CHAPTER-TWO

Time value of money

Time value of money

- The time value of money explains the change in the amount of money over time.
- The time value of money means that we can’t compare amounts of money from two different periods without adjusting for this difference in value.
- This is the most important concept in engineering economy .
- Money has a value
  - ✓ It can be leased or rented same way as apartment
  - ✓ The payment is called interest

Time value of money cont.....

Time Value of money is related to:

- ❑ Labor
- ❑ Materials
- ❑ Machinery & Equipment
- ❑ External service
- ❑ All these factors that have direct relationship with our products or services will be affected by time value of money.

Cont.….

Time Value of money examples

- If a given sum of money is deposited in a savings account; it earns **interest**.
- If it is used to start a business, it earns **profit**
- If it is used to purchase a share in a business, it earns **dividends**.
- If it is used to purchase an office building or apartment house, it earns **rent**.

Definitions of terms wrt Time Value of money

- **Interest**: the money earned by the original sum of money, regardless of whether the earned money is referred to as "interested", "profits", "dividends", or "rent" in ordinary

commercial parlance.

- **Interest rate**: the time rate at which a sum of money earns interest (it is usually expressed in percentage form).
  - ✓ There are always two perspectives to an amount of interest— interest paid and interest earned.
  - ✓ Interest = amount owed now — principal

  Interest rate is the interest paid expressed as a percentage of the principal over a specific time unit.

$$\textbf{Interest rate (\%)} = \frac{\textbf{interest accrued per time unit}}{\textbf{principal}} * \mathbf{100\%}$$

- The time unit of the rate is called the interest period.

Definitions of terms wrt Time Value of money

Example 1 An employee borrows $10,000 on May 1 and must repay a total of $10,700 exactly 1 year later. Determine the interest amount and the interest rate paid.

Solution The perspective here is that of the borrower since $10,700 repays a loan. to determine the interest paid.

Interest paid = $10,700 - 10,000 = $700

- **determines the interest rate paid for 1 year .**

Interest rate (%) = $\frac{700}{10000}$ ***100% = 7%**

Definitions of terms wrt Time Value of money

Example 2

(a) Calculate the amount deposited 1 year ago to have \$1000 now at an interest rate of 5% per year .

(b) Calculate the amount of interest earned during this time period.

Solution

(a) The total amount accrued (\$1000) is the sum of the original deposit and the earned interest.

If X is the original deposit, Total accrued = deposit + deposit(interest rate)

$\$1000 = X + X(0.05) = X(1 + 0.05) = 1.05X$ The original deposit is

$$X = \frac{1000}{1.05} = \$952.38$$

(b) Interest = \$1000 - 952.38 = \$47.62

Definitions of terms wrt Time Value of money

- **Investment**: the productive use of money to earn interest.
- **Capital**: the money that earns interest.
- The interest earned by the original capital can itself be invested to earn interest, and this process can be continued indefinitely. This capacity of money to enlarge itself with the passage of time is referred to as the **time value of money**.

Elements of transactions involving interest

- **P** = value or amount of money at a time designated as the present or time0.
- **F** = value or amount of money at some future time.
- **A** = series of consecutive, equal, end-of-period amounts of money .
- **n** = number of interest periods; years, months, days
- **i** = interest rate per time period; percent per year, percent per month
- **t** = time, stated in periods; years, months, days

Why money loses value over time

There are several reasons.

- Most obviously, there is inflation which reduces the buying power of money.
- But quite often, the cost of receiving money in the future rather than now will be greater than just the loss in its real value on account of inflation.
- The opportunity cost of not having the money right now also includes the loss of additional income that you could have earned simply by having received the cash earlier.
- Moreover, receiving money in the future rather than now may involve some risk and uncertainty regarding its recovery. For these reasons, future cash flows are worth less than the present cash flows.

Cont…..

- Time Value of Money concept attempts to incorporate the above considerations into financial decisions by facilitating an objective evaluation of cash flows from different time periods by converting them into present value or future value equivalents. This ensures the comparison of 'like with like'.

Cash Flow and Cash-Flow Diagrams

- **Cash flows** are the amounts of money estimated for future projects or observed for project events that have taken place.

- All cash flows occur during specific time periods.

- **Cash inflows** are the receipts, revenues, incomes, and savings generated by project and business activity .

- **A plus sign** indicates a cash inflow .

