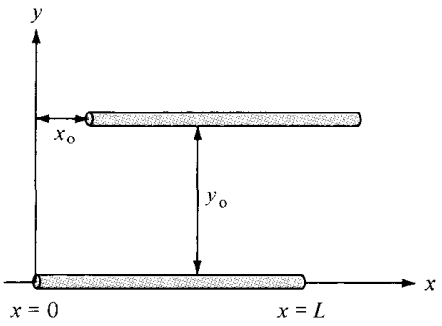


Figure 15.25 Parallel conducting wires of Practice Exercise 15.5.

TABLE 15.2 Capacitance for Practice Exercise 15.5

x_o (m)	C (pF)
0.0	4.91
0.2	4.891
0.4	4.853
0.6	4.789
0.8	4.71
1.0	4.643

15.5 THE FINITE ELEMENT METHOD

The finite element method (FEM) has its origin in the field of structural analysis. The method was not applied to EM problems until 1968.[3] Like the finite difference method, the finite element method is useful in solving differential equations. As noticed in Section 15.3, the finite difference method represents the solution region by an array of grid points; its application becomes difficult with problems having irregularly shaped boundaries. Such problems can be handled more easily using the finite element method.

The finite element analysis of any problem involves basically four steps: (a) discretizing the solution region into a finite number of subregions or *elements,* (b) deriving governing equations for a typical element, (c) assembling of all elements in the solution region, and (d) solving the system of equations obtained.

A. Finite Element Discretization

We divide the solution region into a number of *finite elements* as illustrated in Figure 15.26 where the region is subdivided into four nonoverlapping elements (two triangular and two quadrilateral) and seven nodes. We seek an approximation for the potential V_e within an

[3]See P. P. Silvester and R. L. Ferrari, *Finite Elements for Electrical Engineers.* Cambridge, England: Cambridge Univ. Press, 1983.

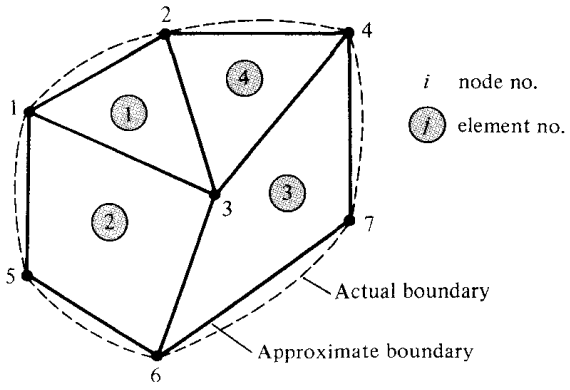


Figure 15.26 A typical finite element subdivision of an irregular domain.

element e and then interrelate the potential distributions in various elements such that the potential is continuous across interelement boundaries. The approximate solution for the whole region is

$$V(x, y) \simeq \sum_{e=1}^{N} V_e(x, y) \tag{15.45}$$

where N is the number of triangular elements into which the solution region is divided.

The most common form of approximation for V_e within an element is polynomial approximation, namely

$$V_e(x, y) = a + bx + cy \tag{15.46}$$

for a triangular element and

$$V_e(x, y) = a + bx + cy + dxy \tag{15.47}$$

for a quadrilateral element. The potential V_e in general is nonzero within element e but zero outside e. It is difficult to approximate the boundary of the solution region with quadrilateral elements; such elements are useful for problems whose boundaries are sufficiently regular. In view of this, we prefer to use triangular elements throughout our analysis in this section. Notice that our assumption of linear variation of potential within the triangular element as in eq. (15.46) is the same as assuming that the electric field is uniform within the element; that is,

$$\mathbf{E}_e = -\nabla V_e = -(b\mathbf{a}_x + c\mathbf{a}_y) \tag{15.48}$$

B. Element Governing Equations

Consider a typical triangular element shown in Figure 15.27. The potential V_{e1}, V_{e2}, and V_{e3} at nodes 1, 2, and 3, respectively, are obtained using eq. (15.46); that is,

$$\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{15.49}$$

The coefficients a, b, and c are determined from eq. (14.49) as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} \tag{15.50}$$

Substituting this into eq. (15.46) gives

$$V_e = [1 \quad x \quad y] \frac{1}{2A} \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}$$

or

$$\boxed{V_e = \sum_{i=1}^{3} \alpha_i(x, y)\, V_{ei}} \tag{15.51}$$

where

$$\alpha_1 = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)\,x + (x_3 - x_2)\,y] \tag{15.52a}$$

$$\alpha_2 = \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)\,x + (x_1 - x_3)\,y] \tag{15.52b}$$

$$\alpha_3 = \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)\,x + (x_2 - x_1)\,y] \tag{15.52c}$$

and A is the area of the element e; that is,

$$2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$
$$= (x_1y_2 - x_2y_1) + (x_3y_1 - x_1y_3) + (x_2y_3 - x_3y_2)$$

or

$$A = 1/2\,[x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)] \tag{15.53}$$

The value of A is positive if the nodes are numbered counterclockwise (starting from any node) as shown by the arrow in Figure 15.27. Note that eq. (15.51) gives the potential at any point (x, y) within the element provided that the potentials at the vertices are known. This is unlike the situation in finite difference analysis where the potential is known at the grid points only. Also note that α_i are linear interpolation functions. They are called the *element shape functions* and they have the following properties:

$$\alpha_i(x_j, y_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{15.54a}$$

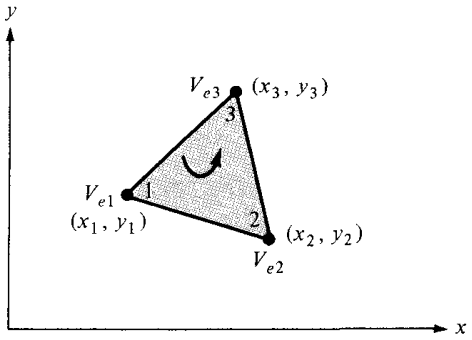


Figure 15.27 Typical triangular element; the local node numbering 1-2-3 must be counterclockwise as indicated by the arrow.

$$\sum_{i=1}^{3} \alpha_i(x, y) = 1 \tag{15.54b}$$

The shape functions α_1 and α_2, for example, are illustrated in Figure 15.28.

