1

HEAT CONDUCTION FUNDAMENTALS

The energy given up by the constituent particles such as atoms, molecules, or free electrons of the hotter regions of a body to those in cooler regions is called *heat*. Conduction is the mode of heat transfer in which energy exchange takes place in solids or in fluids in rest (i.e., no convective motion resulting from the displacement of the macroscopic portion of the medium) from the region of high temperature to the region of low temperature due to the presence of temperature gradient in the body. The heat flow cannot be measured directly, but the concept has physical meaning because it is related to the measurable scalar quantity called *temperature*. Therefore, once the temperature distribution $T(\mathbf{r}, t)$ within a body is determined as a function of position and time, then the heat flow in the body is readily computed from the laws relating heat flow to the temperature gradient. The science of heat conduction is principally concerned with the determination of temperature distribution within solids. In this chapter we present the basic laws relating the heat flow to the temperature gradient in the medium, the differential equation of heat conduction governing the temperature distribution in solids, the boundary conditions appropriate for the analysis of heat conduction problems, the rules of coordinate transformation needed to write the heat conduction equation in different orthogonal coordinate systems, and a general discussion of various methods of solution of the heat conduction equation.

1-1 THE HEAT FLUX

The basic law that gives the relationship between the heat flow and the temperature gradient, based on experimental observations, is generally named after the

French mathematical physicist Joseph Fourier [1], who used it in his analytic theory of heat. For a homogeneous, isotropic solid (i.e., material in which thermal conductivity is independent of direction) the *Fourier law* is given in the form

$$\mathbf{q}(\mathbf{r},t) = -k\nabla T(\mathbf{r},t) \quad \mathrm{W/m^2} \tag{1-1}$$

where the temperature gradient is a vector normal to the isothermal surface, the *heat flux vector* $\mathbf{q}(\mathbf{r},t)$ represents heat flow per unit time, per unit area of the isothermal surface in the direction of the decreasing temperature, and k is called the *thermal conductivity* of the material which is a positive, scalar quantity. Since the heat flux vector $\mathbf{q}(\mathbf{r},t)$ points in the direction of decreasing temperature, the minus sign is included in equation (1-1) to make the heat flow a positive quantity. When the heat flux is in W/m^2 and the temperature gradient in °C/m, the thermal conductivity k has units W/(m·°C). In the rectangular coordinate system, for example, equation (1-1) is written as

$$\mathbf{q}(x,y,z,t) = -\hat{\mathbf{i}}k\frac{\partial T}{\partial x} - \hat{\mathbf{j}}k\frac{\partial T}{\partial y} - \hat{\mathbf{k}}k\frac{\partial T}{\partial z} \tag{1-2}$$

where $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the unit direction vectors along the x, y, and z directions, respectively. Thus, the three components of the heat flux vector in the x, y, and z directions are given, respectively, by

$$q_x = -k\frac{\partial T}{\partial x}, \qquad q_y = -k\frac{\partial T}{\partial y}, \qquad \text{and} \qquad q_z = -k\frac{\partial T}{\partial z} \tag{1-3a,b,c}$$

Clearly, the heat flow rate for a given temperature gradient is directly proportional to the thermal conductivity k of the material. Therefore, in the analysis of heat conduction, the thermal conductivity of the material is an important property, which controls the rate of heat flow in the medium. There is a wide difference in the thermal conductivities of various engineering materials. The highest value is given by pure metals and the lowest value by gases and vapors; the amorphous insulating materials and inorganic liquids have thermal conductivities that lie in between. To give some idea of the order of magnitude of thermal conductivity for various materials, Fig. 1-1 illustrates the typical ranges. Thermal conductivity also varies with temperature. For most pure metals it decreases with temperature, whereas for gases it increases with increasing temperature. For most insulating materials it increases with increasing temperatures. Figure 1-2 illustrates the effect of temperature on thermal conductivity of materials. At very low temperature approaching absolute zero, thermal conductivity first increases rapidly and then exhibits a sharp descent as shown in Fig. 1-3. A comprehensive compilation of thermal conductivities of materials may be found in references 2–4.

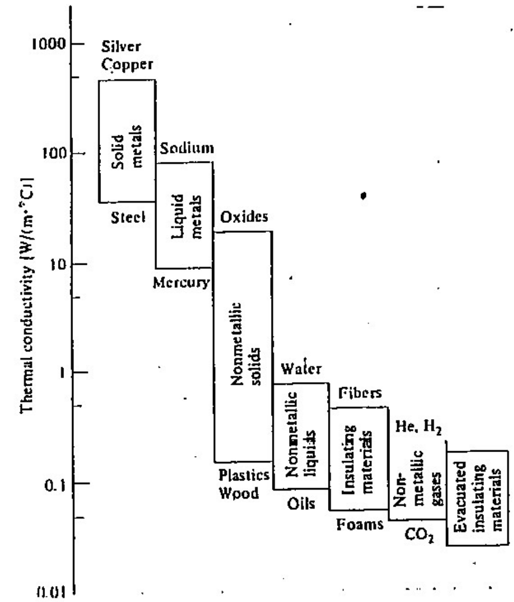


Fig. 1-1 Typical range of thermal conductivity of various materials.

We present in Appendix I the thermal conductivity of typical engineering materials together with the specific heat C_p, density ρ, and the thermal diffusivity α.

1-2 THE DIFFERENTIAL EQUATION OF HEAT CONDUCTION

We now derive the differential equation of heat conduction for a stationary, homogeneous, isotropic solid with heat generation within the body. Heat generation may be due to nuclear, electrical, chemical, γ-ray, or other sources that may be a function of time and/or position. The heat generation rate in the medium, generally specified as heat generation per unit time, per unit volume, is denoted by the symbol $g(\mathbf{r},t)$, and if SI units are used, is given in the units W/m^3.

