_k\}$ conditioned on the m[th] signal being transmitted are also statistically independent random variables with probability density function given by

$$P\{\mathbf{r}|\mathbf{s_m}\} = \prod_{k=1}^{N} P\{r_k|s_{mk}\} \qquad m = 1,2,\ldots\ldots N$$

Correlation Demodulator …

- Where

$$P\{r_k | s_{mk}\} = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(r_k - s_{mk})^2}{N_0} \right]; \quad k = 1,2,\ldots\ldots N$$

- So that the joint conditional PDF is

$$P\{\mathbf{r} | \mathbf{s_m}\} = \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\sum_{k=1}^{N} \frac{(r_k - s_{mk})^2}{N_0} \right]$$

- Example: Consider an M-ary baseband PAM signal set in which the basic pulse shape is rectangular

  g(t) = a *for* $0 \le t \le T$ *and zero otherwise*

  PAM signal set is one dimensional (N=1) and thus there is only one basis function, given by

Correlation Demodulator …

$$f(t) = \frac{1}{\sqrt{a^2 T}} g(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

- The output of the correlator type demodulator is

$$r = \int_0^T r(t) f(t)\, dt = \frac{1}{\sqrt{T}} \int_0^T r(t)\, dt$$

$$= \frac{1}{\sqrt{T}} \int [s(t) + n(t)]\, dt = S_m + n \quad where\ E[n] = 0\ and\ \sigma_n = \frac{N_0}{2}$$

- The probability density function of the sampled output is thus given by

$$P(r|s_m) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(r - s_m)^2}{N_0} \right]$$

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  - Matched filter demodulator
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Matched Filter Demodulator

- Here we employ a bank of N-linear filters instead of the N correlators to generate $\{r_k\}$
- The impulse response of these filters is matched to the basis functions such that

$$h_k(t) = \begin{cases} f_k(T-t) & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

- The output of these filters are given by

$$y_k(t) = \int_0^t r(\tau) h_k(t-\tau)\, d\tau = \int_0^t r(\tau) f_k(T-t+\tau)\, d\tau \qquad k = 1,2,.....N$$

Matched Filter Demodulator …

- Sampling the output of the filters at t = T

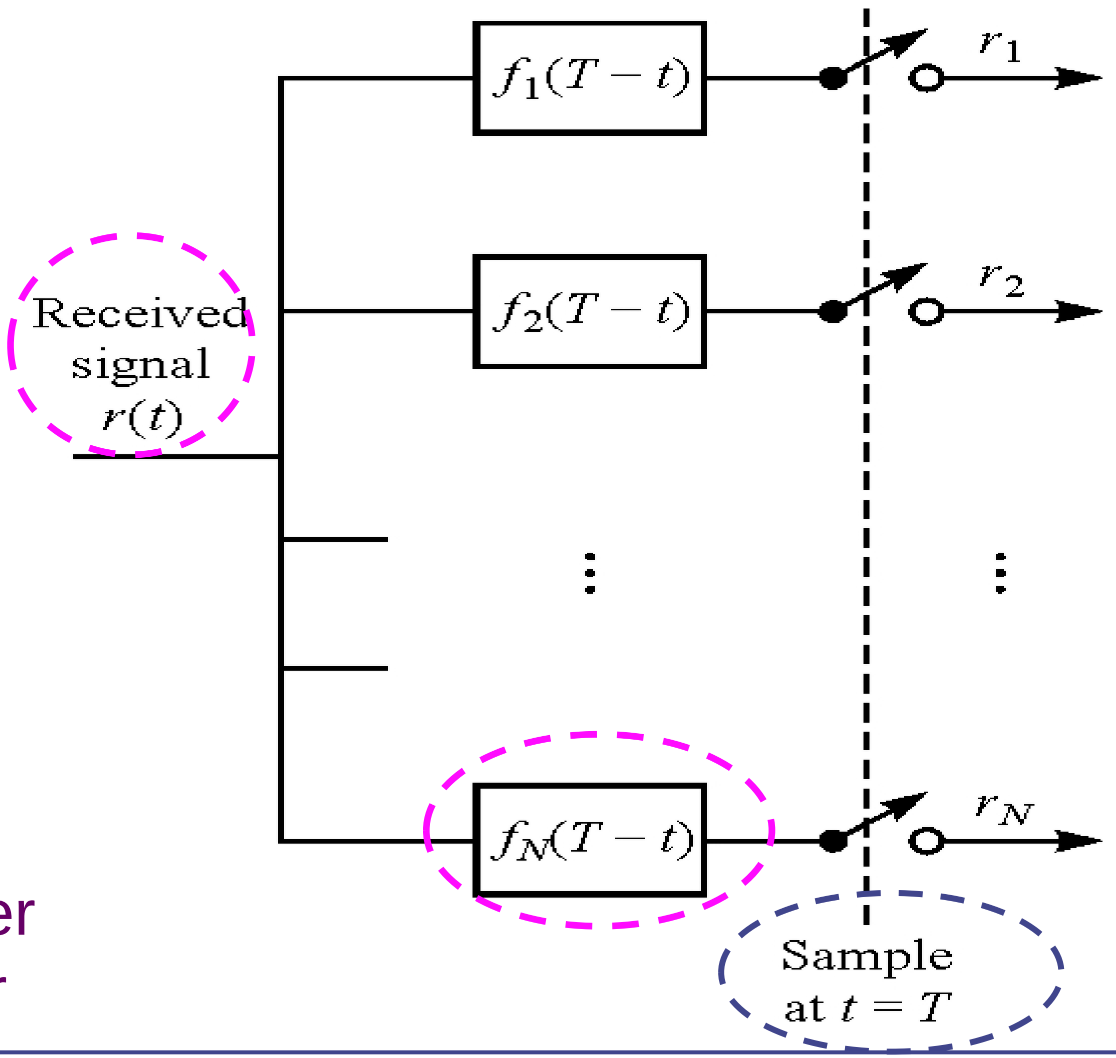
$$y_k(T) = \int_0^T r(t) f_k(\tau)\, d\tau = r_k \qquad k = 1,2,\ldots.N$$

- Hence, the samples outputs of the filter at time t = T are the set of values $\{r_k\}$ obtained from the N linear correlators

- In general, for a signal S(t) and matched filter h(t)= S(T-t)

$$y(t) = \int_0^t S(\tau) S_k(T - t + \tau)\, d\tau$$

Matched Filter Demodulator …



Matched filter demodulator

Properties of the Matched filter

1. If a signal *S(t)* is corrupted by AWGN, the filter with impulse response matched to *S(t), i.e., h(t) = S(T-t)* maximizes the output signal-to-noise ratio (SNR)

$$SNR_{\max} = \frac{\varepsilon_s}{N_0/2} = \frac{2\varepsilon_s}{N_0} \quad \text{for an AGWN with zero mean}$$

and spectral density

$$\Phi_{nn}(f) = \frac{N_0}{2}\; W/Hz \quad \textit{and}\; \varepsilon_s \text{ is the energy of } S(t),$$

$$\varepsilon_s = \int_0^T S^2(t)\,dt$$

Note that the maximum SNR depends only on the energy of the waveform *S(t)* but not on other characteristics of *S(t)*

Properties of the Matched filter …

2. Frequency domain interpretation:

$$H(f) = \int_0^T S(T-t)e^{-j2\pi ft}dt = e^{-j2\pi fT}\int_0^T S(\tau)e^{j2\pi f\tau}\,d\tau; \quad (\tau = -t)$$

$$= S^*(f)e^{-j2\pi fT} \qquad Note\ that\ |H(f)| = |S(f)|\ and\ \theta_S = -\theta_H$$

$$Y(f) = |S(f)|^2 e^{-j2\pi fT} \qquad and$$

$$y_s(t) = \int_{-\infty}^{\infty} Y(f)e^{-j2\pi fT}\,df = \int_{-\infty}^{\infty}|S(f)|^2 df = \int_0^T S^2(T)\,dt$$

The power spectral density of the output is given by

$$\Phi_0(f) = \frac{1}{2}N_0|H(f)|^2 = \frac{1}{2}N_0\varepsilon_s$$

Signal to noise ratio $\quad SNR_{\max} = \dfrac{2\varepsilon_s}{N_0}$

Properties of the Matched filter …

- Example: The M=4 biorthogonal signals have dimension N=2; and we need two basis functions to represent them
- Choose the two basis functions such that