- **Cash outflows** are costs, disbursements, expenses, and taxes caused by projects and business Cash flow activity .

- **A negative or minus sign** indicates a cash outflow .

Cash Flow and Cash-Flow Diagrams….

❖ Cash flow is nothing but the set of payments associated with an investment; and cash flow diagram is a diagram that shows these payments.

In the cash flow diagram:

✓ Time is plotted on a horizontal axis

✓ The payments are represented by vertical bars

✓ The amount of each payment is recorded directly above or below the bar representing it.

✓ The bars are generally not drawn to scale

Cash Flow and Cash-Flow Diagrams….

E.g. assume that a project has the following cash flow:

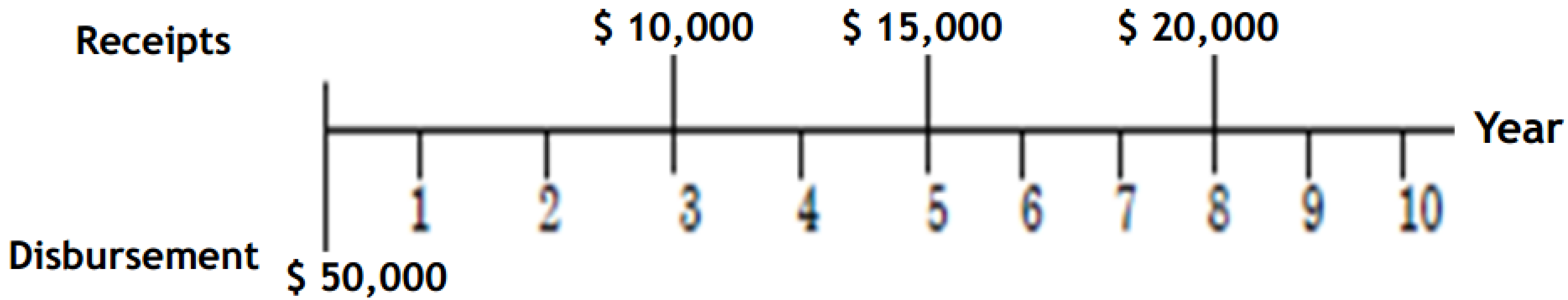
1) a disbursement of $ 50,000 now

2) a receipt of $ 10,000 after three years

3) a receipt of $ 15,000 after five years and

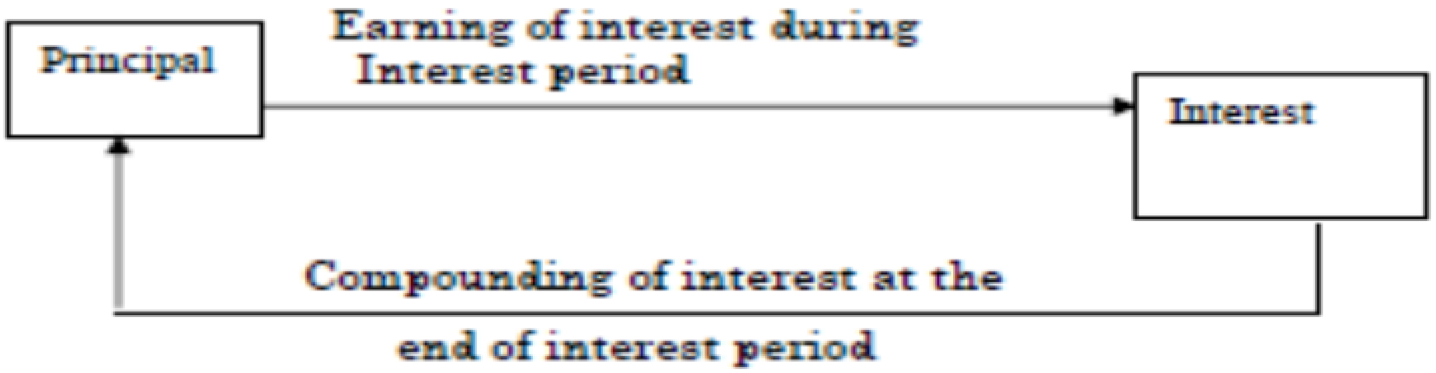
4) a receipt of $ 20,000 eight years hence

Taking the unit of time one year, the cash flow diagram is represented as in figure.



