Integer Linear Programming (ILP)

Introduction

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Types of ILP Models

ILP:

A linear program in which some or all variables are restricted to integer values.

Types:

1. All-integer LP or a pure ILP
2. Mixed-Integer LP
3. 0-1 integer LP

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An All-Integer IP or Pure ILP

Max. $Z = 2x_1 + 3x_2$

Subject to

$$3x_1 + 2x_2 \leq 12$$
$$\tfrac{1}{4}x_1 + 1x_2 \leq 4$$
$$1x_1 + 2x_2 \leq 6$$
$$x_1, x_2 \geq 0 \text{ and integer}$$

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Example 1: All-Integer LP

Example 1 : Company X considering investing in townhouses (T) and apartments(A). Determine the number of T's and A's to be purchased. (Integers)

Funds available: $2 million
Cost: $282k / T and $400k / A
Numbers available: 5 T's and any number of A's.

Management time available: 140 hrs/mo
Time needed: 4 hrs/mo for T and 40 hrs/mo for A.

Contribution: $10k for T and $15k for A.

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Example 1: ILP Model

Maximise 10T + 15A

Subject to: 282T + 400A ≤ 2000

4T + 40A ≤ 140

T ≤ 5

T, A ≥ 0 and integer

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Relaxed LP and Rounded Solution

- If we relax the integer restriction, the optimum solution is
  T=2.48, A = 3.25, Z = 73.57
- Not acceptable (T and A cannot be fractions)
- Rounding off gives T = 2, A = 3 and Z = 65 : Such a solution sometimes yield an optimum solution, but has two problems:
  - Solution may not be feasible (impractical)
  - May not be optimal solution
- The optimal solution is T = 4, A = 2, Z = 70
- An integer solution can never be better than the LP solution and is usually a lesser solution.

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Example 2

- Max $Z = 3x_1 + 4x_2$
- ST

$3x_1 - x_2 \leq 12$

$3x_1 + 11x_2 \leq 66$

$x_1, x_2 \geq 0$ and are integers

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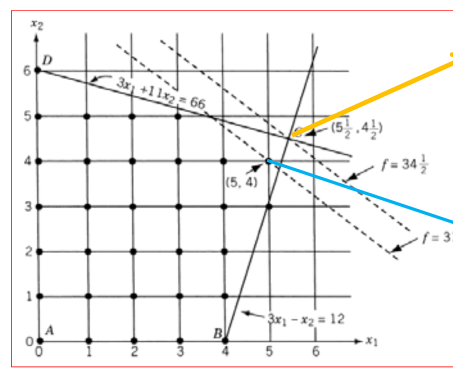
Graphical solution



solution :
x1 = 5.5
x2 = 4.5
Z = 34.5

Round down solution
x1 = 5,
x2 = 4,
Z= 31
Is optimal solution

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Example 2.1

- Same as Example 2 (changes in Red)
- Max $Z = 3x_1 + 4x_2$
- ST

$3x_1 - x_2 \leq 12$

$7x_1 + 11x_2 \leq 88$

$x_1, x_2 \geq 0$ and are integers

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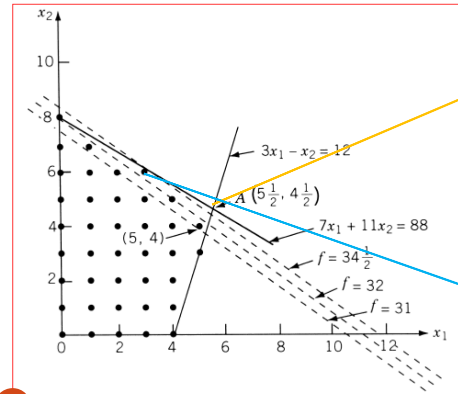
Solution with modified constraint



Optimum solution is of relaxed ILP identical

optimum solution is
x1 = 3,
x2 = 6,
Z = 33

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Techniques for solving IP

- No single technique for solving IP
- A number of procedures have been developed, and the performance of any particular technique appears to be highly problem-dependent.
- Methods classified broadly as :
  - i) enumeration techniques, including the **branch-and-bound procedure**;
  - ii) cutting-plane techniques/ **Gomory's Method**;

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ILP Algorithms

The ILP algorithms are based on exploiting the tremendous computational success of LP.