We consider the energy-balance equation for a small control volume V, illustrated in Fig. 1-4, stated as

$$\begin{bmatrix}\text{Rate of heat entering through}\\ \text{the bounding surfaces of } V\end{bmatrix} + \begin{bmatrix}\text{rate of energy}\\ \text{generation in } V\end{bmatrix} = \begin{bmatrix}\text{rate of storage}\\ \text{of energy in } V\end{bmatrix} \tag{1-4}$$

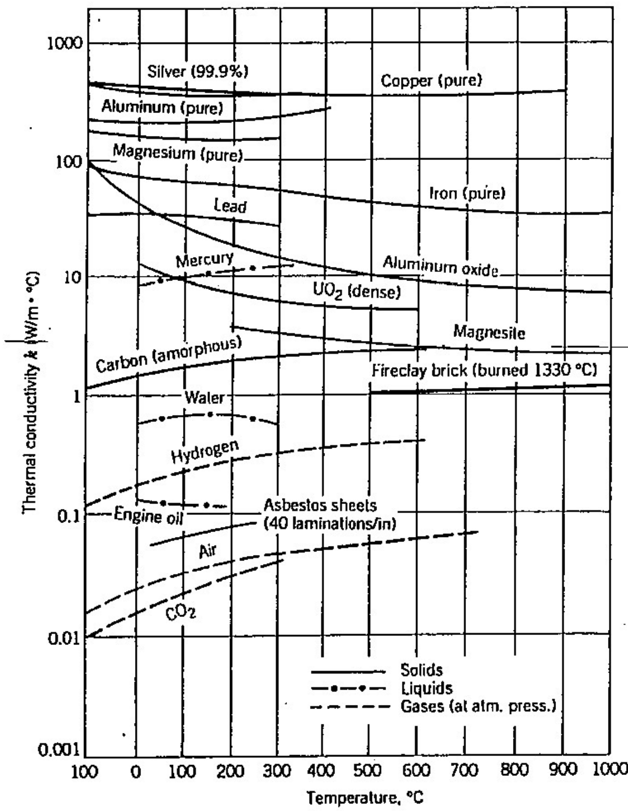


Fig. 1-2 Effect of temperature on thermal conductivity of materials.

Various terms in this equation are evaluated as

$$\left[\begin{array}{c}\text{Rate of heat entering through}\\ \text{the bounding surfaces of } V\end{array}\right] = -\int_A \mathbf{q}\cdot\hat{\mathbf{n}}\,dA = -\int_V \nabla\cdot\mathbf{q}\,dv \qquad \text{(1-5a)}$$

where A is the surface area of the volume element V, $\hat{\mathbf{n}}$ is the outward-drawn normal unit vector to the surface element dA, $\mathbf{q}$ is the heat flux vector at dA; here, the minus sign is included to ensure that the heat flow is into the volume element V, and the divergence theorem is used to convert the surface integral to volume integral. The remaining two terms are evaluated as

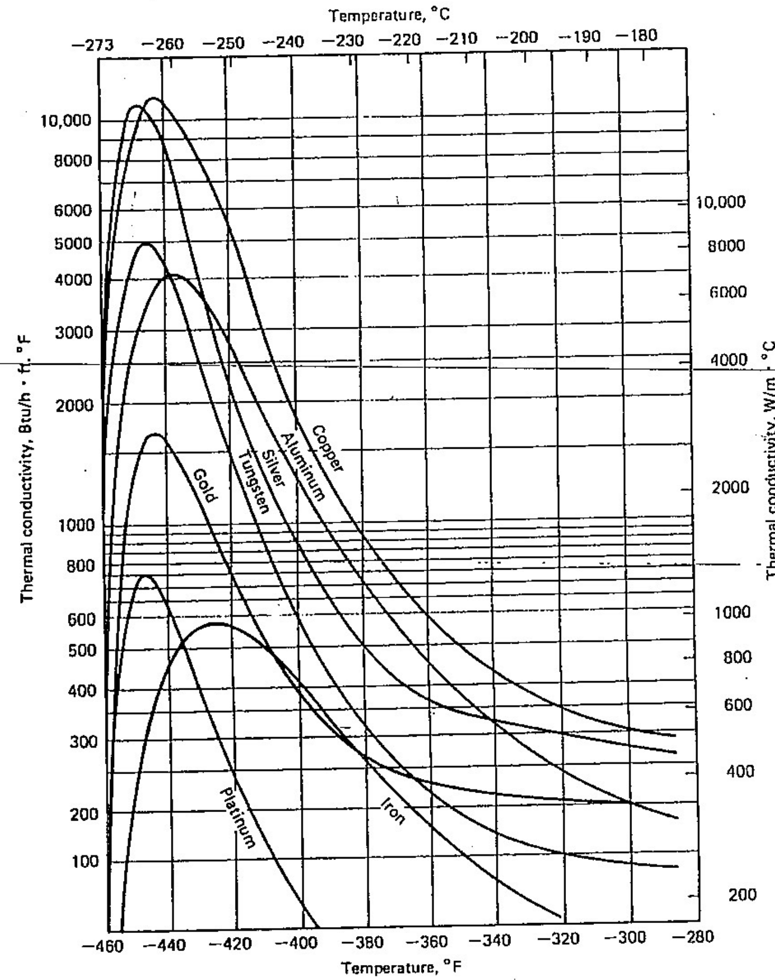


Fig. 1-3 Thermal conductivity of metals at low temperatures. (From Powell et al. [2])

$$(\text{Rate of energy generation in } V) = \int_V g(\mathbf{r}, t)\,dv \qquad \text{(1-5b)}$$

$$(\text{Rate of energy storage in } V) = \int_V \rho C_p \frac{\partial T(\mathbf{r}, t)}{\partial t}\,dv \qquad \text{(1-5c)}$$

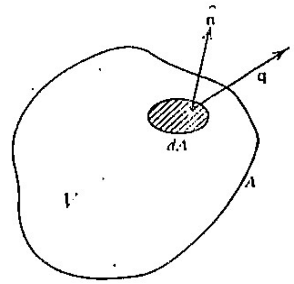


Fig. 1-4 Nomenclature for the derivation of heat conduction equation.