The energy per unit length associated with the element e is given by eq. (4.96); that is

$$W_e = \frac{1}{2} \int \varepsilon \, |\mathbf{E}|^2 \, dS = \frac{1}{2} \int \varepsilon \, |\nabla V_e|^2 \, dS \tag{15.55}$$

where a two-dimensional solution region free of charge ($\rho_S = 0$) is assumed. But from eq. (15.51),

$$\nabla V_e = \sum_{i=1}^{3} V_{ei} \, \nabla \alpha_i \tag{15.56}$$

Substituting eq. (15.56) into eq. (15.55) gives

$$W_e = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \varepsilon V_{ei} \left[\int \nabla \alpha_i \cdot \nabla \alpha_j \, dS \right] V_{ej} \tag{15.57}$$

If we define the term in brackets as

$$C_{ij}^{(e)} = \int \nabla \alpha_i \cdot \nabla \alpha_j \, dS \tag{15.58}$$

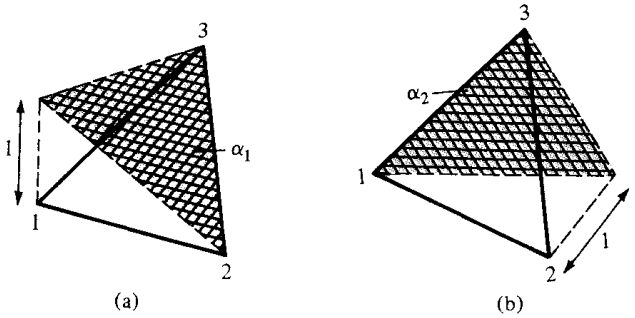


Figure 15.28 Shape functions α_1 and α_2 for a triangular element.

we may write eq. (15.57) in matrix form as

$$W_e = \frac{1}{2}\varepsilon\,[V_e]^T\,[C^{(e)}]\,[V_e] \tag{15.59}$$

where the superscript T denotes the transpose of the matrix,

$$[V_e] = \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} \tag{15.60a}$$

and

$$[C^{(e)}] = \begin{bmatrix} C_{11}^{(e)} & C_{12}^{(e)} & C_{13}^{(e)} \\ C_{21}^{(e)} & C_{22}^{(e)} & C_{23}^{(e)} \\ C_{31}^{(e)} & C_{32}^{(e)} & C_{33}^{(e)} \end{bmatrix} \tag{15.60b}$$

The matrix $[C^{(e)}$ is usually called the *element coefficient matrix.* The matrix element $C_{ij}^{(e)}$ of the coefficient matrix may be regarded as the coupling between nodes i and j; its value is obtained from eqs. (15.52) and (15.58). For example,

$$\begin{aligned} C_{12}^{(e)} &= \int \nabla\alpha_1 \cdot \nabla\alpha_2\, dS \\ &= \frac{1}{4A^2}\,[(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)] \int dS \\ &= \frac{1}{4A}\,[(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)] \end{aligned} \tag{15.61a}$$

Similarly:

$$C_{11}^{(e)} = \frac{1}{4A}\,[(y_2 - y_3)^2 + (x_3 - x_2)^2] \tag{15.61b}$$

$$C_{13}^{(e)} = \frac{1}{4A}\,[(y_2 - y_3)(y_1 - y_2) + (x_3 - x_2)(x_2 - x_1)] \tag{15.61c}$$

$$C_{22}^{(e)} = \frac{1}{4A}\,[(y_3 - y_1)^2 + (x_1 - x_3)^2] \tag{15.61d}$$

$$C_{23}^{(e)} = \frac{1}{4A}\,[(y_3 - y_1)(y_1 - y_2) + (x_1 - x_3)(x_2 - x_1)] \tag{15.61e}$$

$$C_{33}^{(e)} = \frac{1}{4A}\,[(y_1 - y_2)^2 + (x_2 - x_1)^2] \tag{15.61f}$$

Also

$$C_{21}^{(e)} = C_{12}^{(e)}, \qquad C_{31}^{(e)} = C_{13}^{(e)}, \qquad C_{32}^{(e)} = C_{23}^{(e)} \tag{15.61g}$$

However, our calculations will be easier if we define

$$P_1 = (y_2 - y_3), \quad P_2 = (y_3 - y_1), \quad P_3 = (y_1 - y_2) \tag{15.62a}$$
$$Q_1 = (x_3 - x_2), \quad Q_2 = (x_1 - x_3), \quad Q_3 = (x_2 - x_1)$$

With P_i and Q_i ($i = 1, 2, 3$ are the local node numbers), each term in the element coefficient matrix is found as

$$\boxed{C_{ij}^{(e)} = \frac{1}{4A}[P_iP_j + Q_iQ_j]} \tag{15.62b}$$

where

$$\boxed{A = \frac{1}{2}(P_2Q_3 - P_3Q_2)} \tag{15.62c}$$

Note that $P_1 + P_2 + P_3 = 0 = Q_1 + Q_2 + Q_3$ and hence $\sum_{i=1}^{3} C_{ij}^{(e)} = 0 = \sum_{j=1}^{3} C_{ij}^{(e)}$. This may be used in checking our calculations.

