Paradoxes

Springer

Trends in Logic

Volume 31

For further volumes:
http://www.springer.com/series/6645

TRENDS IN LOGIC

Studia Logica Library

VOLUME 31

SCOPE OF THE SERIES

Trends in Logic is a bookseries covering essentially the same area as the journal *Studia Logica* – that is, contemporary formal logic and its applications and relations to other disciplines. These include artificial intelligence, informatics, cognitive science, philosophy of science, and the philosophy of language. However, this list is not exhaustive, moreover, the range of applications, comparisons and sources of inspiration is open and evolves over time.

Volume Editor
Ryszard Wójcicki

Piotr Łukowski

Paradoxes

Springer

Piotr Łukowski
Department of Cognitive Science
University of Łódź
Smugowa 10/12
91-433 Łódź
Poland
e-mail: lukowski@uni.lodz.pl

Translated by Marek Gensler

ISBN 978-94-007-1475-5 e-ISBN 978-94-007-1476-2

DOI 10.1007/978-94-007-1476-2

Springer Dordrecht Heidelberg London New York

Cover design: eStudio Calamar, Berlin/Figueres

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

To my Parents,
Maria and Zdzisław

Contents

Chapter 1
Introduction

Because of their attractive intellectual form, it is tempting to treat paradoxes lightly, as if they were only an intellectual entertainment. Admittedly, when presented in interesting and non-trivial way, they can offer an attractive diversion at a party. Reducing them to entertainment only, however, is disrespectful not only to their authors but also to ourselves, since paradoxes should be treated as alarm signals informing of danger resulting from some errors in our thinking. This absolutely serious function which they play ought to be recognized especially by people who are interested in logic and philosophy. It is for this reason that paradoxes deserve not only attention but also serious consideration of the conclusions following from them. Regrettably, it is not so. One can still encounter logical and, especially, philosophical ideas, which seem to ignore the existence of paradoxical arguments, or at best question the sense of dealing with them. Many philosophical ideas can survive only because they disregard completely such paradoxes as sorites, paradoxes of change, motion or identity. If treated seriously, the conclusions following from those paradoxes could be death blows to a number of metaphysical or ontological ideas.

Paradox is a thought construction, which leads to an unexpected contradiction.[1]

The essence of this definition lies in the word "unexpected". We often reach contradictions, but not every such situation is seen as paradoxical. True, a contradiction understood as a conjunction of two propositions/sentences/statements/claims, of which one is the negation of the other, is something that we should reject. But every such conjunction is paradoxical for us: if I consciously accept some *P* together with *non-P*, I should not be surprised that I have to accept a contradiction *P* and *non-P*. A paradox appears only when we get a contradiction in spite of:

- Using logically correct rules of inference
- Being convinced that we formulated our thoughts precisely
- Certitude that the opinions we have so far are rational.

[1] For stylistic reasons, the word "paradox" will be sometimes replaced by the word "dilemma" in order to avoid repetitions.

P. Łukowski, *Paradoxes*, Trends in Logic, 31,
DOI: 10.1007/978-94-007-1476-2_1, © Springer Science+Business Media B.V. 2011

Then, indeed, the contradiction which appears is unexpected, since the procedure applied in our reasoning was devised specially to prevent it. Logical checks were functioning and yet we obtained a contradiction, a very serious error.

By tradition, a paradox (gr. *παραδοξοζ*, *paradoxos*) is an opinion, conclusion etc., which is contrary to common belief. According to Aristotle (384–322 BC), a statement paradoxical when it is contrary to the belief of a group of people. One and the same opinion can be a paradox for one group of people but not for another. The word "infinity" is often used in its every day sense of something without an end, something inexhaustible. For a common man, the mathematical understanding of infinity and its properties would be a surprise or even something unbelievable. Yet that knowledge is nothing strange for a person with mathematical education. In a narrower sense, a paradox is an antinomy (Greek *αντινομια*, *antinomia*; *anti*—against, *νομοζ*, *nomos*—law). We shall call a reasoning antinomic when, despite its logically correct form, it leads to a contradiction: *P and non-P*. In a still narrower sense, an antinomy is a logically correct reasoning, which justifies an equivalence: *P if and only if non-P*. It is in this sense that antinomy is understood most often. A special kind of paradoxes are sophisms (Greek: *σοφισμα*, *sophisma*—false conclusion). Aristotle uses his name for every argument, which seems to be correct but is not. Usually, the word "sophism" is understood to include an intention to deceive the listener. For this reason, one distinguishes a paralogism (Greek: *παραλογισμοζ*, *paralogismos*—false conclusion), which is an incorrect reasoning without an intention to deceive. A person, who presents such a reasonings, is either unaware that it is erroneous or knows that it is wrong but treats the presentation as a joke or a didactic trick to be explained soon.

Classification of paradoxes is rather difficult, since it is not clear according to which criteria it should be done. In this book, we have accepted the essence of a paradox as the fundament of classification. If the essence of a paradox is an error of ambiguity, it is classified as a paradox ambiguity, regardless of the character of its narration. This can be manifestly seen in the case of such paradoxes as Newcomb's and Protagoras's. Accordingly, the book has four chapters. The first one analyzes those paradoxes, which result from a clash between a logically correct reasoning and our previously accepted opinions. Here we can place e.g., Georg Cantor's set theory paradoxes, but because of their strictly mathematical character they will not be discussed in this book. A reader interested in those issues can find references to the literature of the subject. The second chapter is devoted to paradoxes resulting from the error of ambiguity. The third one, in turn, analyzes reasonings, whose paradoxical character originates in self-referent language constructions. The paradoxes discussed in those three chapters can be commonly called intra-linguistic ones. They are dilemmas, which do not require going beyond the natural language either in their formulation or in analysis, since their existence does not depend on extra-linguistic reality at all: they belong to the world created in and by the language. Another group is formed by paradoxes which have been called ontological ones. Their existence results from a confrontation between the language description of reality and that reality itself. The paradoxes show lack of adequacy of the description and, what is worse, lack of adequacy of a natural language as such. Here we find paradoxes such as sorites, paradoxes of change, motion and identity, as well as the problem of the many. Ontological paradoxes are ones which deserve an especially serious examination, since they are helpful in recognizing the true value of some philosophical opinions. The light they shed makes some ontological or metaphysical proposals look silly and undeserving any attention.

In most cases, the author provides his own proposals of solution, viz. for the Liar antinomy, Buridan's paradox, generalized form of the Liar antinomy, Aristotelian circles paradox, Holy Trinity paradox, Fitch's paradox, Protagoras's paradox, paradox of a stone, unexpected examination (Hangman's) paradox, and Crocodile's paradox. In the cases of the Barber's, Richard's, Berry's and Grelling's antinomies, the author argues that those paradoxes do not require a solution, i.e., finding an error. They are only evidence how unusual and interesting constructions can be built with help of a natural language. In the case of sorites paradoxes, the author shows that some approaches to vagueness fail (ones which do not respect the phenomenon), while other ones have merely technical character (ones which respect the phenomenon). The author's general opinion about that type of paradoxes is in agreement with the opinion of Henri Bergson (1859–1941).

The bibliography provided at the end of the book is not limited to the sources of quotation and includes various texts concerning paradoxes with the exception of na set theory dilemmas. Its aim is helping the reader in finding interesting positions in the literature of the subject. The book not only presents and analyzes the most important proposals concerning paradoxes but also provides the authors' own opinions of them. This was possible for those issues, where the literature of the subject was rich enough. Because of the incremental stage of the discussion concerning contextual approach to vagueness [1], this area has not been included in the presentation. Unlike other approaches to vagueness, which apparently used up their potential, contextualism is still waiting to manifest its own one, not only in the context of vagueness.

Both in its subject and content, this book is close to a Polish publication [2]. It is not its translation, however, but a revised version of a large part of its material. Some parts, like the paradoxes of na set theory, including the famous Russell's antinomy, have not been included here. The author thanks Professor Jacek Malinowski for his encouragement to write the book.

References

1. Shapiro, S. (2006). *Vagueness in context*. Oxford.
2. Łukowski, P. (2006). *Paradoksy* (p. 533). Łódź: Wydawnictwo Uniwersytetu Łódzkiego.

Chapter 2
Paradoxes of Wrong Intuition

Getting to know reality we usually follow tried methods of thinking, we use concepts we have learnt elsewhere, and ever readymade solutions. Our experience makes us form a certain scheme of thinking, which is good as long as it brings good results. Regrettably, reality is always richer and goes beyond all schemes of thought. This is the prime cause of cognitive conflicts, which result from the fact that our necessarily schematic expectations and ready made solutions do not fit in the description of the dynamic reality, which is always richer than all our visions of it. In this chapter, we shall analyze these very problems which originate on the border between reality and our intuition.

2.1 Bottle Imp Paradox (Stevenson's Bottle), or the Unintuitive Character of a Conclusion Following a Sufficient Multiplication of a Simple Reasoning

Bottle imp paradox is sometimes believed to be another form of the unexpected examination paradox, discussed in the third chapter. It turns out, however, that apparently formal, relatively strict similarity between the argument structures of both paradoxes does not suffice to consider them as two versions of one and the same problem. The fundamental difference is that Stevenson's bottle paradox does not have a solution, while the executioner's paradox does, and more than one. The difference is caused by the fact that in the unexpected examination paradox we encounter a specifically circular, self-referring argument, which does not stop in the moment bottle imp paradox argument does. The name of the latter paradox comes from Robert Louis Stevenson's (1850–1894). In his 1893 story *The bottle imp*, he tells a fantastic (unusual) story about the life of Mr. Keawe. On a visit to San Francisco, Mr. Keawe, a happy man from Hawaii, bought a miraculous bottle, which instead of a drink contained an imp that fulfilled all wishes. The condition of purchase, which he accepted to his later despair, was a resale of the bottle for a

P. Łukowski, *Paradoxes*, Trends in Logic, 31,
DOI: 10.1007/978-94-007-1476-2_2, © Springer Science+Business Media B.V. 2011

lower price after some time. A breach of the condition would bring eternal damnation. Naturally, subsequent sales and purchases of the bottle with the imp were more and more difficult, since each buyer knew the conditions of purchase required by the spirit. First, there was a possibility of lowering the price in one currency. When the price reached the lowest value, there was a possibility of changing the currency and the process of price lowering could start again. However, the whole procedure had to end up in the moment when yet another careless buyer was finally unable to sell the bottle. Mr. Keawe found himself in a situation, in which no one sensible would buy the cursed bottle from him. Fortunately, his wife managed to find a drunken sailor, who agreed to buy the bottle that gave him a possibility of drinking all kinds of alcohol until death willingly accepting his eternal damnation.

The problem created by the condition of Stevenson's bottle paradox is clear. We cannot buy the bottle for a penny, for we would not be able to sell it. Nor can we buy it for two pence, as someone would have to buy it from us for a penny, and then he would not be able to sell it. It is difficult to hope we could find a buyer in such a situation. We will not buy the bottle with a genie for three pence either, as we would have to sell it for two pence or a penny, and it has been proved that in neither case the seller can find a buyer. Since we cannot buy the bottle for three pence, we cannot buy it for four pence, and so on. Repeating the reasoning a sufficient number of times we can prove that the bottle cannot be bought for any price, even a very high one. Once the sale takes place, someone must pay for it with his eternal suffering.

Obviously, such a reasoning requires an additional condition that someone irresponsibly buys the bottle with the imp and thus sets the chain of events that must subsequently occur. These events are consecutive purchases of the bottle ended with the tragedy of the last buyer. From a psychological point of view, the more expensive is the bottle the easier is the first decision to buy it. The actual impossibility to sell the bottle seems to be so distant then that one sees it as unreal: "It is something that will never happen to me and I do not care for others". However high is the first price of the bottle, the misfortune must strike someone who will not be able to find a buyer for it. The first purchase is a situation resembling the moment of knocking down the first in the line of domino blocks. Here too, successive events are unstoppable: the bricks must fall one after another.

Bottle imp paradox is therefore an example of a dilemma, whose only source is a sufficiently multiple repetition of a simple reasoning, which, though simple, leads to an unexpected conclusion after a number of runs. It is easy to note that Stevenson's bottle paradox neither hides any mystery nor results from an error, just like blocks of domino falling on one another in a long row have no other secret than the precision and determination of the man who arranged them. If we are surprised by the conclusion that it is impossible to buy the bottle for any price, it is only because in everyday life we are not used to reasoning that involves sufficiently repeated application of one and the same inference. Unavoidable character of the ultimate misfortune that befalls the final buyer of the bottle makes the paradox insoluble. If anyone buys the bottle, the tragedy cannot be averted and it comes sooner or later. This means that if we do not want anyone's doom, we should prevent the sale of the bottle no matter how high was its price.[1]

2.2 Newcomb's Paradox, or the Unintuitive Character of a Conclusion Drawn from Premises with Non-intuitive Assumption

It is believed that this paradox has played an important role in the discussion about the validity of principles that should govern rational decision making. Without questioning that fact it should be noted that the paradox itself can hardly be regarded as one of the most important problems of our thinking. It happens sometimes that a falling apple may lead to a great discovery. An apple, however, remains only an apple. Let us focus on it.

Newcomb's paradox was first published in 1969 in *Newcomb's problem and two principles of choice* by Robert Nozick. The author of the paper claims, however, that the true discoverer of the dilemma is William Newcomb, a physicist from Livermore Radiation Laboratories in California.

Newcomb's Paradox

In the experiment, we have to make a rational choice. An important role is played by an infallible predictor. The rationality of the choice is measured by maximization of profit. Two boxes are marked "A" and "B", respectively. Box A contains $1,000. Box B contains no money so far. We have to decide whether we choose B or A and B together. Our win is, therefore, the content of box B only or the sum of contents of A and B together. It turns out that the content of the box B is dependent on our choice. If we choose only B, then the predictor, who knows that in advance, will put 1 million dollars into it. If we choose both boxes, then the predictor, who knows that in advance, will not put any money to box B. Naturally, we know the rules. It is obvious that we should only choose box B, since then we win 1 million dollars. If we chose both boxes, we would win only $1000. Our desire to maximize gain makes us choose only box B. However, there is a dilemma, since on the other hand the choice of both boxes is more rational. Let c be the content of box B. If we choose only box B, our win is c dollars; if, however, we choose both boxes, then our win is $1{,}000 + c$. No matter what the predictor does, $1{,}000 + c > c$. Then, following the principle of maximal gain, we should choose both boxes.

[1] We have omitted here the question of bottle buyer's wishes. If any of the buyers, not necessarily the last one, wishes that the person who is unable to sell the bottle is spared the punishment, then one can consider the possibility of purchase of the bottle for any price with no unpleasant consequences. When we take this additional factor into account, Stevenson's bottle problem no longer resembles the unavoidably falling domino blocks.

It is believed that Newcomb's paradox occurs as an effect of a conflict of two praxeological principles: *maximize expected utility principle* (MEU) and *dominance principle* (DP).[2] The first of them says that we should choose the option, which gives us greater gains. If we follow the MEU principle, we come to a conclusion that we should only choose box B. Justifying this reasoning does not pose any doubts. It is more difficult to justify the alternative choice of boxes A and B. This is done with help of the DP principle. According to DP, if an option is prevalent in any possible situation, we should choose this very option. If we are granted c dollars in any event, we should choose the option, which gives us something more than c, i.e., 1,000 + c. So if we follow DP, we should choose both boxes. The sensibility of application of dominance principle can be illustrated in another way, more often used in the literature of the subject, sc. in a table, presented below.[3] Let us consider four situations that are resultants of two alternatives. The first alternative is determined by our choice: either we choose box B, or two boxes. The second alternative is determined by the action of the predictor: either he has put 1 million dollars to box B, or he has put no money there. Our gain is therefore a resultant of these two choices:

	the Predictor has put money in *B*	the Predictor has not put money in *B*
we open *B*	we win 1000000 \$	we win 0 \$
we open *A* i *B*	we win 1001000 \$	we win 1000 \$

With such a presentation of the problem, it seems that the choice of both boxes is more appropriate, since it brings a greater gain, both when the predictor puts 1 million dollars into *B* (for \$1,001,000 > \$1,000,000) and when he does not put any money into *B* (for \$1,000 > \$0). The choice of both boxes seems to be well justified, if we consider one more fact associated with the concept of a "*rigid designator*". Let us assume that the constant "c" represents the amount of money in the box B. Now it is rather commonly believed that c is such a "*rigid designator*", i.e., it is one and the same unchanging, constant object.[4] Such an understanding of the content of B apparently results from the following fact. Knowing in advance which of the boxes we open the predictor puts in 1 million dollars to box B or not. Making no important changes in the conditions of Nowcomb's paradox let us assume that the predictor puts in 1 million dollars to box B on the day before we make the choice and then does not touch the box any more. If we are to choose between the contents of box B or boxes A and B together on Friday, then the infallible predictor will put 1 million dollars to box B or not on Thursday. Thus, making a decision on Friday, we are facing the fact that the content of box B has already been settled for many hours (e.g., 24). This means that with no risk of losing 1 million dollars we can decide for both

[2] Sainsbury [1], s. 51–64; Hurley [2]; Kiekeben [3].

[3] Sainsbury [1], 54–55.

[4] E.g., Priest [4].

boxes, for the predictor cannot change anything then. In this sense, the content of box B is considered a rigid designator. Naturally, this opinion is false, for the infallible predictor who can see the future knows what we shall do on Friday and makes a suitable unerring decision depending on our future action. If we open only box B on Friday, the predictor knows it and puts in 1 million dollars on Thursday. In any other case the predictor will put no money into B. From our point of view, the content of box B is not fixed in advance by the predictor, but depends on our Friday choice and in this sense is no rigid designator. Moreover, our Friday choice is completely free. Naturally, we can feel certain awkwardness, which most probably results from the fact that we do not encounter infallible predictors every day. Therefore, it is rather strange for us that the content of box B has been fixed for 24 h and yet is dependent on what we shall do in a moment, to the extent that we can do whatever we may wish making use of our free will and hesitating so much that almost to the last moment we do not know what we shall do. As long as we hesitate between opening only box B or both boxes taking the opportunity of our free choice, we cannot say that from our point of view the content of box B is fixed in advance.

Consequently, in the case of Newcomb's problem, the constant character of designator c is out of question, for the value of c depends on the decision taken by the person who chooses between the content of B and the combined contents of A and B together and the choice is yet to be made. Thus, from the point of view of the person who makes the choice, c is not anything constant, but it is the point of view of the decision maker should be crucial in the assessment of correctness and rationality of the decision. A correct, i.e., a precise formalization of the analyzed problem, should operate not with one symbol "c" but with two "c_1" and "c_2". If we choose box B, then B will contain 1 million dollars, i.e., it will contain "c_1" $=$ \$1,000,000. If we choose both boxes, B will contain no money, so it will have "c_1" $=$ \$0. Instead of an inequality leading to a paradoxical conclusion $1{,}000 + c > c$, we shall have another one that causes no surprise $c_1 > 1{,}000 + c_2$, i.e., \$1,000,000 > \$1,000 + \$0. One can claim, therefore, that Newcomb's problem is a result of an error of equivocation: two different values c_1 and c_2 are substituted with one and the same c.

The incongruence of this conclusion with the analysis based on a table presented above is illusory, for two out of four cases, the shaded ones cannot occur in the accepted conditions:

w1. The predictor has put 1 million dollars into box B only when we have chosen B alone;
w2. The predictor has put no money into box B only when we have chosen both boxes.

Naturally, these two conditions exclude both our winning \$1,001,000 and winning no money at all.[5]

[5] We should note here a considerable inconsistency in justifying the rationality of choosing both boxes. If one admits the possibility of winning \$1,000,1000 (either according to the DP principle or without it), why then one does not admit the possibility of winning \$0? After all, the latter case is as inadmissible according to the principles of the problem as the former one. Therefore, if it can

It can be seen that in the case of Newcomb's problem the application of *maximize expected utility principle* should not pose any difficulties. Consequently, we should choose only box B. Does it mean that we really reject the dominance principle? Obviously, not. Moreover, it should be stressed that a logical conflict between application of MEU and DP is not possible. After all, DP is a special case of MEU. It means that we can apply MEU without applying DP but not the other way round! Indeed, if we apply *the principle that makes us choose a procedure which maximizes gains in every possible case*, then we apply especially, whether we want or not (by virtue of logic alone), *the principle that makes us choose a procedure offering greater gain*. It is no different in the case of Newcomb's problem. If the two shaded cases from the table are excluded, we are left with the two unshaded ones. Hence there is no conflict between MEU and DP, since DP has no application here, as there are no choices within two different cases, but only a choice between the two cases.

If we treat it as an actual rational decision-making problem, Newcomb's problem resembles another one that involves rationality in decision making. Let us imagine that we want to know a reply to the question, what decision I should make not to be hungry. It seems that the only answer is to eat something. The banal problem gets complicated when we realize that following four cases are possible:

z1. I shall not eat a sufficient meal and I shall stay hungry;
z2. I shall not eat a sufficient meal and I shall be no longer hungry;
z3. I shall eat a sufficient meal and I shall stay hungry;
z4. I shall eat a sufficient meal and I shall be no longer hungry.

Naturally, as in the case of Newcomb's problem, we have two excluded cases here: numbers 2 and 4. What sort of analyses can we perform if we turn a blind eye to it?

An important question has to be raised here. All discussions in the literature of the subject devoted to the problem of the way, in which someone has to put 1 million dollars into box B, are pointless, especially from the point of view of the paradox. Moreover, no infallible have to be employed for the problem. After all, it is easy to imagine a simple toy, which has two boxes, A and B, and two buttons, X and Y. Pressing X starts a mechanism that inserts a stack of banknotes and opens only box B. Pressing Y starts a mechanism that opens both boxes but does not insert any money. On web pages, one can find toys of this kind, which allow you to play many times, thanks to the "refresh" option. On such pages, there are pictures of chests. One is open and you can see a little gold on its bottom. The other is locked. Under the pictures, there are two "buttons". One is marked "I choose box B", the other "I choose both chests". When you click on the first button, chest A

(Footnote 5 continued)
happen that the predictor, knowing that we choose both boxes, puts in 1 million dollars to B anyway, it can also happen that knowing that we choose only box B he will still not put in any money. In the light of this possibility, what do the principles of rational action DP and MEU suggest to us?

closes and the locked chest B opens showing lots of gold inside. At the same time you can read a comment: "I knew you would do so! It is a sensible choice!" Pressing the other button opens the so far closed chest B, which turns to be completely empty. Usually, this option is accompanied with a scathing comment or a roar of laughter. These easy games show clearly that there is nothing interesting in Newcomb's paradox from the point of view of rational decision making.

We have shown above that Newcomb's problem results from the error of equivocality. However, it is accompanied by yet another error, which is generated by our "standard" understanding of the relation between cause and effect. This is a situation in which we are influenced by our intuition, which sometimes leads us astray. The standard understanding of the cause–effect relation assumes that cause precedes effect in time. Here, however, we have a situation, in which a cause is temporally later than effect, because of the action of the infallible predictor. As we can see, introducing a fairy tale assumption leads to fairy tale effects. There is nothing surprising, and so nothing paradoxical, that strange assumptions generate strange conclusions. It can be seen, too, that the error of equivocality we have exposed above, results from our "standard" intuitions that involve temporal conditions of cause–effect relations. This erroneous intuition is thus the source of equivocality error, and so is primary with respect to the error. This is why we have placed Newcomb's paradox in this chapter rather than in the next one.

Finally, it is impossible to overlook the psychological aspect of Newcomb's problem. Naturally, we do not intend to accept the opinion of Martin Gardner, who, in his 1982 book *Aha! Gotcha. Paradoxes to puzzle and delight*,[6] describes the choice of box B as characteristic for men and the choice of both boxes as typical of women. What we mean is that Newcomb's problem really touches on human frailties: greed, underestimating other people, tendency to cheat, etc. From this perspective, the paradox can be associated with the agony of a monkey caught in an Indian trap: a box containing a banana has a hole so small that it can pass only an empty hand through it. When a monkey wants to get the fruit, it puts its hand into the box, grabs it and cannot take it out; it gets caught because its greed does not allow it to give up the banana. Similarly, a man, who wants to win 1 million dollars, which is the equivalent of monkey's freedom, loses it, because he cannot give up $1,000 standing for the banana in the Indian trap.

2.3 Paradox of Common Birthday, or the Unintuitive Character of Some Results in Probability Calculus

Determining the probability of events by virtue of the rules of probability calculus usually leads to intuitive conclusions. It is difficult not to agree with the evidence of the fact that getting heads after one throw of a symmetric coin has the

[6] Gardner [5].

probability of ½. Usually, the results of calculating probability of events that have been obtained with recourse to the theory of probability calculus are pretty intuitive. There are cases, however, that can be startling. The following problem, known under the name of "Common birthday paradox", shows that even in such an intuitive theory one can find examples of unintuitive results.

Paradox of Common Birthday

Let us assume that there are 23 people at a party. If we were to estimate the probability of such event that two people out of the group have birthdays on the same day, then, remembering that there are 365 days in a year, we would probably say that it is not too big. And without knowledge of the calculated result of this probability, we certainly would not say that it is more probable that two out of 23 people have birthday on the same day than that there is no such pair. Yet, it turns out that the probability of such event is as high as 0.507297, so greater than 0.5. This means that the opposite event is marginally less probable[7]:

$$P_{23} = 1 - \frac{365 \cdot 364 \cdot \ldots \cdot 343}{365^{23}} = 0.507297234\ldots$$

Such a surprisingly high probability results from the fact that the group of people is sufficiently high. If, however, we calculate the probability of the event that two randomly chosen people have birthday on the same day, it turns out that it is surprisingly low:

$$P_2 = 1 - \frac{365 \cdot 364}{365^2} = 0.002739726\ldots$$

One can only suspect that this rather intuitive and hardly surprising fact of low probability of the latter event in question has some influence upon our estimation of the former, which we also treat as very improbable. All the more so that the probability of events that two people out of groups of three and four have birthday on the same day is also very low and is respectively: $P_3 = 0.0082\ldots$ and $P_4 = 0.0163\ldots$ Even if we raise the number of people to ten, the probability of the event in question will not be significantly higher: $P_{10} = 0.1169\ldots$ We can assume then that in the case of the problem of common birthday the paradox will not appear as long as the group of people we consider is not sufficiently large.

Like the paradox of Stevenson's bottle, the paradox of common birthday does not entail any specific mathematical or philosophical question, nor does it result from any error. Its only source is the fact that some mathematical calculations are

[7] For the sake of simplicity the calculation disregards the possibility of someone out of 23 being born in a leap year.

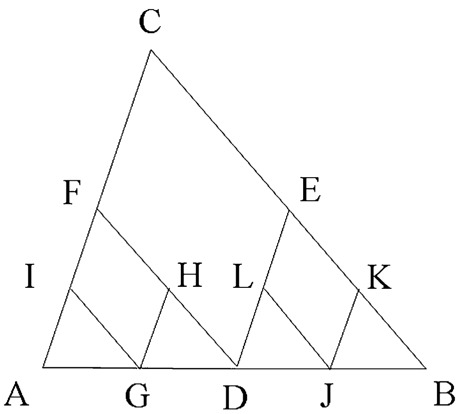
done so rarely that their conclusions seem to us unintuitive. One could suppose that for someone who for some reasons, e.g., professional ones, would have to make such calculations regularly, their conclusions would become not only intuitive but also obvious.

2.4 Paradox of Approximation and Paradox of the Equator, or the Unintuitive Character of Some Results in Euclidean geometry

One could think that in contrast to non-Euclidean geometries, the intuitive character of the Euclidean one does not raise any doubts. There are, however, instances that question the validity of this rather common opinion. The disagreement between intuition and results of precise mathematical reasoning based on simple facts of Euclidean geometry is clearly shown by the two following dilemmas. The first is known under the name of approximation paradox:

Approximation Paradox[8]

Let *ABC* be a given triangle. Let, moreover, points *D*, *E* and *F* bisect the respective sides *AB*, *BC* and *AC*. Two triangles, *ADF* and *DBE*, obtained in this way, are congruent to the triangle *ABC*. Let us assume further that points *G*, *H*, *I*, *J*, *K*, *L* bisect respective sides *AD*, *DF*, *FA*, *DB*, *BE*, *ED*. Four new triangles *AGI*, *GDH*, *DJL* and *JBK* obtained in this way are also similar to the triangle *ABC*.



[8] The argument of this paradox may serve "proving" that the sum of lengths of two sides of any undegenerated triangle equals the length of the third side.

Bisecting the sides of ever smaller triangles can be continued in infinity. Let us consider then an infinite sequence of curves *ACB*, *AFDEB*, *AIGHDLJKB* and all other ones that are obtained through further bisection of consecutive sides of ever smaller triangles. Each of these curves together with the side *AB* limits the surfaces of ever smaller triangles. The areas of surfaces limited with consecutive curves and segment *AB* form a sequence tending to zero. It could follow there from that the lengths of consecutive curves form a sequence leading to the length of *AB*. However, it results from the very construction of triangles that every curve has equal length as the *ACB* curve. Hence the lengths of the curves constitute a constant sequence. This means that there exists such a sequence of areas diminishing to zero that every area of this sequence has a circumference of identical length.[9]

Approximation paradox shows how our intuitions can delude us. Although the areas of the triangles form a sequence tending to zero, the curves that limit them still have the same length. As we can see, approximation paradox contradicts our intuitive opinion that areas forming a sequence tending to zero must have circumferences, whose lengths form a sequence tending also to zero, or at least diminishing. Paradox of the equator is another example showing that precise reasoning based on Euclidean geometry is not always intuitively predictable, even though Euclidean geometry itself is believed to be a particularly intuitive theory.

Paradox of the Equator

Let us consider two spheres K_1 and K_2. The length of the radius of the first one is $R_1 = 6300$ km, the length of the second one is $R_2 = 3$ cm. One can say then that the former is approximately the size of Earth, while the latter, the size of a tennis ball. The lengths of the greatest circumferences of spheres K_1 and K_2 are $2\pi R_1$ and $2\pi R_2$, respectively. Let us girdle both spheres along their equators with two strings, whose lengths are $2\pi R_1 + 1$ m and $2\pi R_2 + 1$ m, respectively. Since both strings are 1 m longer than the greatest circumferences of spheres R_1 and R_2, a gap will form between each of the strings and the surface of the respective sphere. Intuition tells us that the distance between the string of the length $2\pi R_1 + 1$ m and the surface of the sphere R_1 measured along the equator should be much smaller than the distance between the string of the length $2\pi R_2 + 1$ m and the surface of the sphere R_2 also measured along the equator. It turns out, however, that the two distances are the same. Let R_1' and R_2' be the lengths of radii of the circles formed by the two strings, respectively. Then,

[9] The Paradox of Approximation does not refer only to the sequence of appropriately constructed triangles. A similarly surprising conclusion can be obtained using an appropriate sequence of semi-circles, trapezoids, etc.

$$R_1' = \frac{2\pi R_1 + 1\ \text{m}}{2\pi} \text{ and } R_2' = \frac{2\pi R_2 + 1\ \text{m}}{2\pi}$$

So,

$$R_1' = R_1 + \frac{1\ \text{m}}{2\pi} \text{ and } R_2' = R_2 + \frac{1\ \text{m}}{2\pi}$$

This means that in both casus the distance between the string and the surface of the sphere is $\frac{1}{2\pi}$ m, that is approximately 16 cm, and apparently is not dependent on the radius of the sphere but only on the added distance.

2.5 Horses' Paradox, or the Intuitive Character of an Erroneous Application of Mathematical Induction

Horses' paradox serves as example that mathematics can be used in an unreasonable though seemingly intuitive way. The following argument, erroneous yet apparently evident, makes use of the principle of mathematical induction.

Horses Paradox

We shall prove that all horses are of the same color. We shall use mathematical induction with respect to the number of horses. Let us check the first inductive step—a set of 1 h is a set of horses of the same color. We assume now that (for a given n) all horses in an n-element set of horses are of the same color. Let us add a new horse to any n-element set. We have a $(n + 1)$-element set. Now let us take away a horse from the set, but not the one we have just added. We get a n-element set of horses. From the inductive assumption, all horses in this set are of the same color. Accordingly, the horse we have added is of the same colour as other ones. Now we can bring back the eliminated horse (which is obviously of the same color as the rest) an we get a $(n + 1)$-element set of horses of the same color. By virtue of mathematical induction we have proved that all horses are of the same color.

It is obviously true that every one-element set of horses is a set of horses of the same color, so the induction check for $n = 1$ is successful. It is an obvious step in reasoning. Yet, the intuitive reasoning fails already at the second step and only on that step. Let us assume that we have a one-element set of horses made up of a black horse. We add to it a white horse and lead away the black one. We have a one-element set of horses again, this time made up of a white horse. Although every one-element set of horses is indeed a set of horses of the same color, adding back the black one to the set made up of the white one gives a two-element set of horses, which is not a set of horses of the same color. Although the inductive assumption is undoubtedly true for $n = 1$, sc. that every n-element set of horses is a set of horses of the same color, it is not true that this assumption results in an inductive proposition that every $(n + 1)$-element set of horses is a set of horses of the same color. Naturally, if we assume an inductive assumption for $n \geq 2$ that every n-element set of horses is a set of horses of the same color, we shall prove an inductive proposition that every $(n + 1)$-element set of horses is a set of horses of the same color. However, we cannot claim truthfully that every two-element set of horses is a set of horses of the same color. Such a proposition would be equivalent to an assumption that all horses are of the same color: since all two-element sets of horses are sets of horses of the same color, then not only $\{a, b\}$ and $\{c \text{ and } d\}$ but also $\{a, c\}$ are sets of horses of the same color. This means that horses a, b, c, d are horses of the same color. It can be seen that a careless though intuitive application of mathematical induction serves to prove a previously assumed proposition, i.e., it suffers from a petitio principi: it is proved that all horses are of the same color on the assumption that every two horses are of the same color.

2.6 Hempel's Paradox (Raven, Confirmation), or the Unintuitive Character of Some Inductive Reasoning Results

The argument of Raven's paradox, based on a simple inductive reasoning has been discovered by a modern philosopher Carl Gustav Hempel (1905–1997), a representative of logical positivism, as one of the effects of his research concerning confirmation of propositions. This is why the other name of this dilemma is *confirmation paradox*.

Black Raven Paradox

It is a paradox concerning reasoning with help of inductive logic. According to the principles of induction every time we see that a given proposition is true, our feeling that it is true is increased. Accordingly, if the proposition is, e.g., "all ravens are black", and we see a raven, which happens to be black indeed, our trust for the truth of the proposition increases. But such a proposition is equivalent to this one: "whatever is not black, is not a raven". So, if we see, e.g., a white shoe, our trust that all ravens are black should increase too, which is a very unintuitive conclusion.

True, if something that is white turns out to be a shoe rather than a raven, our trust that all ravens are black increases. From the point of view of the proposition, its confirmation is both the fact that every observed raven is black and the fact that every observed nonblack object is a non-raven. This is because the proposition is refuted both when we see a raven that is not black and when we see that the nonblack object turns out to be a non-raven. In both cases the falsification can be reduced to finding the very same object, i.e., a nonblack raven.[10] This fact results directly from classical logic.

Let predicates R and B stand for being a raven and being black, respectively. The proposition

$$\forall x\,(R\,(x) \rightarrow B\,(x))$$

is equivalent of a proposition

$$\forall x\,(\neg B\,(x) \rightarrow \neg R\,(x)).$$

The negation of the former is

$$\exists x\,(R(x)\ \wedge \neg B\,(x))$$

and of the latter is

$$\exists x\,(\neg B\,(x) \wedge \neg\neg R\,(x)).$$

With respect to the double negation and commutability principles both generally quantified propositions are falsified by the same object a_0, for which it is true that

$$R\,(a_0) \wedge \neg B\,(a_0).$$

From the logical point of view, there is no difference if we confirm the proposition "all ravens are black" by checking if an observed raven is black, or by checking if an observed nonblack object is a non-raven. Either way, we make sure of the same, i.e., a nonblack raven does not exist. The fact, however, that our intuition suggests an inequality of confirmations of both kinds, probably results from a reflection that it is much easier to concentrate on a smaller number of verified cases than on a great one. In the world, there are many more nonblack objects than ravens. It is a serious argument for the rationality of one of the checks of confirmation. This rationality evaluation, however, does not follow from the fact that one verification is logical, and the other is illogical. Both are equally logical because they are logically equivalent.

[10] Let us assume that in some box, there are only spheres and cubes. Moreover, we know that every solid has one of two colors: black or white. In order to verify the truth of the statement "All spheres are black", it is enough to check if all spheres are black; it may also be sufficient if we check whether all non-black, i.e., white, objects are non-spheres, i.e., cubes. Every checking, no matter which of the two methods we take, will be equally good. It is also possible to join both methods.

Thus it can be noted that in the case of Raven's paradox our intuition is wrong if it suggests a logical superiority of one confirmation over another one but it is rational if it finds one of the confirmations to be easier in execution.[11] In the argument of Hempel's problem, however, we do not say that some confirmations are more valuable than others, but only that some observations happen to be valid confirmations of universal propositions in general. The problem is thus seen from the point of view of logicality and not practicality. In this sense Hempel's paradox is a result of our imperfect intuition.

It must be stressed that the author of this solution is Hempel himself. The white shoe of the analysis above is directly linked to Hempel's proposal.[12]

It turns out that this opinion is not universally shared by philosophers. For instance, Sainsbury, in his *Paradoxes*, rejects Hempel's argumentation and offers his own. He claims that the paradox is the result of a simultaneous approval of two principles responsible for rationality of opinions. They are[13]:

$E1$: *If two hypotheses can be known* a priori to be equivalent, then any data that confirm one confirm the other.

$G1$: *A generalization is confirmed by any of its instances.*

According to Sainsbury, the solution of Hempel's problem may lie in the rejection of one of the principles: $E1$ or $G1$. He also admits that one can, as Hempel did it, accept the final conclusion of the argument as a correct and de facto non-paradoxical. He thinks, however, that such a standpoint requires necessarily inventing some complicated story that would justify the final proposition, like the one that claims that seeing a white shoe confirms the truth of a proposition "All ravens are black". Discussing the sensibility of rejecting a rather obvious principle $E1$ and no less obvious principle $G1$ Sainsbury finally opts for the latter one but admits that the price he pays for solving of Hempel's paradox in this way is very high.[14]

As we can see, the case of Hempel's paradox is a good illustration of two opposite attitudes. The first is a constant development of intuition through analysis of a given problem and keeping the pace of the progress of science. The other lies in the defense of one's own intuition, i.e., feelings and impressions for the steep price of questioning the principles that are evident from the logical point of view.

[11] When we look for house keys, it is more rational to search the pockets rather than the fridge. It does not mean, however, that it is logically impossible that the keys are in the fridge.

[12] Sainsbury [1], p. 80–81.

[13] Sainsbury [1], p. 75, 78–80.

[14] Sainbury [1], p. 81–82.

2.7 Paradoxes of Infinity, or Unintuitive Character of Some Results Obtained in Set Theory

The awareness of at least potential existence of infinite magnitudes was a natural consequence of operations on natural numbers and points that formed such geometrical objects as a line, segment, surface, etc. It is no wonder then that it appeared already in the Antiquity, giving rise to many understandable, as it later turned out, controversies. An interesting, popular historical survey of man's struggles with the concept of infinity can be found in such books as, e.g., *The Mystery of the Aleph* by Amir D. Aczel, or *Achilles in the Quantum Universe. The Definitive History of Infinity* by Richard Morris.[15] Let us leave aside, however, this otherwise interesting historical trait and focus on some concrete difficulties noticed by philosophers and logicians. Even Aristotle himself encountered a problem, which he was unable to solve. Now we know why. The solution of the Aristotelian Circles Paradox required a definition of an infinite set.

2.7.1 Aristotelian Circles Paradox, or the Definition of an Infinite Set

Studying the literature devoted to paradoxes one can hardly find even a mention of a very interesting problem referring to two coaxial circles moving with the same radial speed. Aristotelian[16] formulation of the paradox brought to light a problem, which had to wait for its solution more than two millennia, i.e., until the development of set theory made it possible to unravel it. Contrary to all appearance, the problem does not belong to geometry and even less so, to classical mechanics, but lies in the realm of set theory. In the single book of Aristotle's *Mechanics*, we can read[17]:

Aristotle's Circles Paradox

"There is a question why a large circle traces out a path equal to that of a smaller circle, when they are placed about the same centre, but when they are rolled separately, their paths are to one another in the proportion of their dimensions. And, further, the centre of both being one and the same, at one time the path which they trace is of the same length as the smaller traces out alone, and at another time of the length which the larger circle traces. Now it is manifest that the larger circle traces out the longer path. For by mere observation it is plain that the angle which the circumference of each

[15] Aczel [6], Morris [7]

[16] Aristotle was not the author of *Mechanics*, but the work is clearly Aristotelian in character.

[17] Aristotle, *Mechanics* (translated by E.S. Foster), 24, 855a–855b.

makeswith its own diameter is greater in the case of the larger circle than in the smaller; so that, by observation, the paths along which they roll will have this same proportion to one another. But, in fact, it is manifest that, when they are situated about the same centre, this is not so, but they trace out an equal path; so that it comes to this, that in the one case the path is equal to that traced by the larger circle, in the other to that traced by the smaller."

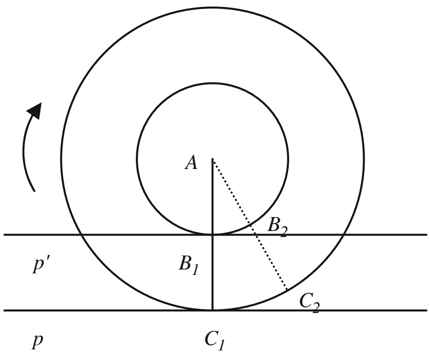
Two circles, both having center in point A and radii r and R ($r < R$) are connected so that they form one object. Consequently, neither can move without the other one. Let us mark a point on the circumferences of both circles[18]: point B on the smaller circle and point C on the larger one, in such a way that C is a tangential point of the larger circle and line p, while B is a common point of the section AC and the smaller circle. Thus, $r = AB$ and $R = AC$. Let us assume now that the larger circle starts to roll along line p in the direction pointed by the arrow. The movement lasts until a complete revolution of the larger circle, i.e., until point C touches line p again. After a complete rotation point C covers point C'. Consequently, the section CC' is the distance covered by the larger circle in the full revolution along line p. This means that $CC' = 2\pi R$. However, the smaller circle, which forms the same object with the larger one, has completed a full revolution along the line p' so that point B has covered point B' in the same moment as C has covered C'. So, $BB' = 2\pi r$. If $BB' = CC'$, then $2\pi r = 2\pi R$, consequently $r = R$, which, however, is contrary to the assumption.

In the reasoning above, we have accepted the revolution of the larger circle as our point of reference and we have proved that the smaller one, in its motion together with the larger one, had to cover the same distance as the larger one, i.e., a path that is longer than the one it would have covered separately. In accordance with the Aristotelian quotation, the whole problem can be reversed, and we can take the revolution of the smaller circle for our point of reference. Then it turns out that the larger circle, moving together with the smaller one covers a distance, which is shorter than $2\pi R$. Naturally, in both instances we come to the same contradiction: $r \neq R$ and $r = R$.

[18] Naturally, circumference is a part of a circle, being its border. I use the words 'circle' and 'circumference' for the sake of precision.

Let us precede the solution of the paradox with a comment that there is no doubt that the conclusion asserting equality of both radii is inadmissible. We can be sure, however, that in the first case, i.e., the revolution of the larger circle only accompanied by the smaller one, both circles cover the distance of $2\pi R$ during a full revolution of the larger circle. In the second case, when the smaller circle revolves accompanied by the larger one, both circles cover the distance of $2\pi r$ after one full revolution of the smaller circle. Now what is the reason why the accompanying circle "adjusts" its full revolution to a different path than the one that results from the length of its own radius? Apparently, the solution to this question cannot be found in the analysis of the problem from a geometrical or mechanical point of view. The cause of this interesting and non-trivial paradox lies neither in geometry nor in mechanics. Its proper solution can be given only by the set theory.

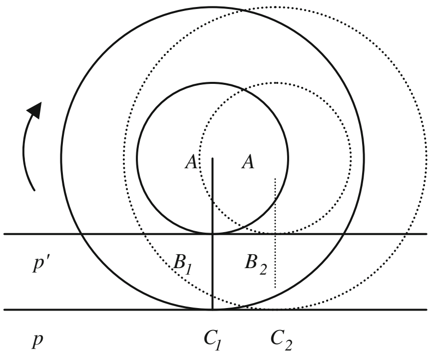
Let us note that simultaneous circular motion of both circles is an actual rotation of one and a motion of the other, which is neither slipping nor friction. To prove this proposition, let us consider two instances. In the first one, the larger circle rolls and the smaller one accompanies, in the second one, the larger one accompanies the revolution of the smaller one.



First instance. Since the smaller circle accompanies the motion of the larger one, it covers longer distance on line p' than it would if it revolved on its own. One could suspect here that it slips while rolling on. We could speak of slipping if a point of that circle were tangential to more than one point of the line p'. Let us assume then that one point of the smaller circle is tangential to points of line p' that form a section, whose length is more than zero and its limits are points B_1 and B_2. Consequently, $B_1 \neq B_2$. A point is tangential to a line when the line is perpendicular to the section that connects the line with the center of the circle. Consequently, both sections AB_1 and AB_2 are perpendicular to line p'. On the crossing of AB_1 and AB_2 with the larger circle we find two different points C_1 and C_2. They must be different because points B_1 and B_2 are different, since it is not possible that with A, B_1 and C_1 being co-linear and A, B_2 and C_2 being co-linear we could have a situation in which $B_1 \neq B_2$ and $C_1 = C_2$. Consequently, $C_1 \neq C_2$.[19] Naturally, in

[19] This argument resembles the one used by John Duns Scotus against the atomistic vision of the world. Cf. paragraph 5.2.3 *Paradoxes of motion: Dychotomy, Achilles and Tortoise, Arrow* and [8].

those points the larger circle is tangential to line p, since sections AC_1 and AC_2 are perpendicular to line p. This means that the larger circle slips on line p. Consequently, slipping of the smaller circle implies the slipping of the larger one.



Second instance. This time the larger circle accompanies the revolution of the smaller one, covering on line p a path, which is smaller than the distance it would cover when revolving on its own. One could suspect here that it slips while moving on. We could speak of slipping of the larger circle if a point of the line p were tangential to more than one point of the circle. Let us assume it is so, i.e., one point on line p is tangential to more than one point of the larger circle and that these points form a section whose length is more than zero. Let the limits of the section be points C_1 and C_2. From this assumption, we know that $C_1 \neq C_2$. Points B_1 and B_2 are situated on the sections AC_1 and AC_2, respectively, marking their crossing of the smaller circle. Naturally, $B_1 \neq B_2$. When AC_1 is perpendicular to line p, AB_1 is perpendicular to line p'. Moreover, since line p is tangential to every point of the section C_1C_2 of the larger circle only in one corresponding point on line p, also line p' is tangential to every point of the non-zero section B_1B_2 of the smaller circle in one corresponding point. This means that one point of line p' is tangential to every point of the non-zero section of the smaller circle. Consequently, the smaller circle would have to experience friction. Thus we have shown that friction of the larger circle implies friction of the smaller one.

These two analyses show that in the Aristotelian problem neither slipping nor friction is possible. The coordination of points from both circles to the points from respective sections has a one-to-one character. Motion of a circle along a respective straight line is a subordination of every single point on a section of the line, the length of which is $2\pi r$ or $2\pi R$, to a corresponding point on the respective circle. Rolling a circle along a respective line is equivalent to defining a one-to-one and "onto" function, in short a 1–1 function, which coordinates all points of a given circle to all points of a section, whose length is $2\pi r$ or $2\pi R$. If we consider a case, in which the larger circle rolls "accompanied" by the smaller one, one complete revolution of the larger one, and the smaller one too, furnishes a proof that the circumferences of both circles have the same number of points as the section of the length $2\pi R$. If, however, the larger circle accompanies the revolution of the smaller one, then we get a proof that both circles have the same number of points as the section of the length $2\pi r$. Taking both proofs together we learn: 1 that a section of the length $2\pi R$ has as many points as a section of the length $2\pi r$ and 2 that there is the same number of points on the circumference of both circles.

It is difficult to blame the author of *Mechanics* for ignorance of the fact that the thesis the proof of which he had formulated would, after over twenty centuries, become an obvious thesis of the set theory, stating that the circumference of every circle has the same number of points (every section has the same number of points), which is to say that any two circles (sections) are equipollent sets of points.[20]

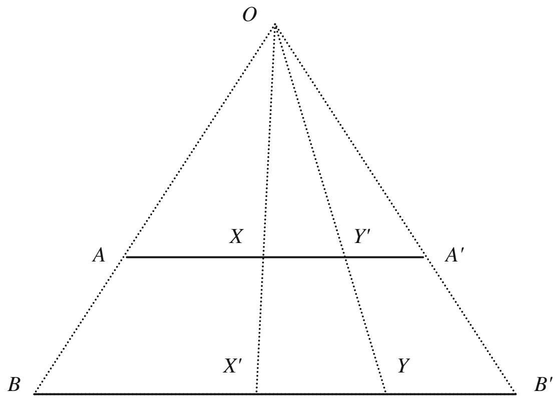
Aristotelian argument resembles another, slightly simpler but also geometric proof that every section has the same number of points. Let us consider two parallel sections AA' and BB' lying on the same plane, such that $AA' < BB'$. Thus lines AB and $A'B'$ have one common point. Let us call it O. By projecting the section AA' onto BB' from the point O we coordinate every point X of section AA' to one and only one point X' of section BB'. In turn, by projecting the section BB' onto AA' also from the same point O we coordinate every point Y of section BB' to one and only one point Y' of section AA'. Defining both of these operations proves a well known thesis of set theory that both sections are equipollent sets of points.[21] Naturally, points of both sections that form equipollent sets are understood as *Dedekind's cut*, i.e., such cuttings of a line, section, etc. into two, which produce no loss, do not use up the line: *everything that was before the cutting is there still, on one or the other side of the cutting*.[22] It is a specific paradox, yet a common belief among mathematicians, that a line, section, etc. are composed of points conceived in such a way. It means that something can be composed of nothing provided that it is repeated sufficiently many times, i.e., infinitely many times - that something that is the cutting of a line is to be its component. Let us take this extraordinary fact to formulate a paradox, which we can call Euclid's Paradox or Dedekind's Cut Paradox:

---

[20] Set A is *equipollent* with set B if and only if there exists a function, which one-to-one translates set A onto set B. We say that such function establishes the equipollence of set A with set B.

[21] This argument, just like other ones proving equipollence of such set of points as section or line can be found in [9], p. 34–39. Traditionally, Hunter "proves" also equipollence of the set of points that form a section with the sets of points forming the interior of a square, semi-plane, plane, interior of a cube, semi-space and space. These proofs are not as evident, for they lack purely geometrical character, but are based on a rather doubtful assumption that the number of points on a line is equal to the number of real numbers. The doubtful character of this proposition results from the fact that the construction of a line is an exact copy of the construction of real numbers. There can hardly be an important difference between sets, which have constructions differing in name only.

[22] We use here an already traditional name "Dedekind's cut". Dedekind, however, described that concept algebraically and in geometric sense it was introduced by Eudoxos of Knidos (408–355 b.Ch.).



Euclid's (Dedekind's Cut) Paradox

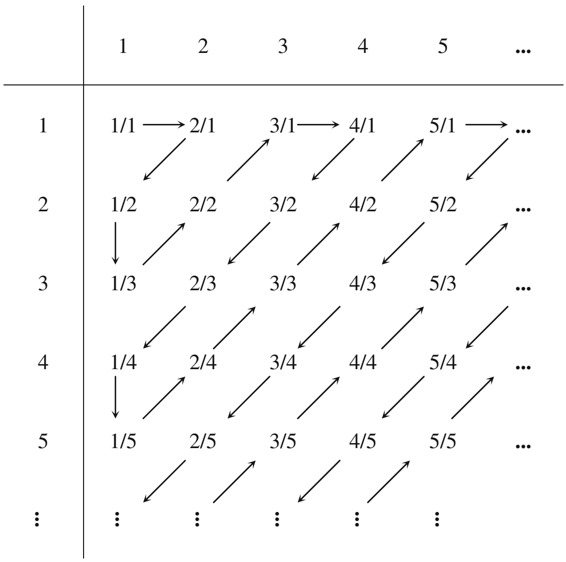
Non-dimensional point is a Dedekind's bisection and a component, i.e., a part, of a line at the same time. In other words, a line is composed of all of its possible sections, each of which divides it into two without loss.

Coming back to the Aristotelian Circles Paradox one can note that it can be substituted with a much simpler argument that will prove equipollence of any two circles. Let us consider a situation presented on the illustration. Point A is the center of two circles. Dotted lines symbolize simple operations of one-to-one projecting of the smaller circle on the larger circle, and thus also the larger circle on the smaller circle. Consequently, the mediation through sections of the same length is eliminated in this argument.

The arguments we encounter in the problem of Aristotelian circles and all other proofs showing equipollence of a set with one of its proper subsets[23] seem paradoxical as long as we believe that every set has such a property that elimination of one of its elements must invariably result in a decrease of the number of elements of that set. Such an intuitive property, however, characterizes finite sets only. True, if we take out one element from an n-element set, where n is any natural number greater than zero, we get a set that has $n-1$ elements, sc. clearly fewer than n. However, if the set is infinite, elimination of some of its elements, sometimes even an infinite number of them, may not affect the number of elements of the original set. One of the easier examples of it is the elimination of every second

[23] Set A is *a subset of set* B, when every element of set A is an element of set B. Set A is *a proper subset of set* B, when A is a subset of B, and, moreover, in set B there exists an element which is not an element of set A. Improper inclusion is an inclusion of a set in itself.

number from the set of all natural numbers $\boldsymbol{N}$, starting from 1. In this way, we shall get a set $A = \{2n: n \in \boldsymbol{N}\}$, which is equipollent with the set $\boldsymbol{N}$. A function that sets this equipollence is $f: \boldsymbol{N} \to A$ such that $f(n) = 2n$, for $n \in \boldsymbol{N}$. Another proof is the one discovered by Cantor, which shows the equipollence of set $\boldsymbol{N}$ with the set of all rational numbers $\boldsymbol{Q}$. Cantor devised a simple method of ordering all rational numbers into a sequence. The very ordering of the set $\boldsymbol{Q}$ is a sufficient proof of the equipollence of this set with the set $\boldsymbol{N}$, for it means a subordination, to every number from $\boldsymbol{Q}$, precisely one and always different index, which is, after all, a number from $\boldsymbol{N}$. An intermediary step is Cantor's ordering of all positive rational numbers. For this purpose, he used a Carthesian product $(\boldsymbol{N} - \{0\})^2$ of the set of natural numbers without zero. Such ordering defines a function, which maps the whole set $\boldsymbol{Q}^+$ on the set $\boldsymbol{N}$. Formulation of a reverse mapping is trivial. The arrows that join rational numbers inform about the order, in which the numbers appear in the formed sequence. Ordering all positive rational numbers into a sequence does not, obviously, end the proof of equipollence of sets $\boldsymbol{Q}$ and $\boldsymbol{N}$. One can, however, order into a sequence the set of all negative rational numbers in a similar way. Next, using both sequences, one can form a new sequence, in which the elements with even indices form the familiar sequence of positive rational numbers and the elements with odd indices form the sequence of negative rational numbers. Adding zero poses no difficulty either. Thus the set of rational numbers is really equipollent with the set of natural numbers.



This way, the research of infinite sets resulted in the formulation of a definition of an infinite set, which can hardly be called intuitive and is sometimes still treated as paradoxical:

Dedekind's and Peirce's Definition of Infinite Set

We call a set infinite if and only if it is equipollent with one of its proper subsets.[24]

Paradoxical character of this definition results from the fact that it contradicts these very intuitions, which make us believe that a part is always smaller than the whole. Yet, according to the definition of an infinite set there such wholes, like infinite sets, which have some parts that are not smaller than they are themselves. Moreover, a proper subset of an infinite set may be identical with that set. Let us imagine a set of identical balls, which is equipollent with the set of natural numbers. Assume that the balls lie before us on an infinitely long table in an infinitely long row: O, O, O, O, O, … Taking out every second ball from the row we form another one. As a result, we get two proper subsets of the original set, which do not differ from one another and, moreover, do not differ from the original set: O, O, O, O, O, … and O, O, O, O, O, …

It is widely believed that the first person to present a proof that an infinite set has such a counterintuitive property, sc. having the same number of elements as its proper subset, was Galileo (1564–1642). He came to this conclusion joining consecutive natural numbers into pairs with their squares: (1,1), (2,4), (3,9), (4,16), (5, 25), … Unfortunately, in his 1638 treatise *Dialogues concerning Two New Sciences* (Macmillan, New York 1914) he did not dare to state directly that the set of "square numbers" has as many elements as the set of natural numbers. In the work, which is a dialogue between the intelligent Salviati and somewhat less brilliant Simplicio, the former says only that the number of squares of natural numbers is not smaller than the number of natural numbers,[25] which is an obvious way of saying that both sets have the same number of elements. Galileo's cautious form of presentation of his discovery makes us hesitate whether the honor for the discovery of contradiction between the concept of infinity and our intuitions should not go to William of Ockham, who had included an interesting and clearly paradoxical argument in his *Quodlibeta* (II 5).[26]

---

[24] This is a definition of a so-called *reflexive set*, also known as an *infinite set in Dedekind's sense*. Dedekind and Peirce were its authors independently from one another, but it is believed that the definition was anticipated by earlier scientists, such as Galileo or Leibniz. Another definition of an infinite set, equivalent, as it turned out much later, to the one proposed by Dedekind and Peirce was given by Russell, who defined a so called *inductive set*: set Z is called inductive when there is a natural number n, such that Z has precisely n elements. Next, Russell proved an inductive theorem that an inductive set is not equipollent with any of its proper subsets, and finantly defined an infinite set as a set that is not inductive; Cf. Marciszewski, *Aksjomatyczne ujęcie teorii mnogości*, [in:] [10], pp. 124–125.