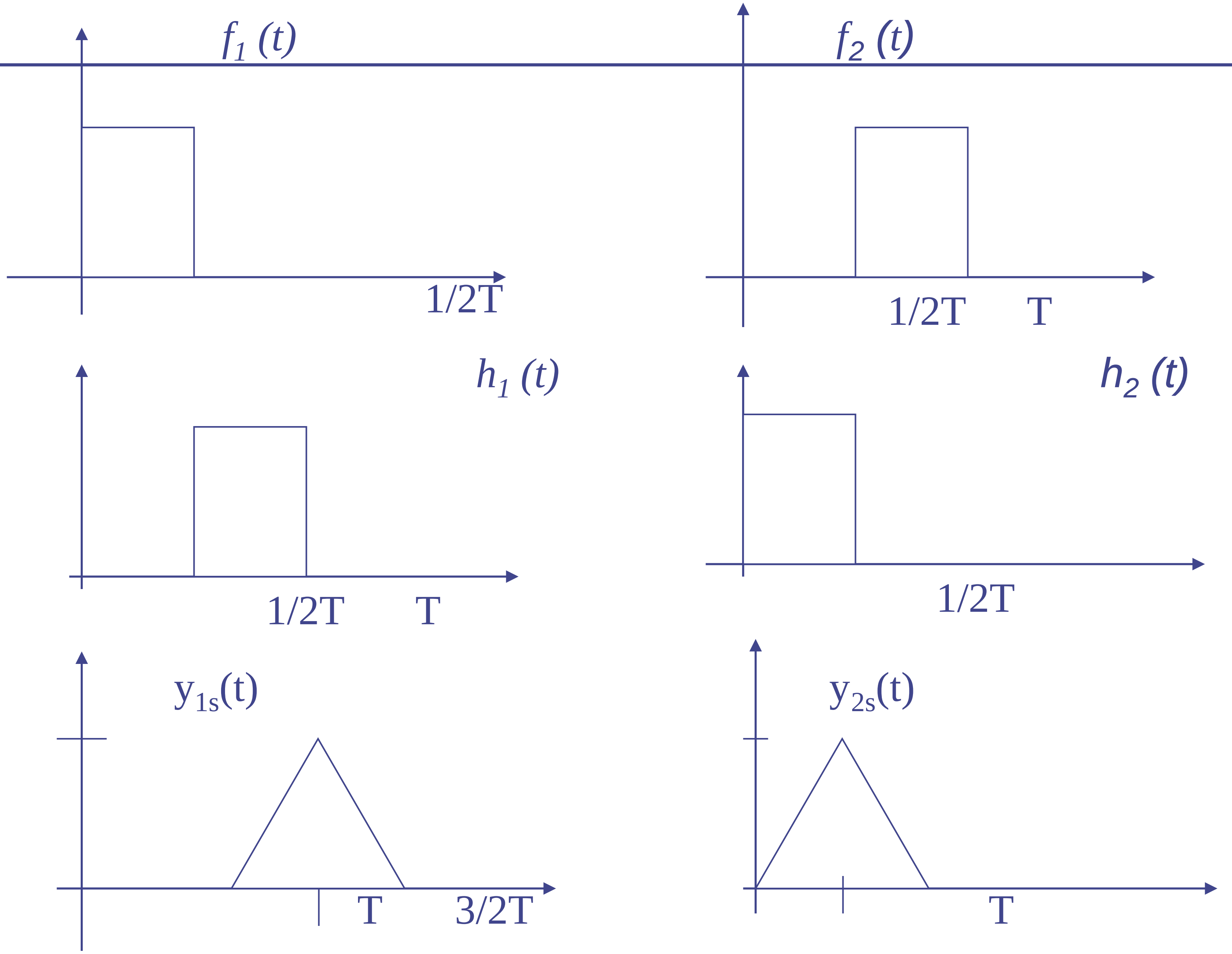
$$f_1(t) = \begin{cases} \sqrt{2/T} & 0 \le t \le \frac{1}{2}T \\ 0 & otherwise \end{cases}$$

$$f_2(t) = \begin{cases} \sqrt{2/T} & \frac{1}{2}T \le t \le T \\ 0 & otherwise \end{cases}$$

- The impulse responses of the two matched filters are

$$h_1(t) = f_1(T-t) = \begin{cases} \sqrt{2/T} & \frac{1}{2}T \le t \le T \\ 0 & otherwise \end{cases}$$

$$h_2(t) = f_2(T-t) = \begin{cases} \sqrt{2/T} & 0 \le t \le \frac{1}{2}T \\ 0 & otherwise \end{cases}$$



f1 (t)
f2 (t)
1/2T
1/2T
T
h1 (t)
h2 (t)
1/2T
T
1/2T
y1s(t)
y2s(t)
T
3/2T
T

Properties of the Matched filter …

- When $s_1(t)$ is transmitted, the noise free response of the two matched filters are $y_{1s}(t)$ and $y_{2s}(t)$ which when sampled at t=T yield

$$y_{1s}(T) = \sqrt{\frac{1}{2}A^2 T} \quad and \;\; y_{2s}(T) = 0$$

- Thus at t =T, the received vector with noise will be

$$\mathbf{r} = [r_1 \, , r_2] = [\sqrt{\varepsilon_s} + n_1 \, , n_2] \quad where \; n_1 = y_{1n}(T) \; and \;\; n_2 = y_{2n}(T)$$

are the noise components at the output of the filters.

$$y_{kn}(T) = \int_0^T n(t) f_k(t)\, dt \quad k = 1,2 \quad and \;\; E[y_{kn}(T)] = E[n_k] = 0$$

- It can be shown that the signal to noise ratio $SNR = \dfrac{2\varepsilon_s}{N_0}$

Properties of the Matched filter …

- The four possible outputs corresponding to the four transmitted signals are

$$[r_1 \, , r_2] = [(\sqrt{\varepsilon_s} + n_1, \; n_2), \; (n_1, \; \sqrt{\varepsilon_s} + n_2), \\ (-\sqrt{\varepsilon_s} + n_1, \; n_2), \; (n_1, \; -\sqrt{\varepsilon_s} + n_2)]$$

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Optimum Detector

- Output of the correlator or the matched filter demodulator produces the vector **r** = [r_1, $r_{2,}$, r_3,… r_N]

- This output vector contains all the relevant information about the received signal waveform

- On the basis of the vector **r,** we need to make a decision on what is transmitted such that the decision is optimal, in some sense, assuming no memory in signals transmitted in successive intervals

Optimum Detector …

- Optimality criterion: Decide on what is transmitted in each interval based on the observation of the vector **r** in each interval such that the ***probability of correct decision is maximized*** **(**or the probability of error is minimized**)**

- Define the a posteriori probability as

$$P\{\text{signal } S_m \text{ was transmitted}\} = P\{\mathbf{s}_m | \mathbf{r}\};\ m = 1,2,\ldots M$$

Optimum Detector …

- Decision criterion: Select the signal corresponding to the maximum of the set of a posteriori probabilities $\{P(\mathbf{s}_m | \mathbf{r})\}$
  - The criterion maximizes the probability of correct decision and, hence, minimizes the probability of error

- This decision criterion is called *maximum a posteriori probability* (MAP) criterion

Optimum Detector …

- Note that:

$$\mathbf{P(s_m|r)} = \frac{\mathbf{P(r|s_m)P(s_m)}}{\mathbf{P(r)}} \quad \textbf{(Baye's Theorem)}$$

- P{$\mathbf{s}_m$} is the apriori probability of the m^{th} signal being transmitted

- Note also that: $$P(\mathbf{r}) = \sum_{m=1}^{M} P(\mathbf{r}|s_m)P(s_m)$$

- The computation of the a posteriori probability P($\mathbf{s}_m$| r) requires knowledge of the:
  - Apriori probability P($\mathbf{s}_m$) and
  - Conditional pdf P($\mathbf{r}|\mathbf{s}_m$)

Optimum Detector …

- For the special case where M signals are equi-probable such that the apriori probability $P(\mathbf{s}_m) = 1/M$ for all M
- Further $P(\mathbf{r})$ is independent of which signal is transmitted
- Then maximizing $P(\mathbf{s}_m | \mathbf{r})$ is equivalent to finding the signal that maximizes $P(\mathbf{r}|\mathbf{s}_m)$

Optimum Detector …

- The conditional pdf $P(\mathbf{r}|\mathbf{s}_m)$, or any monotonic function of it, is called the ***likelihood function***
- The criterion of maximizing $P(\mathbf{r}|\mathbf{s}_m)$ is referred to as the ***maximum likelihood*** (ML) ***criterion***
- Note that the MAP and ML criteria are equivalent when the apriori probabilities $P(\mathbf{s}_m)$ are all equal
  - I.e $\{s_m\}$ are equiprobable

Optimum Detector …

- For the AWGN channel, the likelihood function becomes

$$P(\mathbf{r}|\mathbf{s_m}) = \left(\frac{1}{\sqrt{\pi N_0}}\right)^N \exp\left[-\sum_{k=1}^{N}\frac{(r_k - s_{mk})^2}{N_0}\right] \quad m = 1,2,\ldots\ldots M$$

- Taking the logarithms of both sides simplifies the computation

$$\ln P(\mathbf{r}|\mathbf{s_m}) = -\frac{1}{2}N\ln(\pi N_0) - \frac{1}{N_0}\sum_{k=1}^{N}(r_k - S_{mk})^2$$

- Finding the maximum of ln (P(**r**|**s**$_m$)) over **s**$_m$ is equivalent to finding the signal **s**$_m$ that minimizes the Euclidean distance

$$D(\mathbf{r}, \mathbf{s}_m) = \sum_{k=1}^{N}(r_k - S_{mk})^2 \qquad \text{(Distance metrics)}$$

Optimum Detector …

- The decision rule on the ML criterion is equivalent to finding the signal $\mathbf{s}_m$ that is closest in distance to the received signal vector **r**
  - Called the ***minimum distance detection***
- Expanding the distance metrics

$$D(r, s_m) = \sum_{k=1}^{N} r_k^2 - 2\sum_{k=1}^{N} r_k\, s_{mk} + \sum_{k=1}^{N} s_{mk}^2$$

$$= \|r\|^2 - 2\, r \,.\, s_m + \|s_m\|^2 \qquad m = 1,2,\ldots\ldots M$$

- Since $\|\mathbf{r}\|^2$ is common to all the decision metrics, and hence it may be ignored in the computation of the metrics

Optimum Detector …

- One may use a modified distance metrics

$$D'(r,s_m) = -2r.s_m + \|s_m\|^2 \qquad m = 1,2,.......M$$

- Note that minimizing $D'(\mathbf{r},\mathbf{S}_m)$ is equivalent to maximizing

$$-D'(\mathbf{r},\mathbf{s}_m) = C(\mathbf{r},\mathbf{s}_m) = 2\mathbf{r}.\mathbf{s}_m - \|\mathbf{s}_m\|^2$$

(correlation Metric)

- Note: $\mathbf{r}.\mathbf{s}_m$ represents the projection of the received signal vector onto each of the M possible transmitted vectors
- Thus, it thus measures the correlation between the received vector and the m^{th} signal