C. Assembling of All Elements

Having considered a typical element, the next step is to assemble all such elements in the solution region. The energy associated with the assemblage of all elements in the mesh is

$$W = \sum_{e=1}^{N} W_e = \frac{1}{2}\varepsilon [V]^T [C] [V] \tag{15.63}$$

where

$$[V] = \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_n \end{bmatrix} \tag{15.64}$$

n is the number of nodes, N is the number of elements, and $[C]$ is called the *overall* or *global coefficient matrix,* which is the assemblage of individual element coefficient matrices. The major problem now is obtaining $[C]$ from $[C^{(e)}]$.

The process by which individual element coefficient matrices are assembled to obtain the global coefficient matrix is best illustrated with an example. Consider the finite element mesh consisting of three finite elements as shown in Figure 15.29. Observe the numberings of the nodes. The numbering of nodes as 1, 2, 3, 4, and 5 is called *global* numbering. The numbering *i-j-k* is called *local* numbering and it corresponds with 1-2-3 of the element in

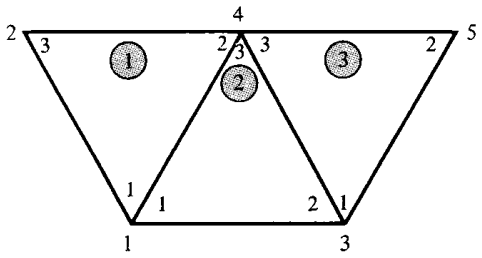


Figure 15.29 Assembly of three elements: *i-j-k* corresponds to local numbering 1-2-3 of the element in Figure 15.27.

Figure 15.27. For example, for element 3 in Figure 15.29, the global numbering 3-5-4 corresponds with local numbering 1-2-3 of the element in Figure 15.27. Note that the local numbering must be in counterclockwise sequence starting from any node of the element. For element 3, for example, we could choose 4-3-5 or 5-4-3 instead of 3-5-4 to correspond with 1-2-3 of the element in Figure 15.27. Thus the numbering in Figure 15.29 is not unique. However, we obtain the same $[C]$ whichever numbering is used. Assuming the particular numbering in Figure 15.29, the global coefficient matrix is expected to have the form

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{bmatrix} \tag{15.65}$$

which is a 5×5 matrix since five nodes ($n = 5$) are involved. Again, C_{ij} is the coupling between nodes i and j. We obtain C_{ij} by utilizing the fact that the potential distribution must be continuous across interelement boundaries. The contribution to the i, j position in $[C]$ comes from all elements containing nodes i and j. To find C_{11}, for example, we observe from Figure 15.29 that global node 1 belongs to elements 1 and 2 and it is local node 1 in both; hence,

$$C_{11} = C_{11}^{(1)} + C_{11}^{(2)} \tag{15.66a}$$

For C_{22}, global node 2 belongs to element 1 only and is the same as local node 3; hence,

$$C_{22} = C_{33}^{(1)} \tag{15.66b}$$

For C_{44}, global node 4 is the same as local nodes 2, 3, and 3 in elements 1, 2, and 3, respectively; hence,

$$C_{44} = C_{22}^{(1)} + C_{33}^{(2)} + C_{33}^{(3)} \tag{15.66c}$$

For C_{14}, global link 14 is the same as the local links 12 and 13 in elements 1 and 2, respectively; hence,

$$C_{14} = C_{12}^{(1)} + C_{13}^{(2)} \tag{15.66d}$$

Since there is no coupling (or direct link) between nodes 2 and 3,

$$C_{23} = C_{32} = 0 \tag{15.66e}$$

Continuing in this manner, we obtain all the terms in the global coefficient matrix by inspection of Figure 15.29 as

$$[C] = \begin{bmatrix} C_{11}^{(1)} + C_{11}^{(2)} & C_{13}^{(1)} & C_{12}^{(2)} & C_{12}^{(1)} + C_{13}^{(2)} & 0 \\ C_{31}^{(1)} & C_{33}^{(1)} & 0 & C_{32}^{(1)} & 0 \\ C_{21}^{(2)} & 0 & C_{22}^{(2)} + C_{11}^{(3)} & C_{23}^{(2)} + C_{13}^{(3)} & C_{12}^{(3)} \\ C_{21}^{(1)} + C_{31}^{(2)} & C_{23}^{(1)} & C_{32}^{(2)} + C_{31}^{(3)} & C_{22}^{(1)} + C_{33}^{(2)} + C_{33}^{(3)} & C_{32}^{(3)} \\ 0 & 0 & C_{21}^{(3)} & C_{23}^{(3)} & C_{22}^{(3)} \end{bmatrix} \tag{15.67}$$

Note that element coefficient matrices overlap at nodes shared by elements and that there are 27 terms (nine for each of the three elements) in the global coefficient matrix $[C]$. Also note the following properties of the matrix $[C]$:

1. It is symmetric ($C_{ij} = C_{ji}$) just as the element coefficient matrix.
2. Since $C_{ij} = 0$ if no coupling exists between nodes i and j, it is evident that for a large number of elements $[C]$ becomes sparse and banded.
3. It is singular. Although this is not so obvious, it can be shown using the element coefficient matrix of eq. (15.60b).