[25] Aczel [6], p. 50.

[26] Murdoch [11] p. 173.

Ockham's Paradox

Let us consider eternal duration of the world, represented by a line without a beginning and end. Let us mark two points on this line: t_1, standing for the beginning of today, and t_2, marking the end of today. Let A stand for this part of infinite time, which lasted since infinite past till the moment t_1; B in turn, this part of infinite time, which starts in the moment t_1 and will last forever. In a similar way, let point t_2 designate infinite parts C and D, such that A is a part of C, and D, a part of B. This way, we get four rays: A and B beginning in point t_1 and C and D beginning in point t_2 such that $D \subset B$.

Let $l(x)$ be the length of x. Let us note now that equality $l(A) = l(B)$ is an obvious fact. Moreover, since $D \subset B$, the inequality $l(B) > l(D)$ is similarly obvious. The direct consequence of these two facts is $l(A) > l(D)$. However, since $l(C) = l(D)$, $l(A) > l(C)$. Yet, $A \subset C$. This means that in case of infinite objects, a part is larger than the whole.

Apparently, Ockham's Paradox seems to be easy to solve, if we apply the definition of an infinite set, from which it clearly follows that the length of a line is equal to the length of its both rays. What is the problem, however, is that we have no knowledge about the infinite magnitude of that length. Infinite numbers we operate with are used to express the number of elements but not measures of length, space, or volume. Consequently, beside the problem that is obvious today, the dilemma concerns also a question, which still awaits a solution. It is easy to note that neither the number representing the quantity of natural numbers, nor the number representing the quantity of real numbers, can signify the length of a line: for any natural number, there are infinitely many real numbers that are greater than it is. Moreover, if we assume that every number, and especially an infinite number, should have its place on the axis, the problem starts looking especially attractive.

The next paradox is closely linked to the definition of an infinite set but goes beyond the scope of mathematics.

2.7.2 Holy Trinity Paradox, or Application of the Definition of Infinite Set in Theology

The problem of Holy Trinity has theological, philosophical and logical dimensions. Its complexity allows to start its analysis from various points of view. One of them is, for instance, the identity of persons of the Holy Trinity.[27] We shall be

[27] See ref [12–17].

interested in another issue, however, viz. the one which concerns the idea of non-contradictory character of three different beings, each of which is included in another one, so that every one includes the remaining two. In other words, we shall analyze a question whether logic allows a possibility that three different beings form a unity that is included in everyone of the three beings. Let us present then a non-contradictory mathematical construction of such an object, leaving aside all other theological and philosophical issues related to the concept of the holy Trinity in order to make the complex problem simpler.

One of the greatest authorities in logic who clearly argued for the non-contradictory character of the concept of the holy Trinity was Jan Łukasiewicz (1878–1956). In his 1910 book *On the Principle of Contradiction in Aristotle*, he writes about his feeling about it [28]:

> "I have read the simple and mighty words of the Athanasian Creed, his wonderful hymn about the Trinity. There is no patent contradiction in it, nor is there a latent one if the words are interpreted in accordance with theology. Yet, whoever yields to the religious-esthetic charm of this poetry without thinking about theological problems, will have a momentary feeling that he believes in two propositions that seem to be contradictory. The awe inspiring words, joined in identically built, metric sentences, ring powerfully: For there is one person of the Father, another of the Son, and another of the Holy Spirit. But the Godhead of the Father, of the Son, and of the Holy Spirit is all one, the glory equal, the majesty coeternal. Such as the Father is, such is the Son, and such is the Holy Spirit. The Father uncreated, the Son uncreated, and the Holy Spirit uncreated. The Father incomprehensible, the Son incomprehensible, and the Holy Spirit incomprehensible. The Father eternal, the Son eternal, and the Holy Spirit eternal. And yet they are not three eternals but one eternal. As also there are not three uncreated nor three incomprehensible, but one uncreated and one incomprehensible. So likewise the Father is almighty, the Son almighty, and the Holy Spirit almighty. And yet they are not three almighties, but one almighty. So the Father is God, the Son is God, and the Holy Spirit is God; And yet they are not three Gods, but one God. So likewise the Father is Lord, the Son Lord, and the Holy Spirit Lord; And yet they are not three Lords but one Lord". The mind of a believer, who takes these words simply and does not analyse their theological content while concentrating on them in reading is filled with the feeling of unfathomable mystery. He believes that there are three persons in God, and each of them is a true God, yet at the same time he believes that there are not three Gods but one uncreated, immeasurable, all-powerful and eternal God. I think that these acts of faith that refer to apparently contradictory propositions, engender the feeling of mystery and awe. Such feelings must have inspired some theologians, like cardinal Nicolas of Cusa, to look for contradiction in the concept of God and see him as *coincidentia oppositorum*".

It is true that the words quoted above were used by Łukasiewicz to question and finally to reject the psychological principle of contradiction. As a result, he is later concentrated only on the ontological and logical principle of contradiction. Let us remind, following Łukasiewicz, that according to the ontological principle of consistency no object can possess one and the same property at the same time, and according to the logical principle of consistency no two propositions of which one attributes to the object the very same property that the other one denies to it can be

[28] Łukasiewicz [18], p. 35–36.

true at the same time.[29] Later, Łukasiewicz notices equivalence of the two principles and claims that it is only possible to accept or reject them both.

The teaching of the Catholic Church shows that the truth of the holy Trinity is clearly at loggerheads with our intuitions. The unintuitive character of the dogma is the reason for including it among the paradoxes analyzed in this chapter, i.e., for treating it as a dilemma resulting from the limitations of our intuitions. Fathoming the truth of the holy Trinity may indeed be a problem exceeding human capabilities, but the problem of non-contradictory character of that truth is a logical challenge, which may encourage people to try to build a non-contradictory mathematical model of the holy Trinity. The existence of such a model would be a proof of the non-contradictory character of the concept. Let us precede the attempt at building such model with a clear and precise formulation of the problem. To express it properly, we shall formulate the paradox of the holy Trinity with help of appropriate sections of the Catechism of the Catholic Church[30]:

Paradox of the Holy Trinity

253 *The Trinity is One*. We do not confess three Gods, but one God in three persons, the "consubstantial Trinity". The divine persons do not share the one divinity among themselves but each of them is God whole and entire: "The Father is that which the Son is, the Son that which the Father is, the Father and the Son that which the Holy Spirit is, i.e., by nature one God." In the words of the Fourth Lateran Council (1215), "Each of the persons is that supreme reality, viz., the divine substance, essence or nature."

254 *The divine persons are really distinct from one another*. "God is one but not solitary." "Father", "Son", "Holy Spirit" are not simply names designating modalities of the divine being, for they are really distinct from one another: "He is not the Father who is the Son, nor is the Son he who is the Father, nor is the Holy Spirit he who is the Father or the Son." They are distinct from one another in their relations of origin: "It is the Father who generates, the Son who is begotten, and the Holy Spirit who proceeds." The divine Unity is Triune.

255 *The divine persons are relative to one another*. Because it does not divide the divine unity, the real distinction of the persons from one another resides solely in the relationships which relate them to one another: "In the relational names of the persons the Father is related to the Son, the Son to the Father, and the Holy Spirit to both. While they are called three persons in view of their relations, we believe in one nature or substance." Indeed "everything (in them) is one where there is no opposition of relationship." "Because of that unity the

[29] Łukasiewicz [18], s. 37.

[30] *Catechism of Catholic Church*, points 253, 254, 255, s. 69.

Father is wholly in the Son and wholly in the Holy Spirit; the Son is wholly in the Father and wholly in the Holy Spirit; the Holy Spirit is wholly in the Father and wholly in the Son."

It follows from point 253 that every person of the Trinity is God whole and entire, so the Trinity should be a being every part of which is itself, i.e., the whole. Naturally, in the finite world it is impossible to find an object, whose proper part, sc. one that results from elimination of something that belongs to the object, would still be the whole object. It is well known, however, that such an unintuitive property not only characterizes but even defines infinite sets. Associating God with infinity has a long established tradition in the Judeo-Christian tradition.[31] The suggestion that we should see God as infinite can be found in the section of the Catechism immediately following the ones we quoted:

256 St. Gregory of Nazianzus, also called "the Theologian", entrusts this summary of Trinitarian faith to the catechumens of Constantinople:

> […] I give you but one divinity and power, existing one in three, and containing the three in a distinct way. Divinity without disparity of substance or nature, without superior degree that raises up or inferior degree that casts down… the infinite co-naturality of three infinites. Each person considered in himself is entirely God… the three considered together… […][32]

A simple division of an infinite set into three sets equipollent with the original one[33] is not sufficient to construct the model, since each person of the Trinity is God whole. This means that every subset should include the whole set. It is easy to note that a rational infinite set of identical balls: $K = \{O, O, O, O, O,\ldots\}$ can be divided into three subsets, identical with the original one. This idea is inappropriate, however, since set K is composed not of three but an infinite number of its repetitions. Accordingly, it is not a model for three objects but for its infinite number. Moreover, each of these objects is indistinguishable from the other ones, which is contrary to the assumption that "He is not the Father who is the Son, nor is the Son he who is the Father, nor is the Holy Spirit he who is the Father or the Son" (CCC 254). Moreover, the distinction has its roots in the mutual relations of the three persons of the Trinity. One should, therefore, construct a three element model, in which the objects are identical with one another, yet their origin is different. In other words, they should be three objects, equal with one another, but different relationally.

---

31 Aczel [6]

32 *Catechism of Catholic Church*, point 256. Quotation from St. Gregory of Nazianzen, *Orationes*, 40, 41: PG 36, 417.

33 For instance, the set of natural numbers $\boldsymbol{N}$ can be divided into three subsets that are equipollent with it: $A = \{k = 3n\text{: } n \in \boldsymbol{N}\}$, $B = \{k = 3n + 1\text{: } n \in \boldsymbol{N}\}$, $C = \boldsymbol{N} - (A \cup B)$. However, none of these three sets contains set $\boldsymbol{N}$. Besides, such construction rather than for three objects would be a model for four ones: A, B, C, $\boldsymbol{N}$.

Let us consider three sequences with elements belonging to a set {1,2,3}, defined as follows:

$$a_n = \begin{cases} 1, & \text{for} = 4\,k \text{ or } n = 4\,k + 1; \\ 2, & \text{for} = 4\,k + 2; \\ 3, & \text{for} = 4\,k + 3; \end{cases}$$

$$b_n = \begin{cases} 1, & \text{for} \quad n = 4\,k; \\ 2, & \text{for} \quad n = 4k + 1 \text{ or } n = 4\,k + 2; \\ 3, & \text{for} \quad n = 4\,k + 3; \end{cases}$$

$$c_n = \begin{cases} 1, & \text{for} = 4\,k; \\ 2, & \text{for} = 4\,k + 1; \\ 3, & \text{for} = 4\,k + 2 \text{ or} \quad n = 4\,k + 3; \end{cases}$$

where k is a natural number ($k \in \boldsymbol{N}$). In a less precise notation, each of the three objects above has the following form:

$$\begin{aligned} a_n &= \{1, 1, 2, 3, 1, 1, 2, 3, 1, 1, 2, 3, \ldots\}, \\ b_n &= \{1, 2, 2, 3, 1, 2, 2, 3, 1, 2, 2, 3, \ldots\}, \\ c_n &= \{1, 2, 3, 3, 1, 2, 3, 3, 1, 2, 3, 3, \ldots\}. \end{aligned}$$

Each of the three sequences can be formed from any other through elimination of appropriate elements. For instance, we can get sequence b_n from a_n if we eliminate the underlined elements of sequence a_n:

$$a_n = \{\underline{1}, 1, \underline{2}, 3, 1, 1, \underline{2}, \underline{3}, 1, \underline{1}, \underline{2}, 3, 1, 1, \underline{2}, \underline{3}, 1, \underline{1}, \underline{2}, 3, 1, 1, \underline{2}, \underline{3}, \ldots\}.$$

We say then that sequence b_n is a proper sub-sequence of a_n, symbolically: $b_n \subset a_n$. Naturally, each of the three sequences is a proper sub-sequence of the remaining two:

$$a_n \subset c_n \subset b_n \subset a_n.$$

Thus each sequence is a proper sub-sequence of every other one, including itself. This means that in some sense three completely different sequences are indeed one and the same sequence. Each of them is the whole and each of them is the same whole. At the same time, each is evidently different from one another, and the difference can be reduced to the relations between the sequences, for the "substance" of the sequences is one, constituted by the three element set {1, 2, 3}. Admittedly, the sequences have different names. Naturally, "a_n", "b_n", "c_n" are not the names of the sequences. These symbols could easily be replaced with other ones. The real names are:

$$``\{1, 2, 3, 1, 1, 2, 3, 1, 1, 2, 3, 1, \ldots\}",$$
$$``\{1, 2, 3, 2, 1, 2, 3, 2, 1, 2, 3, 2, \ldots\}",$$
$$``\{1, 2, 3, 3, 1, 2, 3, 3, 1, 2, 3, 3, \ldots\}".$$

These names define a single sequence in the way shown above. Since the substance of these sequences is one, the differences between them can be described as ones expressed in the relational character of names. This meets the condition set in point 255 of *CCC*. In every case, the relationality is an appropriate inclusion. We should note that each inclusion is different. Although each of them entails elimination of the same elements, in each inclusion those elements form a sequence that is proper for a particular inclusion but different from the other two: "They are distinct from one another in their relations of origin" (254 *CCC*).

From the theological point of view, it would not be difficult, probably, to show that the trinity of sequences $\{a_n, b_n, c_n\}$ is too simple to be a complete and adequate model of the Holy Trinity. The aim of this presentation, however, was to show that there is a possibility of reconciling with one another the paradoxically sounding conditions characterizing the Trinity. What is more, we have demonstrated that the concept of Trinity can be conceived of not only in theology but also in the most precise of sciences that is available for man, i.e., mathematics.

The above construction does not prove the dogma of the Holy Trinity. It shows only that the dogma is not an absurd. Such a construction cannot replace the faith.

2.8 Fitch's Paradox, or the Conflict of Two Intuitions

All of the paradoxes presented so far have been a result of the influence of some inadequate intuition on our minds. Against this background, Fitch's paradox is quite an interesting case. Apparently, it is an effect of substitution of an erroneous but scientific intuition for a correct common sense one. What is worth noting is that the common sense intuition is supported by strong, evident arguments in the form of appropriate theses.

The Paradox of Knowability stems from the realists' argument with anti-realists concerning truth, in which the former claim that it is a non-epistemic concept, i.e., that there are true propositions, which for various reasons escape our cognition completely, while the latter accept the existence of propositions that cannot be completely known, but reject their general incognoscibility. One of anti-realists' favorite examples is the set of all true propositions in the theory of numbers. Although we shall never know many of them, we should not suppose that they are naturally beyond our cognition. We are able to get to know every one of them separately, even though in many cases it would require an enormous effort. Thus we can see that an anti-realist stance assumes *The Knowability Principle* according to which *every true proposition is knowable*.

Fitch's Paradox seems to undermine this principle, for it argues, starting from evident premises concerning knowledge, that: if there is a true proposition whose truthfulness is not known, there is a true proposition whose truthfulness will not be known by anyone. In a 1963 paper, A Logical Analysis of Some Value Concepts, Frederic Fitch formulated his famous fifth theorem, according to which if there is an unknown truth, the fact that there is an unknown truth is an unknowable truth.[34] Fitch's reasoning can be presented as follows:

Fitch's Paradox (Paradox of Knowability)

Let us assume that q is a true proposition, whose truthfulness is not known. Let us consider another proposition:

(1) q and it is not known that q It is a true proposition but an unknowable one. Let us suppose that there is a situation, in which:
(2) It is known that: q and it is not known that q Since knowledge is distributed with respect to conjunction, we obtain:
(3) It is known that q and
(4) It is known that it is not known that q Since knowledge implies truth[35], it follows from (4) that:
(5) It is not known that q, which is in contradiction with (3). Therefore, the situation mentioned in point (2) is excluded. It means that proposition (1) is unknowable

The presentation of this paradox makes frequent use of predicate K, for which it is a tautology that: (*) $K(\alpha \wedge \beta) \rightarrow K(\alpha) \wedge K(\beta)$ and (**) $K(\alpha) \rightarrow \alpha$. Then the steps of the above reasoning are the following[36]:

Let us assume that	(1) $q \wedge \neg K(q)$.
Let, moreover,	(2) $K(q \wedge \neg K(q))$.
Then on virtue of (*) and (2) we obtain	(3) $K(q)$
and	(4) $K(\neg K(q))$.
With respect to (**) and (4) we obtain	(5) $\neg K(q)$.
From (3) to (5)	(6) $K(q) \wedge \neg K(q)$ - contradction.
Therefore,	(7) $\neg K(q \wedge \neg K(q))$.

It could seem that no matter whether we analyse the above reasoning in a more or less formalized form we cannot help noticing that it contains an elementary mistake, i.e., it gives a concrete proposition q in the assumptions (1) and (2), even though out of principle it should be unknown, so it should not be represented by

[34] Fitch [19]; Brogaard and Salerno, [SEPh].

[35] That our knowledge implies truth is expressed in a simple rule: *from fact that we know that p, follows p*.

[36] Other versions of Fitch's paradox can be found in: Brogaard and Salerno, [SEPh].

any concrete symbol. For using symbol q means here that we speak of a concrete proposition identified as q. In other words, accepting that the formula $q \wedge \neg K(q)$ represents the existence of an unknown true proposition is absurd—q is a true proposition and we do not know that q is that proposition, even though we think of q and only q all the time. Therefore, q is such an unusual true proposition that we know and not know it is true at the same time. Evidently, the contradiction lies in the very assumption. No wonder, then, that we can draw a contradictory conclusion: (6) $K(q) \wedge \neg K(q)$ from such a contradictory assumption. The conclusion from line (7): $\neg K(q \wedge \neg K(q))$ is completely understandable and in no way paradoxical. It is not true that in any concrete proposition q we know that it is true and do not know that it is true at the same time.

One can agree that Fitch's Paradox presents no paradoxical reasoning. What is paradoxical is linking Fitch's Paradox with the problem of existence of unknowable true propositions. A proper reasoning should start with the reconstruction of both assumptions. It is clear that the realists' standpoint concerning unknowable true propositions is not formula (1) but:

$$\exists x \,(x \wedge \neg K\,(x)).$$

If we assume now that we are aware of the truthfulness of his assumption, i.e. instead of (2) we shall assume that:

$$K\,(\exists x\,(x \wedge \neg K\,(x)));$$

then we shall deduce no contradiction from such a natural assumption. Clearly, it is evident and thus also known to us that there exist true propositions of which we do not know that they are true.

A fairly good example showing aptness of this diagnosis of Fitch's problem can be found in a simple analysis of two contradictory propositions: p_0—*God exists* and $\neg p_0$—*It is not true that God exists*. It seems that it is impossible not to accept that one of the two is true.[37] At the same time, we have to accept that both $\neg K(p_0)$ and $\neg K(\neg p_0)$ represent a true proposition. Thus, precisely one out of two conjunctions: $p_0 \wedge \neg K(p_0)$, or $\neg p_0 \wedge \neg K(\neg p_0)$ represents a true proposition. Naturally, we do not know not only which of them represents a true proposition but have good grounds to believe that we shall never know it.

Evidently, this case is an argument in support of realists, for it provides an example of two propositions, for which we are unable to make a truthful assessment. Whether p_0 is true or $\neg p_0$ is true is merely the question of belief. The example shows yet another thing, i.e., the truthfulness of our assumptions, which are revised versions of (1) and (2). Although we do not know whether $p_0 \wedge \neg K(p_0)$ is true or $\neg p_0 \wedge \neg K(\neg p_0)$ is true, we know by virtue of logic and not belief that one of them is true. And so, it is true that both $\exists x(x \wedge \neg K(x))$ and $x\ (x \wedge \neg K(x))$.

[37] Let us note that the truthfulness of the opinion that one of these propositions must be true does not depend on the way we understand the word "God".

Moreover, the truthfulness of the latter proposition, $K(\exists x\ (x \wedge \neg K(x)))$ does not lead to contradiction.

References

1. Sainsbury, R.M. (1988). *Paradoxes*. Cambridge University Press: Cambridge, 1991.
2. Hurley, S.L. (1994). A new take from Nozick on Newcomb's problem and prisoners dilemma. *Analysis, 54*(2), 65–72.
3. Kiekeben, F. Newcomb's Paradox. http://members.aol.com/kiekeben/home.html, 1996.
4. Priest, G. (2004). *Odrzucanie*: Przeczenie a Dylematy [Rejection: Denial and Dilemmas], (trans: Łukowski, P., and Rybarkiewicz, D.), *Folia Philosophica* 18 (pp. 131–148).
5. Gardner, M. (1982). *Aha! Gotcha Paradoxes to puzzle and delight*. San Francisco: Freeman & Co.
6. Aczel, A.D. (2002). *Tajemnica alefów*. Matematyka, Kabała i poszukiwanie nieskończoności, (trans Hornowski, T.) seria Nowe Horyzonty, Dom Wydawniczy Rebis, Poznań, 2002. The Mystery of the Aleph Washington Square Press (2000).
7. Morris, R. (1997). *Krótka historia nieskończoności*. Achilles i żółw w kwantowym Wszechświecie [Achilles in the Quantum Universe. The definitive history of infinity], (trans Jerzy Kowalski-Glikman) seria Nauka u progu trzeciego tysiąclecia, Wydawnictwo CiS, Warszawa, 1999.
8. Podkoński, R. (2004). Al-Ghazali's "Metaphysics" as a source of anti-atomistic proofs in John Duns Scotus "Sentences' commentary".
9. Hunter, G. (1982). *Metalogic. An introduction to the metatheory of standard first order logic*, Macmillan Press, 1971.
10. Marciszewski, W. (1987). *Logika formalna. Zarys encyklopedyczny z zastosowaniem do informatyki i lingwistyki [Formal Logic. An encyclopedic introduction for computer studies and linguistics] red*. Warszawa: PWN.
11. Murdoch, J.E. (1982). William of Ockham and the logic of infinity and continuity. In Norman Kretzmann (Ed.), *Infinity and continuity in ancient and medieval thought* (pp. 165–206). New York.
12. Geach, P.T. (1972). *Logic Matters*. Oxford: Basil Blackwell.
13. Hill, E. (1985). *The Mystery pf the Trinity*. London: Geoffrey Chapman.
14. Martinich, A. P. (1987). Identity and Trinity. *The Journal of Religion, 58*, 169–181.
15. van Inwagen, P. (1988). And yet they are not three Gods but one God. In T. V. Morris (Ed.), *Philosophy and the Christian faith* (pp. 241–278). Notre Dame Press, Indiana: University of Notre Dame Press.
16. Williams, C.J.F. (1994). Neither confronting the persons nor dividing the substance. In A. G. Padgett (Ed.), *Reason and the Christian religion essays in honour of Richard Swinburne.* (pp. 227–243). Oxford: Clarendon Press.
17. Ziemiński, I. (1999). Spójność chrześcijańskiego pojęcia Boga the coherence of the notion of Christian good. In Krzysztof Mech (Ed.), *Człowiek wobec religii* (pp. 200–213). Kraków: Zakład Wydawniczy 'Nomos.
18. Łukasiewicz, J. (1910). O zasadzie sprzeczności u Arystotelesa [On Aristotle's Principle of Contradiction], Biblioteka Współczesnych Filozofów. Warszawa: PWN, 1987.
19. Fitch, F. (1963). A logical analysis of some value concepts. *The Journal of Symbolic Logic, 28*, 135–142.

Chapter 3
Paradoxes of Ambiguity

Language is a tool, which we use every time we want to say something about reality, one or another. Because reality transcends the limits of language, we are bound to use the same terms in meanings that can vary, from similar to completely different. Ambiguity of natural language expressions is thus a phenomenon both natural and very common. Unfortunately, we tend to forget about it. Even scientific discourse is sometimes poisoned by the error of ambiguity. In this chapter, we hall focus on the most important problems, which have their source in ambiguity.

3.1 Protagoras' (Law Teacher's) Paradox

Some logicians, e.g., Kazimierz Ajdukiewicz (1890–1963), the leading representative of the Lvov–Warsaw School, believe Protagoras' Paradox to be another version of the Crocodile's Paradox. Ajdukiewicz even claims that both paradoxes are solved in the same way.[1] This opinion is not universal.

Protagoras' Paradox

The Greek sophist Protagoras had a pupil called Euathlos, whom he taught court rhetoric. They agreed that Euathlos would pay Protagoras for the education only after he had won his first case. However, Euathlos did not take any case for a long time after he had concluded his lessons with Protagoras. Protagoras finally lost his patience and sued Euathlos for his fees. Before the jury, he presented his case as follows: *Euathlos will either win or lose this case, which is his first one. If he wins, he should pay me, because of our agreement that he pays after winning his first case. If he loses, he should pay me, because of the court verdict.* But Euathlos must have been a clever pupil, for he replied

1 Ajdukiewicz [1], p. 143.

P. Łukowski, *Paradoxes*, Trends in Logic, 31,
DOI: 10.1007/978-94-007-1476-2_3, © Springer Science+Business Media B.V. 2011

Protagoras in these words: *Either I, Euathlos, will win the case or I will lose it. If I win, it means that the verdict will free me from payment, if I lose, I will be absolved from payment by virtue of our agreement that I pay the fees only if I win my first case.*

Traditionally, two assumptions are made here: 1. the lawsuit between the teacher and the pupil can be won only by one of them; 2. the teacher's win is the pupil's loss and vice versa. According to Ajdukiewicz, it has to be admitted that the source of contradiction is a faulty formulation of the agreement, since it is impossible to meet its conditions.

A similar approach has been presented by Raymond Smullyan, Lennart Åqvist and W. Lenzen. They acknowledge the faulty character of the agreement and wonder what Protagoras should do to get the money from Euathlos. Regrettably, this is how they substitute a legal or pragmatic problem for a logical one. It is no longer contradiction but debt repayment that becomes the main problem. In his 1978 popular book, *What is the Name of This Book*?—*The Riddle of Dracula and Other Logical Puzzles*, Smullyan frankly confesses that he is not sure if he knows the solution of this dilemma and thinks that the best one is that proposed by his lawyer friend (!): the case should be won by Euathlos, since the agreement did not oblige him to pay for education. But after the court victory Euathlos should pay Protagoras. If he did not, Protagoras should sue him again, and that time he should win. Analyzing this solution one can hardly exclude the situation, in which one must bring another case, and another, etc. Here we must ask a question: what is the subject of such analysis? Is it the contradiction, which puzzled the Ancients so much, of any importance anymore? Smullyan's proposal loses the logical aspect altogether.

Regrettably, Lenzen[2] and Åqvist[3] propose a formalization of such a "legal" approach. In his 1977 paper *Protagoras versus Euathlos : Reflections on a so-called Paradox*, Lenzen formalizes the dilemma using the so-called *base logic*, which is defined by the axioms of the classical sentence calculus, axioms of necessity operator of the modal system S5, and axioms of identity predicate. The only inference rule is *Modus Ponens*. Lenzen accepts following four postulates[4]:

$P1.$ $A \Rightarrow (OZb \leftrightarrow G(b,p))$;

$P2.$ $(G(b, p^{+}) \leftrightarrow (V(p^{+}) \Rightarrow -OZb))$ and $(G(a, p^{+}) \leftrightarrow (V(p^{+}) \Rightarrow OZb))$;

$P3.$ $p = p^{+}$;

$P4.$ $(V(p^{+}) \Rightarrow OZb) \leftrightarrow -(V(p^{+}) \Rightarrow -OZb)$.

2 Lenzen [2].

3 Åqvist [3].

4 Lenzen's formalization is presented here in Åqvist's symbolization (cf. Åqvist [3]).

Symbols present in the formulae $P1$–$P4$ are introduced by definitions[5]:

$D0$. $a =$ Protagoras;
$D1$. $b =$ Euathlos;
$D2$. $A =$ the agreement reached between Protagoras and Euathlos;
$D3$. Valid $A \leftrightarrow A$ is valid;
$D4$. $p =$ the first court-case in which Euathlos gets involved;
$D5$. $p^+ =$ the court-case brought by Protagoras against Euathlos;
$D6$. $V(p^+) =$ the verdict passed in p^+;
$D7$. Correct $V(p^+) \leftrightarrow V(p^+)$ is correct[6];
$D8$. $OZb \leftrightarrow$ Euathlos is obliged to pay the fee (it is obligatory that Euathlos pay the fee);
$D9$. $G(b,p) \leftrightarrow$ Euathlos wins case p;
$D10$. $G(b,p^+) \leftrightarrow$ Euathlos wins case p^+;
$D11$. $G(a,p^+) \leftrightarrow$ Protagoras wins case p^+;
$D12$. $(A \Rightarrow \Phi) \leftrightarrow \Box(\text{Valid } A \rightarrow \Phi)$;
$D13$. $(V(p^+) \Rightarrow \Phi) \leftrightarrow \Box(\text{Correct } V(p^+) \rightarrow \Phi)$.

Consequently, it follows, by virtue of $P1$, from the agreement A that Euathlos is obliged to pay for education if and only if he wins his first case p. $P2$ explains how to understand Euathlos's winning the case and Protagoras's winning of it: Euathlos's win is tantamount with the court's decision of absolving him from paying fee to Protagoras; Protagoras's win is tantamount with the court's decision of obliging Euathlos to pay Protagoras his fee. $P3$ stands for the fact that Protagoras's case against his student is the first case, in which Euathlos stands. It follows from $P4$ that either the court will oblige Euathlos to pay, or it will absolve him from this obligation.

From the set $\{P1, P2, P3$ and $P4\}$, Lenzen induces the following formulae:

$C0$.	$G(a,p^+) \leftrightarrow -G(b,p^+)$	$P2$, $P3$, $P4$;
$C1$.	$G(b,p) \leftrightarrow G(b,p^+)$	$P3$;
$C2$.	Valid $A \rightarrow (OZb \leftrightarrow G(b,p))$	$P1$, $D12$;
$C3$.	Valid $A \rightarrow (OZb \leftrightarrow (V(p^+) \Rightarrow -OZb))$	$C2$, $C1$, $P2$.

By means of $C0$–$C3$ Lenzen proves the theorem:

Theorem 1 *The statement –Valid $A \vee -$Valid $V(p^+)$ is deducible in our basic logic from the set* $\{P1, P2, P3$ *and* $P4\}$.

[5] Åqvist specifies that in the accepted definitions, the symbols "=", "→", "↔"and "□" should be read as: "… is identical to…", "if… then…", "… if and only if…", "is necessary that…", respectively.

[6] Correctness of a court verdict stands, most probably, for its unambiguous and non-contradictory character.

Proof

1.	Valid A	hypothesis
2.	$OZb \leftrightarrow (V(p^{+}) \Rightarrow -OZb)$	1, $C3$
3.	OZb	hypothesis
4.	$V(p^{+}) \Rightarrow -OZb$	2, 3
5.	Correct $V(p^{+}) \rightarrow -OZb$	4, $D13$
6.	$-$Correct $V(p^{+})$	3, 5
7.	$OZb \rightarrow -$Correct $V(p^{+})$	3–6
8.	$-OZb$	hypothesis
9.	$-(V(p^{+}) \Rightarrow -OZb)$	2, 8;
10.	$V(p^{+}) \Rightarrow OZb$	9, $P4$;
11.	Correct $V(p^{+}) \rightarrow OZb$	10, $D13$;
12.	$-$Correct $V(p^{+})$	8, 11;
13.	$-OZb \rightarrow -$Correct $V(p^{+})$	8–12;
14.	$OZb \vee -OZb$	tautology
15.	$(OZb \vee -OZb) \rightarrow -$Correct $V(p^{+})$	7, 13;
16.	$-$Correct $V(p^{+})$	14, 15;
17.	Valid $A \rightarrow -$Correct $V(p^{+})$	1–16;
18.	$-$Valid $A \vee -$Correct $V(p^{+})$	17.

Immediate by Theorem 1: □

Theorem 2 *The statement—Correct $V(p^{+})$ is deducible from the postulate-set* $\{P1, P2, P3, P4$ *and* $P5\}$ *and the statement—Valid A is deducible from the postulate-set* $\{P1, P2, P3, P4$ *and* $P6\}$,

where,

$P5$. Valid A;

$P6$. Correct $V(p^{+})$.

By theorem 2:

Theorem 3 *The set of postulates* $\{P1, P2, P3, P4, P5$ *and* $P6\}$ *is inconsistent.*

By the same token, Lenzen showed that accepting the truth of the agreement implies incorrectness of the verdict, no matter what it is, and that accepting any verdict implies the falseness of the agreement. Consequently, accepting the truthfulness of the agreement and the correctness of the verdict at the same time implies contradiction. Since both Protagoras's and Euathlos's arguments are logically correct, they lead to contradiction if taken together.

Protagoras's Argument:

1.	$G(a,p^{+})$	hypothesis
2.	OZb	1, $P2$, $D13$, $P6$;
3.	$G(b,p^{+})$	hypothesis
4.	$G(b,p)$	3, $P3$;
5.	OZb	4, $P1$, $D12$, $P5$;
6.	$(G(a,p^{+}) \vee G(b,p^{+})) \rightarrow OZb$	1–2, 3–5;
7.	OZb	6, $C0$.

Euathlos's Argument:

1.	$G(b,p^{+})$	hypothesis
2.	$-OZb$	1, $P2$, $D13$, $P6$;
3.	$G(a,p^{+})$	hypothesis
4.	$-G(b,p)$	3, $C0$;
5.	$-OZb$	4, $P1$, $D12$, $P5$;
6.	$(G(a,p^{+}) \vee G(b,p^{+})) \rightarrow -OZb$	1–2, 3–5;
7.	$-OZb$	6, $C0$.

It is easy to see that up to this point the symbolically advanced Lenzen proposal is hardly different materially from Ajdukiewicz's single sentence diagnosis. According to Lenzen, the proper solution of the paradox is the following construction, which uses base logic, supplemented with the classical general quantifier ∀ and ten following definitions:

*D*14. $Z^{\leq t}b \leftrightarrow b$ had paid the fee by t
*D*15. $G^{\leq t}(b,p) \leftrightarrow b$ has won his first court-case p by t
*D*16. $G^{\leq t}(b,p^{+}) \leftrightarrow b$ has won the case p^{+} by t
*D*17. $O^{t}Zb \leftrightarrow$ at t it is obligatory for b to pay the fee
*D*18. $t^{+} =$ the (a certain) time before the verdict in p^{+} is pronounced
*D*19. $p^{++} =$ the second case brought by Protagoras against Euathlos
*D*20. $t^{++} =$ the (a certain) time before the verdict in p^{++} is pronounced[7]
*D*21. $V(p^{++}) =$ the verdict passed in p^{++}
*D*22. $G(b,p^{++}) \leftrightarrow$ Euathlos wins case p^{++}
*D*23. $G(a,p^{++}) \leftrightarrow$ Protagoras wins case p^{++}

[7] For the postulate $P7$ not to raise any doubts the definition $D20$ should specify the moment t^{++} as a moment which precedes the moment of proclaiming the verdict in the case p^{++} but later than the moment of proclaiming the verdict in the case p^{+} (author's comment).

As it can be seen, Lenzen introduces time coordinate and thanks to it he can consider two rather than one court cases. On the basis of the above definitions, he analyzes the altered set of postulates, which he uses for a new description of the situation from the paradox:

$P1a.$ $A \Rightarrow \forall t\ (O^tZb \leftrightarrow (G^{\leq t}(b,p) \text{ and } -Z^{\leq t}b))$;
$P2a.$ $(G(b, p^+) \leftrightarrow -O^{t+}Zb)$ and $(G(a, p^+) \leftrightarrow O^{t+}Zb)$;
$P2b.$ $(G(b, p^{++}) \leftrightarrow -O^{t++}Zb)$ and $(G(a, p^{++}) \leftrightarrow O^{t++}Zb)$;
$P3.$ $p = p^+$;
$P5.$ Valid A;
$P7.$ $-G^{\leq t+}(b,p^+)$;
$P8.$ $G^{\leq t++}(b,p^+)$;
$P9.$ $-Z^{\leq t++}b$.

According to Åqvist, the most important postulate is $P1a$, which says: *It follows from the agreement A that for any moment t it is so that Euathlos is obliged to pay for his education in t if and only if Euathlos has won his first case until t and still has not paid Protagoras until t*. Postulates $P2a$ and $P2b$ refer to the first and second-courtcase, respectively. According to $P2a$, *Euathlos will win his first case if and only if he is not obliged to pay his fee in the moment preceding the verdict in the first case*. According to $P2b$, *Protagoras will win the first case if and only if Euathlos is obliged to pay his fee in the moment preceding the verdict in the first case*. Postulate $P7$ says that *it is not true that Euathlos has won his first case before the verdict in the first case*. According to $P8$, *Euathlos won his first case before the verdict in the second case*. The last postulate $P9$ assumes that *Euathlos did not pay for his education until the moment of the verdict in the second case*.

The last step of reasoning in the alleged solution of Protagoras's Paradox is the following theorem proved by Lenzen:

Theorem 4 *Let* $L_1 = \{P1a, P2a, P3, P5, P7\}$ *and* $L_2 = \{P1a, P2b, P3, P5, P8, P9\}$. *Then*:

(i) *The conclusion* $-O^{t+}Zb$ *and* $G(b, p^+)$ *is deducible from the set* L_1.
(ii) *The conclusion* $O^{t++}Zb$ *and* $G(a, p^{++})$ *is deducible from the set* L_2.
(iii) *The conclusion* $-O^{t+}Zb$ *and* $G(b, p^+)$ *and* $O^{t++}Zb$ *and* $G(a, p^{++})$ *is deducible from the postulate-set* $L_1 \cup L_2$.

Proof ad (i):

1.	$\forall t\ (O^tZb \leftrightarrow (G^{\leq t}(b,p) \text{ and } -Z^{\leq t}b))$	$P1a$, $D12$, $P5$;
2.	$O^{t+}Zb \leftrightarrow (G^{\leq t+}(b,p) \text{ and } -Z^{\leq t+}b)$	1;
3.	$-G^{\leq t+}(b,p)$	$P3$, $P7$;
4.	$-O^{t+}Zb$	2, 3;
5.	$G(b,p^+)$	4, $P2a$;
6.	$-O^{t+}Zb$ and $G(b,p^+)$	4, 5.

Proof ad (ii):

1.	$\forall t\ (O^{t}Zb \leftrightarrow (G^{\leq t}(b,p)$ and $-Z^{\leq t}b))$	$P1a$, $D12$, $P5$;
2.	$O^{t++}Zb \leftrightarrow (G^{\leq t++}(b,p)$ and $-Z^{\leq t++}b)$	1;
3.	$G^{\leq t++}(b,p)$	$P3$, $P8$;
4.	$O^{t++}Zb$	2, 3, $P9$;
5.	$G(a,p^{++})$	4, P2b;
6.	$O^{t++}Zb$ and $G(a,p^{++})$	4, 5.

(*iii*) *is a direct conclusion of* (*i*) *and* (*ii*). *This ends the proof of the Theorem* 4.

Sets L_1 and L_2 characterize the first and second case, respectively. Regrettably, this tedious, seemingly precise, formalized reasoning tells us only as much as the laconic comment by Smullyan, quoted above, according to which Euathlos was not obliged to pay the fee until the verdict in the first case, but immediately after the verdict in the first case was given, he was obliged to pay for his education.[8]

Quite naturally, an obvious question appears here: *On what grounds do Lenzen and Smullyan consider another court case?* The problem is entirely different. It is not Protagoras getting his fee but a solution of a contradiction. Introducing the other court case Lenzen and Smullyan, however, lose the logical problem of the first case from sight. A lawyer can feel the need to solve the problem from the point of view of the social perception of justice. A logician can be interested in such issues as well. But in the first place, he should solve the problem of contradiction, which is the result of coexistence of the first agreement and the first court-case. Getting on to the second case without solving the contradiction is an example of evading the problem. It trivializes the ancient problem, especially from logical point of view. Lenzen's and Smullyan's approach is thus unusual for the traditional method of logical paradox solving.

From the logical point of view, Lenzen's proposal has yet another serious drawback. The impossibility of deducing contradiction from Lenzen's assumptions is the result of ignoring one of the two, equally important parts of the ancient narrative. As a result, the contradiction is not eliminated but only passed over. Actually, none of the sets L_1 and L_2 is contradictory. Yet, none of them corresponds to the problem of the paradox and captures the whole problem, and in this sense they are incomplete.

Firstly since the time before the verdict in the first case (t^{+}) is taken into account, just as the time before the verdict in the second case (t^{++}), it is quite natural to take into account the time of proclaiming the verdict in the first case (t');

Secondly what was ignored was the essence of actual verdicts in the first case and in the second one. Postulates $P2a$ and $P2b$ state that only when the

[8] Smullyan's book, in which he spoke about the solution of the paradox by means of two court cases, was published a year later than Lenzen's paper presenting the analysis we recall here.

jury rules for Protagoras, and Euathlos. However, there are no postulates that would say what it means when the verdict is in favor of Euathlos, and what it means when it is in favor of Protagoras.

It can be seen that only half of the argument was taken into account from the paradoxical narrative. The first comment has only technical importance: it is to facilitate talking about what is going to happen when the verdict is given in the first case.

Let us present the omitted assumptions:

1. $t^+ < t' < t^{++}$;	Additional technical assumption;
2. $G(b, p^+)$	From *th* 4 (i);
3. $O^{t++}Zb$	From *th* 4 (ii);
4. $O^{t'}Zb$	From 1, 3;
5. $G(b,p^+) \leftrightarrow -O^{t'}Zb$	Missing postulate explaining the meaning of Euathlos's winning/losing;
6. $-O^{t'}Zb$	From 2, 5;
7. $O^{t'}Zb$ and $-O^{t'}Zb$	From 4, 6.

The added elements are not only in agreement with the assumptions accepted by Lenzen but supplement them. The fourth step of the above proof states that since the moment of the favorable verdict Euathlos is obliged to pay for his education. In the fifth step, a postulate is accepted stating that for Euathlos winning the first case means that he is not obliged to pay for education, at least in the moment when the verdict is given (if the verdict obliged him to pay, it would mean his loss). Thus, the contradiction still exists.

Regrettably, in his critique of Lenzen's solution, presented in the 1981 paper *The Protagoras Case: An Exercise in Elementary Logic for Lawyers*, Åqvist does not notice the above problem. He formulates objections against two postulates *P*2*a* and *P*2*b*. He seems to be right observing that although both postulates should be fulfilled, it can happen that both the first and the second will be falsified by the decision of the jury. In the first court-case, it can happen contrary to *P*2*a* the jury will conclude that Euathlos should pay for education. Then both parts of the conjunction, which makes up the postulate, should be rejected. In a similar way, Åqvist questions the value of *P*2*b*. But the new form of the postulates ignores, as it was the case in Lenzen, the part of argument that is essential for the paradox, i.e., it is silent about the meaning of Euathlos's win/loss. Presenting his version of Lenzen's solution Åqvist exchanges only the two doubtful postulates *P*2*a* and *P*2*b* for two even more doubtful ones, even though they resemble the earlier postulate *P*2:

*P*2.0 $(G(b, p^+) \leftrightarrow (V(p^+) \Rightarrow -O^{t+}Zb))$ and $(G(a, p^+) \leftrightarrow (V(p^+) \Rightarrow O^{t+}Zb))$;
*P*2.1 $(G(b, p^{++}) \leftrightarrow (V(p^+) \Rightarrow -O^{t++}Zb))$ and $(G(a, p^{++}) \leftrightarrow (V(p^+) \Rightarrow O^{t++}Zb))$.

Postulate $P2.0$ says only that: *Euathlos will win his first case if and only if the correct verdict results in Euathlos's lack of obligation to pay for education until the moment of the verdict in the first case and Protagoras will win the first case if and only if the correct verdict results in Euathlos being obliged to pay for education until the moment of the first verdict.* Consequently, as in the case of Lenzen's solution we do not know here what it means that the verdict (in the first case) is favorable for Euathlos. We only know that it is favorable and this is about all. Similarly, it is not known what consequences a negative verdict of the jury can have for Euathlos, it is unfavorable and this is about all. What is known are the conditions, which must be met so that the jury can give a verdict favorable or unfavorable for Euathlos. In the case of the postulate $P2.1$ we face the very same problems.

This way, also postulates $P2.0$ and $P2.1$ will make it possible to avoid talking about what really causes contradiction in the paradox. Refraining from commenting the content of the verdict helps to pass over the contradiction, yet it does not mean it is not there. Introducing a postulate coherent with Åqvist's assumptions we shall presently infer a contradiction without any difficulty, just as it was with Lenzen's formalization, thus demonstrating that it is still an inalienable element of the situation formalized by Åqvist.

Åqvist substitutes two new postulates for $P4$:

$P4.0$ $\quad (V(p^+) \Rightarrow O^{t+}Zb) \leftrightarrow -(V(p^+) \Rightarrow -O^{t+}Zb)$;
$P4.1$ $\quad (V(p^{++}) \Rightarrow O^{t++}Zb) \leftrightarrow -(V(p^{++}) \Rightarrow -O^{t++}Zb)$.

Moreover, he proposes considering the version of the postulate $P6$, which refers to the second court case:

$P6.1$ $\quad$ Correct $V(p^{++})$.

Application of new postulates makes it possible for Åqvist to prove the theorem:

Theorem 4 *Let* $L^+ = \{P1a, P2.0, P3, P4.0, P5, P7\}$, $L^{++} = \{P1a, P2.1, P3, P5, P8, P9\}$. *Then,*

(i) *The conclusion*: *Correct* $V(p^+) \rightarrow G(b, p^+)$ *is deducible from* L^+.
(ii) *The conclusion*: *Correct* $V(p^{++}) \rightarrow G(a, p^{++})$ *is deducible from* L^{++}.

Proof Ad (i):

1.	$\forall t\ (O^tZb \leftrightarrow (G^{\leq t}(b,p)$ and $-Z^{\leq t}b))$	$P1a, D12, P5$
2.	$O^{t+}Zb \leftrightarrow (G^{\leq t+}(b,p)$ and $-Z^{\leq t+}b)$	1
3.	$-G^{\leq t+}(b,p)$	$P3, P7$
4.	$-O^{t+}Zb$	2, 3
5.	*Correct* $V(p^+)$	hypothesis
6.	$V(p^+) \Rightarrow O^{t+}Zb$	hypothesis
7.	$-$*Correct* $V(p^+)$	4, 6, $D13$;

8.	*Correct* $V(p^+)$ and $-$*Correct* $V(p^+)$	5, 7;
9.	$-(V(p^+) \Rightarrow O^{t+}Zb)$	6–8;
10.	$V(p^+) \Rightarrow -O^{t+}Zb$	9, $P4.0$;
11.	$G(b, p^+)$	10, $P2.0$;
12.	Correct $V(p^+) \rightarrow G(b, p^+)$	5–11.

Ad (ii):

1.	$\forall t\ (O^tZb \leftrightarrow (G^{\leq t}(b,p)$ and $-Z^{\leq t}b))$	$P1a$, $D12$, $P5$;
2.	$O^{t++}Zb \leftrightarrow (G^{\leq t++}(b,p)$ and $-Z^{\leq t++}b)$	1;
3.	$G^{\leq t++}(b,p)$	$P3$, $P8$;
4.	$O^{t++}Zb$	2, 3, $P9$;
5.	Correct $V(p^{++})$	hypothesis
6.	$V(p^{++}) \Rightarrow -O^{t++}Zb$	hypothesis
7.	$-$Correct $V(p^{++})$	4, 6, $D13$;
8.	Correct $V(p^{++})$ and $-$Correct $V(p^{++})$	5, 7;
9.	$-(V(p^{++}) \Rightarrow -O^{t++}Zb)$	6–8;
10.	$V(p^{++}) \Rightarrow -O^{t++}Zb$	9, $P4.1$;
11.	$G(a, p^{++})$	10, $P2.1$;
12.	Correct $V(p^{++}) \rightarrow G(a, p^{++})$	5–11.

A direct conclusion of the Theorem 5 is the final conclusion of Åqvist's proposal:

Corollary
(i) *$G(b, p^+)$ is deducible from the set $L^+ \cup \{P6\}$.*
(ii) *$G(a, p^{++})$ is deducible from the set $L^{++} \cup \{P6.1\}$.*
(iii) *$G(b, p^+)$ and $G(a, p^{++})$ is deducible from $L^+ \cup L^{++} \cup \{P6, P6.1\}$.*

In the end Åqvist introduces yet another, rather cosmetic, correction to weaken the postulate $P8$:

$P8^w$. $\quad G(b, p^+) \rightarrow G^{\leq t++}(b,p^+)$;

and build a set (non-contradictory) L^w that is a sum $L^+ \cup L^{++} \cup \{P6, P6.1\}$, in which $P8$ is substituted by $P8^w$. It turns out from the set L^w as follows $G(a, p^{++})$.

As in the case of Lenzen's proposal, there is no danger here that any contradiction could be deduced from Åqvist's postulates. They are not satisfactory, since they do not specify the meaning of Protagoras's/Euathlos's win/loss in the first court-case. Similar to Lenzen, Åqvist omits half of the ancient arguments. It is no wonder then that the set of premises is non-contradictory. Filling in the gaps left by Åqvist's postulates by adding others, which are consistent with them (as was the case with Lenzen's construction), one can easily show that contradiction has not been eliminated here. It has only been passed over in silence. It is enough, then, to accept the same additions as in the case of Lenzen's proposal to make the contradiction evident. From the postulate $P1a$ and the first part (i) of the conclusion we immediately obtain that Euathlos is obliged to pay Protagoras for education as soon as the verdict is given. On the other hand, if the verdict is favorable for Euathlos, he is apparently not obliged to pay for education. This way the contradiction emerges.

Finally, let us mention one more side question, formulated by Åqvist himself, which refers to the postulates $P2a$ and $P2b$. In order to analyze Protagoras's Paradox rationally, according to the ideas of Lenzen, Åqvist and Smullyan, one would rather have to accept that Protagoras, who was an eminent thinker, realized the faulty character of his agreement, albeit a little late. To avoid being ridiculed for his mistake, he sued Euathlos for something that was in no way related to their hapless agreement. For the teacher, the whole situation is easy in the sense that Protagoras must sue in the case he is bound to lose only to fulfill the condition, which obliges Euathlos to pay for his education.

Summing up it must be noted that among the proposals of solving Protagoras's Paradox one finds a dominant opinion that it is the agreement which is the cause of (inalienable!) contradiction, and the real problem lies in making Euathlos pay for his education. Moreover, the core of the dilemma is how Protagoras should get his fee from Euathlos and not the fact that the situation is paradoxical. Entering into the legal details results in substituting a pseudo-problem for the ancient dilemma.

One can accept, like Ajdukiewicz, that the whole evil of contradiction lies in the unfortunate formulation of the agreement and accept that fact as the final solution of the paradox. Is it not possible, however, to find a solution in which the paradoxical contradiction simply does not occur?

From this point of view, a proposal presented by a praxeologist Tadeusz Pszczołowski (1922–1999) in his 1962 book *Umiejętność przekonywania i dyskusji*[9] seems to be of special interest. According to him today we would distinguish the obligation to pay resulting from an agreement and from a court order and, consequently, we could reason as follows: Let us suppose that the case is won by Euathlos. The court absolves him from paying his teacher but the agreement stays in force. If he is a man of honor, the pupil should pay the teacher for his successful work. If Protagoras won, the student should pay by the court order, but Protagoras, if he respected his words, he should give the money back as soon as he got it, since the student lost his first case.

Of all the solutions analyzed so far, the one proposed by Pszczołowski is the only one to distinguish between obligation to pay resulting from an agreement and resulting from the court order. Regrettably, regardless of the distinction Pszczołowski solves the paradox as an ethical problem. For him the fundament of the solution is dignity in the attitude toward oneself and other man, in that case

[9] Pszczołowski [4], p. 35.

represented by Euathlos and Protagoras. Let us then take Pszczołowski's distinction and solve the paradox treating it, like Ajdukiewicz, as a logical one, rather than legal, financial or ethical, and eliminate the contradiction through such corrections of the narrative that it will cease to be paradoxical.

Let us use two expressions: "pay the agreed *fee*" and "pay the court ordered *damages*" rather than one "pay for the education". It is easy to see that such a simple operation eliminates the unwanted contradiction. If Euathlos wins the court case, he must pay the agreed fee, even though he does not pay the damages. If Protagoras is the winner, the situation will be quite opposite: Euathlos will not pay the fee but will have to pay the damages. It can be concluded, therefore, that in both cases the argument between Protagoras and Euathlos is won by the former. No matter what verdict is given, he will always get the money from his pupil; what is unknown is whether it will be his fee or damages. This means that Euathlos must pay the money in both cases and contradiction vanishes.

3.2 Electra's (the Veiled One's) Paradox and Other Equivocations

It is almost universally believed that the author of Electra's Paradox, just like the paradoxes of the Horny one, the Liar and the Bald one, is Eubulides of Miletus (4th c. BC), a fierce opponent of Aristotle educated in the School of Megara, who lived in the fourth century BC.[10] Diogenes Laertios (ca. 3rd c. AD) adds that according to some philosophers the author of the veiled one's and horny one's paradoxes was Diodorus of Iazos, a pupil of Apollonios Cronos, who in turn was a pupil of Eubulides himself.

Electra's Paradox

We ask if Electra knows that Orestes is her brother. A positive answer results from the state of affairs that is evident for Electra. But Orestes is standing veiled in front of her. Electra does not know that the veiled man is her brother; but since that is Orestes, we also get a negative answer that Electra does not know that Orestes is her brother. Consequently, Electra knows and does not know that Orestes is her brother at the same time.

The reasoning in the story about Electra and Orestes starts from a hardly questionable fact that two sentences: "Electra knows that Orestes is her brother" and "Electra does not know that the veiled man is her brother" are true. Since Orestes and the veiled man is one and the same person, we get two true

[10] Diogenes Laertios, *De vitis et moribus philosophorum*, II, 108–112, pp. 137–139.

contradictory sentences: "Electra knows that Orestes is her brother" and "Electra does not know that Orestes is her brother".

A solution proposed by Kotarbiński[11] is based on an intentional function: "*A* knows that *x* is *N*". In contrast to extensional functions, such as negation, conjunction and alternative, intentional functions have the property that sentences formed from them through substitution for a variable of an invariable that belongs to a particular range can have another logical value than another sentence formed from the same function through substitution for the same variable of another invariable from the same range. This way, two names: "Orestes" and "this veiled man" cannot always be substituted in all possible contexts, even though they have the same range: the sentence "Electra knows that Orestes is her brother" is true, even though the sentence "Electra knows that the veiled man is her brother" is false. In his solution, Kotarbiński identified Electra's Paradox with another one, known as the *Morning Star Paradox*:

Morning Star Paradox

Looking on the sky at dawn Plautus sees a bright star. He calls it a Morning Star. In the evening, he also sees a bright star. He calls it an Evening Star. He does not know that in both cases he is looking at the planet Venus. This way, he does not know that the Morning Star is the Evening Star. Consequently, Plautus does not know that the planet Venus is the planet Venus.

One can easily give many other versions of this paradox. Ajdukiewicz gives an example of a name invented by Plato (ca. 429–ca. 347 BC), "featherless biped", as a name equivalent to but not cosignificant with the name "man".[12] All paradoxes of this type are solved according to the above scheme proposed by Kotarbiński.

