Chapter 2

Ordinary Differential Equations

2.1 Introduction

A differential equation is an equation involving an unknown function and its derivatives. A differential equation is an *ordinary differential equation* (ODE) if the unknown function depends on only one independent variable. If the unknown function depends on two or more independent variables, the differential equation is a *partial differential equation* (PDE).

An ODE can be represented as

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0 \tag{2.1}$$

where $y = y(x)$ is the sought-for function.

The function $y = \phi(x)$, which converts (2.1) into an identity, is called the *solution* of the equation. If the solution is represented implicitly, $\Phi(x, y) = 0$, then it is called an *integral*.

The *order* of a differential equation is the order of the highest derivative appearing in the equation.

Example 2.1 Check that $y = \sin x$ is a solution of the equation $y'' + y = 0$. ◄

The integral

$$\Phi(x, y, c_1, c_2, \dots, c_n) = 0 \tag{2.2}$$

of the differential equation (2.1), which contains n independent arbitrary constants $c_1, c_2, \dots, c_n$ and is equivalent (in the given region) to equation

(2.1), is called the *general solution*. By assigning definite values to the constants $c_1, c_2, \ldots, c_n$ in (2.2), we get *particular solutions*.

Conversely, if we have a family of curves (2.2) and eliminate the parameters $c_1, c_2, \ldots, c_n$ from the system of equations

$$\Phi = 0, \quad \frac{d\Phi}{dx} = 0, \ldots, \frac{d^n\Phi}{dx^n} = 0,$$

we, generally speaking, get a differential equation of type (2.1) whose general solution in the corresponding region is the relation (2.2).

Example 2.2 Find the differential equation of the family of parabolas $y = c_1(x - c_2)^2$. ◀

A differential equation along with subsidiary conditions on the unknown and its derivatives, all given at the same value of the independent variable, constitutes an *initial-value problem*. The subsidiary equations are *initial conditions*. If the subsidiary conditions are given at more than one value of the independent variable, the problem is *boundary-value problem* and the conditions are *boundary conditions*.

For example, the problem $y'' + 2y' = e^x, y(\pi) = 1, y'(\pi) = 2$ is an initial value problem, because the two subsidiary conditions are given at $x = \pi$. While the problem $y'' + 2y' = 2e^x, y(0) = 1, y'(1) = 1$ is a boundary-value problem because the two subsidiary conditions are given at $x = 0$ and $x = 1$.

Example 2.3 Find the curve of the family $y = c_1e^x + c_2e^{-2x}$ for which $y(0) = 1, y'(0) = -2$.

Exercise 2.1 Show that for the given differential equations the indicated relations are integrals (solutions)

1. $(x - 2y)y' = 2x - y, \qquad x^2 - xy + y^2 = c^2$
2. $(x - y + 1)y' = 1, \qquad y = x + ce^y$
3. $(xy - x)y'' + xy'^2 + yy' - 2y' = 0, \qquad y = \ln(xy)$

Exercise 2.2 Form the differential equations of the given families of curves

1. $y = cx$
2. $\ln \frac{x}{y} = 1 + cy$
3. $x^3 = c(x^2 - y^2)$

4. $y = c_1 \cos 2x + c_2 \sin 2x$
5. $(c_1 + c_2 x)e^x + c_3 = y$

[*ans.* $y - xy' = 0$; $y = xy' \ln \frac{x}{y}$; $3y^2 - x^2 = 2xyy'$; $y'' + 4y = 0$; $y'' - 2y' + y = 0$.]

Exercise 2.3 Form the differential equation of all circles in the xy-plane. [*ans.* $(1 + y'^2)y''' - 3y'y''^2 = 0$]

Exercise 2.4 For the given families of curves find the lines that satisfy the given initial conditions

1. $y = c_1 \sin(x - c_2)$, $y(\pi) = 1, y'(\pi) = 0$
2. $y = c_1 e^{-x} + c_2 e^x + c_3 e^{2x}$, $y(0) = 0, y'(0) = 1, y''(0) = -2$

[*ans.* $y = -\cos x$; $y = \frac{1}{6}(-5e^{-x} + 9e^x - 4e^{2x})$.] ◀

2.2 First Order Differential Equation

A differential equation of the first order in an unknown function y solved for the derivatives y', is of the form

$$y' = f(x, y) \tag{2.3}$$

Taking into account that $y' = dy/dx$, the differential equation (2.3) may be written in the symmetric form

$$P(x, y)dx + Q(x, y)dy = 0 \tag{2.4}$$

where $P(x, y)$ and $Q(x, y)$ are known functions.

2.2.1 Separable Equation

First-order equations with variables separable are of the type

$$y' = f(x)g(y) \tag{2.5}$$

or

$$X_1(x)Y_1(y)dx + X_2(x)Y_2(y)dy = 0 \tag{2.6}$$

Dividing both sides of (2.5) by $g(y)$, multiplying by dx and integrating, we get

$$\int \frac{dy}{g(y)} = \int f(x)dx + c$$

Similarly, dividing both sides of (2.6) by $X_2(x)Y_1(y)$ and integrating we obtain

$$\int \frac{X_1(x)}{X_2(x)}dx + \int \frac{Y_2(y)}{Y_1(y)}dy = 0$$

Example 2.4 Solve $y' = y/x$.

Exercise 2.5 Solve the differential equations

1. $\tan x \sin^2 y\, dx + \cos^2 x \cot y\, dy = 0$
2. $xy' - y = y^3$
3. $y - xy' = a(1 + x^2 y')$
4. $y' \tan x = y$

[*ans.* $\cot^2 y = \tan^2 x + c$; $x = cy/\sqrt{1+y^2}$; $y = a + \frac{cx}{1+ax}$; $y = c \sin x$]

Exercise 2.6 Solve the differential equations by changing the variables

1. $y' = (x+y)^2$ (Hint: set $u = x + y$)
2. $(2x - y)dx + (4x - 2y + 3)dy = 0$ (Hint: set $u = 2x - y$)

[*ans.* $\arctan(x+y) = x + c$; $5x + 10y + c = 3\ln|10x - 5y + 6|$.] ◀

2.2.2 Homogeneous Equations

A function $f(x, y)$ is a homogeneous function of degree n if, for any λ, it obeys

$$f(\lambda x, \lambda y) = \lambda^n f(x, y) \tag{2.7}$$

The differential equation

$$P(x, y)dx + Q(x, y)dy = 0$$

with homogeneous functions $P(x, y)$ and $Q(x, y)$ of equal degree can be reduced to

$$y' = f\left(\frac{y}{x}\right)$$

by means of the substitution $y = xu$, where u is a new unknown function. It is transformed to an equation with variables separable. We can also apply the substitution $x = yu$.

Example 2.5 Solve $y' = e^{\frac{y}{x}} + \frac{y}{x}$.