The substitution of equations (1-5) into equation (1-4) yields

$$\int_V \left[-\nabla \cdot \mathbf{q}(\mathbf{r}, t) + g(\mathbf{r}, t) - \rho C_p \frac{\partial T(\mathbf{r}, t)}{\partial t} \right] dv = 0 \qquad (1\text{-}6)$$

Equation (1-6) is derived for an arbitrary small-volume element V within the solid; hence the volume V may be chosen so small as to remove the integral. We obtain

$$-\nabla \cdot \mathbf{q}(\mathbf{r}, t) + g(\mathbf{r}, t) = \rho C_p \frac{\partial T(\mathbf{r}, t)}{\partial t} \qquad (1\text{-}7)$$

Substituting $\mathbf{q}(\mathbf{r}, t)$ from equation (1-1) into equation (1-7), we obtain the *differential equation of heat conduction* for a stationary, homogeneous, isotropic solid with heat generation within the body as

$$\nabla \cdot [k \nabla T(\mathbf{r}, t)] + g(\mathbf{r}, t) = \rho C_p \frac{\partial T(\mathbf{r}, t)}{\partial t} \qquad (1\text{-}8)$$

This equation is intended for temperature or space dependent k as well as temperature dependent C_p. When the thermal conductivity is assumed to be constant (i.e., independent of position and temperature), equation (1-8) simplifies to

$$\nabla^2 T(\mathbf{r}, t) + \frac{1}{k} g(\mathbf{r}, t) = \frac{1}{\alpha} \frac{\partial T(\mathbf{r}, t)}{\partial t} \qquad (1\text{-}9a)$$

where

$$\alpha = \frac{k}{\rho C_p} = \text{thermal diffusivity} \qquad (1\text{-}9b)$$

TABLE 1-1 Effect of Thermal Diffusivity on the Rate of Heat Propagation

Material	Silver	Copper	Steel	Glass	Cork
$\alpha \times 10^6\ m^2/s$	170	103	12.9	0.59	0.155
Time	9.5 min	16.5 min	2.2 h	2.00 days	7.7 days

For a medium with constant thermal conductivity and no heat generation, equations (1-9) become the diffusion or the Fourier equation

$$\nabla^2 T(\mathbf{r}, t) = \frac{1}{\alpha} \frac{\partial T(\mathbf{r}, t)}{\partial t} \qquad (1\text{-}10)$$

Here, the thermal diffusivity α is the property of the medium and has a dimension of length^2/time, which may be given in the units m^2/h or m^2/s. The physical significance of thermal diffusivity is associated with the speed of propagation of heat into the solid during changes of temperature with time. The higher the thermal diffusivity, the faster is the propagation of heat in the medium. This statement is better understood by referring to the following specific heat conduction problem: Consider a semiinfinite medium, $x \geqslant 0$, initially at a uniform temperature T_0. For times $t > 0$, the boundary surface at $x = 0$ is kept at zero temperature. Clearly, the temperature in the body will vary with position and time. Suppose we are interested in the time required for the temperature to decrease from its initial value T_0 to half of this value, $\frac{1}{2}T_0$, at a position, say, 30 cm from the boundary surface. Table 1-1 gives the time required for several different materials. It is apparent from these results that the larger the thermal diffusivity, the shorter is the time required for the applied heat to penetrate into the depth of the solid.

1-3 HEAT CONDUCTION EQUATION IN CARTESIAN, CYLINDRICAL, AND SPHERICAL COORDINATE SYSTEMS

The first step in the analytic solution of a heat conduction problem for a given region is to choose an orthogonal coordinate system such that its coordinate surfaces coincide with the boundary surfaces of the region. For example, the rectangular coordinate system is used for rectangular bodies, the cylindrical and the spherical coordinate systems are used for bodies having shapes such as cylinder and sphere, respectively, and so on. Here we present the heat conduction equation for an homogeneous, isotropic solid in the rectangular, cylindrical, and spherical coordinate systems.

Equations (1-8) and (1-9) in the *rectangular coordinate system* (x, y, z), respectively, become

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + g = \rho C_p \frac{\partial T}{\partial t} \qquad (1\text{-}11a)$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{1}{k}g=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1-11b}$$

Figure. 1-5*a*,*b* show the coordinate axes for the cylindrical (r, ϕ, z) and the spherical (r, ϕ, θ) coordinate systems. In the cylindrical coordinate system equations (1-8) and (1-9), respectively, become

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+g=\rho C_p\frac{\partial T}{\partial t} \tag{1-12a}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial z^2}+\frac{1}{k}g=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1-12b}$$

$\frac{1}{r}\left(\frac{\partial T}{\partial r}\right)+\frac{1}{r}\, r\,\frac{\partial^2 T}{\partial r^2}$

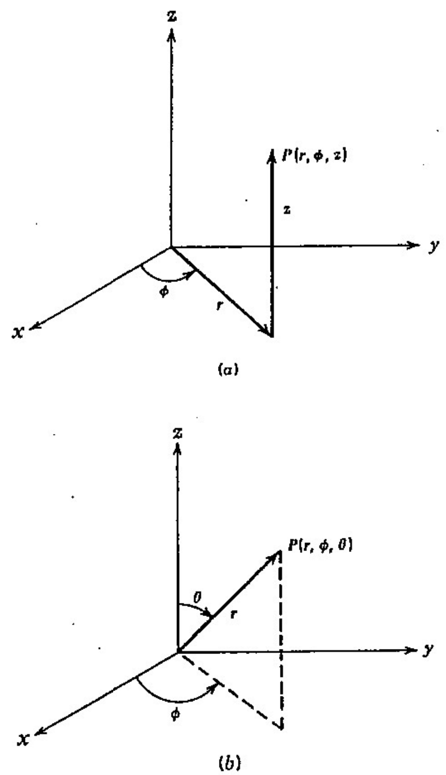


Fig. 1-5 (*a*) Cylindrical coordinate system (r, ϕ, z); (*b*) Spherical coordinate system (r, ϕ, θ).

and in the spherical coordinate system they take the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+g=\rho C_p\frac{\partial T}{\partial t} \tag{1-13a}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}+\frac{1}{k}g=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1-13b}$$

1-4 HEAT CONDUCTION EQUATION IN OTHER ORTHOGONAL COORDINATE SYSTEMS

In this book we shall be concerned particularly with the solution of heat conduction problems in the rectangular, cylindrical, and spherical coordinate systems; therefore, equations needed for such purposes are immediately obtained from equations (1-11)–(1-13) given above. The heat conduction equations in other *orthogonal curvilinear coordinate systems* (i.e., a coordinate system in which the coordinate lines intersect each other at right angles) are readily obtained by the coordinate transformation. Here we present a brief discussion of the transformation of the heat conduction equation into a general *orthogonal curvilinear coordinate system*. The reader is referred to references 5–7 for further details.