D. Solving the Resulting Equations

From variational calculus, it is known that Laplace's (or Poisson's) equation is satisfied when the total energy in the solution region is minimum. Thus we require that the partial derivatives of W with respect to each nodal value of the potential be zero; that is,

$$\frac{\partial W}{\partial V_1} = \frac{\partial W}{\partial V_2} = \cdots = \frac{\partial W}{\partial V_n} = 0$$

or

$$\frac{\partial W}{\partial V_k} = 0, \qquad k = 1, 2, \ldots, n \tag{15.68}$$

For example, to get $\partial W/\partial V_1 = 0$ for the finite element mesh of Figure 15.29, we substitute eq. (15.65) into eq. (15.63) and take the partial derivative of W with respect to V_1. We obtain

$$0 = \frac{\partial W}{\partial V_1} = 2V_1C_{11} + V_2C_{12} + V_3C_{13} + V_4C_{14} + V_5C_{15} + V_2C_{21} + V_3C_{31} + V_4C_{41} + V_5C_{51}$$

or

$$0 = V_1C_{11} + V_2C_{12} + V_3C_{13} + V_4C_{14} + V_5C_{15} \tag{15.69}$$

In general, $\partial W/\partial V_k = 0$ leads to

$$0 = \sum_{i=1}^{n} V_i C_{ik} \tag{15.70}$$

where n is the number of nodes in the mesh. By writing eq. (15.70) for all nodes $k = 1, 2, \ldots, n$, we obtain a set of simultaneous equations from which the solution of $[V]^T = [V_1, V_2, \ldots, V_n]$ can be found. This can be done in two ways similar to those used in solving finite difference equations obtained from Laplace's (or Poisson's) equation.

Iteration Method:

This approach is similar to that used in finite difference method. Let us assume that node 1 in Figure 15.29, for example, is a free node. The potential at node 1 can be obtained from eq. (15.69) as

$$V_1 = -\frac{1}{C_{11}} \sum_{i=2}^{5} V_i C_{1i} \tag{15.71}$$

In general, the potential at a free node k is obtained from eq. (15.70) as

$$\boxed{V_k = -\frac{1}{C_{kk}} \sum_{i=1, i\neq k}^{n} V_i C_{ik}} \tag{15.72}$$

This is applied iteratively to all the free nodes in the mesh with n nodes. Since $C_{ki} = 0$ if node k is not directly connected to node i, only nodes that are directly linked to node k contribute to V_k in eq. (15.72).

Thus if the potentials at nodes connected to node k are known, we can determine V_k using eq. (15.72). The iteration process begins by setting the potentials at the free nodes equal to zero or to the average potential.

$$V_{ave} = 1/2\,(V_{min} + V_{max}) \tag{15.73}$$

where V_{min} and V_{max} are the minimum and maximum values of the prescribed potentials at the fixed nodes. With those initial values, the potentials at the free nodes are calculated using eq. (15.72). At the end of the first iteration, when the new values have been calculated for all the free nodes, the values become the old values for the second iteration. The procedure is repeated until the change between subsequent iterations becomes negligible.

Band Matrix Method:

If all free nodes are numbered first and the fixed nodes last, eq. (15.63) can be written such that

$$W = \frac{1}{2} \varepsilon \begin{bmatrix} V_f & V_p \end{bmatrix} \begin{bmatrix} C_{ff} & C_{fp} \\ C_{pf} & C_{pp} \end{bmatrix} \begin{bmatrix} V_f \\ V_p \end{bmatrix} \tag{15.74}$$

where subscripts f and p, respectively, refer to nodes with free and fixed (or prescribed) potentials. Since V_p is constant (it consists of known, fixed values), we only differentiate with respect to V_f so that applying eq. (15.68) to eq. (15.74) yields

$$C_{ff}V_p + C_{fp}V_p = 0$$

or

$$[C_{ff}]\,[V_f] = -[C_{fp}]\,[V_p] \tag{15.75}$$

This equation can be written as

$$[A]\,[V] = [B] \tag{15.76a}$$

or

$$\boxed{[V] = [A]^{-1}\,[B]} \tag{15.76b}$$

where $[V] = [V_f]$, $[A] = [C_{ff}]$, and $[B] = -[C_{fp}]\,[V_p]$. Since $[A]$ is, in general, nonsingular, the potential at the free nodes can be found using eq. (15.75). We can solve for $[V]$ in eq. (15.76a) using Gaussian elimination technique. We can also solve for $[V]$ in eq. (15.76b) using matrix inversion if the size of the matrix to be inverted is not large.

Notice that as from eq. (15.55) onward, our solution has been restricted to a two-dimensional problem involving Laplace's equation, $\nabla^2 V = 0$. The basic concepts developed in this section can be extended to finite element analysis of problems involving Poisson's equation ($\nabla^2 V = -\rho_v/\varepsilon$, $\nabla^2 \mathbf{A} = -\mu\mathbf{J}$) or wave equation ($\nabla^2\phi - \gamma^2\phi = 0$). A major problem associated with finite element analysis is the relatively large amount of computer memory required in storing the matrix elements and the associated computational time. However, several algorithms have been developed to alleviate the problem to some degree.

The finite element method (FEM) has a number of advantages over the finite difference method (FDM) and the method of moments (MOM). First, the FEM can easily handle complex solution region. Second, the generality of FEM makes it possible to construct a general-purpose program for solving a wide range of problems. A single program can be used to solve different problems (described by the same partial differential equations) with different solution regions and different boundary conditions; only the input data to the problem need be changed. However, FEM has its own drawbacks. It is harder to understand and program than FDM and MOM. It also requires preparing input data, a process that could be tedious.