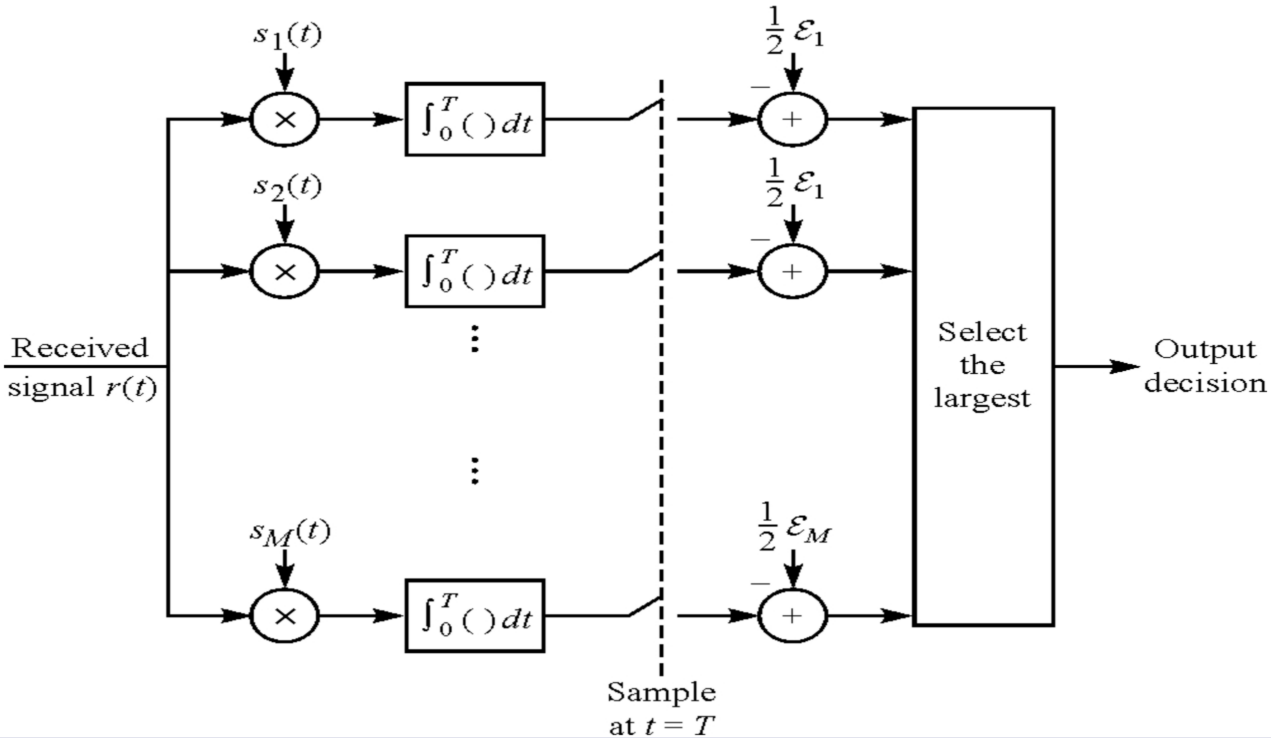
Optimum Detector …

- Note also that $||s_m||^2 = \varepsilon_{sm}$ can be thought of as a bias term that serves as compensation for signal sets that have unequal energies such as PAM
- If all signals have equal energy, it may be ignored in the computation of the correlation metrics and the distance metrics D and D'
- The correlation metrics can thus be computed as

$$C(\mathbf{r}, \mathbf{s}_m) = 2\int_0^T r(t) s_m(t)\, dt - \varepsilon_{sm} \qquad m = 1,2,.....M$$

Optimum Detector …

- These metrics can be generated by a demodulator that
  - Cross-correlates the received signal r(t) with each M possible transmitted signal and
  - Adjust each correlator output for the bias
  - ***Select the signal corresponding to the largest correlation metrics***



Optimum Detector …

- Note that for signals of unequal energies the output of the correlators are adjusted by ½ ε_{sm}
- Alternatively, *r*(t) could be passed through a bank of N matched filters and sampled at $t=T$
- For non-equiprobable signal we apply MAP based on the probabilities $P(\mathbf{s_m} | \mathbf{r})$; m = 1,2,…..M
- Or alternatively on the metrics

$$PM(\mathbf{r}, \mathbf{s}_m) = P(\mathbf{r} | \mathbf{s}_m) P(\mathbf{s}_m)$$

Optimum Detector …

- Example: Consider a binary PAM with $s_1 = -s_2 = \sqrt{\varepsilon_b}$ where ε_b is the energy per bit
- Let the a priori probabilities be $P(s_1) = p$ and $P(s_2) = 1 - p$
- Question: Determine the metrics for an optimum MAP detector when the transmitted signal is corrupted by AWGN?
- Solution:
  - The received signal vector (one-dimensional) for binary PAM is $r = \pm\sqrt{\varepsilon_b} + y_n(T)$
  - Where $y_n(T)$ is a zero-mean Gaussian random variable with variance $\sigma^2 = 1/2\ N_0$

Optimum Detector …

- Consequently, the conditional PDFs of $p(r/S_m)$ for the two signals is

$$P(r|s_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(r-\sqrt{\varepsilon_b})^2}{2\sigma^2} \right];$$

$$P(r|s_2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(r+\sqrt{\varepsilon_b})^2}{2\sigma^2} \right]$$

$$PM(r, s_1) = p\,P(r|s_1) \quad and \quad PM(r, s_2) = (1-p)\,P(r|s_2)$$

- If $PM(r, S_1) > PM(r, S_2)$ then select S_1 as the transmitted signal; otherwise select S_2

Optimum Detector …

- The decision rule may be expressed as

$$\frac{PM(r, s_1)}{PM(r, s_2)} \underset{s_2}{\overset{s_1}{\gtrless}} 1$$

- Which upon substitution for PM(r, S_1) and PM(r, S_2) gives

$$\frac{(r+\sqrt{\varepsilon_b})^2 - (r-\sqrt{\varepsilon_b})^2}{2\sigma^2} \underset{s_2}{\overset{s_1}{\gtrless}} \ln\frac{(1-p)}{p} \quad \text{or}$$

$$r\sqrt{\varepsilon_b} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{1}{2}\sigma^2 \ln\frac{(1-p)}{p} = \frac{1}{4}N_0 \ln\frac{(1-p)}{p} = \tau_h$$

Optimum Detector …

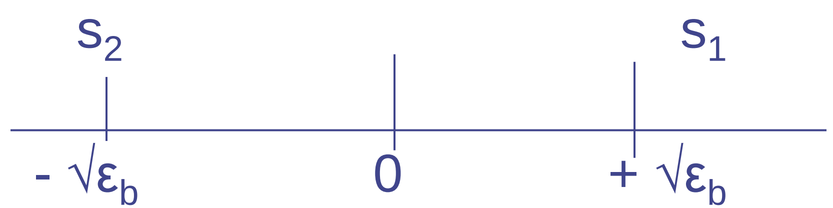
- For the PAM case, the optimum detector computes the product $r\sqrt{\varepsilon_b}$ and compares it with the threshold τ_h
  - If $r\sqrt{\varepsilon_b} > \tau_h$ the s_1 is transmitted or
  - If $r\sqrt{\varepsilon_b} < \tau_h$ then decides on s_2
- Note that the threshold depends on p and N_0
  - For $p = ½$ $\tau_h = 0$
  - If $p > ½$, then s_1 is more probable and hence $\tau_h < 0$
- The above ideas can be generalized for any number of equiprobable signals when using the ML or MAP criterion

Overview

- Optimum Receivers
- Performance of optimum receiver (memoryless modulation)
  - Probability of Error for Binary Modulation
  - Probability of Error for M-ary orthogonal signals
  - Probability of Error for M-ary PSK
  - Probability of Error for QAM
- Comparison of Digital Modulation Methods

Probability of Error for Binary Modulation

- Consider PAM signals $s_1(t) = g(t)$ and $s_2(t) = -g(t)$ where g(t) is an arbitrary pulse which is non-zero in the interval $0 \le t \le T_b$ and zero elsewhere
- Energy of the pulse g(t) be ε_b and the signals may be represented geometrically as



- The received signal from the output of (the matched filter or correlator) demodulator

$$r = s_1 + n = \sqrt{\varepsilon_b} + n$$

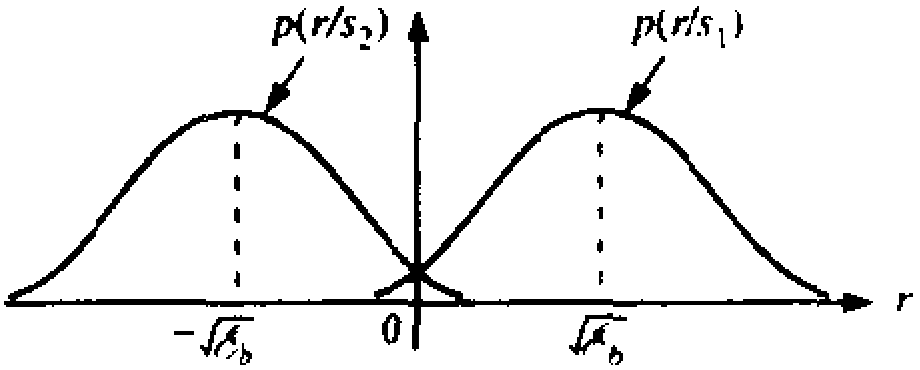
- Note that when $\tau_h = 0$ and $r > 0 \rightarrow s_1(t)$ and $r < 0 \rightarrow s_2(t)$

Probability of Error for Binary ….

- Conditional probability density functions of **r** are given by

$$p(r|s_1) = \frac{1}{\sqrt{\pi N_0}} exp\left[-\frac{(r-\sqrt{\varepsilon_b})^2}{N_0}\right]$$

$$p(r|s_2) = \frac{1}{\sqrt{\pi N_0}} exp\left[-\frac{(r+\sqrt{\varepsilon_b})^2}{N_0}\right]$$



Conditional PDFs of two signals

Probability of Error for Binary ….

- Given that $s_1(t)$ is transmitted, the probability of error is given by P{r < 0}, i.e.,

$$P(e|s_1) = P\{r<0\}$$

$$= \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{0} \exp\left[-\frac{(r-\sqrt{\varepsilon_b})^2}{N_0}\right] dr$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{\frac{2\varepsilon_b}{N_0}}} \exp^{-\frac{x^2}{2}} dx \qquad where\ \frac{(r-\sqrt{\varepsilon_b})}{\sqrt{N_0}} = \frac{x}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2\varepsilon_b}{N_0}}}^{\infty} \exp^{-\frac{x^2}{2}} dx = Q\left(\sqrt{\frac{2\varepsilon_b}{N_0}}\right)$$

Probability of Error for Binary ….