Let u_1, u_2, and u_3 be the three space coordinates, and $\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2$, and $\hat{\mathbf{u}}_3$ be the unit direction vectors in the u_1, u_2, and u_3 directions in a general orthogonal curvilinear coordinate system shown in Fig. 1-6. A differential length dS in the rectangular coordinate system (x, y, z) is given by

$$(dS)^2=(dx)^2+(dy)^2+(dz)^2 \tag{1-14}$$

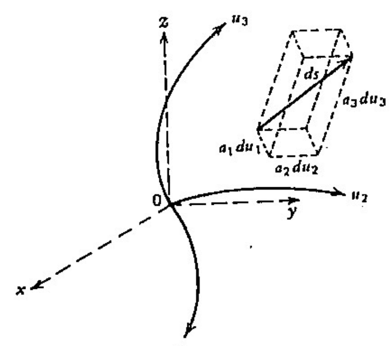


Fig. 1-6 A differential length ds in a curvilinear coordinate system (u_1, u_2, u_3).

Let the functional relationship between orthogonal curvilinear coordinates (u_1, u_2, u_3) and the rectangular coordinates (x, y, z) be given as

$$x = X(u_1, u_2, u_3), \qquad y = Y(u_1, u_2, u_3) \qquad \text{and} \qquad z = Z(u_1, u_2, u_3) \tag{1-15}$$

Then, the differential lengths dx, dy, and dz are obtained from equations (1-15) by differentiation

$$dx = \sum_{i=1}^{3} \frac{\partial X}{\partial u_i} du_i \tag{1-16a}$$

$$dy = \sum_{i=1}^{3} \frac{\partial Y}{\partial u_i} du_i \tag{1-16b}$$

$$dz = \sum_{i=1}^{3} \frac{\partial Z}{\partial u_i} du_i \tag{1-16c}$$

Substituting equations (1-16) into equation (1-14), and noting that the dot products must be zero when u_1, u_2, and u_3 are mutually orthogonal yields the following expression for the differential length dS in the orthogonal curvilinear coordinate system u_1, u_2, u_3

$$(dS)^2 = a_1^2(du_1)^2 + a_2^2(du_2)^2 + a_3^2(du_3)^2 \tag{1-17}$$

where

$$a_i^2 = \left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2, \qquad i = 1, 2, 3 \tag{1-18}$$

Here, the coefficients a_1, a_2, and a_3 are called the *scale factors*, which may be constants or functions of the coordinates. Thus, when the functional relationship between the rectangular and the orthogonal curvilinear system is available [i.e., as in equation (1-15)], then the scale factors a_i are evaluated by equation (1-18).

Once the scale factors are known, the gradient of temperature in the orthogonal curvilinear coordinate system (u_1, u_2, u_3) is given by

$$\nabla T = \hat{\mathbf{u}}_1 \frac{1}{a_1}\frac{\partial T}{\partial u_1} + \hat{\mathbf{u}}_2 \frac{1}{a_2}\frac{\partial T}{\partial u_2} + \hat{\mathbf{u}}_3 \frac{1}{a_3}\frac{\partial T}{\partial u_3} \tag{1-19}$$

The expression defining the heat flux vector $\mathbf{q}$ becomes

$$\mathbf{q} = -k\nabla T = -k \sum_{i=1}^{3} \hat{\mathbf{u}}_i \frac{1}{a_i}\frac{\partial T}{\partial u_i} \tag{1-20}$$

and the three components of the heat flux vector along the u_1, u_2, and u_3 coordinates are given by

$$q_i = -k\frac{1}{a_i}\frac{\partial T}{\partial u_i}, \qquad i = 1, 2, 3 \tag{1-21}$$

The divergence of the heat flux vector $\mathbf{q}$ in the orthogonal curvilinear coordinate system (u_1, u_2, u_3) is given by

$$\nabla \cdot \mathbf{q} = \frac{1}{a}\left[\frac{\partial}{\partial u_1}\left(\frac{a}{a_1}q_1\right) + \frac{\partial}{\partial u_2}\left(\frac{a}{a_2}q_2\right) + \frac{\partial}{\partial u_3}\left(\frac{a}{a_3}q_3\right)\right] \tag{1-22a}$$

where

$$a = a_1 a_2 a_3 \tag{1-22b}$$

The differential equation of heat conduction in a general orthogonal curvilinear coordinate system is now obtained by substituting the results given by equations (1-21) and (1-22) into equation (1-7)

$$\frac{1}{a}\left[\frac{\partial}{\partial u_1}\left(k\frac{a}{a_1^2}\frac{\partial T}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(k\frac{a}{a_2^2}\frac{\partial T}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(k\frac{a}{a_3^2}\frac{\partial T}{\partial u_3}\right)\right] + g = \rho C_p \frac{\partial T}{\partial t} \tag{1-23}$$

The heat conduction equations in the cylindrical and spherical coordinates given previously by equations (1-12) and (1-13) are readily obtainable as special cases from the general equation (1-23) if the appropriate values of the scale factors are introduced.

Length, Area, and Volume Relations

In the analysis of heat conduction problems integrations are generally required over a length, an area, or a volume. If such an operation is to be performed in an orthogonal curvilinear coordinate system, expressions are needed for a differential length dl, a differential area dA, and a differential volume dV. These relations are determined as now described.

