Linear Programming

Chapter 3

Why LP?

- Most popular optimization technique
- LP software packages are readily available
- A lot of work on specialized algorithms for solving specific LP problems (EXCEL-SOLVER, XPRESS-MP, GAMS, LINDO, LINGO, AMPL, MINOS, TORA, etc.)
- Many problems can be converted to a LP formulation

History of LP

1928 – John von Neumann published related central theorem of game theory

1944 – Von Neumann and Morgenstern published Theory of Games and Economic Behavior

1936 – W.W. Leontief formulated a linear model without objective function.

1939 – Kantoravich (Russia) actually formulated and solved a LP problem

1941 – Hitchcock poses transportation problem (special LP)

WW II – Allied forces formulate and solve several LP problems related to military

A breakthrough occurred in 1947…

History of LP Contd…

- US Air Force investigate applying mathematical techniques to military budgeting and planning
- George Dantzig proposed LP model
- Air Force initiated project SCOOP (Scientific Computing of Optimum Programs) and SCOOP began in June 1947, Dantzig and associates developed:
  - An initial mathematical model of the <u>general</u> linear programming problem
  - A general method of solution called the <u>simplex method.</u>

Simplex Today

- A large variety of Simplex-based algorithms exist
- Other algorithms have been developed for solving LP problems:
  - Khachian algorithm (1979)
  - Kamarkar algorithm (AT&T Bell Labs, mid 80s)
  - Etc..
- Simplex (in its various forms) is and will most likely remain the most dominant LP algorithm in actual practical applications for at least the near future

LP Assumption

- a definite objective that can be mathematically represented in an equation format exist.
- Constraints are always limiting the use of the available resources.
- There different alternative or solutions for the problem at hand, and for each solution there is a specific value for the objective function.
- The preferred solution is the one that optimizes the objective and satisfies the constraints.
- All relationships between variables are linear.
- Linear programming assumes confident in all gathered data.

Linear Programming

- Mathematical Model
  - Decision variables
  - Linear objective function
    - maximization
    - minimization
  - linear constraints
    - equations =
    - Inequalities LE or GE
  - Non-negativity constraints

Guideline for Model Formulation

1. Understand the problem thoroughly.
2. Write a verbal statement of the objective function and each constraint.
3. Define the decision variables.
4. Write the objective function in terms of the decision variables.
5. Write the constraints in terms of the decision variables.

Formulation of LP Problems

- The key terms of linear programming model **are *resources, m,*** *and* ***activities, n,*** *where m* denotes the number of different kinds of resources that can be used and *n denotes the* number of activities being considered.
- Assume: *Z= value of overall measure of performance*
  - x_j *= level of activity j (j=1, 2, ….. , n)*
  - c_j *= increase in Z that result from each unit increase in activity j*
  - b_i *= amount of resource i that is available to activity j (i=1, 2, . . . , m)*
  - a_{ij} *= amount of resource i consumed by each unit of activity j.*

General mathematical model of LP

- The general form of allocating resources to activities

Resource	*Resources usage per unit of activity*				*Amount of resource available*
	1	*2*	……	*n*	
1	a_{11}	a_{12}	……	a_{1n}	b_1
2	a_{21}	a_{22}	……	a_{2n}	b_2
.	….	….	……	…	…
m	a_{m1}	a_{m2}	…….	a_{mn}	b_m
Contribution to Z	c_1	c_2	…….	c_n	

Typical resources are money, equipment, personnel, etc.
Sample activities include specific products, investing in particular projects, shipping goods, etc.

Formulation of LP Problem

- Formulation of LP Problems : clearly define the decision variables, objective, and constraints.
- An Example of LP model:

$$
\begin{array}{lrcrcrcl}
\text{maximize} & -x_1 & + & 3x_2 & - & 3x_3 & & \\
\text{subject to} & 3x_1 & - & x_2 & - & 2x_3 & \leq & 7 \\
 & -2x_1 & - & 4x_2 & + & 4x_3 & \leq & 3 \\
 & x_1 & & & - & 2x_3 & \leq & 4 \\
 & -2x_1 & + & 2x_2 & + & x_3 & \leq & 8 \\
 & 3x_1 & & & & & \leq & 5 \\
 & & & x_1, & x_2, & x_3 & \geq & 0.
\end{array}
$$

Standard form

- maximize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$
- constraints

s.t. $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$

Note: b_1, b_2, ..b_m are non negative RHS values

- Non-negative variables

e.g. x1, x2 ≥ 0

Other forms

- Can be rewritten in standard form

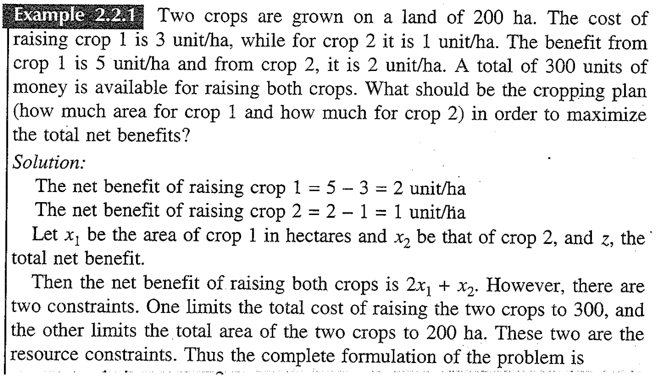
1. Minimization problems
   - Convert by changing the signs of the variables of the objective function from min to max problems.
     - *Min* $z = 0.4x_1 + 0.5x_2$ *is equivalent to*
     - *Max* $-z = -0.4x_1 - 0.5x_2$
2. Problems with constraints on alternative forms,
   - The direction of an inequality is reversed by multiplying both sides by (-1)
3. Problems involving negative RHS variables
   - Multiplying both sides by (-1), makes the right-hand side positive

Graphical Method

Graphical method

- For a model with only two variables, it is possible to solve the problem by drawing the feasible region and determining how the objective is optimized on that region
- gives you intuition and understanding of linear programming models and their solution.
- A **feasible solution** is a solution for which all the constraints are satisfied. An **infeasible solution** is a solution for which at least one constraint is violated.

Example-1 LP model formulation



Example 2.2.1 Two crops are grown on a land of 200 ha. The cost of raising crop 1 is 3 unit/ha, while for crop 2 it is 1 unit/ha. The benefit from crop 1 is 5 unit/ha and from crop 2, it is 2 unit/ha. A total of 300 units of money is available for raising both crops. What should be the cropping plan (how much area for crop 1 and how much for crop 2) in order to maximize the total net benefits?

Solution:

The net benefit of raising crop 1 = 5 – 3 = 2 unit/ha
The net benefit of raising crop 2 = 2 – 1 = 1 unit/ha

Let x_1 be the area of crop 1 in hectares and x_2 be that of crop 2, and z, the total net benefit.

Then the net benefit of raising both crops is $2x_1 + x_2$. However, there are two constraints. One limits the total cost of raising the two crops to 300, and the other limits the total area of the two crops to 200 ha. These two are the resource constraints. Thus the complete formulation of the problem is

Graphical method

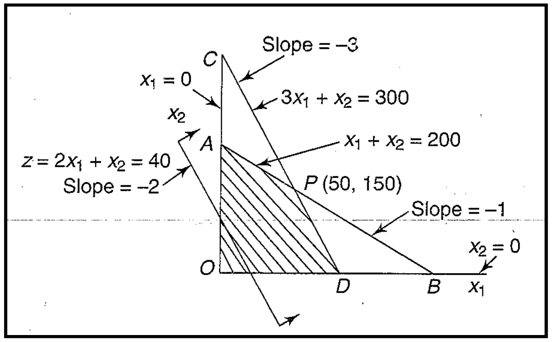
- Problem is to maximize revenue from two crops, given constraints on available land and capital
- LP model formulation:
- **OF max. $Z = 2x_1 + x_2$** (maximize the net benefit)

  s.t. $3x_1 + x_2 \leq 300$ (limit on total cost)

  $x_1 + x_2 \leq 200$ (limit on land)

  $x_1 >= 0, x_2 \geq 0$ (cannot plant a negative area)

Solution



In general, the optimal solution lies at one of the corner points of the feasible region.