- Similarly, if $s_2(t)$ was transmitted, **$r = -\sqrt{\varepsilon_b} + n$** and

$$\mathbf{P}(\mathbf{e}|\mathbf{s}_2) = Q\left(\sqrt{\frac{2\varepsilon_b}{N_0}}\right)$$

- When $s_1(t)$ & $s_2(t)$ are equally probable with probability ½, the average probability of error will be

$$P_b = \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) = Q\left(\sqrt{\frac{2\varepsilon_b}{N_0}}\right)$$

- Note that:
  - Probability of error depends only on the ratio of ε_b/N_0 and not on any other characteristics of the signals
  - ε_b/N_0 = signal-to-noise ratio (SNR) per bit
  - $2\,\varepsilon_b/N_0 = SNR_{max}$ from the matched filter or correlator demodulator

Probability of Error for Binary ….

- Probability of error may also be expressed in terms of the distance between the two signals s_1 and s_2 such that

$$d_{12} = 2\sqrt{\varepsilon_b} \text{ where } \varepsilon_b = \tfrac{1}{4} d_{12}^2$$

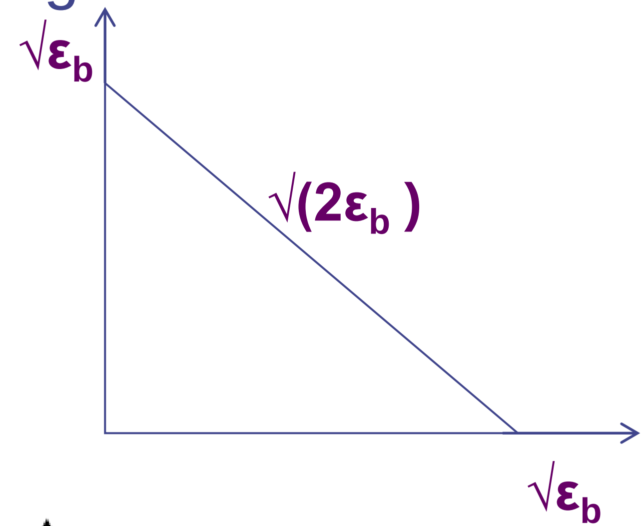
$$P_b = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) = Q\left(d_{12}\sqrt{\frac{1}{2N_0}}\right)$$

Probability of Error for Binary ….

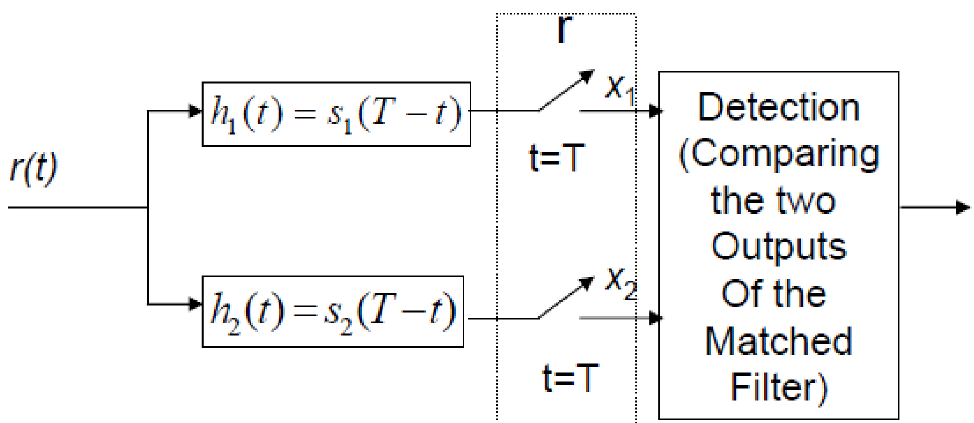
- Now consider the binary orthogonal signals

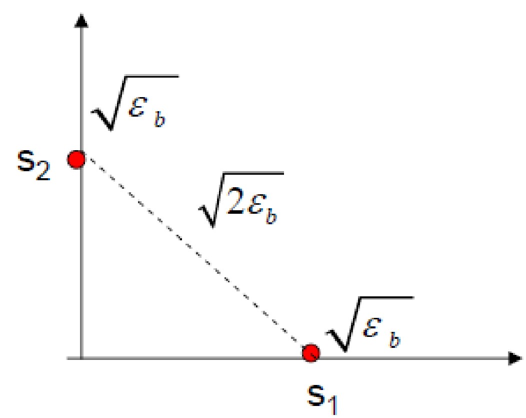
  $S_1 = [\sqrt{\varepsilon_b},\ 0]$ and

  $S_2 = [0,\ \sqrt{\varepsilon_b}]$



- where $\sqrt{\varepsilon_b}$ = energy for each of the waveforms





Probability of Error for Binary ….

- When s_1 is transmitted, the received vector at the output of the demodulator is **r** = $[\sqrt{\varepsilon_b}+n_1, n_2]$
- Based on the correlation metrics, the probability of error

$$P(e|s_1) = P\left[C(r, s_2) > C(r, s_1)\right]$$

$$P[(n_2 - n_1) > \sqrt{\varepsilon_b}\,]$$

- Random variable X = $(n_2 - n_1)$ is zero-mean Gaussian with variance N_0; and n_1 and n_2 are also independent

$$P[(n_2 - n_1) > \sqrt{\varepsilon_b}\,] = \frac{1}{\sqrt{2\pi N_0}} \int_{\sqrt{\varepsilon_b}}^{\infty} e^{-\frac{x^2}{2N_0}}\, dx = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{\varepsilon_b}{N_0}}}^{\infty} e^{-\frac{x^2}{2_0}}\, dx = Q\left(\sqrt{\frac{\varepsilon_b}{N_0}}\right)$$

Probability of Error for Binary ….

- Similarly for s_2

$$P[(n_1 - n_2) > \sqrt{\varepsilon_b}] = Q\left(\sqrt{\frac{\varepsilon_b}{N_0}}\right)$$

- Average error probability for binary orthogonal signals is

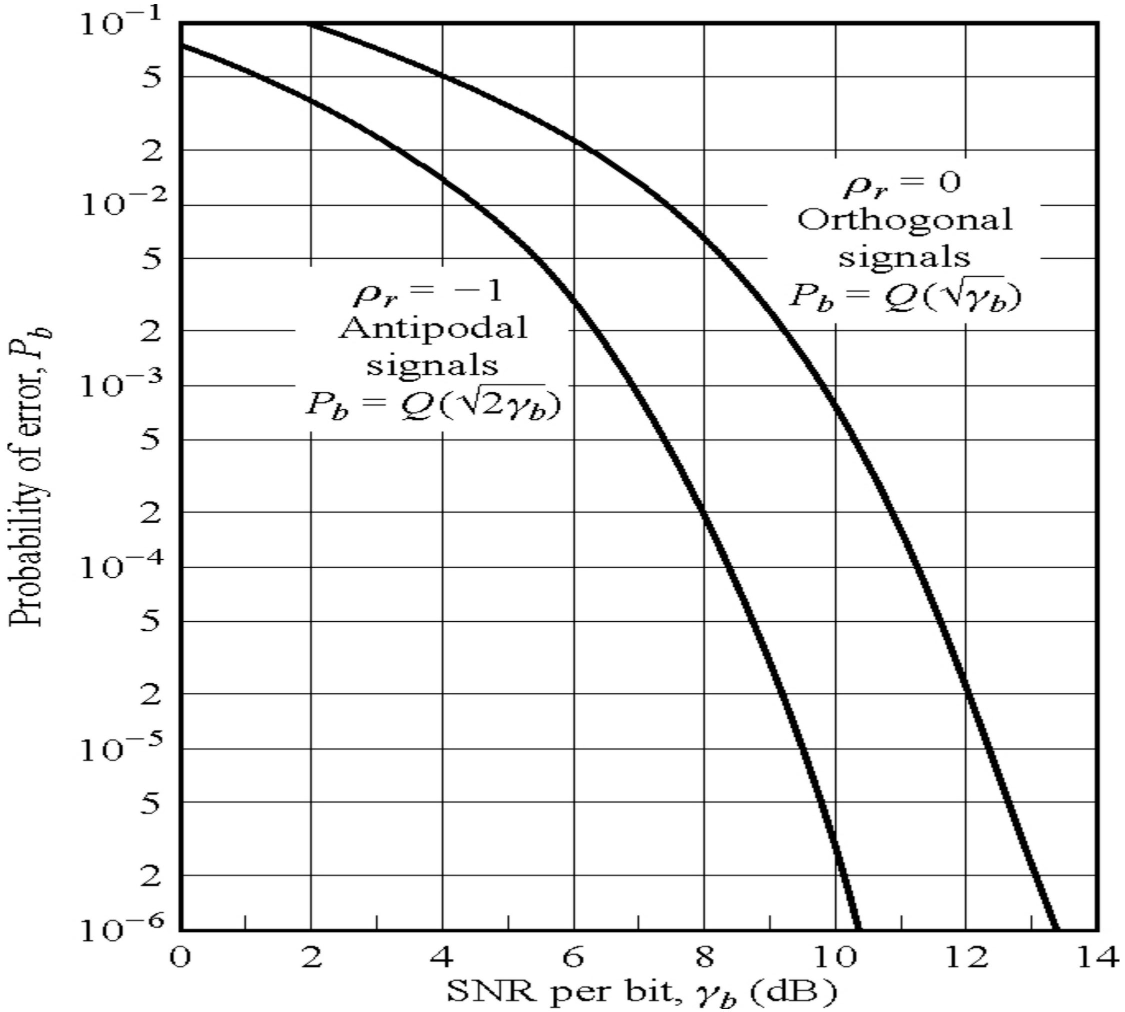
$$P_b = Q\left(\sqrt{\frac{\varepsilon_b}{N_0}}\right) = Q(\gamma_b)$$

where γ_b is the SNR per bit

Probability of Error for Binary ….

- Note that the orthogonal signal requires twice the energy to achieve the same P_b as that of the antipodal signals; i.e it is 3dB poorer than the antipodal signals
- This 3 dB difference is due to the distance between s_1 and s_2 where $d_{12}^2 = 2\varepsilon_b$ for orthogonal signals whereas for the antipodal signals it is $4\varepsilon_b$

Probability of Error for Binary ….