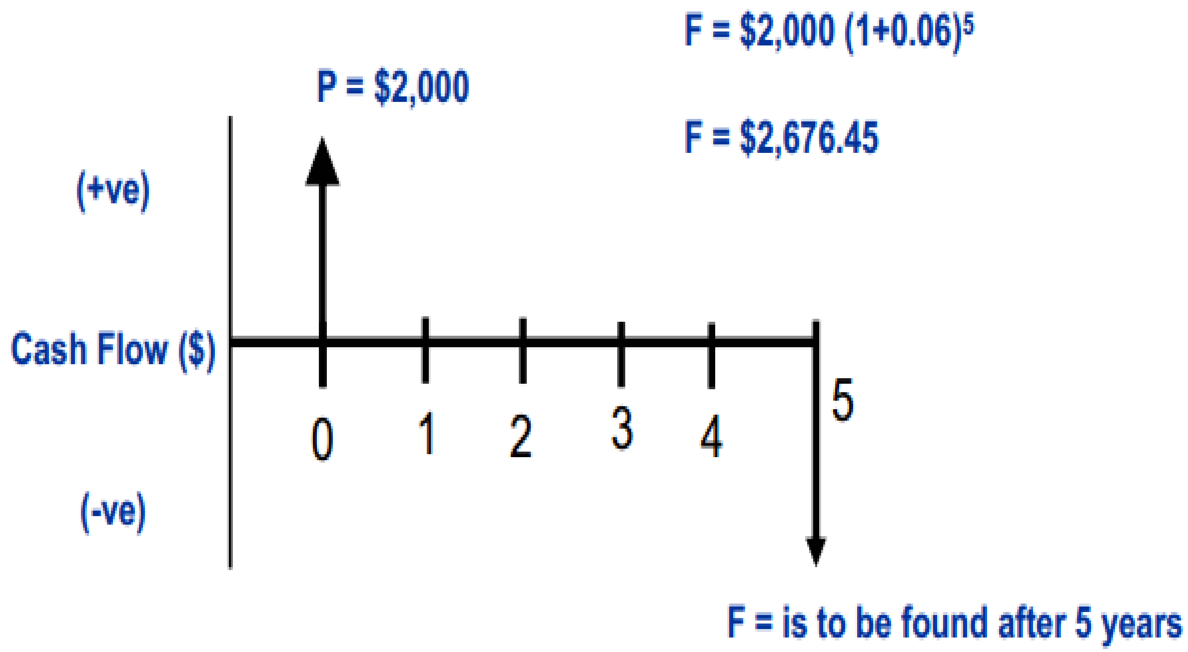
Basic relationship between Money & Time

- The sum of money that is earning interest at a given instant is known as the principal in the account.
- At the expiration of a time interval called the interest period, the interest that has been earned up to that date is converted to principal there by causing it to earn interest during the remainder of the investment.
- This process of converting interest to principal is referred to as the compounding of interest; it represents an investment of the interest in the same investment.



Computing Cash Flows

Example1:If you borrow $2,000 now and must repay the loan plus interest (at rate of 6% per year) after five years. Draw the cash flow diagram. What is the total amount you must pay?

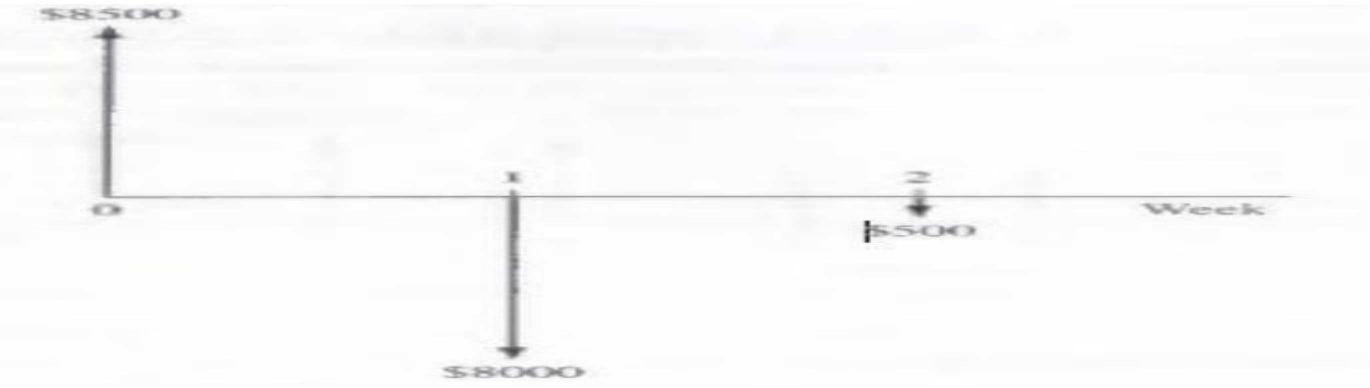


Cash flows…..

