CHAPTER 14

Theory of Small Oscillations and Coupled Oscillators

14.1 INTRODUCTION

In Chapters 3 and 4, we discussed the oscillatory motion of undamped, damped, and forced oscillators. As in the previous chapters where we extended the motion of single particles to the motion of rigid bodies, we now investigate the oscillatory motion of system of particles. One of the most deeply investigated concepts in modern physics is that of oscillatory motion of atoms in the field of molecular physics and solids in the field of solid state physics.

We will describe the oscillatory motion of many coupled oscillators in terms of normal coordinates and normal frequencies. Theory of small oscillations in analyzing coupled oscillatory motion uses methods of Lagrange's equations together with matrix tensor formulation. We will close the chapter with the discussion of vibrations and beats in the vibrating systems. Also, we will briefly touch the topic of dissipative systems under forced oscillators.

14.2 EQUILIBRIUM AND POTENTIAL ENERGY

To understand the general theory of vibrations, it is essential to know the relation between potential energy and equilibrium that leads to the conditions of stable or unstable equilibrium of a given system. To start, let us consider a system with n degrees of freedom, and let its configuration be specified by the generalized coordinative: $q_1, q_2, \ldots, q_n$. Furthermore, let us assume that the system is conservative; hence the potential energy V is a function of the generalized coordinates; that is,

$$V = V(q_1, q_2, \ldots, q_n) \tag{14.1}$$

The generalized forces Q_k are given by

$$Q_k = -\frac{\partial V}{\partial q_k}, \qquad k = 1, 2, \ldots, n \tag{14.2}$$

If the system is in such a configuration that it is in equilibrium, it implies that all the generalized forces Q_k must be zero. Thus the condition for an equilibrium configuration is

$$Q_k = -\frac{\partial V}{\partial q_k} = 0 \tag{14.3}$$

The system will remain at rest in this configuration if no external force is applied. Now let us displace this system slightly from its equilibrium configuration. After displacement, the system may or may not return to its equilibrium configuration. If after a small displacement the system does return to its original equilibrium configuration, the system is said to be in a *stable equilibrium.* If the system does not return to its equilibrium configuration, it is in an *unstable equilibrium.* On the other hand, if the system is displaced and it has no tendency to move toward or away from the equilibrium configuration, the system is said to be in *neutral equilibrium.*

We are interested in finding a relation between the potential energy function V and the stability of the system. Suppose, when the system is in an equilibrium configuration, it has kinetic energy T_0 and potential energy V_0. Now the system is given a small displacement (by a small impulsive force), and at any subsequent time the system has kinetic energy T and potential energy V. Since total energy is conserved, we may write

$$T_0 + V_0 = T + V$$

$$T - T_0 = -(V - V_0) \tag{14.4}$$

Let us assume an arbitrary form of a potential function V versus q, as shown in Fig. 14.1. The points A and B, where $\partial V/\partial q$ is zero, are equilibrium points. Let us consider the nature of stability at these points.

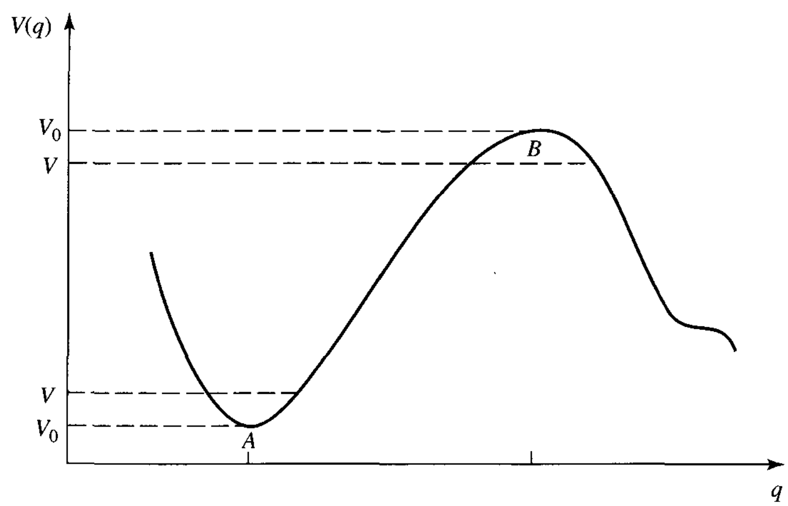


Figure 14.1 Arbitrary form of a potential function V versus q.

Suppose initially the system is in equilibrium corresponding to the configuration at B where the potential energy V_0 is maximum. Any displacement from this equilibrium will lead to a potential energy V that is less than V_0. Thus $V - V_0$ is negative, and from Eq. (14.4) $T - T_0$ will be positive; that is, T increases. Since T increases with displacement, the system never returns to the equilibrium point B; hence B is a position of *unstable equilibrium.* Now let us consider point A, where the equilibrium potential V_0 is minimum. If the system is displaced slightly, the potential energy V_0 increases to V; hence $V - V_0$ is positive. From Eq. (14.4), $T - T_0$ will be negative; hence T decreases with displacement. Since T cannot be negative, it will decrease till it becomes zero at some limiting configuration near the equilibrium configuration; the system will start coming back to an equilibrium configuration. Thus the system is in stable equilibrium. We conclude that for small displacements the condition for *stable equilibrium* is that the potential energy V_0 be minimum at the equilibrium configuration. Furthermore, at equilibrium dV/dq is zero, $V - V_0$ being positive means that d^2V/dq^2 is positive at stable equilibrium. At a position of unstable equilibrium, d^2V/dq^2 will be negative because $V - V_0$ is negative.

Applying the preceding discussion to a system with one degree of freedom, we may write

$$V = V(q) \tag{14.5}$$

and at an equilibrium configuration

$$F = -\frac{dV}{dq} = 0 \tag{14.6}$$

The stability condition may be written as

$$\text{Stable equilibrium: } V_0 \text{ is minimum} \qquad \frac{d^2V}{dq^2} > 0 \tag{14.7}$$

$$\text{Unstable equilibrium: } V_0 \text{ is maximum} \qquad \frac{d^2V}{dq^2} < 0 \tag{14.8}$$

For $d^2V/dq^2 = 0$, we must examine the higher-order derivatives. If the first nonvanishing derivative is odd, the system must be in unstable equilibrium. If, on the other hand, the nonvanishing derivative is of an even order, then the system may be in a stable or unstable equilibrium depending on the value of the derivative (whether it is greater than zero or less than zero).

$$\text{If } \frac{d^nV}{dq^n} \neq 0, \quad n > 2 \text{ and odd} \qquad \text{system is unstable} \tag{14.9}$$

$$\text{If } \frac{d^nV}{dq^n} > 0, \quad n > 2 \text{ and even} \qquad \text{system is stable} \tag{14.10}$$

$$\text{If } \frac{d^nV}{dq^n} < 0, \quad n > 2 \text{ and even} \qquad \text{system is unstable} \tag{14.11}$$

A more general case of this situation will be discussed shortly.

Example 14.1

Show that a bat of length l suspended from point O with a center of mass at a distance d from O is in a stable equilibrium position as in (a) and an unstable equilibrium position as in (b).

Solution

The situation is as shown in Fig. Ex. 14.1. When the bat is displaced, the line OC makes an angle θ with the vertical as in Fig. Ex. 14.1(a). The center of mass is raised a distance h and the potential energy is given by

Potential energy when the bat is displaced.