Exercise 2.7 Solve the differential equations

1. $y' = \dfrac{y}{x} - 1$
2. $(x - y)y\, dx + x^2\, dy = 0$
3. $y' = -\dfrac{x+y}{y}$

4. $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$
5. $y' = \dfrac{x+2y+1}{2x+4y+3}$

[ans. $y = \ln\frac{c}{x}$; $x = ce^{x/y}$; $y = \frac{c}{x} - \frac{x}{2}$; $y = \frac{c}{2}x^2 - \frac{1}{2c}$; $\ln|4x+8y+5| + 8y - 4x = c$.] ◀

2.2.3 Exact Equations

An exact first-order differential equation is of the form

$$P(x,y)dx + Q(x,y)dy = 0 \tag{2.8}$$

for which

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{2.9}$$

In this case $Pdx + Qdy$ is an exact differential

$$Pdx + Qdy = dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

from which

$$P = \frac{\partial U}{\partial x}, \quad Q = \frac{\partial U}{\partial y}$$

Since $\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x}$, we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

If (2.9) holds, the general solution is

$$U(x,y) = c \qquad (\because \quad Pdx + Qdy = dU = 0 \quad \Rightarrow \quad U(x,y) = c.)$$

Example 2.6 Find the solution of $(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0$. ◀

If for (2.8), $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, the equation (2.8) is inexact. In such cases, (2.8) can be made exact by an integrating factor $\mu(x,y)$ such that

$$\mu(Pdx + Qdy) = dU$$

so that

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q).$$

Exercise 2.8 Solve

1. $(x+y)dx + (x+2y)dy = 0$
2. $(x+y^2)dx - 2xy\,dy = 0$

[*ans.* $\frac{1}{2}x^2 + xy + y^2 = c; \quad \ln|x| - y^2/x = c.$] ◄

2.2.4 Linear Equations

A differential equation of the form

$$y' + P(x)y = Q(x) \tag{2.10}$$

is called *linear*. If $Q(x) \equiv 0$, the equation (2.10) takes the form

$$y' + P(x)y = 0 \tag{2.11}$$

which is called a homogeneous linear differential differential equation of order one. In this case, the variables may be separated and we get

$$y = ce^{-\int P(x)dx}$$

The inhomogeneous equation (2.10) can be solved by having the integrating factor

$$\mu(x) = e^{\int P(x)dx} \tag{2.12}$$

so that

$$\mu y' + P\mu y = \mu Q \tag{2.13}$$

is exact. (2.13) can be written as

$$\frac{d}{dx}(\mu y) = \mu Q \quad \Rightarrow \quad \mu y = \int \mu Q dx + c$$

so we have

$$y = \frac{\int \mu Q(x)dx + c}{\mu(x)}.$$

Example 2.7 Solve $y' + \frac{4}{x}y = x^4$. ◄

The *Bernoulli equation* is a first-order equation of the form

$$y' + P(x)y = Q(x)y^n \tag{2.14}$$

where n is a real number. The substitution

$$z = y^{1-n}$$

transforms (2.14) into a linear differential equation in the unknown function $z(x)$. It is also possible to apply directly the substitution $y = ux$.

Example 2.8 Solve $y' = \frac{4}{x}y + x\sqrt{y}$.

Exercise 2.9 Solve the differential equations

1. $y' - \frac{y}{x} = x$
2. $(1 + y^2)dx = (\sqrt{1 + y^2}\sin y - xy)dy$
3. $2xyy' - y^2 + x = 0$.

[*ans.* $y = cx + x^2$; $x\sqrt{1 + y^2} + \cos y = c$; $y^2 = x \ln \frac{c}{x}$.] ◀

2.3 Second Order D.E. with Constant Coefficients

A second order homogeneous equation with constant coefficients p and q is of the form

$$y'' + py' + qy = 0 \tag{2.15}$$

If λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda^2 + p\lambda + q = 0,$$

then the general solution to (2.15) is written in one of the following three ways

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}, \quad \text{if } \lambda_1 \text{ and } \lambda_2 \text{ are real and } \lambda_1 \neq \lambda_2 \tag{2.16}$$

$$y = e^{\lambda_1} x(c_1 + c_2 x), \quad \text{if } \lambda_1 = \lambda_2 \tag{2.17}$$

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) \quad \text{if } \lambda_1 = \alpha + \beta i \text{ and } \lambda_2 = \alpha - \beta i \tag{2.18}$$

Exercise 2.10 Verify the above statements. ◀

The solution of the second order inhomogeneous equation with constant coefficients

$$y'' + py' + qy = f(x) \tag{2.19}$$

may be written in the form

$$y = y_c + y_p \tag{2.20}$$

where y_c is the general solution of the corresponding homogeneous equation (2.15), called the *complementary solution*, and y_p is a particular solution of (2.19).

The function y_p may be found by the method of *undetermined coefficients*. But in a general case, the *method of variation of parameters* can be used. The method consists in first finding the general solution of the respective homogeneous equation, i.e., $y_c(x)$. Then, assuming the constants are functions of x, we seek the solution of the inhomogeneous equation (2.19).

Example 2.9 Solve $2y'' - y' - y = 4xe^{2x}$.

Exercise 2.11 Solve

1. $y'' + y = \cos x$
2. $y'' - 2y' + y = e^x + e^{-x}$

[*ans.* $c_1 \cos x + c_2 \sin x + \frac{1}{2}x \sin x$; $(c_1 + c_2 x + \frac{1}{2}x^2)e^{-x} + \frac{1}{4}e^x$.] ◀

2.4 Series Solution of Ordinary Differential Equations

For a linear differential equation

$$b_n(x)y^{(n)} + b_{n-1}(x)y^{(n-1)} + \ldots + b_0(x)y = R(x) \tag{2.21}$$

with polynomial coefficients, the point $x = x_0$ is called an *ordinary point* of the equation if $b_n(x_0) \neq 0$. A *singular point* of (2.21) is any point $x = x_1$ for which $b_n(x_1) = 0$.

Example 2.10 List the singular points, if any, of the equations

1. $(1 - x^2)y'' - 6xy' - 4y = 0$
2. $y'' + 2xy' + y = \sin x$
3. $y'' + \dfrac{y}{x^2}y' + \dfrac{y}{x} = 0$ ◀

2.4.1 Solution about an Ordinary Point

If $x = x_0$ is an ordinary point of (2.21), then it may be shown that *every* solution $y(x)$ of the equation is also analytic at $x = x_0$. Since every solution is analytic, $y(x)$ can be represented by a power series

$$y(x) = \sum_{n=0}^{\infty} (x - x_0)^n \tag{2.22}$$

Usually we will take $x_0 = 0$. If this is not already the case, then a substitution $x = x - x_0$ will make it so.

Murad Ridwan,
School of Electrical & Computer Engineering
AAiT, Addis Ababa University.