Solution (some notes)

- Map the feasible region (region OAPD)
- A corner-point feasible (CPF) solution is a solution that lies at a corner of the feasible region.
- Any point within or on the boundary of the feasible region is a feasible solution
- Solutions:
  - P (0,200) Z = 200
  - P(50,150) Z = 250
  - P (100,0) Z = 200
  - P(0,0) Z = 0
- **An optimal solution** is a feasible solution that has the most favorable value of the objective function. (largest value for maximization and the smallest value for minimization problems).

Solution (some notes)

- Plot the objective function, Z, on the same graph.
- Determine the direction for moving Z within the feasible range
- Shift the objective function line in the direction of improvement until it last intersected the feasible region
- Consider a line for the OF for an arbitrary value of c Say c=40
- P(50,150) is the farthest point from the origin representing the optimal solution Z=250

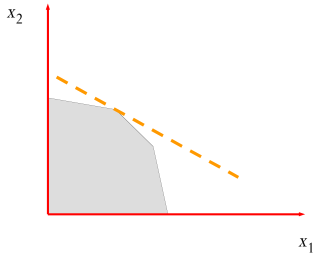
LP assumptions

- Proportionality
  - The contribution to the objective function from each decision variable is proportional to the value of the decision variable
- Additivity
  - The value of objective function is the sum of the contributions from each decision variables
- Divisibility
  - Each decision variable is allowed to assume fractional values.
- Certainty
  - Each parameter is known with certainty

LP Solutions

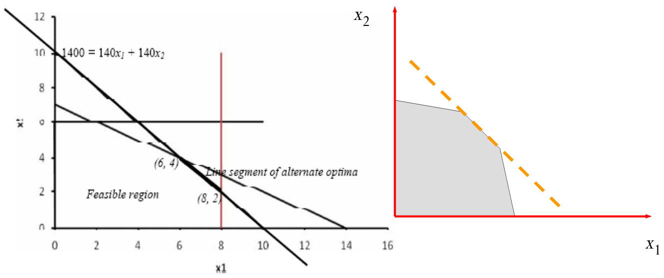
- Whenever a linear programming model is formulated and solved, the result will be one of four characteristic solution types:
  - 1) unique optimal solution,
  - 2) alternate optimal solutions,
  - 3) no-feasible solution, and
  - 4) unbounded solutions.

Unique optimal solution



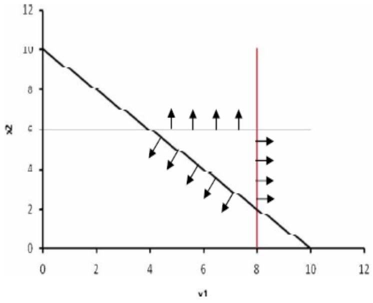
Alternate optimal solutions

- The intersection of the objective function line and the feasible region at optimality becomes a line segment



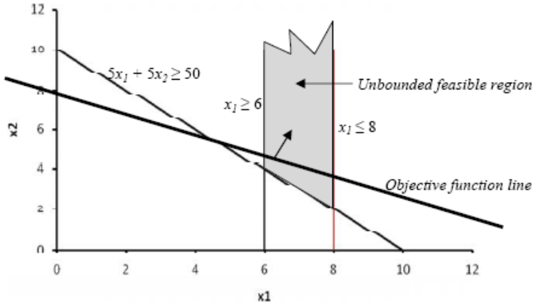
No feasible solution

- This may occur when constraints conflict with one another. (over constrained)
- Assume the following set of constraints
  - $5x_1 + 5x_2 \leq 50$
  - $x_1 \geq 8$
  - $x_2 \geq 6$
- No feasible region formed



Unbounded solutions

- A situation where the problem is under constrained.
- Assume the following set of constraints
  - $5x_1 + 5x_2 \geq 50$
  - $x_1 \leq 8$
  - $x_1 \geq 6$



Example 2

- An aggregate mix of sand and gravel must contain no less than 20% no more than 30% of gravel. The in situ soil contains 40% gravel and 60% sand. Pure sand may be purchased and shipped to site at 5 units of money /m^3. A total mix of at least 1000 m^3 is required. There is no charge for using in situ material.
- The objective is to ***minimize*** the cost

  Draw the feasible region

  Determine the optimum solution by the graphical method

Solution

- Total quantity of material needed = 1000 m^3
- Min. quantity of gravel in the mix = 0.20 x 1000 = 200 m^3
- Max. quantity of gravel in the mix = 0.30 x 1000 = 300 m^3
- Let the decision variables be as follows:

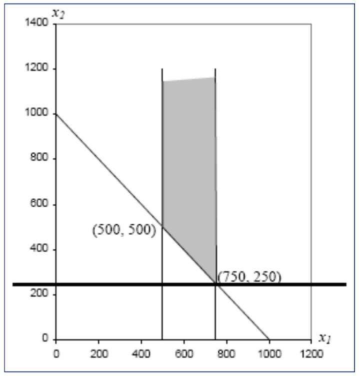
  x_1 : *Quantity of material from in situ*
  x_2 : *Quantity of material from outside*
- The objective is to minimize the cost, z,

  $Min\ z = 5*x_2$
- The constraints are:

  $x_1 + x_2 \geq 1000$
  $0.4x_1 \geq 200$
  $0.4x_1 \leq 300$
  $x_1, x_2 \geq 0$

Solution

- *Optimum solution:*
  $x_1 = 750$
  $x_2 = 250$
- Amount of gravel = 300 m3 from in situ
- Amount of sand = 700 m^3; 450 m^3 from in situ and 250 m^3 from outside.



Simplex Method

Understanding Simplex Method

- Useful in several ways
- Give insights into what commercial linear programming software packages actually do.
- Able to identify when a problem has alternate optimal solutions, unbounded solution, etc.

Gauss-Jordan Elimination for Solving Linear Equations

- It works one variable at a time, eliminating it in all rows but one, and then moves on to the next variable. Example
  - $x_1 + 2x_2 + x_3 = 4$ (1)
  - $2x_1 - x_2 + 3x_3 = 3$ (2)
  - $x_1 + x_2 - x_3 = 3$ (3)
- In the first step of the procedure, we use the first equation to eliminate x_1 *from the other* two. Specifically, in order to eliminate x_1 *from the second equation, we multiply the first* equation by 2 and subtract the result from the second equation. Similarly, to eliminate x_1 from the third equation, we subtract the first equation from the third.

Gauss-Jordan Elimination

- Such steps are called *elementary row operations.We keep the first equation and the modified second* and third equations.
- The resulting equations are:
  - $x_1 \quad + 2x_2 + x_3 = 4$ (1)
  - $-5x_2 + x_3 \quad = -5$ (2)
  - $-x_2 \quad - 2x_3 \quad = -1$ (3)
- Note that only one equation was used to eliminate x_1 *in all the others.This guarantees* that the new system of equations has exactly the same solution(s) as the original one.