(a)

$$V = m \cdot g \cdot d \cdot (1 - \cos(\theta))$$

$$\frac{dV}{d\theta} = \frac{d}{d\theta}(m \cdot g \cdot d \cdot (1 - \cos(\theta)))$$

$$\frac{dV}{d\theta} = m \cdot g \cdot d \cdot \sin(\theta)$$

$$\theta = 0 \qquad \frac{dV}{d\theta} = 0$$

$$\left(\frac{dV^2}{d\theta^2}\right) = \frac{d}{d\theta} m \cdot g \cdot d \cdot \sin(\theta)$$

(b)

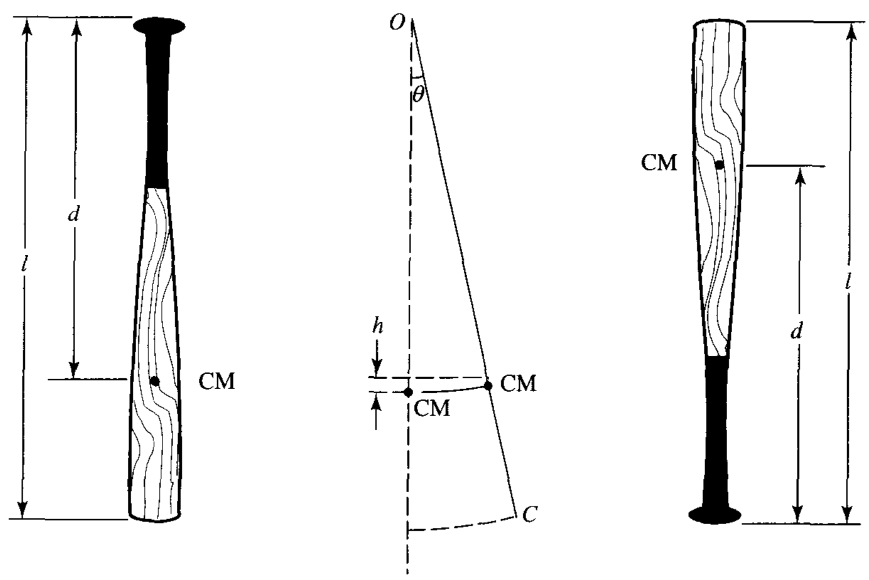
$$V = -m \cdot g \cdot d \cdot (1 - \cos(\theta))$$

$$\frac{dV}{d\theta} = \frac{d}{d\theta}(-m \cdot g \cdot d \cdot (1 - \cos(\theta)))$$

$$\frac{dV}{d\theta} = -m \cdot g \cdot d \cdot \sin(\theta)$$

$$\theta = 0 \qquad \frac{dV}{d\theta} = 0$$

$$\left(\frac{dV^2}{d\theta^2}\right) = \frac{d}{d\theta} - m \cdot g \cdot d \cdot \sin(\theta)$$



(a) (b)

Figure Ex. 14.1

Taking the second derivative and evaluating at $\theta = 0$ reveals that
(a) is in stable equlibrium while
(b) is in unstable equilibrium

$$\frac{dV^2}{d\theta^2} = m \cdot g \cdot d \cdot \cos(\theta) \qquad \frac{dV^2}{d\theta^2} = -m \cdot g \cdot d \cdot \cos(\theta)$$

$$\theta = 0 \quad \frac{dV^2}{d\theta^2} = m \cdot g \cdot d > 0 \qquad \theta = 0 \quad \frac{dV^2}{d\theta^2} = -m \cdot g \cdot d < 0$$

Stable unstable

From our discussion, we may conclude that *if the center of mass lies below the point of suspension, the system will be in stable equilibrium; and if the center of mass lies above the center of suspension, the system will be in unstable equilibrium.*

EXERCISE 14.1 The spherical or cylindrical object shown in Fig. Exer. 14.1 is placed on a plane horizontal surface. The radius of curvature is a, and the center of mass is at a distance d, as shown. Show that the system is in stable equilibrium.

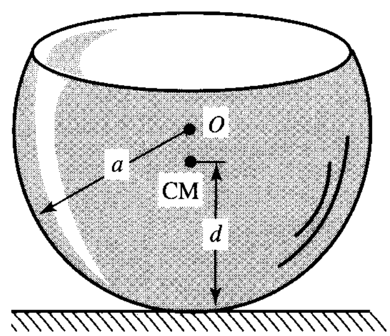


Figure Ex. 14.1

14.3 TWO COUPLED OSCILLATORS AND NORMAL COORDINATES

As a simple example of a coupled system, let us consider two harmonic oscillators coupled together by a spring, as shown in Fig. 14.2. Each harmonic oscillator has a particle of mass m, and the spring constant of one is k_1 and that of the other is k_2. The two are coupled together by another spring of spring constant k'. The motion of the two masses is restricted along the line joining the two masses, say along the X-axis. Thus the system has two degrees of freedom represented by the coordinates x_1 and x_2. The configuration of the sytem is represented by the displacements measured from the equilibrium positions O_1 and O_2, respectively. The displacements to the right are positive and those to the left are negative. If the two oscillators were not connected, each would vibrate independently of the other with frequencies

$$\omega_{10} = \sqrt{\frac{k_1}{m}} \quad \text{and} \quad \omega_{20} = \sqrt{\frac{k_2}{m}} \tag{14.12}$$

When these oscillators are connected by a spring of spring constant k', the system vibrates with different frequencies, which we wish to calculate now.

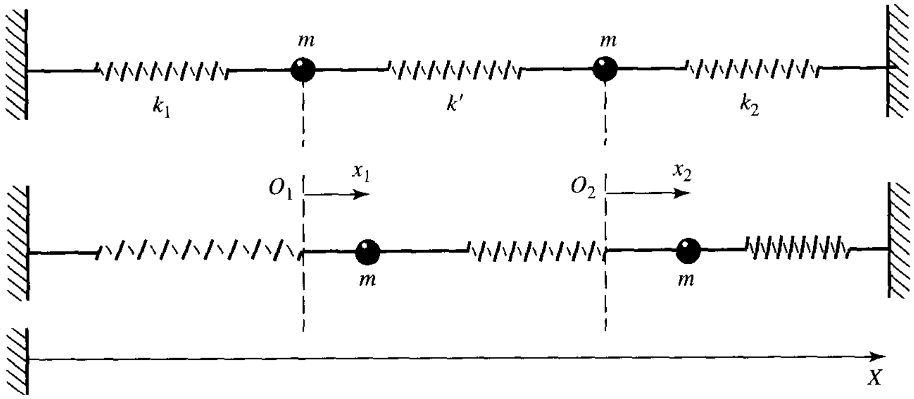


Figure 14.2 Two harmonic oscillators coupled together by a spring of spring constant k'.

The kinetic energy of the system is

$$T = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 \tag{14.13}$$

and the potential energy of the system is

$$V = \tfrac{1}{2}k_1x_1^2 + \tfrac{1}{2}k_2x_2^2 + \tfrac{1}{2}k'(x_1 - x_2)^2 \tag{14.14}$$

Hence the Lagrangian function L of the system is

$$L = T - V = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 - \tfrac{1}{2}k_1x_1^2 - \tfrac{1}{2}k_2x_2^2 - \tfrac{1}{2}k'(x_1 - x_2)^2 \tag{14.15}$$

The Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \tag{14.16}$$

take the form

$$m\ddot{x}_1 + k_1x_1 + k'(x_1 - x_2) = 0 \tag{14.17a}$$

$$m\ddot{x}_2 + k_2x_2 + k'(x_2 - x_1) = 0 \tag{14.17b}$$

The third term in each of these two equations is the result of coupling between the two oscillators. If there were no coupling, these oscillators would vibrate with frequencies given by Eq. (14.12). The preceding second-order linear differential equations may be written as

$$m\ddot{x}_1 + (k_1 + k')x_1 - k'x_2 = 0 \tag{14.18a}$$

$$m\ddot{x}_2 + (k_2 + k')x_2 - k'x_1 = 0 \tag{14.18b}$$

These equations will be independent of each other if the third term in each equation is not present. That is, if we hold the second mass at rest, $x_2 = 0$, and the frequency of vibrations of the first oscillator, from Eq. (14.18a), will be

$$\omega_1' = \sqrt{\frac{k_1 + k'}{m}} \tag{14.19a}$$

On the other hand, if the first mass is at rest, that is, $x_1 = 0$, then the frequency of vibration of the second oscillator, from Eq. (14.18b), will be

$$\omega_2' = \sqrt{\frac{k_2 + k'}{m}} \tag{14.19b}$$

The frequencies ω_1' and ω_2' given by Eqs. (14.19) are higher than ω_{10} and ω_{20} given by Eq. (14.12). The reason is that each mass is tied to two springs, not just one.