In the case of rectangular coordinate system, a differential volume element dV is given by

$$dV = dx\,dy\,dz \tag{1-24a}$$

and the differential areas dA_x, dA_y, and dA_z cut from the planes $x =$ constant, $y =$ constant, and $z =$ constant are given, respectively, by

$$dA_x = dy\,dz, \qquad dA_y = dx\,dz, \qquad \text{and} \qquad dA_z = dx\,dy \tag{1-24b}$$

In the case of an orthogonal curvilinear coordinate system, the elementary lengths dl_1, dl_2, and dl_3 along the three coordinate axes u_1, u_2, and u_3 are given, respectively, by

$$dl_1 = a_1\,du_1, \qquad dl_2 = a_2\,du_2, \qquad \text{and} \qquad dl_3 = a_3\,du_3 \tag{1-25a}$$

Then, an elementary volume element dV is expressed as

$$dV = a_1 a_2 a_3\,du_1\,du_2\,du_3 = a\,du_1\,du_2\,du_3, \qquad \text{where} \qquad a \equiv a_1 a_2 a_3 \tag{1-25b}$$

The differential areas dA_1, dA_2, and dA_3 cut from the planes $u_1 =$ constant, $u_2 =$ constant, and $u_3 =$ constant are given, respectively, by

$$dA_1 = dl_2\,dl_3 = a_2 a_3\,du_2\,du_3, \qquad dA_2 = dl_1\,dl_3 = a_1 a_3\,du_1\,du_3 \qquad \text{and}$$
$$dA_3 = dl_1\,dl_2 = a_1 a_2\,du_1\,du_2 \tag{1-25c}$$

Example 1-1

Determine the scale factors for the cylindrical coordinate system (r, ϕ, z) and write the expressions for the heat flux components.

Solution. The functional relationships between the coordinates (r, ϕ, z) and the rectangular coordinates (x, y, z) are given by

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z$$

Let

$$u_1 \equiv r, \qquad u_2 = \phi, \qquad \text{and} \qquad u_3 \equiv z \tag{1-26}$$

The scale factors $a_1 \equiv a_r, a_2 \equiv a_\phi$, and $a_3 \equiv a_z$ for the (r, ϕ, z) coordinate system are determined by equation (1-18) as

$$a_1^2 \equiv a_r^2 = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = \cos^2\phi + \sin^2\phi + 0 = 1$$

$$a_2^2 \equiv a_\phi^2 = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = (-r\sin\phi)^2 + (r\cos\phi)^2 + 0 = r^2$$

$$a_3^2 \equiv a_z^2 = \left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2 = 0 + 0 + 1 = 1$$

Hence the scale factors for the cylindrical coordinate system become

$$a_r = 1, \qquad a_\phi = r, \qquad a_z = 1, \qquad \text{and} \qquad a = r \tag{1-27a}$$

and the three components of the heat flux are given as

$$q_r = -k\frac{\partial T}{\partial r}, \qquad q_\phi = -\frac{k}{r}\frac{\partial T}{\partial \phi}, \qquad \text{and} \qquad q_z = -k\frac{\partial T}{\partial z} \tag{1-27b}$$

Example 1-2

Determine the scale factors for the spherical coordinate system (r, ϕ, θ).

Solution. The functional relationships between the coordinates (r, ϕ, θ) and the rectangular coordinates (x, y, z) are given by

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

Let

$$u_1 \equiv r, \qquad u_2 \equiv \phi, \qquad \text{and} \qquad u_3 = \theta \tag{1-28}$$

Then, by utilizing equation (1-18), the scale factors $a_1 \equiv a_r, a_2 \equiv a_\phi$, and $a_3 \equiv a_\theta$ are determined as

$$a_1^2 \equiv a_r^2 = (\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2 = 1$$
$$a_2^2 \equiv a_\phi^2 = r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi + 0 = r^2\sin^2\theta$$
$$a_3^2 \equiv a_\theta^2 = r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\phi = r^2$$

Hence the scale factors become

$$a_r = 1, \qquad a_\phi = r\sin\theta, \qquad a_\theta = r, \qquad \text{and} \qquad a = r^2\sin\theta \tag{1-29}$$

1-5 GENERAL BOUNDARY CONDITIONS

The differential equation of heat conduction will have numerous solutions unless a set of boundary conditions and an initial condition (for the time-dependent problem) are prescribed. The initial condition specifies the temperature distribution in the medium at the origin of the time coordinate (that is, $t = 0$), and the boundary conditions specify the temperature or the heat flow at the boundaries of the region. For example, at a given boundary surface, the temperature distribution may be prescribed, or the heat flux distribution may be prescribed, or there may be heat exchange by convection and/or radiation with an environment at a prescribed temperature. The boundary condition can be derived by writing an energy balance equation at the surface of the solid.

We consider a surface element having an outward-drawn unit normal vector $\hat{\mathbf{n}}$, subjected to convection, radiation, and external heat supply as illustrated in Fig. 1-7. The physical significance of various heat fluxes shown in this figure is as follows.

The quantity q_{sup} represents energy supplied to the surface, in W/m^2, from an external source.

The quantity q_{conv} represents heat loss from the surface at temperature T by convection with a heat transfer coefficient h into an external ambient at a temperature T_∞, and is given by

$$q_{conv} = h(T - T_\infty), \qquad W/m^2 \tag{1-30a}$$

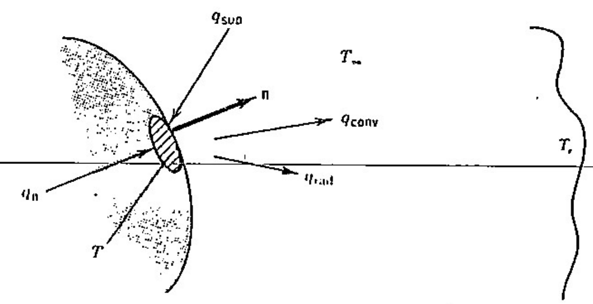


Fig. 1-7 Energy balance at the surface of a solid.