Gauss-Jordan Elimination

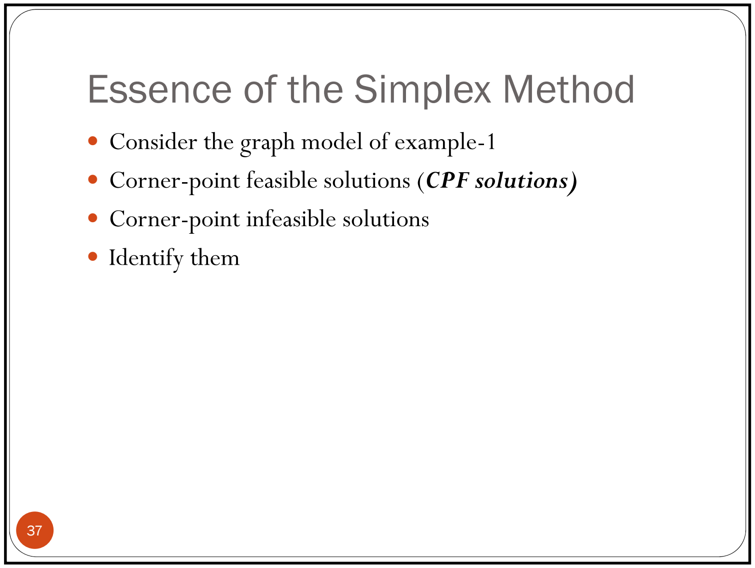
- Second step: divide the second equation by -5 to make the coefficient of x_2 *equal to 1*.
- *Then, use this equation to eliminate* x_2 *from equations 1* and 3.
- This yields the following new system of equations:
  - $x_1 \quad + 7/5x_3 \quad = 2$ (1)
  - $X_2 \quad - 1/5x_3 \quad = 1$ (2)
  - $-11/5x_3 \quad = 0$ (3)

Gauss-Jordan Elimination

- Only one equation was used to eliminate x_2 *in all the others and that* guarantees that the new system has the same solution(s) as the original one.
- In the last step, we use equation 3 to eliminate x_3 *in equations 1 and 2.*
    - $x_1 = 2$ (1)
    - $x_2 = 1$ (2)
    - $x_3 = 0$ (3)
- So, there is a unique solution.
- Sometimes, linear systems of equations do not always have a unique solution (no solution, multiple solution)

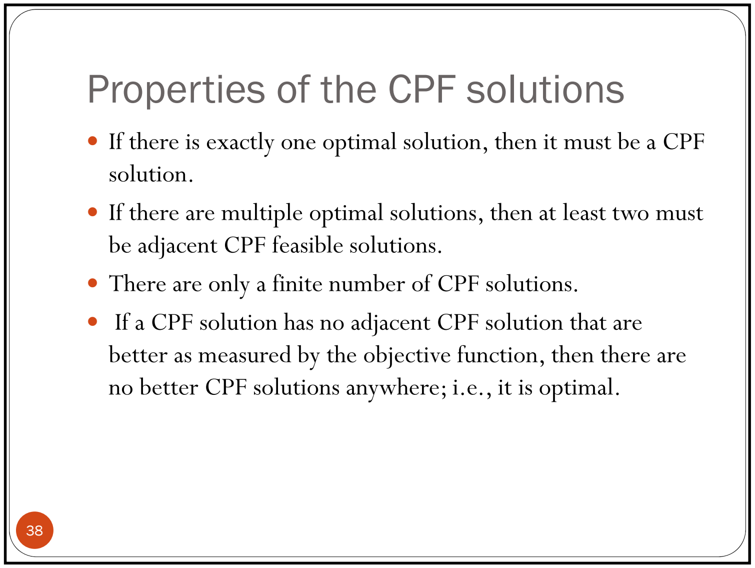
Gauss-Jordan Elimination

- *Example: (No solution)*
  - $x_1 + 2x_2 + x_3 \quad = 4$ (1)
  - $x_1 + x_2 + 2x_3 \quad = 1$ (2)
  - $2x_1 + 3x_2 + 3x_3 = 2$ (3)
- Example : (infinitely many solutions)
  - $x_1 + 2x_2 + x_3 \quad = 4$ (1)
  - $x_1 + x_2 + 2x_3 \quad = 1$ (2)
  - $2x_1 + 3x_2 + 3x_3 = 5$ (3)



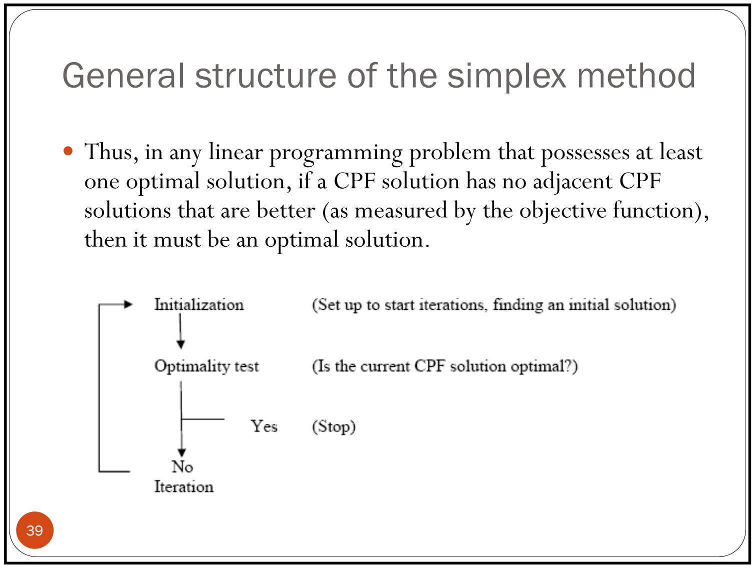
Essence of the Simplex Method

- Consider the graph model of example-1
- Corner-point feasible solutions (***CPF solutions)***
- Corner-point infeasible solutions
- Identify them



Properties of the CPF solutions

- If there is exactly one optimal solution, then it must be a CPF solution.
- If there are multiple optimal solutions, then at least two must be adjacent CPF feasible solutions.
- There are only a finite number of CPF solutions.
- If a CPF solution has no adjacent CPF solution that are better as measured by the objective function, then there are no better CPF solutions anywhere; i.e., it is optimal.



General structure of the simplex method

- Thus, in any linear programming problem that possesses at least one optimal solution, if a CPF solution has no adjacent CPF solutions that are better (as measured by the objective function), then it must be an optimal solution.

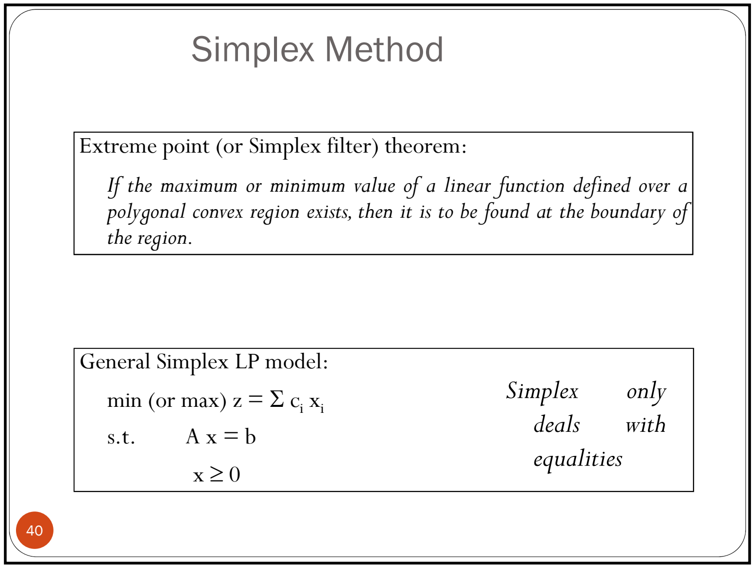
Initialization (Set up to start iterations, finding an initial solution)

Optimality test (Is the current CPF solution optimal?)

Yes (Stop)

No

Iteration



Simplex Method

Extreme point (or Simplex filter) theorem:

If the maximum or minimum value of a linear function defined over a polygonal convex region exists, then it is to be found at the boundary of the region.