Probability of error for binary signals

Overview

- Optimum Receivers
- Performance of the optimum receiver (Memoryless Modulation)
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  - Probability of Error for M-ary orthogonal signals
  - Probability of Error for M-ary PSK
  - Probability of Error for QAM
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Probability of Error for M-ary Orthogonal Signals

- For equal energy orthogonal signals, the optimum detector selects the signal resulting in the largest cross correlation between **r** and each of the possible M transmitted vectors $\{s_m\}$; i.e,

$$C(\mathbf{r},\mathbf{s_m}) = \mathrm{r}\cdot \mathrm{s}_m = \sum_{k=1}^{M} r_k s_{mk}; \quad m = 1, 2, \ldots\ldots\ldots, M$$

- When s_1 is transmitted, the received vector is

$$\mathbf{r} = \left[\sqrt{\varepsilon_s} + n_1,\ n_2,\ n_3, \ldots\ldots\ldots, n_M\right]$$

- Where $n_1,\ n_2,\ n_3, \ldots\ldots\ldots, n_M$ are zero mean, mutually independent Gaussian random variables with equal variances $\sigma_n^2 = \frac{N_0}{2}$

Probability of Error for M-ary …

- The outputs of the M correlators are

$$C(r,s_1)=\sqrt{\varepsilon_s}(\sqrt{\varepsilon_s}+n_1)$$
$$C(r,s_2)=\sqrt{\varepsilon_s}\,n_2$$
$$\vdots$$
$$C(r,s_M)=\sqrt{\varepsilon_s}\,n_M$$

- Removing the scale factor $\sqrt{\varepsilon_s}$ (normalization), the pdf of the first correlator output $r_1 = \sqrt{\varepsilon_s} + n_1$ is

$$p_{r1}(x_1)=\frac{1}{\sqrt{\pi N_0}}exp\left[-\frac{(x_1-\sqrt{\varepsilon_s})^2}{N_0}\right]$$

- And the pdf's of the other M-1 correlators outputs are

$$p_{rm}(x_1)=\frac{1}{\sqrt{\pi N_0}}exp\left[-\frac{x_m^2}{N_0}\right]$$

Probability of Error for M-ary …

- Probability of making a correct decision is

$$P_c = \int_{-\infty}^{\infty} P(n_2 < r_1, n_3 < r_1, \ldots\ldots n_M < r_1 | r_1)\, p(r_1)\, dr_1$$

- Since {n_m} are statistically independent, the joint probability factors as a product of (M-1) marginal probabilities of the form

$$P[(n_m < r_1) | r_1] = \int_{-\infty}^{r_1} p_{rm}(x_m)\, dx_m \qquad m = 2,3,\ldots\ldots, M$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_1\sqrt{2/N_0}} e^{-x^2/2}\, dx,$$

- Which are identical for m=2, 3, …, M

$$P_c = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_1\sqrt{2/N_0}} e^{-x^2/2}\, dx \right)^{M-1} p(r_1)\, dr_1$$

Probability of Error for M-ary …

- Thus, probability of a (k-bit) symbol error is $P_M = 1 - P_c$ and

$$P_M = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[1 - \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_1\sqrt{2/N_0}} e^{-x^2/2}\, dx \right)^{M-1} \right] \exp\left[-\frac{1}{2}\left(y - \sqrt{\frac{2\varepsilon_s}{N_0}} \right)^2 \right] dy$$

- This could be expressed in terms of probability of error per bit instead of per (k-bit) symbol by substituting $k\varepsilon_b$ for ε_s
- For equiprobable orthogonal signals, all symbol errors are equiprobable and occur with probability

$$\frac{P_M}{M-1} = \frac{P_M}{2^k - 1}$$

- And there are $\binom{k}{n}$ ways in which n bits out of k may be in error, the average number of bit error per k-bit symbol is

Probability of Error for M-ary …

$$\sum_{n=1}^{k} n \binom{k}{n} \frac{P_M}{2^k - 1} = \frac{k\, 2^{k-1}}{2^k - 1} P_M$$

- The average bit error probability y is

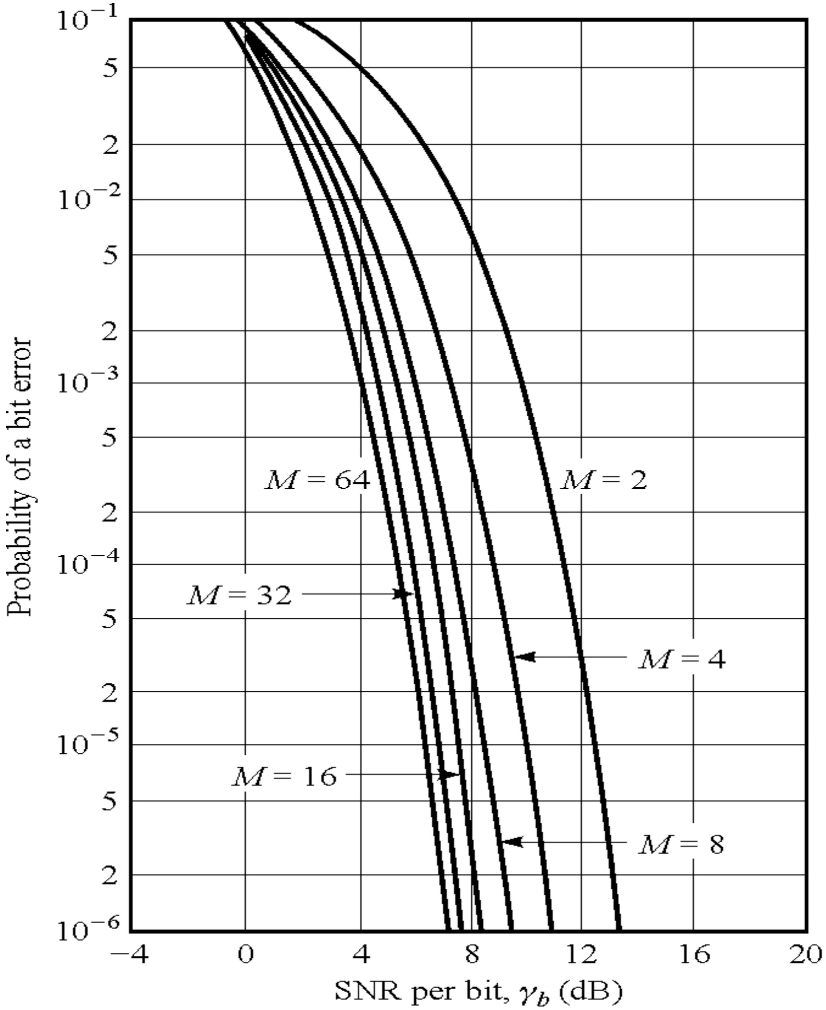
$$P_b = \frac{2^{k-1}}{2^k - 1} P_M \approx \frac{P_M}{2} \qquad for\ k >> 1$$

- By plotting P_b as a function of SNR per bit, ε_b/N_0, we can obtain performance comparison for different values of M
- For example, to achieve a bit error probability of 10^{-5}

$$\text{SNR per bit} \approx \begin{cases} 12\,\text{dB} & \text{for M} = 2\ \ (\text{k} = 1) \\ 6\,\text{dB} & \text{for M} = 64\ \ (\text{k} = 6) \end{cases}$$

- A saving in power (by a factor of 4) can be achieved by increasing M from 2 to 64

Probability of Error for M-ary …



Probability of bit error for coherent detection of orthogonal signals

Probability of Error for M-ary …

- Union bound on the probability of error
- *We now consider the effect of increasing M on the probability of error for orthogonal signals*
- Consider a detector for M orthogonal signals as one that makes (M-1) binary decision between the correlator output $C(r,s_1)$, that contains the signal of interest, and the other (M-1) correlator outputs $C(r,s_m)$; m = 2,3,……M
- The probability of error is bounded by the union of the (M-1) events
- Thus, if E_i represents such an event the bound on the probability of error can be determined as follows

Probability of Error for M-ary …

$$C(\mathbf{r}, s_i) > C(\mathbf{r}, s_1) \ \text{ for i} \neq 1\text{, then}$$

$$P_M = P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum P(E_i);\ \text{hence}$$

$$\mathrm{P_M} < (M-1)P_2 = (M-1)\,Q\left(\sqrt{\mathcal{E}_s / N_0})\right) < M\,Q\left(\sqrt{\mathcal{E}_s / N_0})\right)$$

- The Q function can also upper bounded by $Q\left(\sqrt{\mathcal{E}_s / N_0})\right) < e^{-\mathcal{E}_s / 2N_0}$
- To simplify the above such that

$$P_M < M e^{-\mathcal{E}_s / 2N_0} = 2^k e^{-k\mathcal{E}_b / 2N_0} < e^{-k(\frac{\mathcal{E}_b}{N_0} - 2\ln 2) / 2}$$

Probability of Error for M-ary …

- Note from the above that as k $\rightarrow\infty$ or as M $\rightarrow\infty$ the probability of error approaches zero exponentially provided $\frac{\varepsilon_b}{N_0}$ is greater than 2 *ln*2 = 1.39 (1.42dB)
- However, this bound is not very tight at sufficiently low SNR
- It can be shown that the upper bound could be made sufficiently tight for $\frac{\varepsilon_b}{N_0} > 4\ln 2$

Probability of Error for M-ary …

- Furthermore, a more tighter upper bound is provided for

$$\frac{\varepsilon_b}{N_0} > \ln 2 = 0.693 = -1.6dB$$

- Under this condition, the upper bound of the probability of error is given by

$$P_M < 2e^{-k\left(\sqrt{\varepsilon_b / N} - \sqrt{\ln 2}\right)^2}$$

- This the Shannon limit for the AWGN channel!

Overview

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Probability of Error for M-ary PSK

- We had earlier seen that digitally phase modulated signal waveforms may be expressed as

$$s_m(t) = g(t)\cos\left[2\pi f_c t + \frac{2\pi}{M}(m-1)\right] \qquad 1 \le m \le M$$

- Which has a vector representation as

$$s_m = \left[\sqrt{\varepsilon_s}\cos\frac{2\pi}{M}(m-1), \quad \sqrt{\varepsilon_s}\sin\frac{2\pi}{M}(m-1)\right]$$

- Where $\varepsilon_s = ½\, \varepsilon_g$ is the energy in each of the waveforms and g(t) is the pulse shape of the transmitted signals

Probability of Error for M-ary PSK ….

