Assignment Problem

Chapter 5

Assignment Problem

- Assignment Problem : involve determining the most efficient assignment of
  - People to projects,
  - Machines to Projects
  - jobs to machines,
  - Space to Departments,
  - Contracts to bidders, etc.
- The objective is most often to **minimize** total costs or total time
- **only one** job or worker is assigned to one machine or project. **(one-to-one correspondence)**

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Assignment problem

- The rows (**n**)contain the objects or people we wish to assign, and the columns (**m**) comprise the tasks or things we want them assigned to.
- The numbers in the table are the costs associated with each particular assignment.
- In an n x m matrix, there are **n!** ways of doing the matrix (permutation matrix).
- For example, if we have five men and five jobs , there are 120 feasible assignments.

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The model

- X_{ij} =1 if a person i is assigned to meet the job j
- X_{ij} = 0 if person i is not assigned to job j
- Assignment problem is **balanced transportation problem** in which all supplies and demands are equal to 1.

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The Assignment Model

In general the LP formulation is given as:

Minimize $$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = 1, \forall i = 1,\dots,n$$ **Each supply is 1**

$$\sum_{i=1}^{n} x_{ij} = 1, \forall j = 1,\dots,n$$ **Each demand is 1**

$$x_{ij} = 0 \text{ or } 1, \forall ij$$

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Example-1

- Three repair persons, each with different talents and abilities, are available to do three jobs. The owner of the shop estimates what it will cost in wages to assign each of the workers to each of the three projects. **(The cost table)**

	PROJECTS		
PERSON	1	2	3
Adams	11	14	6
Brown	8	10	11
Cooper	9	12	7

- There are six assignment options. **3!**

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Example-1

- The least-cost solution is a total cost of $25. The assignment is :
  - Cooper to project 1,
  - Brown to project 2,
  - Adams to project 3,

PROJECT ASSIGNMENT				
1	2	3	LABOUR COSTS ($)	TOTAL COSTS ($)
Adams	Brown	Cooper	11 + 10 +7	28
Adams	Cooper	Brown	11 + 12 +11	34
Brown	Adams	Cooper	8 + 14 + 7	29
Brown	Cooper	Adams	8 + 12 + 6	26
Cooper	Adams	Brown	9 + 14 + 11	34
Cooper	Brown	Adams	9 + 10 + 6	25

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Lindo formulation and solution

MIN 11 X1A + 14 X1B + 6 X1C + 8 X2A + 10 X2B + 11 X2C + 9 X3A + 12 X3B + 7 X3C

ST

X1A + X1B + X1C = 1

X2A + X2B + X2C = 1

X3A + X3B + X3C = 1

X1A + X2A + X3A = 1

X1B + X2B + X3B = 1

X1C + X2C + X3C = 1

END

INTEGER (X1A, X1B, X1C, X2A, X2B, X2C, X3A, X3B, X3C)

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Lindo solution

LP OPTIMUM FOUND AT STEP 5

OBJECTIVE FUNCTION VALUE

1) 25.00000

VARIABLE	VALUE	REDUCED COST
X1A	0.000000	3.000000
X1B	0.000000	4.000000
X1C	**1.000000**	0.000000
X2A	0.000000	0.000000
X2B	**1.000000**	0.000000
X2C	0.000000	5.000000
X3A	**1.000000**	0.000000
X3B	0.000000	1.000000
X3C	0.000000	0.000000

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Example-2

- Four men are available to do four different jobs. From past records the time that each man takes to do each job is known and the information is shown in Table. If you want to minimize the total time needed to complete the four jobs.

	Jobs			
Person	**1**	**2**	**3**	**4**
1	14	5	8	7
2	2	12	6	5
3	7	8	3	9
4	2	4	6	10

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LP Model

- The LP problem can be formulated as follows

$$\min Z = 14X_{11} + 5X_{12} + 8X_{13} + 7X_{14} + 2X_{21} + 12X_{22} + 6X_{23} + 5X_{24}$$
$$+ 7X_{31} + 8X_{32} + 3X_{33} + 9X_{34} + 2X_{41} + X_{42} + 6X_{43} + 10X_{44}$$
$$\text{s.t.} X_{11} + X_{12} + X_{13} + X_{14} = 1$$
$$X_{21} + X_{22} + X_{23} + X_{24} = 1$$
$$X_{31} + X_{32} + X_{33} + X_{34} = 1$$
$$X_{41} + X_{42} + X_{43} + X_{44} = 1$$
$$X_{11} + X_{21} + X_{31} + X_{41} = 1$$
$$X_{12} + X_{22} + X_{32} + X_{42} = 1$$
$$X_{13} + X_{23} + X_{33} + X_{43} = 1$$
$$X_{14} + X_{24} + X_{34} + X_{44} = 1$$
$$X_{ij} = 0 \text{ or } X_{ij} = 1$$