TABLE 1-2 Typical Values of the Convective Heat Transfer Coefficient *h*

Type of Flow	h, W/(m^2 °C)
Free Convection, $\Delta T = 25$°C	
0.25-m vertical plate in	
Atmospheric air	5
Engine oil	37
Water	440
0.02-m-OD horizontal cylinder in	
Atmospheric air	8
Engine oil	62
Water	741
Forced Convection	
Atmospheric air at 25°C with $U_\infty = 10$ m/s over $L = 0.1$-m flat plate	40
Flow at 5 m/s across 1-cm-OD cylinder of	
Atmospheric air	85
Engine oil	1,800
Water flow at 1 kg/s inside 2.5-cm-ID tube	10,500
Boiling of Water at 1 atm	
Pool boiling in a container	3,000
Pool boiling at peak heat flux	35,000
Film boiling	300
Condensation of Steam at 1 atm	
Film condensation on horizontal tubes	9,000–25,000
Film condensation on vertical surfaces	4,000–11,000
Dropwise condensation	60,000–120,000

Here the heat transfer coefficient h varies with the type of flow (laminar, turbulent, etc.), the geometry of the body and flow passage area, the physical properties of the fluid, the average temperature, and many others. There is a wide difference in the range of values of the heat transfer coefficient for various applications. Table 1-2 lists the typical values of h, in W/m^2°C, encountered in some applications.

The quantity q_{rad} represents heat loss from the surface by radiation into an ambient at an effective temperature T_r, and is given by

$$q_{rad} = \epsilon\sigma(T^4 - T_r^4) \quad \text{W/m}^2 \tag{1-30b}$$

where ϵ is the emissivity of the surface and σ is the Stefan–Boltzmann constant, that is, $\sigma \equiv 5.6697 \times 10^{-8}$ W/(m^2·K^4).

The quantity q_n represents the component of the conduction heat flux vector normal to the surface element and is

$$q_n = \mathbf{q}\cdot\hat{\mathbf{n}} = -k\nabla T\cdot\hat{\mathbf{n}} \tag{1-31a}$$

For the Cartesian coordinates we have

$$\nabla T = \hat{\mathbf{i}}\frac{\partial T}{\partial x} + \hat{\mathbf{j}}\frac{\partial T}{\partial y} + \hat{\mathbf{k}}\frac{\partial T}{\partial z} \tag{1-31b}$$

$$\hat{\mathbf{n}} = \hat{\mathbf{i}}l_x + \hat{\mathbf{j}}l_y + \hat{\mathbf{k}}l_z \tag{1-31c}$$

Introducing equations (1-31b,c) into (1-31a), the normal component of the heat flux vector at the surface becomes

$$q_n = -k\left(l_x\frac{\partial T}{\partial x} + l_y\frac{\partial T}{\partial y} + l_z\frac{\partial T}{\partial z}\right) = -k\frac{\partial T}{\partial n} \tag{1-32}$$

where l_x, l_y, and l_z are the direction cosines (i.e., cosine of the angles) of the unit normal vector $\hat{\mathbf{n}}$ with the x, y, and z coordinate axes, respectively. Similar expressions can be developed for the cylindrical and spherical coordinate systems.

To develop the boundary condition, we consider the energy balance at the surface as

$$\text{Heat supply} = \text{heat loss}$$

or

$$q_n + q_{sup} = q_{conv} + q_{rad} \tag{1-33}$$

Introducing the expressions (1-30a,b) and (1-32) into (1-33), the boundary condition becomes

$$-k\frac{\partial T}{\partial n} + q_{sup} = h(T - T_\infty) + \epsilon\sigma(T^4 - T_r^4) \tag{1-34a}$$

which can be rearranged as

$$k\frac{\partial T}{\partial n}+hT+\epsilon\sigma T^4=hT_\infty+q_{sup}+\epsilon\sigma T_r^4 \tag{1-34b}$$

where all the quantities on the right-hand side of equation (1-34b) are known and the surface temperature T is unknown.

The general boundary condition given by equations (1-34) is nonlinear because it contains the fourth power of the unknown surface temperature T^4. In addition, the absolute temperatures need to be considered when radiation is involved. If $(|T-T_r|)/T_r \ll 1$, the radiation term can be linearized and equation (1-34a) takes the form

$$-k\frac{\partial T}{\partial n}+q_{sup}=h(T-T_\infty)+h_r(T-T_r) \tag{1-35a}$$

where the heat transfer coefficient for radiation is defined as

$$h_r \equiv 4\epsilon\sigma T_r^3 \tag{1-35b}$$

1-6 LINEAR BOUNDARY CONDITIONS

In this book, for the analytic solution of linear heat conduction problems, we shall consider the following three different types of linear boundary conditions.

1. *Boundary Condition of the First Kind.* This is the situation when the temperature distribution is prescribed at the boundary surface, that is

$$T=f(\mathbf{r},t) \quad \text{on} \quad S \tag{1-36a}$$

where the prescribed surface temperature $f(\mathbf{r},t)$ is, in general, a function of position and time. The special case

$$T=0 \quad \text{on} \quad S \tag{1-36b}$$

is called the *homogeneous boundary condition of the first kind.*

2. *Boundary Condition of the Second Kind.* This is the situation in which the heat flux is prescribed at the surface, that is

$$k\frac{\partial T}{\partial n}=f(\mathbf{r},t) \quad \text{on} \quad S \tag{1-37a}$$

where $\partial T/\partial n$ is the derivative along the outward drawn normal to the surface.