Example2: Assume you borrow $8500 from a bank today to purchase an $8000 used car for cash next week, and you plan to spend the remaining $500 on a new paint job for the car two weeks from now . Construct the cash flow diagram

Soln.

Perspective	Activity	Cash flow($)	Time (week)
you	borrow	8500	0
	buy car	8000	1
	Paint Job	500	2



Methods of Calculating Interest

Simple Interest(constant interest)

- Interest paid (earned) on only the original amount, or principal, borrowed (lent).
- the interest earned during each interest period does not earn additional interest in the remaining periods.

Formula $I = P(i)(n)$

I: Simple Interest

P: Deposit today (t=0)

i: Interest Rate per Period

n: Number of Time Periods

- The total amount available at the end of N periods, F thus would be
- $\mathbf{F = P + I = P\ (l + i\,N)}$

Methods of Calculating Interest

Compound Interest

- the interest accrued for each interest period is calculated on the principal plus the total amount of interest accumulated in all previous periods.
- In general, if you deposited (invested) P dollars at an interest rate i, you would have
- P + iP = P(1 + i) dollars at the end of one interest period.
- This interest-earning process repeats, and after N periods, the total accumulated value (balance) F will grow to
- $F= (1+i)^n$
- Therefore, the Total interest earned = $I_n = p(1+i)^n - p$

Methods of Calculating Interest

Example1:

- Assume that you deposit $1,000 in an account earning 7% simple interest for 2 years. What is the accumulated interest at the end of the 2nd year?

Solution:

- $I = P(i)(n) = \$1,000(.07)(2) = \$140$

Example2: Suppose you deposit $1,000 in a bank savings account that pays interest at a rate of 8% per year. Assume that you don't withdraw the interest earned at the end of each period (year), but instead let it accumulate. (a) How much would you have at the end of year three with simple interest? (b) How much would you have at the end of year three with compound interest?

Solution:

Given: P = $1,000. n = 3 years. and i = 8% per year.

(a) Simple interest: We can calculate F as

F= $1000[1 + (0.08)3]= $1,240

Cont'd

The interest-accruing process shown as follows:

End of Year	Beginning Balance	Interest Earned	Ending Balance
0			$1,000
1	$1,000	$80	$1,080
2	$1,080	$80	$1,160
3	$1,160	$80	$1,240

Cont'd

(b) Compound interest: Applying the Eq. $F = P(1+i)^n$ to a three-year, 8% case, we obtain

$F = 1000(1+0.08)^3 = \$1259.71$

The interest-accruing process shown as follows:

Period (*n*)	Amount at Beginning of Interest Period	Interest Earned for Period	Amount at End of Interest Period
1	$1,000.00	$80.00	$1,080.00
2	$1,080.00	$86.40	$1,166.40
3	$1,166.40	$93.31	$1,259.71

Cont'd

In general, let

P = sum deposited in savings account at the beginning of an interest period

F = Principal in account at expiation of n interest periods

i = interest rate

The principal at the end of the first period is P + Pi = P (1+i)

The principal at the end of the second period is P (1+i) + P (1+i)I

$= P (1+i) (1+i) = P (1+i)^2$

The principal at the end of the 3rd period is

$= P (1+i) (1+i) + P (1+i) (1+i)i$

$= P (1+i) (1+i) (1+i) = P (1+i)^3$

From this we conclude that the principal is multiplied by the factor (1+i) during each period. Therefore, the principal at the end of the nth period is

$F = P (1+i)^n$

Significance of Time Value of Money

- To analyze an investment or compare alternative investments, it is necessary to consider the timing as well as the amount of each payment. To simplify this, let us consider the following question? Is it preferable to receive $ 1000 today or $ 1000 one year hence?
- Ans: it is preferable to receive $1000 now, since it can be invested immediately to earn interest. For instance, if the money is invested at 20% per year, it will have grown to $ 1200 one year hence.
- Similarly, is it preferable to expend $ 1000 now or $ 1000 one year hence?
- Ans: It is preferable to expend $ 1000 one year hence, since by retaining the money for 1 year, we earn interest for that year . Therefore, clearly the significance of time value of money is unquestionable.