Electra's Paradox, however, can be defused in another way by pointing to the error of ambiguity hidden in it. In the story of Electra's problem, there are two already quoted sentences: "Electra knows that Orestes is her brother" and "Electra does not know that the veiled man is her brother". The key word for understanding the problem is the word "knows", present in both sentences. This word is notorious for the number of its meanings and can easily compete with "can" and "be" in this respect. In our case this word has at least two different meanings, different in each sentence. Thanks to the word taken in the first meaning the sentence "Electra knows that Orestes is her brother" stands, more or less, for "Electra knows her brother Orestes". The other sentence from the analyzed story, "Electra

[11] Kotarbiński [5], p. 187.

[12] This definition was ridiculed by Diogenes of Sinope (ca. 412–323 BC). Diogenes Laertios tells the following story: "When Plato gave a definition 'Man is a two-legged, featherless animal' and was proud of the definition for which he received acclaim, Diogenes [of Sinope] plucked a rooster and took it to the school for Plato's lecture saying: 'Here is Plato's man'. Since then the definition was supplemented with words: 'with wide claws'." Diogenes Laertios, *De vitis et moribus philosophorum*, VI, 40; p. 331.

does not know that the veiled man is her brother" signifies the same as "Electra cannot recognize her brother Orestes in the veiled man". It is a different meaning of "know" from the one used in the former sentence. Consequently, the reasoning behind Electra's Paradox is actually based on the truth of two premises: "Electra knows her brother Orestes" and "Electra cannot recognize her brother Orestes in the veiled man". Both premises are true. After all, everyone has difficulties in recognizing someone he or she knows, because of dim light or long distance. Obviously, from such reconstructed premises we cannot infer any paradoxical conclusion. We only know that in a given moment Electra cannot recognize someone she otherwise knows very well, which is nothing unusual.

Another solution treats Electra's Paradox as an equivocation. Let us remember that a reasoning is suffering from the error of equivocation (*fallacia aequivocationis*), when three conditions are met at the same time:

1. a key term of reasoning appears at least in two premises;
2. the premises, which contain the term, are both true only when the term has different meanings in at least two premises, in a suitable way;
3. the inference is correct if and only if the term has the same meaning in every premise it is present.

The inference is correct if it is also materially correct, i.e. all premises are true sentences, and formally correct, i.e., it confirms the laws of logic. As it can be seen, in the case of equivocation, material and formal correctness are mutually exclusive.

Here are some other examples of paradoxes with the fallacy of equivocation:

Drunkard's Paradox[13]

> Qui bibit, dormit; qui dormit, non peccat; qui non peccat, sanctus est; ergo: qui bibit, sanctus est.

That is:

> Who drinks, sleeps; who sleeps, does not sin; who does not sin, is a saint; consequently: who drinks, is a saint.

Thief's Paradox[14]

Since no thief wants to take what is bad, he only wants what is good; who only wants what is good, is good; consequently, every thief is good.

[13] An ambiguous term taken in one sense here is "does not sin". In the potential sense someone "does not sin", because he wants to live a saintly life. In the actual sense, one "does not sin", because sleeping prevents him from sinning.

[14] An ambiguous term taken in one sense here is "desire for what is good"—whether one wants only that which has material value, or wants that which is essential for good life.

A special case of equivocation is the *four terms* fallacy (*quaternio terminorum*). It refers to a syllogistic inference, in which instead of three terms: middle (M), smaller (S) and greater (P); there are four: smaller (S), greater (P) and two that substitute the middle term (M_1 and M_2). An example of a syllogism suffering from *quaternio terminorum* is the following reasoning:

Whoever helps criminals, is a criminal.
Every barrister helps criminals.
Consequently, every barrister is a criminal.

Seemingly, the inference is a representative of the following scheme:

$$\text{Consequently:} \frac{\begin{array}{c}\text{Every } M \text{ is } P\\ \text{Every } S \text{ is } M\end{array}}{\text{Every } S \text{ is } P}$$

Regrettably, in the place of the middle term M, "Whoever helps criminals", we have two different terms: M_1—"Whoever helps criminals in crimes" and M_2—"Whoever gives legal help to criminals". Accordingly, the actual scheme of inference is the following:

$$\text{Consequently:} \frac{\begin{array}{c}\text{Every } M_1 \text{ is } P\\ \text{Every } S \text{ is } M_2\end{array}}{\text{Every } S \text{ is } P}$$

Obviously, this scheme is formally incorrect: the conclusion does not result from premises.

3.3 The Horny One's Paradox

As we have said, this paradox, almost universally ascribed to Eubulides of Miletus, may have been invented by Diodorus of Iazos.[15] Most often, the paradox has the form of a short dialogue.

The Horny One's Paradox[16]

A: Have you lost horns?
B: Of course not! I have not lost any horns!
A: So, since one, who has not lost a thing, still has it, you have horns.

[15] Cf. the preceding paragraph.

[16] Diogenes Laertios tells about a common sense reaction of Diogenes of Sinope, the above mentioned eccentric philosopher, who having heard the argument of the Horned One, addressed to him, touched his forehead and said that he did not notice anything. He is also said to have similarly reacted to the argument in the Achilles and the Tortoise Paradox—he made a step forward, Diogenes Laertios, *De vitis et moribus philosophorum*, VI.2, 38, p. 330.

Naturally, ambiguity does not have to be reserved to single words. In our case the expression "the person B has not lost the thing C" may signify both the situation, in which B had C and still has it, and the one in which B never had and still does not have C. In the paradoxical dialogue of the horned one, each person is reasoning having another situation in mind. The person A thinks of the former situation, while the person B knows it is the latter. The problem lies in the fact that the abbreviate form of utterances making up the dialogue makes both understandings possible. Naturally, the person A knows that the former situation never occurred, i.e., B never had any horns, but makes use of the ambiguity to pull his leg. The ambiguity of that reasoning is caused by the fact that the *enthymematic premise*[17] is different in every version of reasoning. Strictly speaking, a reasoning, which proves that B has horns, should include a branching, because of an alternative that is a logical truth (line 3):

1. B did not lose horns until t (proof's assumption)
2. If B had horns until t and B did not lose horns until t, then B has horns in t (proof's assumption)
3. B had horns until t or B did not have horns until t (logical truth)

4.1 B had horns until t (from 3)	4.2 B did not have horns until t (from 3)
5.1 B had horns until t and B did not lose horns until t (from 1 and 4.1)	
6.1 B has horns in t (from 3 and 5.1)	

As it can be seen, the sentence "B has horns in t" is reached only in the case 4.1. In the case 4.2 no such conclusion can be reached. This is why the branching of the proof cannot be closed with a common line saying that B has horns in t. This means that the above reasoning is not a proof of the sentence "B has horns in t".

3.4 Nameless Club Paradox

In his book *Paradoksy semantyczne*, Eugeniusz Grodziński presents an anecdote titled "The Night Club Paradox".[18]

Nameless Club Paradox

In a town, there are several night clubs called *Eldorado*, *Alhambra*, *Helios*, *Poseidon* etc. A group of business people opens a new club in the town. However, the owners cannot make up their minds as to its name, so the patrons simply call it

[17] An enthymematic premise is one, which is not uttered, because its truth is evident both for the speaker and for the listener.

[18] Grodziński [6], pp. 65–66.

the nameless club. This way the expression "the nameless club" becomes the name of the club, at least temporarily. Now since the club already has a name, the expression "the nameless club" cannot apply to it any longer, for it can only apply to a club that has no name. If the expression "the nameless club" is not the name of the club we talk about, and it has no other name, then it has no name at all. If this is a correct conclusion, then the expression "the nameless club" defines it correctly, so it is its name after all, and the argument becomes looped.

Grodziński proposes to solve the problem by accepting the reasoning presented above as an equivocation. In the sentence: "The name of this club is the nameless club", the equivocal expression is the word "name". This reasoning, however, seems to be wrong, for it does not take into account the use of parentheses. In the conclusion of his argument, Grodziński claims that no one will think of claiming that the expression nameless club is the proper name of that club. This is an erroneous conclusion. It is enough to see how nicknames function in everyday life. Someone is called "shaggy", because he is actually shaggy, but another one is called "shaggy", because he is bald. If we use parentheses, we do cause contradiction saying: "'Shaggy' is bald". However, if we forget about the necessary parentheses, then immediately we get a contradiction resulting from the error of ambiguity: "Shaggy is bald" stands for "Shaggy is not shaggy". In the analyzed sentence, the word "shaggy" is used in two meanings: in the first instance, it is a proper name, in the second one it predicates about a property of the designate, whose name is the first instance of the word "shaggy". Using parentheses prevents the error of ambiguity, since "a" $\neq$ a. Thus, "the nameless club" is not a club without a name, but a club "the nameless club".

Here is a similar dilemma:

Beth's Paradox[19]

Since
a. 343 contains three digits
and
b. $343 = 7^3$,
so
7^3 contains three digits.

The paradox disappears when we notice that "343" $\neq$ 343, and "7^3" $\neq 7^3$. It is enough to put the first premise in the correct form: a'. "343" contains three digits; and the conclusion does not follow from a' and b.[20]

Distinguishing the parenthetic and ordinary use is possible thanks to the so-called *supposition of a name*, i.e., its meaning function. A name without

[19] Grodziński [6], p. 64.

[20] There is no difference what form the conclusion takes: "7^3 contains three digits", or "$7^{3'}$ contains three digits".

parentheses is a sign of either a designate (simple supposition) or a species (formal supposition). A name in parentheses is a of itself (material supposition). Using a name in one supposition in a single utterance, when it should be used in different suppositions is a variant of the error of ambiguity.

3.5 Paradox of a Stone, or an Attempt at Disproving God's Omnipotence

The Paradox of a Stone is an attempt at proving inexistence of an omnipotent being, thus especially the inexistence of God. Logic allows for only one form of proof of inexistence of an object. It is an indirect proof, which shows that the assuming existence of such object leads to contradiction. There is, for instance, a logical proof of inexistence of a married bachelor, or wooden iron, but there is no proof of inexistence of, e.g., a dwarf. The proof of inexistence, for instance, of a married bachelor can have the following form:

1. There exists a married bachelor (assumption of the indirect proof)
2. *A* is a married bachelor (from 1)
3. *A* is married and *A* is a bachelor (from 2)
4. *A* is married (from 3)
5. *A* is a bachelor (from 3)
6. *A* has a wife (from 4)
7. *A* has no wife (from 5)
8. It is not true that *A* has a wife (from 7)
9. *A* has a wife and it is not true that *A* has a wife (from 6 and 8)
10. It is not true that there exists a married bachelor ($1 \rightarrow 9$)

If the premise 1 is true, every sentence of the proof is true, also the sentence 9. But the sentence 9, as contradictory, is false. Consequently, the premise 1 is also false, which means that married bachelors do not exist.

The argument in the Paradox of a Stone has a similar form, though the proof is branched.

Paradox of a Stone

1. There exists an omnipotent God (assumption of the indirect proof)
2. God either can create a stone that he cannot lift, or God cannot create a stone that he cannot lift (logical truth)
   3.1. God can create a stone that he cannot lift
   3.2 There is something that God cannot do (from 3.1)[21]

[21] Presentation of this paradox repeats a banal mistake, which is quite common in the literature of the subject. It will be discussed later in the paragraph, for its elimination is at the same time the solution of the paradox.

3.3 God is not omnipotent (from 3.2)
4.1 God cannot create a stone that he cannot lift
4.2 There is something that God cannot do (from 4.1)
4.3 God is not omnipotent (from 4.2)

5. God is not omnipotent (from 2, 3.1 → 3.3, 4.1 → 4.3)
6. It is not true that there exists an omnipotent God (1 → 5)

Is every step of this argument correct?

A surprisingly simple and exceptionally logical solution for this dilemma, as well as for any other one referring to the concept of divine omnipotence, can be found in the ideas of philosophers such as Peter Damian (1007–1072), John Duns Scotus (ca. 1265–1308),[22] or René Descartes (1596–1650). God's omnipotence is not limited to non-contradictory actions, i.e., such that their effects are described by a set of non-contradictory sentences. That a human being is unable to imagine a contradictory situation does not mean that God is similarly limited too. Specifically, God can make two events, which are described by mutually contradictory sentences, come true at the same time. The assumption that God is above the principle of non-contradiction is a banal solution for any logical dilemma concerning the concept of omnipotence, together with the Paradox of the Stone. Rejecting the principle of non-contradiction, however, we give up all discursive thinking. With such an assumption, we have to replace talking about God with reverent silence. For this reason, let us assume that God honors the principle of non-contradiction, i.e., that he acts only in a non-contradictory way.

Although Thomas Aquinas (1225–1274) did not address this paradox himself, let us analyze the dilemma in a way that is concordant with his views. In the *Summa theologiae*, question 25, article 3, we read[23]: "We say that what is in the power of man, is possible for man. It cannot be claimed, therefore, that God is called omnipotent, because he can do everything that is possible for a created nature". According to Aquinas there are things that a man can do and God cannot, even though it does not mean at all that God is not omnipotent. For instance God cannot sense, move locally, etc. Moreover, God cannot do anything that would cause contradiction in his nature.[24] Taking all his into consideration, Olszewski, inspired by Ralph MacInerny,[25] states[26]: "Thomas himself, if he came across such

[22] Duns Scotus is not so unequivocal in his approach to the problem of God's omnipotence as Peter Damian. His distinction of *potentia ordinata* and *potentia absoluta* make the former part of his views resemble those, which accept the opinion that omnipotence is not opposed to the principle of contradiction.

[23] Thomas Aquinas, *Summa theologiae* (*Traktat o Bogu*), p. 372.

[24] Olszewski, *Komentarz do Kwestii 25 "O mocy Boga"*, [in:] Thomas Aquinas, *Summa theologiae*, (*Traktat o Bogu)*, p. 845.

[25] MacInerny [7] pp. 440–444.

[26] Olszewski, *Komentarz do Kwestii 25 "O mocy Boga"*, [in:] Thomas Aquinas, *Summa theologiae*, (*Traktat o Bogu*), p. 846.

a statement [of the Paradox of the Stone], would immediately consider applying a more general thesis, i.e. that God cannot do any thing, doing which would cause contradiction in his nature, thus eliminating the whole type of paradoxes, similar to the one with a stone". This means that all variants of the Paradox of a Stone, including the one considered below (a very difficult but soluble mathematical problem), do not pose any difficulty in the light of Aquinas's teaching. According to Thomas, God cannot create a stone that he could not lift, because it would be contrary with his nature.[27]

It seems that the essence of the Paradox of a Stone was reached by John L. Mackie in his 1955 work *Evil and Omnipotence*. He tries to prove there that God's omnipotence and infinite goodness cannot be compossible with evil existing in the world. Mackie gradually comes to a conclusion that in the analysis of the problem human free will play a pivotal role and this makes him face the Paradox of a Stone[28]: "This leads us to what I call the Paradox of Omnipotence: can an omnipotent being make things which he cannot subsequently control? Or, what is practically equivalent to this, can an omnipotent being make rules which then bind himself? (These are practically equivalent because any such rules could be regarded as setting certain things beyond his control and vice versa). The second of these formulations is relevant to the suggestions that we have already met, that an omnipotent God creates the rules of logic or causal laws, and is then bound by them". Later, Mackie states that "It is clear that this is a paradox: the question cannot be answered satisfactorily either in the affirmative or in the negative. If we answer 'Yes', it follows that if God actually makes things which he cannot control, or makes rules which bind himself, he is not omnipotent once he has made them: there are *then* things which he cannot do. But if we answer 'No', we are immediately asserting that there are things which he cannot do, that is to say that he is already not omnipotent"[29] Mackie tries to solve this paradox similarly to another paradox, which he calls the Paradox of Sovereignty.[30] According to Mackie, one has to distinguish between the first order sovereignty ($1'$) and the second order sovereignty ($2'$). Analogously, an omnipotent being possessing omnipotence ($1'$) has unlimited power to act, while the one possessing omnipotence ($2'$) has an unlimited power to determine what power to act things shall have. If we assume that God always has omnipotence ($1'$), we cannot claim, without contradiction, that there is any being which can act independent of God. If we claim, however, that in a

---

[27] According to Olszewski every dilemma based on the Paradox of a Stone would erroneously assume that God must have the power proper to his creatures or would amount to performing a task contradictory to the nature of God. Since the immaterial nature of God need not have any relation with the power of lifting stones, we assume, contrary to Olszewski and MacInerny, that the Paradox of a Stone presents God with a task which is contrary with his nature, which brings us back to the problem of God doing something evil.

[28] Mackie [8], reprinted in: Brian Davis (ed.), *Philosophy of Religion*, Oxford 2000, p. 589.

[29] Mackie [8], p. 589.

[30] Mackie [8], p. 590: "Can a legal sovereign make a law restricting its own future legislative power?".

given moment God has omnipotence (2′), and that he uses it to give some beings an ability to act independently, we must come to the conclusion that God thereafter has no sovereignty (1′). Mackie thinks that the core of his dilemma is God's continuing being. The problem could be overcomed if we reject the claim that God is a continuing being and that his actions can have temporal determination. Such an approach, however, faces another difficulty[31]: "No meaning can be [then] given to the assertion that God made men with wills so free that he could not control them. The Paradox of Omnipotence can be avoided by putting God outside time, but the free will solution of the problem of evil cannot be saved in this way, and equally it remains impossible to hold that an omnipotent God *binds himself* by causal or logical laws".

In this way, Mackie comes to a conclusion completely concordant with the argument from the Paradox of a Stone, stating that the concept of "omnipotence" is contradictory[32]: "Quite apart from the problem of evil, the Paradox of Omnipotence has shown that God's omnipotence must in any case be restricted in one way or another, that unqualified omnipotence cannot be ascribed to any being that continues through time. And if God and his actions are not in time, can omnipotence, or power of any sort be meaningfully ascribed to him?"

Mackie's above approach can hardly be called a solution, because it is, in a way, a very precise repetition. The short and simple argumentation of the paradox is replaced with a precise reasoning leading to the same statement that the concept of an omnipotent being cannot be reconciled with the principle of non-contradiction. Although he was correct in specifying the details of the problem he analyzed, Mackie failed to notice certain threads which, when taken into account, can throw new light on the dilemma. The solution presented later in this chapter takes Mackie's observations into account, but goes a little further noticing the need to consider yet another important ambiguity, related not so much to the concept of omnipotence, but with the argumentation in the Paradox of a Stone. To complete the analysis of the paradox, let us remind the existing proposals of its solution.

Modern authors of these proposals were unanimous that it is the paradox's formulation that is the key to its solution. For this reason, we shall relate the arguments as they stand the versions of particular authors who analyzed the paradox. Disappointingly, Mackie's logical distinctions were not taken into consideration by the authors of subsequent proposals, so in some aspects they are rather a step backward then a new insight into the paradox. For instance, those authors hardly notice something which was obvious for Mackie, i.e., that the power to create things God cannot control means only that if God creates such a thing, he will not be able to control it thereafter. Unfortunately, contrary to this apparently obvious understanding of the sentence "God can create a stone which he cannot [sc. will not be able to] lift", the very truth of this sentence is believed by them to mean that God is not omnipotent at all. It is clear then that the authors of papers written in reply to Mackie do not address the problem of the Paradox of a Stone properly.

[31] Mackie [8], pp. 590–591.

[32] Mackie [8], p. 591.

Since the early 1960s papers were published, whose authors tried to approach the Paradox of a Stone in such a way as to show the non-contradictory character of the concept of God's omnipotence. In reply to Mackie's paper, G. B. Keene published his own solution of the paradox in 1960. He formulated it as follows[33]:

A

> Either an omnipotent being can make things which he cannot control, or an omnipotent being cannot make things which he cannot control. If he can make such things then there is something which he cannot control; in which case an omnipotent being is not omnipotent.

The above formulation of the Paradox of a Stone is itself paradoxical, since it includes a logical error, by the way, one that frequently appears in texts by other authors writing about the Paradox of a Stone.[34] As other authors quoted below, Keene clearly does not notice this error. Moreover, he claims that he does not want to criticize that part of the argument, which is expressed by the unfortunate sentence: "If the being in question can create such things, there exists something, over which it has no sovereignty". Ostensibly, this sentence raises no doubts in him. By contrast, Keene questions the observation that the sentence Z_1 = "*X* cannot create things, over which *X* has no sovereignty" causes any limitation of *X*'s power to create things. Noticing that the sentence Z_1 can be replaced with other sentences, e.g., Z_2 = "*X* has sovereignty over whatever *X* can create", or Z_3 = "There is no such thing about which it can be truthfully predicated that *X* can create it and that *X* has no sovereignty over it", which preserve the sense of Z_1, he comes to a conclusion that nothing follows from the sentence Z_1 (consequently, also from sentences Z_2 and Z_3) that could refer to the omnipotence of *X*. He believes that his very observation contradicts the above quoted opinion of Mackie.

Keene's view was opposed by Bernard Mayo, who in 1961 repeated Keene's reasoning applying it to another sentence: "I cannot make a paper plane". Although this sentence has two equivalents, which preserve its sense: "Whatever I can make is not a paper plane" and "There is no such thing about which it can be truthfully predicated that I can make it and that it is a paper plane", it would be absurd to claim that any of these three sentences implies that my powers of making things are unlimited.[35] Later Mayo wonders whether there is any reason why there should be no analogy between his sentences and Keene's. It turns out that Mayo notices such a condition, the meeting of which makes the two examples cease to be analogous. This condition is substituting the expression "omnipotent being" for

---

33 Keene [9], p. 74.

34 That God can create something does not mean that this thing exists. Hardly any modal system accepts that the formula $\diamond\alpha \rightarrow \alpha$ is a thesis. Accepting the formula as one is related to the so-called trivialization of the possibility operator. If $\alpha \rightarrow \diamond\alpha$ is also a thesis of a system, then the equality $\alpha \leftrightarrow \diamond\alpha$ is a thesis of a system too, which means that the possibility of truth of a sentence has been identified with the truth of the sentence.

35 Mayo [10].

the variable *X* in Keene's example. Then, according to Mayo, the question whether *X* can create a thing over which it has no sovereignty becomes a question whether God can cause that an internally contradictory sentence can be a true sentence. The above argument *A* is thus no counterargument against the truthfulness of the sentence stating the existence of an omnipotent being. In Mayo's opinion, other concepts of omnipotence can be formulated following the argument *A*[36]: "An omnipotent being can either create a square circle or it cannot. If it can, some squares are circular, if it cannot, then it is not omnipotent". Again, we have the same unnoticed problem: is it really so that if an omnipotent being can create a square circle, square circles already exists, i.e., they have been created only because an omnipotent being can create them?

In his 1967 paper, *The Paradox of the Stone*, C. Wade Savage presents two versions of the Paradox of a Stone, one by G. I. Mavrodes from 1963, and his own.[37] Let us start the presentation from the version of Mavrodes quoted by Savage[38]:

B

(1) Either God can create a stone which He cannot lift, or He cannot create a stone which He cannot lift.
(2) If God can create a stone which He cannot lift, then He is not omnipotent (since He cannot lift the stone in question).[39]
(3) If God cannot create a stone which He cannot lift, then He is not omnipotent (since He cannot create the stone in question).
(4) Therefore, God is not omnipotent.

Savage relates[40] the proposal of Mavrodes[41] starting from an assumption that God exists. Naturally, an existing God is omnipotent or not. Next, Mavrodes states that if God is not omnipotent, the task of creating a stone such that he could not lift is not internally contradictory. If God is omnipotent, the sentence is internally contradictory. This remark has key importance for Mavrodes's reasoning. The above presented argument *B* of the Paradox of a Stone "proves" that God is not omnipotent in both possible cases described in points *B*(2) and *B*(3). Now Mavrodes thinks that the proof is erroneous, so actually it is no proof at all for the claim that God is not omnipotent because the assumption made in point *B*(3) does not imply lack of omnipotence in God. Since the task of creating a stone that being *X* cannot lift is internally contradictory if and only if *X* is an omnipotent being, the

---

36 Mayo [10], p. 250.

37 Savage [11].

38 Savage [11], p. 74.

39 Keene's error concerning the expression "can" is clearly repeated here. Allegedly God is not omnipotent, because he cannot lift a stone. Actually, the stone cannot be lifted, because it simply does not exist. But can this simple fact be evidence of lack of omnipotence? Or is it simply an example of an ill formulated problem?

40 Savage [11], pp. 74–75.

41 Mavrodes [12].

point $B(3)$ does not imply the point $B(4)$. As it can be seen, the assumption that God's omnipotence is limited to non-contradictory actions is essential here. This way the branched argument B is not concluded with a statement $B(4)$, common for both cases, and stating that God is not omnipotent. We can see that Mavrodes's proposal clearly refers to Mayo's idea of destroying the argument against non-contradictory character of the concept of omnipotence.

Savage strongly criticizes Mavrodes's proposal, calling it simply erroneous, and points to four dubious questions. Firstly, Savage notices that in the version B, the argument has the form of a dilemma, i.e., inference from three premises, of which one is an alternative and two, implications. Mavrodes, however, mistakenly identifies the reasoning presented in B with a *reductio ad absurdum* and consequently believes that he must accept an assumption that God is either omnipotent or not. Secondly, Savage stresses that a sentence assuming feasibility of a task of creating a stone that cannot be lifted is internally contradictory only when the sentence "God is omnipotent" is true in a necessary way. Thirdly, Savage questions the assumption made by Mavrodes that God exists, when the whole argument B of the Paradox of a Stone is to show the opposite. Finally, Savage questions the statement that inability to perform an internally contradictory task is no limitation for the agent, invoking the authority of Descartes.

Argument B can be developed in such a way as to refute at least some of Savage's objections, especially the ones of formal–logical character. Let us consider an argument B', which is such a version of the argument B that takes into account Mavrodes's reasoning:

B'

1. Omnipotent God can perform tasks if and only if these tasks are not internally contradictory (assumption of God's omnipotence)
2. God is omnipotent if and only if the task of creating a stone such that he could not lift is internally contradictory (Mavrodes's assumption)
3. Either God can create a stone which He cannot lift, or He cannot create a stone which He cannot lift. (logical truth)
   - 4.1 God can create a stone which He cannot lift
   - 4.2 The task of creating a stone which God cannot lift is not internally contradictory (from 1, 4.1)
   - 4.3 God is not omnipotent (from 2, 4.2)
   - 4.4 God is omnipotent or is not omnipotent (from 4.3)
   - 5.1 God cannot create a stone which He cannot lift
   - 5.2 The task of creating a stone which God cannot lift is internally contradictory (from 1, 5.1)
   - 5.3 God is omnipotent (from 2, 5.2)
   - 5.4 God is omnipotent or is not omnipotent (from 5.3)
6. God is omnipotent or is not omnipotent (from 3, 4.1 → 4.4, 5.1 → 5.4)

It is clear that similar to the sentence $B'(3)$ the conclusion $B'(6)$ is a logical truth of classical logic. As it can be seen, version B' of the argument does not prove anything important, nor is it a proof that God is not omnipotent. With respect to the Paradox of a Stone it has a destructive role, as it shows pointlessness of its argument. This reconstructed version shows, moreover, which assumptions are the core of Mavrodes's approach. Naturally, they are the assumptions $B'(1)$ and $B'(2)$. Savage questions the sense of using either. However, it follows from our earlier considerations that the first premise can be defended on the basis of an assumption that God has established the principle of non-contradiction and consistently observes it (*potentia absoluta*, *potentia ordinata*). Yet, the status of the premise $B'(2)$, or to be more precise: one of two implications that constitute it, seems to be far more questionable. For if we assume that God is omnipotent, are we really forced to accept the statement that God's creation of a stone which he could not lift is an internally contradictory sentence? It seems that assuming God's omnipotence alone is not enough to accept such an inference. Let us postpone the justification of this doubt until we present our own solution.

Savage's critique is not limited to Mavrodes's solution of the dilemma. He gives his own proposal using another version of the solution for the Paradox of a Stone, which according to him addresses the key problem of the paradox better[42]:

C

(1) Either x can create a stone which x cannot lift, or x cannot create a stone which x cannot lift.
(2) If x can create a stone which x cannot lift, then, necessarily, there is at least one task which x cannot perform (namely, lift the stone in question).[43]
(3) If x cannot create a stone which x cannot lift, then, necessarily, there is at least one task which x cannot perform (namely, create the stone in question).
(4) Hence, there is at least one task which x cannot perform.
(5) If x is an omnipotent being, then x can perform any task.
(6) Therefore, x is not omnipotent.

Since x is any being, this argument proves that the existence of an omnipotent being, God or any other, is logically impossible.

Savage notices that what distinguishes versions B and C is using the variable x in place of the name "God". One cannot blame the argument C for using erroneously an internally contradictory expression "a stone which God cannot lift", since the word "God" does not appear in that reasoning and therefore omnipotence of the one who is to create a stone which he cannot lift is not assumed. Moreover, in Savage's opinion, the version C is neutral with respect to the question whether inability to do something internally contradictory is a limitation of the power of an agent. The properties of the version C mentioned here have no influence on the solution proposed by Savage, which does not depend upon accepting or rejecting the statement saying that the task of creating a stone which its creator cannot lift is internally contradictory. Although Savage's position

[42] Savage [11], p. 76.

[43] Savage repeats here an error well known from the arguments of Keene and Mavrodes. The task which x cannot perform is lifting a stone, which, after all, does not exist. That x can create a stone does not mean that x creates it. Consequently, that x can create a stone does not mean that the stone is too heavy for x to lift. The expression "necessarily" used by Savage makes the argument even more piquant.

is different from that of Mavrodes, both question the same premise and differ only in its formulation. Savage believes namely that the false element of the argument is the sentence $C(3)$. He states adamantly that the sentence "there is at least one task which x cannot perform" follows from the sentence "x can create a stone which x cannot lift". Yet, he thinks that the sentence "there is at least one task which x cannot perform" does not follow from the sentence "x cannot create a stone which x cannot lift". He claims namely that the expression "x cannot create a stone" suggests mistakenly that there is something which x cannot do. Repeating Keene's idea he argues for it in the following way[44]: "x cannot create a stone which x cannot lift" can only mean "If x can create a stone, then x can lift it". Further analysis is done with the help of symbols: $S =$ stone, $C =$ can create and $L =$ can lift. Then, the version C becomes formalized thus:

D
(1) $(\exists y)(Sy \wedge Cxy \wedge -Lxy) \vee -(\exists y)(Sy \wedge Cxy \wedge -Lxy)$.
(2) $(\exists y)(Sy \wedge Cxy \wedge -Lxy) \supset (\exists y)(Sy \wedge -Lxy)$.[45]
(3) $-(\exists y)(Sy \wedge Cxy \wedge -Lxy) \supset (\exists y)(Sy \wedge -Cxy)$.

In accordance with the earlier comment, the formula $-(\exists y)(Sy \wedge Cxy \wedge -Lxy)$ is equivalent, by virtue of classical logic, with a classic formula $(y)[(Sy \wedge Cxy) \supset Lxy]$. Moreover, $D(2)$ is a classic tautology, while $D(3)$ is not. Consequently, in Savage's opinion, a sentence "There is a task which x cannot perform" does not follow from a sentence "x cannot create a stone which x cannot lift". Savage ponders over yet another question. Now he thinks that in spite of the analysis above, we have a certain propensity which makes us infer, from the fact that x cannot create a stone which x cannot lift, that x cannot perform any tasks, because that fact is the evidence of some limitation of x's power. To dispel this intuitive doubt, Savage proposes separating the creator of the stone from its lifter. Then we have to consider a following problem: *does the fact that x cannot create a stone which y cannot lift signify a limitation of x's power?* If y is omnipotent, he will lift any stone. Then, if x cannot create a stone which y cannot lift, does not necessarily mean that x's power is limited: x can create a stone of any poundage and y can lift a stone of any poundage. An analogous conclusion can follow from this analysis if we repeat it for the case when x and y is one and the same person.

[44] Savage [11], p. 77.

[45] Interpretation of this formula is not easy, even though the formula is a tautology of the classical calculus of quantifiers. That something can be created does not mean that it already exists. The antecedent of the implication, which is a formula $D(2)$, has rather strange sense, especially its fragment $(\exists y)Cxy$: "There is something which it will be possible to create in the future". The problem can be solved by giving up the existential sense of the quantifier. However, if we accept that the fragment of the formula in question should be read: "A stone can be created which x cannot lift", then we have another problem, this time with the consequent of the formula $D(2)$, which should be read similarly: "A stone can exist which x cannot lift". The problem still consists in the same, namely that a stone can exist does not mean it exists.

Consequently, Savage concludes as follows[46]: Whether $x = y$ or $x \neq y$, x's inability to create a stone which y cannot lift constitutes a limitation on x's power only if (*i*) x is unable to create stones of any poundage, or (*ii*) y is unable to lift stones of any poundage.

Since omnipotent God can both create stones of any poundage and lift stones of any poundage, the fact that God cannot create a stone which he cannot lift is for Savage a necessary consequence of both mentioned manifestations of omnipotence. Savage ends his paper saying that from the assumption that God can create stones of any poundage and lift stones of any poundage, follows a conclusion that God cannot create a stone which he cannot lift. In a note, Savage admits that in the final section of his paper Mavrodes notices that fact himself, but his earlier errors have distorted his intuitions about it.

Apparently, Savage's solution is simple: God can create stones of any poundage and lift stones of any poundage, which is to testify for his omnipotence. Still, does the Paradox of a Stone question the ability of God to create or lift stones of any poundage? Is it really the problem here? Clearly not. The core of the paradox is a problem which Savage did not address. As in the case of Mavrodes's proposal, let us leave pinpointing the problem until we present our solution.

Another proposal, different from the two discussed above, was presented in a 1979 paper *A solution to the stone paradox* by David E. Schrader. He formulates the Paradox of a Stone in a new way[47]:

E

(1) Either God can make it the case that there exists a stone such that it is not the case that God can make it the case that the stone is lifted, or it is not the case that God can make it the case that there exists a stone such that it is not the case that God can make it the case that the stone is lifted.
(2) If God can make it the case that there exists a stone such that it is not the case that God can make it the case that the stone is lifted, then it is not the case that God is omnipotent.[48]
(3) If it is not the case that God can make it the case that there exists a stone such that it is not the case that God can make it the case that the stone is lifted, then it is not the case that God is omnipotent.
(4) Therefore, it is not the case that God is omnipotent.

Next, he proposes considering two possibilities, either (I) it is logically necessary that God is omnipotent, or (II) it is not logically necessary that God is omnipotent. Using the semantics of possible worlds he comes to a conclusion that in the first case it is not logically possible that in any of the possible worlds there

[46] Savage [11], p. 78.

[47] Schrader [13], pp. 260–261.

[48] Schrader repeats here the error of Keene, Mavrodes and Savage, analyzed already several times, unlike them, however, he notices it and that is why he criticizes the second point of the argument *E*.

exists a stone which God cannot lift. *Eo ipso*, it is not logically possible for God to cause that such a stone exists. This does not mean, however, that God is not omnipotent, for Schrader accepted the following definition of a being which is not omnipotent:

$$x \text{ is omnipotent} \overset{df}{=} \text{for any sentence } p, \text{ if it is logically possible that } x \text{ make it the case that } p \text{ is true, then } x \text{ can make it the case that } p \text{ is true.}$$

According to Schrader, he proves the falsity of the sentence $E(3)$. The above definition is the consequence of the one he accepted earlier for an omnipotent being:

$$x \text{ is not omnipotent} \overset{df}{=} \text{there is some sentence } p \text{ such that it is logically possible that } x \text{ make it the case that } p \text{ is true, and } x \text{ cannot make it the case that } p \text{ is true.}$$

In the case (II), when it is not logically necessary that God is omnipotent, there is such a world in which there is a stone he cannot lift. Consequently, there is a logically possible world in which God is not omnipotent, which means that it is not logically necessary that God is omnipotent. In Schrader's opinion, this conclusion does not say anything about God's omnipotence in the real world. God will be omnipotent as long as he does not decide to create a stone which he cannot lift. This way, Schrader avoided the mistake of incorrect understanding of the expression "can create": that God can create something does not mean that it has already been created by him. Finally, Schrader states the inadequacy of the argument *E*, with respect to both the assumption (I) and (II). Since it is always so that either (I) or (II) is true, the argument *E* does not prove the paradox. Schrader's approach resembles that of Mavrodes, who also pointed to the pointlessness of the argument in the Paradox of a Stone.

The above solutions seem to be interesting not only from the logical point of view. They also show how deep meaning can be found in the concept of "omnipotence". Savage's proposal seems to be especially valuable here, for it shows the correct sense of the sentence: "God cannot create a stone which he cannot lift". The truth of this sentence does not have to imply the truth of a sentence, which predicates about the existence of a task which God cannot perform. Moreover, all solutions can be an introduction to interesting analyses concerning the being known as God and referring to such concepts as "omnipotence", "logically necessary omnipotence", "power limiting action" or "internal contradiction". Regrettably, all of the solutions seem to repeat the same error, although the conclusion from Schrader's analysis may suggest that the last of the solutions which we discussed is free from that error. As we have already noted several times the error consists in ignoring the importance of time, or sequence of events, for the whole argument. Swinburne, the author of the book *The Coherence of Theism* presenting the position of Aquinas, believes that an omnipotent being cannot change the past. Thanks to it, Swinburne introduces the parameter of time to the considerations of omnipotence, defining an omnipotent being in the following way[49]:

[*def. D*] A person *P* is omnipotent in time *t* if and only if he is capable of causing every logically contingent state of affairs after time *t*, whose description does not result in *P* not having caused it in *t*.

In Swinburne's opinion, this definition is to exclude from considerations of omnipotence all sentences which are related to states of affairs that are unfeasible for logical reasons. Consequently, if a state of affairs has occurred, even an omnipotent being cannot cause something to happen if it has already happened. Accordingly, Swinburne substitutes the expression "logically contingent state of affairs" for "logically possible state of affairs". Secondly, Swinburne is afraid that an unmarried spirit *P* is unable to get divorced. This is why he strengthens the omnipotence definition *C* with an additional condition: "whose description does not result in *P* not having caused it in *t*".[50]

It seems that thanks to the inclusion of time in his considerations of omnipotence, Swinburne avoids the error of all his predecessors, even though he does not notice it in their proposals. Criticizing their proposals he pays no attention to the fact that one can hardly claim that God is not omnipotent because he cannot lift a stone, when the stone, as it is assumed, does not yet exist. As his predecessors, Swinburne presents a new formulation for the argument in the Paradox of Stone[51]:

F

(1) Either *P* can in [the moment] *t* cause the existence of a stone, whose lifting *P* cannot subsequently cause, or *P* cannot cause the existence of a stone, whose lifting *P* cannot subsequently cause.
(2) If in *t P* can cause the existence of a stone, whose lifting *P* cannot subsequently cause, then after *t*, there necessarily exists at least one logically contingent state of affairs, whose description does not require that *P* did not cause it in *t*, and that *P* is not able to cause it (sc. lifting the stone) in *t*.
(3) If in *t P* cannot cause the existence of a stone, whose lifting *P* cannot subsequently cause, then after *t*, there necessarily exists at least one logically contingent state of affairs, whose description does not require that *P* did not cause it in *t*, and that *P* is not able to cause it (sc. lifting the stone) in *t*.
(4) Consequently, there exists at least one logically contingent state of affairs after *t*, whose description does not require that *P* did not cause it in *t*, and which *P* is not able to cause in *t*.
(5) If *P* is an omnipotent being, then after *t*, *P* is able to cause every logically contingent state of affairs, whose description does not require that *P* did not cause it in *t*.
(6) Consequently, *P* is not omnipotent.

49 Swinburne [14], p. 212.
50 Swinburne [14], pp. 211–212.
51 Swinburne [14], pp. 216–217.

In this version of the paradox, Swinburne states that the sentence *F*(2) does not have to be true for every *P*[52]*:* "We assume that *P* is able to cause the existence of a stone with such properties that *P* is then unable to cause its lifting. Now what is the state of affairs that *P* cannot cause? Lifting the stone in question. Lifting of which stone cannot *P* cause? Lifting the next stone created by *P*? There is no reason to assume that *P* will create more stones, and even if it does, there is no reason to suppose that *P* will be unable to cause their lifting. […] There is no reason to suppose that *P* will cause the existence of such a stone; from the fact that he can do it, it does not follow that he will. It is true that if an omnipotent being actually uses his ability (in contrast to his mere possession of it) to cause the existence of a stone which is too heavy for him to cause its lifting, then he will cease to be omnipotent. […] The person can remain omnipotent for ever, since he will never use his power to create stones which are too heavy to lift, forces too strong to resist, or worlds too capricious to control". One can say that this analysis of Swinburne is really close to the solution we propose below.

As far as the sentence *F*(3) is concerned, Swinburne's opinion, according to which it can be true with certain understanding of the being *P*, contradicts Savage's claim. First Swinburne shows the falsity of Savage's position; although he admits that the sentence "*P* cannot create a stone which *P* cannot subsequently lift" is logically equivalent to the sentence "If *P* can create a stone, then *P* can lift it", he adds that it does not follow from this fact that the being *P* is omnipotent. He rightly observes that[53]: "These propositions state that if *P* creates a stone, it must be that it is later capable of lifting it. This means that *P* cannot endow any being it creates with the power which later resists *P* lifting it". Swinburne's opinion can be strengthened with the following, rather unquestionable, example: it is true that "I cannot write a letter, which I cannot subsequently read". Obviously, this sentence is logically equivalent to the one: "If I can write a letter, I can subsequently read it". Yet, does it really follow from the truthfulness of both sentences that I am omnipotent in letter writing? Are there not foreign languages, in which I cannot write even a shortest letter? Of course, there are letters which I cannot write. Regrettably, this discussion vividly resembles the old argument between Keene and Mayo.

Swinburne does not limit himself to criticizing Savage's approach. He also proposes his own analysis showing when the sentence *F*(3) is true and when it is false[54]: "*F*(3) will be true if we assume that the only kinds of being, which can cause states of affairs, are beings that are such and such individuals in such and such time, regardless of whether they gain or lose power, knowledge, or other properties, and whether they subsequently cease to exist or not. Our every day understanding of 'person' is such that a person *P* remains the same person if it gains or loses power, knowledge, or other properties; that it later ceases to exist

[52] Swinburne [14], pp. 217–218.

[53] Swinburne [14], p. 215.

[54] Swinburne [14], pp. 218–219.

has no influence upon what it was earlier. […] If beings of this kind are the only kind of being, which can cause states of power, *F*(3) will be true for every *P*. For then, in the time later than *t*, the existence of the stone, which *P* cannot subsequently cause to be lifted, will be a logically contingent state of affairs after *t*, the description of which does not require that *P* did not cause it in *t*. But perhaps there can be such beings, which would not be such and such individuals if they subsequently lost some of their power or ceased to exist, and such beings can cause states of affairs. If such beings can exist and *P* is one of them, then *F*(3) will be false about *P*. Then, after the time *t*, the existence of a stone, the subsequent lifting of which cannot be caused by *P*, is not necessarily a logically contingent sate of affairs after *t*, whose description does not require that *P* did not cause it in *t*. For if *P* is omnipotent in *t*, it would not be the same individual in *t* if it subsequently lost that omnipotence (or ceased to exist). The existence of the stone would not be logically consistent with the way the world developed until the time *t*, since it had included the existence of *P*". Naturally, the person who cannot remain himself if it changes is, in Swinburne's opinion, God. Swinburne summarizes his argument in the following way[55]: "God cannot cause a stone of the described character, for the existence of such a stone requires that God has not caused it".

Summing up Swinburne's analysis, he comes to a conclusion that sentences *F*(2) and *F*(3) are both false. *F*(2) is false no matter who the person *P* is, *F*(3) is false when *P* is God. One could agree with the argument for the falsity of *F*(2), but with respect to the other argument some doubts arise, especially that its author is Swinburne, for he repeats there, although in different words, the claim of Savage, which he has criticized earlier: if God creates a stone, he will lift if, for if he could not lift it, it means he could not create it; consequently, God is omnipotent.

This way Swinburne has proved God's omnipotence through limiting his power, claiming that he cannot create a stone which he cannot subsequently lift. For if we limit omnipotence through a suitable definition, we can then show that a being which possesses such an imperfect omnipotence can be unable to do something and still remain omnipotent in the new, limited sense. Formally, the paradox disappears. It is worthwhile, however to pose a question whether elimination of the paradox eliminates doubts concerning such a solution too. Is the solution not as paradoxical as the paradox itself? For instance, let us imagine that according to the new definition of omnipotence, an omnipotent being is one which can do anything except creating stones, any stones, both light and heavy. Then the argument of the Paradox of a Stone ceases to be paradoxical. If God cannot create any stone and keeps being omnipotent, he also stays omnipotent when he cannot create a stone which he cannot subsequently lift. The paradox disappears but the problem stays. This problem is the hardly intuitive concept of "omnipotence". Naturally, Swinburne did not accept such a radically limited definition of omnipotence. It seems, however, that his analysis falls under that scheme.

---

[55] Swinburne [14], p. 219.

Let us analyze once again the argumentation of the Paradox of a Stone using as broad concept of omnipotence as logically possible. Naturally, the omnipotence suspending the principle of contradiction would be the broadest concept of all. Analyzing that case, however, would lead to trivialization of the problem of the paradox we discuss in this chapter. Let us assume, therefore, that an omnipotent being is limited by the principle of non-contradiction only. Thus we allow that an omnipotent being can change the events which have already occurred. It would be strange to accept that omnipotent God can annihilate the world but cannot alter it, which after all is nothing but a partial or relative annihilation. It is difficult to tell why an omnipotent being should be limited by time, say, that in 1973 it would not be able to do something that was to happen in 1935. If a being can do everything which is not contradictory, it can, *par excellence*, change all consequences of altered facts, including the changes in consciousnesses of whole generations with respect to the "new" facts,[56] so that their knowledge were consistent with the present correction of the past. Moreover, let us assume that God can sin, though the word "can" does not refer to his desires or any kind of aptitude or inclination. This word expresses only the fact of existence of objective conditions which make sinning possible. To avoid any interpretative doubts and in accordance with our plans, outlined above, we shall understand God's omnipotence as his ability to do everything without any extra-logical limitations, bound only by the principle of non-contradiction.

Another problem, important for our solution, is finding the actual sense of the sentence: "God can create a stone which he cannot lift". Here we encounter the error which we have found in all the above mentioned proposals except the argument *F* presented by Swinburne and, obviously, the solution following the opinions of Peter Damian and René Descartes. The error can be expressed in the words of Savage, which we have already quoted, namely the claim that the sentence "there is at least one task which *x* cannot perform" follows from the sentence "*x* can create a stone which *x* cannot lift".[57] Let us assume that God can do everything that is logically possible. Let us also assume that God can create a stone which he cannot lift. What is the task that according to Savage God cannot do with this assumption? Shall it be lifting a stone? But which stone? The one which does not yet exist, because God has not created it yet, even though he can do it? Indeed, it is impossible to lift something that does not exist. So if Savage thinks that God cannot lift a stone which he has not created yet, he is right. But is a sentence understood in that way logically possible? Since as long as God does not create a stone, the stone does not exist, its lifting is logically impossible. For if an object *K* does not exist, the sentence "*X* can lift *K*" expresses a situation which cannot occur.[58] It has to be accepted that the sentence "God can create a stone which he cannot lift" should be, following the

[56] Of course, according to a new consciousness those fact would be old.

[57] Savage [11], p. 77.

[58] Already Aristotle in the *Topics*, IV, 126a, warned against an error of identifying the potency of doing something with the act of doing it.

quoted above opinion of Mackie, replaced with this one "God can create a stone which he will not be able to lift if he creates it".

Taking into account both analyzed questions, let us formulate again, this time correctly, the Paradox of a Stone according to the first version given in the preceding paragraph.

G

1. Either God can create a stone which he cannot lift or God cannot create a stone which he cannot lift.
   2.1. God can create a stone which he will not be able to lift if he creates it.
   2.2. There is something which God cannot do. (from 2.1)
   2.3. God is not omnipotent. (from 2.2)
   3.1. God cannot create a stone which he will not be able to lift if he creates it.
   3.2. There is something which God cannot do. (from 3.1)
   3.3. God is not omnipotent. (from 3.2)
4. God is not omnipotent. (from 1, 2.1 → 2.3, 3.1 → 3.3)

Already the correct formulation of the paradox makes clear that its argument is broken, for the sentence $G(2.2)$ does not follow from the sentence $G(2.1)$. In this place, the argument is similarly broken in all other versions of the paradox. Starting with version *A* by Keene and ending with the above presented version *G*, we find a false sentence "If the being in question can create such things, there is something over which it has no sovereignty". In Mavrodes's version *B*, the second sentence is false, similarly as in Savage's *C* and Schrader's *E*. The second sentence in Savage's formalized argument *D* is an interesting case. It might seem that the sentence $D(2)$, $(\exists y)(Sy \wedge Cxy \wedge -Lxy) \supset (\exists y)(Sy \wedge -Lxy)$ is true, that its scheme is a tautology of the classical logic of predicates. Yet even here there is a mistake, which causes the antecedent of this implication $(\exists y)(Sy \wedge Cxy \wedge -Lxy)$ does not have to be true. x can create a stone y $(Sy \wedge Cxy)$, which does not mean that there is something which x can or cannot lift. One could free the condition $D(2)$ from this error by accepting another understanding of the predicate L, e.g.: $L(x,y) =$ "x could not lift y (in the future)". The meaning of L not withstanding there is another difficulty here. In the formula $D(2)$, the first existential quantifier has one sense and the second one, another. In its first instance, the quantifier is understood potentially: there is something which x can create. In the second instance, we already have a "hard" statement of existence of the respective stone. It can be seen, therefore, that all the above expressions of the Paradox of a Stone are marked by the error which Aristotle warned about.

Let us also observe that the task for God to create a stone such that he cannot lift it can be substituted with another example, for instance, to create a flower with such a delicate smell that he could not sense it. Naturally, it is possible to formulate many other tasks modeled on this one, out of the Paradox of a Stone.[59]

[59] Swinburne [14], p. 213, gives two other possible versions of this paradox, which does not assume any corporeity of God: 1. Can God create a planet too great for him to divide?; 2. Can God make the universe too dependent on himself to destroy it?

What is most interesting in all those possible tasks is that which joins them and makes them copies of the problem analyzed in this paragraph. Though in different ways, they all express one and the same question, namely, whether it is possible for God to limit his power in some particular way or aspect. Thus in all possible questions to be analyzed, so especially in the question concerning the memorable stone, it all boils down to one and the same problem, i.e.,

Can God in any particular aspect limit his omnipotence?

As it can be seen, Mackie's observations are most appropriate also in this question. Creating a stone God will, naturally, limit his omnipotence with respect to lifting this particular stone only. Other stones, even heavier ones than this, will be lifted by God without difficulty in accordance with his power, unlimited with respect to other stones.[60] Similarly, creating the flower mentioned above God will limit his power with respect to recognizing the scent of this particular flower only. The smells of all other flowers, even the most delicate ones, will be distinguishable for God, since he preserves his power, unlimited in all other respects, to recognize all other flowers. For this reason, Savages proposal cannot be considered to be a solution of the Paradox of a Stone, since the core of the paradox is not questioning the statement that God can lift infinitely heavy stones, or stones of any poundage. The paradox poses a question whether it is possible for God to limit his omnipotence in any respect. The founding statement of Mayo's and Mavrodes's proposals, according to which the assumption of God's omnipotence implies an internal contradiction of God's task of creating a stone which he cannot lift, also loses its evidence. It is not clear, after all, why the being, which we assume only to be omnipotent, could not give up its omnipotence or limit it in some strictly defined range.

A negative answer to the question whether God can create a stone which he cannot lift when he creates it would mean that there is something which God cannot do. This unfeasible task would be precisely the creation of the stone and the limitation of his power with respect to lifting it.[61] Consequently, if God can do everything, then performing that task should be all the more possible for him. That God can create a stone which he will not be able lift or that he will limit his power with respect to lifting an already existing stone does not prove that he has already done so. So as long as God has not done it, he cannot be denied omnipotence, for no limitation of it has taken place yet. Now let us assume that God has created a stone which he cannot lift. Since the stone already exists, it makes sense to speak of impossibility of its lifting. There is a task, then, which God cannot perform. Yet, is there any contradiction with God's omnipotence? Now there is no such contradiction. After all, creating the stone is a form of renouncing omnipotence, by the way, a very precise one. Omnipotence is limited with respect to lifting this

[60] It is clear then that the opinion according to which the stone which God could not lift is the heaviest in the world is mistaken.

[61] It can even be assumed that the limitation of Gods power refers to an already existing stone.

particular stone only. The possibility for God to act in all other respects remains the same as before. Still it seems that omnipotence has such character that any limitation of it is de facto its annihilation. Therefore, by creating a stone which he cannot lift God has limited his omnipotence, i.e., he renounced it. God is no longer an omnipotent being. He is not even though he was. Consequently, the statement that the existence of a stone which God cannot lift is the evidence of God not being omnipotent is true. It is not true, however, that the existence of that stone is in contradiction with God's antecedent omnipotence. Creation of that exceptional stone is equivalent with the fact that in God occurred a qualitative change, important from the point of view of the Paradox of a Stone argument. Now there is a quality which God had but has no more. This quality is, or was, omnipotence. But undergoing a change God ceases to be God. This fact explains a possible doubt whether the expressions like "before", "after" or "before the change", "after the change" make any sense when applied to God, who exists outside time.

We should note, however, that ceasing to be God, God undergoes a change and if so, he becomes a being contained in time, for physics assumes that one can speak of time when two different states can be distinguished. Losing the attribute of omnipotence, God undergoes a change and thus ceases to exist outside time. Moreover, creation of a stone, whose existence limits omnipotence seems to be an irreversible act of renouncing omnipotence. For if the created stone is to be impossible to lift, it means that its creation has caused a new situation, such that there is no way to lift the stone. One of those ways would be the return to omnipotence. This means that by limiting his power of lifting the stone God has done it irrevocably, once and for all.

Our analysis shows that the core of the Paradox of a Stone is the question about the understanding of the concept of "God". The reply to the question:

Can God create a stone such that he cannot lift?

depends on the answer to another question:

Can God change, i.e., destroy himself in some respect?

Consequently, a positive reply to the former question is at the same time a positive reply to the latter. Of course, our reply can be "No", but also in this case it will be a reply to both questions. Here we should return to our doubts concerning the proposals of Mayo and Mavrodes, and referring to their statement that omnipotence of a being *X* implies an internal contradiction of a task of creating a stone which *X* could not lift. It can be seen that the task is internally contradictory if beside omnipotence we also assume immutability of *X*. Now since both omnipotence and immutability are almost universal, not just in Medieval philosophy, to be the attributes of God, it seems that the task in question is indeed internally contradictory. God remains omnipotent regardless of his inability to create a stone such that he cannot lift just like he remains omnipotent regardless of his inability to create a square circle. Such a solution of the Paradox of a Stone can be summarized in the following points:

H

1. Omnipotent God can create a stone which he will not be able to lift;
2. Omnipotent God remains omnipotent as long as he has not created this stone;
3. Creating this stone limits the power of God, or precisely the existence of this stone limits the power of God;
4. Limiting God's power causes that God ceases to be omnipotent;

Indeed, with such argumentation the Paradox of a Stone disappears. God was omnipotent and ceased to be omnipotent—there is no contradiction here, unless God cannot change, but then the analyzed task is internally contradictory.

This conclusion notwithstanding, our solution does not end the discussion of God's omnipotence. Let us assume that God has created a stone *K* which he cannot lift. Does the existence of the stone *K* really limit the omnipotence of God? According to the solution presented above, yes. Yet lifting of the stone is the only task God cannot perform. He can take all other actions, for instance change the stone's color, chemical composition, shape, position, finally …he can destroy it. Let us make another assumption that God destroys the stone *K*. Is there anything that limits God's omnipotence in this situation? Of course, there is no such thing. Does it mean that God has changed again, this time returning to omnipotence? Or is the fourth sentence of the argument *H* simply wrong?

The problem caused by the Paradox of a Stone can be compared to another well-known philosophical problem, the problem of human free will. Has God limited his omnipotence by endowing man with free will? If man has free will, he can oppose God, i.e., he can take actions against the will of God. Yet God does not stop man, for otherwise he would deprive him of free will. Moreover, God gives man his free will for ever, just like his power of lifting the stone *K* has been limited for ever. Thus the power of God becomes limited. Does such limitation of God's power mean that God deprives himself of his omnipotence, or at least a part of it? Not really. After all, God can, if so wishes, annihilate man, as he did with the stone *K* in our example. The close resemblance of the problem of a stone to the question of human free will result in an equivalent paradox:

Paradox of Human Free Will[62]

1. Either God can endow man with free will, or God cannot endow man with free will. (logical truth)
   2.1. God can endow man with free will.
   2.2. If God endows man with free will, he will have to honor man's decisions, i.e., it will be impossible for him not to honor them.

[62] Because of the close relation of this paradox to the Paradox of a Stone, the Paradox of Human Free Will will not be discussed separately, since the solution we propose for the Paradox of a Stone applies to the Paradox of Human Free Will as well.