Example 2.11 Solve the following equations near the ordinary point $x = 0$

1. $y' - y = 0$
2. $y' = 2xy$
3. $y'' + 4y = 0$
4. $(1 - x^2)y'' - 6xy' - 4y = 0$

Exercise 2.12 Find the general solutions near the origin

1. $xy' - 3y = 6$
2. $(1 - x^2)y' = 2xy$
3. $y'' + 3xy' + 3y = 0$
4. $y'' - \dfrac{2y}{(1-x)^2} = 0$
5. $(1 - x^2)y'' - 2xy' + 2y = 0$
6. $(1 + x^2)y'' - 4xy' + 6y = 0$

Answers

1. $y = -2 + a_3x^3$
2. $y = a_0(1 + x^2 + x^4 + \ldots) = a_0/(1 - x^2)$
3. $y = a_0\left[1 + \sum_{n=1}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!}\right] + a_1\left[x + \sum_{n=1}^{\infty} \frac{(-3)^n x^{2n+1}}{3\cdot5\cdot7\cdots(2n+1)}\right]$
4. $y = a_0\frac{1}{1-x} + a_1(1 - x)^2$
5. $y = a_0x + a_1(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \frac{1}{7}x^8 - \ldots)$
6. $y = a_0(1 - 3x^2) + a_1(1 - \frac{1}{3}x^3)$

Exercise 2.13 Find the general solutions about the indicated points

1. $y'' - 2(x + 3)y' - 3y = 0$ about $x = -3$
2. $y'' + (x - 1)^2y' - 4(x - 1)y = 0$ about $x = 1$

Answer

1. $y = a_0\left[1 + \sum_{n=1}^{\infty} \frac{3\cdot7\cdot11\cdots(4n-1)(x+3)^{2n}}{(2n)!}\right] + a_1\left[(x + 3) + \sum_{n=1}^{\infty} \frac{5\cdot9\cdot13\cdots(4n+1)(x+3)^{2n+1}}{(2n+1)!}\right]$
2. $y = a_0\sum_{n=0}^{\infty} \frac{4(-1)^n(x-1)^{3n}}{3^n(3n-1)(3n-4)n!} + a_1\left[(x - 1) + \frac{1}{4}(x - 1)^4\right]$ ◀

2.4.2 Solution about a Regular Singular Point

Suppose that the point $x = x_0$ is a singular point of the equation

$$b_2(x)y'' + b_1(x)y' + b_0(x)y = 0 \tag{2.23}$$

with polynomial coefficients. Then $b_2(x_0) = 0$, so $b_2(x_0)$ has a factor $(x-x_0)$ to some power. Let us put (2.23) into the form

$$y'' + p(x)y' + q(x)y = 0 \tag{2.24}$$

If $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are both analytic at $x = x_0$, we have the necessary and sufficient conditions to have a finite solution about x_0. Singular points that have this property are called *regular singular* points, whereas any point not satisfying both these criteria is termed an *irregular* or *essential* singularity.

Example 2.12 Discuss the singularities of

1. $x^4(x^2 + 1)(x - 1)^2 y'' + 4(x - 1)y' + (x + 1)y = 0$
2. $(1 - x^2)y'' - 2xy' + l(l + 1) = 0$

Example 2.13 Show that the the equation $x^2y'' + (x^2 - x)y' + 2y = 0$ has no series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ at the singular point $x = 0$. ◀

If $x = 0$ is a regular singular point of (2.24), then it can be shown that there exists *at least one solution* of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n \tag{2.25}$$

where the exponent r may be any real or complex number, and $a_0 \neq 0$. Such a series is called a generalized power series or *Frobenius series*.

Substitute (2.25) in (2.24) and equate the coefficient of the *lowest power* of x. This gives a quadratic equation in r, which is known as the *indicial equation*.

Equate to zero the coefficients of other powers of x to find $a_1, a_2, \ldots$ in terms of a_0. Substitute the values of $a_1, a_2, \ldots$ in (2.25) to get the series solution of (2.24) having a_0 as arbitrary constant. Obviously this is not the complete solution of (2.23), since the complete solution must have two independent arbitrary constants. The method of complete solution depends on the nature of roots of the indicial equation.

Case I Distinct roots r_1, r_2 not differing by an integer (i.e., $r_1 - r_2$ is not an integer). The complete solution is

$$y(x) = c_1 y|_{r=r_1} + c_2 y|_{r=r_2} \tag{2.26}$$

Case II When the roots are equal $r_1 = r_2$. The complete solution is

$$y(x) = c_1 y|_{r=r_1} + c_2 \left.\frac{\partial y(x,r)}{\partial r}\right|_{r=r_1} \tag{2.27}$$

Murad Ridwan,
School of Electrical & Computer Engineering
AAiT, Addis Ababa University.

Case III When the roots r_1 and r_2 ($r_1 > r_2$) are distinct and differ by an integer (i.e., $r_1 - r_2$ is an integer). The complete solution is

$$y(x) = c_1 y|_{r=r_1} + c_2 \left. \frac{\partial[(r-r_2)y(x,r)]}{\partial r} \right|_{r=r_2} \tag{2.28}$$

Example 2.14 Solve $y'' - \dfrac{y'}{2x} + \dfrac{x^2+1}{2x^2}y = 0$ (Case I)

Example 2.15 Differentiation of a product function: Suppose that

$$u = u_1 u_2 \cdots u_n$$

each of the u's being a function of the parameter r. Show that

$$u' = u\left\{\frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \cdots + \frac{u_n'}{u_n}\right\}$$

where $u' = \dfrac{du}{dr}, u_k' = \dfrac{du_k'}{dr}$ $(k = 1, 2, \dots, n)$.

Example 2.16 Differentiate with respect to r.

1. $y = (ar + b)^k$
2. $y = \dfrac{r^2(r+1)}{(4r-1)^3(7r+2)^6}$
3. $y = \dfrac{r+n}{r(r+1)(r+2)\cdots(r+n-1)}$
4. $y = \dfrac{r^3}{[(r+2)(r+3)\cdots(r+n+1)]^2}$

Example 2.17 Solve $x^2y'' + xy' + x^2y = 0$ (Case II)

Example 2.18 Solve $x^2y'' + (x^2 - 2x)y' + 2y = 0$ (Case III).

Exercise 2.14 Solve the following

1. $9x(1-x)y'' - 12y' + 4y = 0$
2. $x^2y'' - xy' + y = 0$
3. $x(1-x)y'' - 3y' - y = 0$
4. $2x(1-x)y'' + (1-x)y' + 3y = 0$

Answers

1. $y = A\left[1 + \frac{1}{3}x + \frac{1\cdot4}{3\cdot6}x^2 + \frac{1\cdot4\cdot7}{3\cdot6\cdot9}x^3 + \cdots\right] +$
   $Bx^{7/3}\left[1 + \frac{8}{9}x + \frac{8\cdot11}{10\cdot13}x^2 + \frac{8\cdot11\cdot14}{10\cdot13\cdot16}x^3 + \cdots\right]$
2. $y = Ax + Bx\ln x$
3. $y = Ax(1 + 2x + 3x^2 + \cdots) + B[y_1 \ln x + C(1 + x + x^2 + x^3 + \cdots)]$
4. $y = A\sqrt{x}(1-x) + B\left(1 - 3x + \frac{3x^2}{1\cdot3} + \frac{3x^3}{3\cdot5} + \frac{3x^4}{5\cdot7}\right)$ ◀

2.5 Numerical Solution of Differential Equations

Most differential equations have no known analytical solution, and even when one can be found it is often difficult to use. As a result, when solutions are required and an analytical solution either is not known or is inconvenient to use, it becomes necessary to use methods that produce a numerical solution directly. However, unlike the general analytical solution of an initial value problem that can be adapted to any appropriate initial conditions, a numerical solution is the solution of a specific initial value problem, so the calculation must be repeated if the initial conditions are changed.