The strategy involves three steps:

1. Relax the ILP: Remove integer restrictions.
2. Solve the relaxed LP as a regular LP.
3. Starting with the relaxed optimum, add constraints that iteratively modify the solution space to satisfy the integer requirements

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1. Gomory's cutting plane method

- Gomory's method is based on the idea of generating a cutting plane
- Consider Example 2:
  - The original feasible region *ABCD* is reduced to a new feasible region *ABEFGD* such that an extreme point of the new feasible region becomes an integer optimal solution to IPP.
  - Note the inclusion of the two arbitrarily selected additional constraints *PQ* and P′Q′ gives the extreme point F($x1 = 5$, $x2 = 4$, $f = 31$) as optimal solution
- Gomory's method is one in which the additional constraints are developed in a systematic manner

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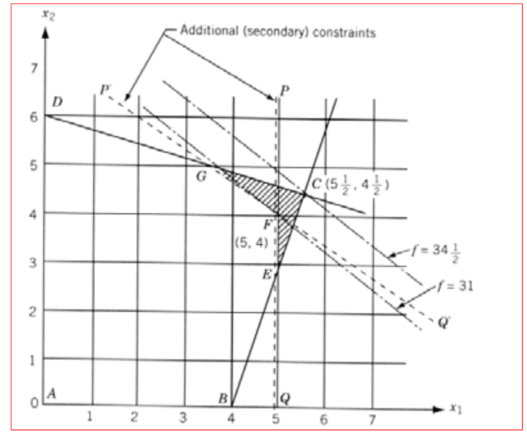
Effect of Additional Constraints



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2. Branch-and-Bound (B&B)

- Simplest method enumerating all integer points, and identifying the point that has the best OF value.
- Computationally expensive even for moderate-size problems.
- The **B&B method** can be considered as a refined enumeration method in which most of the non-promising integer points are discarded without testing them.

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Branch-and-Bound Steps

- 1. Solve the original problem using LP. If the answer satisfies the integer constraints, we are done. If not, this value provides an initial upper bound.
- 2. Find any feasible solution that meets the integer constraints for use as a lower bound. Usually, rounding down each variable will accomplish this.
- 3. Branch on one variable from step 1 that does not have an integer value. Split the problem into two sub-problems based on integer values that are immediately above or below the non-Integer value.

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Branch-and-Bound Steps

- 4. Create nodes at the top of these new branches by solving the new problem.
- 5.
  - (a) If a branch yields a solution to the LP problem that is not feasible, terminate the branch.
  - (b) If a branch yields a solution to the LP problem that is feasible, but not an integer solution, go to step 6.
  - (c) If the branch yields a feasible integer solution, examine the value of the objective function. If this value equals the upper bound, an optimal solution has been reached. If it not equal to the upper bound, but exceeds the lower bound, set it as the new lower bound and go to step 6. finally, if it's less than the lower bound terminate this branch.

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Branch-and-Bound Steps

- 6. Examine both branches again and set the upper bound equal to the maximum value of the objective function at all final nodes. If the upper bound equals the lower bound, stop. If not, go back to step 3.
- Minimization problems involved reversing the roles of the upper and lower bounds.

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Example B& B

- Company produces two products (x1 and x2)
- Maximize profit = 7X1 + 6X2
- subject to
  - 2X1 + 3X2 ≤ 12
  - 6X1 + 5X2 ≤ 30
  - where X1 and X2 are integers
- The optimal non-integer solution is :
  - X1 = 3.75 and , X2 = 1.5 and a max. profit = $35.25
- Since X_1 and X_2 are not integers, this solution is not valid.

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Example B& B

- The profit value of $35.25 will provide the initial *upper bound.*
- We can round down to X_1 = 3, X_2 = 1, profit = $27, which provides a feasible *lower bound.*
- Branching on X1 gives

Subproblem A

$$\begin{aligned} \text{Maximize profit} = \; & \$7X_1 + \$6X_2 \\ \text{subject to} \; & 2X_1 + 3X_2 \le 12 \\ & 6X_1 + 5X_2 \le 30 \\ & X_1 \ge 4 \end{aligned}$$

[X1 = 4, X2 = 1.2, profit = $35.20]

Subproblem B

$$\begin{aligned} \text{Maximize profit} = \; & \$7X_1 + \$6X_2 \\ \text{subject to} \; & 2X_1 + 3X_2 \le 12 \\ & 6X_1 + 5X_2 \le 30 \\ & X_1 \le 3 \end{aligned}$$

X1 = 3, X2 = 2, profit = $33.00

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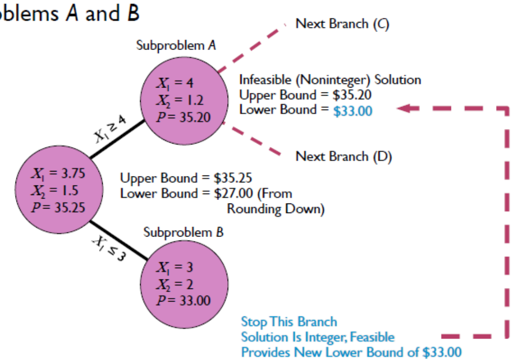
Example B& B

subproblems *A* and *B*



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Example B& B

- Subproblem A has branched into two new subproblems, C and D.