Notation for Compound Interest Factors

- We shall define and apply several compound-interest factors. Each factor will be represented symbolically in the following general

format:

- (A/B, n, i)
  - A & B denoted two sums of money
  - A/B denotes the ratio of A to B
  - n denotes the number of interest periods
  - i denotes the interest rate
- For brevity (shortness in time) the interest rate can be omitted, and the expression will be given simply as (A/B, n).

Calculations of Future Worth, Present Worth, Interest Rate, and Required Investment Duration

$F = P (1+i)^n$

P is referred to as the present worth of the given sum of money

- F is referred to as the future worth of the given sum of money
- The terms "present" and "future" are applied in a purely relative sense as a means of distinguishing between the beginning and end of the time interval consisting of n periods.

The factor $(1+i)^n$ is termed as the **single-payment future-worth factor**. Conventionally we introduce the following notation.

$(F/P, n, i) = (1+i)^n$ ----- 2.1a

Can't

- Thus, equation (1.1) can be rewritten as F = P (F/P, n, i) ----2.1b

Similarly $P = F(1+i)^{-n}$

- The factor $(1+i)^{-n}$ is termed as the **single-payment present-worth factor**.

- N.B when a future sum is converted to its present worth, it is said to be **discounted**.

Can't

- **Example3**: If $ 5,000 is invested at an interest rate of 10% per annum, what will be the value of this sum of money at the end of 2 years?

Solution:

Given: P = $5,000, n = 2, i = 0.10

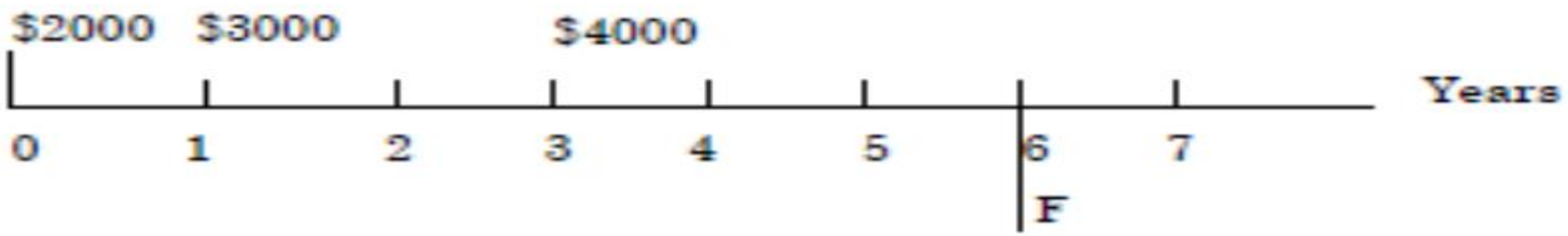
$F = P\ (F/P, n, i) = 5{,}000\ (F/P, 2, 10\%) = 5{,}000\ (1.1)^2 = \$6{,}050$

Can't

Example5: Smith loaned Jones the sum of $2000 at the beginning of year 1, $3000 at the beginning of year 2 and $4000 at the beginning of year 4. The loans are to be discharged by a single payment made at the end of year 6. If the interest rate of the loans is 6% per annum, what sum must Jones pay?

Solution:

- ✓ Refer to figure below. To maintain consistency and there by simplify the calculation of time intervals, convert the date of payment to the beginning of year 7.



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Solution: Applying equation 2.1b

F = 2000 (F/P, 6, 6%) + 3000 (F/P, 5,6%) + 4,000 (F/P, 3, 6%)

= 2000 (1.41852) + 3000 (1.33823) + 4000 (1.19102)

= $ 11,616

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Example6: If $5000 is deposited in an account earning interest at 8% per year compound quarterly, what will be the principal at the end of 6 years?

Solution:

p = $ 5000 n = 6 x 4 = 24

True interest rate = nominal rate/the no of interest periods contained 1 year

= 8%/4 = 2%

F = 5000 (F/P, 24, 2%)

= 5000 (1.60844) = $ 8,042

Meaning of Equivalence

- In general two alternative payments are equivalent to one another if the monetary worth of the firm will eventually be the same regardless of which payment is made.
- Consider that a firm received the sum of $10,000 at the beginning of year 6 and immediately invested this at 8%, by the beginning of year 9, this sum of money has expanded to 10,000 (F/P, 3, 8%) = 12,597.
- Therefore, if the firm had received the sum of $12,597 at the beginning of year 9 rather than the sum of $10,000 at the beginning of year 6, its monetary worth at the beginning of year 9 and at every instant thereafter would have been the same.