Consider the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{2.29}$$

We start from $y_0 = y(x_0)$ and proceed stepwise. In the first step we compute an approximate value y_1 of the solution y of (2.29) at $x = x_1 = x_0 + h$. In the second step we compute an approximate value of y_2 of the solution at $x = x_2 = x_0 + 2h$, etc. Here h is a fixed number.

In each step the computations are done by the same formula, usually such formulas are suggested by the Taylor series

$$\begin{aligned} y(x+h) &= y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \cdots \\ &= y(x) + hf + \frac{h^2}{2!}f'' + \frac{h^3}{3!}f''' + \cdots \end{aligned} \tag{2.30}$$

where $f', f'', f''', \ldots$ are computed at $(x, y(x))$.

2.5.1 Euler's Method

For small h in (2.30), we can approximate

$$\begin{aligned} y(x+h) &\approx y(x) + hf, \quad \text{or} \\ y_{n+1} &= y_n + hf(x_n, y_n), \quad n = 0, 1, 2, \ldots \end{aligned}$$

where $x_n = x_0 + nh$. This is called Euler's method. Geometrically it is an approximation of the curve $y(x)$ by a polygon whose first side is tangent to the curve at x_0 (Fig. 2.1).

Example 2.19 Find $y(1.5)$ for $y' = 2xy, y(1) = 1$ using $h = 0.10$ and $h = 0.05$. Compare with actual values. Verify Figures 2.2 and 2.3 ◀

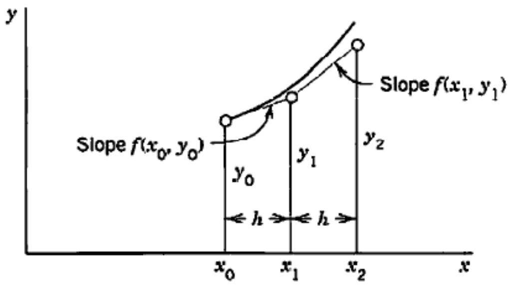

Figure 2.1: Euler method.

x_n	y_n	True Value	|Error|	% Error
1.00	1.0000	1.0000	0.0000	0.00
1.10	1.2000	1.2337	0.0337	2.73
1.20	1.4640	1.5527	0.0887	5.71
1.30	1.8154	1.9937	0.1784	8.95
1.40	2.2874	2.6117	0.3243	12.42
1.50	2.9278	3.4903	0.5625	16.12

Figure 2.2: Euler method, $h = 0.1$.

2.5.2 Improved Euler Method

Taking more terms in (2.30) would produce more accurate numeric results but the difficulty is that we introduce higher order derivatives. Hence the general strategy is to avoid the computation of higher derivatives and to replace it by computing f for one or several suitably chosen auxiliary values of (x, y).

In the improved Euler method (also called Heun method), in each step we compute first the auxiliary value

$$y^*_{n+1} = y_n + hf(x_n, y_n) \tag{2.31}$$

and the new value

$$y_{n+1} = y_n + h\frac{f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})}{2} \tag{2.32}$$

Geometrically, in the interval from x_n to $x_n + \frac{1}{2}h$ we approximate the solution by a straight line through (x_n, y_n) with slope $f(x_n, y_n)$, and then we continue along the line with slope $f(x_{n+1}, y^*_{n+1})$ until x reaches x_{n+1}.

x_n	y_n	True Value	|Error|	% Error
1.00	1.0000	1.0000	0.0000	0.00
1.05	1.1000	1.1079	0.0079	0.72
1.10	1.2155	1.2337	0.0182	1.47
1.15	1.3492	1.3806	0.0314	2.27
1.20	1.5044	1.5527	0.0483	3.11
1.25	1.6849	1.7551	0.0702	4.00
1.30	1.8955	1.9937	0.0982	4.93
1.35	2.1419	2.2762	0.1343	5.90
1.40	2.4311	2.6117	0.1806	6.92
1.45	2.7714	3.0117	0.2403	7.98
1.50	3.1733	3.4903	0.3171	9.08

Figure 2.3: Euler method, $h = 0.05$.

The improved Euler method is a *predictor-corrector method*, because in each step we first predict a value by (2.31) and then correct it by (2.32).

Example 2.20 Use the improved Euler's method to obtain an approximate value of $y(1.5)$ for the solution

$$y' = 2xy, \quad y(1) = 1$$

Compare the results for $h = 0.1$ and $h = 0.05$. Verify Figures 2.4 and 2.5 ◀

x_n	y_n	True Value	|Error|	% Error
1.00	1.0000	1.0000	0.0000	0.00
1.10	1.2320	1.2337	0.0017	0.14
1.20	1.5479	1.5527	0.0048	0.31
1.30	1.9832	1.9937	0.0106	0.53
1.40	2.5908	2.6117	0.0209	0.80
1.50	3.4509	3.4903	0.0394	1.13

Figure 2.4: Improved Euler method, $h = 0.1$.

2.5.3 The Runge-Kutta Method

Probably on of the most popular as well as accurate numerical procedures used in obtaining approximate solution to differential equations is the fourth-order Runge-Kutta method.

The fourth-order Runge-Kutta method consists of determining appropirate constants so that a formula such as

$$y_{n+1} = y_n + ak_1 + bk_2 + ck_3 + dk_4$$

x_n	y_n	True Value	|Error|	% Error
1.00	1.0000	1.0000	0.0000	0.00
1.05	1.1078	1.1079	0.0002	0.02
1.10	1.2332	1.2337	0.0004	0.04
1.15	1.3798	1.3806	0.0008	0.06
1.20	1.5514	1.5527	0.0013	0.08
1.25	1.7531	1.7551	0.0020	0.11
1.30	1.9909	1.9937	0.0029	0.14
1.35	2.2721	2.2762	0.0041	0.18
1.40	2.6060	2.6117	0.0057	0.22
1.45	3.0038	3.0117	0.0079	0.26
1.50	3.4795	3.4903	0.0108	0.31

Figure 2.5: Improved Euler method, $h = 0.05$.

agrees with a Taylor expansion out to h^4 or the fifth term. The constants a, b, c, d and the auxiliary quantities k_1, k_2, k_3, k_4 are given as

$$\begin{aligned} y_{n+1} &= y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1) \\ k_3 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2) \\ k_4 &= hf(x_n + h, y_n + k_3) \end{aligned} \tag{2.33}$$

The method is well suited to the computer because it needs no special starting procedure, makes light demand on storage, and repeatedly uses the same straight forward computational procedure. It is numerically stable.

Example 2.21 Derive the second order Runge-Kutta method by finding the constants a, b, α, and β such that the formula

$$y_{n+1} = y_n + ak_1 + bk_2$$

where

$$k_1 = hf(x_n, y_n), \quad k_2 = hf(x_n + \alpha h, y_n + \beta k_1)$$

agrees with a Taylor series expansion out to h^3.

Exercise 2.15 Derive (2.33).