To obtain different possible modes of vibrations, we must solve simultaneously the second-order linear differential equations (14.18). The problem can be made somewhat simpler if we assume the two oscillators to be completely identical, that is,

$$k_1 = k_2 = k \tag{14.20}$$

and Eqs. (14.18) take the form

$$m\ddot{x}_1 + (k + k')x_1 - k'x_2 = 0 \tag{14.21}$$

$$m\ddot{x}_2 + (k + k')x_2 - k'x_1 = 0 \tag{14.22}$$

The trial solution of these equations can take any one of the following three forms:

$$x = A\cos(\omega t + \phi) \tag{14.23}$$

$$x = A_1 \cos \omega t + A_2 \sin \omega t \tag{14.24}$$

$$x = Ae^{i(\omega t + \delta)} \tag{14.25}$$

where δ is the initial phase factor. Let us assume Eq. (14.25) to be a trial solution, so that

$$x_1 = Ae^{i(\omega t + \delta_1)} \quad \text{and} \quad x_2 = Be^{i(\omega t + \delta_2)}$$

If we assume the initial phase factors to be zero, that is, $\delta_1 = \delta_2 = 0$, then these two solutions take the form

$$x_1 = Ae^{i\omega t} \tag{14.26}$$

and

$$x_2 = Be^{i\omega t} \tag{14.27}$$

Substituting these in Eqs. (14.21) and (14.22), we obtain, after rearranging,

$$(k + k' - m\omega^2)A - k'B = 0 \tag{14.28}$$

$$-k'A + (k + k' - m\omega^2)B = 0 \tag{14.29}$$

We have two algebraic equations with three unknowns A, B, and ω. These equations can be solved for the ratio A/B; that is,

$$\frac{A}{B} = \frac{k'}{k + k' - m\omega^2} = \frac{k + k' - m\omega^2}{k'} \tag{14.30}$$

We could solve for ω from the last equality in Eq. (14.30); or we could solve directly Eqs. (14.28) and (14.29) by assuming that the determinant of the coefficients of A and B is zero; that is,

$$\begin{vmatrix} k + k' - m\omega^2 & -k' \\ -k' & k + k' - m\omega^2 \end{vmatrix} = 0 \tag{14.31}$$

This is called the *secular equation*. This may be written as

$$(k + k' - m\omega^2)^2 - k'^2 = 0 \tag{14.32}$$

or

$$\left(\frac{k}{m} - \omega^2\right)\left(\frac{k + 2k'}{m} - \omega^2\right) = 0 \tag{14.33}$$

which yields the following two roots:

$$\omega = \pm\omega_1 = \pm\left(\frac{k}{m}\right)^{1/2} \tag{14.34a}$$

and

$$\omega = \pm\omega_2 = \pm\left(\frac{k + 2k'}{m}\right)^{1/2} \tag{14.34b}$$

In terms of the roots ω_1 and ω_2, the general solutions of Eqs. (14.21) and (14.22) may be written as

$$x_1 = A_1 e^{i\omega_1 t} + A_{-1} e^{-\omega_1 t} + A_2 e^{i\omega_2 t} + A_{-2} e^{-i\omega_2 t} \tag{14.35}$$

$$x_2 = B_1 e^{i\omega_1 t} + B_{-1} e^{-\omega_1 t} + B_2 e^{i\omega_2 t} + B_{-2} e^{-i\omega_2 t} \tag{14.36}$$

There are eight arbitrary constants for two differential equations, but these are not all independent. Substituting Eqs. (14.34a) and (14.34b) in Eqs. (14.28) and (14.29) or in Eq. (14.30), we can obtain the ratios of A/B for different values of ω to be

$$\text{If } \omega = \omega_1, \qquad A = +B \tag{14.37}$$

$$\text{If } \omega = \omega_2, \qquad A = -B \tag{14.38}$$

Combining Eqs. (14.37) and (14.38) with Eqs. (14.35) and (14.36), we obtain

$$x_1 = A_1 e^{i\omega_1 t} + A_{-1} e^{-i\omega_1 t} + A_2 e^{i\omega_2 t} - A_{-2} e^{i\omega_2 t} \tag{14.39}$$

$$x_2 = A_1 e^{i\omega_1 t} + A_{-1} e^{-i\omega_1 t} - A_2 e^{i\omega_2 t} - A_{-2} e^{i\omega_2 t} \tag{14.40}$$

Thus we have only four arbitrary constants, A_1, A_{-1}, A_2, A_{-2}, as expected from the general solution of two second-order differential equations. The actual values of the constants can be determined from initial conditions.

Normal Coordinates

After determining the constants in Eqs. (14.39) and (14.40), each coordinate (x_1 and x_2) may depend on two frequencies, ω_1 and ω_2. Hence it may not be so simple to interpret the type of motion with which the system is oscillating. It is possible to find new coordinates X_1 and X_2, which are linear combinations of x_1 and x_2, such that each new coordinate oscillates with a single frequency. In the present situation, the sum and difference of x_1 and x_2 [using Eqs. (14.39) and (14.40) give us the new coordinates; that is,

$$X_1 = x_1 + x_2 = 2(A_1 e^{i\omega_1 t} + A_{-1} e^{-i\omega_1 t}) = Ce^{i\omega_1 t} + De^{-i\omega_1 t} \tag{14.41}$$

$$X_2 = x_1 - x_2 = 2(A_2 e^{i\omega_2 t} + A_{-2} e^{-i\omega_2 t}) = Ee^{i\omega_2 t} + Fe^{-i\omega_2 t} \tag{14.42}$$

where C, D, E, and F are the new constants. The new coordinates X_1 and X_2 correspond to new modes of oscillation, each mode oscillating with a single frequency. These are called the *normal modes,* and the corresponding coordinates are called the *normal coordinates.* One outstanding characteristic of normal modes is that, for any given normal modes (X_1 or X_2), all the coordinates (x_1 and x_2 in this case) oscillate with the same frequency. Normally, all the normal coordinates are excited simultaneously, except under special circumstances. If, however, one mode is initially not excited, it will remain so throughout the motion.

The nature of any one of the normal modes can be investigated if all the other normal modes can be equated to zero. In the present situation, to study the appearance of the X_1 mode, we must have $X_2 = 0$; that is, if $X_1 \neq 0$,

$$X_2 = 0 = x_1 - x_2 \quad \text{or} \quad x_1 = x_2 \tag{14.43}$$

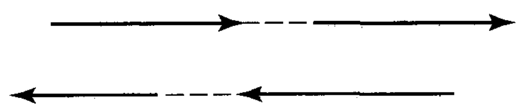
Thus X_1 is a *symmetric mode,* and, as shown in Fig. 14.3(a), both masses have equal displacements, have the same frequency $\omega_1 = (k/m)^{1/2}$, and are in phase. On the other hand, the appearance of the X_2 mode is made possible by letting $X_1 = 0$; that is, if $X_2 \neq 0$,

$$X_1 = 0 = x_1 + x_2 \quad \text{or} \quad x_1 = -x_2 \tag{14.44}$$

Thus X_2 is an *antisymmetric mode* and is as shown in Fig. 14.3(b). Both masses have equal and opposite displacement (out of phase), but vibrate with the same frequency $\omega_2 = [(k + k')/m]^{1/2}$. In short,

$$\textit{Symmetric mode } X_1: \qquad \omega_1 = \sqrt{\frac{k}{m}}, \qquad X_2 = 0: x_1 = x_2 \tag{14.45}$$

$$\textit{Antisymmetric mode } X_2: \qquad \omega_2 = \sqrt{\frac{k + 2k'}{m}}, \qquad X_1 = 0: x_1 = -x_2 \tag{14.46}$$



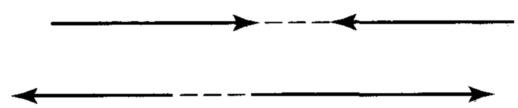


Figure 14.3 Modes of vibration of the two coupled oscillators in Fig. 14.2: (a) symmetrical mode, and (b) antisymmetrical mode.

It is clear that in a symmetric mode the two oscillators vibrate as if there were no coupling between them, and their frequency is the same as the original frequency. In the antisymmetric mode, the result of the coupling is such that the oscillators oscillate out of phase, and their frequency is higher than their individual uncoupled frequency. In general, *the mode that has the highest symmetry will have the lowest frequency, while the antisymmetric mode has the highest frequency.* As the symmetry is destroyed, the springs must work harder, thereby increasing the frequency.

To excite a symmetric mode, the two masses should be pulled from their equilibrium positions by equal amounts in the same direction and released so that $x_1 = x_1(t)$ and $x_2 = x_2(t)$ take the form

$$x_1(0) = x_2(0) \quad \text{and} \quad \dot{x}_1(0) = \dot{x}_2(0) \tag{14.47}$$

For the excitation of an antisymmetric mode, the two masses are pulled apart equally in opposite directions and then released, so that

$$x_1(0) = -x_2(0) \quad \text{and} \quad \dot{x}_1(0) = -\dot{x}_2(0) \tag{14.48}$$

In general, the motion of the system will consist of a combination of these two modes.

