Chapter 3

Partial Differential Equations

A differential equation which involves partial derivatives is called a *partial differential equation* (PDE). For example,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \tag{3.1}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \tag{3.2}$$

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)^4 = \left(\frac{\partial u}{\partial z}\right)^3 \tag{3.3}$$

are partial differential equations.

The *order* of a partial differential equation is the order of the highest partial derivative in the equation. The *degree* of a partial differential equation is the degree of the highest partial derivatives occurring in the equation. Thus, (3.1) is of first order, (3.2) and (3.3) are of second order. (3.1) and (3.2) are of degree one while (3.3) is of degree four.

Partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions. If the number of arbitrary constants to be eliminated is equal to the number of independent variables, the partial differential equations that arise are of the first order. If the number of arbitrary constants to be eliminated is more than the number of variables, the partial differential equations obtained are of second order or higher order. If the partial differential equation is obtained by elimination of arbitrary functions, the order of the partial differential equation is, in general, equal to the number of arbitrary functions eliminated.

Example 3.1 Form partial differential equations from the following equations

1. $z = ax + by + ab$
2. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$
3. $z = y^2 + 2f(\frac{1}{x} + \ln y)$

Exercise 3.1 Form partial differential equations from the following equations ($p \equiv \frac{\partial z}{\partial x}, q \equiv \frac{\partial z}{\partial y}$)

1. $z = ax + by + a^2 + b^2$
2. $z = xy + y\sqrt{x^2 - a^2} + b$
3. $z = f(x^2 - y^2)$
4. $z = f(x + ay) + g(x - ay)$
5. $\phi\left(\dfrac{z}{x^3}, \dfrac{y}{x}\right) = 0$

Answer:
(1) $z = px + qy + p^2 + q^2$, (2) $px + qy = pq$, (3) $py + qx = 0$, (4) $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$, (5) $px + qy = 3z$

Exercise 3.2 Find the differential equation of all spheres whose centers lie on the z-axis. (*ans.* $py - qx = 0$) ◀

Equations which contain *only one* partial derivative can be solved by direct integration.

Example 3.2 Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, given that $z_y = -2\sin y$ when $x = 0$ and $z = 0$ when y is odd multiple of $\frac{1}{2}\pi$.

Exercise 3.3 Solve

1. $\ln\left(\dfrac{\partial^2 z}{\partial x \partial y}\right) = x + y$
2. $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 2y)$

Answers: (1) $z = e^{x+y} + g(y) + \phi(x)$, (2) $z = f(x) + g(y) - \frac{1}{12}\sin(2x + 3y)$. ◀

3.1 Linear PDEs of the First Order

A linear partial differential equation of the first order, involving a dependent variable z and two independent variables x, y is of the form

$$Pp + Qq = R \tag{3.4}$$

where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$ and P, Q, R are functions of x, y, z.

The general solution of (3.4) is

$$\phi(u, v) = 0, \quad \text{or} \quad u = f(v) \tag{3.5}$$

where $u \equiv u(x, y, z) = a$ and $v \equiv v(x, y, z) = b$ are two independent solutions of *Lagrange's auxiliary equations*

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{3.6}$$

Here, a and b are constants and at least one of u, v must contain z.

Example 3.3 Prove the above statement.

Example 3.4 Solve

1. $\dfrac{y^2 z}{x} p + xzq = y^2$
2. $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = xyz$

Exercise 3.4 Solve

1. $2p + 3q = 1$
2. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$
3. $y^2 p - xyq = x(z - 2y)$
4. $\dfrac{y - z}{yz} p + \dfrac{z - x}{zx} q = \dfrac{x - y}{xy}$

Answers: (1) $\phi(3x - 2y, y - 3z) = 0$, (2) $\sqrt{x} - \sqrt{y} = f(\sqrt{x} - \sqrt{y})$, (3) $\phi(x^2 + y^2, yz - y^2) = 0$, (4) $\phi(x + y + z, xyz) = 0$. ◀