Here $f(\mathbf{r},t)$ is the prescribed heat flux, W/m^2. The special case

$$\frac{\partial T}{\partial n}=0 \quad \text{on} \quad S \tag{1-37b}$$

is called the *homogeneous boundary condition of the second kind.*

3. *Boundary Condition of the Third Kind.* This is the convection boundary condition which is readily obtained from equation (1-35a) by setting the radiation term and the heat supply equal to zero, that is

$$k\frac{\partial T}{\partial n}+hT=hT_\infty(\mathbf{r},t) \quad \text{on} \quad S \tag{1-38a}$$

where, for generality, the ambient temperature $T_\infty(\mathbf{r},t)$ is assumed to be a function of position and time. The special case

$$k\frac{\partial T}{\partial n}+hT=0 \quad \text{on} \quad S \tag{1-38b}$$

is called the *homogeneous boundary condition of the third kind.* It represents convection into a medium at zero temperature. Clearly, the boundary conditions of the first and second kind are obtainable from the boundary condition of the third as special cases if k and h are treated as coefficients. For example, by setting $k=0$ and $T_\infty(\mathbf{r},t)\equiv f(\mathbf{r},t)$, equation (1-38a) reduces to equation (1-36a). Similarly, by setting $hT_\infty(\mathbf{r},t)\equiv f(\mathbf{r},t)$ and then letting $h=0$ on the left-hand side, equation (1-38a) reduces to equation (1-37a).

4. *Interface Boundary Condition.* When two materials having different thermal conductivities k_1 and k_2 are in imperfect contact and have a common boundary as illustrated in Fig. 1-8, the temperature profile through the solids experiences a sudden drop across the interface between the two materials. The physical significance of this temperature drop is envisioned better if we consider an enlarged view of the interface as shown in this figure and note that actual metal-to-metal contact takes place at a limited number of spots and the void between them is filled with air, which is the surrounding fluid. As thermal conductivity of air is much smaller than that of metal, a steep temperature drop occurs across the gap. To develop the boundary condition for such an interface, we write the energy balance as

$$\begin{pmatrix}\text{Heat conduction}\\ \text{thru. solid 1}\end{pmatrix}=\begin{pmatrix}\text{heat transfer}\\ \text{across the gap}\end{pmatrix}=\begin{pmatrix}\text{heat conduction}\\ \text{thru. solid 2}\end{pmatrix} \tag{1-39a}$$

$$-k_1\left.\frac{\partial T_1}{\partial x}\right|_i=h_c(T_1-T_2)_i=-k_2\left.\frac{\partial T_2}{\partial x}\right|_i \tag{1-39b}$$

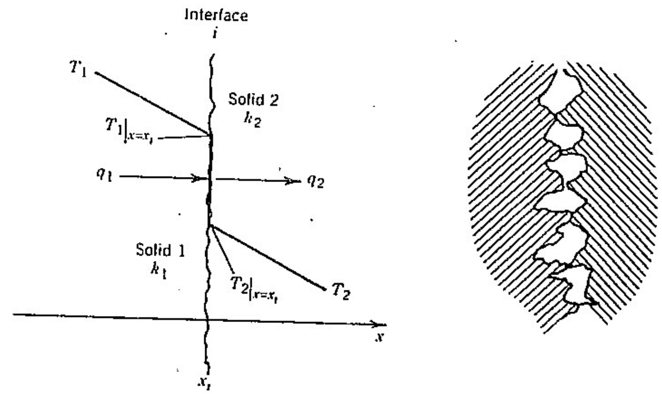


Fig. 1-8 Boundary condition at the interface of two contacting surfaces.

where subscript i denotes the inferface and h_c, in W/(m^2·°C), is called the *contact conductance* for the interface. Equation (1-39b) provides two expressions for the boundary condition at the interface of two contacting solids, and it is generally called the *interface boundary conditions.*

For the special case of *perfect thermal contact* between the surfaces, we have $h_c \to \infty$, and equation (1-39b) reduces to

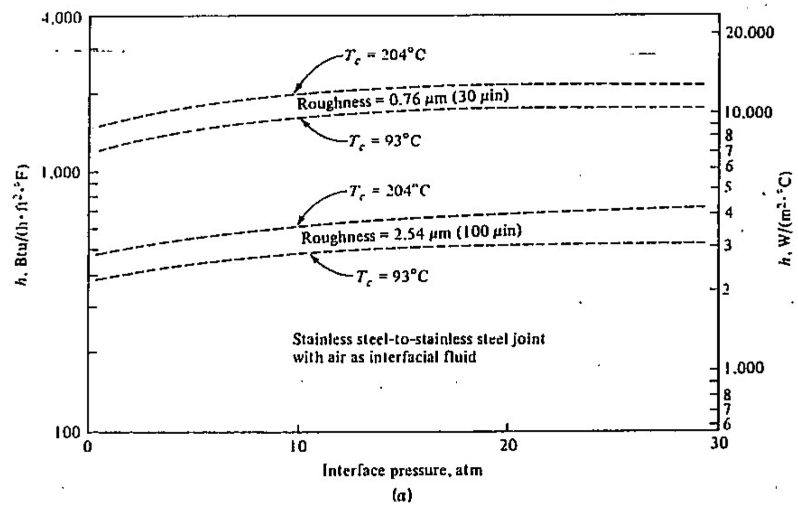
$$T_1 = T_2 \quad \text{at} \quad S_i \tag{1-40a}$$

$$-k_1 \frac{\partial T_1}{\partial x} = -k_2 \frac{\partial T_2}{\partial x} \quad \text{at} \quad S_i \tag{1-40b}$$

where equation (1-40a) is the continuity of temperature, and equation (1-40b) is the continuity of heat flux at the interface.

The experimentally determined values of contact conductance for typical materials in contact can be found in references 8–10. The surface roughness, the interface pressure and temperature, thermal conductivities of the contacting metal and the type of fluid in the gap are the principal factors that affect contact conductance.

To illustrate the effects of various parameters such as the surface roughness, the interface temperature, the interface pressure, and the type of material, we present in Fig. 1-9*a,b* the *interface thermal contact conductance h* for stainless steel-to-stainless steel and aluminum-to-aluminum joints. The results on these figures show that interface conductance increases with increasing interface pressure, increasing interface temperature, and decreasing surface roughness. The interface conductance is higher with a softer material (aluminum) than with a harder material (stainless steel).