EXAMPLE 15.6 Consider the two-element mesh shown in Figure 15.30(a). Using the finite element method, determine the potentials within the mesh.

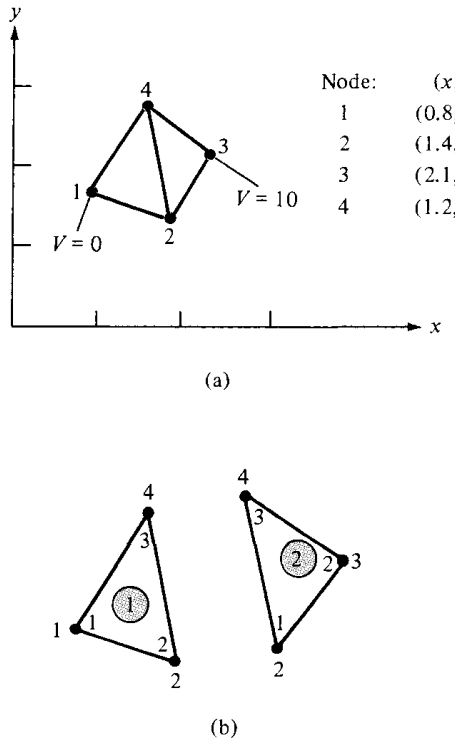


Figure 15.30 For Example 15.6: **(a)** two-element mesh, **(b)** local and global numbering of the elements.

Solution:

The element coefficient matrices can be calculated using eq. (15.62). For element 1, consisting of nodes 1-2-4 corresponding to the local numbering 1-2-3 as in Figure 15.30(b),

$$P_1 = -1.3, \qquad P_2 = 0.9, \qquad P_3 = 0.4$$

$$Q_1 = -0.2, \qquad Q_2 = -0.4, \qquad Q_3 = 0.6$$

$$A = 1/2\,(0.54 + 0.16) = 0.35$$

Substituting all these into eq. (15.62b) gives

$$[C^{(1)}] = \begin{bmatrix} 1.236 & -0.7786 & -0.4571 \\ -0.7786 & 0.6929 & 0.0857 \\ -0.4571 & 0.0857 & 0.3714 \end{bmatrix} \tag{15.6.1}$$

Similarly, for element 2 consisting of nodes 2-3-4 corresponding to local numbering 1-2-3 as in Figure 15.30(b),

$$P_1 = -0.6, \qquad P_2 = 1.3, \qquad P_3 = -0.7$$

$$Q_1 = -0.9, \qquad Q_2 = 0.2, \qquad Q_3 = 0.7$$

$$A = 1/2\,(0.91 + 0.14) = 0.525$$

Hence,

$$[C^{(2)}] = \begin{bmatrix} 0.5571 & -0.4571 & -0.1 \\ -0.4571 & 0.8238 & -0.3667 \\ -0.1 & -0.3667 & 0.4667 \end{bmatrix} \tag{15.6.2.}$$

Applying eq. (15.75) gives

$$\begin{bmatrix} C_{22} & C_{24} \\ C_{42} & C_{44} \end{bmatrix} \begin{bmatrix} V_2 \\ V_4 \end{bmatrix} = -\begin{bmatrix} C_{21} & C_{23} \\ C_{41} & C_{43} \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix} \tag{15.6.3}$$

This can be written in a more convenient form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_{22} & 0 & C_{24} \\ 0 & 0 & 1 & 0 \\ 0 & C_{42} & 0 & C_{44} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -C_{21} & -C_{23} \\ 0 & 1 \\ -C_{41} & -C_{43} \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix} \tag{15.6.4a}$$

or

$$[C]\,[V] = [B] \tag{15.6.4b}$$

The terms of the global coefficient matrix are obtained as follows:

$$C_{22} = C_{22}^{(1)} + C_{11}^{(2)} = 0.6929 + 0.5571 = 1.25$$

$$C_{42} = C_{24} = C_{23}^{(1)} + C_{13}^{(2)} = 0.0857 - 0.1 = -0.0143$$

$$C_{44} = C_{33}^{(1)} + C_{33}^{(2)} = 0.3714 + 0.4667 = 0.8381$$

$$C_{21} = C_{21}^{(1)} = -0.7786$$

$$C_{23} = C_{12}^{(2)} = -0.4571$$

$$C_{41} = C_{31}^{(1)} = -0.4571$$

$$C_{43} = C_{32}^{(2)} = -0.3667$$

Note that we follow local numbering for the element coefficient matrix and global numbering for the global coefficient matrix. Thus the square matrix $[C]$ is obtained as

$$[C] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.25 & 0 & -0.0143 \\ 0 & 0 & 1 & 0 \\ 0 & -0.0143 & 0 & 0.8381 \end{bmatrix} \tag{15.6.5}$$

and the matrix $[B]$ on the right-hand side of eq. (15.6.4a) is obtained as

$$[B] = \begin{bmatrix} 0 \\ 4.571 \\ 10.0 \\ 3.667 \end{bmatrix} \tag{15.6.6}$$

By inverting matrix $[C]$ in eq. (15.6.5), we obtain

$$[V] = [C]^{-1}[B] = \begin{bmatrix} 0 \\ 3.708 \\ 10.0 \\ 4.438 \end{bmatrix}$$

Thus $V_1 = 0$, $V_2 = 3.708$, $V_3 = 10$, and $V_4 = 4.438$. Once the values of the potentials at the nodes are known, the potential at any point within the mesh can be determined using eq. (15.51).