- Note: The signal waveforms have equal energies
- The optimum detector for an AWGN channel simply computes the correlation metrics

$$C(\mathrm{r}, \mathrm{s}_\mathrm{m}) = \mathrm{r} \cdot \mathrm{s}_\mathrm{m} \qquad m = 1, 2, \ldots\ldots, M$$

- Accordingly, the received signal **r** = [r_1, r_2] is projected onto each of the M possible signal vectors & the decision is made in favor of the signal with the highest projection

- Alternatively, a phase detector that computes the phase of the received signal from **r** and selects the signal vector $\mathbf{s}_m$ whose phase is closest to that of **r**
- Phase of **r** is $\Theta_{\mathbf{r}} = \tan^{-1}\left(\frac{r_2}{r_1}\right)$

Probability of Error for M-ary PSK ….

- Now compute the PDF of Θ_r from which the probability of error may be computed
- Assume that the transmitted signal is $s_1(t)$ having a phase of $\Theta_r = 0$ such that

$$s_1 = [\sqrt{\varepsilon_s}, \quad 0] \quad \text{and} \quad r_1 = [\sqrt{\varepsilon_s} + n_1]; \qquad r_2 = n_2$$

- n_1 and n_2 are jointly normal each with zero mean and variance ½ N_0
- r_1 and r_2 are also jointly normal with means $E\{r_1\} = \sqrt{\varepsilon_s}$ and $E\{r_2\} = 0$ and equal variances $\sigma_r^2 = ½ N_0$
- Hence,

$$p(r_1, r_2) = \frac{1}{2\pi\sigma_r^2} \exp\left(-\frac{(r_1 - \sqrt{\varepsilon_s})^2 + r_2^2}{2\sigma_r^2} \right)$$

Probability of Error for M-ary PSK ….

- The PDF of Θ_r can be obtained by change of variables from (r_1, r_2) to

$$V = \sqrt{r_1^2 + r_2^2} \quad \textbf{and} \quad \Theta_{\mathbf{r}} = \tan^{-1}\left(\frac{r_2}{r_1}\right)$$

- Such that

$$p(V, \Theta_{\mathbf{r}}) = \frac{1}{2\pi\sigma_r^2} \exp\left(-\frac{(V^2 + \varepsilon_s - 2\sqrt{\varepsilon_s} V \cos\Theta_{\mathbf{r}}}{2\sigma_r^2}\right)$$

- Integrating this joint PDF over the range of values of V yields the marginal PDF of Θ_r

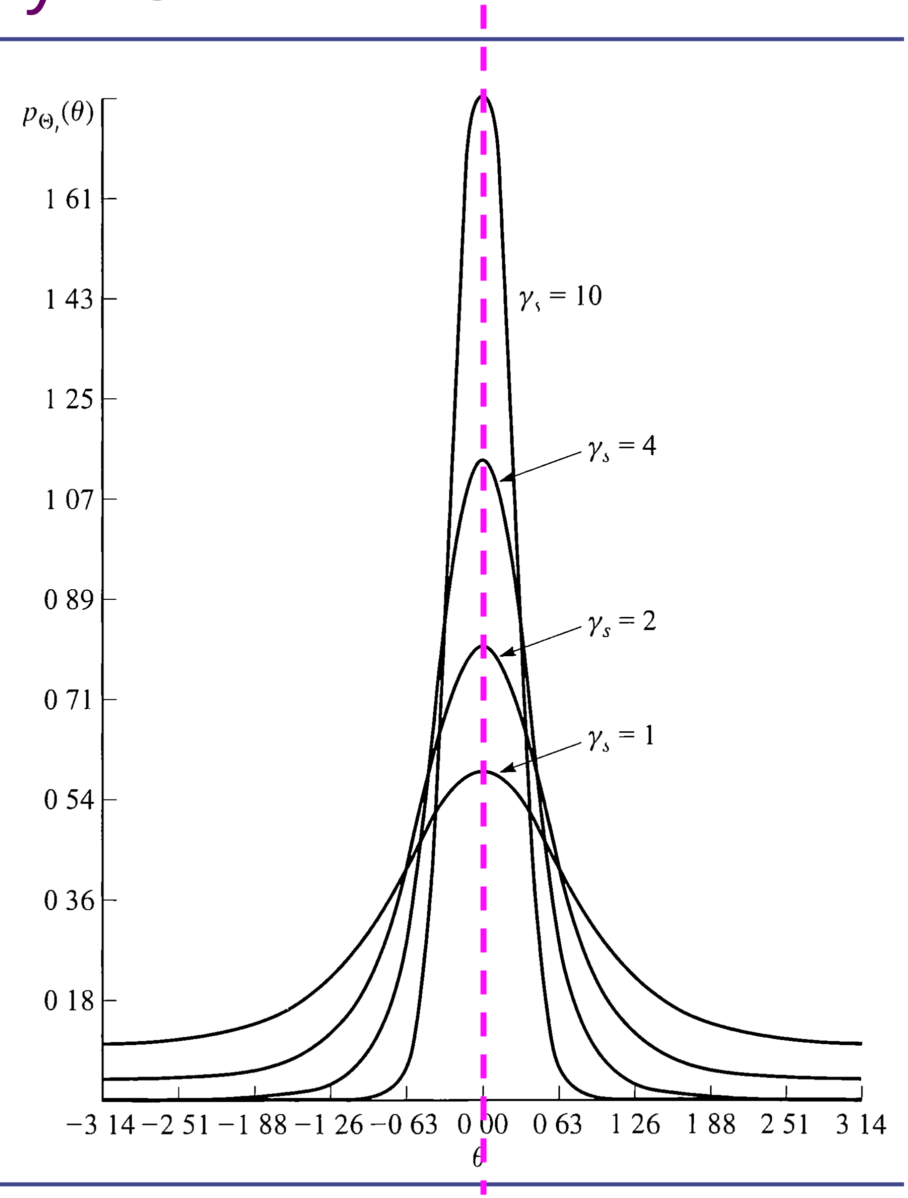
$$p(\Theta_r) = \int_0^\infty p(V, \Theta_r) dV = \frac{1}{2\pi} e^{-\gamma_s \sin^2\Theta_r} \int_0^\infty V e^{-\left(\frac{V - \sqrt{2\gamma_s} \cos\Theta_r}{2}\right)^2} dV$$

- Where $\gamma_s = \varepsilon_s / N_O$ is the symbol SNR

Probability of Error for M-ary PSK ….

- The figure shows plot of the probability density function of Θ_r for different values of γ_s
- When $s_1(t)$ is transmitted, a decision error is made if the noise causes the phase to fall outside $-\pi/M \leq \Theta_r \leq \pi/M$
- Symbol error probability is

$$P_M = 1 - \int_{-\pi/M}^{\pi/M} p(\Theta_r)\, d\Theta_r$$



Probability of Error for M-ary PSK ….

- In general, the integral $p(\Theta_r)$ does not reduce to a simple form and evaluated numerically (except for M=2 or 4)
- For M=2, binary phase modulation, the signals $s_1(t)$ & $s_2(t)$ are antipodal and hence the probability of error is given by

$$P_2 = Q\left(\sqrt{\frac{2\varepsilon_b}{N_0}}\right)$$

- For M=4, we have two binary phase modulation signals in phase quadrature and since there is no cross talk between the signals on the quadrature carriers, the bit error probability is identical to the one given for M=2 above

Probability of Error for M-ary PSK ….

- On the other hand the symbol error is different
- The probability of correct decision for the 2-bit symbol is $P_c = (1 - P_2)^2$

$$P_c = (1 - P_2)^2 = \left[1 - Q\left(\sqrt{\frac{2\varepsilon_b}{N_0}}\right)\right]^2$$

- And the probability of symbol error

$$\mathrm{P}_4 = 1 - P_c = 2Q\left(\sqrt{\frac{2\varepsilon_b}{N_0}}\right)\left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\varepsilon_b}{N_0}}\right)\right]$$

Probability of Error for M-ary PSK ….

- An approximation to the error probability for large values of M and large SNR can be made when $\varepsilon_s/N_0 \gg 1$ and $|\Theta_r| \le 1/2\pi$ such that

$$p(\Theta_r) \approx \sqrt{\frac{\gamma_s}{\pi}} \cos\Theta_r e^{-\gamma_s \sin^2\Theta_r}$$

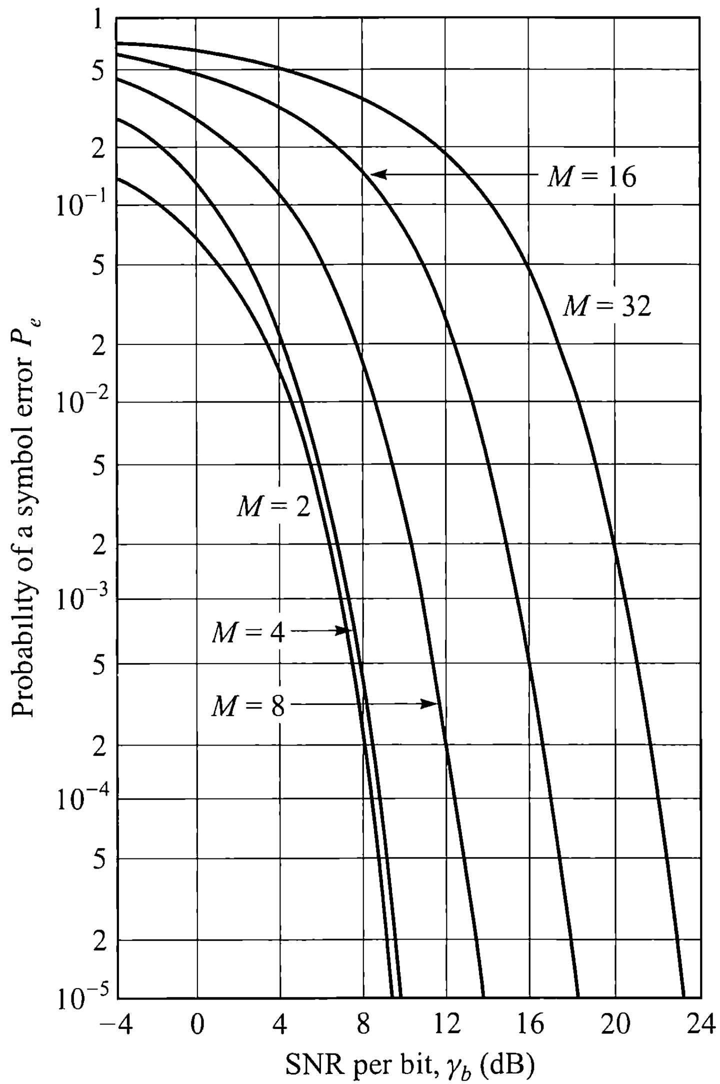
- Substituting this approximation, the symbol error P_M can be expressed as

$$P_M \approx 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} \sqrt{\frac{\gamma_s}{\pi}} \cos\Theta_r e^{-\gamma_s \sin^2\Theta_r} d\Theta_r$$

$$\approx 2Q\left(\sqrt{2\gamma_s}\sin\frac{\pi}{M}\right) = 2Q\left(\sqrt{2k\gamma_b}\sin\frac{\pi}{M}\right)$$

- Where $k=\log_2 M$ and $\gamma_s = k\gamma_b$

Probability of Error for M-ary PSK ….