General Simplex LP model:

min (or max) $z = \Sigma c_i x_i$

s.t. $A x = b$

$x \geq 0$

Simplex only deals with equalities

Slack/surplus variables

- Each of the inequality constraints can be converted to an equality constraint by adding a slack variable to the LHS
- The coefficient of this slack variable in the OF will be zero
- *slack,* if x ≤ b, then x + slack = b
- *surplus,* if x ≥ b, then x - surplus = b

Example of LP

$$\begin{aligned} &\textbf{Maximize} \quad 5x_1 + 7x_2 \\ &\textbf{s.t.} \quad x_1 \le 6 \\ &2x_1 + 3x_2 \le 19 \\ &x_1 + x_2 \le 8 \\ &x_1, x_2 \ge 0 \end{aligned}$$

Standard form with equality constraints:

$$\begin{aligned} &\textbf{Max} \quad 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3 \\ &\textbf{s.t.} \quad x_1 + s_1 = 6 \\ &2x_1 + 3x_2 + s_2 = 19 \\ &x_1 + x_2 + s_3 = 8 \\ &x_1, x_2, s_1, s_2, s_3 \ge 0 \end{aligned}$$

Standard form

- A total of *n*+*m* variables (*n* decision variables and m slack variables) and a constraint set of *m* equations
- These equations can be solved uniquely for any set of *m* variables
- Simplex method : the starting solution start by assuming all decision variables to be zero => Z=0
- Iterations are performed on this starting solution for better values of OF till optimality reached

Some definitions

- **Feasible and infeasible solutions**:
- **Basis and basic variables**: the number of basic variables is equal to the number of constraints. The variables in the basis only can be non negative values.
- **Non basic variables:** variables which are outside the basis
- **Basic feasible solution:** Assume there are a total of n + m variables (n decision and m slack variables). Then a basic solution is one that has m number of basic variables and n number of non-basic variables. All non basic variables are zeros.
- **Basic feasible solution:** a basic solution which is also feasible is a basic solution.

Basic feasible solution: Example

- Find all basic feasible solutions of the following system:

$$\text{Max } P = 5x_1 + 6x_2$$

$$\text{S.t.} \quad 4x_1 + 2x_2 \leq 200$$

$$x_1 + 3x_2 \leq 150$$

$$x_1 \geq 0 \quad x_2 \geq 0$$

- First add slack variables so that our new constraints are

$$4x_1 + 2x_2 + s_1 \quad = 200$$

$$x_1 + 3x_2 \quad + s_2 \quad = 150$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad s_1 \geq 0 \quad s_2 \geq 0$$

Basic feasible solution

- In this example we have 2 equations and 4 variables. We find basic solutions by setting 2 variables at a time equal to zero.

0	0	200	150	1. feasible
0	100	0	-150	2. Not feasible
0	50	100	0	3. feasible
50	0	0	100	4. feasible
150	0	-400	0	5. Not feasible
30	40	0	0	6. feasible

Basic feasible solution

- To solve the L.P. problem we need to evaluate the objective function at each of the basic feasible solutions.
- However, in practice this becomes impractical. Say for example we had an L.P. problem with 3 decision variables and 3 constraints (hence 3 slack variables). The number of basic feasible solutions: $\frac{6!}{3!3!} = 20$
- For 4 decision variables and 5 constraints, we have:

$$\frac{9!}{5!4!} = 126$$

- ….. and so on

Solution of Example-1

- **Maximize $Z = 2x_1 + x_2$**

  s.t. $3x_1 + x_2 \leq 300$

  $x_1 + x_2 \leq 200$

  $x_1 >= 0, x_2 \geq 0$

- **max $Z = 2x_1 + x_2 + 0x_3 + 0x_4$**

  s.t. $3x_1 + x_2 + x_3 = 300$

  $x_1 + x_2 + x_4 = 200$

  $x_1, x_2, x_3, x_4 \geq 0$

- **(x_1, x_2, x_3, x_4)**
- **(25, 25, 200, 150) is feasible but not a basic solution**
- **(100, 25, 0, 0) is basic but infeasible**
- **(0, 0, 300, 200) is basic and feasible solution**

All slack basic feasible solution

- Models involving ≤ (LE inequality) with non-negative RHS offer convenient all slack starting basic feasible solution
- Models involving ≥ and = constraints have different solution procedure. (not discussed here)
- **Read** the Book by Taha for problems involving ≥ and = constraints

Solution using Simplex tableau

- In principle one can start from any basic feasible solution
- Let's identify x_3 and x_4 as basic and x_1 and x_2 as non-basic variables (assumes zero value)
- We shall now start with the initial basic feasible solution (0, 0, 300, 200) with z=0

Table 2.1 *Starting Solution*

		Coefficient of					
	Basis	x_1	x_2	x_3	x_4	*RHS*	*Ratio*
Row 1	x_3	3	1	1	0	300	300/3 = 100
Row 2	x_4	1	1	0	1	200	200/1 = 200
Row z	z	−2	−1	0	0	0	

←Departing variable

↑ Entering variable

Note that OF as basic variable: $\mathbf{Z - 2x_1 - x_2 - 0x_3 - 0x_4} = 0$

Entering and Departing variable

Given any basis we move to an adjacent extreme point (another basic feasible solution) of the solution space by **exchanging one of the columns that is in the basis for a column that is not in the basis**

Two things to determine:

1) which (non-basic) column of should be brought into the basis so that the solution improves?

2) which column can be removed from the basis such that the solution stays feasible?

Entering and Departing variable

- Entering variable: the variable entering the basis is the one with the most negative coefficient in the z-row X_1. It will contribute to the increase of OF most. The column x_1 is now the *pivotal column*.
- The one basic variable to leave is the one which gives the minimum ratio test by applying those pivot column coef. That are strictly positive..

Solution Contd..

We determine that x_1 replaces x_3 in the new solution which has (x_1, x_4) as the basis. However, the coefficients in the Simplex table should be worked out using Gauss-Jordan transformation:

The new pivot row (row 1) is obtained:

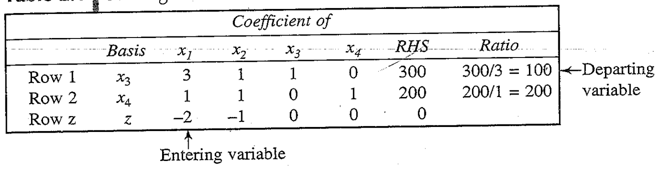
New pivot row = old pivot row/pivot coefficient

The rows other than the pivot row are transformed in the iteration:

New row = old row – (pivot column coeff)*(New pivot row)

Solution Contd..

Table 2.1 *Starting Solution*

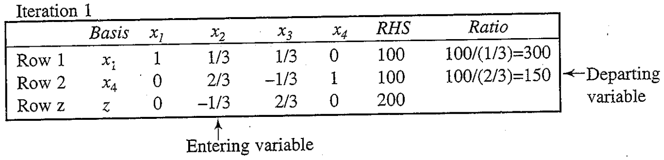


		Coefficient of					
	Basis	x_1	x_2	x_3	x_4	*RHS*	*Ratio*
Row 1	x_3	3	1	1	0	300	300/3 = 100
Row 2	x_4	1	1	0	1	200	200/1 = 200
Row z	z	–2	–1	0	0	0	

←Departing variable

↑ Entering variable

Iteration 1



	Basis	x_1	x_2	x_3	x_4	*RHS*	*Ratio*
Row 1	x_1	1	1/3	1/3	0	100	100/(1/3)=300
Row 2	x_4	0	2/3	–1/3	1	100	100/(2/3)=150
Row z	z	0	–1/3	2/3	0	200	

←Departing variable

↑ Entering variable

Solution Contd..