Equations of Motion in Normal Coordinates

We obtain expressions for kinetic energy and potential energy in terms of normal coordinates. From Eqs. (14.41) and (14.42),

$$x_1 = \frac{X_1 + X_2}{2} \tag{14.49}$$

and

$$x_2 = \frac{X_1 - X_2}{2} \tag{14.50}$$

Substituting this in Eqs. (14.13) and (14.14),

$$T = \frac{m}{2}\left(\frac{\dot{X}_1 + \dot{X}_2}{2}\right)^2 + \frac{m}{2}\left(\frac{\dot{X}_1 - \dot{X}_2}{2}\right)^2 = m\left(\frac{\dot{X}_1}{2}\right)^2 + m\left(\frac{\dot{X}_2}{2}\right) \tag{14.51}$$

$$V = \frac{k}{2}\left(\frac{X_1 + X_2}{2}\right)^2 + \frac{k}{2}\left(\frac{X_1 - X_2}{2}\right)^2 + \frac{k'}{2}X_2^2$$

or

$$V = \frac{k}{2}\left(\frac{X_1^2}{2}\right) + \left(\frac{k + 2k'}{2}\right)\left(\frac{X_2^2}{2}\right) \tag{14.52}$$

whereas

$$L = T - V = \frac{m}{4}\dot{X}_1^2 + \frac{m}{4}\dot{X}_2^2 - \frac{k}{4}X_1^2 - \left(\frac{k + 2k'}{4}\right)X_2^2 \tag{14.53}$$

Note that the expressions for T, V, and L do not contain cross terms. Thus the Lagrange equations of motion in normal coordinates,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_1}\right) - \frac{\partial L}{\partial X_1} = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_2}\right) - \frac{\partial L}{\partial X_2} = 0 \tag{14.54}$$

yield

$$\ddot{X}_1 + \omega_1^2 X_1 = 0, \quad \text{where } \omega_1 = \left(\frac{k}{m}\right)^{1/2} \tag{14.55}$$

and

$$\ddot{X}_2 + \omega_2^2 X_2 = 0, \quad \text{where } \omega_2 = \left(\frac{k + 2k'}{m}\right)^{1/2} \tag{14.56}$$

That is, an X_1 mode vibrates with frequency ω_1, and an X_2 mode vibrates with frequency ω_2 in agreement with the results derived previously. [Note that these equations can be obtained by directly substituting Eqs. (14.49) and (14.50) into Eq. (14.18).]

From our discussion, we can conclude the following about normal coordinates: *No cross terms are present when the kinetic and potential energies are expressed in terms of normal coordinates; that is, both T and V are homogeneous quadratic functions.* The differential equations are automatically separated; that is, there is one differential equation for each normal coordinate. The solution of each differential equation represents a separated mode of vibration. In the following, we shall establish the general procedure of transferring to normal coordinates and hence to normal modes of vibrations.

14.4 THEORY OF SMALL OSCILLATIONS

Consider a system of N interacting particles with $3n$ degrees of freedom and described by a set of generalized coordinates $(q_1, q_2, \ldots, q_{3n})$. Furthermore, let us assume that frictional forces are absent and that the forces between particles are conservative. We shall demonstrate that the method of Lagrange's equations can be used for the determination of the frequencies and amplitudes of small oscillations about positions of stable equilibrium in conservative systems.

For such a conservative system, let us express the potential energy by $V(q_1, q_2, \ldots, q_{3n})$. Small oscillations take place about an equilibrium point whose generalized coordiantes are $(q_{10}, q_{20}, \ldots, q_{3n0})$. Expanding the potential energy about an equilibrium point in a multidimensional Taylor series, we have

$$V(q_1, q_2, \ldots, q_{3n}) = V(q_{10}, q_{20}, \ldots, q_{3n0}) + \frac{1}{1!}\sum_{l=1}^{3n}\left(\frac{\partial V}{\partial q_l}\right)\bigg|_{q_l=q_{l0}}(q_l - q_{l0})$$

$$+\frac{1}{2!}\sum_{l=1}^{3n}\sum_{m=1}^{3n}\left(\frac{\partial^2 V}{\partial q_l\,\partial q_m}\right)\bigg|_{\substack{q_l = q_{l0}\\ q_m = q_{m0}}}(q_l - q_{l0})(q_m - q_{m0}) + \cdots \tag{14.57}$$

Since the zero of the potential energy is arbitrary, the first term on the right is constant and may be equated to zero without affecting the equations of motion. Also, because the system is in equilibrium, the generalized forces Q_l must vanish; that is,

$$Q_l = -\frac{\partial V}{\partial q_l} = 0, \qquad l = 1, 2, \ldots, 3n \tag{14.58}$$

and the second term in the expansion vanishes. Thus, keeping the second-order term and dropping the higher-order terms, we may write the potential energy to be

$$V(q_1, q_2, \ldots, q_{3n}) = \frac{1}{2!}\sum_{l=1}^{3n}\sum_{m=1}^{3n}\left(\frac{\partial^2 V}{\partial q_l\,\partial q_m}\right)\bigg|_{\substack{q_l = q_{l0}\\ q_m = q_{m0}}}(q_l - q_{l0})(q_m - q_{m0}) \tag{14.59}$$

Introducing a new set of generalized coordinates η_l that represent the displacements from the equilibrium,

$$V = V(\eta_l) = \frac{1}{2!}\sum_{l=1}^{3n}\sum_{m=1}^{3n} V_{lm}\,\eta_l\,\eta_m \tag{14.60}$$

where $\qquad \eta_l = (q_l - q_{l0}) \quad \text{and} \quad \eta_m = (q_m - q_{m0})$

and

$$V_{lm} = \left(\frac{\partial^2 V}{\partial q_l\,\partial q_m}\right)\bigg|_{\substack{q_l = q_{l0}\\ q_m = q_{m0}}} = V_{ml} = \text{constant} \tag{14.61}$$

The constants V_{lm} form a symmetric matrix $\mathbf{V}$. Since we are considering motioins about *stable* equilibrium, the potential energy must be minimum; that is, $V(\eta_l) > V(0)$; hence the homogeneous quadratic form of V given by Eq. (14.60) must be positive. [That is, for a one-dimensional case, $(\partial^2 V/\partial q^2)_{q=q_0} > 0$, the second derivative evaluated at equilibrium is greater than zero.] Thus for a multidimensional system the necessary and sufficient conditions that a homogeneous quadratic form be positive definite are (derivatives are evaluated about equilibrium)

$$\frac{\partial^2 V}{\partial q_l^2} > 0, \qquad l = 1, 2, \ldots, 3n \tag{14.62a}$$

$$\begin{vmatrix} \dfrac{\partial^2 V}{\partial q_1^2} & \dfrac{\partial^2 V}{\partial q_1\,\partial q_m} \\ \dfrac{\partial^2 V}{\partial q_1\,\partial q_m} & \dfrac{\partial^2 V}{\partial q_m^2} \end{vmatrix} > 0, \qquad \begin{matrix} l = 1, 2, \ldots, 3n \\ m = 1, 2, \ldots, 3n \\ l \neq m \end{matrix} \tag{14.62b}$$

$$\begin{vmatrix} \dfrac{\partial^2 V}{\partial q_1^2} & \dfrac{\partial^2 V}{\partial q_1 \partial q_2} & \cdots & \dfrac{\partial^2 V}{\partial q_1 \partial q_{3n}} \\ \dfrac{\partial^2 V}{\partial q_2 \partial q_1} & \dfrac{\partial^2 V}{\partial q_2^2} & \cdots & \dfrac{\partial^2 V}{\partial q_2 \partial q_{3n}} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial^2 V}{\partial q_{3n} \partial q_1} & \dfrac{\partial^2 V}{\partial q_{3n} \partial q_2} & \cdots & \dfrac{\partial^2 V}{\partial q_{3n}^2} \end{vmatrix} > 0 \tag{14.62c}$$

or, in terms of matrix notation, the coefficients $V_{lm} = V_{ml}$ must satisfy the conditions

$$V_{11} > 0$$

$$\begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} > 0$$

$$\begin{vmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{vmatrix} > 0$$

$$\begin{vmatrix} V_{11} & V_{12} & \cdots & V_{1m} \\ V_{21} & V_{22} & \cdots & V_{2m} \\ \vdots & \vdots & & \vdots \\ V_{1l} & V_{l2} & \cdots & V_{1m} \end{vmatrix} > 0 \tag{14.63}$$

where V_{lm} are given by Eq. (14.61) and each individual V_{lm} need not be positive.