2.3. Man has free will (additional assumption)
2.4. There is something which God cannot do (from 2.2 and 2.3)
2.5. God is not omnipotent (from 2.4)
3.1. God cannot endow man with free will.
3.2. There is something which God cannot do (from 3.1)
3.3. God is not omnipotent (from 3.2)
4. God is not omnipotent (from 1, 2.1 → 2.5, 3.1 → 3.3)

God can create a stone which, if he creates it, he will not be able to lift. If God creates such a stone, he will limit his power with respect to lifting it and only in this respect. This has nothing to do with omnipotence, which remains unchanged. It is proved by the fact that God can destroy the stone. Then the situation is exactly the same as before the creation of the stone, i.e., there is nothing which could limit God's omnipotence. This means that for God creation of the stone which he could not lift is analogous to creation of the world alongside with establishment of the principle of non-contradiction. God can annihilate the world at the same time revoking the principle of non-contradiction, he can also create a new world replacing this principle with another fundamental law of logic, which would be impossible for man to imagine. Thus the problem of God's creation of a stone which he could not lift is solved by recourse to the distinction between God's power and omnipotence, similar to the Medieval distinction between *potentia dei absoluta* and *potentia dei ordinata*. The Paradox of a Stone can thus constitute an important argument supporting the Medieval distinction of the two powers.

This means that the Paradox of a Stone is yet another problem resulting from the error of ambiguity: limiting of God's power is not the same as limiting his omnipotence. God can limit his power in many ways without diminishing his omnipotence through it.

We should not forget the lessons from the proposed solutions, which multiplied the same error which Aristotle warned against: identifying the power of doing something with doing it as such. One can dare to say that everyone, who ignores this warning in his struggles with the paradox, actually analyzes another paradox, or in fact a pseudo-paradox, which can be expressed in a question: Can God lift a stone which does not exist? It is a pseudo-problem, because the task mention in the question is logically contradictory.

References

1. Ajdukiewicz, K. (1931). Paradoksy starożytnych [Paradoxes of ancients]. In *Język i Poznanie* (vol. 1, pp. 135–144). Warszawa: PWN, 1985.
2. Lenzen, W. (1977). Protagoras versus Euathlus: Reflections on a so-called paradox. *Ratio, XIX*, 176–180.
3. Åqvist, L. (1981). *The Protagoras case: An exercise in elementary logic for lawyers*. In Tankar och tankefel tillägnade Zalma Puterman. Filosofiska Studier utgivna av Filosofiska Föreningen och Filosofiska Institutionen vid Uppsala Universitet, Uppsala, pp. 211–224.

4. Pszczołowski, T. (1962). *Umiejętność przekonywania i dyskusji [The skill of persuasion and discussion]*. Warszawa: Wiedza Powszechna.
5. Kotarbiński, T. (1957). Paradoksy starożytne i antynomie nowoczesne u podstaw semantyki, logiki formalnej, teorii mnogości i kinematyki (teorii ruchu) [Ancient paradoxes and modern antinomies as fundaments of semantics, formal logic, set theory and kinematics], In: Wykłady z Dziejów Logiki [Lectures in History of Logic], Zakład Narodowy im. Ossolińskich we Wrocławiu, Łódzkie Towarzystwo Naukowe, Łódź, pp. 186–192.
6. Grodziński, E. (1983). *Paradoksy semantyczne [Semantical paradoxes]*. Wrocław: IFiS PAN, Zakład Narodowy im. Ossolińskich we Wrocławiu.
7. MacInerny, R. (1986). *Aquinas on divine omnipotence*. In L'homme et son univers au Moyen-Ñge. Actes 7ème congrès international de philosophie médiévale (1982), Louvain-la-Neuve, pp. 440–444.
8. Mackie, J. L. (1955). Evil and omnipotence. *Mind, 64*, 200–213.
9. Keene, G. B. (1960). A simpler solution to the paradox of omnipotence. *Mind, 69*, 74–75.
10. Mayo, B. (1961). Mr Keene on omnipotence. *Mind, 70*, 249–250.
11. Savage, C. W. (1967). The paradox of the stone. *Philosophical Review, 76*, 74–79.
12. Mavrodes, G. I. (1963). Some puzzles concerning omnipotence. *Philosophical Review, LXXII*, 221–223.
13. Schrader, D. E. (1979). A solution to the stone paradox. *Synthese, 42*, 255–264.
14. Swinburne, R. (1977). *Spójność teizmu* [*The coherence of theism: Introduction*] (transl. Tadeusz Szubka). Kraków: Wydawnictwo "Znak", 1995.

Chapter 4
Paradoxes of Self-Reference

4.1 Möbius Ribbon and Klein Bottle, or Self-Reference in Mathematics[1]

When we think of problems related to self-reference, we usually refer to Liar Antinomy and Russell's Antinomy.[2] This is probably why we are inclined to see self-referent constructions as ones that generate contradictions. It is easy to notice, however, that it is our thinking (talking) about these constructions rather than they themselves that is the source of contradiction. A spectacular case can be found in Möbius ribbon—a construction that is not contradictory, because …it exists.[3] Naturally, if we make mistakes in its description, a contradiction will appear as their inevitable consequence. Still, it will be the consequence of description and not of the construction as such. Hence, the sense of paradoxical character in we have encountering some self-referent constructions is an argument showing that our intuition is not infallible, since it denies existence of structures that apparently

[1] Möbius's ribbon is a construction, which has rightly become the symbol of self-reference. In the analysis of the Liar Antinomy below, Möbius's ribbon will play a key role. Therefore, it merits a special interest.

[2] Russell's Antinomy can be presented as follows: *Let Z be the set of all sets, which are not their own elements. Is the set Z its own element? Let us assume it is. Then the set Z is outside the set Z, since it contains all and only those sets, which are their own elements. Consequently, the set Z does not belong to itself. Thus we have proved that if the set Z is its own element, it is not its own element. Then the set Z belongs to the set Z, because it contains all sets, which are not their own elements. Consequently, the set Z is its own element. Thus we have proved that if the set Z is not its own element, it is its own element. Both of the proved inverted implications give an equivalence: the set Z is its own element if and only if the set Z is not its own element.* This is the result of Cantor's accepting the so called *axiom* of abstraction, which identifies the existence of a property φ with the existence of a set $A = \{x : \varphi(x)\}$, of all and only these objects x, which possess this property. Contradiction is generated by the so called Russell's set $R = \{x : x \notin x\}$. Cantor's set theory paradoxes go beyond the scope of this book and are, therefore, not included.

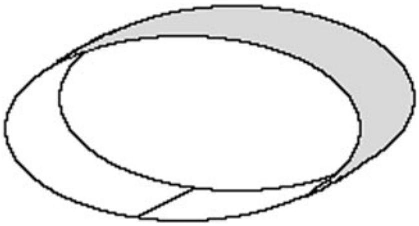
[3] Following tradition we accept that everything, whose existence does not imply contradiction, can exist.

P. Łukowski, *Paradoxes*, Trends in Logic, 31,
DOI: 10.1007/978-94-007-1476-2_4,

can exist. Möbius ribbon and Klein's bottle are examples of such structures existing in mathematics.[4] What is more, Möbius ribbon exists not only as a mathematical object but can be produced as a material object out of a simple strip of paper.

Let us consider a strip of paper, one side of which is white while the other is grey. Let us call the white side *a* and the grey one, *b*.

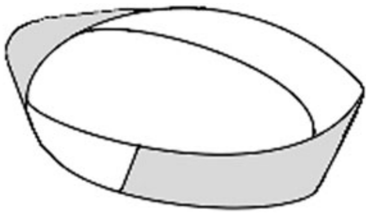
Both ends of the strip can be joined together in a simple way, where side *a* on one end is joined with side *a* on the other end. Naturally, side *b* will then be joined in the same way and the line of junction will not stand out, since it will join two grey ends on side *b* and two white ends on side *a*:



The ends of the strip can be joined in another way, however: side *b* at one end can be joined with side *a* at the other, so that side *a* is continued as *b* after the junction. Then the line of junction stands out as a border between the grey side and the white side. The construction we get in this way is an example of a one-sided surface, known as "Möbius ribbon".[5] Indeed, its side *a* becomes side *b*, so side *b* becomes side *a*:

[4] It is clear that this paragraph could be a part of a chapter devoted to paradoxes resulting from our imperfect intuition. However, the essence of Möbius's ribbon is so close to the Liar's Paradox that presenting this self-referent construction in the chapter on paradoxes of self-reference seems sufficiently justified.

[5] August Ferdinand Möbius (1790-1868), a German mathematician and astronomer, professor of the University of Leipzig, laid foundations for projective geometry and topology. He won the greatest renown for his 1858 discovery of a ribbon that has only one side, later known as "Möbius ribbon".



In case of Möbius ribbon, the names “side a of the strip” and “side b of the strip” cease to denote two different objects, for the peculiar way of joining of the strip has made side a one with b. Consequently:

$$a = b$$

Apparently, Möbius ribbon is a relatively simple construction, free from any contradiction in spite of its surprising one-sidedness. It is possible, however, that our speaking (thinking) about this structure shall lead us to a conclusion that it does contain something contradictory after all, despite its evident existence and, hence, non-contradictoriness. It is enough to call side b “side not-a” and side a “side not-b”, which is admissible, since the strip does not have a third side. Then joining a twisted strip of paper results in a Möbius ribbon, which has an astounding property: its side a is side not-a. Hence:

$$a = \text{not-}a.$$

If we make yet another assumption that what is not-a is not a, we immediately obtain a conclusion that is a contradictory proposition: side a is not side a. Similarly, we shall conclude that side b is not side b:

$$a \neq a \quad \text{and} \quad b \neq b.$$

What is then side a if it is no longer side a and what is b when it ceases to be b? Moreover, the names “side not-a” and “side not-b” lose their sense too, for we also get:

$$\textit{not-a} \neq \textit{not-a} \quad \text{and} \quad \textit{not-b} \neq \textit{not-b}.$$

His is a patent absurd. That side a is still side a can be best seen in the fact that it is still white. Similarly, side b is still side b. But the fact that both white and grey can be seen on the same side is an evidence that the two sides which used to be separate have become one and the same side. None of these inequalities is true: $a \neq a$, $b \neq b$, $\textit{not-a} \neq \textit{not-a}$, $\textit{not-b} \neq \textit{not-b}$. Undoubtedly, whatever a is, if it is, it is a. Consequently, $a = a$, $b = b$. To treat joining a strip of paper into a Möbius ribbon as equivalent of creating a construction in which $a = b$ and $a \neq a$ at the same time is to misunderstand the construction. For if we accept that an a is from now on a b, we clearly start from two absolutely fundamental assumptions that a is a and b is b. Otherwise, the equality $a = b$ would lose its sense. For how could we say that an a is a b, if a is not

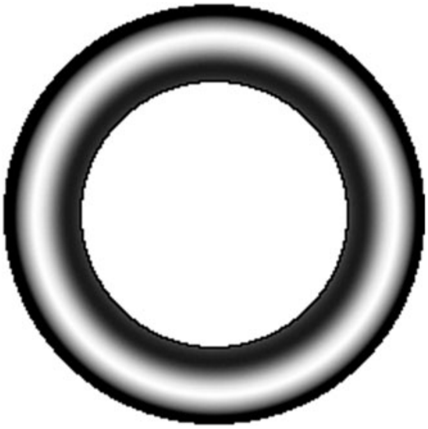
a and b is not b? If $a \neq a$, i.e. a is not a, then it is not a which is b, but something quite different from a equals b, so it cannot be true that $a = b$. Still worse, b is not b any more, either. So, equaling a with b becomes incomprehensible, since it means that something that is not a becomes something that is not b. How can one express it with symbols "a" and "b"? Definitely, not with the equality $a = b$. Thus, there is no other way than accepting that Möbius ribbon is a construction that is founded upon the propositions $a = a$, $b = b$ and $a = b$, i.e. that a and b remain themselves and are identified as such. It seems then that in the narration that uses names "side *not-a*" and "side *not-b*" the source of contradiction is a tacit assumption that being *not-a*(*not-b*) is tantamount to not being a (b).

Our solution of the Liar Antinomy is based on the assumption that it is the mistaken choice of names for relevant self-referent propositions that is responsible for the possibility of inferring contradiction. The analysis of the Möbius ribbon presented here is an introductory illustration of our solution of the antinomy.

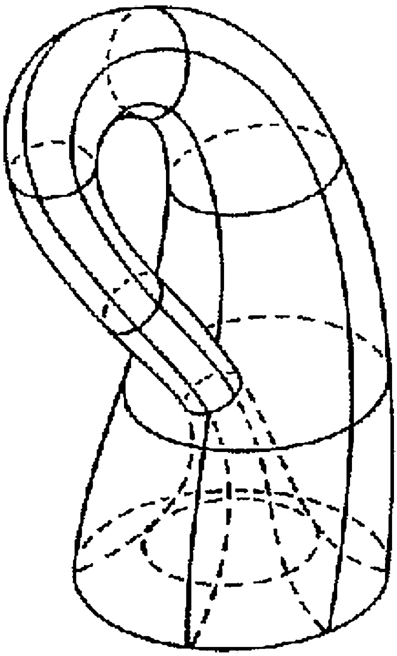
The essence of Möbius ribbon lies in the fact that a two-dimensional object, i.e. a rectangle, may be used to construct a three-dimensional object, i.e. that very ribbon. It is enough to twist appropriately the two-dimensional space, i.e. the rectangle, to enter the space of three dimensions. The 'twist' is possible only when we have another dimension (here the third one) at our disposal. The same operation, however, can be performed on such an object of a three-dimensional space as a cylinder.

Here one has to 'twist' a three-dimensional space into another, fourth one. What we get is a four-dimensional object called Klein's bottle.[6] What is interesting about this object is that its inside is at the same time its outside. Such a property results from the fact that the 'twist' causes the inside of the cylinder at one end to attach to its outside at the other. A simple attachment of one end of a cylinder to the other does not require an additional dimension and such a construction is a three-dimensional object known as torus, which can be easily depicted thus:

[6] Felix Klein (1849–1925), a German mathematician, professor of the University of Leipzig, fellow of the Berlin Academy of Sciences. He specialized in non-Euclidean geometries, group theory, theory of algebraic equations and in the theory of elliptic and automorphic functions.



It is impossible to see the Klein's bottle—our imagination cannot operate in a four-dimensional space.[7] However, it is possible to represent a four dimensional figure in two dimensional space (on a sheet of paper) similarly to the presentation of e.g. square in the two dimensional picture.



[7] A more precise description in topological categories of both Möbius ribbon and Klein's bottle can be fund in mathematical monographs devoted to topology.

Möbius ribbon and Klein's bottle could be discussed in the first chapter as examples of constructions involving the conflict between our intuitions and mathematics.

4.2 Great Semantical Paradoxes

The ages old Liar's Paradox is believed to be the most potent and most important of all possible paradoxes of our thought. Its very form shows that the dilemma is caused by self-reference of the proposition, which is also called self-reflexivity of an utterance. Such looping of an utterance may have the form of a single sentence and then it is a direct vicious circle. It can also have the form of two or more sentences and then it is an indirect vicious circle. Traditionally, the name "Liar's Paradox" has been used with respect to "direct" utterances. The other, "indirect" form, composed of two mutually related propositions, is known by the name of its medieval discoverer as "Buridan's Paradox". Buridan's idea can be generalized so that it refers to any finite number of mutually related propositions. Since Liar's Paradox leads to a situation in which two contradictory propositions imply one another, it is called an antinomy.

4.2.1 Liar Antinomy

Self-referential paradoxes point to an important property of all systems that include semantic terms, which refer to expressions of the language of the system, or terms, which allow to define such terms, and are governed by the laws and rules of classical logical calculus. As was shown by Alfred Tarski (1901–1983) in his 1933 work *The Concept of Truth in Languages of Deductive Sciences*,[8] it appears that such a system is contradictory. One of the methods for solving the problem was proposed by Stanislaw Leśniewski (1886–1939) and later developed by Tarski. It distinguishes between a language and a meta-language, i.e. a language, in which one speaks of the propositions of a language.[9] A similar opinion about these paradoxes was expressed by Ramsey, who is famous for the division, presented in his 1931 book *The foundations of mathematics and other logical essays*, of paradoxes into two groups that are now known as first and second Ramsey's groups. The first one includes so-called logical paradoxes, or set theory paradoxes. The other one contains paradoxes that involve the concepts of signification and reference. These dilemmas are usually called semantic ones,

[8] Tarski [32].

[9] Borkowski [4], p. 356.

occasionally syntactic or epistemological ones. The greatest semantic paradox, for some the greatest paradox ever, is the Liar Antinomy.[10]

The reputation of Ancient inhabitants of Crete seems to have been low. The principal reason of it was their alleged propensity for lying. An interesting explanation of their permanent lies can be found in the myth of Medea. Idomeneus, the unfortunate king of Crete, was asked by Medea, the daughter of the king of Colchis, and Thetis, the mistress of Zeus and Poseidon, to decide which of them is more beautiful. He found the beauty of Thetis more appealing, which caused a great anguish in Medea, who uttered the memorable words: "All Cretans always lie" and cast a spell on the inhabitants of the island that made it impossible for them to tell the truth. Medea's comment became better known when it was repeated by Epimenides, a famous Cretan from Knossos who lived in the sixth century BC. He was a hal-legendary theologian, seer and thaumaturgos. Diogenes Laertios puts him on the list of first *sophoi* together with Thales of Miletus, Solon of Athens, Chilo of Sparta and Pittacus of Mytilene.[11] Epimenides was an author of poetic and prose works.[12] However, he is best remembered for recalling Medea's comment. While the utterance of an inhabitant of Colchis, a far away land beyond the Black Sea, causes no serious logical problems, the very same comment (All Cretans always lie) uttered by a Cretan seems to be evidently false. If Cretans always lie, Epimenides, who says that all Cretans always lie, must lie too. Accordingly, the proposition "All Cretans always lie" cannot be the truth if it is uttered by a Cretan. It should be remembered that though Medea's comment became a logical problem when it was repeated by a Cretan, Epimenides himself may have not realized the paradox, for, as noticed by Bocheński, he did not analyse any logical paradoxes. Liar Antinomy was not known even by Plato, though he analysed a similar problem of self-referent propositions in his dialogue *Euthydemus*.[13] Nor did Aristotle notice the importance of the problem, even though he was wondering whether one can tell the truth while telling lies[14]: "The argument is similar for the problem whether the same man can at the same time say what is false and what is true; but it appears to be a troublesome question because it is not easy to

[10] Our opinion is completely different. The Liar Antinomy, a great paradox that it is, must cede its priority to such profound paradoxes as the heap, the bald, etc., which are discussed in the final chapter devoted to ontological paradoxes.

[11] Diogenes Laertios, *Lives and Opinions of Great Philosophers*, I, 13, pp. 13–14; I, 41–42, p. 31.

[12] Diogenes Laertios, *Lives and Opinions of Great Philosophers*, I, 111–112, p. 69.

[13] Plato, *Complete Works*, ed. J. M. Cooper, Hackett Publishing Company 1997, *Euthydemus*, 283E–286E. Cf. also Bocheński [3], p. 131.

[14] Aristotle, *Sophistical Refutations* (transl. W. A. Pickard-Cambridge), in: *The Complete Works of Aristotle. The Revised Oxford Translation,* ed. J. Barnes, Princeton University Press 1984, vol. I, 180b. Cf. also, Bocheński [3], p. 132.

see whether it is saying what is true or saying what is false which should be stated without qualification."[15]

Although a question close to Liar Antinomy was apparently considered by philosophers before Aristotle, the problem was recognized as a logical dilemma only by an Aristotle's slightly older contemporary, Eubulides (fourth century BC), who is now credited to be the discoverer of this unusual paradox.

Eubulides, known to be a fierce opponent of Aristotle, was not only the author of Liar Antinomy but also of a number of extremely important paradoxes of vagueness and other logical problems of lesser importance, such as *The Hidden One*, *Electra*, *Cuckold*.[16] Unfortunately, Aristotle must have ignored the great logical discoveries of Eubulides, for he dismissed Liar Antinomy and, regrettably, remained silent about the paradoxes of vagueness. Bocheński notes that the problem of Liar was later analyzed by Theophrastus of Eresos (ca. 370–287 BC) and Chrysippus of Soloi (ca. 277–208 BC). The former two books to the problem, the latter, twenty-eight. In his attempt to reconstruct Chrysippus' solution, Bocheński had to overcome the difficulty of the language and fragmentary character of the evidence, yet he managed to show that Chrysippus probably believed liar's proposition to be senseless.[17] If this analysis is correct, Chrysippus proposed a solution, which is now reckoned as an important one. We should also remember Philetas from Kos (ca. 340–285 BC), who died in despair unable to solve the paradox.[18]

The ill reputation of Cretans must have been both widespread and lasting, for its traces can be seen even in the New Testament, written several centuries later. In the epistle to Titus, St. Paul repeats an evidently common opinion[19]: "It was one of them, their very own prophet, who said, 'Cretans are always liars, vicious brutes, lazy gluttons' That testimony is true." Naturally, the poet he had in mind was Epimenides of Knossos.

Regrettably, Eubulides's formulation of Liar Antinomy has not survived to our times. It should be noted that antinomic character of liar's proposition could not be limited by Eubulides to the form that is associated with Epimenides. It is easy to notice that if a Cretan says that all Cretans lie, it is not a dilemma. To put it precisely, Epimenides's statement should be formulated: "All Cretans always lie". Thus it is always understood. Then it contains two general quantifiers: one points to all Cretans, the other, to all moments in which any Cretan utters a proposition or to all propositions uttered by any Cretan. This means that Epimenides's proposition predicates about every proposition uttered by any Cretan. Assuming the truth

---

[15] As can be seen, there is no visible reference to the Liar's Paradox here. Moreover, Bocheński rightly observes that one can hardly accept that Aristotle presented any solution of the paradox there. Bocheński [3], p. 132.

[16] Diogenes Laertios, *Lives and Opinions of Great Philosophers*, II, 108, p. 137.

[17] Bocheński [3], pp. 132–133.

[18] Bocheński [3], p. 131.

[19] *The Bible*, *Titus*, 1, 12–13.

of Epimenides's proposition we come to a conclusion that it must be false. Consequently, the truth of the proposition implies its falsity, which, in accordance with the law of classical logic $(p \rightarrow \neg p) \rightarrow \neg p$, means that the proposition is false. What is of key importance here is that the assumption of its falsity does not imply the truth of the proposition. Actually, if Epimenides's statement: "All Cretans always lie" is false, it does not mean that the very proposition is true. The falsity of the proposition implies only that there is a proposition uttered by a Cretan that is true. Our analysis shows then that Epimenides's proposition is a false one and so we are not faced with any dilemma here. Accordingly, one cannot claim that the famous Liar Antinomy has the form given to it by Epimenides.

There are many ways in which Liar Antinomy can be formulated. Each of them serves the same purpose: the proposition should predicate about its own falsity or, putting it more precisely, truthlessness. It is often believed that the falsity in liar's proposition is related to the principle of bivalence: *every sentence is true or false*, so the falsity of the proposition is identical with its truthlessness.[20] To tell the truth, if a proposition is false, it is difficult to think that it is not truthless. Having one logical value, it cannot have another one. However, to eliminate every shadow of doubt one should use such a version of liar's proposition which predicates about its own truthlessness: if a proposition is truthless, it is obvious that it is not true. Naturally, it is even better if such a proposition predicates about itself being not true.

One of the easier ways of expressing antinomic self-reference, and a relatively precise one, is that which uses the name given to a proposition; in our case it is "*L*" (for 'liar').

Liar Antinomy
L. Proposition marked as L is not true.

Let us assume that proposition L is true. This means that it is as it predicates, so it is not true. Accordingly, the proposition is not true. However, this is the very state of affairs it predicates about. Accordingly, the proposition is true, because things are precisely as it predicates. This way we get the famous dilemma: the proposition is true when and only when it is not true. The form of the paradox is such that two mutually exclusive possibilities imply one another. For this reason Liar's Paradox is called an antinomy. Formally, if $L \rightarrow \neg L$ and $\neg L \rightarrow L$, so:

$$L \leftrightarrow \neg L.$$

[20] Martin proves that both the sentence asserting its own falseness and the sentence asserting its truthlessness merit being called Liar sentences, since they lead to a contradiction with the Tarski convention (reminded later). Moreover, he also proves that a sentence predicating about its own falseness is a Liar sentence no matter whether we assume a two-valued logic or not. Such a sentence is sometimes called *ordinary Liar*, while the sentence predicating its truthlessness is called *Strengthened Liar*. Martin [20], pp. 1–2.

Liar Antinomy is inseparably bound with Tarski's definition of truth, also called *Tarski biconditionals*. In his famous 1933 book, *The Concept of Truth in Languages of Deductive Science*, a fundamental work in the history of logic,[21] Tarski proposed his definition of truth. Let us remind that according to this definition[22]: "It snows" is a true proposition if and only if it snows. Generally, a proposition p is true if it is as it predicates[23]:

$$\underline{T}.\quad v(p) = 1 \text{ if and only if } p$$

The original form of the definition is the following[24]:

$$x \text{ is a true proposition if and only if } p.$$

In this formula, the symbol "p" is substituted for any proposition, while "x" is substituted for a particular name of that proposition.[25]

The relation between Liar Antinomy and Tarski's definition $\underline{T}$ is so important that some solutions of the antinomy involve substituting another definition of truth for definition $\underline{T}$. Feferman lists three classes of possible solutions for Liar Antinomy. The first one involves language limitation. Here we find Tarski's idea of distinguishing separate language orders. The second class contains logic limiting solutions, i.e. approaches in which some other logic, which uses truth-value gaps, e.g. paraconsistent or three-valued logic, is substituted for the classical one. Finally, the third class includes solutions that limit Tarski's definition of truth.[26] Another classification of possible solutions of Liar Antinomy is presented by Robert L. Martin. He distinguishes four diagnoses of the problem[27]:

Diagnosis 1. Both L and $\underline{T}$ seem correct, as genuine consequences of our intuitive semantic concepts of truth, reference, etc. The incompatibility resulting from the set $\{L, \underline{T}\}$ is also correct. Such incompatibility proves only that it is not possible to join the two propositions in a non-contradictory opinion. The solution of Liar Antinomy should involve substituting L' for L or $\underline{T}'$ for $\underline{T}$. At the same time, L' or $\underline{T}'$ should be the result of a "rational reconstruction" of L or $\underline{T}$, respectively. Moreover, the new conjunction, i.e. L and $\underline{T}'$, or L' and $\underline{T}$, or L' and $\underline{T}'$, would have to be non-contradictory.

Diagnosis 2. It is incorrect to claim that L and $\underline{T}$ are contradictory, because of the equivocal character of the concept "true". In order to show that s is not true we cannot use definition $\underline{T}$, since the sense of the word "true" in the proposition

---

[21] Tarski [32].

[22] Tarski [32], p. 19.

[23] Tarski [32], pp. 17–19. Tarski limited the application of this definition to the languages, in which the objective level is separated from the upper level. Tarski [32], pp. 165–166.

[24] Naturally, the sentence $\underline{T}$ is understood as generally quantified with respect to p. Thus, $\underline{T}$ has actually the form: $\forall p\ (v(p) = 1 \text{ if and only if } p)$.

[25] Tarski [32], p. 18. This form of the definition is known as "Tarski biconditionals".

[26] Feferman [9], pp. 240–241.

[27] Martin [20], pp. 3–4.

stating that *s* is not true is different from the one we find in $\underline{T}$. A variant of this solution is indexing the predicate of truthfulness with an assumption that the change of extension of the predicate produced by the indicated level does not result in the change of meaning of the predicate.[28]

Diagnosis 3. Convention $\underline{T}$ is incorrect, since the predicate "is true" is only partly defined, as there are propositions, which cannot be given any logical value. In such cases, we speak of *truth-value gaps*. By assumption, $\underline{T}$ refers to all propositions. Yet, some of them have no logical value. This means that $\underline{T}$ alone is an example of a proposition, which has no logical value, when we put a proposition with no logical value in place of *p*. Naturally, this proposition is the liar sentence, for otherwise $\underline{T}$ could be applied to liar's proposition, and that would cause a contradiction. However, there is a trap in this approach, called *revenge problem*. It follows from it that *L* is not true, for it could only be true by virtue of $\underline{T}$. Accordingly, it is precisely as predicated by proposition *L*, i.e. *L* is true by virtue of $\underline{T}$—contradiction. According to Martin, every doctrine of truth-value gaps refers to the revenge problem.

Diagnosis 4. Proposition *L* is incorrect, because it is a proposition of natural language, and natural languages have no concept of truth. The only languages that have concepts of truth, falsity, un-truth and un-falsity are these, which possess the following reservation: propositions that include the above mentioned truth concepts do not belong to those languages. Thus, there is a separation of subjective language from the language about subjective language.

4.2.1.1 Tarski's Proposal

Tarski himself supported the fourth type of solutions. He presented his proposal in the same work in which he gave the above mentioned definition of truth. The test of correctness for the definition is Liar Antinomy, the analysis of which made Tarski introduce the above mentioned restriction[29]: "For every formalized language we can construct a formally correct and materially right metalingual definition of a true proposition by means of general logical expressions, expressions of a language itself, and morphological terms of the language only, on condition, however, that the metalanguage is of a higher level than the language that is analyzed. If the level of the metalanguage is equal, at best, to the level of the language, such a definition cannot be constructed." Alonzo Church (b. 1903), in his 1976 work *Comparison of Russell's Resolution of the Semantical Antinomies with That of Tarski* showed that

[28] Every object, predication of which produces a true sentence, belongs to *extension* of a predicate. Every object, predication of which produces a false sentence, belongs to *anti-extension* of a predicate. If a predicate is an *n*-argument one, its extension contains ordered *n*s, predication of which produces true sentences, and its anti-extension contains ordered *n*s, predication of which produces false sentences. In this chapter, extension is called positive extension, while anti-extension is called negative extension.

[29] Tarski [32], p. 165.

the original form of the branched types' theory, proposed by Bertrand Arthur William Russell (1872–1970) in 1908,[30] that liberates the set theory from Russell's Antinomy, is a special case of Tarski's proposal to divide language into levels.[31]

What follows from Tarski's analyses is that the concept of truth is inexpressible in a natural language, understood as universal, i.e. containing languages of all possible levels beside the main one. Accordingly, talking about truth in a natural language seems senseless.[32] Consequently, there cannot be a predicate of the language that could make it possible for a sentence, in which it is the main functor, to express its own truth. This opinion is shared e.g. by Prior in *Correspondence Theory of Truth* (1967) and Wallace in *On the Frame of Reference* (1972).[33] Such a conclusion, however, is strongly non-intuitive. Following Tarski's opinion, to operate with a concept of truth in a natural language, one has to divide a natural language into levels of language, separated through applicability of definition $\underline{T}$. Other proposals suggest weakening of convention $\underline{T}$. They include also the ones, which assume truth-value gaps.

4.2.1.2 Parsons' Proposal

Tarski's idea to take into account existence of languages with various levels inspired Charles Parsons, who presented his solution of Liar Antinomy in a 1974 paper *The Liar Paradox*.[34] Martin rightly notes that Parson's approach represents another view. Indeed, the elimination of contradiction between L and $\underline{T}$ is achieved for the price of accepting the hierarchical form of proposition L. This new, more complex form of L makes it possible to distinguish a proposition and a statement expressed by the proposition. Definition $\underline{T}$ remains in the same form, but it takes into account the 'two-level' character of the proposition. Adding definition $\underline{T}$ to so reformed liar's proposition does not lead to contradiction, for it turns out that in its new form liar's proposition does not express any statement. Thus it is impossible to apply definition $\underline{T}$ here.

Parsons starts the presentation of his approach with the truth analysis of two propositions that are versions of liar's proposition:

(1P) The sentence written in the upper left-hand corner of the blackboard in Room 913-D South Laboratory, The Rockefeller University, at 3:15 p.m. on December 16, 1971 expresses a false proposition.
(2P) The sentence written in the upper right-hand corner of the blackboard in Room 913-D South Laboratory, The Rockefeller University, at 3:15 p.m. on December 16, 1971 does not express a true proposition.

30 Russell [30].

31 Church [7], pp. 301–302.

32 Field [10], p. 374.

33 Prior [26] and Wallace [33].

34 Parsons (1974).

Let us assume that precisely two sentences were written at that time on the blackboard in question: (1P) in the upper left-hand corner and (2P) in the upper right-hand corner. Moreover, let "*A*" be the abbreviation for (1P), and "*B*" for (2P). Then:

(3P) $A =$ the sentence written in the upper left-hand corner of the blackboard in Room 913-D South Laboratory, The Rockefeller University, at 3:15 p.m. on December 16, 1971.
(4P) B $=$ the sentence written in the upper right-hand corner of the blackboard in Room 913-D South Laboratory, The Rockefeller University, at 3:15 p.m. on December 16, 1971.

Considering conditions for truth Parsons analyses the first possibility, assuming that '"p" expresses a true proposition $\leftrightarrow p$' and '"p" expresses a false proposition $\leftrightarrow \neg p$'. Then, by virtue of standard propositional logic we can infer a following proposition from these assumptions: 'p' expresses a true proposition $\vee$ 'p' expresses a false proposition. Thus we get: 'p' expresses a proposition. This means that every proposition expresses a statement, which is in contradiction with Parson's idea. Consequently, he proposes replacing the earlier assumptions with new ones, which describe truth in such a way that not every proposition has to express a statement:

(5P) $\forall x$ ((x is a proposition $\wedge$ 'p' expresses x) $\rightarrow$ x is true $\leftrightarrow p$); and

(6P) $\forall x$ ((x is a proposition $\wedge$ 'p' expresses x) $\rightarrow$ x is false $\leftrightarrow \neg p$).

Let us assume now that x is a statement and that A expresses x. Directly from (5P) we get:

x is true $\leftrightarrow$ the sentence written in the upper left-hand corner of the blackboard in Room 913-D South Laboratory, The Rockefeller University, at 3:15 p.m. on December 16, 1971 expresses a false proposition.

With respect to (3P) we get:

(7P) x is true $\leftrightarrow A$ expresses a false proposition.

Let us assume that x is not true. Then,

$$\exists x\ (x \text{ is a sentence} \wedge \neg(x \text{ is true }) \wedge \text{'}A\text{' expresses } x),$$

so A expresses a false proposition. Consequently, by virtue of (7P), x is true. Since x is arbitrary:

(8P) $\forall x$ (x is a proposition $\wedge$ 'A' expresses x) $\rightarrow$ (x is true); hence

(9P) $\neg\exists x$ (x is a proposition $\wedge$ 'A' expresses x) $\wedge \neg$(x is true).

Let us assume now that y is a proposition and, moreover, that A expresses y. By virtue of (8P), y is true. Yet we get from (7P) that A expresses a false proposition, which contradicts (9P). This means that it has been proved that no proposition is expressed by A. Consequently, (1P) does not express any proposition. Similarly, (2P) does not express any proposition. This results in falsity of antecedents of main implications in conditions (5P) and (6P). Thus it is impossible to infer any contradiction. Liar Antinomy disappears.[35]

4.2.1.3 Burge's Proposal

The approach proposed by Burge, in his 1979 paper *Semantical Paradox*,[36] resembles Parson's proposal and is another attempt that can be seen as belonging to the second category. The key role here is also played by Tarski's idea of hierarchisation that separates the levels of truthfulness in the language with respect to the language order. Burge, however, does not apply it to liar sentence, but to the definition of truth $\underline{T}$. Truth is expressed through a class of indexed truth predicates T_i. Naturally, accepting such assumptions makes it impossible to infer a contradiction from liar sentence and Tarski's convention. Indeed, liar sentence rejects its truthfulness but this truthfulness must have a particular level k. Accordingly, the sentence is not $true_k$. At the same time, applying definition $\underline{T}$ to that very sentence of the liar makes it possible to state that it is true on the level $k + 1$, i.e. liar sentence is $true_{k+1}$.[37] Moreover, Burge shows differences between his proposal and the one that uses truth-value gaps. In his case, no correctly formed sentence can have logical value in the absolute sense. True, a sentence can be neither true$_i$, nor false$_i$, but there must be such a k for which the sentence is either true$_k$, or false$_k$.[38] According to Martin, such a dependence excludes the possibility of a revenge problem.[39] Burge uses the concept of *pathologicality*: a sentence is *pathological* when it is intuitively empty or leads to a paradox.[40] Naturally, pathologicality of a sentence is relative to a given level k, which results in *pathologicality*$_k$ or *nonpathologicality*$_k$. A sentence may be pathological$_k$ and then it is false$_k$, at the same being *nonpathological*$_n$ for some $n > k$, and so a particular true$_n$. Burge introduces pathologicality in three ways, proposing more or less formal definitions of the concept and thus develops three constructions, which overcome the Liar Antinomy.[41] It is worth noting that Burge's understanding of pathologicality of a sentence preserves the law of excluded middle in every

35 Parsons (1974), pp. 15–16.

36 Burge [5].

37 Burge [5], pp. 93–95.

38 Burge [5], pp. 96–97.

39 Martin [20], p. 7.

40 Burge [5], p. 102.

41 Burge [5], pp. 101–105.

construction, for every level i—for every i every sentence is either true$_i$, or false$_i$. What is especially interesting is Burge's final remark, where he admits his considerations assume an idealization through ignoring the phenomenon of vagueness when assessing the logical value of sentences.[42] It is clear that in many non-mathematical cases valuation of a sentence faces an insurmountable problem of vagueness. Burge's accurate remark becomes obvious in the light of the analyses of the phenomenon of vagueness, presented in the last chapter. This question apparently more serious than Liar Antinomy is given due analysis in the chapter *Ontogical Paradoxes*.

4.2.1.4 Martin's and Woodruff's Proposal

The third position is represented by the proposal of Martin and Woodruuff. Their approach, presented in a 1975 paper *"True-in-L" in L*,[43] makes use of Kleene's three valued logic.[44] The third value, traditionally marked by u (*unknown*), is essentially either truth or falsity, but it is unknown for us, which of them it actually is, so it cannot be substituted by either truth or falsity. It can be seen then that using u value makes it possible to express the truth-value gap. The formal language of S is the traditional language of logic of predicates with a difference consisting in individual variables being relative to a finite number of sorts, expressed with indices 1, …, k. Consequently, the domain of interpretation is a sum of domains, each of which corresponds to another sort of variables: $U = U_1 \cup \ldots \cup U_k$; for any $i \in \{1, \ldots, k\}, U_1 \neq \emptyset$. Moreover, language S contains a truth predicate T, which plays the key role in specifying partial order on the class of valuations interpreting language S. Valuations ascribing one of three logical values to sentences $P(a_1, \ldots, a_n)$, where P is a n-argument predicate, are partially ordered in such a way that $v_1 < v_2$, if a following condition is met: if v_1 ascribed the value of truth (falsity) to a sentence $T(a_1, \ldots, a_n)$, then v_2 ascribed to the same sentence $T(a_1, \ldots, a_n)$ the value of truth (falsity), too. Let us assume now that for a valuation v of the domain D and for some $j \in \{1, \ldots, k\}$, $S = U_j$. Then, v *partially represents truth for a language S* if and only if two conditions are met for some sentence $A \in S$: (1) if $vT(A) = 1$, then $v(A) = 1$, and (2) if $vT(A) = 0$, then $v(A) = 0$. In Martin's and Woodruff's construction, there is a possibility, then, that on a certain level sentences with a truth predicate T have no logical value of truth or falsity, so they have the value u. In turn, sentences that do not contain this predicate can be valued according to intuitions—some can be accepted as true,

---

[42] Burge [5], pp. 114–115.

[43] Martin and Woodruuff [21].

[44] As is known, in this logic, apart from truth (1) and falsity (0) there is the third logical value (u), which intuitively corresponds to the unknown logical value. Thus, if $v(p) = 1$ and $v(q) = u$, then the logical value of the conjunction $p \wedge q$ (i.e., $v(p \wedge q) = u$) is unknown, since everything depends on what is the unknown to us value of the sentence q: if q is true, then the conjunction is true, but if q is false, then the conjunction is false.

some other as false, as long as they meet conditions of Kleene's table of logical values. At the same time, the higher is the level, greater is the number of sentences with a logical value different from u. Thanks to the partial order specified on the set of valuations, Martin and Woodruff can use Kuratowski-Zorn Lemma on existence of maximal element in partially ordered set, in which every chain has an upper bound. This maximal level is the aim of the whole construction for no longer specifies only a partial representation of truth, but a complete one, called *representation*, which entails simultaneous meeting of two conditions: (1) $vT(A) = 1$ iff $v(A) = 1$, and (2) $vT(A) = 0$ iff $v(A) = 0$.

Naturally, a reasoning that leads to a contradiction, which is characteristic for the Liar Antinomy, cannot be performed on the language level S, i.e. the jth level of Martin's and Woodruff's model interpreting truth. On the jth level, sentences containing predicate T are valued in u, i.e. they are neither true nor false.[45]

4.2.1.5 Kripke's Proposal

Kripke's work, the *Outline of a Theory of Truth*,[46] also published in1975, presents the same approach to the liar sentence and definition $\underline{T}$ as the proposal of Martin and Woodruff, even though the two papers were written independently. It may accepted that Kripke's proposal, as well as those of Skyrms, Herzberger, or Gupta, which are modeled on it, belong to the third approach.

With a given domain D, Kripke interprets one-argument predicate $P(x)$ with help of a pair of disjunct sets (E_1,E_2), being extension and anti-extension of predicate P, respectively.[47] Predicating P of an object from E_1 is a true sentence, predicating it of an object E_2 is a false sentence, and predicating it of an object from outside $E_1 \cup E_2$ is an unspecified sentence.[48] The distribution of logical values is exactly the same as in Kleene's three valued logic. Moreover, Kripke proposes to extend the language S with help of a new, one-argument predicate T, whose interpretation is, naturally, a pair R_1,R_2), which is its extension and anti-extension. (Naturally, outside the set $R_1 \cup R_2$ the predicate T is unspecified. Let S be the interpretation of the language S, and $S(R_1,R_2)$, the interpretation of the language S extended with T. The new interpretation is an extension of the old one in the sense that (R_1,R_2) is an interpretation of T while the remaining predicates have the same interpretations as in S. Let $R_1{}'$ be the set of true sentences, and $R_2{}'$ a set of false sentences in $S(R_1,R_2)$. If T is understood as expressing the truth of sentences from the language S, which already contains the predicate T,

[45] Martin [20], p. 7.

[46] Kripke [16].

[47] If P is an unspecified predicate, neither its extension nor anti-extension is a set. As can be seen, Kripke tacitly assumes this idealization, mentioned by Parsons. According to this idealization, extension and anti-extension is assigned in a specified way in case of every predicate, also the unspecified one—cf. *Parsons's Proposal* above.

[48] Kripke [16], p. 64.

then $R_1 = R_1'$ and $R_2 = R_2'$. This means that for any sentence A of the language S: 1. A satisfies T iff A is true and 2. A falsifies T iff A is false. The pair (R_1,R_2), which meets his condition, is called by Kripke a *fixed point*. In the next step, Kripke proves the existence of fixed points by constructing one of them. Using Kuratowski-Zorn Lemma he proves that every fixed point can be extended to a maximal fixed point.

Similarly as in Martin and Woodruff, at the first stage of Kripke's construction, i.e. its first interpretation, the predicate T is completely unspecified—its extension and anti-extension is an empty set. Each consecutive extension of the interpretation somehow narrows down the description of the predicate. Each consecutive interpretation extends not only the set of true sentences but also the set of false sentences. The end of those extensions is determined by the maximal fixed point. Naturally, the solution of the Liar Antinomy is the same as in Martin's and Woodruff's construction. Kripke shows also the mutual equivalence between Tarski's language hierarchy and his own model.

Kripke's idea to use partial valuations was applied by Brian Skyrms in his 1982 paper *Intensional Aspects of Semantical Self-Reference*, published in 1984.[49] Another form of Kripke's model is the construction by Herzberger. He presented his attempt to solve the Liar's Paradox in a 1982 paper *Notes on Naive Semantics*.[50] Yet another version of Kripke's approach is the proposal of Anil Gupta, analysed below. We remind this construction because of its classical character, since it is based on traditional models for the first order logic.

4.2.1.6 Gupta's Proposal

In his 1982 paper *Truth and Paradox*,[51] Anil Gupta gave up the inductive way of specifying the predicate, which expresses the truth of a sentence, characteristic of the two above constructions and proposed a variant of the rule for establishing the truth of sentences remaining within the standard set by Tarski, who related truth to the levels of language. Gupta based his approach on four assumptions:

1. The language has, aside from the concept of truth, none of the other complicating factors: indexicals, vagueness, ambiguity, intensional constructions, truth-value gaps, etc. Gupta assumes that the language he considers is a classical first-order quantificational language.
2. The theory of meaning of a class of sentences gives the assertibility conditions of sentences belonging to that class. This means that for every sentence of the class the theory tells us the conditions under which the sentence is correctly assertible and when it is not correctly assertible. In other words, the theory of meaning provides truth conditions for sentences. Gupta stresses here that the

---

49 Martin (1982), pp. 119–131.

50 Herzberger [15].

51 Gupta [14].

assumption has nothing to do with the argument between realists with anti-realists concerning the status of semantics.
3. Truth is (expressed by) a predicate.
4. The only objects truth can be predicated about are sentences.

The assumptions accepted here define the space, in which Gupta intends to specify the truth conditions for sentences of language S. He notes that in case of sentences without the truth functor T these conditions are defined by standard, classical models. Naturally, it is the sentences with the truth functor T that are important for Gupta's considerations of the problem. He notices, however, that not all sentences containing the predicate T are problematic, i.e. paradoxical. An example of a sentence causing no interpretation problems is, e.g. the Cretan's sentence that all Cretans lie.[52] As we have already shown, this sentence is false. Similarly, a sentence asserting its truth raises hardly any doubts. So Gupta narrows down his search for an adequate way of interpreting sentences to the ones he himself defines as paradoxical. He assumes that language S equipped with a tool for creating names for all sentences. Then, if "*b*" is a name of a sentence "*b* is not true", then by virtue of Tarski's convention we obtain:

"*b* is not true" is true if and only if *b* is not true,

which clearly leads to a contradiction if only we use the classical interpretation of language S. From this point of view, Tarski's approach, which evidently is an inspiration for the following construction, seems justified.[53]

Gupta uses the analysis of the problem of self-reference for an introduction to his solution of Liar Antinomy. His language S includes, beside the predicate T, two parentheses 'and', the application of which is limited to closed formulae, i.e. sentences. So, if $(\forall x)(Fx \rightarrow Tx)$ is a sentence of language S, then '$(\forall x)(Fx \rightarrow Tx)$' is a name of that sentence.[54] Let $M = \langle D, I \rangle$ be any model of language S. Naturally, D is the domain of the model M, and I its interpretative function. Moreover, set Zd of sentences in language S is included in D and[55]:

(i) I assigns to a quotation name 'A' the sentence A.
(ii) If a is not a quotation name than $I(a) \notin Zd$.
(iii) If F is an n-place predicate and $d_i \in Zd(1 \leq i \leq n)$, then $\langle d_1, \ldots, d_j, \ldots, d_n \rangle \in I(F)$ iff for all $d'_i \in Zd$, $\langle d_1, \ldots, d'_i, \ldots, d_n \rangle \in I(F)$.
(iv) If f is an n-place function symbol then the range of $I(f)$ does not contain any sentences. Further if $d_i, d'_i \in Zd(1 \leq i \leq n)$ then, $I(f)(d_1, \ldots, d_i, \ldots, d_n) = I(f)\big(d_1, \ldots, d'_i, \ldots, d_n\big)$.

52 Gupta [14], p. 181.

53 Gupta [14], pp. 181–183.

54 Gupta himself admits that in this way he makes use of symbols characteristic for the way of distinguishing the expressions of the language and those of metalanguage [14], p. 184f.

55 Gupta [14], pp. 180–181.

Gupta shows that the model which satisfies such conditions may be extended to the so-called *standard model*, in which Tarski's definition no longer leads to paradoxical consequences. $M' = \langle D', I' \rangle$ is a *standard extension* of the model M if and only if $D = D'$ and I' is a function I, with the only difference being that I' coordinates a subset of set D to the predicate T. We shall say then that M' is a model generated by M and $I'(T)$, symbolically $M' = M + I'(T)$. Later, Gupta proposes that the name "standard model" mean a standard extension of a model of the language S.[56]

In his construction of a standard model for a given model of language S that satisfies conditions (i)–(iv), Gupta uses the concept of a "set of true sentences of language S at ordinal level α for the set U", in short $Tr(\alpha, U)$. To define it, he uses transfinite recursion:

Let M be a model of language S, satisfying conditions (i)–(iv). *Moreover, let* $U \subseteq D$. *Then,*

(Tr i) *If* $\alpha = 0$ *then* $Tr(\alpha, U) = U$.

(Tr ii) *If* $\alpha = \beta + 1$ *then* $Tr(\alpha, U)$ *is a set of sentences true in the standard model* $M + Tr(\beta, U)$.

(Tr iii) *If* α *is a limit ordinal then* $Tr(\alpha, U) = \{d : \exists \beta < \alpha (d \in \cap_{\beta \leq \gamma < \alpha} Tr(\gamma, U))\}$.

One can clearly see that the solution of Liar Antinomy is found in the introduction of successive levels of true sentences: a sentence asserting the truth of a sentence from level α belongs to the set of true sentences from levels higher than α. Thus only the standard model eliminates the truth-value gaps of a given model of language S. It is no wonder, then, that for the model M of language S and for $U \subseteq D$, $M + Tr(\omega, U)$ is that standard model of language S, in which all substitutions of Tarski's definition are true, and so they no longer lead to paradoxical consequences.[57] If b is a name of a sentence $Tb \vee \neg Tb$, then, according to Gupta's concept of truthfulness of sentences, the substitution of Tarski's definition $T(\text{'}Tb \vee \neg Tb\text{'}) \leftrightarrow Tb \vee \neg Tb$ is a true sentence, and so the sentence $Tb \vee \neg Tb$ is also true.[58] Then there is no paradox, for the sentence $Tb \vee \neg Tb$ is true not being false.

4.2.1.7 Feferman's Proposal

In his 1982 paper *Towards Useful Type-Free Theories, I*,[59] Feferman presented an approach based on the idea of truth-value gaps, which, however, in no way realized Tarski's idea of language levels hierarchy and in this sense was different from the ones presented above. Equivalence between semantic paradoxes, especially

56 Gupta notices that a possibility of extending the model M to a standard model is guaranteed also by conditions weaker than (i)–(iv).

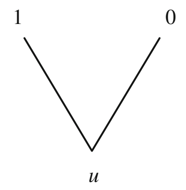
57 Gupta [14], p. 186.

58 Gupta [14], p. 191.

59 Feferman [9].

Liar Antinomy, and paradoxes of na set theory, especially Russell's Antinomy, is a well known fact. Feferman rightly observes that Russell's type theory has its semantic equivalent in Tarski's postulate that language be divided into successive orders. This postulate is realized in many ways; the more important of them have been cited above. Feferman's intention is to obtain such a solution that could be a semantic equivalent of those approaches in set theory which depend on its non-contradictory axiomatic character. The key concept of the new approach is the so-called *partial predicate*.

Let M be a set. $R^{\sim}$ is a partial, k-argument predicate on M, for $1 \leq k \leq \omega$, if it is a partial function mapping M^k into the set $\{1,0\}$ two logical values: truth and falsity. The introduction of u, the third logical value of unspecifiedness makes it possible to identify this partial function with a complete function mapping M^k into the set $\{1, u, 0\}$. In both cases, Feferman uses the same symbol "$R^{\sim}$". Partial ordering of the three logical values



makes it possible to use this order for specifying the order of predicates: $R_1^{\sim} \leq R_2^{\sim}$ if and only if any $m_1, \ldots, m_k \in M, R_1^{\sim}(m_1, \ldots, m_k) \leq R_2^{\sim}(m_1, \ldots, m_k)$. Then, $R_1^{\sim}$ is a subfunction of function $R_2^{\sim}$. Naturally, if $R_1^{\sim} \leq R_2^{\sim}$ and $R_2^{\sim} \leq R_1^{\sim}$, then $R_1^{\sim} = R_2^{\sim}$. Now let $M_0 = (M, \ldots)$ be an ordinary structure interpreting language S_0; the structure is ordinary because its function is not partial. $M = (M_0, R_1^{\sim}, \ldots, R_n^{\sim}, \ldots)$ is then a partial structure, because of the partial character of predicates $R_1^{\sim}, \ldots, R_n^{\sim}, \ldots$ Feferman limits his further considerations to simple structures $M = (M_0, R^{\sim})$ with one partial predicate $R^{\sim}$. $(M_0, R_1^{\sim}) = M_1 \leq M_2 = (M_0, R_2^{\sim})$, if $R_1^{\sim} \leq R_2^{\sim}$.

Let K be a class of structures $M = (M_0, R^{\sim})$, ordered by the relation $\leq$, for a certain structure M_0. Every structure $M = (M_0, R^{\sim})$ is coordinated by the operator Γ with a new structure $\Gamma(M) = (M_0, \Gamma(R^{\sim}))$. Γ is a monotonic operator: $M_1 \leq M_2$ implies $\Gamma(M_1) \leq \Gamma(M_2)$. Later, Feferman proves a theorem, according to which for any monotonic operator Γ and any structure $M \leq \Gamma(M)$ there is a smallest structure M^*, such that $M \leq M^*$ and $\Gamma(M^*) = M^*$.[60]

Feferman's intentions are clear. The accepted ordering of three logical values makes every partial predicate $R^{\sim}$ a 'sub'-predicate, maybe also a partial one, which is formed from $R^{\sim}$ when at least one instance of the value u is substituted with the value 1 or 0. Such a gradual specification of the partial predicate is secured by the monotonic operator Γ. A sequence of partial structures that are

[60] Feferman [9], pp. 250–252.

formed in this way is closed by a structure, which cannot be specified by operator Γ, since it is already completely specified: its predicate is no longer a partial predicate. Naturally, this more general theory is applicable to structures, in which the partial, gradually specified predicate is the truth predicate. Thus, its originality notwithstanding, Feferman's proposal is apparently situated in the within the series of solutions originated by Martin and Woodruff, and Kripke.

Most of constructions presented above accept an assumption that the liar sentence has no logical value of truth or falsity and ascribe to it some third logical value, thus treating the liar sentence L as a logical one. There is, however, an opinion denying that sentence L has the logical value of truth or falsity, or any other, and altogether refuses to accept L as a sentence in logical sense. In Poland, this standpoint is presented for instance by Gumański in his 1992 paper *Logical and semantical antinomies*.[61] Regrettably, such standpoint practically stops, or at least limits, some research approaches to the Liar's Paradox, especially the ones focused on understanding truth and inference.

Outside Martin's classification, we find the proposal of Graham Priest, which can be classified in Feferman's second group of solutions, which weaken logic. Weakening of logic in such a way that it makes it possible to accept contradictions at the same time avoiding making the consequences of accepted sentences trivial is a fundamental idea of contradiction tolerating logics, known as paraconsistent logics.

4.2.1.8 Priest's Proposal

Martin's classification of four types of solutions for Liar Antinomy does not include *dialetheism*, an opinion, which accepts the existence of sentences that true and false at the same time.[62] As we have shown above, for the liar sentence L the equality $L \leftrightarrow \neg L$ is true. Yet, when the principle of identity is applied to the liar sentence, $L \leftrightarrow L$, we get $L \leftrightarrow L \wedge \neg L$. Consequently, the liar sentence is false, for it is equivalent to a contradictory sentence. However, the principle of identity can be applied to the negation of the liar sentence: $\neg L \leftrightarrow \neg L$. This means that we have another equivalence: $\neg L \leftrightarrow L \wedge \neg L$, which proves the negation of the liar sentence. Consequently, the liar sentence L is true and false at the same time and so is its negation. Dialetheism's creator and advocate is Graham Priest.[63]

[61] Gumański [13].

[62] Priest, e.g. [23–25].

[63] This reasoning does not have to be accepted. $L \leftrightarrow \neg L$ is, namely, a conjunction of two implications: $L \rightarrow \neg L$ and $\neg L \rightarrow L$. From the first implication follows the rejection of the truthfulness of L, from the other, the rejection of the truthfulness of $\neg L$. Such an interpretation leads to the justification of a proposition that the liar sentence L is an example of a sentence that has no logical value, so it illustrates the truth value gap. Yet, in the light of the first reasoning, which proves that L is a sentence that is true and false at the same time, it is clear that the argument leading to a conclusion that L has no logical value or is not based on the classical logic.