Con’t

- Thus, these two alternative events receipt of **$10,000 at the beginning of year 6, and receipt of $12,597 at the beginning of year 9- yield an identical monetary** worth if money is worth 8%.
- We therefore can say that these two events are **equivalent to one another.**

Change of interest rate

- The interest rate changes at a particular date.
- This date serves to divide time into two intervals, each characterized by a specific interest rate.
- If a given sum of money is to be carried from one interval to the other, it is necessary to find its value at the boundary point. Mathematically,

$$F = p(1+i)^{n_1} * (1+i)^{n_2}$$

Example8:The sum of $5000 will be required 10 years hence. To ensure its availability, a sum of money will be deposited in a reserve fund at the present time. If the fund will earn interest at the rate of 4% for the first three years & 6% thereafter, what sum must be deposited?

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Solution:

The problem requires that we find the present worth of the specified sum of money . We move this sum of $5000 backward 7 years at 6% and then backward another 3 Years at 4 %. Then

$F = P(P/F, 7, 6\%)\ (P/F, 3, 4\%) = P\ (1+i)^7\ (1+i)^3$

$5000 = P\ (1+i)^7\ (1+i)^3$

$= P\ (1+0.06)^7\ (1+0.04)^3$

P = $ 2956

Equivalent & Effective Interest Rates

If a given rate applies to a period less than a year, its equivalent rate for an annual period is referred to as its effective rate.

Consider that at the beginning of a given year the sum of $100,000 is deposited in each of three funds designated A, B & C. The interest rates are as follows .

- Fund A, 8% per year compounded quarterly
- Fund B, 8.08% per year compound semi-annually
- Fund C, 8.243% per year compounded annually
- The principal in each fund at the beginning of the following year is as follows .

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- Fund A: 100,000 (1.02)4

= $ 108,243

- Fund B: 100,000 (1.0404)2

= $ 108,243

- Fund C: 100,000 (1.08243) = $ 108,243
- Thus, the effective rate corresponding to a rate of 2% per quarterly period (or 8% per year compound quarterly) is 8.243%.

What is the similarity & difference between true interest and effective interest rate? (Reading A.)

Opportunity Costs and Sunk Costs

- Consider that an organization has a choice of two alternative investments, A and B. If it undertakes investment A, it forfeits the income that would accumulate under B. Therefore, the income associated with investment B is referred to as an **Opportunity Cost of investment A.**

- **A Sunk Cost** is an expenditure that was made in the past and that exerts no direct influence on future cash flows. (is it relevant in economic analysis ?)

It is irrelevant in an economy analysis.

Effect of taxes on investment rate

- ❖ If the rate of taxation varies according to the amount of the income, it is necessary to establish the rate at which the income from a specific venture will be taxed.

Definitions:

- ❑ **Original income**: income received by a firm before taxation.
- ❑ **Residual income**: the income that remains after the payment of taxes (income after tax).
- ❑ **Before-tax rate**: investment rate calculated on the basis of original income.
- ❑ **After-tax rate**: investment rate calculated on the basis of residual income.
- ❑ Assume **Q** as the sum of money received & this income is taxed at the rate **t**.

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The tax payment = Qt

Residual income = original income- tax payment

= Q – Qt = Q (1-t)

- We shall now develop the relationship between the before-tax & after-tax investment rates.
- Assume that a given investment pays a constant annual dividend during its life & that the firm recovers the money originally invested when the venture terminates.

Let

C = amount invested

I = annual income from investment rate

ib = before-tax investment rate

ia = after-tax investment rate

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We have the following:

Original income = I, &

i_b = I/C= original income/amount investment

Residual Income = Net taxable income – Tax payment

= I – It = I (1 – t),

& i_a = I (1-t)/C = residual income/amount invested

Then

$i_a = i_b (1 - t)$

Excerscise: A firm expended $560 to have an employee attend a seminar. If this expenditure is tax deductible and the tax rate of the firm is 45%, what was the after tax value of the expenditure?