PRACTICE EXERCISE 15.6

Calculate the global coefficient matrix for the two-element mesh shown in Figure 15.31 when: (a) node 1 is linked with node 3 and the local numbering $(i - j - k)$ is as indicated in Figure 15.31(a), (b) node 2 is linked with node 4 with local numbering as in Figure 15.31(b).

Answer: (a)
$$\begin{bmatrix} 0.9964 & 0.05 & -0.2464 & -0.8 \\ 0.05 & 0.7 & -0.75 & 0.0 \\ -0.2464 & -0.75 & 1.5964 & -0.6 \\ -0.8 & 0.0 & -0.6 & 1.4 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1.333 & -0.7777 & 0.0 & -1.056 \\ -0.0777 & 0.8192 & -0.98 & 0.2386 \\ 0.0 & -0.98 & 2.04 & -1.06 \\ -1.056 & 0.2386 & -1.06 & 1.877 \end{bmatrix}$$

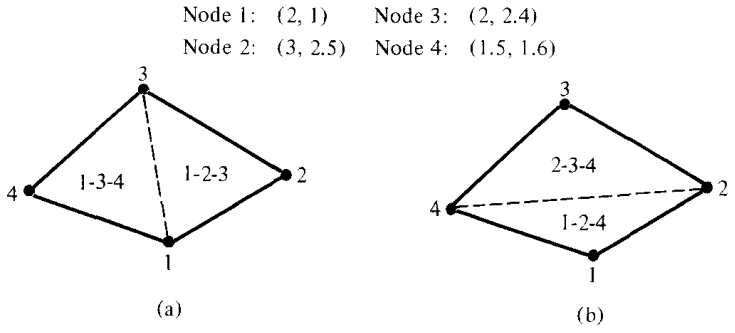


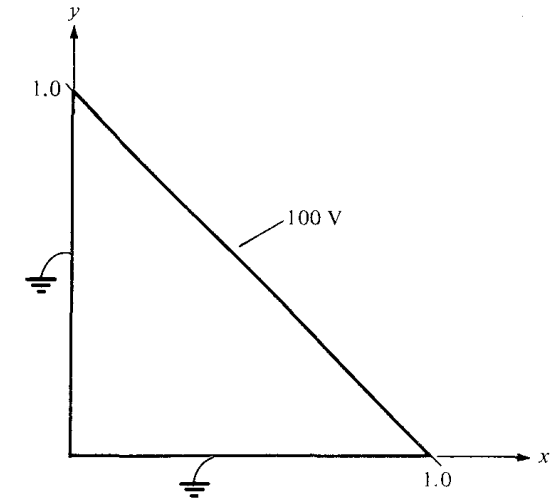
Figure 15.31 For Practice Exercise 15.6.

EXAMPLE 15.7

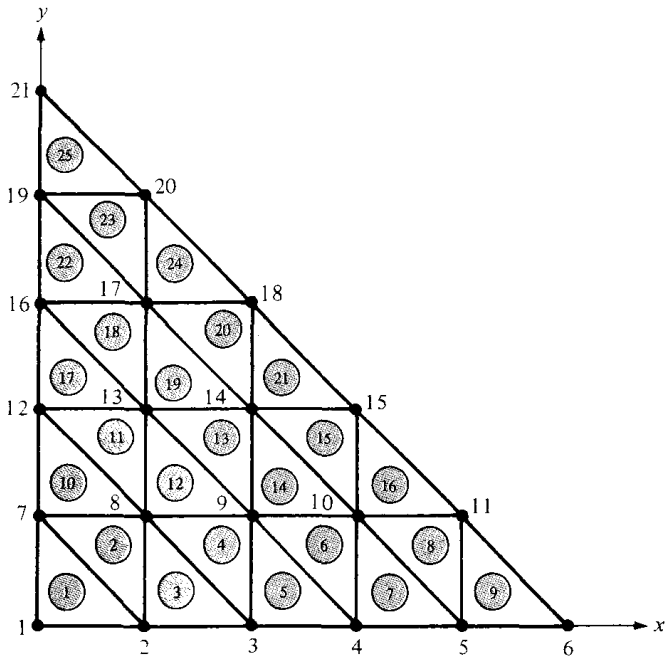
Write a program to solve Laplace's equation using the finite element method. Apply the program to the two-dimensional problem shown in Figure 15.32(a).

Solution:

The solution region is divided into 25 three-node triangular elements with the total number of nodes being 21 as shown in Figure 15.32(b). This is a necessary step in order to have input data defining the geometry of the problem. Based on our discussions in Section 15.5, a general Matlab program for solving problems involving Laplace's equation using three-node triangular elements was developed as in Figure 15.33. The devel-



(a)



(b)