Note:

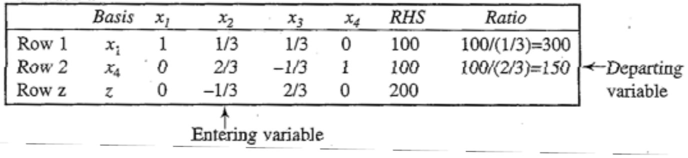
In Iteration 1 the OF value increased from 0 to 200

This solution would have been optimal if all the coeff. of the Z row were non-negative

Another iteration is needed. X_2 is entering and X_4 is the departing variables

Solution Contd..

Iteration 1

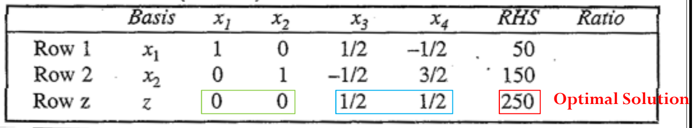


	Basis	x_1	x_2	x_3	x_4	*RHS*	*Ratio*
Row 1	x_1	1	1/3	1/3	0	100	100/(1/3)=300
Row 2	x_4	0	2/3	–1/3	1	100	100/(2/3)=150
Row z	z	0	–1/3	2/3	0	200	

←Departing variable

↑ Entering variable

Iteration 2 (*solution*)



	Basis	x_1	x_2	x_3	x_4	*RHS*	*Ratio*
Row 1	x_1	1	0	1/2	–1/2	50	
Row 2	x_2	0	1	–1/2	3/2	150	
Row z	z	0	0	1/2	1/2	250	Optimal Solution

Models involving "=" and '≥ ' constraints

- Simplex method for LP problem with 'greater-than-equal-to' (≥) and 'equality' (=) constraints needs a modified approach.
- Big-M method
- The LPP is transformed to its standard form by incorporating a large coefficient M

Big-M method

Step 1 One 'artificial variable' is added to each of the (≥) and (=) constraints to ensure an initial basic feasible solution

Step 2 Artificial variables are 'penalized' in the objective function by introducing a large negative (positive) coefficient for maximization (minimization) problem.

Step 3 Cost coefficients, which are supposed to be placed in the Z-row in the initial simplex tableau, are transformed by 'pivotal operation' considering the column of artificial variable as 'pivotal column' and the row of the artificial variable as 'pivotal row'.

If there are more than one artificial variables, the last step is repeated for all the artificial variables one by one **(repeat step 1 to 3)**

Example – 2

Maximize

$$Z = 3x_1 + 5x_2$$

s.t.

$$x_1 \le 4$$

$$2x_2 \le 12$$

$$3x_1 + 2x_2 = 18$$

Constraints, note one of them is equality constraint

$$x_1 \ge 0 \quad \}$$

$$x_2 \ge 0$$

Non-negativity of decision variables

Example – 2 (Contd.)

The problem is converted to standard LP form

Maximize $Z = 3x_1 + 5x_2$

s.t.

$$x_1 \le 4 \longrightarrow x_1 + x_3 = 4$$

$$2x_2 \le 12 \longrightarrow 2x_2 + x_4 = 12$$

$$3x_1 + 2x_2 = 18 \longrightarrow 3x_1 + 2x_2 = 18$$

$$x_1 \ge 0 \qquad x_1 \ge 0;\ x_2 \ge 0$$

$$x_2 \ge 0 \qquad x_3 \ge 0;\ x_4 \ge 0$$

n= no. of variables = 4; m = no. of constraints =3

Example – 2 (Contd.)

No initial basic feasible solution is available for this problem.

Add artificial variable to constraint 3

$$Z - 3x_1 - 5x_2 + M \times A_1 = 0$$

$$3x_1 + 2x_2 + A_1 = 18$$

Transformation of coefficients in row-Z

Example – 2 (Contd.)

$$Z - 3x_1 - 5x_2 + M \times A_1 = 0$$

$$3x_1 + 2x_2 + A_1 = 18$$

	x_1	x_2	x_3	x_4	A	b_i
E_1	-3	-5	0	0	M	0
E_2	3	2	0	0	1	18

Pivotal operation $E_1 - M \times E_2$

-3M-3	-2M-5	0	0	0	-18M

Example – 1 (Contd.)

Iteration-1

Entering variable

Departing variable

Basis	Row	x_1	x_2	x_3	x_4	A_1	b_i	b_i/a_{ij}
Z	0	-3M-3	-2M-5	0	0	0	-18M	–
x_3	1	①	0	1	0	0	4	4
x_4	2	0	2	0	1	0	12	–
A_1	3	3	2	0	0	1	18	6

Pivot point

Example – 1 (Contd.)

Iteration-2

Entering variable

Departing variable

Basis	Row	x_1	x_2	x_3	x_4	A_1	b_i	b_i/a_{ij}
Z	0	0	-2M-5	3M+3	0	0	-6M+12	–
x_1	1	1	0	1	0	0	4	–
x_4	2	0	2	0	1	0	12	6
A_1	3	0	2	-3	0	1	6	3

Example – 1 (Contd.)

Iteration-3

Entering variable

Departing variable

Basis	Row	x_1	x_2	x_3	x_4	A_1	b_i	b_i/a_{ij}
Z	0	0	0	-9/2	0	M+5/2	27	–
x_1	1	1	0	1	0	0	4	4
$\rightarrow x_4$	2	0	0	3	1	-1	6	2
x_2	3	0	1	-3/2	0	1/2	3	–

Example – 1 (Contd.)

Iteration-4

Basis	Row	Z	x_1	x_2	x_3	x_4	A_1	b_i
Z	0	1	0	0	0	3/2	M+1	36
x_1	1	0	1	0	0	-1/3	1/3	2
x_3	2	0	0	0	1	1/3	-1/3	2
x_2	3	0	0	1	0	1/2	0	6

Example – 1 (Contd.)

Since all coefficients in the Z-row are non-negative this is the optimal solution.

$$Z = 36$$
$$x_1 = 2$$
$$x_2 = 6$$
$$x_3 = 2$$
$$x_4 = 0$$
$$A_1 = 0$$

Note that this is the same solution with the constraint $3x_1 + 2x_2 \leq 18$

$\downarrow$

Binding (tight) constraint

Multiple artificial variables

- In case of multiple artificial variables, carryout the transformation one by one.
- Use the transformed Z-row in the initial simplex table.

Example-2

Consider the following problem

$$\begin{aligned} &\text{Maximize} && Z = 3x_1 + 5x_2 \\ &\text{subject to} && x_1 + x_2 \ge 2 \\ & && x_2 \le 6 \\ & && 3x_1 + 2x_2 = 18 \\ & && x_1, x_2 \ge 0 \end{aligned}$$

Example-2

After incorporating the artificial variables

$$\begin{aligned} &\text{Maximize} && Z = 3x_1 + 5x_2 - Ma_1 - Ma_2 \\ &\text{subject to} && x_1 + x_2 - x_3 + a_1 = 2 \\ & && x_2 + x_4 = 6 \\ & && 3x_1 + 2x_2 + a_2 = 18 \\ & && x_1, x_2 \ge 0 \end{aligned}$$

where x_3 is surplus variable, x_4 is slack variable and a_1 and a_2 are the artificial variables

Example-2

Considering the objective function and the first constraint

$$\begin{aligned} Z - 3x_1 - 5x_2 + Ma_1 + Ma_2 &= 0 && (E_1) \\ x_1 + x_2 - x_3 + a_1 &= 2 && (E_2) \end{aligned}$$