Example 2.22 Use the Runge-Kutta method to obtain an approximation to $y(1, 5)$ for the solution

$$y' = 2xy, \quad y(1) = 1$$

use $h = 0.1$ and verify Figure 2.6. ◀

Murad Ridwan,
School of Electrical & Computer Engineering
AAiT, Addis Ababa University.

x_n	y_n	True Value	|Error|	% Error
1.00	1.0000	1.0000	0.0000	0.00
1.10	1.2337	1.2337	0.0000	0.00
1.20	1.5527	1.5527	0.0000	0.00
1.30	1.9937	1.9937	0.0000	0.00
1.40	2.6116	2.6117	0.0001	0.00
1.50	3.4902	3.4903	0.0001	0.00

Figure 2.6: Runge-Kutta method, $h = 0.1$.

2.5.4 Milne's Method

In a *one-step method* we compute y_{n+1} using only a single step, namely, the previous value y_n. One-step methods are 'self-starting', they need not help to get going because they obtain y_1 from the initial value y_0, etc. The Euler, Improved Euler and Runge-Kutta are examples of a one-step method.

In contrast, a *multi-step method* uses in each step values from two or more previous steps. These methods are motivated by the expectation that the additional information will increase accuracy and stability. But to get started, one needs values, say, y_0, y_1, y_2, y_3 in a 4-step method, obtained by Runge-Kutta or other accurate methods. Thus, multi-step methods are not self-starting.

Milne's method is an example of multi-step method. The predictor is

$$y^*_{n+1} = y_{n-3} + \tfrac{4}{3}h(2y'_n - y'_{n-1} + 2y'_{n-2}) \qquad \text{for } n \geq 3 \tag{2.34}$$

where

$$y'_n = f(x_n, y_n), \quad y'_{n-1} = f(x_{n-1}, y_{n-1}), \quad y'_{n-2} = f(x_{n-2}, y_{n-2})$$

and the corrector is

$$y_{n+1} = y_{n-1} + \tfrac{1}{3}h(y'_{n+1} + 4y'_n + y'_{n-1}) \tag{2.35}$$

where $y'_{n+1} = f(x_{n+1}, y^*_{n+1})$.

Exercise 2.16 For the following differential equations, construct a table computing the indicated values of $y(x)$ using Euler, improved Euler and Runge-Kutta methods. Use $h = 0.1$ and compute to four rounded decimal places.

1. $y' = 2\ln(xy), \quad y(1) = 2.$
   $y(1.1),\ y(1.2),\ y(1.3),\ y(1.4),\ y(1.5)$

2. $y' = \sin x^2 + \cos y^2, \quad y(0) = 0.$
   $y(0.1),\ y(0.2),\ y(0.3),\ y(0.4),\ y(0.5)$
3. $y' = \sqrt{x+y}, \quad y(0.5) = 0.5$
   $y(0.6),\ y(0.7),\ y(0.8),\ y(0.9),\ y(1.0)$
4. $y' = xy + y^2, \quad y(1) = 0.$
   $y(1.1),\ y(1.2),\ y(1.3),\ y(1.4),\ y(1.5)$

Exercise 2.17 Use the Euler method to obtain the approximate value of $y(0.2)$ where $y(x)$ is the solution of

$$y'' + xy' + y = 0$$
$$y(0) = 1, \quad y'(0) = 2.$$

Exercise 2.18 Use Milne's method to approximate the value of $y(0.4)$ where

$$y' = 4x - 2y, \quad y(0) = 2$$

Use the Runge-Kutta formula and $h = 0.1$ to obtain the values of y_1, y_2 and y_3. ◀

2.6 Special Functions

2.6.1 Gamma and Beta Functions

The *gamma function* denoted by $\Gamma(n)$ is defined as

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \tag{2.36}$$

It may be regarded as a generalization of $n!$ for n positive integers. In particular $\Gamma(1) = 1$.

A recursive formula for the gamma functions is

$$\Gamma(n+1) = n\Gamma(n) \tag{2.37}$$

In particular if n is a positive integer, then

$$\Gamma(n+1) = n!, \quad n = 1, 2, 3, \ldots \tag{2.38}$$

For this reason $\Gamma(n)$ is sometimes called the *factorial function.*

Example 2.23 Verify (2.37) and (2.38). ◀

From (2.37), $\Gamma(n)$ can be determined for all $n > 0$ when the values for $1 \leq n < 2$ (or any other interval of unit length) are known. For instance,

$$\Gamma\left(\frac{11}{4}\right) = \Gamma\left(\frac{7}{4} + 1\right) = \frac{7}{4}\Gamma\left(\frac{7}{4}\right) = \frac{3}{4}\Gamma\left(\frac{3}{4}\right)$$

The value of $\Gamma\left(\frac{3}{4}\right)$ can be obtained from the table of gamma functions.

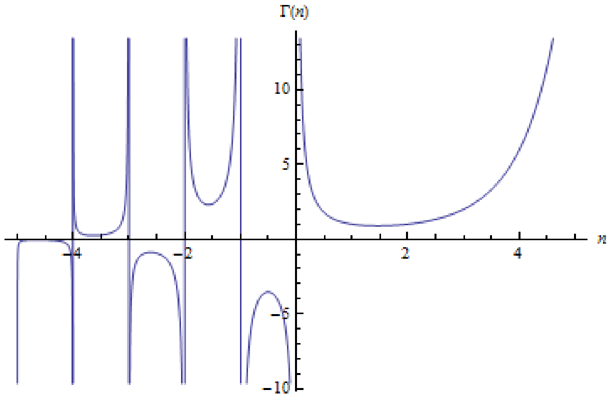

Figure 2.7: The gamma function.

Example 2.24 Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Example 2.25 Evaluate the following integrals

1. $\int_0^\infty x^3 e^{-x} dx$
2. $\int_0^\infty x^6 e^{-2x} dx$
3. $\int_0^\infty 3^{-4z^2} dz$
4. $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$
5. $\int_0^\infty x^m e^{-ax^n} dx$, m, n, a positive constants. ◀

The *beta function* is defines as

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \qquad m > 0, n > 0 \tag{2.39}$$

The beta function is symmetric, i.e.,

$$B(m, n) = B(n, m) \tag{2.40}$$

The beta function can be expressed through gamma function

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{2.41}$$

Example 2.26 Verify (2.40) and (2.41). ◀

Many integrals can be expressed through beta and gamma functions. Two of special interest are

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}B(m,n)$$
$$\int_0^{\infty} \frac{x^{p-1}}{1+x}dx = \Gamma(p)\Gamma(p-1) = \frac{\pi}{\sin \pi p}, \quad 0 < p < 1$$