Figure 15.32 For Example 15.7: **(a)** two-dimensional electrostatic problem, **(b)** solution region divided into 25 triangular elements.

```
%  FINITE ELEMENT SOLUTION OF LAPLACE'S EQUATION FOR
%  TWO-DIMENSIONAL PROBLEMS
%  TRIANGULAR ELEMENTS ARE USED
%  ND = NO. OF NODES
%  NE = NO. OF ELEMENTS
%  NP = NO. OF FIXED NODES (WHERE POTENTIAL IS PRESCRIBED)
%  NDP(I) = NODE NO. OF PRESCRIBED POTENTIAL, I=1,2,...,NP
%  VAL(I) = VALUE OF PRESCRIBED POTENTIAL AT NODE NDP(I)
%  NL(I,J) = LIST OF NODES FOR EACH ELEMENT I, WHERE
%            J=1,2,3 REFERS TO THE LOCAL NODE NUMBER
%  CE(I,J) = ELEMENT COEFFICIENT MATRIX
%  C(I,J) = GLOBAL COEFFICIENT MATRIX
%  B(I) = RIGHT-HAND SIDE MATRIX IN THE SYSTEM OF
%  SIMULTANEOUS EQUATIONS; SEE EQ. (15.6.4)
%  X(I), Y(I) = GLOBAL COORDINATES OF NODE I
%  XL(J), YL(J) = LOCAL COORDINATES OF NODE J=1,2,3
%  V(I) = POTENTIAL AT NODE I
%  MATRICES P(I) AND Q (I) ARE DEFINED IN EQ. (15.62a)

%  ******************************************************
%  FIRST STEP - INPUT DATA DEFINING GEOMETRY AND
                BOUNDARY CONDITIONS
%  ******************************************************

clear
input('Name of input data file = ')

% ******************************************************
%  SECOND STEP - EVALUATE COEFFICIENT MATRIX FOR EACH
                 ELEMENT AND ASSEMBLE GLOBALLY
% ******************************************************
B = zeros(ND,1);
C = zeros(ND,ND);
for I=1:NE
% FIND LOCAL COORDINATES XL(J), YL(J) FOR ELEMENT I
   K = NL(I,[1:3]);
   XL = X(K);
   YL = Y(K);
P=zeros(3,1);
Q=zeros(3,1);
   P(1) = YL(2) - YL(3);
   P(2) = YL(3) - YL(1);
   P(3) = YL(1) - YL(2);
   Q(1) = XL(3) - XL(2);
   Q(2) = XL(1) - XL(3);
   Q(3) = XL(2) - XL(1);
   AREA = 0.5*abs( P(2)*Q(3) - Q(2)*P(3) );
```

Figure 15.33 Computer program for Example 15.7.

```
%  DETERMINE COEFFICIENT MATRIX FOR ELEMENT I
   CE=(P*P'+Q*Q')/(4.0*AREA);
%  ASSEMBLE GLOBALLY - FIND C(I,J) AND B(I)
   for J=1:3
      IR = NL(I,J);
      IFLAG1=0;
%  CHECK IF ROW CORRESPONDS TO A FIXED NODE
   for K = 1:NP
      if (IR == NDP(K))
         C(IR,IR) = 1.0;
         B(IR) = VAL(K);
         IFLAG1=1;
      end
   end % end for K = 1:NP
   if(IFLAG1 == 0)
   for L = 1:3
     IC = NL(I,L);
     IFLAG2=0;
%  CHECK IF COLUMN CORRESPONDS TO A FIXED NODE
     for K=1:NP
        if ( IC == NDP(K) ),
          B(IR) = B(IR) - CE(J,L)*VAL(K);
          IFLAG2=1;
        end
      end % end for K=1:NP
    if(IFLAG2 == 0)
      C(IR,IC) = C(IR,IC) + CE(J,L);
      end
    end  % end for L=1:3
  end   %end if(iflag1 == 0)
end  % end for J=1:3
end % end for I=1:NE
% ******************************************************
%   THIRD STEP - SOLVE THE SYSTEM OF EQUATIONS
% ******************************************************

V = inv(C)*B;
V=V';
% ******************************************************
%   FOURTH STEP - OUTPUT THE RESULTS
% ******************************************************
diary exam147.out
[ND, NE, NP]
[ [1:ND]' X' Y' V']
diary off
```

Figure 15.33 (Continued)

opment of the program basically involves four steps indicated in the program and explained as follows.

Step 1: This involves inputting the necessary data defining the problem. This is the only step that depends on the geometry of the problem at hand. Through a data file, we input the number of elements, the number of nodes, the number of fixed nodes, the prescribed values of the potentials at the free nodes, the x and y coordinates of all nodes, and a list identifying the nodes belonging to each element in the order of the local numbering 1-2-3. For the problem in Figure 15.32, the three sets of data for coordinates, element-node relationship, and prescribed potentials at fixed nodes are shown in Tables 15.3, 15.4, and 15.5, respectively.

TABLE 15.3 Nodal Coordinates of the Finite Element Mesh of Figure 15.32

Node	x	y	Node	x	y
1	0.0	0.0	12	0.0	0.4
2	0.2	0.0	13	0.2	0.4
3	0.4	0.0	14	0.4	0.4
4	0.6	0.0	15	0.6	0.4
5	0.8	0.0	16	0.0	0.6
6	1.0	0.0	17	0.2	0.6
7	0.0	0.2	18	0.4	0.6
8	0.2	0.2	19	0.0	0.8
9	0.4	0.2	20	0.2	0.8
10	0.6	0.2	21	0.0	1.0
11	0.8	0.2			

TABLE 15.4 Element-Node Identification

	Local Node No.				Local Node No.		
Element No.	1	2	3	Element No.	1	2	3
1	1	2	7	14	9	10	14
2	2	8	7	15	10	15	14
3	2	3	8	16	10	11	15
4	3	9	8	17	12	13	16
5	3	4	9	18	13	17	16
6	4	10	9	19	13	14	17
7	4	5	10	20	14	18	17
8	5	11	10	21	14	15	18
9	5	6	11	22	16	17	19
10	7	8	12	23	17	20	19
11	8	13	12	24	17	18	20
12	8	9	13	25	19	20	21
13	9	14	13				

TABLE 15.5 Prescribed Potentials at Fixed Nodes

Node No.	Prescribed Potential	Node No.	Prescribed Potential
1	0.0	18	100.0
2	0.0	20	100.0
3	0.0	21	50.0
4	0.0	19	0.0
5	0.0	16	0.0
6	50.0	12	0.0
11	100.0	7	0.0
15	100.0		