If the derivative $V_{lm} = \partial^2 V/\partial q_l\, \partial q_m = 0$ for all values of l and m, stable equilibrium is still possible provided the first nonzero derivative of the potential is of an even order.

Let us now consider the kinetic energy of the system. In terms of Cartesian coordinates, the kinetic energy of the system is

$$T = \frac{1}{2} \sum_{j=1}^{3n} m_j \dot{x}_j^2 \tag{14.64}$$

The transformation equations from Cartesian to generalized coordinates may be utilized to express T in terms of generalized coordinates; that is,

$$x_j = x_j(q_1, q_2, \ldots, q_l, t)$$

and
$$\dot{x}_j = \sum_{l=1}^{3n} \frac{\partial x_j}{\partial q_l}\dot{q}_l + \frac{\partial x_j}{\partial t}$$

Hence the kinetic energy given by Eq. (14.64) may be written as

$$T = \frac{1}{2}\sum_{j=1}^{3n} m_j\left(\sum_{l=1}^{3n} \frac{\partial x_j}{\partial q_l}\dot{q}_l + \frac{\partial x_j}{\partial t}\right)\left(\sum_{m=1}^{3n} \frac{\partial x_j}{\partial q_m}\dot{q}_m + \frac{\partial x_j}{\partial t}\right) \tag{14.65}$$

Upon expanding the right side, we find that T contains three types of terms: (1) terms that are quadratic in generalized velocities, (2) terms linear in generalized velocities, and (3) terms independent of generalized velocities. We are interested in transformation equations that do not contain time explicitly (terms such as $\partial x_j/\partial t$ contain time explicitly). This means that T from Eq. (14.65) should contain only those terms that are quadratic in generalized velocities. (The transformation equations involving other terms occur, for example, in rotating coordinate systems.) Hence Eq. (14.65) for kinetic energy takes the form

$$T = \frac{1}{2}\sum_{l=1}^{3n}\sum_{m=1}^{3n}\left(\sum_{j=1}^{3n} m_j \frac{\partial x_j}{\partial q_l}\frac{\partial x_j}{\partial q_m}\right)\dot{q}_l\dot{q}_m \tag{14.66}$$

For small oscillations about equilibrium, the term in parentheses may be expanded and written as

$$\sum_{j=1}^{3n} m_j \frac{\partial x_j}{\partial q_l}\frac{\partial x_j}{\partial q_m} = \sum_{j=1}^{3n} m_j\left(\frac{\partial x_j}{\partial q_l}\right)_{q_{l0}}\left(\frac{\partial x_j}{\partial q_m}\right)_{q_{m0}} + \sum_{j=1}^{3n} m_j \sum_{k=1}^{3n}\frac{\partial}{\partial q_k}\left(\frac{\partial x_j}{\partial q_l}\frac{\partial x_j}{\partial q_m}\right)_{\substack{q_{l0}\\ q_{m0}}}\eta_k + \cdots \tag{14.67}$$

where $\eta_k = (q_k - q_{k0})$. Since we are interested in small oscillations, we need keep only those $\dot{q}$ terms in T that are of the same order as q in V. Hence, from Eqs. (14.66) and (14.67), remembering that $\dot{q}_l = \dot{\eta}_l$ and $\dot{q}_m = \dot{\eta}_m$, we may write

$$T \simeq \frac{1}{2}\sum_{l=1}^{3n}\sum_{m=1}^{3n} T_{lm}\,\dot{\eta}_l\,\dot{\eta}_m \tag{14.68}$$

where
$$T_{lm} = \frac{1}{2}\sum_{j=1}^{3n} m_j\left(\frac{\partial x_j}{\partial q_l}\right)_{q_{l0}}\left(\frac{\partial x_j}{\partial q_m}\right)_{q_{m0}} = T_{ml} \tag{14.69}$$

and T_{lm} are the elements of a symmetric matrix $\mathbf{T}$.

After obtaining the expressions for potential energy given by Eq. (14.60) and kinetic energy, Eq. (14.68), we are now in a position to write the Lagrangian:

$$L = T - V = \frac{1}{2}\sum_{l=1}^{3n}\sum_{m=1}^{3n}(T_{lm}\,\dot{\eta}_l\,\dot{\eta}_m - V_{lm}\,\eta_l\,\eta_m) \tag{14.70}$$

Hence the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}}\right) - \frac{\partial L}{\partial \eta} = 0 \tag{14.71}$$

take the form

$$\sum_{m=1}^{3n} (T_{lm}\,\ddot{\eta}_m + V_{lm}\,\eta_m) = 0, \qquad l = 1, 2, \ldots, 3n \tag{14.72a}$$

or

$$T_{l1}\,\ddot{\eta}_1 + V_{l1}\,\eta_1 + T_{l2}\,\ddot{\eta}_2 + V_{l2}\,\eta_2 + \cdots + T_{l3n}\,\ddot{\eta}_{3n} + V_{l3n}\,\eta_{3n} = 0 \tag{14.72b}$$

Equations (14.72) represent $3n$ linear, coupled, second-order differential equations. From our experience with the solution of a one-dimensional case, we may write the solution of Eq. (14.72) to be

$$\eta_m = A_m \cos(\omega t + \phi_m) \tag{14.73}$$

where the amplitude A_m and the phase angle ϕ_m are to be determined from initial conditions, while the natural frequency ω is determined from the system's constants. Substituting Eq. (14.73) into Eq. (14.72a), we obtain

$$\sum_{m=1}^{3n} [V_{lm} A_m \cos(\omega t + \phi_m) - \omega^2 T_{lm} A_m \cos(\omega t + \phi_m)] = 0, \qquad l = 1, 2, \ldots, 3n \tag{14.74}$$

For a given ω, all ϕ_m must be the same, $\phi_m = \phi$; hence $\cos(\omega t + \phi)$ can be factored out; that is,

$$\cos(\omega t + \phi) \sum_{m=1}^{3n} [V_{lm} A_m - \omega^2 T_{lm} A_m] = 0, \qquad l = 1, 2, \ldots, 3n \tag{14.75}$$

Since $\cos(\omega t + \phi)$ is *not,* in general, equal to zero, we must have

$$\sum_{m=1}^{3n} [V_{lm} A_m - \omega^2 T_{lm} A_m] = 0, \qquad l = 1, 2, \ldots, 3n \tag{14.76}$$

Thus we have a total of $3n$ linear, homogeneous, algebraic equations in A_m and ω represented as

$$\begin{matrix} (V_{11} - \omega^2 T_{11})A_1 & + & (V_{12} - \omega^2 T_{12})A_2 & + \cdots + & (V_{1.3n} - \omega^2 T_{1.3n})A_{3n} & = 0 \\ \vdots & & & & \vdots & \vdots \\ (V_{3n.1} - \omega^2 T_{3n.1})A_1 & + & (V_{3n.2} - \omega^2 T_{3n.2})A_2 & + \cdots + & (V_{3n.3n} - \omega^2 T_{3n.3n})A_{3n} & = 0 \end{matrix} \tag{14.77}$$

For a nontrivial solution, the determinant of the coefficients of A_m in Eq. (14.77) must be zero; that is,

$$\begin{vmatrix} (V_{11} - \omega^2 T_{11}) & (V_{12} - \omega^2 T_{12}) & \cdots & (V_{1.3n} - \omega^2 T_{1.3n}) \\ \vdots & \vdots & & \vdots \\ (V_{3n.1} - \omega^2 T_{3n.1}) & (V_{3n.2} - \omega^2 T_{3n.2}) & \cdots & (V_{3n.3n} - \omega^2 T_{3n.3n}) \end{vmatrix} = 0 \tag{14.78a}$$

$$|\mathbf{V} - \omega^2 \mathbf{T}| = 0 \tag{14.78b}$$

This results in a secular equation of a $3n$-degree polynomial in ω^2. Each of the $3n$ roots of this equation represents a different frequency. Thus the general solution, for small amplitude of oscillations, is

$$\eta_l = \sum_{k=1}^{3n} A_{kl} \cos(\omega_k t + \phi_k) \tag{14.79}$$

where the values of ω_k are known from the secular equation, Eq. (14.78), while A_{kl} and ϕ_k are determined from initial conditions.