Probability of a symbol error for PSK Signals

Probability of Error for M-ary PSK ….

- When gray coding is used in mapping the signal points, probability of bit error can be approximated by P_b=1/k P_M

- The above assumes that the carrier phase is estimated accurately by the detector
- However, this is not always true (*we will see this later*) and there may be ambiguity in phase estimation
- To avoid the problem of phase ambiguity in the estimation of carrier phase, the information is encoded in the phase difference between successive transmission as opposed to the absolute phase encoding
- This type of modulation technique is called *differential PSK (DPSK).(See section 5.2.8 in text)*

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Probability of Error for QAM

- QAM signal waveforms can in general be expressed as

$$s_m(t) = A_{mc}\, g(t) \cos 2\pi f_c t - A_{ms}\, g(t) \sin 2\pi f_c t$$

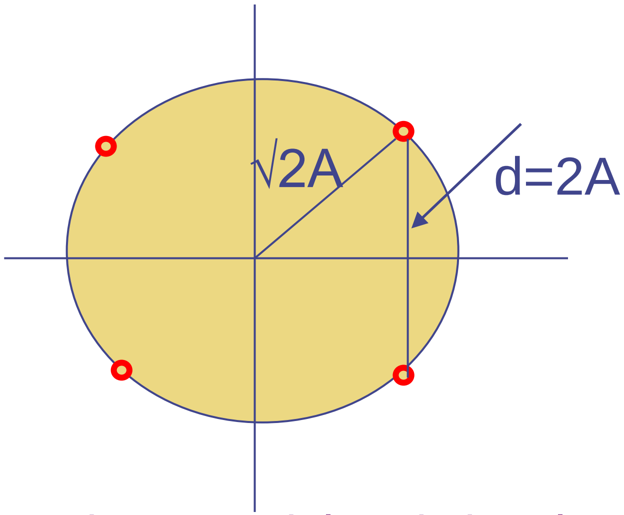
- Where A_{mc} and A_{ms} are amplitudes of the information bearing quadrature carriers and g(t) is the signal pulse
- The vector representation of these waveforms is

$$\boldsymbol{S_m} = \left[A_{mc}\sqrt{\frac{1}{2}\varepsilon_g}\,, \quad A_{ms}\sqrt{\frac{1}{2}\varepsilon_g} \right]$$

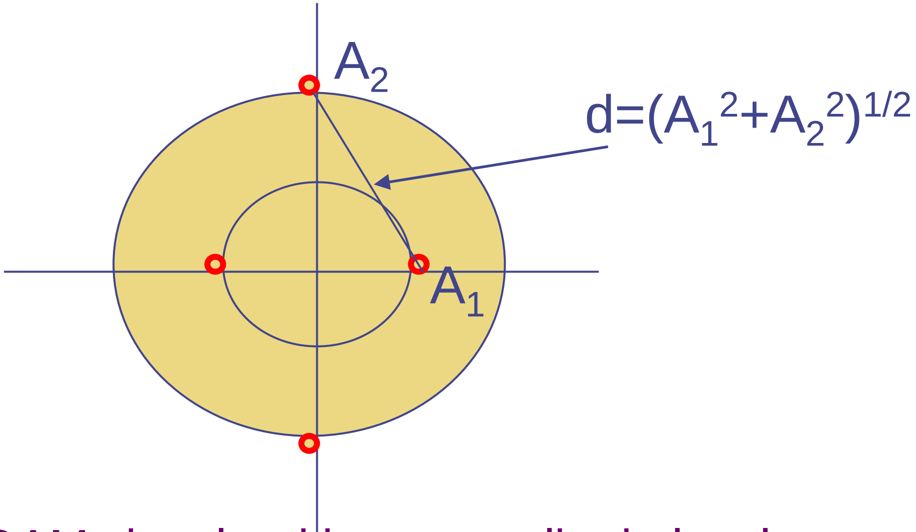
- The probability of error depends on how the signal constellations are arranged as we will demonstration below

Probability of Error for QAM ….

- Consider a QAM signal set for M=4 arranged in the two different ways as shown below
- Note that the distance between any two signal points is constrained to be the same such that $d_{min}=2A$
- Recall that the probability of error is essentially determined by the minimum distance between pairs of signals



4-phase modulated signals



QAM signals with two amplitude levels

Probability of Error for QAM ….

- If the signal points are equally probable, the average transmitter power for the four-phase modulation case is

$$P_{av} = \frac{1}{4} x 4 x 2A^2 = 2A^2$$

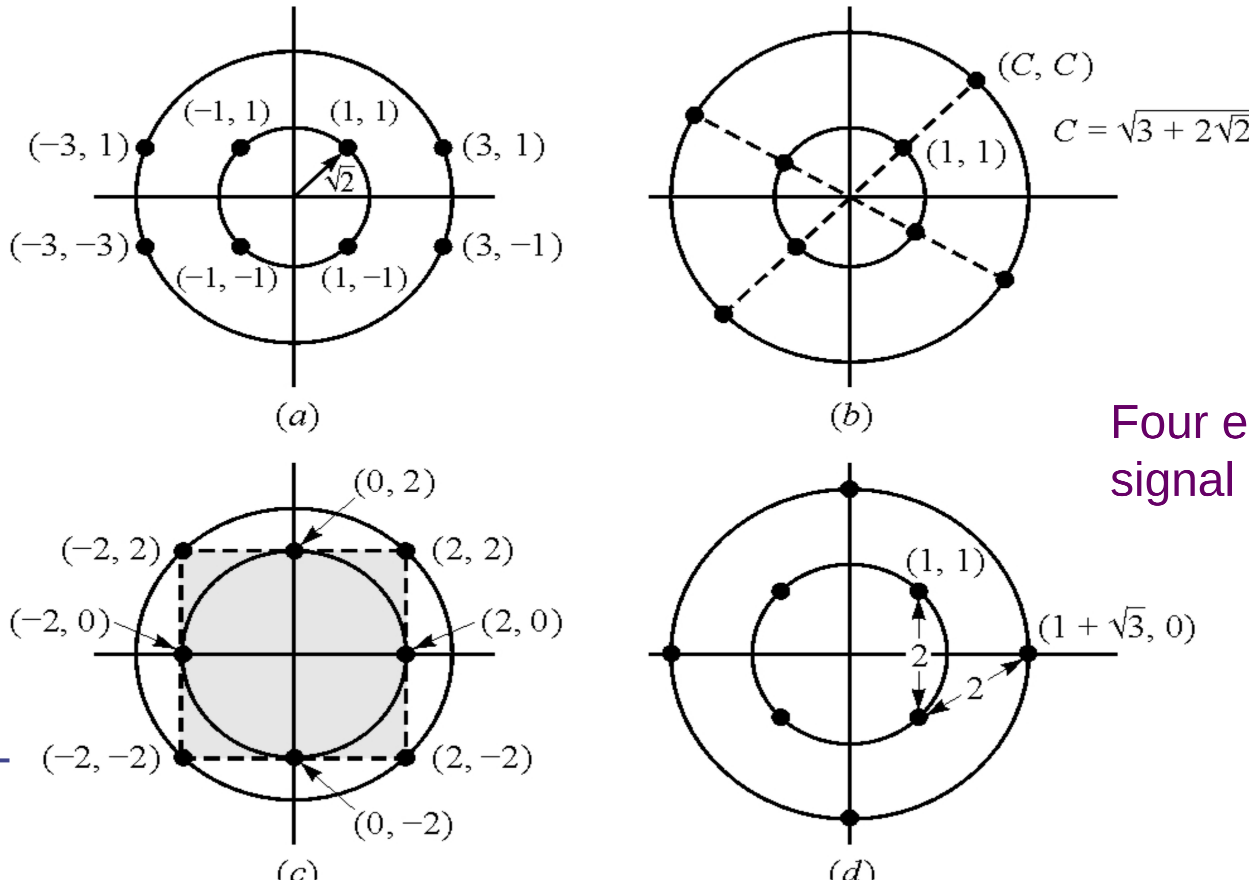
- For the QAM, the points must be placed on circles of radii A_1=A and A_2=√3A such that $d_{min}=\sqrt{(A^2 + 3A^2)}$ =2A, so that

$$P_{av} = \frac{1}{4} x[2(3A^2) + 2A^2] = 2A^2$$

- Thus for all practical purposes for the same average power, the error rate for the two signal sets are the same
- There is no advantage in using the two amplitude QAM signal set over the M = 4-phase modulated signals

Probability of Error for QAM ….

- Consider the case for M = 8 QAM shown below with
  - Four different constellations
  - All having two amplitudes levels and
  - A minimum distance between signal points of 2 *(Normalized by A)*



$$C = 2 + \sqrt{2}$$

Four eight-point QAM signal constellations

Probability of Error for QAM ….