3.2 Second-Order Equations

A general second-order partial differential equation in two independent variables x and y can be written as

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \tag{3.7}$$

The partial differential equations are classified into three groups:

- *Elliptic*: if $B^2 - 4AC < 0$
- *Parabolic*: if $B^2 - 4AC = 0$

- *Hyperbolic*: if $B^2 - 4AC > 0$

These three types of partial differential equations are associated with equilibrium state, diffusion state, and oscillating systems, respectively. Moreover the classification is of fundamental importance for the following reasons:

1. The classification of a partial differential equation is *independent* of the choice of coordinate system used when formulating the equation. Expressed differently, the classification is such that it does not depend on the choice of independent variables. So, for example, if a partial differential equation is of elliptic type when expressed in terms of the cartesian coordinates x and y, it will still be of elliptic type when expressed in terms of any other coordinate system like the cylindrical polar coordinates r, θ, and z.

2. The nature of an appropriate domain D and the associated auxiliary conditions (initial and/or boundary conditions) that must be imposed on the partial differential equation in order to ensure a unique solution throughout D differ according to the classification.

Physical problems whose solution is governed by the partial differential equation (3.7) are formulated in some region D of the xy-plane on a boundary C of which suitable auxiliary conditions, called *boundary conditions*, are imposed that serve to identify a particular problem. The most important types of boundary conditions are as follow:

1. **Dirichlet conditions**: $u(x, y)$ is specified at each point of the boundary C.
$$u(x, y) = \Phi(x, y) \quad \text{for } (x, y) \text{ on } C$$
where $\Phi(x, y)$ is a given function.

2. **Neumann conditions**: The normal derivative of u, $\frac{\partial u}{\partial n} = \nabla \cdot \hat{\mathbf{n}}$, is specified at each point of the boundary.
$$\frac{\partial u(x, y)}{\partial n} = \Psi(x, y) \quad \text{for } (x, y) \text{ on } C$$
where $\Psi(x, y)$ is a given function and $\frac{\partial}{\partial n}$ is the directional derivative normal to the boundary C.

3. **Mixed condition**: Both u and $\frac{\partial u}{\partial n}$ are specified at each point of the boundary.
$$u(x, y) = \Phi(x, y) \text{ and } \frac{\partial u(x, y)}{\partial n} = \Psi(x, y) \quad \text{for } (x, y) \text{ on } C$$
where $\Phi(x, y)$ and $\Psi(x, y)$ are given functions and $\frac{\partial}{\partial n}$ is the directional derivative normal to the boundary C. The conditions are also termed *Cauchy conditions*.

Some important linear partial differential equations of the second order:

Name	**Equation**	**Type**
1-dimensional wave equation	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	hyperbolic
1-dimensional heat equation	$\frac{\partial u}{\partial t} = \kappa^2 \frac{\partial^2 u}{\partial x^2}$	parabolic
2-dimensional Laplace equation	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	elliptic
1-dimensional diffusion equation	$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$	parabolic

3.3 Separation of Variables Method

In this method we assume that the solution to be product of functions, each of which involves only one of the independent variables.

The success of the method rests on the following results:

1. **Superposition principle**: If u_1 and u_2 are any solutions of a linear homogeneous partial differential equation in some region, then

$$u = c_1 u_1 + c_2 u_2 \tag{3.8}$$

where c_1 and c_2 are any constants, is also a solution of that equation in the region.

The above assumption can be extended to that fact that if $u_1, u_2, \ldots,$ is an infinite sequence of linearly independent solution of the partial differential equation, then

$$u = c_1 u_1 + c_2 u_2 + \ldots \tag{3.9}$$

is also a solution of the partial differential equation.

2. **Orthogonality**: The orthogonality properties of the eigenfunctions associated with the partial differential equation can be used to determine the coefficients $u_1, u_2, \ldots$ in the linear superposition $u = c_1 u_1 + c_2 u_2 + \ldots$ to make it satisfy the boundary conditions imposed on the partial differential equation, and so become the solution of the boundary value problem.