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Lindo solution

LP OPTIMUM FOUND AT STEP 6

OBJECTIVE FUNCTION VALUE

1) 15.00000

VARIABLE	VALUE	REDUCED COST
X1A	0.000000	11.000000
X1B	**1.000000**	0.000000
X1C	0.000000	5.000000
X1D	0.000000	1.000000
X2A	0.000000	0.000000
X2B	0.000000	8.000000
X2C	0.000000	4.000000
X2D	**1.000000**	0.000000
X3A	0.000000	4.000000
X3B	0.000000	3.000000
X3C	**1.000000**	0.000000
X3D	0.000000	3.000000
X4A	**1.000000**	0.000000
X4B	0.000000	0.000000
X4C	0.000000	4.000000
X4D	0.000000	5.000000

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Lindo model

MIN 14X1A + 5X1B + 8X1C + 7X1D + 2X2A + 12X2B + 6X2C + 5X2D + 7X3A + 8X3B + 3X3C + 9X3D + 2X4A + 4X4B + 6X4C + 10X4D

ST

X1A + X1B + X1C + X1D = 1

X2A + X2B + X2C + X2D = 1

X3A + X3B + X3C + X3D = 1

X4A + X4B + X4C + X4D = 1

X1A + X2A + X3A + X4A = 1

X1B + X2B + X3B + X4B = 1

X1C + X2C + X3C + X4C = 1

X1D + X2D + X3D + X4D = 1

END

INTEGER (X1A, X1B, X1C, X1D, X2A, X2B, X2C, X2D, X3A, X3B, X3C, X3D, X4A, X4B, X4C, X4D)

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Solving Assignment Problem

- US Air Force has used this for assigning thousands of people to jobs. Imagine the procedure of direct comparison !!!
- Special algorithms exist to solve assignment problems.
- The efficient means of finding the optimal solution is Hungarian solution method.
- It operates on a ***principle of matrix reduction***, reducing the problem to a matrix of opportunity costs.

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Solving Assignment Problem

- Opportunity costs : show the relative penalties associated with assigning any person to a project as opposed to making the best or least-cost assignment.
- Make assignments such that the opportunity cost for each assignment is zero.
- Use Hungarian Method

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Hungarian Method –Step 1

- ***Find the opportunity cost table*** by :
  - (a) Subtracting the smallest number in each row of the original cost table or matrix from every number in that row.
  - (b) Then subtracting the smallest number in each column of the table obtained in part (a) from every number in that column.

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Hungarian Method –Step 2

- ***Test the table resulting from step 1 to see whether an optimal assignment can be made.***
- The procedure is to draw the minimum number of vertical and horizontal straight lines necessary to cover all zeros in the table.
  - If the number of lines equals either the number of rows or columns, an optimal assignment can be made.
  - If the number of lines is less than the number of rows or columns, we proceed to step 3.

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Hungarian Method –Step 3

- ***Revise the present opportunity cost table***.
- This is done by subtracting the smallest number not covered by a line from every other uncovered number. This same smallest number is also added to any number(s) lying at the intersection of the horizontal and vertical lines.
- We then return to step 2 and continue the cycle until an optimal assignment is possible.

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Example-1 (Hungarian Method)

- Apply three steps to assignment example-1.

	PROJECTS		
PERSON	1	2	3
Adams	**11**	**14**	**6**
Brown	**8**	**10**	**11**
Cooper	**9**	**12**	**7**

the row reduction (Step 1 part a)

	PROJECTS		
PERSON	1	2	3
Adams	**5**	**8**	**0**
Brown	**0**	**2**	**3**
Cooper	**2**	**5**	**0**

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Example-1

- The **column reduction** of Step 1 part (b),

	PROJECTS		
PERSON	1	2	3
Adams	**5**	**8**	**0**
Brown	**0**	**2**	**3**
Cooper	**2**	**5**	**0**

	PROJECTS		
PERSON	1	2	3
Adams	**5**	**6**	**0**
Brown	**0**	**0**	**3**
Cooper	**2**	**3**	**0**

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Example-1

- Draw vertical and horizontal straight lines (**Step 2**) to cover all the zeros in last Table.

	PROJECTS		
PERSON	1	2	3
Adams	5	6	0
Brown	0	0	3
Cooper	2	3	0

- Since the number of lines is less than the number of rows or columns an optimal assignment cannot be made.
- No optimal solution, revise the table.