- Assuming the signal points are equiprobable, the average transmitted signal power is given by

$$P_{av} = \frac{1}{M}\sum_{m=1}^{M}(A_{mc}^2 + A_{ms}^2) = \frac{A^2}{M}\sum_{m=1}^{M}(a_{mc}^2 + a_{ms}^2)$$

- Where (a_{mc} ,a_{ms}) are the coordinates of the signal points normalized by A

- For the signal constellations shown in Figures (a) and (c)

  $P_{av} =$ $A^2/8$ [4x2 +4x10] = $A^2/8$[4x4 + 4x8} = $6A^2$ (7.78 dB)

- For those in Figure (b)

  P_{av} = $A^2/8$ [4x2 + 4x(2 + √2)] = $6.83A^2$ (8.34 dB)

- And for those in Figure (d)

  P_{av} = $A^2/8$ [4x2 + 4(1 + √3)^2] = $4.732A^2$ (6.75 dB)

Probability of Error for QAM ….

- The signal set in (d) requires approximately 1dB less that sets in (a) and (c) and 1.6 dB less than the signal set in (b)
- The signal constellation in (d) is the best QAM constellation since it requires least power for a given minimum distance between signal points
- For M ≥ 16, there are many possibilities for selecting the two dimensional signal space
  - However, *multi-amplitude circular constellation* is not necessarily the best for AWGN channel
- Rectangular QAM signal constellations are advantageous as easily being generated as two PAM signals impressed on phase-quadrature carriers & can be easily demodulated
  - Thus, rectangular M-ary QAM signals are most frequently used

Probability of Error for QAM ….

- For rectangular constellation in which $M = 2^k$, where k is even, the QAM signals are equivalent to two PAM signals on quadrature carriers each having $\sqrt{M} = 2^{k/2}$ signal points
- The probability of error for QAM can be determined from the probability of error for PAM

- The probability of correct decision for the M-ary QAM system is thus given by

$$P_c = (1 - P_{\sqrt{M}})^2$$

where $P_{\sqrt{M}}$ is probability of error of $\sqrt{M}$-ary PAM with half the average power in each quadrature signal of equivalent QAM

$$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\varepsilon_{av}}{(M-1)N_0}}\right)$$

Where ε_{av}/N_O = average SNR per symbol

Probability of Error for QAM ….

- Probability of symbol error for the M-ary QAM is then

  $P_M = 1 – (1 - P_{\sqrt{M}})^2$

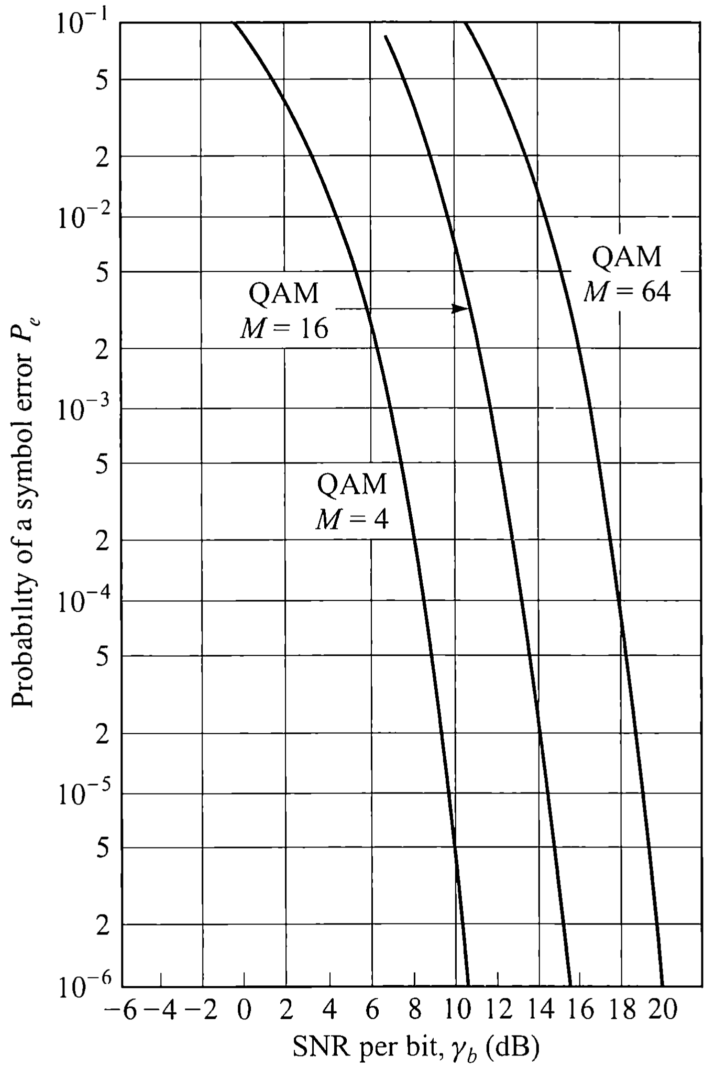
- For k odd, there is no equivalent expression, but a tight upper bound can be established as

$$P_M \le 1 - \left[1 - 2\,Q\left(\sqrt{\frac{3\varepsilon_{av}}{(M-1)N_0}} \right) \right]^2$$

$$\le 4Q\left(\sqrt{\frac{3k\,\varepsilon_{bav}}{(M-1)N_0}} \right); \quad k \ge 1$$

- Where ε_{bav}/N_O = average SNR per bit

Probability of Error for QAM ….



Probability of a symbol error for QAM

Probability of Error for QAM ….

- Let us compare the performance of QAM with that of PSK for a given signal size M
  - Since both are signal types having two dimensions
- For M-ary PSK

$$P_M \approx 2Q\left(\sqrt{2\gamma_s}\sin\frac{\pi}{M}\right)$$

- Since the error probabilities are dominated by the Q-function it would be sufficient to compare the arguments of the Q-function for the two signal formats
- Ratio of the arguments give

$$\mathrm{R_m} = \frac{3/(\mathrm{M}-1)}{2\sin^2\frac{\pi}{\mathrm{M}}}$$

Probability of Error for QAM ….

- M-ary QAM yields better performance for M > 4 as the following values of R_m indicates

M	R_m (dB)
4	1.00
8	1.65
16	4.20
32	7.02
64	9.95

Overview

- Optimum receivers
- Performance of the optimum receiver (Memoryless Modulation)
- Comparison of digital modulation methods

Comparison of Digital Modulation Methods

- Digital modulation methods may be compared in a number of ways
- One option is on the basis of SNR required to achieve a specified probability of error
  - Not useful unless constrained on the basis of fixed rate of transmission or fixed bandwidth
  - The required channel bandwidth depends on the bandwidth of the equivalent low-pass signal, g(t)
  - Assume g(t) is a pulse of duration T and its bandwidth is approximately $W \approx 1/T$
  - Since $T = k/R = \log_2 M / R$, it follows that $W = R/\log_2 M$
  - As M is increased the channel bandwidth required, for a fixed bit rate R, decreases

Comparison of Digital Modulation Methods …

- Bandwidth efficiency is measured by the bit rate to bandwidth ratio which is $R/W = \log_2 M$
- For transmitting PAM, bandwidth efficiency is achieved for single sideband where $W \approx 1/2T$ and $R/W = 2 \log_2 M$
  - *Better than PSK by a factor of 2*
- For QAM, the two orthogonal carriers are each PAM
- Thus the rate is doubled relative to PAM; but each PAM is transmitted using double sidebands
- Thus QAM and PAM have the same bandwidth efficiency when the bandwidth is referred to the bandpass signal

Comparison of Digital Modulation Methods …

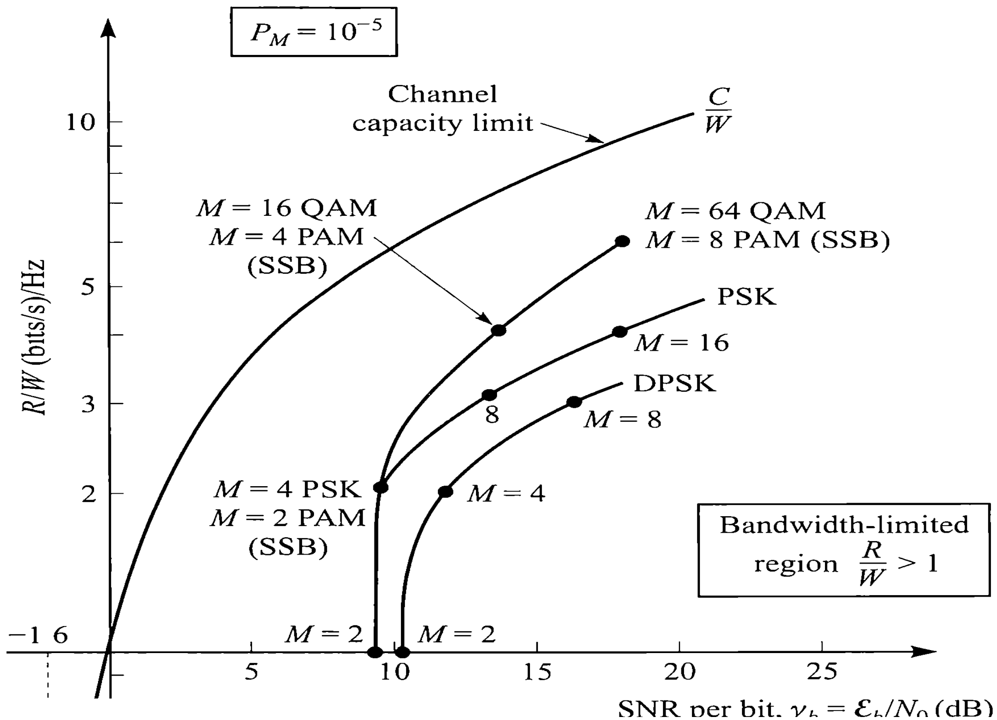
- Orthogonal signals: $M = 2^k$ orthogonal signals are transmitted using orthogonal carriers with minimum frequency separation of 1/2T for orthogonality
- The bandwidth required:

$$W = \frac{M}{2T} = \frac{M}{2(k/R)} = \frac{M}{2\log_2 M} R$$

- Note that W increase as M increases
- Similar relationships hold for simplex and biorthogonal signals for which W is half of that for orthogonal signals

Comparison of Digital Modulation Methods …

- A compact and meaningful comparison of modulation methods is the normalized data rate R/W (***Bits per second per hertz of bandwidth***) versus the SNR (ε_b/N_0) required to achieve a given error probability



Comparison of Digital Modulation Methods …