If ω^2 is negative ($\omega^2 < 0$), ω will be complex and there will be no small oscillations. If $\omega^2 = 0$, the coordinate η remains constant, hence with no oscillations, only translation or rotation of the whole system. Only if $\omega^2 > 0$ will there be oscillation about the stable equilibrium. Thus

$$\text{If } \omega_k^2 > 0, \qquad \eta_k = A_k e^{i\omega_k t} + B_k e^{-i\omega_k t} \tag{14.80}$$

$$\text{If } \omega_k^2 = 0, \qquad \eta_k = C_k t + D_k \tag{14.81}$$

$$\text{If } \omega_k^2 < 0, \qquad \eta_k = E_k e^{\omega_k t} + F_k e^{-\omega_k t} \tag{14.82}$$

We have found the frequencies, while the task of calculating the amplitudes still remains. The amplitudes A_{kl} are related by the algebraic equations (14.77). Substituting each value of ω_k separately in Eq. (14.77), it is possible to determine all the coefficients A_{kl} except one, say A_{k1}. Thus it is possible to determine the coefficients A_{kl} in terms of A_{k1} in the form of ratios:

$$\frac{A_{k2}}{A_{k1}}, \frac{A_{k3}}{A_{k1}}, \ldots, \frac{A_{k,3n}}{A_{k1}} \tag{14.83}$$

We must determine $6n$ constants ($3n$ are A_{k1} and $3n$ are ω_k) from initial conditions.

14.5 SMALL OSCILLATIONS IN NORMAL COORDINATES

Let us once again consider an arbitrary system with r degrees of freedom. The system has small oscillations about some stable equilibrium point. The potential energy is described in terms of generalized coordinates $(q_1', q_2', \ldots, q_l')$, while the equilibrium configuration is described by the coordinates $(q_{10}', q_{20}', \ldots, q_{l0}')$, where $l = 1, 2, \ldots, r$. As explained in the previous section, for stable equilibrium the only nonzero coefficient in the expansion of the potential energy $V(q_1', q_2', \ldots, q_l')$ is V_{lm} given by

$$V = \frac{1}{2} \sum_{l=1}^{r} \sum_{m=1}^{r} V_{lm}\, \eta_l'\, \eta_m' \tag{14.84}$$

where

$$\eta_l' = q_l' - q_{l0}'$$

$$\eta_m' = q_m' - q_{m0}'$$

and

$$V_{lm} = \left(\frac{\partial^2 V}{\partial q_l'\, \partial q_m'} \right)\Bigg|_{\substack{q_l' = q_{l0}' \\ q_m' = q_{m0}'}} = V_{ml} = \text{constant} \tag{14.85}$$

Thus the potential energy expression, as stated earlier, is not only a homogeneous quadratic but is also positive definite for stable equilibrium. It cannot be negative and is zero only if all the coordinates are zero. For such a system, the potential energy V may be written as

$$V = a_{11}\, \eta_1'^2 + a_{22}\, \eta_2'^2 + \cdots + a_{rr}\, \eta_r'^2 + 2a_{12}\, \eta_1'\eta_2' + \cdots \tag{14.86}$$

where every term is quadratic in the coordinates and the coefficients $a_{11}, a_{22}, \ldots, a_{rr}, a_{12}, \ldots$, and so on, are all constant. Note the presence of square terms as well as cross terms.

Similarly, we have seen that, if the kinetic energy T does not contain time explicitly, it will be homogeneous in velocity coordinates and may be written as

$$T = b_{11}\,\dot{\eta}_1'^2 + b_{22}\,\dot{\eta}_2'^2 + \cdots + b_{rr}\,\dot{\eta}_r'^2 + b_{12}\,\dot{\eta}_1'\dot{\eta}_2' + \cdots \tag{14.87}$$

For small oscillations, the quantities $b_{11}, b_{22}, \ldots, b_{rr}, b_{12}, \ldots$, and so on, are constant; hence T is positive definite. Once again, note the presence of cross terms.

It is possible to cause a linear transformation to new generalized coordinates $\eta_1, \eta_2, \ldots, \eta_r$, in which V and T will not contain cross terms. The original coordinates $\eta_1', \eta_2', \ldots, \eta_r'$ by their linear combination can result in new generalized coordinates $\eta_1, \eta_2, \ldots, \eta_r$.

$$\begin{aligned} \eta_1 &= e_{11}\,\eta_1' + e_{12}\,\eta_2' + \cdots + e_{1r}\,\eta_r' \\ \eta_2 &= e_{21}\,\eta_1' + e_{22}\,\eta_2' + \cdots + e_{2r}\,\eta_r' \\ &\;\;\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ \eta_r &= e_{r1}\,\eta_1' + e_{r2}\,\eta_2' + \cdots + e_{rr}'\,\eta_r' \end{aligned} \tag{14.88}$$

so that V and T will take the following forms that do not contain cross terms:

$$V = \tfrac{1}{2}(\lambda_1^2\,\eta_1^2 + \lambda_2^2\,\eta_2^2 + \cdots + \lambda_r^2\,\eta_r^2) \tag{14.89}$$

and

$$T = \tfrac{1}{2}(m_1\dot{\eta}_1^2 + m_2\,\dot{\eta}_2^2 + \cdots + m_r\,\dot{\eta}_r^2) \tag{14.90}$$

where λ's and m's are constants. The new linear combination $\eta_1, \eta_2, \ldots, \eta_r$ is called the *normal coordinates* of the system.

Now the Lagrangian equations for the normal coordinates η_l are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_l}\right) - \frac{\partial L}{\partial \eta_l} = 0 \tag{14.91}$$

where $L = T - V$. If V and T are given by Eqs. (14.89) and (14.90), the resulting equations of motion for η_l are

$$\ddot{\eta}_l + \omega_l^2\eta_l = 0 \tag{14.92}$$

where ω_l are the normal frequencies given by

$$\omega_l^2 = \frac{\lambda_l^2}{m_l} \tag{14.93}$$

The quantities $\eta_1, \eta_2, \ldots, \eta_r$ are *normal coordinates,* and $\omega_1, \omega_2, \ldots, \omega_r$ are the corresponding *normal frequencies*. The solution of Eq. (14.92) is

$$\text{If } \omega_l^2 > 0, \qquad \eta_l = A_l e^{i\omega_l t} + B_l e^{-i\omega_l t} \tag{14.94a}$$

$$\text{or } \eta_l = A_l' \cos(\omega_l t + \phi_l) \tag{14.94b}$$

$$\text{If } \omega_l^2 = 0, \qquad \eta_l = C_l t + D_l \tag{14.95}$$

$$\text{If } \omega_l^2 < 0, \qquad \eta_l = E_l e^{\omega_l t} + F_l e^{-\omega_l t} \tag{14.96}$$

where A_l, B_l, A_l', ϕ_l, C_l, D_l, E_l, and F_l are all constants.

As pointed out earlier and as is clear from Eq. (14.92), each normal coordinate varies with only one normal frequency ω_l; hence these are called *normal modes* of vibration and each normal coordinate η_l is given by Eq. (14.94). It is necessary to note that if a normal coordinate η_l for which the associated frequency ω_l^2 is not greater than zero, such a coordinate does not correspond to oscillatory motion about the equilibrium. Thus, if $\omega_l^2 = 0$, as is obvious from the solution in Eq. (14.95), the mode of motion is that of translation motion; that is, if the particle is slightly displaced, there will be no restoring force, and the particle will simply translate about the center of mass. On the other hand, if $\omega_l^2 < 0$, as is clear from Eq. (14.96), the motion is nonoscillatory; it consists of increasing and decreasing exponentials, with the result that the motion grows without bounds.

14.6 TENSOR FORMULATION FOR THE THEORY OF SMALL OSCILLATIONS

The problems of small oscillations discussed in the two previous sections can be presented and solved more elegantly by using the techniques of tensor analysis similar to the one used in describing rigid body motion in Chapter 13.