Naturally, though such conclusion was reached in accordance with the rules of classical logic, it is impossible to reconcile the stance that accepts it with any logic that upholds the principle of contradiction and Duns Scotus's Law. For this reason dialetheism is found in paraconsistent logics, which suspend the principle of contradiction and weaken Duns Scotus's Law thus tolerating contradiction. Paraconsistent theories are not trivial even when they include a sentence and its negation. Liar Antinomy has a special importance for dialetheists, because the liar sentence is apparently the only example of a sentence that is true and false at the same time. Naturally, this view goes far beyond the conclusions of the liar sentence, for the only thing that can be inferred about it is that the sentence is not true if it is true and vice versa. From this point of view, dialetheism, seen as a solution of Liar Antinomy, seems to go too far. What is worse, it is difficult to find another, truly philosophical justification of it apart from the liar sentence.[64]

Though technically advanced, the above proposals do not seem specially attractive, mostly for the reason of their artificial and non-intuitive character. Can the introduction, more or less manifest, of language hierarchies explain this problem, admittedly great yet surprisingly simple in its structure? Isn't the truth only one? Can introduction of a third logical value, no matter how we understand it, be any solution if also the following sentence, analogous to the liar's one, is paradoxical: "This sentence is false or has a third logical value"?[65] Finally, is acknowledging that there are sentences in logical sense, e.g. the liar's one, which have the logical value of truth and falsity at the same time, not an excessively bold move? Can the sense of classical logic, extraordinary and exceptional as it is, be actually undermined in such a simple way?

The next proposal, presented by Barwise and Etchemendy in their famous 1987 book *The Liar. An Essay on Truth and Circularity*,[66] gives a new solution to Liar Antinomy, different from all other ones.

4.2.1.9 Barwise's and Etchemendy's Proposal

This approach belongs to the so-called situation theory, which takes into account the context of utterances. According to Barwise and Etchemendy, neither the self-reference of the utterance, nor improper understanding of truth is the source of Liar Antinomy, but neglecting the context.

---

[64] Sometimes it is pointed out that such justifications are provided by sentences with vague terms. This opinion, however, seems difficult to accept. Cf. this chapter.

[65] If this sentence is true, it is false or has the third logical value, so it is untrue. If it is false, then things are as it predicates, so it is true and thus not false. If it has the third logical value, things are as it predicates, so it is true and thus does not have the third logical value. Naturally, this reasoning is yet another versions of the Revenge Paradox, mentioned above.

[66] Barwise and Etchemendy [2].

In their analysis of the liar sentence, Barwise and Etchemendy start with the traditional distinction between a sentence and a proposition.[67] As a sequence of symbols, graphic or acoustic, a sentence is neither true nor false. Sentences do not fall under the criterion of truth and falsity, just like material objects, e.g. a pencil, table, etc. Accordingly, if a person a says: $L =$ "What I am stating here is not true", the expression "What I am stating here" refers to a proposition stated by the sentence. Thus the sentence L uttered by a refers to a proposition p. This means that by uttering sentence L, the person a states a proposition "p is not true". However, since the proposition stated by a is marked with a symbol "p", p and "p is not true" are the same:

$$p = [p \text{ is not true}] \tag{4.1}$$

In the situation theory, assessing the truth of the proposition is dependent on the context of an utterance. Often a person uttering a proposition is unaware that his utterance refers to the context of the situation, which determines the truthfulness of the proposition. Still, regardless of the person's awareness, the context is the key to the truthfulness of the proposition he utters. Let us mark the context of uttering sentence L by the person a by a symbol "c". The utterance: "What I am stating here" refers to a proposition that p is true in context c. In formal notation of situation theory we put it: $c| = p$. Since it has been noticed that the utterance "What I am stating here" refers also to a proposition p, then p must be the same as $c| = p$. Thus we get another equality:

$$p = [c| = p] \tag{4.2}$$

Let us consider two cases now: one when proposition p is true and another one when proposition p is not true.

1. Let us assume that proposition p is true. Then, p is this very true context, which states the truth of the proposition p, so we get:

$$c = p. \tag{4.3}$$

   Yet, in the light of (4.1), proposition p is at the same time the falsehood of proposition p. This means that:

$$c| = [p \text{ is not true}], \tag{4.4}$$

   which gives us a contradiction. Indeed, by virtue of (4.3), p is a true proposition in the context c, yet it follows from (4.4) that p is not true in the same context c. To explain the problem Devlin uses the example of seasons in various parts of the globe: true, it is summer and it is not at the same time, but summer is in the USA, while it is not in Australia. So, the difference of

[67] We summarise Barwise's and Etchemendy's proposal following its clear and elegant presentation by Devlin [8], pp. 338–343.

context, in this case the USA or Australia, makes a given proposition acceptable in one context and inadmissible in another. It is not so, however, when the proposition analyzed above in the very same context turned out to be true and not true. It remains to analyze the other instance of falseness of proposition p.

2. Let us assume that p is not true. According to an earlier assumption, c is the context of the utterance of person a. Thus we obtain a condition (4.4), which is contradictory with respect to equality (4.1). The conclusion is easy: c cannot be the context of person's a false proposition p. Devlin supports the recognition of this conclusion with the following example: if someone says that June is a winter month in country X, then the USA cannot be country X. He cautions us, however, not to jump to the conclusion that Australia is country X; what is only certain is that the USA cannot be country X.

Finishing his presentation of Barwise's and Etchemendy's solution of Liar Antinomy, Devlin states that the antinomy disappears when we pay proper attention to the context. When person a says: "What I am stating here is false", he is uttering a proposition implicitly referring to a certain context c, i.e. to the context in which the sentence was uttered. If the proposition is true, it is true in context c, which, as it turns out, leads to a contradiction. Consequently, the proposition must be false. Yet c cannot be the context, in which it is stated that the proposition is false, since the supposition that c is this context also leads to contradiction. Person a who says (in context c) "What I am stating here is false", utters a false statement. The fact that it is false, however, cannot be stated in the same context c. This seems a weird conclusion; similarly, it is most unusual for someone to utter the original sentence, on which the liar's "paradox" is based.[68] The weirdness of this conclusion, noticed by Devlin, does not have to be explained by the strangeness of the liar sentence. The sentence is strange indeed. The approach to it, however, even though it is illustrated with appropriate and persuasive examples does not manage to keep the analogy to the problem of the liar sentence until the final conclusion. What is unusual is that falsity of proposition p cannot be stated in the context c, while the fact that June is not a winter month can actually be stated in the USA, i.e. in the context, in which the proposition: "June is a winter month" is false. This means that although the postulate to take into account the context of an utterance in order to properly assess the truthfulness of the proposition stated in it is well justified, its application by Barwise and Etchemendy is rather faulty in the case of the liar sentence.

The opinion that context o fan utterance influence the logical evaluation of a proposition seems to be well justified. Moreover, one can suspect that failing to take this context into account should make it impossible to assess the logical value of a proposition. Our own proposal of a solution for the Liar Antinomy seems to justify both claims. Since our approach involves a fairly simple analysis on the level of sentence calculus, the distinction between a sentence and a proposition stated in it is not necessary.

[68] Devlin [8], p. 342.

4.2.1.10 The Author's Own Proposal[69]

Our approach is based on three fundamental assumptions:

1. Liar Antinomy can be analyzed on the ground of sentence language;
2. Logic, which we use when thinking and speaking is the key element of the context that affects the logical value of a sentence;
3. Every sentence conveys more information than what is conveyed by the words constituting that sentence.

The first assumption is to protect us against the trap of unnecessarily complex analyses, which obscure the real problem of the liar sentence, the essence of which is actually simple.

The second assumption is to make us aware of something evident, which, unfortunately, usually escapes our attention. It is the problem of logic, which we tacitly assume every time we utter a proposition. Recognizing the laws of that logic is one of the more important problems of contemporary logical and philosophical research. We shall limit ourselves here to an indubitable observation: the logic we use when formulating thoughts is the logic of truth. Therefore, it is not the logic of falsehood. In simplest words, the logic of truth (falsehood) is a logic in which the designated value is truth (falsehood). This means that every our utterance is treated as one that expresses a true proposition. It does not matter here if we want to tell the truth or not in a given moment. In both cases, we intend the sentences we utter to be treated as true by our listeners. It is important because a sentence with no logical value carries no information. For instance, what information can we get from a sentence "Milk is good for children" if we do not assume any logical value of the sentence? Naturally, none. Without taking into account the logical value we cannot know whether it follows from this sentence that children should drink milk or not. The problem becomes obvious when we substitute "drugs" for "milk" in the sentence. Taking into account the logical value of a sentence plays a crucial role in the case of the liar sentence. As it was noted by Jan Woleński, in the logic of falsehood, the paradoxical sentence is not "This sentence is false" but "This sentence is true".[70]

Our third assumption makes us aware that every sentence can yield information that is not identical with the sequence of words which form the sentence. For example, let us take a following sentence: $Z =$ "The Trench street garage belongs to brothers". The proposition stated in the sentence is clear and related to the sequence of words forming the sentence. Now if we accept sentence Z, we should also accept many other ones, e.g. $Z_1 =$ "There is an Trench street in the town", $Z_2 =$ "Some brothers are owners of a garage", $Z_3 =$ "The garage in Trench street has at least two owners", $Z_4 =$ "The owners of the garage in Trench street are men", $Z_5 =$ "Cars can be repaired in Trench street", etc. We can clearly see that

[69] This solution is more precisely presented in [17].

[70] Woleński [34], pp. 91–97.

sentence Z predicates much more than merely the proposition expressed by the sequence of words which form sentence Z. The sentence predicates also about everything that is predicated by the sentences we have mentioned, which we would have to accept once we have accepted sentence Z. Consequently, if sentence Z is true, all above sentences must be true too. However, if sentence Z is false, then we only know that one of those sentences must be false. Many of them may still be accepted as true sentences. For instance, let us assume that sentence Z is accepted as false, because the only garage in Trench street is owned by a woman and a man, who are siblings having no other brothers or sisters. Then sentences Z_2 and Z_4 must be accepted as false but sentences Z_1, Z_3, Z_5 can still be believed to be true. Summing up, we shall say that in accordance with the third assumption every sentence "says" more than it actually follows from the sequence of words making up the sentence. Moreover, a false sentence, regardless of its falsehood, can say something that is true.

From the first assumption it follows that a sentential language should be sufficient for a symbolic expression of the liar sentence. It is not said, however, that this must be the language of the classical sentence calculus. Moreover, a simple reasoning illustrating Priest's proposal shows that the language of the classical sentence calculus is not subtle enough for expressing the liar sentence L. Naturally, the problem lies in the implication conjunction, the use of which simplifies everything to such an extent that contradiction seems to be the only possible consequence of accepting the liar sentence as a sentence in the logical sense.

Let us consider then the sentence calculus with an additional implication conjunction which would be subtle enough to express the questions explained above. Let us extend the language of the classical sentence calculus to accept a new two-argument conjunction : (colon), which should stand for citing someone's utterance. Our intention is for this conjunction to express the fact that when analyzing a sentence one can infer other sentences with specified meaning from it. Therefore, the conjunction : should not be the well known classical implication, since the intended sense of a sentence in the form $p : q$ is that the state of affairs expressed by sentence q must have taken place if the state of affairs expressed by sentence p took place. The content of sentence q is then somehow included in the content of sentence p. We can also say that the content of sentence q is, somehow, the content of sentence p, which obviously does not mean that the content of sentence q is the content of sentence p. For the sake of simplicity we shall say then that "sentence p says that q".

In the example above, the sentences we should accept are $(Z : Z_1)$, $(Z : Z_2)$, $(Z : Z_3)$, $(Z : Z_4)$, $(Z : Z_5)$, no matter whether sentence Z itself is accepted or not. Even if we accept the falsity of Z, we have to accept the truth of all five of the sentences quoted above. For even if we think Z is false for the reason stated above, it cannot be denied that it says, e.g. that two brothers are owners of a garage in Trench street.

Specifying the new conjunction, one has to take into account that sentence Z "speaks" of many other sentences, e.g. Z_1 and Z_2 and Z_3 and Z_4 and Z_5.

Evidently, the key for defining the new conjunction will be the logical conjunction. Let us accept the following set of axioms:

Ax1. $\alpha : \alpha$
Ax2. $((\alpha : \beta) \wedge (\beta : \delta)) \rightarrow (\alpha : \delta)$
Ax3. $(\alpha \wedge \beta)\ : \alpha$
Ax4. $(\alpha \wedge \beta)\ :\ (\beta \wedge \alpha)$
Ax5. $\alpha :\ (\alpha \wedge \alpha)$
Ax6. $((\alpha : \beta) \wedge (\beta : \alpha)) \rightarrow ((\neg\alpha : \neg\beta) \wedge (\neg\beta : \neg\alpha))$
Ax7. $((\alpha : \beta) \wedge (\beta : \alpha) \wedge (\delta : \gamma) \wedge (\gamma : \delta)) \rightarrow (((\alpha \S \delta)\ :\ (\beta \S \gamma)) \wedge ((\beta \S \gamma)\ :\ (\alpha \S \delta)))$, for $\S \in \{\rightarrow, \leftrightarrow, :\}$
Ax8. $((\alpha : \beta) \wedge (\delta : \gamma)) \rightarrow ((\alpha \S \delta)\ :\ (\beta \S \gamma))$, for $\S \in \{\wedge, \vee\}$
Ax9. $(\alpha : \beta) \rightarrow (\alpha \rightarrow \beta)$

We shall say that α is a consequence of a set of premises X, which we shall put down as

$$X| - \alpha$$

if the formula α is deducible by means of *Modus Ponens* from a set which is a sum of sets X and $\{\text{Ax1}, \ldots \text{Ax9}\}$ and the set of axioms of the classical sentence logic.

Naturally, it is easy to see that the logic proposed here does not make it possible to analyze the content of sentences. However, it opens a possibility of arbitrary formulation of sentences stating content relation between the sentences, which is completely sufficient from the point of view of our objective. Tautologies with : as the main conjunction are rather trivial. More specifically, the sense of the new conjunction is such that the formulae of the form $(\alpha_1 \wedge \cdots \wedge \alpha_k) : \alpha_i$, for $i \in \{1, \ldots, k\}$ be tautologies of the logic proposed here. It is justified in the sense that coming into being of the state expressed by a sentence $p_1 \wedge \cdots \wedge p_k$ implies coming into being all states of affairs expressed by sentences $p_1, \ldots, p_k$. The difference between this conjunction and a classical implication lies, among others, in the fact that no formula of the form $\alpha_i : (\alpha_1 \vee \cdots \vee \alpha_k)$, where $i \in \{1, \ldots, k\}$ is a tautology. The difference is still more serious. For if we have a classical implication $p \rightarrow q$, we can infer the truth of sentence q from the truth of sentence p and if p is false, we can hardly say anything about the logical value of q. Naturally, the truth of a classical implication $p \rightarrow q$ depends solely on the truth of sentences p and q. In the case of sentence $p : q$, the truth of sentence p necessitates the truth of sentence q, while the falsity of sentence p is no ground for assessment of the truth of sentence q. However, the truth of sentence $p : q$ does not depend on the truth of sentences p and q. More precisely, the difference between classical implication and sentence $p : q$ lies in the falsity condition. The possible cases are the following:

$p:q$ - true	p - true	q - true
$p:q$ - true	p - false	q - true
$p:q$ - true	p - false	q - false
$p:q$ - false	p - true	q - true
$p:q$ - false	p - true	q - false
$p:q$ - false	p - false	q - true
$p:q$ - false	p - false	q - false

In the case of classical implication, only the fourth one, out of last four, occurs. Thus it can be seen that the new conjunction is not a truth conjunction.

The sense of axioms defining the new conjunction is easy to read. Ax1—the content of a given sentence is included in the content of that sentence. Ax2—if the occurrence of the state of affairs expressed by sentence p requires the occurrence of the state of affairs expressed by sentence q and the occurrence of the state of affairs expressed by sentence q requires the occurrence of the state of affairs expressed by sentence r, then the occurrence of the state of affairs expressed by sentence p requires the occurrence of the state of affairs expressed by sentence r. Axioms Ax3 and Ax4 refer to the very core sense of the new conjunction. As it was noted before, there is a specially close relation between it and logical conjunction, which is reflected by the semantics proper for the new logic. Ax5—the repetition of the same sentence does not say anything more than its single utterance. Ax6 is the invariance law because of the negation for the equivalence conjunction defined by logical conjunction $(\alpha : \beta) \wedge (\beta : \alpha)$. Three axioms Ax7 are invariance laws because of $\rightarrow$, $\leftrightarrow$ and : for the conjunction of equivalence defined by logical conjunction $(\alpha : \beta) \wedge (\beta : \alpha)$. Ax8 is the invariance law because of the logical conjunction and alternative for the new conjunction. The last axiom joins the new conjunction with implication in order to eliminate a situation in which sentences $p : q$ and p are true while sentence q is false. Indeed, when the occurrence of the state of affairs expressed in sentence p necessitates the occurrence of the state of affairs expressed in sentence q, then if p is true, q must also be true. Natural character of such formalization is manifested in its ability to accept the unquestionable conclusions from sentences, whose content may sometimes seem only loosely connected with the content of those conclusions. This fact apparently has a great influence on the evaluation of self-reflective sentences, to which our sentence of the liar belongs.

The adequate semantics for the new sentence calculus will be the class of all matrices $M = (A, D)$, in which $A = (A, -, \cap, \cup, \Rightarrow, \Leftrightarrow, \supset)$ is an algebra similar to the language $S_{:}$ - language of classical sentence logic enlarged with a two-argument conjunction :, D is a non-empty subset A, and for any given $a, b \in A$,

$c1.$	$a = a \cap a$		
$c2.$	$a \cap b = b \cap a$		
$c3.$	$-a \in D$	iff	$a \notin D$
$c4.$	$a \cap b \in D$	iff	$a \in D$ and $b \in D$
$c5.$	$a \cup b \in D$	iff	$a \in D$ or $b \in D$
$c6.$	$a \Rightarrow b \in D$	iff	$a \notin D$ or $b \in D$
$c7.$	$a \supset b \in D$	iff	$a = b \cap c,$ for some $c \in A$

Every matrix $M = (A, D)$ meeting these conditions will be called :-model. Let *Hom*(*S*,*A*) be a set of all homomorphisms mapping S on A.[71] Then,

$$X| - \alpha \quad \text{iff} \quad \text{for any :-model } M = (A, D) \text{ and for any homomorphism } v \in Hom(S, A) \text{ (if for any } \beta \in X, v(\beta) \in D, \text{ to } v(\alpha) \in D).$$

It is easy to see that the interpretation of the conjunction: expressed by the seventh condition of the :-model is in agreement with earlier comments explaining the desired sense of the new conjunction.

The existence of conjunction : in the sentence calculus makes it possible to express the liar sentence in a new way:

$$L : \neg L.$$

Now the liar sentence L says that "It is not true that L". We shall say then that sentence L expresses the state of affairs consisting in L not being true.

Making use of the semantics adequate for the new sentence logic let us try to assess the truth of sentence L knowing that the sentence speaks of its own falsity, i.e. assuming that it is true that $L : \neg L$. Let us note that the truth of sentence L and the truth of sentence $L : L$ are two different issues, even though there is undoubtedly some relation between them. On the whole, the truth of sentence $p : q$ depends upon the relation between the content of sentence p and the content of sentence q, but does not depend on whether they are true or not. If p is a sentence "Humpty Dumpty settled in Sherwood with his family" and q is a sentence "Humpty Dumpty has a family", then regardless of both p and q being considered false $p : q$ should certainly be accepted as true. Naturally, in the case of the liar's paradoxical sentence L we do not have complete independence of truth of sentences L, $\neg L$ and $(L : \neg L)$, for the simple reason that the content of these sentences is the truth of sentence L.

[71] A summary proof of this theorem is presented in [17], p. 72.

Let $M = (A, D)$ be any model and $v \in Hom(S, A)$, any homomorphism, such that $v(L) = a_0 \in A$. To acknowledge the liar sentence as true we have to assume that sentence $L : \neg L$ is satisfied in the model M and valuation v, i.e. that sentence L actually speaks of its own falsity. Thus we assume that

$$a_0 \supset -a_0 \in D$$

which, in view of condition ($c7$) of :-model, means that $a_0 = -a_0 \in c$, for some $c \in A$.

Let us assume that $a_0 \in D$. Then,$-a_0 \cap c \in D$. From ($c4$) we get that $-a_0 \in D$. and $c \in D$. But, in view of ($c3$), $a_0 \notin D$. Consequently, $a_0 \notin D$. Therefore, sentence L cannot be valid in model M and valuation v.

It must be seen, if invalidity of sentence L in model M and valuation v does not lead to a contradiction. Let us assume, then, that $a_0 \notin D$. Z ($c3$), $-a_0 \in D$. From assumption, $a_0 = (-a_0 \cap c) \notin D$, for a $c \in A$. Consequently, $-a_0 \notin D$ or $c \notin D$. Now, since $-a_0 \in D$, $c \notin D$. As we can see, in his case no contradiction has been reached.

This means that sentence L can be accepted as false if we can find such a false sentence z for which sentence $L : z$ is true. This conclusion is in accordance with our earlier findings: a sentence expressing a state of affairs, the occurrence of which necessitates some other states of affairs, is false if at least one of those states of affairs does not really occur. What remains is finding the sentence that is false and expresses such a state of affairs that would have to occur when the state of affairs expressed by sentence L has occurred.

From assumption $L : \neg L$. Now it follows from axiom Ax1 that $L : L$. Consequently, in view of axiomAx8, $(L \wedge L) : (L \wedge \neg L)$. From Ax5 we get $L : (L \wedge L)$. By virtue of Ax2 we obtain

$$L : (L \wedge \neg L).$$

As can be seen, our proposal points to an apparent similarity between the Liar Antinomy and Möbius' Ribbon. As in Möbius' Ribbon, where side a is identified with side b, or *not-a*, still remaining side a, so in the Liar Antinomy, sentence L is identified with sentence $\neg L$, while still remaining sentence L. Stating that L is $\neg L$, no longer being L does not agree with the assumption, for L, which by its very nature will always be L, becomes identified with $\neg L$. If L were no longer L, it would be impossible to say that L is identified with $\neg L$, since L would no longer be itself. Consequently, some other sentence would be identified with $\neg L$, but certainly it would not be L. Our proposal is built on this obvious fact. Thus it cannot be accepted that $L = \neg L$ and $L \neq L$. What ca be accepted is the equality $L = L \wedge \neg L$. Sentence L is sentence $\neg L$ only as sentence L.

It is evident that on the ground of our extension of classical logic, sentence $L \wedge \neg L$ is false. Moreover, it expresses a contradictory state of affairs, which would have to occur once the state of affairs expressed by sentence L occurred. Consequently, the liar sentence L must be a false sentence and, moreover, in no way can it be a true sentence.

The key axiom for his construction is apparently the first axiom, evident in its meaning. It is responsible for respecting the obvious statement that no matter what something is identified with, it always remains itself. Indeed, every sentence p says what it says, so it says p. This trivial axiom is a guarantee that we shall not forget that the logic we use when thinking and speaking is the logic of truth. Therefore, every sentence we utter should be understood, according to our intentions, as one stating the truth; thus, if we utter a sentence p, we want the recipient of our utterance to accept p, because this is the only condition for his accepting the content of p. A similar state of affairs occurs in the case of uttering or writing the liar sentence L. If we are to take into account the fact that the sentence predicates $\neg L$, we have to treat the sentence as true, so we have to accept L. Only on this condition we can understand and accept that sentence L says $\neg L$. The sense of sentence L is clear for us only in the context of that sentence. Thus the liar sentence L predicates not only its own falseness but also its own truth, as does every sentence uttered in the logic of truth.

The necessity of taking into account the utterance context, pointed out by Barwise and Etchemendy, has a concrete realization in our proposal. The context is defined by the logic of truth. Moreover, contrary to Barwise and Etchemendy but in agreement with our intuitions, one can specify the falsity of sentence L by recognizing the context, in which sentence L is not true. In our proposal this context is no secret.

4.2.1.11 Summing up of the Proposed Solution

These simple considerations hale show that the liar sentence must be false. If we confront this conclusion with the content of sentence L, however, we may feel astonished. Now the sentence says it is false, so, consequently, it means that L must be true after all. Was the time spent on the above formalization wasted? It does not seem so. Let us return for a while to our example, in which the sentence $Z =$ "The garage in Trench street belongs to brothers" is false, because according to the assumption it belongs to a woman and a man, who are siblings with no other brothers or sisters. Let us recall another sentence $Z_3 =$ "The garage in Trench street has at least two owners". It is undoubtedly true that sentence $(Z : Z_3)$ is a true sentence. Moreover, sentence Z is false Z_3 while is true. Consequently, the example clearly shows that a false sentence can predicate about something that is true. After all, a sentence is false not only when all sentences it predicates about are false. Sentence Z is false only because it states that brothers are owners of a garage. All other facts following from that sentence are true. It can be stated that in the light of our interpretation a sentence can be false even though it says something true on the sole condition that additionally the sentence states something that is not true.

We should note that from the formal point of view everything is unequivocal in our analysis: sentence L is false, so the evident part of what it predicates about is true. At the same time, the sentence states that L and his is the less evident part of what sentence L says. Consequently, L is false, while $\neg L$ is true. If we identified one sentence with the other, $L = L$, we would obtain a contradiction. Yet in our approach, sentence L is not sentence $\neg L$. Now beside saying that $\neg L$, sentence L says that L. Consequently, sentence L predicates especially that $L \wedge \neg L$. Thus, if we were to show what the liar sentence L is in an equation, we should rather write

$$L = (L) \wedge (\neg L) \wedge (L \wedge \neg L) \wedge \ldots$$

Naturally, the dots are justified, since it is really difficult to be sure that the content of sentence L has been fully used. Consequently, since sentence L is sentence $(\neg L) \wedge (L) \wedge (L \wedge \neg L) \wedge \ldots$, then L turns out to be true only when every part of the conjunction, which is what sentence L is, is a true sentence. Now this cannot be the case, for one of the parts of the conjunction is $L \wedge \neg L$. Consequently, L must be a false sentence. Yet if L is false, then $\neg L$ is evidently true. Indeed, $\neg L = (\neg L) \vee (\neg\neg L) \vee \neg(L \vee \neg L) \vee \ldots$ However, it is not sufficient to state that L is true, because it says that $\neg L$. For L does not speak only about it. This very fact allows us to state that in the light of the accepted formalization, sentence calculus and semantics adequate to it

the liar sentence has the logical value of falsehood, even though this is what it says.

4.2.2 Buridan's Paradox

We have noticed above that the Liar Antinomy, as a direct form of a self-reflexive utterance, has an indirect (not immediate) equivalent in the form of Burian's Paradox. Jean Buridan (before 1300–1361), a French medieval logician and philosopher presented, in his famous logic manual *Summulae de dialectica*, a number of interesting logical questions and problems, among which one can find the Liar's Paradox in the form of a vicious circle[72]:

> **Buridan's Paradox**
> Socrates in Troy says, "What Plato is now saying in Athens is false". At the same time, Plato in Athens says, "What Socrates is now saying in Troy is false".

Indeed, let us assume that the sentence uttered by Socrates, $S =$ "What Plato is now saying in Athens is false", is true. Then it is as it says, so the sentence uttered by Plato $P =$ "What Socrates is now saying in Troy is false" is not true. What results from the falsehood of sentence P is the truth of sentence S, i.e. lack of contradiction. We get a similar lack of contradiction if we assume the truth of sentence P. Then it is sentence S that is false. This way, Buridan's problem is hardly a paradox. A paradox appears only when we state that both sentences should be treated

[72] Buridan [6], p. 200.

in the same way. Then it is impossible to assume the truth of one of the sentences while refusing it to the other. This, however, is an extra-logical difficulty; logically, there is no paradox here. To see it better let us assume that we substitute Charles for Socrates and Adam for Plato. Moreover, let us assume that Charles is an honest person, who has never been found on a lie or other treachery, while Adam is a notorious liar and crook. Then we shall easily accept the truth of a sentence uttered by Charles, so shall have to accept the falsehood of a sentence uttered by Adam. What is more, the situation will be in no way paradoxical—we cannot equally trust the word of a an honest person and a liar. Regrettably, we seldom have similarly clear situations. Sometimes Buridan's problem is difficult to solve, because the two informers are almost equally trustworthy.

The example with Charles and Adam shows fairly well that Buridan's Paradox is not a logical dilemma but a psychological one, which we encounter more often than we would wish. It is not difficult, however, to reshape the paradox in such a way that it really becomes a dilemma of self-reflexivity:

Corrected Version of Buridan's Paradox
Socrates in Troy says, "What Plato is now saying in Athens is false". At the same time, Plato in Athens says, "What Socrates is now saying in Troy is true".

If Socrates utters a true sentence $S =$ "What Plato is now saying in Athens is false", this means that things are as it is predicated by it, i.e., Plato's sentence $P' =$ "What Socrates is now saying in Troy is true" is not true. It follows therefrom that sentence S is not true, in contradiction to the assumption. Let us assume now that sentence S is not true. Consequently, things are as it is predicated by it, i.e., sentence P' is true. But then also sentence S is true, in contradiction to the assumption. Consequently, sentence S is neither true nor false, which means that sentence P' is neither true nor false, too.

Another simple analysis, which makes use of classical sentence calculus shows that both sentences are true and false at the same time. Since $S \leftrightarrow \neg P'$ and $P' \leftrightarrow S$, so $P' \leftrightarrow \neg P'$, which means that $P' \leftrightarrow P' \wedge \neg P'$ and $\neg P' \leftrightarrow P' \wedge \neg P'$.[73] Consequently, sentence P' is true and false at the same time. However, by virtue of equality $P' \leftrightarrow S$, sentence S is true and false at the same time, too.

4.2.2.1 Proposed Solution of Buridan's Paradox

Naturally, the solution we proposed for the Liar Antinomy can be easily applied to the new form of Buridan's Paradox.[74] Let us assume the truth of two sentences, $S : \neg P'$ and $P' : S$. Thus, for a certain model $M = (A, D)$ and a certain

[73] Cf. *Priest's Proposal* in the previous paragraph.

[74] Naturally, the sentence calculus with the conjunction : may also be applied to the original form of Buridan's Paradox; however, this would have no sense, for in its original form Buridan's problem is not a logical paradox.

homomorphism $v \in Hom(L, A)$ such that $v(S) = p$, $v(P') = a$ we get: $p \supset -a \in D$ and $a \supset p \in D$. Consequently, $p = -a \cap c$ and $a = p \cap d$, for certain c and d.

Let us assume now that $p \in D$, so that $-p \notin D$, $-a \in D$ and $c \in D$. Then, since $-a = -p \cup -d$, so, consequently $-d \in D$. Let $d = a$ and $c = p$. With his New assumption: $p = -a \cap p$ and $a = p \cap a$. This is in agreement with the assumption we accepted in the previous paragraph: sentence S states falsehood of sentence P', at the same time, silently but evidently, stating its own truth. If we had not assumed that sentence S states its own truth, we could not claim that it states the falseness of sentence P'. Similarly, sentence P' states the truth of sentence S, as well as its own. Earlier semantic analysis shows that sentences S and $\neg P'$ are true, while sentences $\neg S$ and P' are not true. As we can see, there is no paradox here.[75]

A number of sets of utterances, fairly easy for assessing their logical value, is considered by Skyrms.[76] Application of classical sentence calculus with the additional conjunction :, which we have proposed above, makes giving those sentences logical value easy and free from contradiction.

4.2.3 Generalized Form of Liar Antinomy

An indirect vicious circle can have more than two links. Similarly, the Liar Antinomy can be translated into a dilemma, in which paradoxical character of self-reflexivity results from n sentences of appropriate form. It is not difficult to generalize Buridan's Paradox in its corrected version to a case of n appropriately looped sentences:

Generalized Form of Liar Antinomy[77]

$L_1.L_2$[78]

$L_2.L_3$

$\vdots$

$L_{n-2}.L_{n-1}$

$L_{n-1}.L_n$

$L_n.\neg L_1$

The above set of sentences Leeds to a paradox, which is analogous to the Liar Antinomy. Indeed, if we accept the truth of sentence L_1, we must consequently accept the truth of all sentences from L_2 to L_n. Consequently, we have to accept

---

[75] It should be addend here that there is no other solution, in which A' would be a true sentence. For then the sentence P would be true, and that implies the falsity of the sentence A'.

[76] Skyrms (1984), pp. 119–131.

[77] It is easy to notice that there are many other, possible forms of this paradox.

[78] The expression "L_1. L_2" means that a sentence L_2 is called L_1.

sentence $\neg L_1$ as true, which, however, is in contradiction with the assumption of truth of L_1. Similarly, starting from falsehood of sentence L_1, we must accept the falsehood of all sentence from L_2 to L_n, so sentence $\neg L_1$ must be false too, which is in contradiction with the assumption of falsehood of sentence L_1.

4.2.3.1 Proposed Solution for Generalized Liar Antinomy

In order to eliminate the paradoxical character of the analyzed set of sentences it is enough to apply the classical sentence calculus with the additional conjunction :. Let us assume the truth of the following sentences:

$$(L_1 : L_2),\ (L_2 : L_3), \ldots,\ (L_{n-2} : L_{n-1}),\ (L_{n-1} : L_n),\ (L_n : \neg L_1).$$

In agreement with our earlier proposal, we find, omitting some steps in reasoning, the following interpretation:

$$(L_1 = L_2 \cap L_1),\ (L_2 = L_3 \cap L_2), \ldots,\ (L_{n-2} = L_{n-1} \cap L_{n-2}),\ (L_{n-1} = L_n \cap L_{n-1}),\ (L_n = \neg L_1 \cap L_n).$$

Now it is enough to assume the falsehood of sentence L_1, i.e. the truth of $\neg L_1$ and the truth of all sentences from L_2 to L_n to see manifestly that it is a non-contradictory interpretation of the set $\{L_1, \ldots, L_n\}$.

4.2.4 Curry's Paradox

The Liar Antinomy has many versions. All of them, however, are usually a simple repetition of the idea that is the essence of the Liar Antinomy in its original version. There is, however, a more inventive version of this antinomy. Let us present it in the following way:

Curry's Paradox
Let C be a sentence of the form: *If sentence C is true, a circle has four equal sides.*

Introducing paradoxical contradiction here can be twofold.

1. *First Reasoning*. If C is a true sentence, a circle has four equal sides. If C is a false sentence, the implication "If C is a true sentence, a circle has four equal sides" has a false antecedent, and so it is true. Consequently, C is a true sentence, so a circle has four equal sides. In this reasoning we "prove" the truth of any sentence, also a contradictory one "a circle has four equal sides".
2. *Second Reasoning*. If C is a true sentence, a circle has four equal sides. Yet it is not true that a circle has four equal sides. This means that the implication "If C is a true sentence, a circle has four equal sides" has a true antecedent and a

false consequent, so it is false—there is a contradiction. If C is a false sentence, the implication "If C is a true sentence, a circle has four equal sides" has a false antecedent, so it is true. Consequently, C is a true sentence. This means that C is true if and only if it is false. In this reasoning we prove contradiction in sentence C following the pattern of the liar sentence L.

As we can see Curry's Paradox leads to contradiction in at least two ways. Naturally, it is one and the same problem: the antecedent of an implication presents a proposition predicating the truth of the whole implication. We can say, therefore, that this is a case of self-reference of an utterance that is entangled in the context of that utterance.

The proposed solution of the Liar Antinomy works well for Curry's Parodox too. The paradoxical sentence has the form:

$$C : C \to \bot.$$

Consequently, in the proposed interpretation (with simplified notation) we obtain e.g. $C = (C \to \bot) \cap X$, i.e. $C = (\neg C \cup \bot) \cap X$. Naturally, the assumption that C has a specified value leads to contradiction. Actually, in such case $\neg C$ has an unspecified value, while "$\bot$" symbolizes a sentence that has an unspecified value in any interpretation. Consequently, $\neg C \cup \bot$ has an unspecified value, and so C has an unspecified value as well. However, if we assume that C has an unspecified value, even though $\neg C \cup \bot$ has a specified value then, it is possible to select such X that will have an unspecified value. Looking for an appropriate form for X let us note that from C and $C \to \bot$ we obtain immediately $\bot$. Indeed, since $C : C \to \bot$ and $C : C$, then $C : \bot$. Consequently, a desired form for X is $\bot$. Let us assume, then, that $C = (\neg C \cup \bot) \cap \bot$, and the paradox disappears: C and $\bot$ are false, while $\neg C \cup \bot$ are true.

Irrespective of the solution, it should be noted that Curry's Paradox is a warning not to argue from false premises. If we are content with a reasoning that starts from the assumption that C is true, we arrive at falsehood or even contradiction, which is nothing unusual. The real problem appears only when argue from C even if we have assumed that it is false.

4.3 Other Semantic Paradoxes

Russell's Antinomy, which we have discussed in the chapter devoted to paradoxes resulting from imperfect intuition, points to the existence of a logical principle, the disregard of which must lead to a contradiction[79]: "No subject of a *given kind* may remain in relation to all and only things of *that kind*, which do not remain in that relation to themselves." However, if we remember about defining the greatest and

[79] Quine [27], p. 131.

smallest elements in an ordered set, we should note that there is a subject of a given kind that remains in a given relation to all and only things of that kind, which, firstly, are not that subject and, secondly, do not remain in that relation to themselves.

4.3.1 Barber's Antinomy

Barber's Antinomy
Let us assume that the town of Barberville had only one barber. When we consider who was shaved by the barber, we come to an evident and apparently the only possible conclusion: the barber shaved all men of the town who did not shave themselves. The problem is, however, the barber himself. If he shaves himself, he belongs to the group of people who shave themselves and those are the very men he does not shave. If he does not shave himself, however, then he is one of those who do not shave themselves and those are the very men he does shave. Thus we get a contradiction: the barber shaves himself if and only if he does not shave himself.

If we accept the symbols: *M*—the set of all men in Barberville, *a*—barber of Barberville, $G(x,y)$—"*x* shaves *y*"; then the sentence leading to a paradoxical conclusion can be presented as follows (G):

$$\forall x \in M(G(a,x) \leftrightarrow \neg G(x,x)).$$

Since $a \in M$, it is enough to substitute the constant a for the variable x to get contradiction:

$$G(a,a) \leftrightarrow \neg G(a,a).$$

This formal structure of the Barber's Antinomy shows that it presents the very same problem, which was diagnosed by Willard van Orman Quine (1908–2000) in the above quotation. It turns out, however, that the Barber's Antinomy can be solved in a quite natural way, i.e. through limiting the scope of the general quantifier. In this sense, it can resemble other problems we have with establishing that a given person is best or worst in some respect. For instance, let us consider a statement saying that with respect to knowledge and contribution to science Albert Einstein is a better physicist than any man, so he is the best physicist in history, while Edwin Stetson is taller than any man in the world, so he is the tallest man on earth. In both cases contradiction in immediate. If Einstein is a better physicist than any man, he is, specifically, a better physicist than himself. Similarly, we can prove that Stetson is taller than himself, since he is taller than any man. Yet these conclusions seem to be contrary to the obvious and hardly dubitable everyday use of such names as "the best physicist" or "the tallest man". Some problems with the use of these expressions may be related to a difficulty in establishing whether a given man meets the conditions for being a designate of one of those names. In other words, finding out which object is the designate of a given name may be a practically insoluble question. However, it is difficult to accept that there is any fundamental logical problem, which would require elimination of all superlative names from the language, especially that even mathematics allows for the concepts of the smallest and the greatest elements of an ordered set: "The greatest element of an ordered set is such an element of the set, which is later than any other element of it".[80] The ordering relation, however, is reflexive, which means that every element is later than itself. Such a reflexive definition of the ordering relation prevents paradoxical utterances concerning the smallest and greatest elements. Indeed, if being a better physicist were understood as a reflexive relation, Einstein would be better than himself and we would have no paradox. Similarly, we would not encounter a paradox in the case of a reflexive definition of being a taller man. Unfortunately, the usual understanding of the superlative seems to be incompatible with reflexive relation, for whose definition it is necessary. For this reason, there is another way of avoiding contradiction in pondering over the best physicist or the tallest man. This solution involves limiting the scope of the general quantifier and seems to be in agreement with the standard usage of names. Einstein will be considered to be the best physicist of all people, if he turns out to be a better physicist than anyone who is not him. Similarly, Stetson will be considered to be the tallest of all people, if he turns out to be taller than any man who is not him.

Thus we do not fall into contradiction if saying that "the barber shaves all men in the town who do not shave themselves" we mean that he shaves every man who is not him and who does not shave himself. Substituting the formula (G') for (G)

$$\forall x \in M - \{a\}\,(G(a,x) \leftrightarrow G(x,x))$$

we make introducing contradiction impossible. It must be added here that apparently this seems to be the common understanding of the sentence "the barber shaves all men in the town who do not shave themselves". Similarly, all other sentences with the superlative are understood with a limited the scope of the general quantifier. This point of view allows other, equally obvious, statements. For instance, a man that helps everyone who cannot help themselves (other than himself), can turn out to be someone who can manage his own affairs well or badly, so it happens in life.

We must stress here, however, that without an additional assumption, clearly omitted in the story of the Barber's Antinomy, limiting the scope of the general quantifier, the antinomy points to a irremovable contradiction, i.e., to the inexistence of the barber described in it. For if we accept that what is assumed in the story does not go beyond what is said there, then according to the rules of classical logic there is no barber who shaves all men in the town that do not shave themselves. The insoluble character of the strictly formulated original version of the

[80] Borkowski [4], p. 275.

Barber's Antinomy makes this paradox closer to other semantic antinomies, which are analyzed in two following chapters.

4.3.2 Richard's and Berry's Antinomies

Since the paradoxes of Richard, Berry, and Grelling do not have even a "compromise" solution similar to the Barber's Antinomy, they seem to be closer to Russell's Antinomy rather than to the former. Namely, they point to certain potentialities of the language we use. These potentialities refer to formulating such expressions that inevitably lead to paradoxes.

The first of these antinomies, the one discovered by Jules Antoine Richarda (1862–1956) and published by him in 1905, we present in its original form[81]:

Richard's Antinomy

"I am going to define a certain set of numbers, which I shall call the set E, through the following considerations. Let us write all permutations of the twenty-six letters of the French alphabet taken two at a time, putting these permutations in alphabetical order; then, after them, all permutations taken three at a time, in alphabetical order; then, after them, all permutations taken four at a time, and so forth. These permutations may contain the same letter repeated several times; they are permutations with repetitions.

For any integer p, any permutation of the twenty-six letters taken p at a time will be in the table; and, since everything that can be written with finitely many words is a permutation of letters, everything that can be written will be in the table formed as we have just indicated.

The definition of a number being made up of words, and these words of letters, some of these permutations will be definitions of numbers. Let us cross out from our permutations all those that are not definitions of numbers. Let u_1 be the first number defined by a permutation, u_2 the second, u_3 the third, and so on. We thus have, written in a definite order, *all numbers that are defined by finitely many words*. Therefore, the numbers that can be defined by finitely many words form a denumerably infinite set.

Now, here comes the contradiction. We can form a number not belonging to this set. "Let p be the digit in the nth decimal place of the nth number of the set E; let us form a number having 0 for its integral part and, in its nth decimal place $p + 1$ if p is not 8 or 9, and 1 otherwise." This number N does not belong to the set E. If it were the nth number of the set E, the digit in its nth decimal place would be the same as the one in the nth decimal place of that number, which is not the case

[81] Richard [28], p. 143.

I denote by G the collection of letters between quotation marks.[82] The number N is defined by the words of the collection G, that is, by finitely many words; hence it should belong to the set E. But we have seen that it does not. Such is the contradiction.

In the same paper, in which he presents the construction of the set E and number N, Richard gives his own solution of the problem.[83] He notices that the definition of the number N is based on the definition of the set E. Now the whole infinite set E is defined by the infinite number of words, so the number N is also defined by the infinite number of words. This means that the number N cannot belong to the set E, since only numbers defined by a finite number of words belong to it, even though the number of those numbers is infinite. For this reason, according to Richard, there is no contradiction here. This solution, expectedly, was accepted by the constructivist Jules-Henri Poincaré.[84]

In the very same paper, Richard modifies his construction in the following way. He suggests expanding the set E to include the number N by putting it on a n^{th} (finite) position on the list, adequate for G, thus increasing by one the position of every following number of the set E. The set E' constructed in this way includes the number N, which, however, is different from the number N' defined by G: the number N has the k^{th} position on the list, so the digit, which has the k^{th} decimal position of the number N', is different from the digit on the k^{th} decimal position of the number N.[85] Contradiction has been saved.

Richard was convinced that the problem, which he discovered had a set-theory nature. To show the analogy between Richard's and Russell's antinomies let us present the former in the words of Witold Marciszewski taken from his *Antynomie w logice*[86]:

Another Form of Richard's Antinomy
"[…] A list of definitions is created in order to define various arithmetical properties, e.g. the property of being a prime number[…]. This list is ordered in a certain way, e.g. from the shortest definition (in the number of letters) to the longest one. Every position of the list ordered in this way is given a consecutive number. This way, every definition is coordinated with precisely one natural number. It can happen now that a number coordinated with a definition may have the very property that the definition presents; for instance, let the definition of a prime number, given above, have number 17. It is a number

[82] Naturally, a sentence corresponding to the set G should be understood as a sentence in French and such expressions as "+", "1", "8", "9", "1.", "2." find their French word substitutes in the set G, i.e., "plus", "un", "huit", "neuf", "première", "seconde".

[83] Richard [28], p. 143.

[84] Poincaré [22], p. 145.

[85] Richard [28], pp. 143–144.

[86] Marciszewski, *Antynomie w logice* [w:] [19], pp. 20–21.

that possesses the property defined by definition 17. Naturally, there are other cases, in which the number coordinated with a definition does not have the property defined by it: such numbers will be called Richard numbers, while the remaining ones, such as 17 from our example, will be called non-Richard numbers. Definition of a Richard number defines a certain property of numbers, so it will be included on the list of definitions and coordinated with a number. Will that number be a Richard number?
This is a question leading to an antinomy. If it is a Richard number, then it does not have the property defined in the definition, so it is not a Richard number. But if it is not a Richard number, then it possesses a property defined in the definition, so it is a Richard number."

How much Richard's Antinomy resembles Russell's Antinomy in its structure can be seen if we note that every definition occurring on the list under a number k marks $\boldsymbol{N}_k$—a subset of natural numbers, which possess a property marked by the number k. For the sake of precision in reasoning let us assume, what has no influence on the paradoxical character of Richard's Antinomy argument, that if two properties generate the same set of numbers, they are treated as the same property. Such a property can occur on the list only once then. Thus we can assume that every natural number k is univocally defined by the set $\boldsymbol{N}_k$, which leads to actual indentifying of the number k with the set $\boldsymbol{N}_k$: $k := N_k$. The natural number k is a Richard number if it does not belong to the set $\boldsymbol{N}_k$, i.e., when $k \notin k$. Now since the set of all Richard numbers possesses a property of being a Richard number, it also defines a natural number k_0. Consequently, the set of all Richard numbers has the form:

$$N_{k0} = \{k : k \notin N_k\},$$

so

$$k_0 = \{k : k \notin k\}.$$

Specifically,

$$k_0 \in N_{k0} \text{ iff } k_0 \notin N_{k0},$$

so

$$k_0 \in k_0 \text{ iff } k_0 \notin k_0.$$

Thus every Richard number is a set of numbers, to which it does not belong itself, while the Richard number of all Richard numbers, since it is a Richard number, simultaneously must and cannot belong to itself. Apparently, the similarity to the Russell's Antinomy is striking. Neither a set of all sets which are not their own elements, nor a Richard number of all Richard numbers.

Small wonder then that the solution of Richard's Antinomy on the ground of the set theory is the same as the solution of Russell's Antinomy. In the ramified theory of types we have a division of formulae into predicative and non-predicative ones.

The postulate, which limits definitions only to predicative formulae, aims at fulfilling the idea of Poincare, who wanted to escape the vicious circle in definition, so it specially protects the set theory against Richard's paradox. It should be noted here that the problem with defining the number N does not depend on the infiniteness of the set E but can be understood merely as a question of self-reference of the definition formula. To show it, it is enough to alter a little Richard's construction. Let us assume then that the definition of the number N, given by the French equivalent of the permutation G, is a permutation with repetitions composed of exactly 281 elements. Now let the list of all n-element permutations of the set F, composed of the 26 letters of French alphabet, be limited to those n-element permutations for which the natural number n fulfils the condition: $2 \leq n \leq 300$. Then the permutation which is the definition of the number N belongs to a new list, this time a finite one, and yet by its very definition the number N cannot belong to the set E. Richard's Antinomy remains in force. Another proof that Richard's Antinomy is independent of the infiniteness of the set E is the existence of Berry's Antinomy, rightly believed to be a variant of Richard's Antinomy.

On the ground of axiomatic set theories, the way to escape Richard's Antinomy is the same as in the case of Russell's Antinomy. Leśniewski's Systems, however, refer to other non-distributive sets, in which both Richard's and Russell's Antinomies simply do not exist. Further considerations of mathematical solutions for Richard's Antinomy were cut short by the intervention of an Italian mathematician and logician, Giuseppe Peano (1858–1932), who stated strongly in 1906[87]: "Richard's case does not belong to mathematics but to linguistics; the element, which is fundamental for the definition of the diagonal number N, cannot be defined in a precise way, i.e. according to the rules of mathematics."

Regardless of our opinions about the mathematical or non-mathematical character of Richard's Antinomy, it must be said that as long as the ways of avoiding Russell's Antinomy are recognized, Richard's Antinomy can be treated as defanged on the ground of mathematics. What remains is the question of the linguistic character of the antinomy. Mathematical theories notwithstanding, there is still the problem of inexpressibility of some problematic questions in the natural language, which was pointed by Richard among others. One could think that the division of natural language into separate orders will solve the problem. Indeed, the definition of a number belonging to E is a sentence of a language that is one order higher than the language order of definitions which make up Richard's list: the definition using the set of definitions making up the set E should be a sentence of order higher by one than that, to which definitions of numbers from the set E belong. Regrettably, this solution does not seem convincing, because every definition, if it defines an expression of the objective language, should belong to its metalanguage. Consequently, all definitions from Richard's list should have the same status as the definition of the number N, for they belong to the same natural

[87] Cf. van Heijenoort's *Introduction* to Richard [28], p. 142.

language order. It seems then that Richard's paradox should be independent of distinguishing strictly separated orders of natural language.[88]

The linguistic aspect of Richard's Antinomy makes us think whether all semantic paradoxes should be solved by all means. After all, we are convinced, quite naturally and obviously, about the richness of expression characterizing natural languages. Consequently, it is no wonder and no paradox that we are able to express all propositions, even absurd, contradictory and antinomian ones in a natural language. Solving semantic paradoxes 'by force' must stem from an opposite view, according to which our natural language is so logical and so perfect that its precise application should prevent any linguistic expression of antinomian constructions. It seems absurd that logicality (lack of contradiction) be necessitated by the set of syntactic rules of a language. How on earth could the possibility of forming contradictions of the type *A* and *not*-*A* be prevented on the level of language syntax? What is interesting and indeed paradoxical is that if it were possible, one could not speak of contradictions in that language, for the expression *A* and *not*-*A* would be inconstructible. Naturally, it is our opinion that for a natural language it is necessary that it not only can but also must allow to express all sorts of unusual and thus also antinomian constructions of thought. This does not mean, however, that attempts at logical solutions of discovered paradoxes should be stopped. Yet, some dilemmas are constructions, which are created with the aim of generating paradoxical conclusions. Richard's Antinomy seems to be one of such problems.

Berry's Antinomy, Publisher by Russell[89] in 1906, is treated as a simplified version of Richard's Antinomy. Apparently, the paradoxical self-reflexivity of this antinomy is simplified to the utmost. We shall recall it in the formulation of Stanisław Krajewski[90]:

Berry's Antinomy
"[…] Let Berry's number be the smallest natural number, which cannot be defined by means of a sentence composed of 30 Polish words at the most. The definition is correct, since only a finite number of numbers can be defined by means of such sentences. Consequently, Berry's number is and is not definable by means of a sentence of less than thirty words."

Commenting Richard's Antinomy we have noticed its independence of the infinity of the set *E*. Berry's Antinomy is the best proof of that statement. The list forming the set *E* has been reduced to a one element set. Naturally, all comments of both mathematic and linguistic nature must be repeated in the analysis of

[88] Naturally, a strict and formal adherence to Tarski's principle of separation of language orders is an effective protection against Richard's Antinomy.

[89] Russell [29].

[90] Krajewski, *Antynomie*, [in:] Marciszewski [18], pp. 177–178.

Berry's Antinomy too. For this reason, we shall now move to the last semantic antinomy analyzed here.

*4.3.3 Grelling's Antinomy (*Vox Non Appellans Se*)*

In the 1908 paper by Kurt Grelling (1886–1942) and Leonard Nelson (1882–1927), we find a presentation of another paradox inspired by Russell's Antinomy.[91] This dilemma is now known as Grelling's Antinomy. According to Borkowski, this antinomy is, however, and old medieval dilemma of a word that does not name itself. The original Latin name of the antinomy is "*vox non appellans se*"[92]:

Grelling's Antinomy (vox non appellans se)
Let us divide predicates in two groups. The first one will contain all predicates, which express the property they themselves possess, such as "polski", "English", "multisyllabic". Let us call them autosemantic predicates. The other group will contain all predicates, which express the property they themselves do not possess, e.g. "German", "Polish", "monosyllabic". The predicates of this group will be called heterosemantic. A problem arises here: where should we place the predicate "heterosemantic". If it is heterosemantic, it means that it expresses the property that it does not possess, but this very predicate expresses a property of not having the property that it expresses. Consequently, it turns out that the predicate "heterosemantic" is autosemantic. Let us assume, then, that "heterosemantic" is autosemantic. This means that it possesses a property that it predicates abort. But the property is not having the property that it predicates about. Consequently, the predicate "heterosemantic" is heterosemantic. Both reasonings imply, therefore, an antinomian balance: the predicate "heterosemantic" is heterosemantic if and only if it is autosemantic.

In his book *Quiddities. An Intermittenly Philosophical Dictionary*, Quine gives the following form of Grelling's Antinomy, which seems to be closer to the medieval original and, moreover, shows its close relation with the Barber's Paradox[93]:

"No adjective denotes all and only those adjectives that do not denote themselves".

As in the casus of the Liar's, Richard's and Berry's Antinomies, one can try solving Grelling's Antinomy in a somewhat artificial way by imposing on the natural language some unnatural limitations that are alien to it. This refers

[91] Grelling and Nelson [11].

[92] Borkowski [4], p. 355.

[93] Quine [27], p. 133.

especially to the antinomies presented in Sect. 3.3, since, as we have shown in Sect. 3.2.1, the Liar Antinomy can be solved in a quite natural way when we take into account things, which are so natural that we do not realize them(like breathing). This obvious element is the assumption that the sentences we utter are true, which in case of the liar sentence is expressed in the simple and evident identity principle: the sentence p says that p; in our formal notation, $p : p$. Unfortunately, the three remaining semantic antinomies seem to resist this solution, because the essence of those dilemmas lies in the fact that natural language allows for creation of all sorts of semantic constructions, which in some interesting cases lead to absurd. Natural language potential is unlimited in this respect and one can be hardly surprised that some seemingly logical language constructions lead to manifestly illogical conclusions. For instance, the Liar Antinomy can be "corrected" in such a way that even our proposed solution will be helpless with it. We can assume that sentence L is "This sentence is true", and sentence $\neg L$, "This sentence is not true". Then we have $L : \neg L$ and $\neg L : L$. It is easy to notice that the sentence calculus we have proposed cannot solve the dilemma created by both formulae put together. Clearly, we can make up most unusual constructions in the language and it is not at all surprising. From this point of view, all artificial attempts to solve Richard's, Berry's and Grelling's Antinomies seem pointless, since we should be able to create any problems in the language we use, also ones that are similar to the three antinomies.

The existence of semantic antinomies is thus a natural phenomenon, which characterizes every language that is rich enough, i.e. one, which contains semantic terms referring to the expressions of that system, or terms which allow to define such terms, and which is ruled by the laws and principles of the classical logical calculus. This position is in agreement with the opinion of Quine, who even gave a "recipe" for antinomies, which we have already mentioned. Surprise should not be caused by mere existence of such antinomies but by their unusual constructions. From this point of view, the aim of logical analysis with respect to many semantic paradoxes should be not so much solving them as learning the actual potential of the language. It is the desire to discover the properties of our language, usually hidden from view, that should be the impulse for the study of semantic paradoxes. Reaching this goal seems to be much more valuable and important than some artificial and sophisticated pseudo-solutions of particular semantic paradoxes.

4.4 Unexpected Examination (Hangman's) Paradox, or Self-Reflexive Reasoning

This paradox is mostly known in two popular forms: Unexpected Examination and Hangman's paradox.