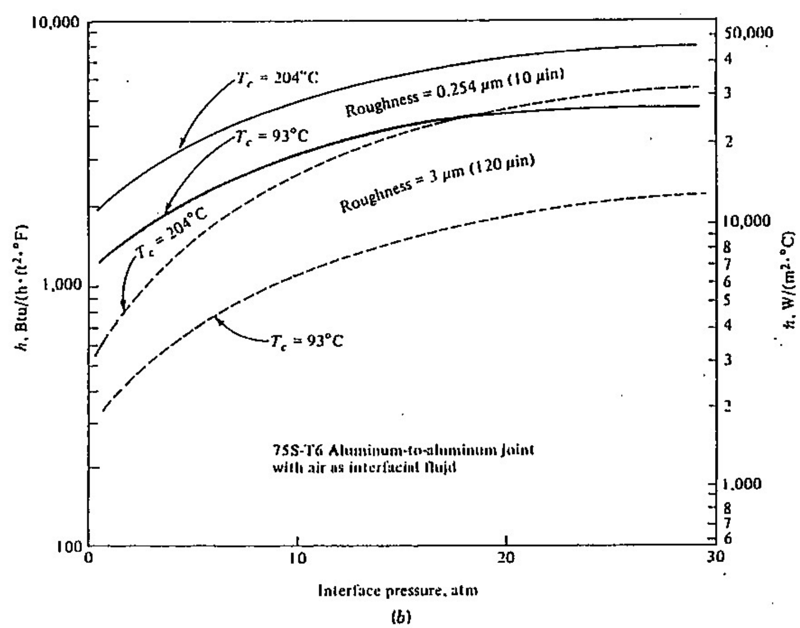


Fig. 1-9 Effects of interface pressure, contact temperature, and roughness on interface conductance h. (Based on data from reference 8).

The smoothness of the surface is another factor that affects contact conductance; a joint with a superior surface finish may exhibit lower contact conductance owing to waviness. The adverse effect of waviness can be overcome by introducing between the surfaces an interface shim from a soft material such as lead.

Contact conductance also is reduced with a decrease in the ambient-air pressure, because the effective thermal conductance of the gas entrapped in the interface is lowered.

Example 1-3

Consider a plate subjected to heating at the rates of f_1 and f_2, in W/m^2, at the boundary surfaces $x=0$ and $x=L$, respectively. Write the boundary conditions.

Solution. The prescribed heat flux boundary condition is given by equation (1-37a) as

$$k\frac{\partial T}{\partial n}=f \qquad \text{on} \qquad S \tag{1-41}$$

The outward-drawn normal vectors at the boundary surfaces $x=0$ and $x=L$ are in the negative x and positive x directions, respectively. Hence the boundary conditions become

$$-k\frac{\partial T}{\partial x}=f_1 \qquad \text{at} \qquad x=0 \tag{1-42a}$$

$$k\frac{\partial T}{\partial x}=f_2 \qquad \text{at} \qquad x=L \tag{1-42b}$$

Example 1-4

Consider a hollow cylinder subjected to convection boundary conditions at the inner $r=a$ and outer $r=b$ surfaces into ambients at temperatures $T_{\infty 1}$ and $T_{\infty 2}$, with heat transfer coefficients $h_{\infty 1}$ and $h_{\infty 2}$ respectively, as illustrated in Fig. 1-10. Write the boundary conditions.

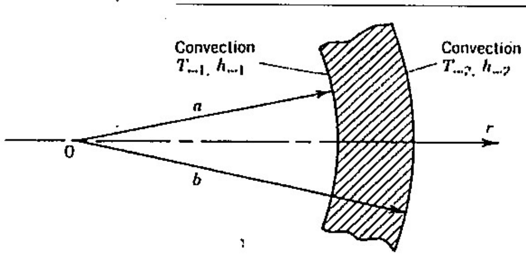


Fig. 1-10 Boundary conditions for Example 1-4.

Solution. The convection boundary condition is given by equation (1-38a) in the form

$$k\frac{\partial T}{\partial n}+hT=hT_\infty \qquad \text{at} \qquad S \tag{1-43}$$

The outward-drawn normal at the boundary surfaces $r=a$ and $r=b$ are in the negative r and positive r directions. Hence the boundary condition (1-43) gives

$$-k\frac{\partial T}{\partial r}+h_{\infty 1}T=h_{\infty 1}T_{\infty 1} \qquad \text{at} \qquad r=a \tag{1-44a}$$

$$k\frac{\partial T}{\partial r}+h_{\infty 2}T=h_{\infty 2}T_{\infty 2} \qquad \text{at} \qquad r=b \tag{1-44b}$$

1-7 TRANSFORMATION OF NONHOMOGENEOUS BOUNDARY CONDITIONS INTO HOMOGENEOUS ONES

In the solution of transient heat conduction problems with the orthogonal expansion technique, the contribution of nonhomogeneous terms of the boundary conditions in the solution generally gives rise to convergence difficulties when the solution is evaluated near the boundary. Therefore, whenever possible, it is desirable to transform the nonhomogeneous boundary conditions into homogeneous ones. Here we present a methodology for performing such transformations for some special cases.

We consider one-dimensional transient heat conduction with energy generation and nonhomogeneous convection boundary conditions for a slab, hollow cylinder and sphere given by

$$\frac{1}{x^p}\frac{\partial}{\partial x}\left(x^p\frac{\partial T}{\partial x}\right)+\frac{1}{k}g(x,t)=\frac{1}{\alpha}\frac{\partial T}{\partial t}, \qquad \text{in} \qquad x_0<x<x_L, \qquad t>0 \tag{1-45a}$$

$$-k\frac{\partial T}{\partial x}+h_1T=h_1f_1(t) \qquad \text{at} \qquad x=x_0 \qquad t>0 \tag{1-45b}$$

$$k\frac{\partial T}{\partial x}+h_2T=h_2f_2(t) \qquad \text{at} \qquad x=x_L \qquad t>0 \tag{1-45c}$$

$$T=F(x) \qquad \text{for} \qquad t=0 \qquad x_0\leqslant x\leqslant L \tag{1-45d}$$