For a system with $3n$ degrees of freedom, the expression for small oscillation about a stable equilibrium, the Lagrangian equations according to Eq. (14.76), are

$$\sum_{m=1}^{3n} [V_{lm} A_m - \omega^2 T_{lm} A_m] = 0, \qquad l = 1, 2, \ldots, 3n \tag{14.97}$$

where

$$V_{lm} = \left(\frac{\partial^2 V}{\partial q_l\, \partial q_m}\right)\Bigg|_{\substack{q_l = q_{l0} \\ q_m = q_{m0}}} = V_{ml} \tag{14.98}$$

$$T_{lm} = \frac{1}{2} \sum m_j \left(\frac{\partial x_j}{\partial q_l}\right)_{q_{l0}} \left(\frac{\partial x_j}{\partial q_m}\right)_{q_{m0}} = T_{ml} \tag{14.99}$$

Equation (14.97) is equivalent to the $3n$ linear equations of the form

$$\begin{array}{l} (V_{11} - \omega^2 T_{11})A_1 + (V_{12} - \omega^2 T_{12})A_2 + \cdots + (V_{1.3n} - \omega^2 T_{1.3n})A_{3n} = 0 \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ (V_{3n.1} - \omega^2 T_{3n.1})A_1 + (V_{3n.2} - \omega^2 T_{3n.2})A_2 + \cdots + (V_{3n.3n} - \omega^2 T_{3n.3n})A_{3n} = 0 \end{array} \tag{14.100}$$

The quantities V_{lm} are the elements of symmetric matrix $\mathbf{V}$ given by

$$\mathbf{V} = \begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1,3n} \\ V_{21} & V_{22} & \cdots & V_{2,3n} \\ \vdots & \vdots & & \vdots \\ V_{3n,1} & V_{3n,2} & \cdots & V_{3n,3n} \end{bmatrix} \tag{14.101}$$

and the quantities T_{lm} are the elements of a symmetric matrix **T** given by

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1,3n} \\ T_{21} & T_{22} & \cdots & T_{2,3n} \\ \vdots & \vdots & & \vdots \\ T_{3n,1} & T_{3n,2} & \cdots & T_{3n,3n} \end{bmatrix} \tag{14.102}$$

while the Lagrange equations, Eqs. (14.97) and (14.100), may be written in tensor form as

$$(\mathbf{V} - \omega^2\mathbf{T})\mathbf{A} = 0 \tag{14.103}$$

where **A** is a column vector:

$$\mathbf{A} = \begin{bmatrix} A_1 \\ \vdots \\ A_{3n} \end{bmatrix} \tag{14.104}$$

For each frequency ω_k, there corresponds a vector A_k: hence, as before, the general solution will be the linear combinations of individual solutions.

The next task is to determine the normal coordinates corresponding to each normal frequency, that is, to determine the normal modes of vibrations. This involves transferring both **V** and **T** to a new set of generalized coordinates in which both **V** and **T** matrices will be diagonal (so that the off-diagonal elements will be zero). The existence of such a coordinate transformation that will cause simultaneous diagonalization of **V** and **T** is possible only if both the **V** and **T** matrices are symmetrical with real elements, and **V** as well as **T** is positive definite (determinant is greater than zero). Such a process of simultaneous diagonalization will change Eq. (14.103) into

$$(\mathbf{V}' - \omega^2\mathbf{T}')\mathbf{A} = 0 \tag{14.105}$$

where

$$(\mathbf{V}' - \omega^2\mathbf{T}') = \begin{bmatrix} V'_{11} - \omega^2 T'_{11} & 0 & 0 & \cdots & 0 \\ 0 & V'_{22} - \omega^2 T'_{22} & 0 & \cdots & 0 \\ 0 & 0 & V'_{33} - \omega^2 T'_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & V'_{3n,3n} - \omega^2 T'_{3n,3n} \end{bmatrix} \tag{14.106}$$

The diagonalization can be achieved in a manner explained in Chapter 13.

For each normal frequency ω_m, there exists a solution of the form

$$\eta_l = C_m a_{lm} \cos(\omega_m t + \phi_m) \tag{14.107}$$

where C_m is the scale factor, a_{lm} is the coefficient, and ϕ_m is the phase angle. This solution is a linear combination of two independent functions cos $\omega_m t$ and sin $\omega_m t$. Thus the most general solution will be

$$\eta_l(t) = \sum_{m=1}^{n} a_{lm} C_m \cos(\omega_m t + \phi_m) \tag{14.108}$$

which is a linear combination of $2n$ functions. Equation (14.108) may be written as

$$\eta_l(t) = \sum_{m=1}^{n} [a_{lm}(C_m \cos\phi \cos\omega_m t - C_m \sin\phi \sin\omega_m t)] \tag{14.109}$$

Defining

$$A_m = C_m \cos\phi \quad \text{and} \quad B_m = -C_m \sin\phi$$

we may write Eq. (14.109) as

$$\eta_l(t) = \sum_{m=1}^{n} [a_{lm}(A_m \cos\omega_m t + B_m \sin\omega_m t)] \tag{14.110}$$

where the coefficients a_{lm} form a set associated with the frequency ω_m or the mth normal mode.

The constants in Eq. (14.110) may now be determined by the following procedure. First, calculate the normal frequencies ω_m from the characteristic equation

$$\det |V - \omega^2 T| = 0$$

Second, replace ω^2 in

$$\sum_{l=1}^{n} [V_{lm} - \omega^2 T_{lm}] a_{lm} = 0, \qquad m = 1, 2, \ldots, n \tag{14.111}$$

by ω_m and calculate the n sets of solutions (a_{lm}), one for each m. (One of the factors a_{lm} must be assigned a unit value; otherwise, only the ratios of the coefficients will be calculated.) Third, A_m and B_m may be calculated by using the initial conditions of the systems.

$$\eta_l(0) \equiv \eta_{l0} = \sum_{m=1}^{n} a_{lm} A_m \tag{14.112}$$

$$\dot{\eta}_l(0) \equiv \dot{\eta}_{l0} (= v_{l0}) = \sum_{m=1}^{n} a_{lm} \omega_m B_m \tag{14.113}$$

In a special case, if the number of degrees is very large and we impose the condition $a_{lm} = \delta_{lm}$; then Eqs. (14.110), (14.112), and (14.113) become

$$\eta_l(t) = A_l \cos\omega_l t + B_l \omega_l t$$

$$\eta_l(0) \equiv \eta_{l0} = A_l$$

and $$\dot{\eta}_l(0) \equiv v_{l0} = \omega_l B_l$$

This holds for normal coordinates; that is, it is possible to find a coordinate transformation such that all $\eta_l(t)$ are normal coordinates as represented by this equation.

Example 14.2

Consider the situation of two coupled pendula, as shown in Fig. Ex. 14.2. Using matrix notation calculate **(a)** the components V_{lm} of **V**, **(b)** the components T_{lm} of **T**, **(c)** the normal frequencies, and **(d)** the normal modes. **(e)** Find the equations of motion and **(f)** the general solution.

Solution

As shown in Fig. Ex. 14.2, each pendulum is of length l and mass m, and equilibrium is where both are vertical in which position $x_1 = x_2 = 0$. The two masses are tied by a spring of spring constant k. The displacements x_1 and x_2 to the right are positive, while θ_1 and θ_2 are positive in a counterclockwise direction.

(a) The potential energy of the system is given by

$$V = mgl(1 - \cos\theta_1) + mgl(1 - \cos\theta_2) + \tfrac{1}{2}k(x_1 - x_2)^2$$

For a small angle,

$$mgl(1 - \cos\theta) = mgl\left[1 - \left(1 - \frac{\theta^2}{2} + \cdots\right)\right] \simeq mgl\frac{\theta^2}{2}$$