Unexpected Examination Paradox

A teacher told her class that she would give them one examination over the following week (from Monday to Friday). The day of examination will be chosen in such a way that the pupils will not be able to find out its precise day. It turns out that the examination cannot be organized any day. For if it were to be on Friday, pupils would know on Thursday that it can only be organized on Friday. Apparently, the last day to organize it is Thursday. However, since the pupils know the teacher cannot organize the examination on Friday, then, if it was not organized until Wednesday, it must be organized on Thursday. This is against the teacher's promise, for the pupils can expect it on Thursday. So the last day for the exam seems to be Wednesday. Repeating the same reasoning for Wednesday excludes that day and similarly the preceding days of the week. Consequently, organizing an unexpected examination is not possible on any day of the week.

The conclusion following from this reasoning is startling: the teacher cannot organize a test any day from Monday to Friday. It is manifestly in disagreement with our natural belief that the teacher can surprise her pupils organizing the examination e.g. on Tuesday. Its timing on Monday, Tuesday or Wednesday is impossible to predict.

In his book *Paradoxes*,[94] Mark Sainsbury proposes to reject the assumption that the class can behave irrationally, i.e. it cannot expect something it should not, and not expect something that it should. Sainsbury points, however, that after the above reasoning the class may come to a conclusion that the teacher does not want to deliver on her promise, not least because it is impossible for her to do it. Sainsbury stresses that the truth of the teacher's promise immediately leads to contradiction and therefore proposes introducing simple symbols, which facilitate the notation of the reasoning that takes the teacher's promise as an assumption. He also suggests limiting the number of days, on which the exam can be organized from five to two, arguing that the reasoning does not depend on the number of days.[95] Let us consider the following abbreviations then:

M for "the examination occurs on Monday";

T for "the examination occurs on Tuesday ";

$K_M(\ldots)$ for "the class knows on Monday morning that…";

$K_T(\ldots)$ for "the class knows on Tuesday morning that…".

The symbolic notation of the teacher's promise receives the following form:

$A1$. Either [M and not-$K_M(M)$] or [T and not-$K_T(T)$].

[94] Sainsbury [31], pp. 93–106.

[95] Naturally, for our problem it is irrelevant whether the number of days is expressed by 5, 10, or 365. Yet, shortening the period to two days may significantly change the conditions of the problem. We only signalize this reservation without going deeper into its analysis.

The reasoning reconstructed by Sainsbury has the following form[96]:

1.	Suppose $A1$	
2.	Suppose not-M	
3.	K_T(not-M)	2, memory
4.	If not-M, then T	$A1$
5.	$K_T(T)$	3, 4
6.	If $K_T(T)$ and not-M, then not-$A1$	$A1$
7.	not-$A1$	2, 5, 6
8.	M (and so not-T)	1, 2, 3–7
9.	$K_M(M)$	8, $A1$
10.	If $K_M(M)$ and not-T, then not-$A1$	$A1$
11.	not-$A1$	8, 9, 10
12.	not-$A1$	1, 2–11

The reasoning, in the form of *reductio ad absurdum*, leads to the rejection of the promise given by the teacher, since assuming the promise leads to contradiction. In the latter case, we have a paradox, for even though we have obtained "not-$A1$", we feel intuitively that $A1$ may be a true sentence. According to Sainsbury, the above construction would prove the falsity of the teacher's promise to her pupils, however, the justification of the fifth step is doubtful. For an inference to be true, the reasoning "If $A1$ is true, then the examination must occur on Tuesday if it does not occur on Monday; so if we knew the examination did not occur on Monday, we would know that it would occur on Tuesday" must be strengthened by the assumption that "We know that the examination must occur on Tuesday if it does not occur on Monday".[97] This way Sainsbury obtained the first of the two solutions he proposed for the paradox[98]:

1.	Suppose $K(A1)$	
2.	Suppose not-M	
3.	$K_T(\textit{not-}M)$	2, memory
4.	If not-M, then T	$A1$
5.	K(If not-M, then T)	1
6.	$K_T(T)$	3, 5
7.	If $K_T(T)$, then not-$A1$	$A1$
8.	If not-$A1$, then not-$K(A1)$	only the truth is known

96 Sainsbury [31], pp. 97–98.

97 Sainsbury [31], p. 99.

98 Sainsbury [31], p. 99.

9.	If $K_T(T)$, then not-$K(A1)$	7, 8
10.	M (and so not-T)	1, 2, 3–9
11.	$K_M(M)$	10
12.	If $K_M(M)$ and M, then not-$A1$	$A1$
13.	not-$A1$	10, 11, 12
14.	If not-$A1$, then not-$K(A1)$	only the truth is known
15.	not-$K(A1)$	13, 14
16.	not-$K(A1)$	1, 15

The author of that construction himself has serious doubts as to the value of such solution. First of all, he notices that for the reasoning to be strict, the sentence 11 should be understood not as one resulting from the sentence "If $K(A1)$, then M", but precisely from the sentence "K[If $K(A1)$, then M]". But then we need a premise "$K[K(A1)]$", instead of "$K(A1)$". Therefore, the above reasoning uses an enthymematic premise: "If $K(\varphi)$, then $K[K(\varphi)]$", which, according to Sainsbury, is of dubious value.[99] The other problem noticed by Sainsbury is that the conclusion achieved in the proposed solution of the paradox seems to be paradoxical as well. Now if the teacher is a trustworthy person, which has to be assumed for the paradox to have sense, the pupils should believe in her words and then the sentence "$K(A1)$" should be an obvious premise of the reasoning. It is difficult to question that sentence, even though we can easily solve the paradox of unexpected examination with help of its negation "not-$K(A1)$". If it is not true that pupils know that $A1$, the teacher has plenty of chances to surprise them. And here Sainsbury has a doubt: since $A1$ cannot be known, it is easy to accept that $A1$ is true. It is enough to assume that the teacher is a trustworthy and resolute to have grounds for accepting the truth of $A1$. And if it is so, the paradox returns. Sainsbury was right then when he rejected the first construction and presented the other one, based on another formalization of the teacher's promise, as a solution of the paradox. Thus, the sentence $A2$ was substituted for $A1$:

$A2$. Either [M and not-K_M(If $A2$, then M)] or [T and not-K_T(If $A2$, then T)].

As can be seen, the definition of a promise refers to itself, so $A2$ is a self-reflexive declaration. The reasoning using this definition is the following[100]:

1. Suppose $A2$
2. Suppose not-M

[99] Sainsbury [31], p. 99.

[100] Sainsbury [31], pp. 100–101.

3.	K_T(not-M)	2, memory
4.	K_T(If not-M, then if $A2$, then T)	the class understands $A2$
5.	K_T(If $A2$, then T)	3, 4
6.	not-$A2$	$A2$, 2, 5
7.	M	1, 2–6
8.	If $A2$, then M	1–7
9.	K_M(If $A2$, then M)	the proved is known
10.	If K_M(If $A2$, then M), then if $A2$, then not-M	$A2$
11.	If $A2$, then not-M	9, 10
12.	not-$A2$	8, 11

In Sainsbury's opinion, the value of this solution lies in the fact that the intuition, which tells us that the sentence $A2$ is true, is not as strong as in $A1$. One could even agree with it, yet it is much more difficult to accept the sentence $A2$ as a formalization of the teacher's promise given to the pupils. In the first place, the promise lacks any trace of reference to itself. Thus it is difficult to accept that in its original form the promise is a self-referring sentence. Secondly, as noted by Sainsbury himself, the essence of this paradox is that a rationally inferred conclusion is contrary to our expectations basing on the intuitions we have about the promise given by the teacher. It is easy to notice that Sainsbury's other solution contains a mistake or am intended trick of methodological character. Its essence is that an intuitively accepted sentence is formalized so as to deprive it of its intuitive character. As a result, it is difficult to accept that proposal as a solution of the Unexpected Examination Paradox.

Below we present a proposal of an unusually simple solution for this paradox, which questions neither the strictness and correctness of the reasoning leading to the conclusion that the examination cannot be organized in accordance with the promise given by the teacher, nor the evident character of the belief that organizing the examination, say, on Wednesday would be a total surprise for the pupils. Let us return to the original version of the paradox, which assumed that the teacher has more than two days to organize the examination. For a better grasp of the problem, let us present the reasoning from the point of view of a pupil:

"The examination cannot be on Friday, since we would know it must be then on Thursday after the classes. So the last day for the exam is Thursday. But it cannot be on Thursday either, since we would know it must be then on Wednesday after the classes… [repeating the reasoning several times the pupil comes to a conclusion that] *I know that the exam can be organized neither on Friday, nor on Thursday, Wednesday, Tuesday or Monday. Consequently, there is no such week day, on which she could organize the examination."*

This is the place, in which the reasoning of the paradox always ends. However, the pupil should notice something else, for his reasoning should be continued as follows:

"Since I know that the examination cannot be organized on any day of the next week, the condition given by the teacher is fulfilled, for there is no such day on which I would know that the examination is next day, as I know that it cannot be on any day. Now the condition for organizing the exam refers only to my knowledge about it and not the facts that are independent of my knowledge. Thus I know that the exam can be organized on any day. But if so, it cannot be on Friday, for we would know about it on Thursday. So the last day for the examination is… [etc.]"

The reasoning has become circular. The pupil can reason on and on, without contradiction and in accordance with the conditions of the problem, going through all the same inference steps *ad nauseam*. This is an instance of self-reference problem affecting the whole reasoning. The reasoning turned to be self-referent, for it became circular and the same inference steps can only be repeated infinitely. Moreover, in this circular reasoning there is no privileged step, which could stop the procedure. This means that if we want to foretell the day of the exam, we cannot exclude any of the conclusions that make up the circle of reasoning.

Finally, let us note a rather startling conclusion resulting from the fact that the condition for organizing the examination depends on the knowledge of pupils. We can obtain it in the reasoning independent of the above ones. Now every day it is so that: *either pupils know that there will be an exam next day or they do not know that there will be an exam next day*. In the latter case, it can be organized for obvious reasons; in the former one, if the condition for organizing the exam is known and the pupils know it will be organized on the next day, then by the same token they know it cannot be organized on the next day, and if so, it may be organized then. Since the reasoning retains its value any day, we come to the following conclusion: *The examination can be organized on any day*. The teacher can do it any time she wishes. This means that it was wrong to assume that her promise was false. It is obvious that the promise is true if it announces the exam, which the pupils really cannot expect.

The above reasonings show clearly that seemingly similar paradoxes: the bottle imp[101] and the unexpected examination are different in an important way. This is the epistemic-psychological factor, which plays a key role in the latter paradox, but does not affect the former. This is the reason why the latter paradox has a solution while the former does not.

4.5 Crocodile's Paradox, or Baron Münchhausen Fallacy

Another example of circular reasoning is the Crocodile's Paradox, known already in the Antiquity.

[101] Cf. Sect. 2.1.

Crocodile's Paradox

A crocodile has stolen a child from its mother. Implored to give it back to her, the beast said shedding crocodile tears: "Your grief has moved me, woman. I will give you a chance to get your child back. Answer my question whether I return the child. If your answer is true, I shall give you the child back, but if it is false, I shan't". After some consideration the mother replied: "You will not give the child back". Said the crocodile: "You have lost your child. For you either told the truth or not. If you told the truth saying that I would not give you the child back, then I will not give it back, otherwise what you said would not be true. But if what you said was false, then the child must stay with me as we have agreed". But the mother was not satisfied with the crocodile's verdict and claimed that the child should stay with her arguing: "if I told the truth, you should give me the child, as you promised to do if I tell the truth. If what I said was false, if it is not true that you will not give me the child back, then you must give it back to me, for otherwise what I said would not be false".

Let us first look at the formal analysis presented by Ajdukiewicz.[102] For the sake of precision, he suggests using the letter *Z* to symbolize one of two mother's possible replies to the crocodile's question about his giving the child back. Thus the sentence *Z* stands for either "The crocodile will give the child back" or for "The crocodile will not give the child back". In the former case *non-Z* symbolizes the sentence "The crocodile will not give the child back", in the latter, "The crocodile will give the child back". Later, Ajdukiewicz proposes putting down the crocodile's promise as follows:

I. *If the mother says that Z, then, if Z, the crocodile will give the child back, and if non-Z, the crocodile will not give the child back.*

If we substitute for *Z* the reply of the mother "*the crocodile will not give the child back*", we shall obtain:

II. *If the mother says that the crocodile will not give the child back, then, if crocodile does not give the child back, it will give it back, but if it gives the child back, it will not give it back.*

The structure of the sentence II is the following:

III. *If the mother says that the crocodile will not give the child back, then (if Z, then non-Z) and (if non-Z, then Z).*

Ajdukiewicz observes later that according to the principles of logic if a sentence *non-A* follows from a sentence *A*, which is contradictory to it, then *non-A* must be true. By analogy, if *A* follows from *non-A*, then *A* must be true. Consequently, from III results:

IV. *If the mother says that the crocodile will not give the child back, then Z and non-Z.*

[102] Ajdukiewicz [1], pp. 141–142.

So,

V. *If the mother says that the crocodile will not give the child back, then the crocodile will give the child back and will not give it back.*

Ajdukiewicz states, therefore, that if the crocodile's promise is true, then a contradiction results from the mother's reply that "the crocodile will not give the child back". Consequently, if the crocodile wants to keep its promise, it cannot allow a situation, in which the mother tells it: "you will not give the child back". Ajdukiewicz puts the whole blame on the crocodile claiming[103]: "The crocodile failed to keep the promise the moment it allowed her to speak. Whoever allows a fact, which excludes the promised state of affairs, does not fulfill the promise".

An interesting interpretation, close to the traditional way of telling the story was presented by Grzegorczyk,[104] who accepts the following six premises:

1_G *It should give it back* $\equiv$ *she has guessed.*

2_G *It should eat it* $\equiv$ *she has not guessed.*

3_G *It will eat* $\rightarrow$ *she has guessed.*

4_G *It will not eat* $\rightarrow$ *she has not guessed.*

5_G (*It should give it back and will eat*) $\rightarrow$ [*it will not (be consistent*)].

6_G [*It should eat it and will not* (*eat*)] $\rightarrow$ [*it will not* (*be consistent*)].

The first two premises express the crocodile's obligation, the next two result from the mother's reply, while the final two are suggested as additional ones in order to evaluate the crocodile's action. Since the formula $(p \rightarrow q) \rightarrow [(r \equiv q) \rightarrow (p \rightarrow r)]$ is a tautology of the classical sentence calculus, it follows from the first and third premises that:

7_G *It will eat* $\rightarrow$ *It should give it back.*

Since the formula $(p \rightarrow q) \rightarrow \{[(q\, and\, p) \rightarrow not\, r] \rightarrow (p \rightarrow not\, r)\}$ is also a tautology of the classical sentence calculus, from 7_G and 5_G we obtain:

8_G *It will eat* $\rightarrow$ *it will not (be consistent*).

An analogous reasoning using premises 2_G, 4_G, 6_G leads to a conclusion that

9_G *It will not eat* $\rightarrow$ *it will not (be consistent*).

By virtue of the tautological formula $\{(p \rightarrow r)\, and\, [(not\, p) \rightarrow r]\} \rightarrow r$ it follows straight from 8_G and 9_G that

10_G *It will not (be consistent*).

With this interpretation, the contradiction hidden in the paradox is replaced by a declaration of impossibility of consistent action.

Both of the presented analyses show that contradiction cannot be eliminated from the Crocodile's Paradox. But is a solution of the problem that would not imply contradiction possible? Maybe, there is a way of interpreting the Crocodile's Paradox in such a way that would involve a non-contradictory description of the situation presented in the story.

Very often, when a problem seems insoluble, it helps to present it once again for oneself. Let us do it then with the problems of the Crocodile's Paradox.

[103] Ajdukiewicz [1], p. 142.

[104] Grzegorczyk [12], pp. 122–127.

The facts are the following: stealing the child by the crocodile, its agreement with the mother, and her reply that it will not give the child back. Beside these facts, there are speculations, which are not so much paradoxical as absurd. The crocodile will give the child back or not if it turns out that the mother gave a true reply to the question if it is going to give the child back or not. Let us note that the crocodile will know what to do for it has done it already! Let us mark the crocodile's action of giving the child back with a symbol *A*. The crocodile must make a decision whether to do *A* or *non-A*. To this end it must recognize the mother's reply as true or false. However, the logical value of the mother's reply will be known to it on condition that it will have done *A* or *non-A* before. And this seems to be the real problem of the paradox. The crocodile will be able to make a decision whether to do *A* or *non-A* only after it has done *A* or *non-A* before. The whole situation seems to resemble that of baron Münchhausen pulling himself out of a swamp by his hair. Apparently, taking into account necessary the sequence of events is a necessary step to solve the Crocodile's Paradox:

1. *The crocodile steals the child.*
2. *The mother asks the crocodile to give the child back.*
3. *The crocodile asks the mother a question: "Shall give you the child back?".*
4. *The mother replies: "You will not give me the child back".*
5. *The crocodile must evaluate the logical value of the mother's reply. It must, therefore, compare her reply with reality.*

And this is the heart of the matter: what event should be the reference point in evaluation of the truth or falsity of the mother's words? As can be seen, it cannot be anything that is only about to happen. It should be a situation, which may and must take place before the evaluation of the logical value of the mother's reply. Naturally, one can try to claim that it is the fact of not returning the child, which has just taken place and lasts since the moment of stealing. Yet it is difficult to accept this solution, for it would mean that the crocodile asks about something absolutely obvious and the mother's reply involves no difficulty.

The analysis of the problem presented above shows that it is reasonable to think that its solution lies in assuming that the crocodile wants the mother to guess its intentions. Then points 3 and 4 of the above sequence of events have the following form:

3′. *The crocodile asks the mother a question: "Do I intend to give you the child back?"*

4′. *The mother replies: "You do not intend to give me the child back".*

It is easy to notice that in this situation there is no paradox. This time, point 5 furnishes a very precise criterion of evaluation for the mother's reply:

5′. *The crocodile must evaluate the logical value of the mother's reply. It must, therefore, compare her reply with its intentions concerning the child. If it wants to give the child back, it means that it must change its mind since the* *mother was wrong. If it does not want to give it back, however, it must change its mind too, since the mother has guessed its plans and gave the correct answer.*

It is easy to notice as well that the other reply of the mother does not lead to a paradox either. If she replied "You intend to give me the child back", the crocodile does not have to change its plans if it intended to do so and what remains for the crocodile is to do so. However, if it did not want to give the child back, it does not have to change its plans but this time it can keep the child.

References

1. Ajdukiewicz, K. (1931). *Paradoksy starożytnych (Paradoxes of Ancients)*. In *Język i Poznanie* (Vol. 1, pp. 135–144). Warszawa: PWN, 1985.
2. Barwise, J., & Etchemendy, J. (1987). *The liar an essay on truth and circularity*. New York: Oxford University Press.
3. Bocheński, J. M. (1961). *A history of formal logic* (I. Thomas, Translated and Edited). Notre Dame: Thomas University of Notre Dame Press.
4. Borkowski, L. (1991). *Wprowadzenie do logiki i teorii mnogości (An introduction to logic and set theory)*. Lublin: Towarzystwo Naukowe Katolickiego Uniwersytetu Lubelskiego.
5. Burge, T. (1979). Semantical paradox. *The Journal of Philosophy*, *76*, 169–198. Cytaty z reedycji. In Martin, R. L. (Ed.) *Recent essays on truth and the liar paradox* (pp. 83–117). Oxford: Clarendon Press (1984).
6. Buridan, J. (1966). *Sophisms on meaning and truth*. Kermit Scott transl., Meredith, New York. Also In G. E. Hughes, *John Buridan on Self-Reference* (pp. 73–79). Cambridge University Press, Cambridge (1982).
7. Church, A. (1976). Comparison of Russell's resolution of the semantical antinomies with that of Tarski. *Journal of Symbolic Logic*, *41*, 747–760. Quotations after: Martin, R. L. (Ed.). *Recent Essays on Truth and the Liar Paradox* (pp. 289–306). Oxford: Clarendon Press (1984).
8. Devlin, K. (1997). *Żegnaj Kartezjuszu. Rozstanie z logiką w poszukiwaniu nowej kosmologii umysłu (Goodbye, Descartes: The End of Logic and the Search for a New Cosmology of the Mind)*, transl. Barbara Skarga, Prószyński i Spółka, Warszawa, 1999.
9. Feferman, S. (1982). Towards useful type-free theories, I. *Journal of Symbolic Logic*, *49*, 75–111. Quotations from Martin, R. L. (Ed.) *Recent essays on truth and the liar paradox* (pp. 237–287). Oxford: Clarendon Press (1984).
10. Field, H. (1972). Tarski's Theory of Truth. *Journal of Philosphy,69*(13), 347–375.
11. Grelling, K., & Nelson, L. (1908). *Bemerkungen zu den Paradoxien von Russell und Burali-Forti*, Abh. der Friesschen Schule II, 301–324.
12. Grzegorczyk, A. (1960). *Logika popularna* (*Popular Logic*), PWN, Warszawa.
13. Gumański, L. (1992). Logical and semantical antinomies. *Ruch Filozoficzny,49*(1), 21–30.
14. Gupta, A. (1982). *Truth and Paradox*, Journal of Philosophical Logic 11 (1982), (pp. 1–60). Quotations from Martin, R. L. (Ed.). (1984). *Recent Essays on Truth and the Liar Paradox* (pp. 175-–35). Oxford: Clarendon Press.
15. Herzberger, H. R. (1982). *Notes on Naive Semantics*, Journal of Philosophical Logic 11 (1982), pp. 61-102. Quotations from Martin, R. L. (Ed.). (1984). *Recent Essays on Truth and the Liar Paradox* (pp. 133–174). Oxford: Clarendon Press.
16. Kripke, S. (1975). *Outline of a Theory of Truth*, The Journal of Philosophy 72 (1975), pp. 690–716. Quotations from Martin, R. L. (Ed.). (1984). *Recent Essays on Truth and the Liar Paradox* (pp. 53-81). Oxford: Clarendon Press.

17. Łukowski, P. (1997). An approach to the liar paradox. In *New Aspects in Non-Classical Logics and Their Kripke Semantics* (pp. 68–80). RIMS, Kyoto University.
18. Marciszewski, W. (1987). Logika formalna. Zarys encyklopedyczny z zastosowaniem do informatyki i lingwistyki [Formal Logic. An Encyclopedic Introduction for Computer Studies and Linguistics], red. Warszawa: PWN.
19. Marciszewski, W. (1988). *Mała encyklopedia logiki (Short Encyclopedia of Logic*). W.Marciszewski (Ed.). Zakład Narodowy im. Ossolińskich we Wrocławiu, Wrocław.
20. Martin, R. L. (Ed.). (1984a). *Introduction*. In Martin, R. L. (Ed.). (1984). *Recent Essays on Truth and the Liar Paradox* (pp. 1–8). Oxford: Clarendon Press.
21. Martin, R. L., & Woodruff, P. W. (1975), "Truth-in-L" in L. *Philosophia*, *5* 213–217 (1975). Quotations from Martin, R. L. (Ed.). (1984). *Recent Essays on Truth and the Liar Paradox* (pp. 47–51). Oxford: Clarendon Press.
22. Poincaré, H. (1908). *Nauka i metoda* (*Science and Method*), transl. L. Silberstein, J. Mortkowicz, 1911.
23. Priest, G. (1993). Can contradictions be true? II. In Proceedings of the Aristotelian Society (supplementary vol. 67, pp. 35–54).
24. Priest, G. (SEPh). Dialetheism, the Stanford encyclopedia of philosophy, E. N. Zalta (Ed.) http://www.plato.stanford.edu/entries/dialetheism. The Metaphysics Research Lab., Center for the Study of Language and Information, Stanford University.
25. Priest, G. (2004). Odrzucanie: Przeczenie a Dylematy [Rejection: Denial and Dilemmas], transl. P. Łukowski and D. Rybarkiewicz, Folia Philosophica 18, pp. 131–148.
26. Prior, A. N. (1967). Correspondence theory of truth. In P. Edwards (Ed.). The encyclopedia of philosophy (Vol. 2, pp. 223–232).
27. Quine, W. van O. (1987). Różności. Słownik prawie filozoficzny [Quiddities. An Intermittenly Philosophical Dictionary], transl. Cezary Cieśliński, Fundacja Aletheia, Warszawa, 2000.
28. Richard, J. A. (1905). *Les Principles de mathématiques et le problème des ensembles*, Revue générale des sciences XVI, p. 541. *The principles of mathematics and the problem of sets* (pp. 142–144) In van Heijenoort (1967).
29. Russell, B. (1906). On some difficulties in the theory of transfinite numbers and other types. *Proceedings of the London Mathematical Society*, *IV*, 29–53.
30. Russell, B. (1908). Mathematical logic as based on the theory of types. *American Journal of Mathematics*, *30*, 222–262.
31. Sainsbury, R. M. (1988). *Paradoxes*. Cambridge: Cambridge University Press, 1991.
32. Tarski, A. (1933). *Pojęcie prawdy w językach nauk dedukcyjnych (The concept of truth in the languages of deductive sciences)*, Prace Towarzystwa Naukowego Warszawskiego, Wydział III Nauk Matematyczno-Fizycznych, nr 34, Warszawa, pp VII + 116. In Alfred Tarski, *Pisma logiczno-filozoficzne*, red. J. Zygmunt (Vol. 1, pp. 13–172), Biblioteka Współczesnych Filozofów, PWN, 1995.
33. Wallace, J. (1972). On the Grame of Reference. In Davidson, D., & Herman, G., (Eds.), *Semantics of Natural Language* (pp. 219–252). Dordrecht and Boston: D Reidel Publishing Company.
34. Woleński, J. (1993). Samozwrotność i odrzucanie (Selfreference and rejecting). *Filozofia Nauki,1*(1), 89–102.

Chapter 5
Ontological Paradoxes

The paradoxes we have analyzed so far have a common property: they do not go beyond the language. The difficulties, which they expose, have intra-linguistic character. Their analysis, and all the more so their solution, does not require any confrontation of thought and reality. It is otherwise with paradoxes analyzed in this chapter. All of them result from such confrontation of the language and propositions we formulate in it with the real world which surrounds us. Ontology is said to be that discipline of philosophy, which shows the possible structure of being. Paradoxes of vagueness, motion, identity, sorites paradoxes and the problem of the many clearly show that no ontology, which uses the concepts of set, Euclidean point, rest, property or thing (understood as something defined by its properties) that are formed under the spell of language, has any logical basis. The use of all of these categories with reference to the world has been questioned by paradoxes. The image of the world they create is precise and static, and thus in disagreement with the reality itself, which by its nature is rich and dynamic in the variety of its phenomena. The effect of this disagreement is an inalienable contradiction, which appears every time we speak (think) of the real world by means of the language. For this reason, paradoxes sorites, motion, identity and the problem of the many deserve the name of *ontological paradoxes*. Just like Russell's Antinomy showed naivety of Cantor's set theory, ontological paradoxes show naivety of ontology and metaphysics of Aristotelian provenance.

5.1 Paradoxes of Difference: Paradox of a Heap

Is this fabric yellow or is its color close to yellow? Is this man middle aged or old? Is this luggage light or heavy or, maybe, neither? Is eight in the morning early or not so early? There are many such questions. The answers to them are always subjective and depend on the experience, age and many other factors we usually do not find important. We know which color is certainly yellow for us and which certainly not yellow but there are colors and hues, make it difficult for us to

P. Łukowski, *Paradoxes*, Trends in Logic, 31,
DOI: 10.1007/978-94-007-1476-2_5,

categorize them as either yellow or not yellow. There are colors different people have different opinions about. Also one and the same person can change his mind in a particular question, not over the period of his life but over a single day. These problems have a specific feature that they do not disappear even if we try to get more information about the issues in question. It makes no difference whether we are good at distinguishing colors or have thorough knowledge about wave lengths and physiology of sight stimuli. There will always be some colors which cause problems, like the one we mentioned, and the problems will be insoluble. The problem of vagueness of natural language expressions is in some essential and fundamental way independent of our knowledge. Vagueness is a common phenomenon, affecting more expressions than the above examples might suggest. The phenomenon of vagueness has been known for ages. The best known paradoxes of vagueness, i.e., the Paradox of a Heap and the Paradox of a Bald One were formulated already in the Antiquity.

Most often, these paradoxes take a light and funny form, which may suggest that the problem that lies at the bottom of their arguments is trivial. It is quite the contrary. Paradoxes of vagueness point to one of the greatest, if not the greatest philosophical problem of inalienable inadequacy between the language picture of the world and the world itself. Its richness and dynamics makes it impossible to express it adequately in words. The picture of reality created (perhaps encoded) in language does not and probably cannot reflect that reality. The rank of paradoxes of vagueness is much higher than all other paradoxes, including the Liar Antinomy, as it is possible to live in such a way as to avoid the difficulties exposed in the Liar Antinomy and other paradoxes discussed above. Yet, it is impossible to live in such a way as to avoid all the difficulties related to vagueness.

The philosopher, who is credited for formulating the paradox of vagueness is Eubulides of Miletus (4th c. BC), a pupil of Euclides of Megara (ca. 400 BC) contemporary to Aristotle.[1] It is probably, thanks to the bright mind of Eubulides, that we owe not only the paradox of vagueness but also the Liar Antinomy. We know from Diogenes Laertios that Eubulides was an opponent of Aristotle and attacked him often.[2] Yet in the works of the latter, it is difficult to find even the slightest remark about the Paradox of a Heap or the Paradox of a Bald one. It is difficult to suppose that such philosopher as Aristotle did not notice the special importance of paradoxes of vagueness. It is puzzling that a thinker who presented his opinions in every philosophical issue that was raised by his predecessors and contemporaries alike made an exception for the problem of vagueness, even though he was personally attacked by Eubulides. Could it be the case that Aristotle was well aware that the argument of the Paradox of a Heap attacked the very core

[1] Over a hundred years before Eubulides, Zeno of Elea (ca. 490–ca. 430 BC), presented an argument aimed at disqualifying the value of empirical evidence. This well-known paradox looks as follows: One grain of corn makes no noise when it falls on the ground. If x grains produce no sound, then $x + 1$ grains fall down soundlessly too. Consequently, no number of grains of corn makes any sound when falling on the ground. This, however, is in disagreement with reality.

[2] Diogenes Laertios, II 109.

of his philosophy? Perhaps, he might have no solution to his problem and his silence is a sign of his helplessness in the question.

Paradox of a Heap

Let us assume that we have stated beyond any doubt that a given aggregation of grain is a heap.[3] Taking into account the fact that one missing grain will not change the heap into not-heap we must use the name "heap" for any aggregation of grain, smaller than the original one. Indeed, taking one grain away we still have a heap. But if we repeat this action a sufficient number of times, we come to a conclusion that one grain forms a heap as well. The other way round, if we state that some number of grains does not form a heap (is a not-heap), then by adding grains one by one we have to admit that no aggregation of grain is a heap (is a not-heap). Consequently, we cannot have precise knowledge what the word "heap" ("not-heap") means.

The argument of the Paradox of a Heap may be a sequence of reasonings, each of which uses Modus Ponens, or it may take the form of mathematical induction, in which, in our case, the inductive test is a sentence: "30,000 grains of wheat make up a heap". The inductive step is then the following: "For any natural number n, if a set of n grains of wheat makes a heap, then a set of $n - 1$ grains of wheat makes a heap". Naturally, from the assumption that 30,000 grains of wheat poured together make a heap, it follows that any number of grains of wheat poured together greater than 30,000 also make a heap. Consequently, the reasoning proves that if a great number of grains are a heap, then every aggregation of appropriately poured grains makes a heap too. Analogical reasoning shows that if some aggregation of grains (even appropriately poured) is a not-heap, then every number of grains (even appropriately poured) is a not-heap, for the inference proving that a given aggregation of wheat grains makes a not-heap can be continued infinitely. Summing up, every number of appropriately poured grains is a heap and not-heap at the same time.

The essence of the problem is that there is no boundary separating a from not-a. In the case of the argument concerning the heap, there is no such number k of appropriately poured grains, which would still be a not-heap while $k + 1$ would already be a heap. We encounter here the so-called tolerance of the expression "heap". An expression is *tolerant* if, from the fact that it can be applied in the case P, it follows that it can be applied in any case appropriately similar to P. Tolerance of expressions is thus the essence of their vagueness. The inductive step of the *sorites* type expresses the tolerance of a given word.

[3] Usually, presentations of the Paradox of a Heap silently assume an appropriate form of aggregation of grains. However, 30,000 grains of wheat may not form a heap if they are spread on sufficiently large area. For this reason, both in this and all following arguments we assume silently that every considered aggregation of grains takes an "appropriate" form, i.e., one that is as similar to a heap as possible.

Argument of the *sorites* type may be performed for expressions such as e.g., “the bald” (two people differing by a single hair are either both bald or both not bald), “child” (two people differing by a single hour of life are either both children or not children), “infant”, the “elderly” (the same case as with “child”), “red color” (adding one drop of green paint to a tin of red paint we do not change the red color), “murder” (two blood spills differing by one drop either both kill or neither does)[4] and “longing” (extending the distance separating two people by one inch does not affect the feeling of longing).

Taking into account the commonness of vague expressions a question should be posed about the existence of expressions, whose precision is sharp and stable. It is worthwhile to consider the reasoning of Hao Wang, which, as it was shown by Michael Anthony Eardley Dummett (b. 1925), does not necessarily have to be a paradox and even though it uses the *sorites* argument to a predicate “is a small number”. This dilemma refers to natural numbers[5]:

Wang’s Paradox

0 is a small number;
If n is a small number, then $n + 1$ is a small number;
Consequently, every natural number is small.

Dummett gives such an interpretation of the predicate “is a small number”, in which not only both premises but also the conclusion are true: *a natural number is small if it is both greater than a finite number and smaller than an infinite number of natural numbers at the same time*. Obviously, every natural number is small in this sense, so the argument of Wang’s Paradox is actually not paradoxical. The conclusion from this analysis is predictable: defining the predicate “is a small number” in a strict, mathematical way changed it into an unvague predicate. Similarly, it is not possible to move, by means of the *sorites* reasoning from a square to a non-square: a figure is either a square or not. This lack of tolerance characterizes all mathematical expressions. Naturally, even though Dummett’s understanding of smallness of a natural number is quite elegant, it has no overlap with the way smallness is understood in everyday language. It is thus an arbitrary, precise interpretation, which changes the vague sense of the expression “a small natural number” into a precise one.

Phenomenon of being and motion. Vague expressions dominate natural language. Almost every expression of natural language which does not have a mathematical sense is vague. “Almost every” does not mean “all” however. One can show that names such as “motion”, “rest”, “being” and “non-being” are precise (non-vague) terms of natural language.

---

4 An example given by Sorensen [1].

5 Dummett [2], pp. 303–308.

Indeed, if motion were to undergo even a slightest change with respect to being motion, it would have to cease being motion and would unavoidably become rest. What is essential here is that the change should occur "with respect to being motion" and not with respect to velocity or direction of motion. Such a small change with respect to being motion would cause that motion only "in a very small degree" would cease to be motion, which seems impossible both to happen and to understand. Consequently, in any mental experiment, it is impossible to move imperceptibly from motion to rest or vice versa by some "small steps". Moreover, it is difficult to know how those "small steps" leading us through intermediary states should be understood. So, in contrast to various "hues" of being a heap, bald or red, there are no "hues" of being motion. Either something is in motion or not. In this sense, motion is one. Similarly, being is one. It is impossible to go from being to non-being by small steps. As motion, being is one. If it is not, it is not being. Something cannot be partly being and partly non-being. The famous arguments of Parmenides of Elea (ca. 540–ca. 470 BC) concerning the unity, immutability, immobility and indivisibility of being are nothing but arguments for the precision, sc. non-vagueness of the name "being".

The history of struggles with paradoxes of vagueness is long but the phenomenon of vagueness seems to have been far from the center of interest of philosophers. An interesting chronological study of reflection over vagueness was presented by Timothy Williamson in his book *Vagueness*. This journey in time acquaints us with opinions, sometimes just isolated comments by a number of philosophers, who paid at least some attention to the phenomenon of vagueness. The list of philosophers included in Williamson's book is rather short not for the fault of its author. We find there: Eubulides, Chrysippus of Soloi (ca. 277–208 BC), Carneades (214–129 BC), Sextus Empyricus (1st–2nd c. AD), Claudius Galenus (Galen) (129–199), Lorenzo Valla (1407–1457), Rudolf Goclenius (1547–1628), Pierre Gassendi (1592–1655), John Locke (1632–1704), Gottfried Wilhelm Leibniz (1646–1716), Georg Wilhelm Friedrich Hegel (1770–1831), Charles Sanders Peirce (1839–1914), Gottlob Frege (1848–1925), Bertrand Arthur William Russell (1872–1970) and Max Black (1909–1988). In most cases, the philosophers' opinions about vagueness are limited to relating more or less appropriate examples, used as basis for their opinions, which were closer to beliefs than logical analyses. The most interesting approaches to vagueness come from the Antiquity. Later, the renaissance of interest in the problem comes only in the nineteenth century. The development of formal systems forced logicians to face the problem of vague expressions. Realizing their specific character Frege was inclined to identify vagueness with incompleteness of definition. He saw that the phenomenon of vagueness is impossible to reconcile with logical calculus and therefore postulated that the latter be applied only to precise terms. According to him vagueness should be avoided. An opposite view was presented by Peirce, the author of a notable definition of vagueness. In the three volume *Dictionary of Philosophy and Psychology*, edited by James Mark Baldwin (1861–1934) and published in 1901–1905, Peirce defines a vague sentence as one for which there are such states of affairs that it is indeed uncertain for a person who thinks of them whether these states are allowed or disallowed by that sentence. He also stresses that the "uncertainty" has its source not in the ignorance of that person but is caused by the fact that rules of language application are so vague that the person may believe a proposition to be disallowed on one day and allowed on another. This opinion has formed the way of perceiving vagueness as a phenomenon independent of man's knowledge, whose source is the language we use.

Vagueness was recognized as important in the twentieth century. The impulse for a specially serious treatment of this phenomenon was Russell's lecture for Jowett Society in Oxford on 25 November 1922. It was published under a short title *Vagueness* in 1923.[6]

Russell prefaces the presentation of his views with a clear statement that vagueness refers solely to representation of reality, not reality itself. We can speak of vagueness only in the context of language, i.e., that which represents, not that which is represented. Consequently, things are neither vague nor precise. "They are what they are and will never be anything else". Russell thinks that the philosophers, who see continuity or fluidity in nature, transport vagueness of the linguistic nature into the extra-linguistic reality. Russell's strong position is in agreement with his definition of *precision* and, at the same time, of *vagueness*, which makes use of the phenomenon of representation of one system in another. If, on one hand, we have a system of natural language and on the other, the objects represented by the language in question, then the assumption of vagueness of the extra-linguistic system is completely unnecessary for Russell. His main thesis is the proclamation of vagueness of the whole language which we use, including scientific and even logical terms (e.g., "truth", "falsity", logical conjunctions, etc.). Vagueness is in conflict with classical logic because it is at loggerheads with the laws of classical logic, especially the law of the excluded middle. Russell points to a particular property of vagueness too. The dubious cases, which form *penumbra* (in Russell's terminology) do not constitute a set in the mathematical (precise) sense. Penumbra, or the area of vagueness is such, because it is impossible for it to be defined precisely. Precisely defined penumbra is opposed to vagueness of a word, for which it is penumbra. If we had precisely defined borders for the penumbra of such a word as "bald", we could introduce a new word, e.g., "half-bald", to denote that penumbra. Then we would have three words: "bald", "half-bald" and "non-bald", which would cover all possible cases together and none of them would be vague. Analyzing commonly occurring vagueness, which affects the expressions from scientific language as well, Russell considers the question of "vague" knowledge, stating paradoxically that a vague belief has a better chance to be true than a precise one. The vagueness of a belief causes that it is confirmed by a greater number of facts than the precise one. A belief is precise if it is confirmed by one fact only. Russell calls a clear and true belief an accurate one. The natural tendency in the language of science to substitute precise beliefs

[6] Russell [3]. Earlier Russell moved the question of vagueness in his 1913 manuscript *Theory of Knowledge*. He returned to the problem in the book *The Analysis of Mind*, London, Paul Kegan, 1921.

for vague ones makes the truth of scientific statements even more difficult to achieve, yet the statements themselves gain greater value. Precision of a sentence reduces the probability of its truthfulness, but precise theses of science are a better source of information than vague, common sense sentences. As a result, it is more likely for the scientific sentences to be false, but when they are true, their utility is greater than that of usual, vague ones.

Fourteen years passed between the publication of Russell's *Vagueness* in 1923 and the next paper important for the study of vagueness. It was only in 1937 that Black published his acclaimed study *Vagueness: an exercise in logical analysis*.[7] From the moment of its publication, philosophers became specially interested in natural language. Their attitude to vagueness started to change and gradually it came to be seen not as a curse but as a virtue of natural language.

According to Black, vagueness is closely related to the use of symbols and should be treated as a sign of deviation of the language model from the empirically established language customs of a society. Vagueness of a term is understood as a tendency to generate borderline cases or doubtful objects, i.e., individuals for which it seems impossible both to apply and not to apply the term. This specific dubiousness characterizing vague expressions functions as a bad definition: no one knows how to use vague terms, for one never knows if a term may be used or not at all. Black recognizes the value of Peirce's definition of a vague sentence from the latter's famous *Dictionary of Philosophy and Psychology*. In the spirit of this definition, he analyzes the vagueness of the word "chair" trying to show the objective character of vagueness. "Chair" is a vague term, because it is possible to present such objects, whose classification as "chairs" is "uncertain" or "dubious".

Paradox of a Chair, or any Object of Inanimate Matter[8]

Let us imagine an unusual exhibition in the unusual Museum of Applied Logic, which presents a row of thousands of objects. Every two neighboring objects differ from one another the least, in an almost imperceptible way. At one end of the row, there is the first object, which is an exquisite Chippendale chair.[9] The second object is a perfect copy of the first object but for a very slight, barely perceptible damage. The third object differs from the preceding one in a similarly small damage, etc. At the other end of this long row, there is a small, unrecognizable piece of wood, which is a fragment of a leg of the perfect copy of the chair in question.[10] If we call the first object a chair, then every next one

[7] Black [4].

[8] It is obvious that the analysis presented for the Paradox of a Chair can be reconstructed for any object of inanimate matter: a building, stone, pen, book, etc.

[9] Thomas Chippendale (1718–1779), a famous British furniture designer, the author of the book *The Gentleman and Cabinet Maker's Director*, published in 1754.

[10] The expressions like "perfectly executed" or "ideal copy" have sense, since they are used in considerations, which are logically admissible mental experiments. We assume, moreover, that successive damages gradually destroy the chair without dividing it into two or more pieces.

must be called a chair too, so a piece of a chair's leg will be called a chair. Now a damage so small as each of the ones we have considered here will not change a chair into a non-chair. Similarly, if we call a piece of a chair's leg a non-chair, then every preceding object must be called a non-chair too, including the exquisite product of Chippendale.

This unusual set of objects illustrates a "fluid", in a sense, metamorphosis of a chair into a non-chair. Black notices that for every normal[11] person visiting the exhibition it is hard to draw a line separating chairs from non-chairs, since "chair" is not a word that would allow for such a sharp distinction. Moreover, inexistence of such a sharp distinction makes words such as "chair" very useful, even though they cause grave problems of logical character. For Black, the illustrative presentation of the problem of vagueness was a starting point for a meticulous empirical study of opinions taken from representative groups of people. He analyzed the evidence he gained in a quasi-mathematical way to obtain, among others the justification of the concept of "gradation of vagueness". His observations, however, did not bring decisive solutions concerning the nature of vagueness.

The first important reaction to Black's ideas was a 1939 paper *Vagueness and logic* by Carl Gustav Hempel (1905–1997). The paper started a debate, whose main issue was the problem whether vagueness is merely a semiotic problem or a question of purely semantic nature. In his critical analysis of Black's proposal, Hempel started with a distinction of two approaches toward language. According to the first one, whose representative was Black, the perspective for viewing language questions is set according to the behavior of language users. According to the other one, language is described by two sets of rules: syntactic and semantic ones. The first approach is risky, since the behavior of a user need not have relation to either syntactic or semantic rules. Observation of this behavior can be performed even without understanding the language. Using the example of chess, Hempel asked whether it is possible to correctly recognize the rules of the game through observation of the players' behavior. He noticed that uttering the words "check and mate" is usually accompanied by signs of satisfaction in one player not reciprocated by the other one. Black's approach would take into account the behavioral regularities, even though the logic of the game is established by extra-behavioral rules only. Hempel stated, therefore, that according to Black vagueness is a semiotic term referring to three elements: the sign itself, the object a sign refers to and the sign's user. Logical truth, however, has semantic character, which means that Black's vagueness does not refer to logic. By noticing the analogy between the three-argument relation: "the user z designates the property y with the term x", and the two-argument one: "the term x signifies the property y", Hempel posed a problem whether a similar reduction is possible, which would make semantic vagueness out of a semiotic one. To a large degree, the solution of this

[11] In Black's understanding, a "normal" observer is anyone, who, in a situation similar to the one of the exhibition, does not expect to discover a precise boundary between the objects called "*a*" and objects called "*non-a*"; Black [4], p. 433.

question should depend on whether semantic vagueness is gradual in a similar way as Black's semiotic one. Regrettably, according to Hempel there is no semantic form of vagueness, since there is no gradation in signifying. A given predicate can be applied to an object if it signifies a property the object possesses. Now, since Hempel believes that possessing a property by an object cannot be gradual, semantic vagueness of a predicate would have to result from gradation in signifying, which leads to hardly acceptable conclusions. To show those difficulties Hempel invokes the question of translation from one language to another. Let us assume that the word "sol" signifies sun in the degree 0.7 and the word "cal" signifies the property of being hot in the degree 0.9. Then the sentence "Sol esti cal" should signify in the degree $0.7 \times 0.9 = 0.63$ the state of affairs consisting in the sun being hot. Now how—asks Hempel—is it possible to translate this sentence exactly into a sentence "The sun is hot"?[12]

Black decided that the vagueness he studied had semiotic character. However, he rejected the view that languages, in which the relations of signification are gradable, cannot be translated. He stated that even if some parts of a language cannot be translated into English, it is always possible to extend English to include those expressions which evade precise translation. For this reason Black believes that semantic vagueness is possible. Consequently, vagueness can affect the evaluation of logic and justify the need for its change. Moreover, the attempts to defend classical logic based on heightening precision of language expressions do not lead the study of vagueness in the right direction, because classical logic requires too high level of abstraction in language.

Nevertheless, Hempel remained convinced that vagueness is a semiotic problem. The phenomenon of vagueness occurs when people communicate but it is difficult to distinguish it among other linguistic faults. Hempel comes to a conclusion, however, that regardless of the semiotic character of vagueness, it cannot be comprehended in behavioral categories and so gradation of vagueness can be taken into account with some words only.

Hempel's criticism turned out to be so important for Black that in his 1963 paper *Reasoning with loose concepts* he no longer mentions profiles of density or any statistical method of vagueness analysis. He stopped postulating a change of logic, even though he still believed that classical logic cannot be applied to border cases. Thus Black came closer to Hempel's standpoint according to which solution of problems resulting from vagueness should not be based on applying a non-classical logic suitable for a given vagueness, but should start from better understanding of the relation between logical systems and the praxis of language use.[13]

Black's mental experiment of the chair is important in the sense that it can be repeated for any object of inanimate matter. Thus it is a representative of all arguments proving vagueness of names of all objects and respective predicates.

[12] Williamson [5], pp. 79–80.

[13] Williamson [5], pp. 82–83.

A similar role with respect to the category of objects of animate matter is played by the mental experiment of Leon Chwistek (1884–1944):

Paradox of Man, or any Animate Being (First)

Can a non-human be a mother of a human being? Of course, not. A mother of a human being is always a human being. Let us take, then, any man and start a journey in time looking for his female ancestors[14]: mother, mother^2, mother^3, …, mother^n, …, etc. If this journey is long enough, it will turn out that for some natural number k, mother^k is no longer a human being but some small, primitive mammal, and for another natural number m, greater than k, it is no longer possible to speak of the gender of the ancestor mother^m. Now, if we have called, and rightly so, the first female ancestor a human being, then in order to be consistent we have to call some primitive, single-cell organism a human being, too.

Naturally, this reasoning can be repeated for any living creature. All such arguments show the vagueness of creatures' names in formal supposition (a name designates a species, not an individual designate). The argument proving vagueness of the name "man" in a simple supposition (a name designates an individual, not the whole species) can have the following form:

Paradox of Man, or any Animate Being (Second)

Is a n-minute fetus a human being? If yes, then a $(n - 1)$-minute fetus is also a human being. Consequently, every fetus is a human being ever since the moment of conception.
Is a n-minute fetus a non-human being (i.e., is not a man)? If yes, then a $(n + 1)$-minute fetus is also a non-human being. Consequently, every fetus up to the moment of birth is a non-human being, i.e., is not a man—so humanity appears suddenly at the moment of birth.[15]

All reasonings of the first type (Chwistek's), referring to the theory of evolution are arguments for vagueness of all names of the so-called *natural species*. Neither "man" nor any other name of an animal or plant understood as a name of a natural species is precise. Similarly, specific continuity of individual development of any living creature belonging to any natural species is in opposition to the precision of that creature's name.

---

[14] The abbreviation used here should be understood as follows: mother^{n+1} = mother of mother^n, for any natural number $n \geq 1$.

[15] Naturally, the so-called moment of birth, or the moment of conception, is not a dimensionless time point either, but a non-zero time sequence.

5.1.1 What is Vagueness?

The above historical survey of the studies in vagueness shows that the phenomenon was most often treated as a linguistic problem. Speaking of vagueness one usually means certain faultiness of names, predicates or sentences. It does not mean that there is no problem of extra-linguistic vagueness. Yet, no matter whether we accept vagueness in material world or not, it remains a separate question whether linguistic vagueness has only pragmatic or, perhaps, also semantic character.

Definitions of vagueness. In his 1958 paper *Nazwy Nieostre* (*Vague Names*) Tadeusz Kubiński (1923–1991) presented three approaches to defining vague names[16]:

Pragmatic definition. A name *a* is vague in the language *J* if and only if there exists an object, which will not be considered a designate of the name *a* or a designate of a name non-*a* by any speaker of the language *J* who understands that name.[17]

Semantic definition. A name *a* is vague if its boundary is not empty.[18]

Syntactic definition. A name *a* is vague in a system *S* if and only if neither the expression "*b* is *a*" nor the expression "*b* is non-*a*", where "*b*" is a proper name (i.e., designating one object only), are statements of the system *S*.

Let us remember that the *positive extension* (*extension*) of a vague name is formed by those objects (cases) of which the name can be truthfully predicated. The *negative extension* (*anti-extension*) of a vague name is formed by those objects (cases) of which the name cannot be truthfully predicated.

It is beyond any doubt that vagueness has pragmatic character. It is also difficult to deny its semantic nature. The syntactic aspect of vagueness, however, is something unusual and is related to Kubiński's approach to vague names, which was based on Leśniewski's ontology.[19] In our subsequent analyses we shall treat vagueness as a semantic phenomenon, which is an obvious cause of difficulties that are encountered, sooner or later, by any user of a language containing vague expressions. Meticulous distinctions between a name and a predicate will not be essential here. After all, a name "*x*" has a counterpart in a predicate "to be *x*" and vice versa. What is important is to take into consideration all the properties which are characteristic for vagueness.

---

[16] Kubiński [6], pp. 121–122.

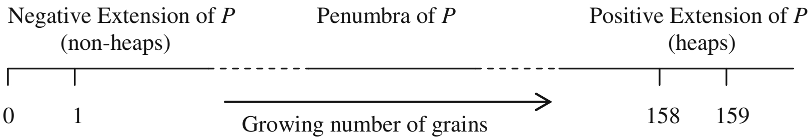
[17] Kubiński also gives another version of this definition, in which the word "no one" is replaced by an expression "a visible majority"; Kubiński [6], p. 122.

[18] A *boundary* of the name *a* is the difference between the universum and the sum of both positive and negative scope of the name. In other words, the boundary of a name is formed by all objects which to not belong either to its positive or to its negative extension; Kubiński [6], p. 119.

[19] It is an interesting construction but it concerns partly defined names rather than vague ones.

As we have shown earlier, vagueness is associated with the so-called border cases. The existence of such cases is a visible proof of vagueness. Such is the opinion of Sorensen in his *Stanford Encyclopedia of Philosophy* (internet edition) entry. Precise determination of border cases requires the concepts of "positive extension" and "negative extension" of a given name or predicate. We say that an object falls within the *positive extension* of a predicate just on condition that the object definitely possesses the relevant property; an object falls within the *negative extension* of a predicate just on condition that the object definitely lacks the property; otherwise it falls within the *penumbra*.[20] Using the concepts of truth and falsity the two extensions can be explained as follows: an object o belongs to the positive extension of a predicate P, when a sentence $P(o)$, which predicates that the object o has a property expressed by P, is true; analogically, an object o belongs to the negative extension of a predicate P, when a sentence $P(o)$ is false. Penumbra is also called border, vagueness area, or border case area.

The very existence of border cases does not have to be a proof of vagueness of a predicate/name. Border cases also characterize unspecified or partly defined expressions. It is important that the vagueness area be vaguely delineated. Vague distinguishing of the penumbra from every extension seems to be the essence of vagueness. An good support for this opinion can be found in the example given by Black, with a row of objects, of which the first is a perfect chair and the last one, a piece of a chair leg coming from a perfect copy of the first object.[21] In this sense, the doubtlessly vague predicate "to be a heap" is a typical example:



The dotted line in the scheme stands for lack of a (precise) border, both between the negative extension and the penumbra and between the penumbra and the positive extension. In other words, the line is to mean a "fluent" move from each extension to the penumbra. It turns out, however, that it is possible to imagine predicates which are most clearly vague and yet do not possess those properties. An interesting predicate was defined by Roy A. Sorensen. He proposed considering a following expression: "to be n-small".[22] Let us assume that $n = 33$. Then, "to be n-small" is defined as follows:

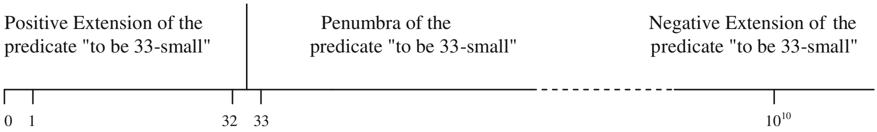
A natural number k is 33-small if and only if
k is a small number or k is smaller than 33.

[20] Sainsbury [7], p. 31.

[21] According to the idea of Peter Unger, in this mental experiment damage can be minimalized to separation of a single atom from the whole; Unger [8], pp. 237–238.

[22] Sorensen [9].

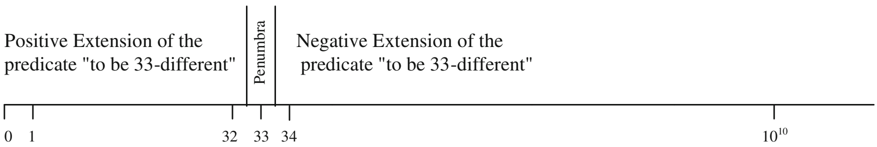
The sketch below illustrates a rather unintuitive situation of existence of a distinct boundary separating the positive extension of a predicate from its penumbra.



Indeed, 33-small(k)[23] are true sentences for $0 \leq k \leq 32$. If the number 33 belongs to the penumbra of the predicate "is a small natural number", it is not decided whether the sentence 33-small(33) has the value of truth or falsehood. Still, the sentence 33-small(10^{10}) is undeniably false. Apparently, there is no clear boundary between the penumbra of Sorensen's predicate and the negative extension of that predicate. Using Sorensen's idea one can define a predicate (vague, perhaps), the penumbra of which has precise boundaries:

A natural number k is n-different if and only if
k is a small number or k is different from n.

If 33 is still a number of the penumbra of the predicate "is a small number", the scheme presenting the extensions and penumbra of the predicate "is 33-different" is the following:



Naturally, a precise delimitation of the penumbra is achieved here thanks to the mathematical component in the definition of the predicate "to be n-different". It seems impossible to make a similar trick with the help of vague predicates of natural language only. We accept, therefore, that a complex predicate is vague if it possesses a vague component and if the meaning of that component essentially influences the meaning of the complex predicate, e.g., the conjunction of an alternative in a definition of a complex predicate may cause the vague component to lose its importance.

Taking the above remarks into account let us replace the definition of vagueness of a predicate/name, which links this phenomenon to having border cases, with a definition, which uses the concept of a *inconclusively gradual property*, i.e., a property that can gradually disappear and vanish altogether in a certain, unspecified moment:

[23] The expression "33-small(k)" is an abbreviated form of the sentence "k is 33-small".

Definition of Predicate Vagueness. A predicate is vague if its penumbra is a non-empty set and, moreover, either designates an inconclusively gradual property itself or its meaning depends on a predicate that expresses an inconclusively gradual property.