Example 3.5 Solve the equation $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, given $u(x,0) = 6e^{-3x}$

Exercise 3.5 Solve

1. $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ where $u(x,0) = 4e^{-x}$ $[ans.\ u = e^{\frac{1}{2}(3y-2x)}]$
2. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$ $[ans.\ z = \left(c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}\right) e^{-ky}]$ ◀

The wave equation: Consider the transverse vibration of a string stretched between two points $x = 0$ and $x = L$. The motion takes place in the xy-plane in such a manner that each point of the string moves in a direction perpendicular to the x-axis. If $u(x,y)$ denotes the displacement of the string measured from the x-axis for $t > 0$, then u satisfies the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, t > 0. \tag{3.10}$$

A typical boundary conditions are

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t \geq 0 \tag{3.11}$$

$$u(x,0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L \tag{3.12}$$

Boundary condition (3.11) states that the string is secured at the end points for all times while (3.12) gives us the initial configuration and initial velocity of each point of the string.

The solution is given by

$$u = \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi c}{L}t + B_n \sin\frac{n\pi c}{L}t\right) \sin\frac{n\pi}{L}x, \quad n = 1, 2, \ldots \tag{3.13}$$

where

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\, dx$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\frac{n\pi}{L}x\, dx$$

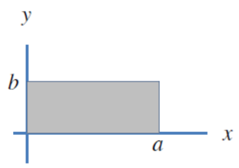
Example 3.6 Derive (3.13).

Exercise 3.6 Find the deflection $u(x,t)$ of the vibrating string ($L = 1$, ends fixed, and $c^2 = 1$) corresponding to initial zero velocity and initial deflection with $k = 0.01$:

1. $k \sin 2\pi x$
2. $kx(1-x)$

Answer:

1. $k \cos 2\pi t \sin 2\pi x$
2. $\frac{8k}{\pi^3}\left(\cos \pi t \sin \pi x + \frac{1}{27}\cos 3\pi t \sin 3\pi x + \frac{1}{125}\cos 5\pi t \sin 5\pi x + \cdots\right)$



Exercise 3.7 Vibrating membrane: Show that the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

satisfying:

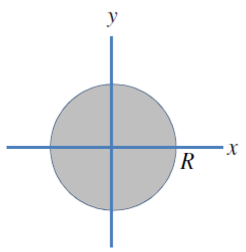
$$\begin{aligned} u &= 0 \quad \text{on the boundary of the membrane for } t \geq 0 \\ u(x,y,0) &= f(x,y) \quad \text{- given initial displacement, and} \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= g(x,y) \quad \text{- given initial velocity} \end{aligned}$$

has the solution

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A_{mn}\cos\lambda_{mn}t + B_{mn}\sin\lambda_{mn}t)\sin\tfrac{m\pi}{a}x\ \sin\tfrac{n\pi}{b}y$$

where

$$\begin{aligned} \lambda_{mn} &= c\pi\sqrt{m^2+n^2}, \quad m,n = 1,2,3,\ldots \\ A_{mn} &= \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\tfrac{m\pi}{a}x\right)\sin\left(\tfrac{n\pi}{b}y\right)dxdy \\ B_{mn} &= \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\left(\tfrac{m\pi}{a}x\right)\sin\left(\tfrac{n\pi}{b}y\right)dxdy \end{aligned}$$



Exercise 3.8 Circular membrane: Solve the wave equation for a circular membrane of radius R. Assume $u(r,t)$ is radially symmetric and does not depend on θ. The boundary conditions are

$$\begin{aligned} u(R,t) &= 0 \quad t \geq 0 \\ u(r,0) &= f(r) \quad \text{- given initial deflection, and} \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= g(r) \quad \text{- given initial velocity} \end{aligned}$$

Note that the wave equation in cylindrical coordinate is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$$