Inflation

- Inflation is the general increase of costs with the passage of time respect to a given commodity . Let

Co = Cost of commodity at the beginning of the first year

Cr = Cost of commodity at the end of the r^{th} year

q = (effective) rate of inflation of r^{th} year

- The **rate of inflation** for a given year is taken as the ratio of the increase in cost of the commodity during that year to the cost at the beginning of the year .

symbolically,

$$q = \frac{C_r - C_{r-1}}{C_{r-1}}$$

If the annual rate of inflation remains constant for a year, the cost of commodity at the end of that period is,

$$C_n = C_o(1+q)^n$$

Uniform Series of Payments

- **Uniform Series/Annuity**: a set of payments each of equal amount made at equal intervals of time.

Payment Period: the interval between successive payments.

For example, if a corporation makes an interest payment of $50,000 to its bondholders at 3-month intervals, these interest payments constitute a uniform series, and **the payment period is 3 months.**

Similarly, if a construction company rents equipment for which it pays $4000 at the end of each month, these rental payments constitute a uniform series, and the payment period is 1 month.

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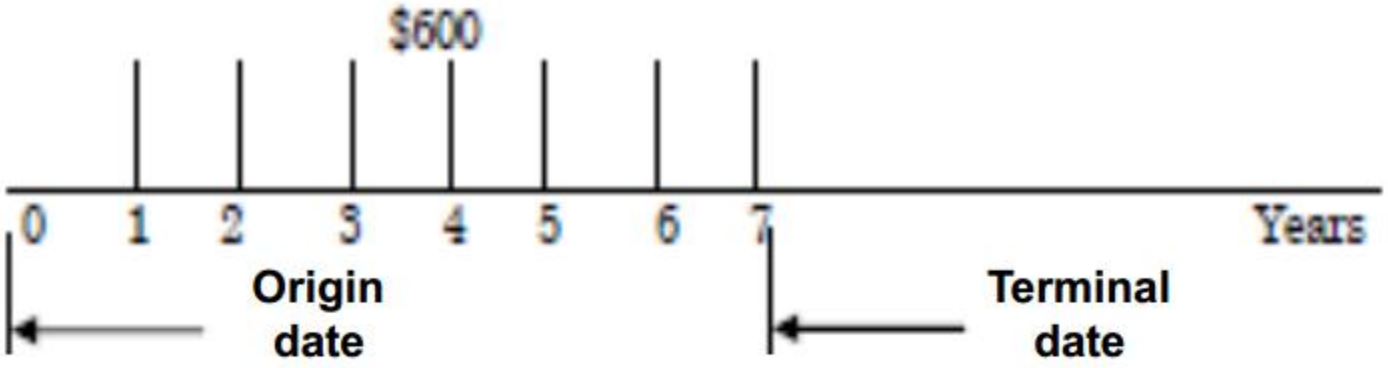
- **Ordinary Uniform Series**: one in which a payment is made at beginning or end of each interest period.
- The payment period and interest period coincide in all respects in ordinary uniform series.



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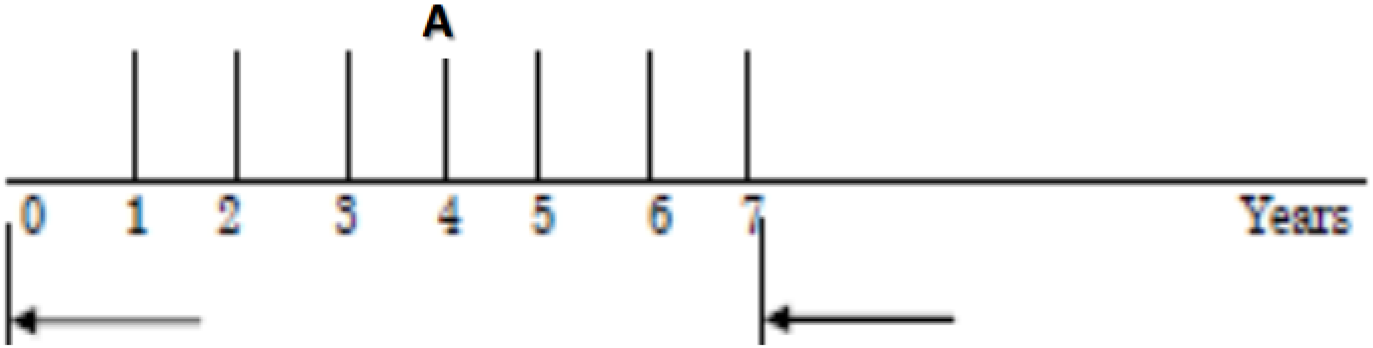
- By convention, the origin date of a uniform series is placed one payment period prior to the first payment, and the terminal date is placed at the date of the last payment.
- The value of the entire set of payments at the origin date is called the **present worth of the series**, and the value at the terminal date is called **the future worth.**