Accordingly, if a predicate does not express an inconclusively gradual property itself, but expresses such a property that can be attributed to an object depending on whether an inconclusively gradual property belongs to that object, we shall say that such a predicate is vague. As it can be seen, accepting the definition of a vague predicate in this form means that we do not intend to conclude whether predicates of the type "to be n-different" designate a gradual property or, perhaps, a non-gradual one. Considering, for the natural number k_0, the truth of sentences which form a series: "0 is k_0-different", "1 is k_0-different", "2 is k_0-different", "3 is k_0-different", etc., we can notice the gradual character of the property expressed by the predicate "to be k_0-different". This could mean that the predicate "to be k_0-different" designates a gradual property. Such a predicate does not constitute a typical case, in contrast to the characteristically vague predicates "to be bald", "to be red", etc. For this reason it seems justified to accept the definition of vagueness in the form presented above.

Another important property of vagueness is that it is neither ambiguity nor universality. In the internet edition of the *Stanford Encyclopedia of Philosophy*, Sorensen gives a useful, easy to remember, example: the name "child" is ambiguous, because it can mean either an offspring or a young offspring; it is vague because there are border cases of the term "young offspring"; it is universal, because it designates both a boy and a girl.[24] It is interesting that in the name "child" vagueness "seeps" into both ambiguity and universality, for neither the boundary between offspring and young offspring nor the boundary between being a boy and a girl is precise.

Achille C. Varzi has a different understanding of vagueness. His definition contains an element showing the impossibility of delineating a boundary for a collection of border cases. Moreover, Varzi points to the possibility of various interpretations of one and the same definition. In his opinion, the expression "possesses border cases" is not precise and thus is responsible for the ambiguity of the earlier definition. Therefore, he suggests two alternative extensions of his definition, calling the first one *de re* and the other, *de dicto*[25]*:*

- The term t designates an entity x such that it is indeterminate whether such-and-such objects fall within the boundaries of x.
- It is indeterminate whether the term t designates an entity x such that such-and-such objects fall within the boundaries of x.

[24] Sorensen [10]. Sorensen does not notice that both ambiguity and universality are related to vagueness too. For instance, a boundary between an offspring and a young offspring is not precise, just like the boundary between being a boy and being a girl.

[25] Cf. Varzi [11].

Varzi relates the *de re* definition to the "ontological" and the *de dicto* one, to the "linguistic" or "conceptual" understanding of vagueness. He illustrates both definitions with the following examples referring to the same predicate "is bald". The predicate "to be bald" is vague in the sense of the first definition, i.e., the ontological one, if it defines a vague set of people: no objective fact decides whether a man who is the border case, i.e., a man whose baldness is questionable, belongs to the set or not. The same predicate is vague according to the second definition, i.e., in the linguistic sense, if no objective fact decides, which of the possible sets of people is defined by the predicate "to be bald". If we assume that n is the number of all people living now, we get exactly 2^n various sets of people, and according to the definition *de dicto* it is not decided, which of those sets corresponds to the predicate "to be bald". Naturally, from the point of view of the definition *de re*, none of the existing sets of people are defined by the predicate in question. Varzi believes that some proposals of identifying the above two forms of the definition of vagueness are unfortunate, since certain forms of vagueness should be recognized as ontological and other ones, as linguistic. He adds, however, that the two understandings are not mutually exclusive. We should note here an important difference between the two forms of definition proposed by Varzi. A vague set is not a set in the sense of traditional mathematics. From this point of view, the definition *de dicto* has certain superiority over the definition *de re*, since, in contrast to it, it uses the concept of a set in a way which is correct from the point of view of classical mathematics.

Fuzziness of a typical vagueness area has some consequences. One can consider the vagueness of the penumbra as well as the vagueness of the new vagueness, and so on in infinity. This means that the vagueness of a given expression generates subsequent vaguenesses, the so-called higher order vaguenesses. This means that meta-language cannot provide tools for dealing with the problem of vagueness.

As it can be seen, the definition of predicate vagueness, which we have proposed, is not precise but vague. This, however, does not constitute any problem; to the contrary, it is good if an expression defined by a vague definition is vague itself. A vague expression cannot have a precise definition, for it would cease to be vague. What remains to be decided is whether "vagueness" is a vague or, perhaps, a precise expression. One should think that "vagueness" is vague. We cannot be satisfied with intuitions, however, for then we should also accept that "preciseness" is a precise expression. Now preciseness is understood as negation of vagueness. This means that the collection of vague expressions is a completion of the collection of precise expressions in the same language. Consequently, either both "vagueness" and "preciseness" are precise expressions, or both are vague. Let us consider this problem now.

Vagueness of vagueness. In order to solve the question whether vagueness is precise, or perhaps vague, some authors pose additional questions, for instance whether the predicate "to be vague" is vaguely vague, or whether something that is vaguely vague is not vague sometimes, etc.[26] Risking circularity of argument

[26] Sorensen [9], Tye [12], Hyde [13, 14] and Varzi [15].

they try to solve the problem through the analysis of the sense of vagueness. Yet, the best proof can be found in constructing a suitable sequence of predicates, which, similarly to Black's set of chairs/non-chairs, leads us imperceptibly from a vague predicate to precise one and the other way round. We owe such a suitable sequence of predicates to Sorensen.[27] For every natural number n, the predicate "to be n-small" is defined as follows:

A natural number k is n-small iff k is a small number or $k < n$.

For $n = 0$, the inequity $k < 0$ is false for every natural number k. Consequently, in this case the predicate "to be 0-small" is simply the predicate "to be a small number" and as such is undoubtedly a vague predicate. Predicates "to be n-small" are similarly vague for $n \in \{1, 2, 3, 4, 5\}$. For instance, when $n = 5$, sentences 5-small(k) are undoubtedly true for $k \in \{1, 2, 3, 4\}$, they should be also true for some other natural numbers greater than 5. At the same time, the predicate "to be 10^{10}-small" is a precise one. It is proved by the fact that for every natural number $k < 10^{10}$, the sentence 10^{10}-small(k) is true, while for every number $k \geq 10^{10}$, the sentence 10^{10}-small(k) is false. This means that the predicate has no border cases, so it is precise. In a very long sequence of predicates between the vague "to be 0-small" and the precise "to be 10^{10}-small", there must be such predicates, for which it is uncertain whether they have border cases or not. This results from the fact that in the whole sequence the key role is played by "smallness" of a natural number, which is definitely a vague term. The situation is illustrated by the following table[28]:

$n = 0$	...	$n = 10^{10}$
0-small(0) = 1	...	10^{10}-small(0) = 1
0-small(1) = 1	...	10^{10}-small(1) = 1
$\vdots$		$\vdots$
0-small(33) = ?	...	10^{10}-small($10^{10} - 1$) = 1
$\vdots$	...	10^{10}-small(10^{10}) = 0
0-small(10^{10}) = 0		10^{10}-small($10^{10}+1$) = 0
$\vdots$	...	$\vdots$

Thus, in this sequence of predicates, there are such, which are neither vague nor precise, for it is neither true that they have nor that they do not have border cases. These predicates are thus border cases for the predicate "to be a vague predicate". This means that vagueness is vague.

[27] Cf. Sorensen [9] and Varzi [15].

[28] Naturally, the predicate 33-small(<33) being vague, it is not a case of a borderline instance. It was given here for better illustration of the sequence of predicates analyzed here.

It must be noted that the very same sequence of predicates proves vagueness of the predicate "to be a precise predicate". Consequently, preciseness is vague too. Moreover, we already know that every definition of vagueness/preciseness should be vague. To finish the subject, let us quote the opinion of John Langshaw Austin (1911–1960), who claimed that it is obvious that the predicate "to be vague" is vague.[29]

The question of extra-linguistic vagueness. In his above quoted lecture,[30] Russell stated that vagueness is a phenomenon which characterizes only that, which represents, and not that, which is represented, so it is a purely linguistic or mental phenomenon. Consequently, looking for vagueness in extra-linguistic objects, i.e., objects which are represented by the language, would be for him a typical error of verbalism: properties of words would be ascribed to objects represented by them. Michael Dummett went on to state that it is impossible to understand the opinion, which claims that things are vague, as well as the one, which claims that they are vaguely described.[31] With respect to the problem of vagueness, David Lewis is close to both Russell and Dummett. He believes that thought and language are the only places where vagueness can be located. In his opinion, the cause of linguistic vagueness is not the existence of vague objects but the great number of precise objects which vary with respect to the position of their boundaries. According to Lewis it is fortunate that no one is so stupid as to take one of those particular cases as an object deciding about the precise understanding of a vague word. Thus vagueness is a sort of semantic hesitation.[32] A similar opinion is presented by Timothy Williamson. He starts his work *Vagueness in Reality* from presenting a revealing observation[33]: "When I take off my glasses, the world looks blurred. When I put them on again, it looks sharpened. I do not think that the world is actually blurred; I know that what has changed is only my relation to objects of the physical world that are in front of my eyes, they themselves have not changed. I am, therefore, all the more willing to believe that the world is and was sharp." As a representative of epistemicism, Williamson not only denies the existence of material vagueness but also believes that the expressions of language are precise, and vagueness is merely a result of the lack of knowledge or ability to recognize the precise boundaries which separate the positive extension of a given expression from the negative one. Such an understanding of vagueness is called *epistemic vagueness*.

Although the opinion denying the existence of vagueness in the material world has supporters among such influential philosophers as Russell, Dummett or Lewis, there are other philosophers who not only admit that such a vagueness is possible but even claim that the objects of material world are vague. Such an understanding

[29] Hyde [14], p. 303.

[30] Russell [3].

[31] Dummett [2], p. 314.

[32] Lewis [16], p. 212.

[33] Williamson [17].

of vagueness is called *objective*, *metaphysical* or *ontological vagueness*. The view, according to which vagueness of language expressions has its source in extra-linguistic vagueness, is represented for example by Tadeusz Pawłowski (1926–1996)[34], who states that "the cause of vagueness is not to be found in a person's lack of knowledge or actions but in the nature of objects signified by vague expressions, in the specific features of those objects. They are, namely, gradual properties, such that moving from the state presence to the state of absence of a given property has successive character. This means that it is impossible to place a precise boundary between the two states." A similar opinion is presented by Tye, though he argues for it in another way: since the predicate "to be red" is a vague one, the property expressed by it is also vague. Now there are objects which are neither purely red nor purely non-red. This vagueness of a property seems to require objects which are *possible borderline instances*.[35] As an advocate of extra-linguistic vagueness, Tye proposes considering the case of Mount Everest. There are some parts of rock that clearly belong to the mountain, some other ones that are equally clearly outside it. There are, however, some parts that have unclear, borderline status. Since it is not decided that they are inside the mountain or outside it, it can be stated that Mount Everest has a borderline temporal–spatial part, and it is due to it that it has some similarity with the cloud. Consequently, Mount Everest is a vague but concrete object.[36] Tye even proposes a definition of a vague object[37]:

A concrete object *o* is vague if and only if
(a) has borderline spatio-temporal parts and
(b) there is no determinate fact of the matter about whether there are objects that are neither parts, borderline parts, nor non-parts of *o*.

There are also extra-material objects which, according to Tye, also deserve the name of vague objects.[38] For these very objects, he proposes another definition[39]:

An object *o* is vague if there is an object o' such that it is indefinite whether *o* is identical with o'.

Vagueness of the object *o* is manifested in the unspecified character of the relation of identity between the object *o* and another object. Quoting Broome [21], Tye recalls the example of D. Parfit, concerning a club, which has its location, list of members, and a statute. Let us assume that meetings of the club are less and less often and finally stop. Suppose that after a few years with no meetings, some old members of the club together with a few new ones organized a meeting in its old

[34] Pawłowski [18], p. 72.
[35] Tye [19], p. 563.
[36] Tye [20], p. 535, also Tye [19], p. 563.
[37] Tye [20], pp. 535–536.
[38] Tye [20], p. 536.
[39] Tye [19], p. 563.

location. Naturally, the name of the club was not changed. Now it is hard to say if the old club and the new one are one and the same, so their identities are unspecified. Consequently, both the old club and the new one are vague objects, even though none of them is an object of temporal–spatial character.

According to Tye some abstract objects are also vague. To prove his point, he proposes considering the set of all tall men[40]:

A set *S* is vague if and only if
(a) it has borderline members and
(b) there is no determinate fact of the matter about whether there are objects that are neither members, borderline members, nor non-members.

Of course, *a vague set* is not a set in the well known, mathematical sense. In an analogical way, one can define vagueness of a property[41]:

A property *P* to be vague if and only if
(a) it could have borderline instances and
(b) there is no determinate fact of the matter about whether there could be objects that are neither instances, borderline instances, nor non-instances.[42]

Referring to the achievements of modern physics Tye pointed not only to the existence of material, i.e., temporal–spatial, vague objects, but also showed that all objects of that type are vague. A similar opinion is presented by Silvio Seno Chibeni. In order to prove his thesis, he takes an argument from the best theory describing the structure of master, i.e., the quantum mechanics. His argument has both a general theoretical aspect and involves analysis of many particular cases. The existence of vague objects seems to be all the more evident in the light of theoretical and experimental achievements of microphysics, which puts serious constraints on every theory claiming that it is possible to reconstruct precision of quantum objects.[43]

The problem of extra-linguistic vagueness became the core of a class of paradoxes referring to the so-called problem of the many.

In the discussion of the question of extra-linguistic vagueness, we cannot ignore the analyses of Sorensen, who claims that everything in the universe, together with the universe itself, is precise. He proves his opinion with an argument showing that a blob of water is a precise object. The analysis of the problem starts with a mental experiment showing precision of a gray ball, whose grayness imperceptibly turns into the whiteness of its surrounding. In other words, he shows that even a vague

---

40 Tye [20], p. 536.

41 Tye [20], pp. 535–536.

42 Following Russell, Tye argues for vagueness of even such seemingly precise properties as having the height of 2,000 feet. Indeed, one can safely assume that the standard from Trafalgar Square in London gives information which is as precise as the one given by the standard from Sèvres near Paris. Naturally, today neither the rod of Sèvres, nor the table from Trafalgar Square are standards for measuring length. Cf. a note above.

43 Chibeni [22].

ball is precise.[44] Let us assume, after Sorensen, that some grayness in the form of a ball turns more or less fluently into the whiteness of the surrounding. Let us assume, moreover, that from the central point of the gray ball, a new white ball starts growing within it, gradually devouring all grayness. The white ball is perceived as a growing white hole in the gray ball, which causes the grayness to be gradually extinguished. The extinction of the gray ball seems to be gradual, in apparent agreement with expectations. But then the gray ball disappears suddenly. Now the boundary of grayness has no width, so it cannot delimit an area which could survive the gray ball. If the boundary of grayness has to vanish immediately, since it is delimited by a precise line and not a strip of non-zero width, the whole ball has to vanish immediately, too. This means that a gray ball, even if it gradually turns into whiteness, is an object with precise boundaries.

On the basis of his reasoning, Sorensen proposes another one, his time concerning a blob of matter, e.g., water. His inference is presented in five steps[45]:

1. The blob must have a boundary.
2. If a spherical cavity grows from the center of the blob, the blob's outer boundary is completely unaffected as long as some of the blob remains.
3. As soon as nothing remains of the blob, the blob's boundary goes out of existence all at once.
4. Lemma: The blob's boundary goes out of existence instantaneously.
5. Conclusion: The blob goes out of existence instantaneously.

Even a simple analysis of the first of the two arguments shows that it cannot be accepted, since Sorensen is guilty of an error of *petitio principii* here, for his argument he uses a thesis that strives to prove. Naturally, this refers to the sentence stating precision of the gray ball. First of all, Sorensen accepted such definition of a ball's boundary in which it is an object without width, so he assumed preciseness of this boundary. Secondly, he assumed silently that there is only one, and hence precise, whiteness, forgetting altogether that for some, e.g., for Eskimos, there are more hues of whiteness than we can distinguish in grayness. If Sorensen's whiteness is delimited precisely, its boundary, i.e., a line with no width, is at the same time a precise boundary of the gray ball, since he understands the gray ball to be everything which is beyond the whiteness delimited by the precise line mentioned above. In other words, for Sorensen, the gray ball is the completion of the precise area, which is its surrounding. Thus the gray ball is a precise object out of principle, even though its saturation with grayness is not the same throughout. It is not difficult to prove that in an object conceived in this way the boundary will disappear suddenly and it will vanish in the same moment its boundary does so, sc. suddenly. All in all, Sorensen proved that a precise object will disappear suddenly.

---

[44] Sorensen uses the word "vague" in parentheses, since he wants to show that vagueness of this ball is impossible; Sorensen [23], p. 275.

[45] Sorensen [23], p. 276.

Sorensen's second argument repeats the error he committed in the first one. Out of principle, the surrounding of a blob is precisely delimited by a precise boundary with no width. What is a blob is that which is not its, precisely understood, surrounding. Consequently, a blob is precise out of principle. No wonder, then, that having assumed precision of a blob Sorensen proved that it disappears suddenly, i.e., that it is a precise object. Quite another thing disqualifying Sorensen's argument is that he proves a fact of empirical character using purely speculative tools taken from topology.

5.1.2 Proposals Substituting Preciseness for Vagueness

Understanding the phenomenon of vagueness and solving the problems generated by it is one of the greatest challenges of philosophy all over centuries. The existing approaches to vagueness are so varied that it is rather difficult to classify them. The division proposed here is simple: proposals which treat vagueness fairly are separated from the ones which in one way or another replace vagueness with preciseness. The essence of the latter type of proposals is that vague names and predicates become precise expressions as a result of application of a particular technique or are treated as precise expressions by virtue of an arbitrary decision. It is an obvious fact that for various reasons, e.g., the privileged position of classical logic, popularity of ontologies of Aristotelian provenance, "natural" presence of mathematical concepts in thinking, the use of precise expressions (allegedly) facilitates philosophizing in the spirit of logical precision, which does not undermine the sacrosanct canons of philosophical thinking. It may be for this reason that "demonstrating", that vague expressions are precise, is such a popular operation: it makes it possible for the illusion of precision of philosophical thinking to persist.

We shall start the discussion of approaches to vagueness from those which substitute preciseness for vagueness, i.e., eliminate vagueness from natural language.

Supervaluations. The author of the *supervaluation theory* is Kit Fine, who, in his work *Vagueness, truth and logic*[46] applied supervaluations to vagueness; they were earlier introduced in formal logic by Bas van Fraassen, without any reference to that question.[47] The class of *admissible valuations* is made by the so-called *precisification*, which meets the following conditions:

- Every precisification sets a precise boundary separating the positive extension from the negative one for every vague term.
- Every precisification v setting a precise boundary honors the positive and negative extension of a given vague term. This means that if P is a vague

[46] Fine [24].

[47] van Fraassen [25–27] and Fine [24]. The origins of the ideas of supervaluation theory can be found in Henryk Mehlberg's 1958 book *The Reach of Science*.

predicate, then $vP(x) = 1$, if x belongs to the positive extension of the predicate P, and $vP(x) = 0$, if x belongs to the negative extension of P. Consequently, every precisification sets a precise boundary between the new positive extension and the new negative extension, which runs across the vagueness area.[48]

- Every precisification sets a precise boundary of a vague term consistently with respect to that term. Consequently, if a sentence "A man, who is x feet tall, is tall", is true for $x = x_1$, then the sentence must be true also when $x > x_1$. If, however, that sentence is false for $x = x_1$, then the sentence must be false also when $x < x_1$.
- Every precisification sets a precise boundary of a vague term consistently with respect to all other terms. This is so as to avoid such cases that with a given precisification, someone is tall and short at the same time, or is very tall not being tall, etc. A true (false) sentence becomes supertrue (superfalse) with every precisification. Other sentences are neither supertrue nor superfalse. In this way, two new logical values have been introduced: supertruth and superfalshood. Naturally, no law of logic is supertrue. If neither p_1 nor $\sim p_1$ are supertrue, then $p_1 \vee \sim p_1, p_1 \wedge \sim p_1, p_1 \rightarrow p_1, p_1 \leftrightarrow \sim p_1$ are not supertrue also. Consequently, all classic laws of metalogic, i.e. the principle of consistency, the law of excluded middle, the law of double negation, etc., are suspended.

The Paradox of the Bald One is evidently avoided here. For a sentence "A man with n hairs on his head is bald", the class of all precisifications designates two natural numbers k_1 and $k_2 (k_1 < k_2)$, such that the sentence is supertrue for $n \leq k_1$, and superfalse for $k_2 < n$. Consequently, the sorites reasoning will "stop" on the number k_1 or k_2 depending on its direction. The Paradox of a Heap disappears but its paradoxical character remains: two above mentioned boundaries are introduced, even though it is known that they do not exist. The boundaries depend on the class of precisification–every class sets different boundaries. Consequently, one would have to consider supersupertruth, supersupersupertruth, etc. Moreover, what is a class of precisification? An expression of the opinion of a community, which uses a given language in the beginning of the twenty-first century? No community uses the values of supertruth and superfalshood. Another problem results from the assumption that there exists a class of *all* precisifications: how should the word "all" be understood? Clearly, the understanding of that word depends on where we draw the boundaries of the area of vagueness. But such boundaries do not exist. Then what is the sense of supervaluations theory? Its only sense seems to be escaping paradoxicality in the reasonings of the sorites type. This success, however, is only apparent, for it is achieved with the assumption that vagueness does not exist. The elimination of vagueness takes place in the phase of defining precisification. As a result, we have a paradoxical situation: for any vague predicate P, sentences "The predicate P has no area of vagueness", "The predicate P is

[48] Because of the fuzzy nature of the vagueness area this condition seems to be an idealization difficult to execute.

a precise predicate" are supertrue (a patent contradiction!). Supervaluation theory is thus more paradoxical than the phenomenon of vagueness.

Subvaluations, dialetheism. The opinion that some sentences are both true and false is held by the advocates of *dialetheism*, who believe in the existence of the so-called *true contradictions*.[49] Usually, dialetheists point to the Liar sentence as an example of such sentences.[50] Moreover, in their opinion, numerous examples confirming the value of dialetheism are provided by sentences with vague terms. Graham Priest considers the case of a person A in the moment, when the person is going through the door between a corridor and a room. In his opinion, the statement that the person A is in the room is both true and false for a moment.[51] The best known form of the idea of dialetheism, however, is the so called *subvaluation theory*. Although its author is Dominic Hyde, who presented his proposal in the 1997 paper *From heaps and gaps to heaps of gluts*,[52] the same view was presented already in 1948 by Stanisław Jaśkowski in the paper *Propositional calculus for contradictory deductive systems* (English version 1969)[53] Jaśkowski's *discussion systems* formalize dialogue of persons, who differ in their opinions about the truth of sentences used in a discussion. The most important system is D_2. Jaśkowski found inspiration in the occurrence of vague terms in natural language: it can happen that one and the same sentence can be differently understood by participants in a discussion, so the logical value of the sentence is different for them too. It is important to remember that then two "seemingly" contradictory sentences A and $\neg A$ can be accepted simultaneously, when A is accepted with one meaning of terms that occur in it, while $\neg A$ with another meaning of the same terms.

As in the case of supervaluation theory, Hyde's theory is founded on a familiar class of precisification. The concepts of supertruth and superfalshood are replaced with the concepts of subtruth and subfalshood. In some precisification, a true sentence is subtrue, a false sentence is subfalse in some precisification. Consequently, some sentences are both subtrue and subfalse. Hyde also uses concepts of *definite truth* and *definite falsehood*, which are identical with supertruth and superfalshood, respectively. Thus, from the point of view of subvaluations theory, there are three types of sentences: definitely true, definitely false and sentences which are both true and false.

It is evident that paradoxes of vagueness can be avoided here. Let us consider the predicate "to be bald" and the kth reasoning, which leads from premises:

(kth categorical premise) a man with k hairs on his head is bald.
(kth conditional premise) If a man with k hairs on his head is bald, then a man with $k + 1$ hairs on his head is bald.

---

[49] Dialetheism is usually associated with the broad current of paraconsistent logics, i.e., ones, which tolerate contradiction. Priest and Tanaka [28].

[50] Cf. e.g., Priest [29], pp. 42–45; Priest [30], pp. 31–37; Priest [31].

[51] Priest [32], p. 202.

[52] Hyde [33].

[53] Jaśkowski [34, 35].

To a conclusion:

(kth conclusion) A man with $k + 1$ hairs on his head is bald.

Obviously, the class of precisification will designate two familiar natural numbers k_1 and k_2 ($k_1 < k_2$). Yet the process of predicating truth of the sentence "A man with n hairs on his head is bald", begun with the number $n = 0$ stops only on the number k_2. Indeed, the kth conclusion will not follow from both of the above premises only when $k = k_2$. As a result the positive extension of the predicate will be "glued" to the penumbra to form a new positive extension. With the opposite direction of predicating, the penumbra is "glued" to the negative extension in a new negative extension. As a result, both new extensions are precise sets with a non-empty intersection. In the case of subvaluations theory, vagueness is also simply replaced with preciseness. For any vague predicate P, sentences "The predicate P has no area of vagueness" and "The predicate P is a precise predicate" are definitely true. The differences between supervaluations and subvaluations theories are of secondary importance, since there is no difference between them with respect to their approach to vagueness. This is the result of applying the same class of precisification.

Regulating definitions. The use of regulating definitions is a necessity in some situations. For instance, application of law requires accurate definition of some patently vague terms, such as mature or juvenile. Naturally, every regulating definition of a vague expression is in disagreement with the natural language understanding of that expression. The aim of those definitions, however, is to catch some truth of the vagueness. This aim is purely practical.

Kubiński's theory of vague names. Tadeusz Kubiński, who created the theory of vague names, assumed that the areas of vagueness are precise, without any explanation how such preciseness can be achieved. This way he identified vague terms with open terms, i.e. ones which can be partly defined.[54] Kubiński presented his complex and precise theory in his 1958 work *Nazwy Nieostre* (*Vague Names*).[55] Let us remind that partial definition, also known as conditional one, of a term Q is a pair of sentences:

$$\begin{array}{l}\forall x(K_1(x) \rightarrow Q(x))\\ \forall x(K_2(x) \rightarrow \neg Q(x))\end{array},$$

where it is not true that $\exists x\ (K_1(x) \wedge K_2(x))$. Predicates K_1 and K_2 set the criterion of applicability of the term Q to objects of a domain D in the following way: if an object a of a domain D fulfils K_1, it means that the term Q can be applied to that object, i.e., if it is true that $K_1(a)$, it is true that $Q(a)$ and if an object a fulfils K_2, it means that the term Q cannot be applied to a, i.e., if it is true that $K_2(a)$ and it is not true that $Q(a)$.

[54] The author of the theory of partial definitions is Rudolf Carnap, who presented its outline in *Testability and Meaning*; Carnap [36].

[55] Kubiński [6].

Partiality of such definition lies in the fact that it does not assume the equality $K_1 = \neg K_2$,[56] i.e., it is not stated that both criteria exhaust all possible cases in which any object of the domain can occur. For if it is possible that an object is neither K_1 nor K_2, it means that the object in question is neither Q nor *not*-Q.[57]

Kubiński's proposal expresses both preciseness and vagueness of names in deductive systems, which he called quasi-ontologies, because they were systems built on the ontology of Stanisław Leśniewski (1886–1939). Formal definition of the boundary of a name B, i.e. that which remains after subtracting both extensions of the name from the universum, is the key to distinguishing a vague name $(B \neq \varnothing)$ from a precise one $(Bx = \varnothing)$ in the syntactic sense. For vague and precise names defined in this way Kubiński proves a string of theorems, which point to rather intuitive relations between the positive extension, the negative one and the boundaries of names.

Halldén's proposal. In 1938 D.A. Bochwar published a Russian paper,[58] in which he presented the characteristics of three values: truth T, falsehood F and nonsense N[59]:

p	$\neg p$
T	F
N	N
F	T

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	N	N	N	N	N
T	F	F	T	F	F
N	T	N	N	N	N
N	N	N	N	N	N
N	F	N	N	N	N
F	T	F	T	T	F
F	N	N	N	N	N
F	F	F	F	T	T

[56] The assumption that $K_1 = \neg K_2$ means that the definition given by the two conditions becomes a complete definition, given in one of the two equivalent forms: either $\forall x(K_1(x) \leftrightarrow Q(x))$, or $\forall x(K_2(x) \leftrightarrow \neg Q(x))$ Przełęcki correctly observes that the same form of the definition does not mean that it defines a vague term. A vague term can also be defined with help of a complete definition, since it is enough for the predicate K_1 (i.e., also K_2), which occurs in it, to be vague. Przełęcki [37], p. 82.

[57] Przełęcki [37], p. 80.

[58] Bochwar [38].

[59] In Kleene's book *Introduction to Metamathematics*, these conjunctions are called *weak*. He calls *strong* the conjunctions, which he introduced earlier, defined with help of the value U. In a straightforward way, Kleene states, however, that in case of *partial predicates* only strong conjunctions, i.e., the ones described in his 1938 work, can be applied; Kleene [39], p. 334. Naturally, all *partial predicates* are predicates, which are partly defined.

In 1949, 3 years before Kleene's monograph, Sören Halldén, who most probably did not know Bochwar's paper then,[60] published a book *The Logic of Nonsense*[61], in which he proposed dividing sentences into two categories on the basis of whether the sentences have a logical value (truth or falsehood) or have neither. He called the former ones, *sensible*, the latter, *senseless*. Moreover, Halldén accepted that both sensibility, i.e., truth or falsity, and senselessness of complex sentences should exhibit certain regularity, which he explained in the same way as Bochwar. Halldén linked vagueness with the logical value of nonsense.

Halldén's proposal gives rise to many difficulties. It is easy to see that in the logic of Bochwar (Halldén), just like in the logic of Kleene, presented below, there is no tautology. Even sentences of the form $p \rightarrow p$, or $p \vee \neg p$ may not be true (since senseless) if p is neither true nor false. All this made Halldén accept two designated values for that logic: P and N. Yet, defining logical implication as one preserving non-falsehood in Halldén's logic made inference difficult to accept, because of its non-intuitive character. For instance, the rule of detachment fails in it. Now if a sentence p has a value N and q has a value F, then two premises p and $p \rightarrow q$ have designated values (N). It means that a sentence with a value of falsehood (non-designated one) follows from two sentences of designated values.[62] Similarly, even when a sentence $p \wedge q$ has a designated value, q may be a false sentence: it is enough that p is senseless. It is clear here that separation of conjunction is not a rule which retains non-falsehood. A similarly difficult situation occurs when the set of designated values is limited to a single-element set containing the value of truth. Now if we have a true sentence p, the alternative $p \vee q$ does not have to be a true sentence. It means that attaching alternative is a rule which does not retain truth. Williamson correctly observes that the problem lies not in the choice of one or another set of designated values, but in the Halldén's tables defining conjunctions. To support this statement he gives an example analyzing the logical value of a sentence "Jim is a bald philosopher", with the assumption that Jim was never, is not, and never will be a philosopher. The logical value of the sentence varies with Jim's growing baldness. First, it is false, then senseless, finally to become false again. Yet, it should be false throughout Jim's life.[63]

In this proposal, eliminating the Paradox of a Heap is illusory. It only exposes the precisification of the vagueness area's boundaries. We can state, then, that every vague predicate P can be replaced with three precisely understood predicates: P, *quasi*-P, and *not*-P. Thus it is clear that Halldén's eliminates vagueness by setting precise boundaries where they have never existed.

---

60 Williamson [5], p. 288.

61 Halldén [40].

62 Williamson [5], pp. 105–106.

63 Williamson [5], p. 107.

Körner's proposal. In 1938, Stephen Cole Kleene (1909–1994) published a paper *On a notation for ordinal numbers*, where he first presented tables of three logical values: truth *T*, falsehood *F* and unknown logical value *U*:

p	$\neg p$
T	F
U	U
F	T

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	U	U	T	U	U
T	F	F	T	F	F
U	T	U	T	T	U
U	U	U	U	U	U
U	F	F	U	U	U
F	T	F	T	T	F
F	U	F	U	T	U
F	F	F	F	T	T

The symbol "*U*" stands for the value of truth or falsehood, but we do not know which. In constructing his three-value logic, Kleene was inspired by partial definitions.

In a series of papers published in 1950s and 1960s, Stephan Körner presented his proposal of *logic of inexact concepts*.[64] The source of inexactness is vagueness or, to be more precise, the existence of borderline cases. Vague expressions are defined by cases. Every vague expression *F* divides objects into *positive, negative and neutral candidates*. Objects are neutral candidates if, depending on free choice, they can be selected as positive or negative cases of the expression *F*. Körner's proposal is thus related to two logics: the three-value *pre-election logic* and two-value *post-election logic*. The three types of candidates reflect three types of sentences: true, false and neutral. The distribution of the three values given to sentences is fixed by Kleene's matrices. Ascribing the value of truth or falsehood to a sentence has stable character, while ascribing the neutral value is provisional.[65] In his considerations, Körner does not use the symbols introduced by Kleene. He interprets Kleene's logic with help of the so-called *vague classes algebra*. Regrettably, Körner's terminology is awkward. For him a precise class is a function, which precisely designates three sets. It means that even though there are three sets corresponding to every vague predicate *P*, which are its both extensions and penumbra, respectively, none of these sets is any of the extensions or the penumbra of *P*. For if *P* is a

64 Körner [41–44].

65 Williamson [5], p. 108.

vague predicate, its extensions are not separated by precise boundaries from its penumbra. In his proposal, Körner identified a vague predicate with one that is defined precisely but partly.

It is easy to see that Körner's proposal, which is a three-value approach to vagueness, causes the same philosophical problems as Halldén's one and neither solves the Paradox of a Heap nor analyzes the phenomenon of vagueness. This means that neither Halldén's nor Körner's proposal can be accepted as a theory of vagueness.

Tye's proposal. Kleene's table was used as a basis for a vagueness theory also by Michael Tye. In 1990, many years after Körner, he published a work *Vague Objects*,[66] where, beside a justification of vagueness in the material world, he proposed giving a third logical value (*neither-true-nor-false*) to sentences whose indefinite character is caused by vagueness. The difference between Körner and Tye lies in the fact that the former treats the third value *U* as one which expresses a temporary suspension of judgment about the truth of the sentences it is ascribed to, while the latter treats it as a permanent lack of logical value. Although Tye starts with the assumption of fuzziness of boundaries of a vagueness area, through the use of a three-value logic he achieves the effect of precisification of that naturally vague boundary.

Epistemicism. The opinions of epistemicists are in a stark contrast with the traditional understanding of vagueness, which treats this phenomenon as independent of the knowledge of a language user. They claim that all difficulties caused by vagueness stem from our ignorance. If we had appropriate knowledge, every term would appear as precise to us. This popular opinion is known under the name of *epistemicism* or *epistemological theory of vagueness*. It was created by Timothy Williamson.[67] He presented his opinions in a number of works, of which *Inexact Knowledge* and *Vagueness and Ignorance* are especially prominent.[68] Roy Sorensen gained fame for his construction of arguments supporting epistemicism.

A sentence "A man with k hairs on his head is bald" has a definite logical value for every natural number *k*. Regrettably, we do not know the logical value of the sentence for every number *k*. Obviously, for epistemicism there is no Paradox of a Heap, just like there is no Paradox of a Square, or Circle. The false

---

[66] Tye [20].

[67] Opinions concordant with Williamson's theory were expressed even earlier. In his 1948 paper *Zmiana i sprzeczność* (*Change and inconsistency*), Ajdukiewicz states: "We shall lack the means to decide about a man in a given age whether he is young or not young. This is an obvious fact. But some people are ready to conclude therefrom that for a man who grows old it is neither true that he is young nor that he is not young. This attack on the principle of the excluded middle errs fundamentally, because it mistakes the situation, in which we are unable to determine between two contradictory sentences for the situation, in which none of the two contradictory sentences is true. That we cannot determine the truth of either "he is young" or 'he is not young" for fundamental reasons, not merely because of technical difficulties, is no proof at all that none of them is true". Ajdukiewicz [45], pp. 105–106.

[68] Williamson [5, 46, 47], Chap. 8 *Inexact knowledge*.

conclusion that a man with 100,000 hairs on his head is bald results from the fact that although it is true that a man with zero hairs on his head is bald, it is, naturally, not true that, for any natural number k, if someone with k hairs is bald, then he is also bald with $k + 1$ hairs. This is because there is a number k_0, unknown to people, such that whoever has less than k_0 or precisely k_0 hairs on his head is bald, and whoever has more than k_0 hairs on his head is non-bald. Borderline cases, in their semantic or ontological sense, no longer exist. Admittedly, we think they do, because of our ignorance. So if they exist, they exist only epistemically. Williamson "solves" the problem of vagueness by assuming that it does not exist. He claims it is simply an illusion. All the more so, the vagueness of objects in extra-linguistic world.

Williamson based his theory on three fundamental assumptions. The first one claims that knowledge refers to facts only, so only facts can constitute knowledge. According to the second one the imperfection of our senses has a direct influence on our inability to get to know precise facts. Every fact appears to us as imprecise. Our memory of facts is even more imprecise, for it diminishes their preciseness further. According to the third assumption the credibility of knowledge in situations, where it is bound to be imprecise, depends on the so called margin for error.

To demonstrate the impreciseness of our knowledge Williamson proposes the following reasoning concerning assessment of one's own knowledge of a particular case. Let us assume that we are watching a crowd of people. We do not know how many people there are in the crowd, and especially:

1. We do not know whether there are not n people in the crowd.

   This assumption means that there are numbers about which we do not know that they do not express the number of people forming the crowd we are looking at in the moment, e.g. we do not know that it does not have 1000 people, but we also do not know that it does not have 800 people, etc. These numbers form a subset of the set of natural numbers, so there is the smallest number in the subset, of which we know that it is not the number of people in the crowd. But this means that a number smaller by one is the one, of which we know that it is not the number of people in the crowd. If, for instance, the smallest number for which the assumption 1 is true is 582, which means that the greatest number, of which we know that it is not the number of people in the crowd is 581. Consequently,

2. We know that there are not precisely $n - 1$ persons here.

   Yet on the basis of his own experience, Williamson states that:

3. We know that if there are precisely n persons here, then we do not know that there are not precisely $n - 1$ persons here.

Williamson claims that 1–3 form a contradictory set of sentences if we do not reject the principle "I know that I know". Then, a negation of the first sentence follows from the second and the third ones, on one condition however. Using a simple sentence calculus with an operator W: "we know that", let us mark as N a sentence "there are precisely n people here" and as $N - 1$, a sentence "there are precisely $n - 1$ people here". Then we have: 1. $\neg W(\neg N)$, 2. $W(\neg(N - 1))$, 3. $W(N \rightarrow \neg W(\neg(N - 1)))$. The detachment rule for implication within the range of the operator W should have the form: if $W(a \rightarrow b)$ and $W(\neg b)$, then $W(\neg a)$. From a set of sentences, the second and the third one, we can, therefore, infer a sentence which is contradictory to the first one if we assume additionally: 3. $W(\neg(N - 1)) \rightarrow WW(\neg(N - 1))$. Then, from 2 and 4 we get: 5. $WW(\neg(N - 1))$, i.e. $W\neg\neg W(\neg(N - 1))$. And only now from 5 and 3 we get $W(\neg N)$, a sentence contradictory to the assumption 1. In this way, Williamson argues against the soundness of the principle *WW*, expressed by the premise 4, and stating that if I know something, I know that I know it. According to Williamson, accepting this principle has led to contradiction.

The formal aspect of contradiction following from three accepted sentences and the *WW* principle may seem correct. Yet the reasoning is erroneous. The problem lies in the three sentences accepted by Wiliamson or, to be more precise, in the second one. It is clear that looking at a crowd of people I can point to several numbers n, for which the first sentence will be true. This does not mean, however, that the pointed set of numbers is defined precisely. Consequently, I cannot accept that it is a precisely defined, classical subset of the set of natural numbers N, so, perhaps, there is the smallest number in that subset. Already a simple observation shows that there are a group of numbers, which will make it difficult for me to say whether the first sentence is true for them or not. This obvious fact is evidence of vagueness of the assessment how many people make up the crowd I am looking at. There is yet another, possibly graver, error here. In order to infer from sentences 1–3 in the way proposed by Williamson, one has to accept that in each of these sentences there is one and the same number n. Consequently, it follows from the first two sentences that n is the smallest number of those which are defined in a precise way and of which we do not know whether they do not express the number of people in the crowd we are looking at. Naturally, $n - 1$ is then the greatest of those numbers, which are smaller than the precise set mentioned above, of which I know that they do not express the number of people in the crowd. As we have shown, this is a very strong assumption of the lack of vagueness. However, if we take into consideration the third sentence, it will turn out that the accepted set of premises becomes a manifest assumption of preciseness, since… n is a number of people in the crowd. Consequently, the assumption of lack of vagueness, sc. assumption of preciseness, which follows from sentences 1–3 is stronger than one which could follow only from sentences 1 and 2. As can be seen, Williamson assumes that if there are n people in the crowd, a viewer does not know that there are not n people in the crowd and he knows that there area not $n - 1$ people in the crowd, when $n - 1$ is the greatest number of those about which he knows that they do not express the number of people in the crowd.

This means the opinion presented by Williamson seems to be internally contradictory.[69] An analysis of this reasoning shows that it makes a tacit assumption that our knowledge is precise in a situation, where preciseness is conspicuously absent. Naturally, with such visible and rather serious errors in reasoning any reference to the law "I know that I know" is senseless. This reasoning does not prove anything.

The example with assessment of the number of people in the crowd we watch was to serve Williamson as an argument showing that knowing something and knowing that we know it are two different things. It is possible that there is knowledge about something without self-knowledge, i.e. awareness that we possess that knowledge.[70] This is to show the imprecision of our knowledge. Now, if our knowledge were precise, it would respect the principle *WW*: *I know that I know*.[71] What is worse, knowledge is inseparably bound to the margin for error. Williamson's starting point in this part of his reasoning is the so called *margin for error principle*. According to it, a sentence A is true in all cases similar to those cases, in which the sentence "I know that A" is true.[72] It means that if we assume that the sentence "I know that A" is true in the case P, then by virtue of the principle, we can come to a conclusion that the sentence A is true in all cases similar to P. Applying the principle to the sentence "I know that A", we obtain a following criterion: the sentence "I know that A" is true in all cases similar to the cases, in which the sentence "I know that I know that A" is true. Let us assume then that in the case P the sentence "I know that I know that A" is true. Then, by virtue of the margin for error principle, the sentence "I know that A" is true in all cases similar to P. Applying the principle again we obtain that the sentence A is true in all cases, which are similar to cases similar to P. Now, because the principle *WW* does not apply, every new "I know that" enhances the risk of error in

[69] Let us assume here that the crowd in question is composed of 1019 people. From the assumption, someone who looks at the crowd knows that there are not 1018 people in it. Moreover, it is the biggest number, of which he knows that it does not express the number of people in the crowd. He must know that, because with respect to the next number, 1019, he does not know that it does not express the number of people in the crowd. This means that so far our observer knows that the area of vagueness (impreciseness) starts with the number 1019. Only up to 1018, he has certitude that those numbers do not express the number of people in the crowd. Naturally, the number, which expresses the actual number of people in the crowd, 1019, belongs to that area. Consequently, our observer knows that the number of people in the crowd equals 1019 or is greater than 1019. However, it enough for our observer to get to know Williamson's reasoning to discover that this is precisely 1019. This contradicts not only the whole reasoning but also, and most of all, the fundamental assumption of epistemicism. What is worst is that the assumptions he accepts are in disagreement with the evident state of affairs.

[70] It should be mentioned here that natural science also distinguishes between *homo sapiens* and *homo sapiens sapiens* accepting without hesitation that we are representatives of the latter.

[71] The problem whether someone is aware of the knowledge he possesses or not seems to belong rather to psychology of mental processes. Moreover, this question is most probably connected with a person's mental faculties, intellectual development, etc., and cannot be globally determined by means of sentence calculus, especially if the solution were to apply to any person.

[72] "A margin for error principle is a principle of the form: 'A' is true in all cases similar to cases in which 'It is known that A' is true", Williamson [5], p. 227.

assessing the truth of a sentence. In other words, every application of the operator "I know that" widens the margin for error.[73] Consequently, it is impossible to precisely assess the trustworthiness of our opinions, since every our act of self-reflection only widens the margin for error. It is another argument for accepting that our knowledge in imprecise and thus untrustworthy.[74]

It cannot escape our attention that Wiliamson's margin for error principle leads to a situation where the truth of a sentence A in a context P depends on the truth of the sentence "I know that A" in a context which somehow resembles P. Leaving aside the question of contexts, it cannot be so that objective truth of the sentence A depends on the subjective truth of the sentence "I know that A". Few ideas are as weird as this one. Moreover, it is difficult to understand how the sentence "I know that A" could actually be influenced by the ever more dubious truth of sentences: "I know that I know that A", "I know that I know that I know that A", etc. Does the fact that we do not know how many grains of sugar start making a cup of tea sweet really result from the fact that we (allegedly) have a problem with the truth of sentences stating knowledge about …sweetness of tea?

Although the solution of the Paradox of a Heap is immediate, it has to be explained that we do not and even cannot know where, i.e. between which cases, runs the precise boundary between the positive and negative extension. What becomes manifest here is the importance of the margin for error principle in epistemicism. We shall explain the issue using the example of a precise, in Williamson's opinion, predicate "to be a heap". According to the principal thesis of epistemicism, for this predicate there exists such a natural number k_0 that:

- even appropriately poured k grains do not form a heap if $k < k_0$; and
- appropriately poured k grains form a heap if $k \geq k_0$.

Consequently, the sentence $Z_0 =$ "$k_0 - 1$ grains do not form a heap and k_0 grains form a heap" is true from the assumption.[75] The truth of Z_0 notwithstanding, however, we do not know that Z_0 is true, so the sentence "I know that Z_0" is not true. The cause of this state of events is impreciseness of our knowledge, bound to depend on the margin for error principle. Because the sentence Z_0 is a conjunction, the sentence "I know that Z_0" would be true if two sentences: "I know that $k_0 - 1$ grains do not form a heap" and "I know that k_0 grains form a heap", were true.

---

[73] "In effect, knowledge that one knows requires two margins of error [in the case described here], More generally, every iteration of knowledge widens the required margin.", Williamson [5], s. 228.

[74] It should be noted that proving a rather commonly accepted thesis about impreciseness of knowledge concerning facts, which are essentially related to vagueness, Williamson paradoxically accepts that such knowledge is more precise than it is generally accepted. Usually, no one assumes that the area of vagueness is precisely delimited. This way, Williamson, who tries to persuade us to accept the thesis about impreciseness of knowledge assumes that knowledge is much more precise than we think.

[75] Although we shall not mention it later, we have in mind only "appropriately arranged quantities of grains".

Now let us assume that the former sentence is true. Then the sentence "I know that $k_0 - 1$ grains form a non-heap" is true. Consequently, by virtue of margin for error principle the sentence "k_0 grains form a non-heap" is true, so the sentence "k_0 grains form a heap" is false. Using the margin for error principle we have inferred the falsehood of the conjunction Z_0 from the truth of the sentence "I know that $k_0 - 1$ grains form a non-heap". Let us assume now that the sentence "I know that k_0 grains form a heap". Then, by virtue of the same principle, the sentence "$k_0 - 1$ grains form a heap" is true, so the sentence "$k_0 - 1$ grains do not form a heap" is false. As in the former case, using the margin for error principle we infer a conclusion about falsehood of the conjunction Z_0 from the truth of the sentence "I know that k_0 grains form a heap". Consequently, if we assume that we know at least one part of the conjunction Z_0, we immediately obtain the falsehood of the whole conjunction, which means that it cannot be known as it appears to be permanently false.[76]

This reasoning seems to be both simple and evident. This, however, is a false appearance. The value, sc. the correctness, of a reasoning is derived not only from its formal correctness but also from the material one, i.e. the truth of its premises.[77] Consequently, a correct reasoning is one which is correct both formally and materially. Let the following reasoning, strictly analogical to the former one, be an illustration for it:

Let us assume a *General principle of life on Earth*[78], which, just like the margin for error principle prevents a simultaneous approval of two respective sentences: *If I know that an animal species lives at the bottom of the sea, it does not live in the desert, and if I know that an animal species lives in the desert, it does not live at the bottom of the sea*. Moreover, let us assume that the sentence $Z_1 =$ "Buffalos live at the bottom of the sea and buffalos live in the desert" is true. Using the *General principle of life on Earth* it is easy to show why no one knows that Z_1 is a true sentence. It is sufficient to repeat Williamson's reasoning for the sentence "I know that Z_0" for the sentence "I know that Z_1".

A presentation of epistemicism cannot overlook some fairly popular arguments allegedly justifying the theory. In his 1994 work, *The epistemic conception of vagueness*, Crispin Wright presents arguments, which in his opinion, are believed to be the strongest in support of epistemicism.[79]

The first argument is Williamson's. Analyzing the theory of supervaluation he showed that giving up the principle of two values while keeping the law of the excluded middle and Tarski's convention leads to contradiction. Consequently, Williamson believes that if rejection of the principle of two values leads to contradiction it must be retained and so every predicate, even a vague one, must have a precise boundary separating its two extensions. Naturally, Williamson's

[76] Williamson [5], pp. 232–233.

[77] After all, for most logicians Z_0 is a manifestly false sentence.

[78] The ridiculous and pompous sound of that name is intended.

[79] Wright [48].

argument is hard to defend, since it is not at all certain that Tarski's convention, the principle of two values, and the law of the excluded middle must actually be preserved in cases of sentences with vague terms.

Another argument is the already discussed "proof" that a vague grey ball is precise; we have shown it suffers from the error of *petitio principii*. Yet another Sorensen's argument is that which allegedly shows inexistence of predicates that are both vague and tolerant. It is puzzling in that it sets vagueness against tolerance in the situation where the essence of vagueness of expressions lies in their tolerance.[80] Sorensen considers the following reasoning[81]:

(1) A Sorites argument concerning "short man" has a false induction step if the step's increment equals or exceeds ten thousand millimetres.
(2) If a Sorites argument concerning "short man" has a false induction step if the step's increment is n millimetres, it also has a false induction step if the step's increment is $n - 1$. Thus,
(3) All Sorites arguments concerning "short man" having induction steps with increments convertible to millimetres have false induction steps.

According to Sorensen this reasoning shows that predicates which are both vague and tolerant cannot exist. Let us remind here that a similar sorites reasoning for the word "bald" proves only the vagueness of the word "bald" and one for the word "chair" proves the vagueness of the word "chair". Consequently, the above reasoning proves only the vagueness of the expression "induction step" and nothing more.

Another Sorensen's argument believed to be a strong support of epistemicism is one called "A Thousand Clones". Sorensen assumes that a man, Mr. Original has a growing clone Mr. Copy. Both grow keeping the same pace in all phases of development according to the principle "the earlier one is first": if something undergoes a finite process of change, then if it started changing earlier, and changed in the same way, it would finish changing earlier. Sorensen assumes thus that not only is one man an ideal copy of the other, but also their respective developments.[82] Later Sorensen claims that if Mr. Original has to stop growing first, he will cease to grow in a moment t and will become a tall man. However, in the same moment t, Mr. Copy is still growing, so he is not a tall man yet. Consequently, the boundary between the height of a tall man and the height of a non-tall man is somewhere between the height of Mr. Original in the moment t and the height of Mr. Copy in the same moment. Since cloning may have taken place in any moment after the birth of Mr. Original, say a second later, the height of Mr. Original in the moment t is of whatever closeness to the height of Mr. Copy. This means that the boundary between tall and non-tall height is precise, since it can be found in a range of numerical values of whatever broadness.

---

[80] Tolerance is explained in the beginning of this chapter.

[81] Sorensen [49].

[82] The research of cloning showed the falsehood of the second of these assumptions.

Naturally, we find a *petitio principii* here, but not just it. First, Sorensen assumes that both gentlemen's processes of growth end suddenly (!) in the moment t and this is the moment of reaching tall height (!). So in any moment before t, Mr. Original was not tall. But this is the assumption of preciseness of the predicate "to be a tall man": a clear *petitio principii*. It is accompanied by yet another error of identifying two completely different expressions: "the final height Mr. Original achieves in the moment t" and "the height of a tall man". Now Mr. Original can be tall in a moment $t_1 < t$ or never be tall. But even if we assume that Mr. Original is the paragon of tall height, the *petitio principii* remains.

We shall leave other arguments of Sorensen, such as "climbing a mountain ends suddenly by standing on it" or "many years of learning English finish suddenly with the command of it" without commentary. All these arguments suggest something contrary to the opinions of epistemicists: to prove preciseness of a vague predicate, one has to assume it first.

The value of epistemicism is undermined also by arguments of historical and geographical character. For instance, the name "tall man" has changed its sense over the ages and nowadays its sense depends on geographical location. Moreover, epistemicitsts' claim that vague expressions can be replaced by precise ones cannot be defended either. For instance, it is impossible to express sweetness of tea by the number of grains of sugar put into it. The expression "grain of sugar" is vague because of "grain" and because of "sugar". Vagueness can be replaced only by another vagueness.

Indeed, it is difficult to show even one serious argument for epistemicism which would respect logic, so, summing up the analysis of the theory, it has to be stated that its application leads to shocking conclusions. Let us assume, as epistemicists want, that the principle "I know that I know" does not hold. Then it is impossible to precisely assess the trustworthiness of our opinions, for each of our self-reflection only widens the margin for error. But all arguments of epistemicists are notorious examples of application of the predicate "I know that". Consequently, epistemicists' arguments and, especially, conclusions are, according to their own theory, instances of imprecise knowledge, crippled by a notorious error, the greater the more times they apply self-reflection. If they claim that vague predicates have precise boundaries of extensions but this knowledge is imprecise and laden with error, it simply means that vague predicates do not have precise boundaries of extensions. Thus, it is not only that the statements of epistemicism do not respect the evidence and their arguments do not respect logic; epistemicism falsifies itself.

In this paragraph, we have passed over numerous arguments that advocates of some theories raised against other ones. For the sake of brevity, we have focused only on the criterion we find to be of primary importance, i.e. whether a proposal in question respects (preserves) vagueness and tries to grasp it, or whether it simply ignores it by replacing it with its negation. The following paragraph is devoted to proposals which treat the phenomenon of vagueness with due honesty and respect.

5.1.3 Vagueness Respecting Proposals

Pragmatic, or perennial approach. Pragmatic doctrine of vagueness is probably the simplest and most natural of all possible approaches to the phenomenon of vagueness. Its natural character results from the fact that it is in agreement with our every day practice in using vague expressions, which is an age old practice. If we are to use a vague predicate, e.g. "to be red in color", we use it for cases, which do not raise our doubts. We usually do not use vague terms in dubious situations or in borderline cases.[83] Well aware of the existence of such cases, we choose other words, the application of which is not controversial. Thus, instead of obstinate insistence on using the predicate even in doubtful cases, we use other ones, such as "to be pale red", "to be dark red", etc. Naturally, this approach does not solve the problem of vagueness; most of all, it is of no help with paradoxes of the Sorites type. Nevertheless, it can be treated as a serious approach, since it shows how to live with the phenomenon of vagueness without distorting reality.

Fuzzy sets and degrees of truth. So far, our analyses have evidently shown, among others, the inappropriateness of those approaches to vagueness, which are based on three-value logics. In those theories, vague predicates have extensions with precisely defined boundaries. Moreover, these approaches generate numerous hardly acceptable consequences. It is easy to see that no n-value logic can be a suitable basis for a correct analysis of vagueness. For instance, a hundred-value logic will introduce a hundred precisely defined sets, which form both extensions and the area of vagueness. The difference between a three-value approach and a hundred-value approach is merely a difference of degree and not a qualitative one. Every theory, which defines the boundaries of vagueness in any way, is in disagreement with vagueness itself, since it attacks the very definition of vagueness. Awareness of this evident fact lied at the fundament of a new proposal, which replaced the set of finite number of logical values with an infinite one, moreover one with the power of a continuum. The rationale for such an idea is very intuitive. Let us consider a road becoming continuously darker from whiteness to blackness. Let us now leave aside the fact that both whiteness and blackness are vague and imagine there could be one whiteness and one blackness. Going down our road we gradually enter into darkness. In every moment, we are in a darker place than the one we were in even a second earlier. Let us assume now that while going along that road we are considering the logical value of a sentence "The road is dark". Second after second the truth of that sentence becomes more and more evident.[84] It is possible, then, to give a numerical expression for the growing evidence of truthfulness. Using values of real numbers from the bilaterally closed interval [0,1], we define a growing function, which subordinates to every real number, expressing the distance from the beginning of the road in a given moment,

[83] That we usually do not do it does not mean at all that we never do it. Using vague expressions outside the range of their preciseness may mean e.g., a joke or irony.

[84] Williamson [5], p. 113.

a successive real number from the interval [0,1] in such a way that the starting point of the road has the value 0 and the final point, the value 1. In the moment we start, the sentence "The road is dark" is evidently false, in the last point of the road, the very same sentence is evidently true. Between the starting point of the road and its final one, the sentence has the value, which is greater than 0 and smaller than 1, depending on the moment of the march in which it is uttered.