Pivotal Row

Pivotal Column

By the pivotal operation $E_1 - M \times E_2$ the cost coefficients are modified as

$$Z - (3+M)x_1 - (5+M)x_2 + Mx_3 + 0a_1 + Ma_2 = -2M$$

Example-2

Considering the modified objective function and the third constraint

$$\begin{aligned} Z - (3+M)x_1 - (5+M)\,x_2 + Mx_3 + 0a_1 + Ma_2 &= -2M && (E_3) \\ 3\,x_1 + 2\,x_2 + a_2 &= 18 && (E_4) \end{aligned}$$

Pivotal Row

Pivotal Column

By the pivotal operation $E_3 - M \times E_4$ the cost coefficients are modified as

$$Z - (3+4M)x_1 - (5+3M)x_2 + Mx_3 + 0a_1 + 0a_2 = -20M$$

Example-2 *Simplex Tableau*

Corresponding simplex tableau

Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	a_1	a_2		
	Z	1	$-3-4M$	$-5-3M$	M	0	0	0	$-20M$	--
1	a_1	0	1	1	-1	0	1	0	2	2
	x_4	0	0	1	0	1	0	0	6	--
	a_2	0	3	2	0	0	0	1	18	6

Pivotal row, pivotal column and pivotal elements are shown as earlier

Example-2

Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	a_1	a_2		
	Z	1	0	0	0	3	M	$1+M$	36	--
4	x_1	0	1	0	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	2	--
	x_2	0	0	1	0	1	0	0	6	--
	x_3	0	0	0	1	$\frac{1}{3}$	-1	$\frac{1}{3}$	6	--

Check using software :
After four iterations Optimality has reached.
Optimal solution is Z = 36 with $x_1 = 2$ and $x_2 = 6$

Special cases

Cases for a tie : Entering variable

- *Entering variable*: tie can be broken by arbitrarily (optimal solution will be reached eventually regardless of the variable chosen)
- *max* $x_1 + x_2$
  - S.t. $2x_1 + x_2 \leq 4$
  - $x_1 + 2x_2 \leq 3$
  - $x_1 \geq 0; x_2 \geq 0$

Table 3.6: Tie of entering basic variables

Basic variables	Coefficient of					Right-hand side (solution)
	z	x_1	x_2	s_1	s_2	
z	1	**-1**	**-1**	0	0	0
s_1	0	**2**	**1**	1	0	4; 4/2=2
s_2	0	**1**	**2**	0	1	3; 3/1=3

Cases for a tie: Departing variable

- *Departing variable*: a tie for the departing variable.
- One variable can be arbitrarily selected as the departing variable.
- This results in a *degenerate solution*. Degeneracy reveals that there is at least one redundant constrain.
- In some cases, degeneracy may lead to "cycling", i.e. a sequence of pivots that goes through the same tableaus and repeats itself indefinitely.

Example

- $max \quad 2x_1 + x_2$
  - S.t. $3x_1 + x_2 \leq 6$
  - $x_1 - x_2 \leq 2$
  - $x_2 \leq 3$
  - $x_1 \geq 0; x_2 \geq 0$

Basic variables	Coefficient of						Right-hand side (solution)
	z	x_1	x_2	s_1	s_2	s_3	
z	1	**-2**	-1	0	0	0	0
s_1	0	**3**	1	1	0	0	6; 6/3 = **2**
s_2	0	**1**	-1	0	1	0	2; 2/1 = **2**
s_3	0	**0**	1	0	0	1	3

Example : Multiple solution

Maximize $Z = 2x_1 + x_2$

s.t $3x_1 + x_2 \leq 300$

$4x_1 + 2x_2 \leq 500$

$x_1 >= 0, x_2 \geq 0$

Initialize, do first iteration and iteration 2 yields optimal solution

X_3 has 0 coeff in z-row= multiple solution

(x_1, x_2) = (50,150) and (0,250) and any point on a line joining the two is a solution

Iteration 2 Optimal solution

Basis	x_1	x_2	x_3	x_4	*RHS*	*Ratio*
x_1	1	0	1	–1/2	50	
x_2	0	1	–2	3/2	150	
z	0	0	0	1/2	250	

Iteration 3 Alternate solution

Basis	x_1	x_2	x_3	x_4	*RHS*	*Ratio*
x_3	1	0	1	–1/2	50	
x_2	2	1	0	1/2	250	
z	0	0	0	1/2	250	

Multiple solutions

- Existence of multiple solution is indicated by *the presence of a zero in the z-row* under a basic variable in the final simplex table. New solution in the next iteration by choosing this non-basic variable as the entering variable.

- $max \quad x_1 + 1/2x_2$
  - S.t.
  - $2x_1 + x_2 \leq 4$
  - $x_1 + 2x_2 \leq 3$
  - $x_1 \geq 0; x_2 \geq 0$

Basic variables	Coefficient of					Right-hand side (solution)
	z	x_1	x_2	s_1	s_2	
z	1	0	**0**	½	0	2
x_1	0	1	0	2/3	-1/3	5/3
x_2	0	0	1	-1/3	2/3	2/3

Sensitivity Analysis

Sensitivity analysis

- A change in the data of original problem may affect optimality or feasibility of the current solution.
- Parameters Sensitivity
  - LP assumes certainty of the model parameters, but are are only estimates.
- Sensitivity analysis is to identify the sensitive parameters, to try to estimate these parameters more closely, and then to select a solution that remains a good one over the range of likely values of the sensitive parameters.

Sensitivity analysis

1. RHS sensitivity analysis
   - measures how sensitive is the optimal solution to the change in the resources values i.e., by changing the resource limits, would the optimal solution be changed and to what limit.
2. OF sensitivity analysis.
   - The coefficients of the OF could be based on uncertain data or subjective judgment of the decision maker.
   - changes in the values of the coefficients that multiply the decision variables in the objective function.

Sensitivity analysis in LP

- *Sensitivity analysis* is an exercise of obtaining a new solution corresponding to a change in the data of the original problem, given the original problem and the final simplex table, without solving afresh the new problem with changed data.
- Example: EXCEL-SOLVER sensitivity outputs

Duality

Dual Problem

- Every primal LP problem will have its dual.
- Sometimes it is easier to formulate the dual problem, rather than the primal problem, and thereby determine the solution of the primal.
- The solution of dual is extremely handy if the primal problem has a small number of decision variables and a large number of constraints

Example

Primal

Maximize $z = 2x_1 + x_2$

subject to

$$3x_1 + x_2 < 300$$
$$x_1 + x_2 < 200$$
$$2x_1 + 5x_2 < 900$$
$$5x_1 + 2x_2 < 600$$
$$x_1, x_2 > 0$$

Dual

Minimize $z' = 300y_1 + 200y_2 + 900y_3 + 600y_4$

subject to

$$3y_1 + y_2 + 2y_3 + 5y_4 > 2$$
$$y_1 + y_2 + 5y_3 + 2y_4 > 1$$
$$y_1, y_2, y_3, y_4 > 0$$

Dual Example -2

- **Maximize Z = $2x_1 + x_2$**

  s.t. $3x_1 + x_2 \leq 300$ (constraint 1)

  $x_1 + x_2 \leq 200$ (constraint 2)

  $x_1, x_2 \geq 0$
- For every primal constraint there is a dual variable and for every primal variable there is a dual constraint
  - Two dual variables y_1 and y_2 corresponding to constraint 1 and 2)
  - There will be two constraints in the dual, one each corresponding to x_1 and x_2.
- Optimization Problem is reversed: Minimization

Example-2 Contd..