Exercise 2.19 Show that

1. $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$
2. $\Gamma\left(-\frac{5}{2}\right) = -\frac{8}{15}\sqrt{\pi}$
3. $\displaystyle\int_0^1 x^m (\ln x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, \quad n$ is positive integer and $m > -1$
4. $\displaystyle\int_{-\infty}^{\infty} e^{-k^2x^2} dx = \frac{\sqrt{\pi}}{k}$
5. $\displaystyle\int_0^{\infty} \frac{x^c}{c^x} dx = \frac{\Gamma(c+1)}{(\ln c)^{c+1}}, c > 1$
6. $\displaystyle\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$
7. $yB(x+1,y) = xB(x,y+1)$
8. $\displaystyle\int_0^a x^{n-1}(a-x)^{m-1}dx = a^{m+n-1}B(m,n)$
9. $\displaystyle\int_a^b (x-a)^m(b-x)^n dx = (b-a)^{m+n+1}B(m+1,n+1)$ [Hint: put $x = a+(b-a)z$]
10. $\displaystyle\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$
11. $B(m,n) = B(m+1,n) + B(m,n+1)$
12. $\displaystyle\int_0^{\infty} \frac{y^{q-1}}{(1+y)^{p+q}}dy = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx = B(p,q)$ ◀

2.6.2 Legendre Functions

The differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{2.42}$$

when n is real number, is called Legendre's differential equation. This equation is of considerable importance, particularly in boundary value problems involving spherical configurations. Any solution of (2.42) is called *Legendre function.*

Dividing (2.42) by $1-x^2$ we see that the coefficients of the resulting equation are analytic at $x = 0$, so that we may apply the power series method $y = \sum_{m=0}^{\infty} a_m x^m$. The resulting solution becomes

$$y(x) = Ay_1(x) + By_2(x) \tag{2.43}$$

where

$$\begin{aligned} y_1(x) &= 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - + \cdots \\ y_2(x) &= x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)n(n+2)(n+4)}{5!}x^5 - + \cdots \end{aligned}$$

These series converge for $|x| < 1$.

Exercise 2.20 Verify (2.43). ◀

In many applications, the parameter n in Legendre's equation is non-negative integer. In this case the series has only finite terms. It is customary to choose $a_m = 1$ when $m = 0$. Then the resulting solution of (2.42) is called the *Legendre polynomial of degree* n and is denoted by $P_n(x)$ which is given by

$$\begin{aligned} P_n(x) &= \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \\ &= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + - \cdots \end{aligned} \tag{2.44}$$

where $M = n/2$ or $(n-1)/2$, whichever is an integer.

Exercise 2.21 Verify (2.44). ◀

In particular

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \tfrac{1}{2}(3x^2 - 1) \\
P_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \tfrac{1}{8}(63x^5 - 70x^3 + 15x) \\
P_6(x) &= \tfrac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) \\
P_7(x) &= \tfrac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)
\end{aligned}
$$

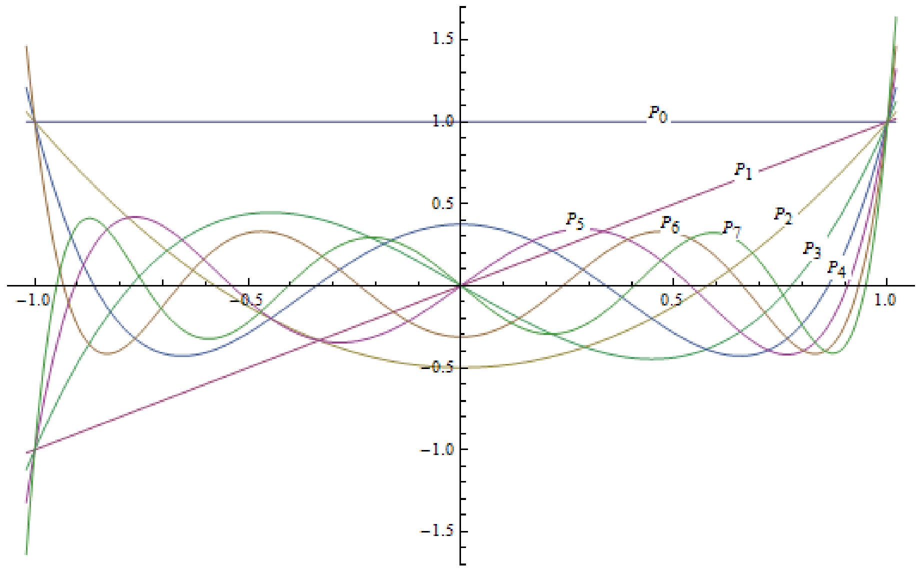

Figure 2.8: Legendre polynomials.

Rodrigues' formula: The Legendary polynomials can be obtained from Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2 - 1)^n \right] \tag{2.45}$$

Exercise 2.22 Verify (2.45) by applying the binomial theorem to $(x^2 - 1)^n$, differentiating n times term by term, and comparing the result with (2.44).

Example 2.27 Express $f(x) = x^3 - 5x^2 + x + 2$ in terms of the Legendre polynomials.

Generating function: It can be shown that $P_n(x)$ is the coefficient of u^n in the expansion of $(1-2xu+u^2)^{-1/2}$ in ascending powers of u, i.e.,

$$\frac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n \tag{2.46}$$

Exercise 2.23 Verify (2.46). ◀

Orthogonality: The Legendre polynomials are orthogonal over $[-1, 1]$, i.e.,

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, & \text{if } m \neq n; \\ \frac{2}{2n+1}, & \text{if } m = n. \end{cases} \tag{2.47}$$

Exercise 2.24 Prove (2.47). ◀

The orthogonality of Legendre polynomials enable us to expand a function $f(x)$, defined from $x=-1$ to $x=1$, in a series of Legendre polynomials.

Let

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \tag{2.48}$$

To determine a_n, multiply both sides of (2.48) by $P_m(x)$ and integrating w.r.t. x from -1 to 1, we have

$$\begin{aligned} \int_{-1}^{1} f(x)P_m(x)dx &= \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x)P_m(x) \\ &= \sum_{n=0}^{\infty} a_n \frac{2}{2n+1}\delta_{mn}, \quad \text{where } \delta_{mn} = \begin{cases} 1, & \text{if } m = n; \\ 0, & \text{otherwise.} \end{cases} \\ &= a_m \frac{2}{2m+1} \end{aligned}$$

which gives us

$$a_n = (n+\tfrac{1}{2}) \int_{-1}^{1} f(x)P_n(x)dx$$

The expansion of $f(x)$ given by (2.48) is known as *Fourier-Legendre series.*

Exercise 2.25 Using (2.46) show that

1. $P_n(1) = 1$

2. $P_n(-x) = (-1)^n P_n(x)$
3. $P_{2n}(0) = (-1)^n \dfrac{2n!}{2^{2n}(n!)^2}$
4. $P_{2n+1}(0) = 0$

Exercise 2.26 Verify the following recurrence relations

1. $(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$
2. $nP_n(x) = xP'_n(x) - P'_{n-1}(x)$
3. $(2n+1)P_n(x) = P'_{n+1}(x) - nP'_{n-1}(x)$
4. $P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x)$
5. $(1-x^2)P'_n(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$
6. $(1-x^2)P'_n(x) = (n+1)\left[xP_n(x) - P_{n+1}(x)\right]$

Exercise 2.27 Show that

1. $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$
2. $P_n(-1) = (-1)^n$
3. $P'_n(1) = \frac{1}{2}n(n+1)$
4. $P'_n(-1) = \frac{1}{2}(-1)^{n-1}n(n+1)$