Subproblem C

Maximize profit = $\$7X_1 + \$6X_2$

subject to
$$2X_1 + 3X_2 \leq 12$$
$$6X_1 + 5X_2 \leq 30$$
$$X_1 \geq 4$$
$$X_2 \geq 2$$

no feasible solution because the all the constraints can not be satisfied (Fathom/terminate this branch

Subproblem D

Maximize profit = $\$7X_1 + \$6X_2$

subject to
$$2X_1 + 3X_2 \leq 12$$
$$6X_1 + 5X_2 \leq 30$$
$$X_1 \geq 4$$
$$X_2 \leq 1$$

X1 = 4.17, X2 = 1, profit = $35.16
This noninteger solution yields a new upper bound of $35.16

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Example B& B

- we create subproblems *E and F*

Subproblem E

Maximize profit = $\$7X_1 + \$6X_2$

subject to
$$2X_1 + 3X_2 \leq 12$$
$$6X_1 + 5X_2 \leq 30$$
$$X_1 \geq 4$$
$$X_1 \leq 4$$
$$X_2 \leq 1$$

Optimal solution to *E*:
$X_1 = 4, X_2 = 1$, profit = $34

Subproblem D

Maximize profit = $\$7X_1 + \$6X_2$

subject to
$$2X_1 + 3X_2 \leq 12$$
$$6X_1 + 5X_2 \leq 30$$
$$X_1 \geq 4$$
$$X_1 \geq 5$$
$$X_2 \leq 1$$

Optimal solution to *F*:
$X_1 = 5, X_2 = 0$, profit = $35

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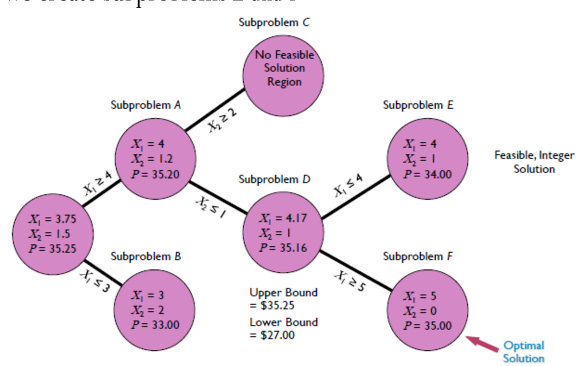
Example B& B

- we create subproblems *E and F*



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Example 1 using B&B method

- Maximize $f = 10x1 + 15x2$
- subject to
  - $282x1 + 400x2 \leq 2000$, $4x1 + 40x2 \leq 140$, $X1 \leq 5$, $x1 \geq 0$, $x2 \geq 0$ (E1)
  - xi = integer, $i = 1, 2$ (E2)
- Step 1: First the problem is solved as a continuous variable problem [without Eq. (E2)] to obtain:
- Problem (E1) : ($x1 = 2.48$, $x2 = 3.25$, $f = 73.57$)

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Example 1 using B&B method

- Step 2: The branching process, with integer bounds on x1, yields the problems:
- Maximize $f = 10x1 + 15x2$
- subject to
  - $282x1 + 400x2 \leq 2000$, $4x1 + 40x2 \leq 140$, $x1 \leq 5$, $x1 \leq 2$, $x2 \geq 0$ (E3) &
- subject to
  - $282x1 + 400x2 \leq 2000$, $4x1 + 40x2 \leq 140$, $x1 \leq 5$, $x1 \geq 3$, $x2 \geq 0$ (E4)
- Problem (E3) : ($x1 = 2$, $x2 = 3.3$, $f = 69.5$) **fathomed**
- Problem (E4) : ($x1 = 3$, $x2 = 2.89$, $f = 73.275$)

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Example 1 using B&B method

- Step 3: The next branching process, with integer bounds on x2, leads to the following problems:
- Maximize $f = 10x1 + 15x2$
- subject to
  - $282x1 + 400x2 \leq 2000$, $4x1 + 40x2 \leq 140$, $x1 \leq 5$, $x1 \geq 3$, $x2 \leq 2$ (E5) &
- subject to
  - $282x1 + 400x2 \leq 2000$, $4x1 + 40x2 \leq 140$, $x1 \leq 5$, $x1 \geq 3$, $x2 \geq 3$ (E6)
- Problem (E5) : ($x1 = 4.26$, $x2 = 2$, $f = 72.55$)
- Problem (E6) : (Infeasible) : **fathomed**