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Example-1

- Apply Step 3:

	PROJECTS		
PERSON	1	2	3
Adams	5	6	0
Brown	0	0	3
Cooper	2	3	0

Revised opportunity cost table

	PROJECTS		
PERSON	1	2	3
Adams	3	4	0
Brown	0	0	5
Cooper	0	1	0

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Example-1

- Test optimality: return to Step 2

Optimality test on the revised opportunity cost table

	PROJECTS		
PERSON	1	2	3
Adams	3	4	0
Brown	0	0	5
Cooper	0	1	0

- Because it requires three lines to cover the zeros (see Table above), an optimal assignment can be made.

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Example-1

- Make the allocation. Note that only one assignment will be made from each row or column.
- (a) Find a row or column with only one zero cell.
- (b) Make the assignment corresponding to that zero cell.
- (c) Eliminate that row and column from the table.
- (d) Continue until all the assignments have been made.

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Example-1

- The final allocation:

	FIRST ASSIGNMENT				SECOND ASSIGNMENT				THIRD ASSIGNMENT		
	1	2	3		1	2	3		1	2	3
Adams	3	4	$\boxed{0}$	Adams	3	4	$\boxed{0}$	Adams	3	4	$\boxed{0}$
Brown	0	0	5	Brown	0	0	5	Brown	0	$\boxed{0}$	5
Cooper	0	1	0	Cooper	$\boxed{0}$	1	0	Cooper	$\boxed{0}$	1	0

- Objective: minimize costs
- (there is only one assignment that Adams can go to where the opportunity costs are $0. That is to assign Adams Project 3.)

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Example-1

- The optimal allocation :
  - assign Adams to Project 3, Brown to Project 2, and Cooper to Project 1.
- The total labor cost of this assignment are computed from the original cost table

ASSIGNMENT	COST ($)
Adams to Project 3	6
Brown to Project 2	10
Cooper to Project 1	9
Total cost	25

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Example 3

- A contractor has three different types of excavation equipment that are needed to be assigned to three excavation sites. The production cost for each 1 m3 of excavated soil is as given in the following table.

		Job sites		
		1	2	3
Machines	1	8	3	7
	2	-	10	3
	3	6	5	4

- Formulate a mathematical model to assign the equipment to excavation sites to minimize total cost.

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Example 3: using Vogel's method

- The assignment problem is solved the same way as the transportation problem

		Assignment			Supply
		1	2	3	1
Assignee	1	8	3	7	1
	2	M	10	3	1
	3	6	5	4	
Demand	1	1	1	1	

- Solution $X_{12}=1$, $X_{23}=1$, $X_{31}=1$ and min. cost =12

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Maximization Assignment Problems

- maximizing the payoff, profit, or effectiveness of an assignment
- An equivalent minimization problem by converting all numbers in the table to opportunity costs; efficiencies to inefficiencies, etc.
- This is achieved through subtracting every number in the original payoff table from the largest single number in the number.

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Maximization Assignment Problems

- The transformed entries represent opportunity costs; it turns out that minimizing the opportunity costs produces the same assignment as the original maximization problem.
- Once the optimal assignment for this transformed problem has been computed, the total payoff or profit is found by adding the original payoffs of those cells that are in the original assignment.

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Example- 4

- British Navy patrol assignments to produce the greatest overall efficiencies
- Convert the maximizing efficiency table into a minimization opportunity cost table.

	SECTOR			
SHIP	A	B	C	D
1	20	60	50	55
2	60	30	80	75
3	80	100	90	80
4	65	80	75	70

Table 32: Efficiencies of British Ships in Patrol sectors

	SECTOR			
SHIP	A	B	C	D
1	80	40	50	45
2	40	70	20	25
3	20	0	10	20
4	35	20	25	30

Table 33: Opportunity Costs of British Ships

- Next, we follow steps 1 and 2 of the assignment algorithm

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Example- 4

- Perform row and column reduction

	SECTOR			
SHIP	A	B	C	D
1	40	0	10	5
2	20	50	0	5
3	20	0	10	20
4	15	0	5	10

Table 34: Row opportunity costs for the British Navy Problem

	SECTOR			
SHIP	A	B	C	D
1	25	0	10	0
2	5	50	0	0
3	5	0	10	15
4	0	0	5	5

Table 35: Total opportunity costs for the British Navy Problem

- The minimum number of straight lines needed to cover all zeros = 4.

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Example- 4

- Hence an optimal assignment can be made.
- The optimal assignment is ship 1 to sector D, ship 2 to sector C, ship 3 to sector B, and ship 4 to sector A.
- The overall efficiency is:

ASSIGNMENT	EFFICIENCY
Ship 1 to Sector D	55
Ship 2 to Sector C	80
Ship 3 to Sector B	100
Ship 4 to Sector A	65
Total Efficiency	300

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