In this example, we have assumed that numbers from the range [0,1] express the degree of evidence of truthfulness of the sentence in question. The degree of truthfulness is something rather difficult to talk about: a sentence is either true or not. In this theory, however, the expression "degrees of truth" has been accepted and the whole approach was named *theory of degrees of truth*. The formal fundaments for it are fuzzy sets calculus and fuzzy logic. Fuzzy sets were discovered by Abraham Kaplan and Hermann Schott, who in 1951 applied the subordination of natural numbers from the range [0,1] to determining the degree of representation of empirical knowledge by sentences. They also defined appropriate calculus for product, sum, complement and inclusion of empirical statements. Fuzzy sets became truly famous thanks to Lofti A. Zadeh, who published his work *Fuzzy* sets in 1965.[85] The great importance of the sets became apparent after they had been applied in advanced technologies that contributed to the rapid development of computer science and navigation theory. It is no wonder, then, that fuzzy sets have found their place not only in the new set theory but also in topology, algebra and calculus of probability.

A fuzzy set A is characterized by a generalized characteristic function U_A: $U \rightarrow [0,1]$, which ascribes to every universe element U a numerical value from the range of real numbers [0,1]. $a \in U$ is an element of the set A in the degree s, if $U_A(a) = s$. Consequently, $a \in U$ is undoubtedly an element of the set A, when $U_A(a) = 1$; but it is evidently not an element of the set A, when $U_A(a) = 0$. For these sets, Zadeh defines inclusion and the operations of completion, sum and product. An algebra defined in this way generates a fuzzy logic with an indenumerable number of logical values. The values of generalized characteristic functions can be identified with the logical values of respective sentences: $U_A(a) = s$ means that s is a logical value of the sentence $a \in A$. Let $[p], [q] \in [0,1]$ be degrees of truthfulness for sentences p and q, respectively. The sentence p is completely true (completely false), when $[p] = 1$ ($[p] = 0$). The sentence p is at least as true as the sentence q, if only $[p] \leq [q]$. Naturally, there are many ways of determining fuzzy logic.

Williamson gives an example of application a particular fuzzy logic to "disarming" the Paradox of a Heap.[86] Let p_k = "A man who has k hairs on his head is bald". The heap argument leading to a contradiction should have the form:

[85] Zadeh [50].

[86] Williamson [5], p. 123.

$$\begin{array}{l} p_0 \\ p_0 \rightarrow p_1 \\ p_1 \rightarrow p_2 \\ \vdots \\ \underline{p_{99999} \rightarrow p_{100000}} \\ \quad p_{100000} \end{array}$$

Usually, by accepting the truth of all premises and applying the *Modus Ponens* one hundred thousand times, we arrive at the truth of the conclusion, which is false after all. On the ground of the theory of degrees of truth, however, we shall never get the truth of the sentence p_{100000}. In this example, the sentence p_k is true in the degree $1 - k/100000$, for $0 \leq k \leq 100000$. Then, especially, p_0 is completely true (true in the degree 1), p_1 is true in the degree 99999/100000, while p_{100000} is completely false (false in the degree 1), since it is true in the degree 0. And every implication is true in the degree 99999/100000. As can be seen, this simple interpretation of the heap argument is concordant with our intuitions, for it turns out that every implication is almost true, since the truth in the degree 99999/100000 is an almost complete truth. No one tries to make us believe that the sentence "If two persons differ by one hair, either both are bald or both are non-bald" is false. This sentence is almost completely true. Moreover, the first premise is completely true and yet the final conclusion is completely false. It turns out, then, that just like a great number of insignificant differences produces a significant difference in the case of vagueness, a completely insignificant falsehood of conditionals (falsehood in the degree 1/100000) produces a complete falsehood (falsehood in the degree 1) as a result of the whole reasoning.

The theory of degrees of truth uses precised sets, out of which it builds the area of vagueness (fuzzy sets can be defined with help of classical mathematics' sets). It does so, however, in a subtle way trying to reconstruct the phenomenon of vagueness with the greatest possible preciseness. In this theory, vagueness is not eliminated but reconstructed. This is why the theory of degrees of truth may be considered a theory of vagueness.

Nihilism. There are few names which are as inappropriate as "nihilism" in characterizing the theory of Peter Unger, presented in 1979 in a series of papers under provocative titles: *I Do Not Exist*, *There Are No Ordinary Things* and *Why There Are No People*.[87] Of course, the titles do suggest that the name "nihilism" is appropriate, but their reading shows something opposite. Unger accepted the heap argument as an *ad absurdum* proof, faultless and completely concordant with the rules of logic. From this perspective, he only had to find out which statement is contradicted by the heap argument. According to Unger rejecting the rules of logic is out of question here. They are too evident to be questioned. Both the rule of

[87] Unger [8, 51, 52].

mathematical induction and the *Modus Ponens* must guarantee the correctness and aptness of reasoning if properly applied. He does not believe in existence of an alternative logic, which would save the heap argument from being paradoxical. As a result, the thesis attacked by the sorites argument is the assumption of inexistence of an object, which is marked by a name susceptible to sorites argument. It must be added here that the inexistence refers to objects in those forms, in which we usually imagine them. A bald person does not exist, a table does not exist, a man does not exist; but all these objects do not exist in the forms, in which we usually understand them. Reality exists but it is not such as its image expressed in words. The problem of vagueness lies in the language we use. Our language is not adequate for the world, which it is to express. Unger shows contradictory character of every expression invented by man. To do so, he presents a generalized form of the heap argument applicable to any expression[88]:

(1*n*) If an "invented expression" refers to some (actual or potential) object *O*, the expression also refers to every (actual or potential) object, which is microscopically different from *O* in the respect *A*.

(2*n*) If an "invented expression" refers to some (actual or potential) object *O*, there exist (actual or potential) objects, essentially different from *O* in the respect *A*, to which the expression does not apply.

The scale of the problem is beyond imagination. The arguments of sorites type show the need for a radical reconstruction of our thinking and expressing ideas. Unger, however, warns against such solutions, which look attractive and resemble the ideas of replacing vague predicates with precise ones.[89] Unger believes that sorites arguments refer not only to words but also to things. Such simple things, as those designated by vague expressions, cannot exist.[90] Unger protests, however, against treating his proofs as attacks on sorts of things. He believes that it is far from certain that things cannot exist in some other way.[91] The problem of vagueness is exceptionally serious and extremely difficult if you want to solve completely. In 1979, Unger confessed that he saw no chance of getting over that difficulty at the moment. He hoped, however, that someone would manage to do it in the future. In his opinion, the problem of vagueness was the most important of all philosophical questions.

Over the following years, Unger denounced nihilism[92] and directed his attention at the so called problem of the many, a paradox, which touches upon the very problems raised in nihilism, i.e. the conflict between naming and reality.[93]

---

[88] Unger [52], pp. 178–182.

[89] "For, if there are no heaps, we can define the word 'heap', for example, so that a heap may consist, minimally, of two items: for example, beans or grains of sand, touching each other", Unger [8], p. 250.

[90] Unger [51], p. 147.

[91] Unger [51], pp. 147–148.

[92] He confesses that he blushes recalling his works of the period. Unger [53], p. 1.

[93] Unger [53, 54].

The year 1980 brought two important works in this field: Peter T. Geach's book *Reference and Generality* and Peter Unger's paper *The Problem of the Many*.[94] In their independent works[95], both authors exposed the very same problem concerning the understanding of names.

The Problem of the Many[96, 97]

Think of a cloud—just one cloud, and around it a clear blue sky. Seen from the ground, the cloud may seem to have a sharp boundary. But it is not so. The cloud is a swarm of water droplets. At the outskirts of the cloud, the density of the droplets falls off. Eventually they are so few and far between that we may hesitate to say that the outlying droplets are still part of the cloud at all; perhaps we might better say only that they are near the cloud. But the transition is gradual. Many surfaces are equally good candidates to be the boundary of the cloud. Therefore many aggregates of droplets, some more inclusive and some less inclusive (and some inclusive in different ways than others), are equally good candidates to be the cloud. Since they have equal claim, how can we say that the cloud is one of these aggregates rather than another? But if all of them count as clouds, then we have many clouds rather than one. And if none of them count, each one being ruled out because of the competition from the others, then we have no cloud. How is it, then, that we have just one cloud? And yet we do.

The problem of the many is a "dynamic" equivalent of the "static" sorites. On one hand, sorites arguments expose the difficulties, which result from application of terms that are tolerant (out of necessity) of the richness of objects from the extra-linguistic reality. On the other, the arguments of the "the problem of the many" type expose the difficulties, which result from the application of terms that are tolerant (out of necessity) of the continuously changing objects from the extra-linguistic reality.

[94] Geach [55] and Unger [54]. Geach presents the problem analyzing the case of a cat or, more precisely, 1001 cats. Cf. Paradox of 1001 cats in the paragraph devoted to paradoxes of identity.

[95] The problem raised by Geach and Unger is not new. Actually, it has an ancient pedigree. The true author of the idea, which is the core of the paradox, is Chrysippus. In the paragraph devoted to paradoxes of identity below, we mention the Paradox of 1001 cats, which is both a repetition of Chrysippus' paradox and the paradox of the many. Chrysippus' paradox as a dilemma of identity is discussed in detail in the same paragraph.

[96] Lewis [56], p. 164.

[97] Those interested in the attempts at solving the paradox with help of nihilism, over-population, brutalism, relative identity, partial identity, may consult Weatherson [57].

5.2 Paradoxes of Change

Another serious difficulty generated by natural language is the problem of change of state (every change), identity and motion. Paradoxes of motion are not special cases of paradoxes of change of state, because they are not connected with the phenomenon of vagueness. After all, "motion" and "rest" are precise names. Like paradoxes of vagueness, they have an ancient pedigree.

5.2.1 Paradox of the Moment of Death (Change of State Paradox)

Paradox of the Moment of Death is not considered to be a serious argument. It is treated as a manifestation of philosophical sophistry. Yet it is a representative of a huge class of paradoxes concerning change of state. Diogenes Laertios quotes Epicurus's letter to Menoikeus, in which we find a well known argument[98]:

> **Paradox of the moment of Death**
>
> And so death, the most horrible of misfortunes, has nothing to do with us because when we exist, it is absent, and when it appears, we are no longer. Thus death has relation neither with the living nor with the dead; it has nothing with the former and the latter no longer exists.

The reasonings in Paradox of the Moment of Death is based on an assumption that if there is a transition from one state to another, there is a precise boundary between the two. This boundary is the final point (!) of the first state and at the same time, the starting point of the other one; as a point it is nothing.

It is easy to see that the paradox results from vagueness of the name "death". Death cannot be reduced as an event occurring in a point of time. It is a complex process of cessation of numerous life functions, which do not cease at the same time. If "death" has a vagueness area, and it does, the moment of death is not a point in time but a non-zero sequence. This is a simple logical justification of the sensibility in fear of death.

Death and life are representatives of two states A and B, such that when A occurs, B does not and vice versa. Every analogical reasoning concerning change of state from A to B results from vagueness of both states (otherwise there would be no paradox). Since in our reality every state is dynamic, sc. undergoes changes, we cannot point to any precise boundaries of either reaching a state or leaving it.[99]

---

[98] Diogenes Laertios, *op.cit.*, X, 125.

[99] From this point of view, the so called natural species look rather funny, since they owe their naturality only to the exceptional slowness of evolutionary changes. A living being A is a representative of a natural species, because we look at it. A discovered relic of a being B is also a

5.2.2 Paradoxes of Identity

Paradoxes of identity point to logical consequences of applying a mathematically conceived identity (out of necessity) to a continuously changing object. Taking into account the fact that everything which exists in reality changes constantly, these paradoxes concern every material object and every living being. They are ever new arguments for the lack of adequacy of the natural language in describing the world.

We remember that on the ground of the first order logic, identity is a two-argument predicate, which meets two conditions:

$$(\text{Id } 1) x = x$$
$$(\text{Id } 2) x = y \rightarrow \varphi(x) = \varphi(y),$$

where φ is any formula of the first order logic. Such an understanding of identity satisfies the so called *Leibniz's Law*, according to which identical objects are undistinguishable in all possible contexts. Identical objects cannot, therefore, differ in any respect. Naturally, such identity is trivial because identity must be trivial. Only A can be identical with A. If equality refers to some aspects but not to all, then we speak of resemblance in those very respects. For this reason, identity and distinctness should satisfy two more conditions[100]:

(N Id) If $a = b$ is true, then it is necessarily true.
(N non-Id) If $a \neq b$ is true, then it is necessarily true.

(N Id) is *necessity of identity*, (N non-Id) is *necessity of distinctness*. The key assumption is that "a" and "b" appearing in those formulae are so called *rigid terms*, i.e. such terms, whose denotations do not change irrespective of the parameter: time, possible world, etc. On the ground of mathematics, using identity and distinctness is well justified. Yet do rigid terms exist outside mathematics?

One of the best known paradoxes of identity is the Paradox of Theseus's Ship, noticed by Plutarchus of Cheronea (ca. 46 - ca. 120) in his *Life of Theseus*. Plutarchus stresses that Theseus's Ship was a model of a logical problem for philosophers wondering whether a changing object preserves its identity or not. He notes that some philosophers believed that it does, others that it does not.[101,102]

(Footnote 99 continued)
representative of a natural species, because we can go describe it and give it a name. But all beings of the evolutionary chain joining A and B are apparently representatives of non-natural species.

[100] Deutsch [58], p. 1. Cf. also: Kripke [59].

[101] Plutarchus, [*Life of Theseus*].

[102] Deutsch [58], p. 2.5.

The Ship of Theseus Paradox[103]

Imagine a wooden ship restored by replacing all its planks and beams (and other parts) by new ones. Plutarch reports that such a ship was "… a model for the philosophers with respect to the disputed arguments… some of them saying it remained the same, some of them saying it did not remain the same". Hobbes added the catch that the old parts are reassembled to create another ship exactly like the original. Both the restored ship and the reassembled one appear to qualify equally to be the original. In the one case, the original is "remodeled", in the other, it is reassembled. Yet the two resulting ships are clearly not the same ship.

The importance of the paradox is strengthened by the fact that every living organism is, in a sense, a Theseus's ship. "In a sense", because every organism, even though it changes (renews) all of its cells, yet continuously grows old, so it is no longer a copy of itself as it was several years ago. For this reason, we can accept that Theseus's ship is a simplified form of the problem of identity of any living organism.

The discussion about the best reply to the question which ship is the original one is long and well documented.[104,105,106,107,108] Some said that no ship is the original (e.g., *best candidate doctrine,* also S. Kripke), others claimed that only the ship reconstructed from the old parts, still others, that only the renewed ship (R. M. Chisholm). The problem, however, lies in the fact that the apparatus of mathematics, which is created to describe static constructions, is inapplicable to changing reality. For this reason speaking of identity or distinctness in the mathematical sense, has no sense outside mathematics. New paradoxes support this claim. The universal character of each of them is manifest.

We shall present an ancient paradox, discovered by Chrysippus, in its version by Harry Deutsch[109]:

Chrysippus' Paradox

Suppose that at some point t' in the future poor Oscar loses his tail. Consider the proper part of Oscar, as he is now (at t), consisting of the whole of Oscar-minus his tail. Call this object "Oscar-minus". Chrysippus wished to know which of these

[103] Theseus's ship has its modern equivalents, e.g., in Buddhist temples in Japan. One of the oldest and most important of them is the Horyuji temple, built by Shotoku-Taishi in the old capital of Japan, Nara, in 607 AD. The temple survives thanks to successive replacement of its wooden parts.

[104] Deutsch [58], p. 2.5. Cf. also: Nozick [60] and Parfit [61].

[105] Cf. Wiggins [62].

[106] Kripke [59], p. 114.

[107] Cf. Chisholm [63].

[108] A detailed presentation of the discussion on that topic can be found in: Deutsch [58], p. 2.5.

[109] Deutsch [58], p. 2.2.

objects—Oscar or Oscar-minus—survives at t'. According to the standard account of identity, Oscar and Oscar-minus are distinct at t. and hence, by N non-Id, they are distinct at t'. (Intuitively, Oscar and Oscar-minus are distinct at t' since Oscar has a property at t'. that Oscar-minus lacks, namely, the property of having had a tail at t. Notice that this argument involves a tacit appeal to N non-Id—or N Id, depending on how you look at it). Hence, if both survive, we have a case of two distinct physical objects occupying exactly the same space at the same time. Assuming that is impossible, and assuming, as commonsense demands, that Oscar survives the loss of his tail, it follows that Oscar-minus does not survive. This conclusion is paradoxical because it appears that *nothing happens* to Oscar-minus in the interval between t and t that would cause it to perish.

Another paradox, formulated by Peter Geach[110]:

The Paradox of 1001 Cats

Let us assume that we have a fluffy cat. In some moment of the future, the cat will lose one of its hairs. If so, then by virtue of (N Id) or (N non-Id), we already have two different cats. Naturally, our cat has lots of hair and can shed it in various configurations, e.g., two or three at a time. This means that by virtue of one of the two conditions, we now have not one but thousand and one cat or, to be true, so many that it is impossible to count them.

Among paradoxes of identity, some place, rather unfortunately, Church's Paradox.[111]

Church's Paradox

Suppose Pierre thinks that London and Londres are different cities, but of course doesn't think that London is different from London, or that Londres is different from Londres. We can apply Leibniz Law to get the result that London and Londres *are* distinct.

It is easy to see that the paradoxical conclusion is drawn here because of inexactness of narration. Instead of "London" and "Londres", we should have "Pierre's belief about London" and "Pierre's belief about Londres". The paradox vanishes. It is clear that the paradox does not speak of London and Londres but about Pierre's beliefs about London and Londres.

[110] Geach [55] and Lewis [56].

[111] In Church's intentions, this argument was directed against Russell's intensional logic, and especially in the assumption that invariants and variables are not loaded with concrete meaning. Cf. Church [64]; Deutsch [58], p. 2.6.

5.2.3 *The Paradoxes of Motion*

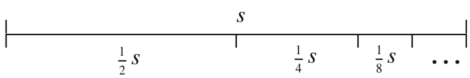
Zeno of Elea is the author of the arguably most famous paradoxes, beside the Liar Antinomy, in the history of European thought: *the dichotomy*, *Achilles and the tortoise*, *the arrow*, *the stadium* and *the grain of millet*. The last one is a dilemma pointing to the existence of audibility threshold. The stadium shows the truth about a simple fact that our speed can be different with respect to different objects, depending on whether the object are in motion or rest. Since neither *the stadium* nor *the grain of millet* contain a contradiction, i.e. a real paradox, these dilemmas will not be discussed here. Other paradoxes: the dichotomy, Achilles and the tortoise and the arrow, are still interesting.

The dichotomy. The dichotomy can be presented as follows:

The Dichotomy

In order to cover the distance of finite length, we first have to cover its half and then a half of the remaining half, etc., in infinity. Covering the distance of the length s is thus tantamount to covering an infinite number of distances, whose lengths are non-zero and form a sequence: $(1/2)s$, $(1/4)s$, $(1/8)s$, $(1/16)s$,... This means that covering a distance of any shortness is tantamount to performing infinite actions in finite time. This, however, is impossible.

Mathematics rejects the claim that adding an infinite number of positive quantities produces an infinite quantity. Consequently, the mathematical solution of the paradox is simple[112,113]:



The divisions described in the paradox form an infinite sequence $\frac{1}{2}s$, $\frac{1}{4}s$, $\frac{1}{8}s$, $\frac{1}{16}s$, ... It is a geometric sequence, whose first element is $a_1 = 0{,}5 \cdot s$, and quotient $q = 0{,}5$. Because the quotient of the sequence satisfies the condition $|q| < 1$, the sequence is convergent, and the infinite sum of all of its words is given in the formula $\sum = a_1/(1-q)$, which in our case means that $\sum = s$:

$$\begin{aligned}\sum &= \frac{1}{2}s + \frac{1}{4}s + \frac{1}{8}s + \frac{1}{16}s + \cdots = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right) \cdot s \\ &= \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots\right) \cdot s = \frac{\frac{1}{2}}{1 - \frac{1}{2}} \cdot s = 1 \cdot s = s < \infty.\end{aligned}$$

112 Whitehead [65], p. 95.

113 Ajdukiewicz [66], p. 140.

Ajdukiewicz notes that a descriptive form of that solution was given already by Aristotle.[114] He distinguished the “infinity of division” from the “infinity of limits”, which, in the case of continuous quantities, allowed for infinite division with respect to limits.[115]

Zeno, however, was interested in motion and not the division of a line. Therefore, a purely mathematical solution is not satisfactory here. The paradox poses the problem of performing infinitely many tasks in a finite time. In his two part paper on the paradoxes of Zeno,[116] Tomasz Placek notes that Zeno’s paradoxes against motion are related to the question of existence of the so called *infinity machine*, i.e. a machine which can perform infinitely many operations in a finite time. Thus Placek refers to a mathematician intuitionist Hermann Weyl (1885–1955).[117] Zeno’s paradox is, therefore, a question whether motion is an infinity machine. Placek recalls a 50 year long search for other infinity machines beside motion.[118] For instance, a ring was invented, such that it rang infinitely many times in a minute, or a bulb, which blinked in a similar manner, or a machine, which I finite time printed full, sc. infinite, decimal expansion of π and many other similar automata.[119] What is the problem is that the machines cannot be built, since their operation requires law of physics that are different from the ones we know or actually infinite physical magnitudes.[120] Moreover, construction of those machines may also contradict the laws of logic.[121]

---

114 Ajdukiewicz [66], p. 93.

115 Aristotle, *Physics*, 232b–234a.

116 Placek [67, 68].

117 Weyl [69].

118 As a matter of fact, similar questions were analyzed already in the Middle Ages. Kilvington considers the possibility of cutting a cone in infinitely many parts. We cut a cone of a finite height h with a plane parallel to the base of the cone at the half of its height. A new cone of the height $h/2$ is similarly cut at the half of its height, and so on infinitely. Kilvington wonders whether it is possible to go on cutting the successively formed cones infinitely. He comes to a conclusion that for technical reasons the cutting will be no longer possible at some point. He observes, nevertheless, that theoretically it should be possible to perform infinitely many cuttings in finite time, because every successive cutting requires less and less time. Podkoński [70].

119 Placek [68], p. 68; Cf. also Grünbaum [71], p. 78.

120 Placek [68], p. 68.

121 An interesting problem is considered by Priest [72], p. 1–2. He calls it *Bernadete Paradox*, from the name of the author of this construction, Bernadete [73], p. 259. In fact, the paradox is an example of an infinity machine, which suspends the laws of logic. Let us assume that a person A moves successively along a line from $-\infty$, through 0, to $+\infty$. However, her every move beyond the point 0 causes that in a double distance from the point 0, there appears an obstacle which A cannot overcome. Thus, if A reached the point x, an obstacle appeared in the point $2x$. Priest proposes a following formalization of this construction: (1) $(Rx \wedge y < x) \rightarrow Ry$, (2) $(By \wedge y < x) \rightarrow \neg Rx$, (3) $\neg\exists x(x < y \wedge Bx) \rightarrow Ry$, (4) $x \leq 0 \rightarrow \neg Bx$, (5) $x > 0 \rightarrow (Bx \leftrightarrow Rx/2)$; where Rx and Bx mean respectively: “the person A reached the point x” and “an obstacle appeared in the point x”. He also notes that the set of premises is non-contradictory $\{1,2,3\}$. To be true, the set $\{1,2,3,4\}$ is non-contradictory too. But a contradiction does follow already from the set $\{1,2,3,4,5\}$: the person A cannot go beyond the point 0 by any non-zero magnitude a (obviously, the point a is identified here with its distance from 0, which is a) because if she did

One can suppose then that motion is the only really existing thing that can pass for an infinity machine.[122] It seems, however, that modern physics does not allow to see motion as an infinity machine. Quantum physics clearly suggests that time and motion are not infinitely divisible. It becomes manifest, then, that Zeno's Dichotomy Paradox concerns the argument between atomists and anti-atomists, which lies outside the scope of this work.

Achilles and the tortoise. Aristotle notes that the Achilles and the Tortoise Paradox is a variant of the dichotomy[123,124]:

Achilles and the Tortoise

The second paradox, the so called "Achilles", can be summarized as follows: the best runner can never overtake the worst one, because the former first has to reach the point, which was left by the latter, who had started a little earlier. In fact, it is the same argument as in the previous paradox, with the only difference lying in the fact that successive sections of the way are not divided into halves as in the Dichotomy. The conclusion of the reasoning states that the slower runner cannot be overtaken and is similar to the one from Dichotomy: in both cases dividing space leads to a conclusion that the end has not been reached, but "Achilles" goes further—it states that even the fastest runner will be vanquished by the slowest one.

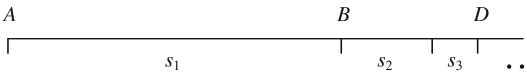
From the mathematical point of view, this is another instance of an infinite sum of a convergent geometric sequence. Let V_1 and V_2 be the speeds of Achilles and the tortoise, respectively. Before Achilles reaches the tortoise, he first has to get to the point B. Achilles covers the path of the length $AB = s_1$ in the time t_1. In the same time, however, the tortoise reaches the point C, covering a non-zero distance $BC = s_2$. This way, after the time t_1, Achilles is in the point B, and the tortoise, in the point C. This sequence is repeated in infinity generating an infinite sequence of sections s_n, with $n \in \{1,2,3,\ldots\}$.

(Footnote 121 continued)
it, she would have to cover the distance of, say $a/4$, much earlier; but that would mean that in the point $a/2$ an obstacle appeared, which A would not be able to overcome; consequently, A would not be able to reach the point a. If so, then the person A cannot go beyond the point 0, which means that no obstacle will appear—contradiction. Naturally, the infinity machine is defined by the premise (5). One can clearly see that Bernadete Paradox is inspired by Zeno's Dichotomy. It attempts to define the infinity machine taking the example from motion. There is, however an important difference between the two paradoxes: Bernadete's machine is contradictory, motion is not.

122 Placek [68], p. 68.

123 Achilles and the Tortoise Argument is commonly ascribed to Zeno of Elea. Diogenes Laertios believes Zeno to be the author of the dilemma too, but adds that according to the opinion of Favorinus, presented in his *Historiae diversae*, it was Parmenides who discovered the paradox, Diogenes Laertios, *op.cit.*, IX.5, 28, p. 532.

124 Aristotle *Physics*, VI, 239b10-33, 405.



This is a geometric sequence with the first element $a_1 = s_1$ and the quotient $q = V_2/V_1$. Consequently, for $n \geq 1, s_n = s_1 \cdot (V_2/V_1)^{n-1}$. Naturally, $|q| < 1$, since from the assumption $V_2 < V_1$. Therefore, the sequence $\{s_n\}_{n \in \{1,2,3,\ldots\}}$ is a convergent sequence, whose sum of all words is defined by the formula $\sum = a_1/(1-q)$. This means that Achilles will catch on with the tortoise covering a finite distance s:

$$
\begin{aligned}
s &= s_1 + s_2 + s_3 + \cdots + s_n + \cdots \\
&= s_1 + s_1 \cdot (V_2/V_1) + s_1 \cdot (V_2/V_1)^2 + \cdots + s_1 \cdot (V_2/V_1)^{n-1} + \cdots \\
&= s_1 \cdot \left(1 + (V_2/V_1) + (V_2/V_1)^2 + \cdots + (V_2/V_1)^{n-1} + \cdots\right) \\
&= s_1 \cdot V_1/(V_1 - V_2) < \infty.
\end{aligned}
$$

As in the case of the Dichotomy, the mathematical solution of Achilles and the Tortoise Paradox is illusory. The remarks concerning the real problem of the dichotomy remain in force also in his case. The difference between the two paradoxes lies in the fact that in the former Zeno tries to show the impossibility of absolute motion, i.e. one without a point of reference, while in the latter, the impossibility of motion in relation to another moving object.

The arrow. The Arrow Paradox joins simplicity of form with original content[125]:

The Arrow

The third is … that the flying arrow is at rest, which result follows from the assumption that time is composed of moments … he says that if everything when it occupies an equal space is at rest, and if that which is in locomotion is always in a now, the flying arrow is therefore motionless.

A correct solution of this problem was given already by Aristotle[126]: "Zeno's reasoning, however, is fallacious, when he says that if everything when it occupies an equal space is at rest, and if that which is in locomotion is always in a now, the flying arrow is therefore motionless. This is false; for time is not composed of indivisible nows any more than any other magnitude is composed of indivisibles. […] if this assumption is not granted, the conclusion will not follow." Indivisible now is a dimensionless point (as in Euclidean geometry)[127]; it is indivisible because

125 Ibidem.

126 Ibidem.

127 Euclid's *Elements* begin with a statement that "We call a point something that has no parts".

it has no dimensions. Aristotle clearly states that not only time but also no other quantity can be composed of indivisible points. Such a point is nothing and as nothing it cannot be material for something that is something. Therefore, if a non-zero section of time, which is something, is composed of points, which are not nothing, as they are sections of *infinitesimal* duration,[128] the arrow does not rest in such points, and the paradox disappears. It should be noted that in a dimensionless time point, an arrow does not rest either, because it does not exist: there is no reality in a zero section of time, only nothingness.[129] Time moments cannot be points, for a dimensionless point is a Dedekind's section, i.e. such a division of a line that does not waste the line, i.e. whatever was in the line before the section is on one or the side of the division after the section. How can a non-zero section of time or any non-zero line be composed of its own Dedekin's section? This would be absurd.

Russell observes dryly that points do not belong to objects of sensory perception and things which we encounter in everyday life do not seem to be composed of them. He can see the difficulty of imagining a mental experiment, which could prove that points are material for the space around us. For this reason it seems reasonable to assume that there are no points in this space.[130] Most of all, time cannot be composed of dimensionless time points. Russell does not limit himself to criticism but gives an outline of a "corrective" research program insisting that no one has ever seen a point or touched it. If there are points in experiential space, they must be a result of reasoning.[131] Russell points to Alfred North Whitehead (1861–1947) as the author of such a logical deduction of dimensionless points from that which is dimensional and to Jean Nicod (1893–1924) as the creator of various types of geometry[132], constructible of the raw materials provided by our perceptions.[133] In two works: *Principles of Natural Knowledge*[134] and *Concept of Nature*[135], Whitehead proposed a pointless funding of geometry. It is based on an assumption that its building materials are dimensional solids (so called regions), whose respective sequences are convergent to limits, which are dimensionless Euclidean points. It is them, however, that are the

---

[128] A non-zero number x is an *infinitesimal number* if and only if its absolute value is greater than zero and smaller than any positive real number. Infinitesimal numbers belong to the so called *hiperreal numbers*. The author of the theory of hiperreal numbers is John Conway. He had, however, many forerunners: Archimedes, Isaac Newton, Gottfried Leibniz, Leonhard Euler, Augustin Louis Cauchy. Conway [74], Conway and Guy [75], Robinson [76].

[129] This claim becomes even more evident, when we look at the example of a shutter in film photo cameras. The opening of the shutter decides about the length of exposition time of the film, which is equivalent to the length of "looking" at the photographed object by the camera. The time is zero if the shutter is completely closed. But then nothing can be seen, not even stillness.

[130] Russell [77].

[131] Russell [77], p. 122.

[132] Nicod [78].

[133] Russell [77], pp. 122–123.

[134] Whitehead [79].

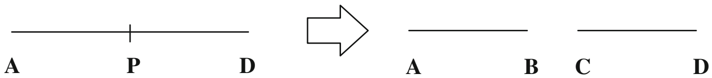
[135] Whitehead [80].

raw material of geometry, and dimensionless points are merely abstracts. Whitehead popularized this idea in his *Process and reality*.[136] It found continuation in the works of Andrzej Grzegorczyk (b. 1922)[137] and Bowman L. Clarke.[138] Another construction of pointless geometries was proposed by Alfred Tarski (1901–1983) in *Les fondements de la géometrié de corps* from 1929, which is a printed version of a lecture presented at the first Polish Mathematical Congress.[139]

One can risk saying that the idea of constructing pointless geometries is an attempt at changing the topsy turvy situation in geometry. Among the scholars who realized what sort of troubles are caused by points were not only Russell, Whitehead, Nicod, Tarski but also Kurt Gödel (1906–1978), Charles Sanders Peirce (1839–1914) and Hilary Putnam (b. 1926). An interesting interpretation of Peirce's mathematical ideas was given by Putnam in *Peirce's Continuum*.[140] In that work, Putnam shows paradoxical consequences of a number of theses *de rigeur* in mathematics. Among others, he presents a paradox which is a consequence of accepting dimensionless points as substrate of a line:

Peirce-Putnam Paradox[141]

Suppose we divide the line interval at a point P and then separate the two broken halves by moving the right half a short distance to the right.



Some obvious questions appear: What happened to the point P? Did it become the point B, or perhaps point C? Maybe, P is neither of them, or both? Still, points B and C exist. No one can say that the point P is none of them, just as no one can claim that it is only one of them. The only solution is to accept that the point P has "become" the two points B,C. Consequently, P cannot be a dimensionless object.

Putnam adds also that although the above reasoning is in agreement with the opinions of Aristotle and Peirce, the latter goes a little further, since he believes that not only line interval but also a point can be "separated" from line in a similar way[142]:

136 Whitehead [65].

137 Grzegorczyk [81]; cf. also Biacino and Gerla [82], and Gorzka [83].

138 Clarke [84, 85].

139 Tarski [86].

140 Putnam [87], pp. 5–6.

141 This name is not used in the literature of the subject but has been proposed by the author of this study.

142 Putnam [87], p. 4.

It is clear that substituting the point understood as a distance of infinitesimal length for the dimensionless point solves not only the "riddle" of separating point from line but also the Peirce-Putnam problem.

5.3 Diagnosis of Paradoxes of Vagueness and Change

The characteristics of causes for troubles resulting from paradoxes of vagueness, change and motion presented in this paragraph is common for all three groups of paradoxes. Moreover, it is convergent with the opinion of Henri Bergson (1859–1941), presented in his *L'évolution créatrice*, and Ludwig Wittgenstein (1889–1951), expressed in the *Blue* and *Brown Books*, as well as in his *Philosophical Investigations*.

The common cause of occurrence of paradoxes of vagueness, change and motion seems to be a specific character of natural language, which is bound to reify everything it predicates about, so also, and especially, that which is not a thing. Another common cause is its poverty and schematic character. The former is responsible or the language's tendency to "freeze" everything that is dynamic, the latter, for ignoring the richness of reality. The consequence of this state of affairs is mathematization of thinking about reality and taking up mathematical (precise) terms, such as set, point, identity, difference, contradiction and non-contradiction. A natural language becomes a "reflection" of such thinking and creates a possibility of reaching preciseness in expressing that which *de facto* is not precise.[143] Language and mathematical concepts/structures cause that the dynamic and elusive reality becomes closed in a static and precise picture. What is an ongoing process becomes something we have reified into a being with constant, precise and immutable properties in certain time sections.[144] Naturally, we want these properties to be described in a precise way. The picture we get "freezes" the world in one, dimensionless time point. This way, we create fiction, which, however, passes

---

[143] The supposed preciseness of philosophical thinking often assumes a naive, and sometimes a ridiculous form. This great and diverse problem can be richly illustrated by various casus. Let us remind just a few. 1. Using the term "natural species", which is natural only because we look at it: other phases of development of the species are transitional phases. 2. Continual failure in defining man. 3. Ideas such as *physicalism*: a "precise", physical description of an emotional state of a man, e.g. disappointment, hope or anger, enumerates a sequence of properties, such as flexing "appropriate" muscles, but finishes up with the magical expression "etc.".

[144] Bergson stated straightforwardly that there are changes but there are no objects which change "under" those changes; change does not require a substrate. There are motions but there is no motionless, unchanging object that could be in motion: motion does not imply something which moves.

a practical exam both in life and in applied science. Nevertheless, immobile and precise models created for practical (technological) purposes cannot be taken for the reality itself. A simple confrontation of such a model with the existing world, performed with help of logical arguments analyzed in this chapter, leads to paradoxes, i.e. to surprising contradictions.

The Case of a Photographer

Fascinated by the beauty of an unusual cloud moving across the sky, a photographer took a picture of it. After a while, the cloud changed a little both its shape and colors. The photographer took another picture. Because the cloud was constantly changing, the photographer found it necessary to take still more photos. When he developed them later, he observed with wonder that he had not known they were the pictures of one and the same cloud, he would have believed they showed different clouds.
When he showed the series of photos to his philosopher friends, they started a long argument about the identity of the cloud shown there. The photographer was bewildered to hear them present all sort of conclusions concerning the understanding of what the photographed cloud is. Someone said the cloud he had photographed was unique and specially privileged of a million other clouds, which, after all, were one cloud. Someone else argued that they are certainly different clouds, since each of them differed from other ones in an essential way by their properties, such as shape or color. Still another person claimed that it was the case of a manifest contradiction: something was and at the same time was not something, so reality was a contradiction came true. There was even someone, who insisted that only the present moment exists and therefore there is no past and no future, perhaps forgetting that there should be something to join all those moments it took him to utter his valuable comment. And only the photographer, who was listening to the discussion, realized with surprise that his friends were not discussing about the cloud but about the images of it, which he had recorded on the photos. Apparently, they must have identified the images with the real cloud. He also noticed that the discussion, heated and noisy as it was, lacked an opinion stating that the photos showed different phases of one and the same process of changes that was the cloud, which all of the disputants had reified without any justification by projecting onto a sequence a row of successive time points. For each of his friends, the cloud was apparently a thing represented on his photos.[145]

In reality, everything that is, is mutable. The cloud, dynamic in every respect, is thus a model of everything that exists in reality.[146] By observing reality we "freeze" it in immobile linguistic images. These still, time point images are then

[145] Cf. Russell's comments on a fading, sc. changing color, wallpaper. Russell [77].

[146] After all, the different speed of change for a cloud and a stone cannot testify to the difference between the two objects *qua* beings.

the basis for ostensive definitions of that which they present. Successive pseudo-logical analysis is to replace those ostensive definitions with other ones. The real problem appears only when we start believing that the images we have are adequate, that they present the truth about reality. This way, the images of reality become for us more real than reality itself. That results in paradoxes of vagueness, change and motion being treated as silly, scientifically insignificant stories, created for amusement. One cannot help here quoting a warning given by Wittgenstein[147] that although philosophers often speak about investigating or analyzing the sense of words, we should remember that a word has no sense that was given to it by some power independent from us, which would make it possible for us to investigate what the word *really* means. A word has a meaning that someone gave to it. It is the creating of senses of words which is the fundament of our mythological thinking about reality and it results from our tendency towards generalization.[148] Wittgenstein points at four human urges, responsible for our striving for what is general.[149] The first of them is the tendency to look for something that is common to all beings, which we subsume together under one general term. The belief that a universal concept is a common property of its particular instances is related to other primitive, oversimplified ideas about the structure of the language. The second tendency is related to the impossibility of complete definitions of non-mathematical names of properties and objects.[150] The third tendency is, in Wittgenstein's opinion, a result of confusion concerning psychical state as some hypothetical psychical mechanism and as a state of consciousness. The fourth tendency should be most embarrassing, for it is our approval for identifying the poor, primitive by its nature, image of reality with the superabundant reality itself.[151]

Wishing to give a metaphoric form to our diagnosis of causes for paradoxes of vagueness, change and motion, we have invented the following story, based on Plato's metaphor of the cave[152]:

Metaphor of a Cinema

Let us imagine a dark room, in which a film is shown, presenting a running sportsman. The viewers can clearly see the runner in motion. They have no reason to doubt that the motion is real. Yet some of the spectators wanted to investigate the mystery of the phenomenon they observed. Since they excelled

[147] Wittgenstein [88], p. 58.

[148] Wittgenstein [88], p. 43.

[149] Wittgenstein [88], p. 44.

[150] Wittgenstein [88], pp. 44–45.

[151] Wittgenstein [88], p. 45.

[152] This story, rather sarcastic towards numerous conclusions in ontology, takes its idea from a fragment of Bergson's 1907 book, *L'évolution créatrice*, where human mind is compared to a cine-camera; Bergson [89], pp. 257–260.

all others in their inquisitiveness, they soon discovered the projection room and the projector, which was beaming the film onto the screen. This was an unusual discovery: motion appeared to be a mere illusion, since it was a sequence made of a number of pictures, each of which was still. Proud of their discovery, they returned to the audience and announced their surprising news to everyone. Now every spectator knew that motion is secondary to stillness. After all, the basis for description of a dynamic picture of a running sportsman is a series of frames. And every frame is perfectly still. This stillness gives a possibility of a precise, mathematical description of the picture. The analyses they made were exceptionally precise, since they could be made in the projection room. The inquisitive gentlemen did not need the screen, it was for the *hoi polloi*. To be closest to the truth, they stayed in the projection room. They knew that the truth lay in the long bands with thousands of still frames. They have already discovered the true nature of changes on the screen.

Their genius notwithstanding the gentlemen did not notice that the films beamed onto the screen are not reality but only an image of it. Unfortunately, every film looked more clear when the room was darker. In order to keep the standard of good projection, they made sure no outside light penetrated into the cinema. In the darkness, it was difficult to remember about the world outside, resplendent in true sunlight. It was also difficult to find the exit door. What was easy was finding the way to the projection room: you only needed to follow the light beam falling on the screen. Naturally, you had to be very diligent too, in order to analyze ever new frames of ever new film with utmost precision. And yet, just leaving the cinema would make the spectators realize that the films they watch are human creation, that they are merely images of the reality, which is accessible to everyone who dares to leave the cinema. They would discover then that it is not so that motion is secondary to stillness, from which it emerges, but that it is the other way round. It is stillness that is abstracted from motion, which really exists. Stillness as such actually does not exist. The frames they obstinately analyze are only human creations, which have only so much in common with actual motion that they were "prepared" from it. Does skill and diligence in analyzing still pictures enclosed in film frames have any real philosophical or scientific importance? Is not the true world beyond the cinema a real challenge for a philosopher?

The paradoxes of vagueness, change and motion are *de facto* ontological paradoxes, since they result from an erroneous perception of being. Being is identified there with its inadequate image, expressed in natural language. The identification causes the properties of the image of reality to be ascribed to reality itself. The properties result from the nature of concepts invented by man and have nothing in common with the nature of reality. Ontological paradoxes uncover a conflict on the borderline between two realities: that of the world and that of its linguistic representation. What distinguishes ontological paradoxes from all other ones is their inevitable character. If one can at least try to solve the paradoxes occurring within language by correcting the language and our thinking based on it, with ontological paradoxes one must face the challenge of a general overhaul of language. Is that feasible? This is hard to tell.[153] But treating language as an adequate tool for reflecting reality is a misunderstanding. Such an approach may have a practical justification but will never have a philosophical one. After all, the etymology of the word "philosophy" points to the truth as its essence. From such a point of view, the attempts at solving ontological paradoxes are de facto attempts to prove that: 1. The richness of language is on par with the richness of reality; 2. Concepts, which are static by their nature, are equivalent to dynamic phenomena. For obvious reasons, analyzed here, neither the first nor the second claim cannot be defended.

References

1. Sorensen, R. A. (1990). Vagueness implies cognitivism. *The American Philosophical Quarterly, 27/1*. Retrieved from http://www.dartmouth.edu/~rasoren/papers/vaguenessimpliescog.pdf
2. Dummett, M. A. E. (1975). Wang's paradox. *Synthese,30*, 301–324.
3. Russell, B. (1923). Vagueness. *The Australasian Journal of Psychology and Philosophy, 1*, 84–92. Retrieved from http://cscs.umich.edu/~crshalizi/Russell/vagueness/
4. Black, M. (1937). Vagueness: An exercise in logical analysis. *Philosophy of Science, 4*, 427–455, also in: *Language and philosophy. studies in method* (pp. 23–58). Ithaca: Cornell University Press, 1949.
5. Williamson, T. (1994). *Vagueness*. London: Routledge.
6. Kubiński, T. (1958). Nazwy nieostre [Vague names]. *Studia Logica, VII*, 115–179.
7. Sainsbury, R. M. (1988). *Paradoxes* (p. 1991). Cambridge: Cambridge University Press.
8. Unger, P. (1979). I do not exist. In G. F. MacDonald (Ed.), *Perception and identity* (pp. 235–251). London: Macmillan.
9. Sorensen, R. A. (1985). An argument for the vagueness of "vague". *Analysis,27*, 134–137.
10. Sorensen, R. A. (SEPh). Vagueness. In E. N. Zalta (Ed.) *The stanford encyclopedia of philosophy* (Fall 2003 Edition). The Metaphysics Research Lab., Center for the Study of Language and Information, Stanford University. Retrieved from http://plato.stanford.edu/contents.html
11. Varzi, A.C. (2003). Vagueness. In L. Nadel (Ed.), *Encyclopedia of cognitive science* (pp. 459–464). London: Macmillan and Nature Publishing Group. Retrieved from http://www.columbia.edu/~av72/papers/ECS_2002.pdf
12. Tye, M. (1994). Why the vague need not be higher-order vague. *Mind,103*, 43–45.
13. Hyde, D. (1994). Why higher-order vagueness is a pseudo-problem. *Mind,103*, 35–41.
14. Hyde, D. (2003). Higher-orders of vagueness reinstated. *Mind,112*(446), 301–305.

[153] Bergson tries to show the way in which the paradox can be avoided. In his opinion, our mind can go another way. It can place itself in a mobile reality, assimilate to its ever changing direction and finally comprehend it intuitively. To do so, it must force itself to reverse the direction of the actions which constitute its thinking, it must constantly change, or rather alter, all of its categories. But in this way, it will finally reach the fluid concepts, which can follow reality in all of its tides and ebbs and take on motion of the very inner life of things. Only in this way one can create a philosophy, which will be freed from disputes between schools and capable of solving all problems in a natural way because it will liberate itself from the shackles of artificial terms that were selected in the times those problems were posed.

15. Varzi, A. C. (2003). Higher-order vagueness and the vagueness of "vague". *Mind,112*, 295–299.
16. Lewis, D. K. (1986). *On the plurality of worlds*. Oxford: Blackwell.
17. Williamson, T. (2003). Vagueness in reality. In M. Loux & D. Zimmerman (Eds.), *The Oxford handbook of metaphysics*. Oxford: Oxford University Press. pp. 690–715
18. Pawłowski, T. (1978). *Tworzenie pojęć i definiowanie w naukach humanistycznych* [*Creating concepts and defining in human studies*]. Warszawa: PWN.
19. Tye, M. (1998). Vagueness. In E. Craig (Ed.), *Routledge Encyclopedia of Philosophy* (vol. 9, pp. 563–566). London: Routledge.
20. Tye, M. (1990). Vague objects. *Mind,99*, 535–557.
21. Broome, J. (1984). Indefiniteness in identity. *Analysis,44*, 6–12.
22. Chibeni, S.S. Ontic vagueness in microphysics. Retrieved from http://www.sorites.org/Issue_15/chibeni.htm
23. Sorensen, R. A. (1998). Sharp boundaries for blobs. *Philosophical Studies,91*, 275–295.
24. Fine, K. (1975). Vagueness, truth and logic. *Synthese,30*, 265–300.
25. van Fraassen, B.C. (1966). Singular terms, truth-value gaps and free logic. *Journal of Philosophy,63*, 481–495.
26. van Fraassen, B. C. (1968). Presupposition, implication and self-reference. *Journal of Philosophy,65*, 136–152.
27. van Fraassen, B. C. (1972). *Formal semantics and logic*. London: Macmillan.
28. Priest, G., & Tanaka, K. (SEPh). Paraconsistent logic. In E. N. Zalta (Ed.), *The stanford encyclopedia of philosophy*. The Metaphysics Research Lab., Center for the Study of Language and Information, Stanford University. Retrieved from http://plato.stanford.edu/contents.html
29. Priest, G. (1993). Can contradictions be true? II. *Proceedings of the Aristotelian Society, Supplementary,67*, 35–54.
30. Priest, G. (2000). *Logic, a very short introduction*. Oxford: Oxford University Press.
31. Priest, G., & Berto, F. (SEPh). Dialetheism. In E. N. Zalta (Ed.), *The stanford encyclopedia of philosophy*. The Metaphysics Research Lab., Center for the Study of Language and Information, Stanford University. Retrieved from http://plato.stanford.edu/contents.html
32. Priest, G. (1987). *In contradiction: a study of the transconsistent*. Dordrecht: Martinus Nijhoff.
33. Hyde, D. (1997). From heaps and gaps to heaps of gluts. *Mind,106*(424), 641–660.
34. Jaśkowski, S. (1948). Rachunek zdań dla systemów dedukcyjnych sprzecznych [Sentential calculus for inconsistent deductive systems]. *Studia Societatis Scientiarum Torunensis, Toruń-Polonia,I*(5), 57–77.
35. Jaśkowski, S. (1969). Propositional calculus for contradictory deductive systems. *Studia Logica,24*, 143–157.
36. Carnap, R. (1936). Testability and meaning. *Philosophy and Science, III/IV*, 47–92.
37. Przełęcki, M. (1958). W sprawie terminów nieostrych [A propos vague terms]. *Filozofia Nauki,1*, 2–3.
38. Bochwar, D. A. (1938). Ob odnom trehznacnom iscislenii i ego primenenii k analizu paradoksov klassiceskovo rassirennovo funkcjonalnovo iscislenia. *Matematiceskij Sbornik, 4*, 287–308.
39. Kleene, S. C. (1952). *Introduction to metamathematics*. Amsterdam: North-Holland.
40. Halldén, S. (1949). *The logic of nonsense*. Uppsala: Uppsala Universitets Arsskrift.
41. Körner, S. (1955). *Conceptual thinking*. Cambridge: Cambridge University Press.
42. Körner, S. (1959). On determinables and resemblance. *Aristotelian Society, Supplementary,33*, 125–140.
43. Körner, S. (1960). *The philosophy of mathematics*. London: Hutchinson.
44. Körner, S. (1968). Reply to Mr Kumar. *British Journal for the Philosophy of Science,18*, 323–324.
45. Ajdukiewicz, K. (1948). Zmiana i sprzeczność [Change and Inconsistency]. In Język i Poznanie (vol. 2, pp. 90–106). Warszawa: PWN 1985.

46. Williamson, T. (1992). Inexact knowledge. *Mind,101*, 217–242.
47. Williamson, T. (1992). Vagueness and ignorance. *Proceedings of the Aristotelian Society, Supplementary,66*, 163–177.
48. Wright, C. (1994). The epistemic conception of vagueness. In T. Horgan (Ed.), *Vagueness.The Southern Journal of Philosophy*, pp. 133–160.
49. Sorensen, R.A. (1988). *Blindspots* (p. 249). Oxford University Press: Oxford.
50. Zadeh, L. A. (1965). Fuzzy sets. *Information and Control,8*, 338–353.
51. Unger, P. (1979). There are no ordinary things. *Synthese,41*, 117–154.
52. Unger, P. (1979). Why there are no people. *Midwest Studies in Philosophy, 4*: Studies in Metaphysics, 177–222.
53. Unger, P. (2004). The mental problems of the many. In D. Zimmerman (Ed.), *Oxford Studies in Metaphysics* (vol. 1, p. 27). Retrieved from http://www.nyu.edu/gsas/dept/philo/faculty/unger/papers/mentalproblem.pdf
54. Unger, P. (1980). The problem of the many. *Midwest Studies in Philosophy, 5*: Studies in Metaphysics, 411–467.
55. Geach, P. T. (1980). *Reference and generality*. Ithaca: Cornell University Press.
56. Lewis, D. K. (1993). Many, but almost one. In J. Bacon (Ed.), *Ontology, causality and mind: Essays in honour of D.M. Armstrong*. New York: Cambridge University Press.
57. Weatherson, B. (SEPh). The problem of the many. In E. N. Zalta (Ed.), *The stanford encyclopedia of philosophy*. The Metaphysics Research Lab., Center for the Study of Language and Information, Stanford University. Retrieved from http://plato.stanford.edu/contents.html
58. Deutsch, H. (SEPh). Relative identity. In E. N. Zalta (Ed.), *The stanford encyclopedia of philosophy*. The Metaphysics Research Lab., Center for the Study of Language and Information, Stanford University. Retrieved from http://plato.stanford.edu/contents.html
59. Kripke, S. (1972). *Nazywanie a konieczność* [*Naming and necessity*] (transl. B. Chwedeńczuk). Warszawa: Instytut Wydawniczy Pax, 1988.
60. Nozick, R. (1982). *Philosophical explanations*. Cambridge: Harvard University Press.
61. Parfit, D. (1984). *Reasons and persons*. Oxford: Oxford University Press.
62. Wiggins, D. (1967). *Identity and spatio-temporal continuity*. Oxford: Blackwell.
63. Chisholm, R. M. (1973). Parts as essential to their wholes. *Review of Metaphysics,26*, 581–603.
64. Church, A. (1982). A remark concerning Quine's paradox about modality, Analisis Filosophico 2 [in Spanish] [in English]. In N. Salmon and S. Soames (Eds.), *Propositions and attitudes*. Oxford: Oxford University Press, 1988.
65. Whitehead, A. N. (1929). *Process and reality.* Cambridge: Cambridge University Press.
66. Ajdukiewicz, K. (1931). Paradoksy starożytnych [Paradoxes of ancients]. In *Język i Poznanie* (vol. 1, pp. 135–144). Warszawa: PWN, 1985.
67. Placek, T. (1989). Paradoksy ruchu Zenona z Elei a problem continuum [Zeno's paradoxes of motion and the problem of continuum], dychotomia. *Studia Filozoficzne,4*(281), 57–73.
68. Placek, T. (1997). Paradoksy ruchu Zenona z Elei a labirynt kontinuum: 'Achilles i żółw', 'Strzała', 'Stadion' [Zeno's paradoxes of motion and the Labyrinth of continuum: 'Achilles and the tortoise', 'arrow', 'stadium']. *Filozofia Nauki,1*(17), 65–77.
69. Weyl, H. (1963). *Philosophy of mathematics and natural science* (p. 42). New York: Atheneum.
70. Podkoński, R. (2004). Al-Ghazali's "Metaphysics" as a source of anti-atomistic proofs in John Duns Scotus "Sentences' commentary".
71. Grünbaum, A. (1967). *Modern science and Zeno's paradoxes*. London: Allen and Unwin.
72. Priest, G. (1999). On a version of one of Zeno's paradoxes. *Analysis,59/1*(261), 1–2.
73. Bernadete, J. (1964). *Infinity: An essay in metaphysics*. Oxford: Clarendon Press.
74. Conway, J. H. (1976). *On numbers and games*. New York: Academic Press.
75. Conway, J. H., & Guy, R. K. (1996). *The book of numbers*. New York: Springer.
76. Robinson, A. (1996). *Nonstandard analysis*. Princeton: Princeton University Press.

77. Russell, B. (1914). *Our knowledge of the external world as a field for scientific method in philosophy*. Routledge, 2002.
78. Nicod, J. (1923). La géométrie dans le monde sensible. Thèse Univ. de Paris, Paris.
79. Whitehead, N. A. (1919). *An inquiry concerning the principles of natural knowledge*. Kessinger Pub.Co., 2008.
80. Whitehead, N. A. (1920). *The concept of nature*. Prometheus Books, 2010.
81. Grzegorczyk, A. (1960). *Logika popularna*[*Popular logic*] . Warszawa: PWN.
82. Biacino, L., & Gerla, G. (1996). Connection structures: Grzegorczyk's and Whitehead's definitions of point. *Notre Dame Journal of Formal Logic,37*(3), 431–439.
83. Gorzka, C. (2003). *Mereologia a topologia i geometria bezpunktowa*[*Mereology in comparison to topology and pointless geometry*]. Toruń: Wydawnictwo Uniwersytetu Mikołaja Kopernika.
84. Clarke, B. L. (1981). A calculus of individuals based on 'connection'. *Notre Dame Journal of Formal Logic, 22*(3), 204–217
85. Clarke, B. L. (1985). Individuals and points. *Notre Dame Journal of Formal Logic, 26*, 61–67
86. Tarski, A. (1929). Les fondements de la géométrie des corps. In *Księga pamiątkowa I Polskiego Zjazdu Matematycznego*. Kraków.
87. Putnam, H. (1995). Peirce's continuum. In K. L. Ketner (Ed.), *Perice and Contemporary Thought* (pp. 1–22). New York: Fordham University Press.
88. Wittgenstein, L. (1934/1935). *Niebieski i brązowy zeszyt* [*The blue and brown books. preliminary studies for the "philosophical investigations"*] (pp. 19–123, pp. 125–274), *Szkice do Dociekań filozoficznych* (transl. Adam Lipszyc and Łukasz Sommer). Warszawa: Wydawnictwo Spacja, 1998.
89. Bergson, H. (1913). *Ewolucya Twórcza* [*L'évolution créatrice*] (transl. F. Znaniecki). Wydawnictwo Towarzystwa Literatów i Dziennikarzy Polskich w Warszawie, Warszawa, 1913.

Names Index

Subject Index