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Example 1 using B&B method

- Step 4: The next branching process, with integer bounds on x1, leads to the following problems:
- Maximize $f = 10x1 + 15x2$
- subject to
  - $282x1 + 400x2 \leq 2000$, $4x1 + 40x2 \leq 140$, $x1 \leq 5$, $x1 \leq 4$, $x2 \leq 2$ (E7) &
- subject to
  - $282x1 + 400x2 \leq 2000$, $4x1 + 40x2 \leq 140$, $x1 \leq 5$, $x1 \geq 5$, $x2 \leq 2$ (E8)
- Problem (E7) : ($x1 = 4$, $x2 = 2$, $f = 70.0$)
- Problem (E8) : ($x1 = 5$, $x2 = 1.48$, $f = 72.125$) can't give better solution than E7 **fathomed!**

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B&B Tree Solution for Example 1

LP0
T= 2.48, A = 3.25, Z = 73.57
Non-integer, non-inferior to current best, branch on T

LP1 = LP0 & T ≤ 2
T = 2, A = 3.3, Z = 69.5
Non-integer, can't give better solution than LP5, fathomed

LP2 = LP0 & T ≥ 3
T = 3, A = 2.89 Z = 73.28
Non-integer, non-inferior to current best, branch on A

LP3 = LP0 & T ≥ 3 & A ≤ 2
T = 4.26, A = 2, Z = 72.55
Non-integer, non-inferior to current best, branch on T

LP4 = LP0 & T ≥ 3 & A ≥ 3
Infeasible, fathomed

LP5 = LP0 & T ∊ [3,4] & A ≤ 2
T = 4, A = 2, Z = 70
Integer, Lower (best) bound

LP6 = LP0 & T ≥ 5 & A ≤ 2
T = 5, A = 1.48, Z = 72.13
Can't give better solution than LP5, fathomed

Note: Z is a multiple of 5 and hence only Z ≥ 75 can be better than z = 70

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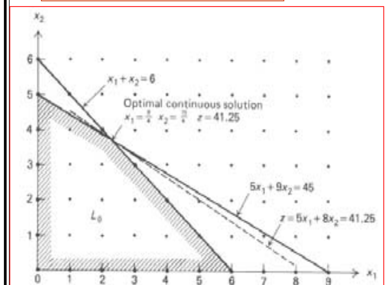
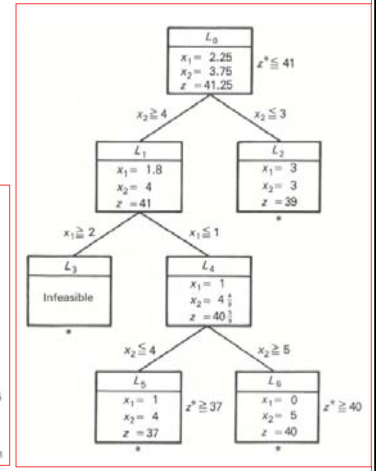
Example 2

max $z = 5x_1 + 8x_2$

Subject to
$x_1 + x_2 <= 6$,
$5x_1 + 9x_2 <= 45$,
$x_1, x_2 >= 0$ *and integers.*





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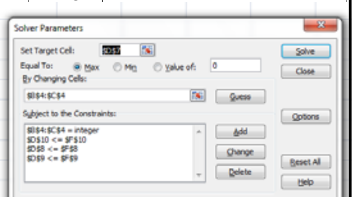
Excel Solution: Example 1

	T	A			
varables	4	2			
coef. &eqn			eqns		
OF	10	15	70		
C1	282	400	1928	LE	2000
C2	4	40	96	LE	140
C3	1	0	4	LE	



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MIP

- In mixed IP, some variables are required to be integers and others are allowed to be either integer or nonintegers.
- To solve a mixed IP by the branch-and-bound method, modify the method by branching only on variables that are required to be integers.
- For a solution to a subproblem to be a candidate solution, it need only assign integer values to those variables that are required to be integers

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Example

- Maximize profit = \$85X + \$1.50Y
- subject to
  - $30X + 0.5Y \leq 2{,}000$
  - $18X + 0.4Y \leq 800$
  - $2X + 0.1Y \leq 200$
- $X, Y \geq 0$ and X integer

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Binary Integer LP

- "0 -1" decision variables are used in problems where an Yes-No decision is to be taken regarding multiple choices.
- If the variable is 1, the corresponding choice is selected; if the variable is 0, it is not selected.

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Some Applications

- **Capital Budgeting problem:** decisions involve the selection of a number of potential investments. (to choose among possible plant locations, equipment, or projects)
- **Warehouse Location** :decisions must be made about tradeoffs between transportation costs and costs for operating distribution centers. As an example, suppose that a manager must decide which of *n warehouses to use for meeting the demands of m customers for a good.*
- **sequencing, scheduling, and routing Problems**: **(**critical-path scheduling with resource constraints, and vehicle dispatching, scheduling of students, faculty, and classrooms, etc..)

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