Exercise 2.28 Prove that

1. $\displaystyle\int_{-1}^{1} P_n(x)dx = 0$ if $n \neq 0$
2. $\displaystyle\int_{-1}^{1} P_0(x)dx = 2$
3. $\displaystyle\int_{-1}^{1} x^m P_n(x)dx = 0$ if m, n are positive integers and $m < n$
4. $\displaystyle\int_{-1}^{1} xP_n(x)P_{n-1}dx = \frac{2n}{4n^2-1}$
5. $\displaystyle\int_{0}^{1} P_n(x)dx = \frac{1}{n+1}P_{n-1}(0)$

Exercise 2.29 Obtain the Fourier-Legendre expansion of $f(x)$ when

1. $f(x) = \begin{cases} 0, & -1 < x < 0; \\ 1, & 0 < x < 1. \end{cases}$
2. $f(x) = \begin{cases} 0, & -1 < x \leq 0; \\ x, & 0 < x < 1. \end{cases}$

Answer

1. $\frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \cdots$
2. $\frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + \cdots$ ◄

2.6.3 Bessel's Function

The differential equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \tag{2.49}$$

is called *Bessel's equation of order n* and its particular solutions are called *Bessel's function of order n*.

Since $x = 0$ is a regular singular point of (2.49), we can use Frobenius series to obtain the solution. The two roots of the resulting indicial equation are $r = n\ (\geq 0)$ and $r = -n$. Based on the value of r we have different solutions:

Case I When $n \neq 0$ or when n is not an integer.

The solution is

$$y(x) = c_1 J_n(x) + c_2 J_{-n}(x) \tag{2.50}$$

where $J_n(x)$, called the *Bessel function of the first kind of order n*, is

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m} \tag{2.51}$$

and $J_{-n}(x)$, called the *Bessel function of the first kind of order −n*, is

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(-n+m+1)} \left(\frac{x}{2}\right)^{-n+2m} \tag{2.52}$$

Case II When $n = 0$.

The Bessel's equation takes the form

$$xy'' + y' + xy = 0$$

This is called Bessel's equation of order zero. The complete solution is

$$y(x) = c_1 J_0(x) + c_2 Y_0(x) \tag{2.53}$$

where $J_0(x)$ is Bessel function of the first kind of order zero. From (2.51),

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}, \qquad [\text{since } \Gamma(m+1) = m!] \tag{2.54}$$

$Y_0(x)$ is called *Bessel function of the second kind of order zero* or Neumann function.

$$Y_0(x) = J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(m!)^2} \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right] \tag{2.55}$$

Murad Ridwan,
School of Electrical & Computer Engineering
AAiT, Addis Ababa University.

Case III When n is an integer.

The two functions $J_n(x)$ and $J_{-n}(x)$ are not independent but are connected by the relation

$$J_{-n}(x) = (-1)^n J_n(x)$$

A solution can be obtained by letting $y = u(x)J_n(x)$ in (2.49). Solving for $u(x)$, we obtain

$$u(x) = c_2 \int \frac{dx}{xJ_n^2(x)} + c_1$$

The complete solution becomes

$$y = \left[c_2 \int \frac{dx}{xJ_n^2(x)} + c_1\right] J_n(x)$$

or

$$y(x) = c_1 J_n(x) + c_2 Y_n(x) \tag{2.56}$$

where

$$Y_n(x) = J_n(x) \int \frac{dx}{xJ_n^2(x)} \tag{2.57}$$

The function $Y_n(x)$ is called the *Bessel function of the second kind of order n* or *Neumann function of order n*.

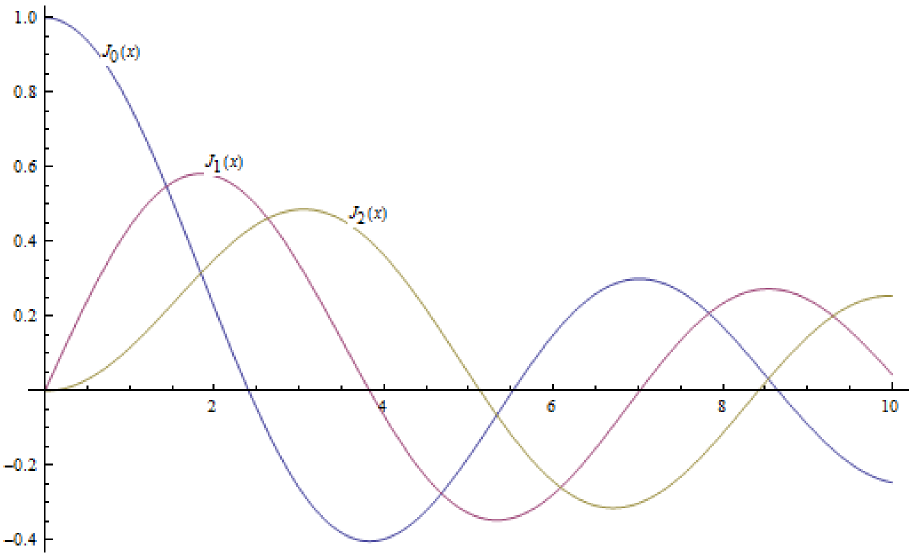

Figure 2.9: The first three integer order Bessel functions of the first kind.

Murad Ridwan,
School of Electrical & Computer Engineering
AAiT, Addis Ababa University.
Nov 2012.

Exercise 2.30 Derive (2.50), (2.53) and (2.56). ◄

Generating function: Bessel functions of various orders can be derived as coefficients of various power in the expansion

$$\exp\left[\frac{x}{2}(t-\tfrac{1}{t})\right] = \sum_{n=-\infty}^{\infty} t^n J_n(x) \tag{2.58}$$

Exercise 2.31 Verify (2.58). ◄

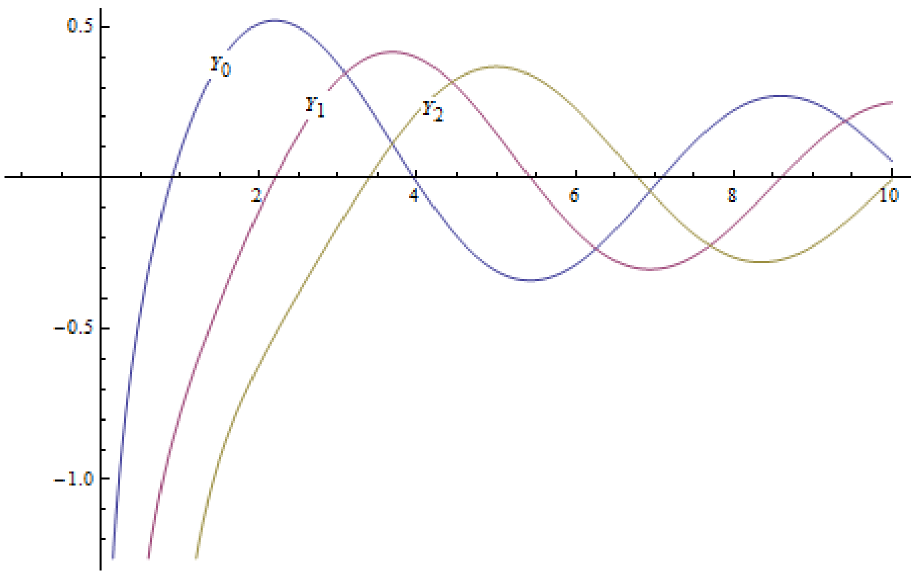

Figure 2.10: The first three integer order Bessel functions of the second kind.