- The OF z' for the dual is:
  - Minimize z' = $300y_1 + 200y_2$
- S.t. $3y_1 + y_2 \geq 2$

  $y_1 + y_2 \geq 1$

  $y_1, y_2 \geq 0$
- some differences between the primal simplex and the dual simplex methods

Dual Simplex method

- The primal simplex method starts from a non optimal feasible solution and moves towards the optimal solution, maintaining feasibility every time
- Dual simplex method starts with an infeasible basic solution and strives to achieve feasibility, while satisfying optimality criterion every time.
- The dual simplex method has rules for the
  - entering variable,
  - departing variable
  - and testing the feasibility of a solution.

Example

- Minimize z' = $300y_1 + 200y_2$
- S.t. $3y_1 + y_2 \geq 2$

  $y_1 + y_2 \geq 1$

  $y_1, y_2 \geq 0$
- ***Solution of the Dual:***
- ***Writing the dual in the standard form with equality*** constraints,

Maximize $(-z') = -300y_1 - 200y_2$

or $(-z') + 300y_1 + 200y_2 = 0$

$3y_1 + y_2 - y_3 = 2$

or $y_1 + y_2 - y_4 = 1$

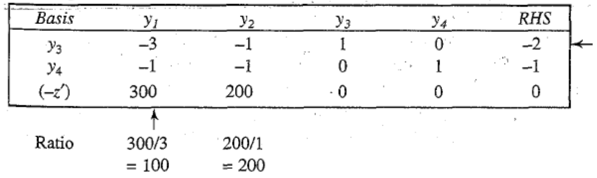
$y_1, y_2, y_3, y_4 \geq 0$

Example

- Writing the problem in a way to facilitate a starting basic infeasible solution for dual simplex method:

$$(-z') + 300y_1 + 200y_1 = 0$$
$$-3y_1 - y_2 + y_3 = -2$$
$$-y_1 - y_2 + y_4 = -1$$
$$y_1, y_2, y_3, y_4 \geq 0$$

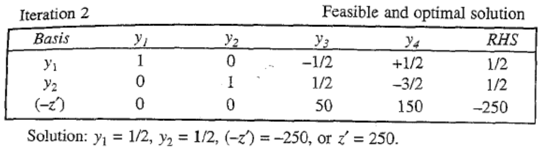
Starting solution



Basis	y_1	y_2	y_3	y_4	*RHS*
y_3	−3	−1	1	0	−2 ←
y_4	−1	−1	0	1	−1
$(-z')$	300	200	0	0	0
	↑				
Ratio	300/3 = 100	200/1 = 200			

Example

- The departing basic variable is identified first as one with the most negative value (Row)
- The entering variable: For each nonbasic variable, determine the **absolute value of the minimum ratio**. (column)
- Iteration 1…..



Iteration 2 Feasible and optimal solution

Basis	y_1	y_2	y_3	y_4	*RHS*
y_1	1	0	–1/2	+1/2	1/2
y_2	0	1	1/2	–3/2	1/2
$(-z')$	0	0	50	150	–250

Solution: $y_1 = 1/2$, $y_2 = 1/2$, $(-z') = -250$, or $z' = 250$.

Example

- Note that the dual variables from the optimal solution are $y_1 = 1/2$ and $y_2 = 1/2$.
- The optimal value of x_1 in the primal can be identified by the coefficient of the slack variable y_3 in the corresponding dual constraint, which is equal to 50.
- Thus $x_1 = 50$ and similarly $x_2 = 150$.

Dual Example-2

- Consider the following primal problem

Maximize

$$Z = 12x_1 + 4x_2$$

subject to:

$$\begin{aligned} 4x_1 + 7x_2 &\leq 56 \\ 2x_1 + 5x_2 &\geq 20 \\ 5x_1 + 4x_2 &= 40 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

The first inequality requires no modification. But the second and the third constraint have to be modified

Dual Example-2 contd..

- The second inequality can be changed to the less-than-or-equal-to type by multiplying both sides by -1 that is,

$$-2x_1 - 5x_2 \leq -20$$

- The equality constraint can be replaced by the following two inequality constraints:

$$\begin{aligned} 5x_1 + 4x_2 &\leq 40 \\ 5x_1 + 4x_2 &\geq 40 \end{aligned}$$

Dual Example-2 contd..

- The primal problem can now take the following standard form:

Maximize

$$Z = 12x_1 + 4x_2$$

subject to:

$$\begin{aligned} 4x_1 + 7x_2 &\leq 56 \\ -2x_1 - 5x_2 &\leq -20 \\ 5x_1 + 4x_2 &\leq 40 \\ -5x_1 - 4x_2 &\leq -40 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Dual Example-2 contd..

- The dual of this problem can now be obtained as follows:

Minimize

$$P = 56y_1 - 20y_2 + 40y_3 - 40y_4$$

subject to:

$$\begin{aligned} 4y_1 - 2y_2 + 5y_3 - 5y_4 &\geq 12 \\ 7y_1 - 5y_2 + 4y_3 - 4y_4 &\geq 4 \\ \text{all } y_i &\geq 0 \end{aligned}$$

Primal –Dual relationship

Primal Problem	*Dual Problem*
Maximize	Minimize
$Z = c_1x_1 + c_2x_2$	$P = b_1y_1 + b_2y_2 + b_3y_3$
subject to:	subject to:
$k_{11}x_1 + k_{12}x_2 \leq b_1$	$k_{11}y_1 + k_{21}y_2 + k_{31}y_3 \geq c_1$
$k_{21}x_1 + k_{22}x_2 \leq b_2$	$k_{12}y_2 + k_{22}y_2 + k_{32}y_3 \geq c_2$
$k_{31}x_1 + k_{32}x_2 \leq b_3$	all $y_i \geq 0$
all $x_i \geq 0$	

LP in Matrix form

Matrix form

- Matrix form expression facilitate understanding of the simplex operations
- maximize $c^T x$

  subject to $Ax \leq b$, $x \geq 0$

$$\begin{array}{ll} \text{Maximize} & z = \mathbf{CX} \\ \text{subject to} & (\mathbf{A}, \mathbf{I})\mathbf{X} = \mathbf{b} \\ & \mathbf{X} \geq 0 \end{array}$$

where **I** is $(m \times m)$ identify matrix, **X** is a column vector and **C**, a row vector given by

$$\mathbf{X} = (x_1, x_2, \ldots, x_{n+m})^T$$
$$\mathbf{C} = (c_1, c_2, \ldots, c_{n+m}),$$

and **A** is $(m \times n)$ matrix, **b** is a column vector given by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Example 2.2.3 Consider the LP problem

$$\begin{array}{ll} \text{Maximize} & z = 4x_1 + 5x_2 \\ \text{subject to} & 2x_1 + 3x_2 \leq 12 \\ & 4x_1 + 2x_2 \leq 16 \\ & x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{array}$$

The problem is written in the standard form first.

$$\begin{array}{ll} \text{Maximize} & z = 4x_1 + 5x_2 + 0.x_3 + 0.x_4 + 0.x_5 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 = 12 \\ & 4x_1 + 2x_2 + x_4 = 16 \\ & x_1 + x_2 + x_5 = 8 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

The problem is expressed in the matrix form as

$$\text{Maximize} \quad z = \begin{pmatrix} 4 & 5 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\text{subject to} \quad \begin{bmatrix} 2 & 3 & 1 & 0 & 0 \\ 4 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 12 \\ 16 \\ 18 \end{bmatrix}$$

$$x_j \geq 0 \quad j = 1, 2, \ldots, 5.$$

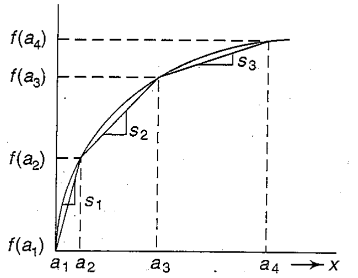
Example in matrix form

Non-linearity

Piecewise Linearization

- LP can be used with some modification to solve non-linear problems, if the nonlinear expression can be expressed as piecewise linear segments.
- Requires additional variables and constraints
- Consider a maximization problem of a concave nonlinear function f(x).
- F(x) can be expressed as a piecewise linear function consisting of segments, with slope of the function in each reducing as x increases.