TABLE 15.6 Input Data for the Finite Element Program in Figure 15.33

```
NE = 25;
ND = 21;
NP = 15;
NL = [ 1 2 7
   2 8 7
   2 3 8
       3 9 8
       3 4 9
       4 10 9
       4 5 10
       5 11 10
       5 6 11
       7 8 12
      8 13 12
       8 9 13
       9 14 13
       9 10 14
       10 15 14
       10 11 15
       12 13 16
       13 17 16
       13 14 17
       14 18 17
       14 15 18
       16 17 19
       17 20 19
       17 18 20
       19 20 21];

   X = [0.0  0.2  0.4  0.6  0.8  1.0  0.0 ...
        0.2  0.4  0.6  0.8  0.0  0.2  0.4 ...
        0.6  0.0  0.2  0.4  0.0  0.2  0.0];

   Y = [0.0  0.0  0.0  0.0  0.0  0.0  0.2 ...
        0.2  0.2  0.2  0.2  0.4  0.4  0.4 ...
        0.4  0.6  0.6  0.6  0.8  0.8  1.0];

   NDP = [ 1 2 3 4 5 6 11 15 18 20 21 19 16 12 7];

   VAL = [ 0.0  0.0  0.0  0.0 0.0 ...
           50.0  100.0  100.0  100.0  100.0
           50.0  0.0   0.0   0.0   0.0];
```

Step 2: This step entails finding the element coefficient matrix $[C^{(e)}]$ for each element and the global coefficient matrix $[C]$. The procedure explained in the previous example is applied. Equation (15.6.4) can be written in general form as

$$\begin{bmatrix} 1 & 0 \\ 0 & C_{ff} \end{bmatrix} \begin{bmatrix} V_P \\ V_f \end{bmatrix} = \begin{bmatrix} 1 \\ -C_{fp} \end{bmatrix} [V_p]$$

or

$$[C]\,[V] = [B]$$

Both "global" matrix $[C]$ and matrix $[B]$ are calculated at this stage.

Step 3: The global matrix obtained in the previous step is inverted. The values of the potentials at all nodes are obtained by matrix multiplication as in eq. (15.76b). Instead of inverting the global matrix, it is also possible to solve for the potentials at the nodes using Gaussian elimination technique.

Step 4: This involves outputting the result of the computation.

The input and output data are presented in Tables 15.6 and 15.7, respectively.

TABLE 15.7 Output Data of the Program in Figure 15.33

Node	X	Y	Potential
1	0.00	0.00	0.000
2	0.20	0.00	0.000
3	0.40	0.00	0.000
4	0.60	0.00	0.000
5	0.80	0.00	0.000
6	1.00	0.00	50.000
7	0.00	0.20	0.000
8	0.20	0.20	18.182
9	0.40	0.20	36.364
10	0.60	0.20	59.091
11	0.80	0.20	100.000
12	0.00	0.40	0.000
13	0.20	0.40	36.364
14	0.40	0.40	68.182
15	0.60	0.40	100.000
16	0.00	0.60	0.000
17	0.20	0.60	59.091
18	0.40	0.60	100.000
19	0.00	0.80	0.000
20	0.20	0.80	100.000
21	0.00	1.00	50.000

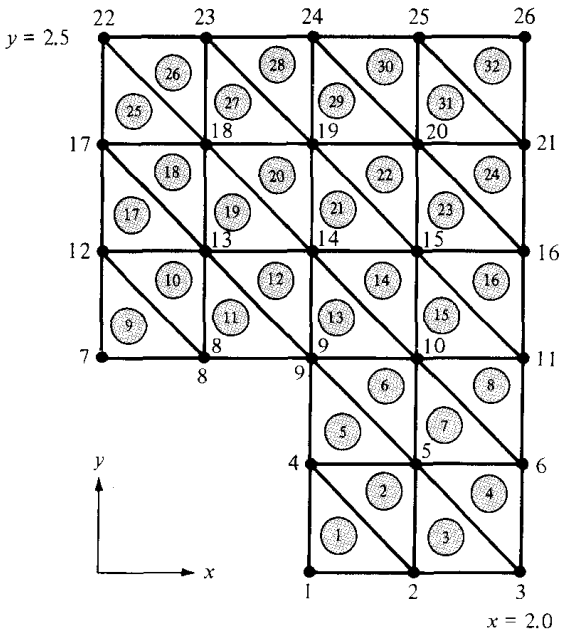


Figure 15.34 For Practice Exercise 15.7.

PRACTICE EXERCISE 15.7

Rework Example 15.3 using the finite element method. Divide the solution region into triangular elements as shown in Figure 15.34. Compare the solution with that obtained in Example 15.3 using the finite difference method.

Answer: See Example 15.3.

SUMMARY

1. Electric field lines and equipotential lines due to coplanar point sources can be plotted using the numerical technique presented in this chapter. The basic concept can be extended to plotting magnetic field lines.
2. An EM problem in the form of a partial differential equation can be solved using the finite difference method. The finite difference equation that approximates the differential equation is applied at grid points spaced in an ordered manner over the whole solution region. The field quantity at the free points is determined using a suitable method.
3. An EM problem in the form of an integral equation is conveniently solved using the moment method. The unknown quantity under the integral sign is determined by matching both sides of the integral equation at a finite number of points in the domain of the quantity.
4. While the finite difference method is restricted to problems with regularly shaped solution regions, the finite element method can handle problems with complex geometries. This method involves dividing the solution region into finite elements, deriving equations for a typical element, assembling all elements in the region, and solving the resulting system of equations.