Orthogonality: If α and β are the roots $J_n(x) = 0$, then

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \begin{cases} 0, & \text{if } \alpha \neq \beta; \\ \frac{1}{2}J_{n+1}^2(\alpha), & \text{if } \alpha = \beta. \end{cases} \tag{2.59}$$

From the orthogonality property of Bessel functions, we can expand a function $f(x)$ in fourier-Bessel series in the range 0 to a. Let

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\lambda_i x) = c_1 J_n(\lambda_1 x) + c_2 J_n(\lambda_2 x) + \dots \tag{2.60}$$

where $\lambda_1, \lambda_2, \dots$ are the roots of the equation $J_n(\lambda a) = 0$.

To determine c_i, we multiply both sides of (2.60) by $xJ_n(\lambda_i x)$ and integrate w.r.t. x between the limits 0 to a. From the orthogonality property, all integrals on the right hand side will vanish except the one containing c_i and we have

$$c_i = \frac{2}{a^2 J_{n+1}^2(\lambda_i a)} \int_0^a x f(x) J_n(\lambda_i x) dx \tag{2.61}$$

Exercise 2.32 Verify (2.59) and (2.61).

Exercise 2.33 Show that the Fourier-Bessel series in $J_2(\lambda_i x)$ for $f(x) = x^2$ $(0 < x < a)$ where $\lambda_i a$ are positive roots of $J_2(x) = 0$, is $x^2 = 2a^2 \sum_{i=1}^{\infty} \frac{J_2(\lambda_i x)}{a\lambda_i J_3(\lambda_i a)}$. ◀

Example 2.28 Show that

1. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$
2. $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

Exercise 2.34 Show that

1. $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$
2. $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$
3. $J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x)$
4. $J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$
5. $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{3-x^2}{x^2} \sin x - \frac{3}{2} \cos x\right]$
6. $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$
7. $\displaystyle\int J_3(x) dx = -J_2(x) - \frac{2}{x} J_2(x)$
8. $\displaystyle\int x J_0^2(x) dx = \frac{1}{2} x^2 [J_0^2(x) + J_1^2(x)] + c$, $c-$ a constant ◀

A number of second order differential equations with variable coefficients can be reduced to Bessel's equation by a suitable transformation and, hence, can be solved in terms of Bessel functions.

Consider the differential equation

$$x^2 y'' + (1 - 2\alpha) x y' + [\beta^2 \gamma^2 x^{2\gamma} + (\alpha^2 - n^2 \gamma^2)] y = 0 \tag{2.62}$$

where α, β, γ and n are constants. Putting $X = \beta x^\gamma$ and $Y = x^{-\alpha} y$, equation (2.62) reduces to

$$X^2 Y'' + XY' + (X^2 - n^2) Y = 0 \tag{2.63}$$

which is Bessel's equation. The solution is

$$Y(X) = \begin{cases} c_1 J_n(X) + c_2 Y_n(X), & \text{for } n\text{- integer;} \\ c_1 J_n(X) + c_2 J_{-n}(X), & \text{for } n\text{- noninteger.} \end{cases} \tag{2.64}$$

In terms of the original variables x and y, the solution becomes

$$y(x) = \begin{cases} x^\alpha [c_1 J_n(\beta x^\gamma) + c_2 Y_n(\beta x^\gamma)], & \text{for } n\text{- integer;} \\ x^\alpha [c_1 J_n(\beta x^\gamma) + c_2 J_{-n}(\beta x^\gamma)], & \text{for } n\text{- noninteger.} \end{cases} \tag{2.65}$$

Example 2.29 Solve

1. $y'' - \frac{2}{x} y' + 4(x^2 - \frac{1}{x^2}) y = 0$
2. $xy'' - 3y' + xy = 0$

Exercise 2.35 Find the solutions of the following differential equations in terms of Bessel functions.

1. $xy'' + y = 0$
2. $y'' + \frac{1}{x} y' + \left(3 - \frac{1}{4x^2} y\right) = 0$
3. $xy'' + y' + \frac{1}{4} y = 0$

Answer:

1. $y(x) = \sqrt{x}\,[c_1 J_1(2\sqrt{x}) + c_2 Y_1 \sqrt{x})]$
2. $y(x) = c_1 J_{1/2}(\sqrt{3}x) + c_2 J_{-1/2}(\sqrt{3}x)$
3. $y(x) = c_1 J_0(\sqrt{x}) + c_2 Y_0(\sqrt{x})$ ◀

2.6.4 Hypergeometric Function

The differential equation

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0 \tag{2.66}$$

where a, b, c are real constants, is called the *(Gauss's) hypergeometric differential equation*. It has regular singular points at $x = 0, 1$. The corresponding indicial equation has roots $r_1 = 0$ and $r_2 = 1 - c$.

For $r_1 = 0$, the Frobenius method yields

$$\begin{aligned} y_1(x) &= F(a, b, c; x) \equiv 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{x^n}{n!} \end{aligned} \tag{2.67}$$

where $c \neq 0, -1, -2, \ldots$. The function $F(a, b, c; x)$ is called the *hypergeometric function* or *hypergeometric series*. The series converges for $|x| < 1$.

For $r_2 = 1 - c$, the Frobenius method yields (where $c \neq 2, 3, 4, \ldots$)

$$\begin{aligned} y_2(x) &= x^{1-c}\left\{1 + \frac{(a-c+1)(b-c+1)}{(-c+2)}\frac{x}{1!} + \right. \\ &\quad \left. \frac{(a-c+1)(a-c+2)(b-c+1)(b-c+2)}{(-c+2)(-c+3)}\frac{x^2}{2!} + \cdots\right\} \\ &= x^{1-c}F(a-c+1, b-c+1, 2-c; x) \end{aligned} \tag{2.68}$$

The complete solution is then

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{2.69}$$

Exercise 2.36 Derive (2.67) and (2.68). ◀

The general nature of the hypergeometric equation allows us to write a large number of elementary functions in terms of the hypergeometric function.

Exercise 2.37 Show that

1. $\frac{1}{1-x} = 1 + x + x^2 + \ldots = F(1, 1, 1; x) = F(1, b, b; x) = F(a, 1, a; x)$
2. If a or b is a negative integer, (2.67) reduces to a polynomial.
3. $(1 + x)^n = F(-n, b, b; -x)$
4. $(1 - x)^n = 1 - nxF(1 - n, 1, 1; x)$
5. $\tan^{-1} x = xF(\frac{1}{2}, 1, \frac{3}{2}; -x^2)$
6. $\ln(1 + x) = xF(1, 1, 2; -x)$
7. $F(a, b, c; x) = (1 - x)^{c-a-b}F(c - a, c - b, c; x)$
8. $F'(a, b, c; x) = \frac{ab}{c}F(a + 1, b + 1, c + 1; x)$ ◀