Piecewise Linearization cont.



Method 2

Let the slopes of the linear segments be $s_1, s_2, \ldots,$ where $s_1 > s_2 > s_3 \ldots$

Then the problem is to

Maximize $f(x) = s_1 x_1 + s_2 x_2 + s_3 x_3 + \ldots = \Sigma s_n x_n$

subject to $a_1 + x_1 + x_2 + \ldots = x$

$x_j \leq a_{j+1} - a_j$ for all segments j.

LP in Construction Management

- Linear programming can be used in construction management to solve many problems such as:
  - **Optimizing use of resources**
  - **Determining most economic product mix**
  - **Transportation and routing problems**
  - **Location of new production plants, offices and warehouses**
  - **Personnel assignment**
  - **Determining Optimum size of bid**

LP applications in other areas

- Developing a production schedule that will satisfy future demands for a firm's product and at the same time minimize total production and inventory costs.
- Selecting the product mix in a factory to make best use of machine- and labor-hours available while maximizing the firm's profit
- Picking blends of raw materials in feed mills to produce finished feed combinations at minimum costs
- Determining the distribution system that will minimize total shipping cost

LP practical applications

- Scheduling school buses to minimize total distance traveled
- Allocating police patrol units to high crime areas in order to minimize response time to (911) calls
- Scheduling tellers at banks so that needs are met during each hour of the day while minimizing the total cost of labor.
- Allocating space for a tenant mix in a new shopping mall so as to maximize revenues to the leasing company
- Etc..

Integer and Mixed-Integer Problems

- An LP problem in which all the decision variables must have integer values is called an **integer programming** problem. (IP)
- A problem in which only some of the decision variables must have integer values is called a **mixed-integer programming** problem. (MIP)
- Sometimes, some (or all) of the decision variables must have the value of either 0 or 1. Such problems are then called **zero-one mixed-integer programming** problems.
- Simplex method cannot be used to such problems. Advanced methods are available for this purpose

Software

- **Numerous Computer programs to solve LP problems are widely available.**
- **Most large LP problems can be solved with just a few minutes of computer time**
- **Most computer-based LP packages use the simplex method**

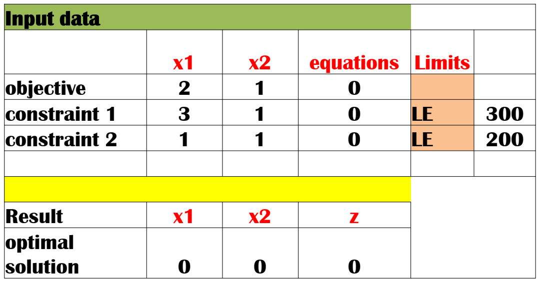
EXCEL-Solver, LINDO/LINGO, GAMS, XPRESS-MP are very popular . Others exist too : TORA , AMPL, etc..

Solving using Excel Solver

- Solver uses standard spreadsheets together with an interface to define variables, objective, and constraints to define a linear program.
- Solver, while not a state of the art code is a reasonably robust, easy-to-use tool for linear programming.
- Excel Solver add-in optimizes linear and integer problems using the simplex and branch and bound methods.
- Solver does sensitivity analysis automatically

Solver

- Start with entering the data into spreadsheet and Create the model in a separate part of the worksheet.
- Solve the previous example-1 using SOLVER

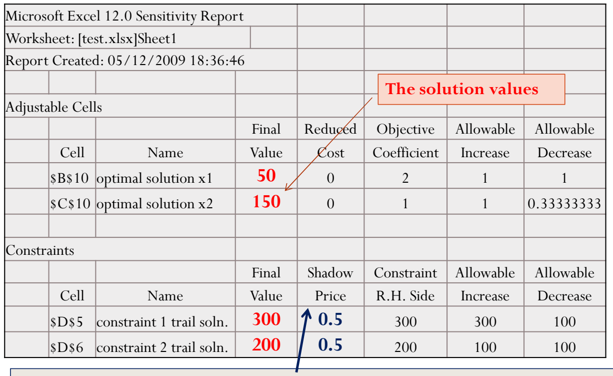


Input data					
	x1	**x2**	**equations**	**Limits**	
objective	**2**	**1**	**0**		
constraint 1	**3**	**1**	**0**	**LE**	**300**
constraint 2	**1**	**1**	**0**	**LE**	**200**
Result	**x1**	**x2**	**z**		
optimal solution	**0**	**0**	**0**		

Sensitivity Analysis

- How sensitive the results are to parameter changes
  - Change in the value of coefficients
  - Change in a right-hand-side value of a constraint
- Trial-and-error approach
- Analytic post-optimality method
- EXCEL-SOLVER Output for Example-1

Sensitivity Report



Microsoft Excel 12.0 Sensitivity Report							
Worksheet: [test.xlsx]Sheet1							
Report Created: 05/12/2009 18:36:46							
Adjustable Cells							
			Final	Reduced	Objective	Allowable	Allowable
	Cell	Name	Value	Cost	Coefficient	Increase	Decrease
	B10	optimal solution x1	50	0	2	1	1
	C10	optimal solution x2	150	0	1	1	0.33333333
Constraints							
			Final	Shadow	Constraint	Allowable	Allowable
	Cell	Name	Value	Price	R.H. Side	Increase	Decrease
	D5	constraint 1 trail soln.	300	0.5	300	300	100
	D6	constraint 2 trail soln.	200	0.5	200	100	100

The solution values

If we use one more Unit of money, the net benefit will increase by 0.5 unit of money. This is true up to 300 more units. Net benefit will fall by 0.5 for each decrease , down as low as 100 units

Sensitivity report

- The solution/course of action changes with a change in values of the objective function coefficients within the range of allowable increase and decrease. The result (course of action) will not change (remains constant) if the coefficients values are outside the range.
- The net benefit changes within the range of allowable increase and decrease with a change of the RHS value of a constraint. The net benefit remains constant for values outside the range. Availing more resource doesn't improve the solution.

Changes in Resources limits

- The RHS values of constraint equations may change as resource availability changes
- The shadow price of a constraint is the change in the value of the objective function resulting from a one-unit change in the right-hand-side value of the constraint
- Shadow prices are often explained as answering the question "How much would you pay for one additional unit of a resource?"

LINDO/LINGO

See presentation

Integer/binary programming

- Assumption of divisibility
- All the software packages in our Courseware (Excel, LINGO/LINDO, and TORA) include an algorithm for solving (pure or mixed) algorithm for solving IP models where variables need to be integer but not binary.
- When using the Excel Solver, the procedure is basically the same as for linear pro
- In a LINDO model, the binary or integer constraints are inserted after the END statement.
- In Excel solver “int” and “bin” options

AMPL

- A Mathematical Programming Language
- algebraic modeling language for linear and nonlinear optimization problems, in discrete or continuous variables.
- Developed at Bell Laboratories http://www.ampl.com
- General and natural syntax for arithmetic, logical, and conditional expressions;

GAMS

- GAMS (General Algebraic Modeling System)
- www.gams.com

TORA

- The Temporary-Ordered Routing Algorithm (TORA) – An Operations Research Software
- TORA is menu-driven and Windows-based (low screen resolution)
- Operation Research Book 8th Edition By Hamdy A. Taha (with CD)
- Old version???

TORA

- TORA software deals with the following algorithms:
  - Solution of simultaneous linear equations
  - Linear programming
  - Transportation model
  - Integer programming
  - Network models
  - Project analysis by CPM/PERT
  - Poisson queuing models
  - Zero-sum games