Calculation of Present Worth and Future Worth (For uniform serious payment)

- Let equal amounts of money, A, be deposited in a savings account (or placed in some other interest-bearing investment) at the end of each year, as indicated in figure .



- If the money earns interest at a rate i, compounded annually, how much money will have accumulated after n years?

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To answer this question, we note that after n years, the first year's deposit will have increased in value to

$F_1 = A\,(1+i)^{n-1}$

Similarly, the second year's deposit will have increased in value to

$F_2 = A\,(1+i)^{n-2}$ and so on

The total amount accumulated will thus be the sum of a progression:

$$Fu = F_1 + F_2 + F_3 + \ldots\ldots\ldots\ldots\ldots + F_n$$

$$= A\,(1+i)^{n-1} + A(1+i)^{n-2} + \ldots\ldots\ldots + A$$

$$= A[(1+i)^{n-1} + (1+i)^{n-2} + \ldots\ldots\ldots + 1\,]$$

$$\text{Then, } Fu = A\,\frac{[(1+i)^n - 1]}{i}$$

$$Fu = A(Fu\,/\,A, n, i)$$

$$where: (Fu\,/\,A, n, i) = \frac{[(1+i)^n - 1]}{i}$$

-is called the uniform-series-future-worth factor.

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Similarly, for present worth: evaluating all payments at the origin of the series and summing the results, we obtain:

$$\mathbf{Pu} = A[1 + (1+i)^{-1} + (1+i)^{-2} + \ldots + (1+i)^{-n}]$$

$$Then: Pu = A[\frac{1-(1+i)^{-n}}{i}] = A[\frac{1 - \frac{1}{(1+i)^{n}}}{i}]$$

$$Pu = A(Pu / A, n, i)$$

$$where: (Pu / A, n, i) = [\frac{1-(1+i)^{-n}}{i}]$$

is called the uniform-series-present-worth factor.

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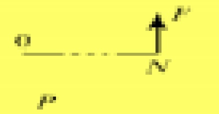
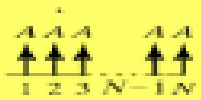
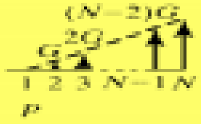
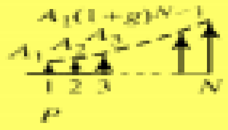
Summary of Discrete Compounding Formulas with Discrete Payments

Flow Type	Factor Notation	Formula	Excel Command	Cash Flow Diagram
SINGLE	Compound amount $(F/P, i, N)$	$F = P(1+i)^N$	$= FV(i, N, P, , 0)$	
	Present worth $(P/F, i, N)$	$P = F(1+i)^{-N}$	$= PV(i, N, F, , 0)$	
EQUAL PAYMENT SERIES	Compound amount $(F/A, i, N)$	$F = A\left[\frac{(1+i)^N - 1}{i}\right]$	$= PV(i, N, A, , 0)$	
	Sinking fund $(A/F, i, N)$	$A = F\left[\frac{i}{(1+i)^N - 1}\right]$	$= PMT(i, N, P, F, 0)$	
	Present worth $(P/A, i, N)$	$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]$	$= PV(i, N, A, , 0)$	
	Capital recovery $(A/P, i, N)$	$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]$	$= PMT(i, N, P)$	
GRADIENT SERIES	Linear gradient Present worth $(P/G, i, N)$	$P = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right]$		
	Conversion factor $(A/G, i, N)$	$A = G\left[\frac{(1+i)^N - iN - 1}{i[(1+i)^N - 1]}\right]$		
	Geometric gradient Present worth $(P/A_1, g, i, N)$	$P = \begin{cases} A_1\left[\frac{1-(1+g)^N(1+i)^{-N}}{i-g}\right] \\ A_1\left(\frac{N}{1+i}\right) \text{ (if } i = g) \end{cases}$		

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TUTORIALS WILL CONTINUE!!!