$$\simeq \frac{mgl}{2}\left(\frac{x}{l}\right)^2 \simeq \frac{mg}{2l}x^2$$

Therefore,

$$V = \frac{mg}{2l}x_1^2 + \frac{mg}{2l}x_2^2 + \frac{1}{2}k(x_1 - x_2)^2$$

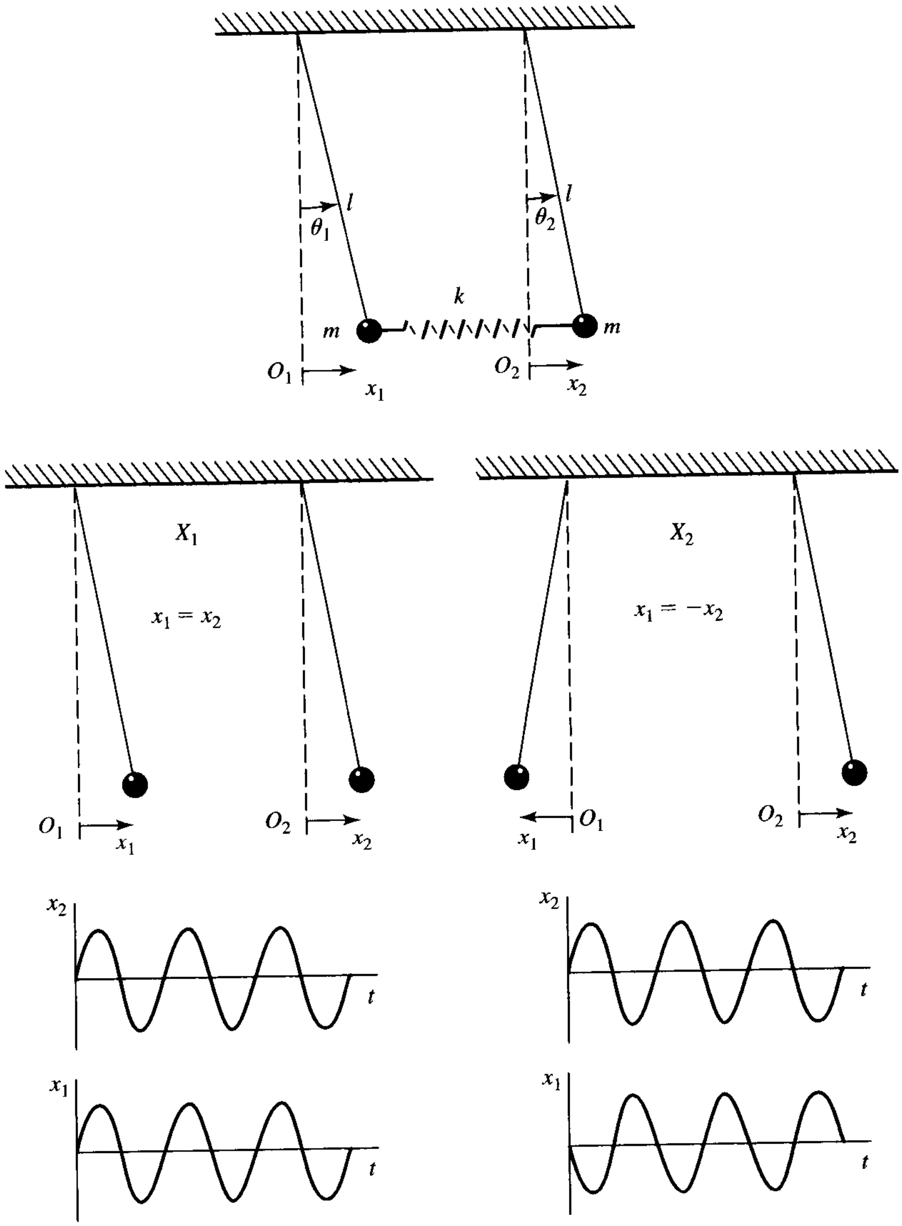
$$= \frac{1}{2}\left(k + \frac{mg}{l}\right)x_1^2 + \frac{1}{2}\left(k + \frac{mg}{l}\right)x_2^2 - kx_1x_2 \qquad \textbf{(i)}$$

$$\left.\frac{\partial V}{\partial x_1}\right|_{\substack{x_1=0\\x_2=0}} = \left(k + \frac{mg}{l}\right)x_1 - kx_2\Big|_{\substack{x_1=0\\x_2=0}} = 0 \qquad \textbf{(ii)}$$

$$\left.\frac{\partial V}{\partial x_2}\right|_{\substack{x_1=0\\x_2=0}} = \left(k + \frac{mg}{l}\right)x_2 - kx_1\Big|_{\substack{x_1=0\\x_2=0}} = 0$$

$$\left.\frac{\partial^2 V}{\partial x_1^2}\right|_{\substack{x_1=0\\x_2=0}} = k + \frac{mg}{l} \quad \text{and} \quad \left.\frac{\partial^2 V}{\partial x_2^2}\right|_{\substack{x_1=0\\x_2=0}} = k + \frac{mg}{l} \qquad \textbf{(iii)}$$

$$\left.\frac{\partial^2 V}{\partial x_2\,\partial x_1}\right|_{\substack{x_1=0\\x_2=0}} = -k \quad \text{and} \quad \left.\frac{\partial^2 V}{\partial x_1 x_2}\right|_{\substack{x_1=0\\x_2=0}} = -k \qquad \textbf{(iv)}$$



Symmetric mode: $x_1 = x_2$

Antisymmetric mode: $x_1 = -x_2$

Figure Ex. 14.2

Thus the required matrix for the potential energy is

$$\mathbf{V} = \begin{bmatrix} k + \dfrac{mg}{l} & -k \\ -k & k + \dfrac{mg}{l} \end{bmatrix} \tag{v}$$

Since this gives

$$\begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} > 0$$

the associated homogeneous quadratic form is positive definite.
(b) The expression for kinetic energy is

$$T = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 \qquad \textbf{(vi)}$$

The components T_{ll} and T_{lm} are coefficients of $\frac{1}{2}\dot{x}_l^2$ and $\dot{x}_l\dot{x}_m$. Hence

$$\mathbf{T} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \textbf{(vii)}$$

Thus the Lagrangian for the system is

$$L = T - V = \sum_{l=1}^{2}\sum_{m=1}^{2} \frac{1}{2}(T_{lm}\dot{x}_l\dot{x}_m - V_{lm}x_lx_m \qquad \textbf{(viii)}$$

while the Lagrangian equations are

$$\sum_{l=1}^{2} (T_{lm}\ddot{x}_l + V_{lm}x_l) = 0, \qquad m = 1, 2 \qquad \textbf{(ix)}$$

That is,

$$T_{11}\ddot{x}_1 + V_{11}x_1 + T_{12}\ddot{x}_2 + V_{12}x_2 = 0$$

$$T_{21}\ddot{x}_1 + V_{21}x_1 + T_{22}\ddot{x}_2 + V_{22}x_2 = 0$$

Using Eqs. (v) and (vii) in the preceding equations, we get

$$m\ddot{x}_1 + \left(k + \frac{mg}{l}\right)x_1 - kx_2 = 0 \qquad \textbf{(x)}$$

$$m\ddot{x}_2 + \left(k + \frac{mg}{l}\right)x_2 - kx_1 = 0 \qquad \textbf{(xi)}$$

These are two coupled equations.
(c) To determine the normal or characteristic frequencies, we use Eq. (14.78b), that is,

$$|\mathbf{V} - \omega^2\mathbf{T}| = 0$$

Thus

$$\begin{vmatrix} k + \dfrac{mg}{l} - m\omega^2 & -k \\ -k & k + \dfrac{mg}{l} - m\omega^2 \end{vmatrix} = 0$$

or

$$\left(k + \frac{mg}{l} - m\omega^2\right)^2 - k^2 = 0 \qquad \textbf{(xii)}$$

$$\left(k + \frac{mg}{l} - m\omega^2 - k\right)\left(k + \frac{mg}{l} - m\omega^2 + k\right) = 0$$

Either
$$k = \frac{mg}{l} - m\omega^2 - k = 0$$

which gives
$$\omega^2 = \omega_1^2 = \frac{g}{l} \quad \text{or} \quad \omega_1 = \pm\left(\frac{g}{l}\right)^{1/2} \tag{xiii}$$

or
$$k + \frac{mg}{l} - m\omega^2 + k = 0$$

which gives
$$\omega^2 = \omega_2^2 = \left(\frac{g}{l} + \frac{2k}{m}\right) \quad \text{or} \quad \omega_2 = \pm\left(\frac{g}{l} + \frac{2k}{m}\right)^{1/2} \tag{xiv}$$

As before, we try the solutions
$$x_1 = Ae^{i\omega t} \quad \text{and} \quad x_2 = Be^{i\omega t} \tag{xv}$$

Substituting these in Eqs. (x) and (xi), we get
$$\left(k + \frac{mg}{l} - m\omega^2\right)A - kB = 0$$
$$\left(k + \frac{mg}{l} - m\omega^2\right)B - kA = 0$$
$$\text{If } \omega^2 = \omega_1^2 = \frac{g}{l}, \qquad \text{we get } A = B \tag{xvi}$$
$$\text{If } \omega^2 = \omega_2^2 = \frac{g}{l} + \frac{2k}{m}, \qquad \text{we get } A = -B \tag{xvii}$$

Hence, using these, the general solution becomes
$$x_1 = A_1e^{i\omega_1 t} + A_{-1}e^{i\omega_1 t} + A_2e^{i\omega_2 t} + A_{-2}e^{-i\omega_2 t} \tag{xviii}$$
$$x_2 = A_1e^{i\omega_1 t} + A_{-1}e^{i\omega_1 t} - A_2e^{i\omega_2 t} - A_{-2}e^{-i\omega_2 t} \tag{xix}$$

These two equations contain four constants, as they should for two linear differential equations. These constants are determined from initial conditions.
(d) We now proceed with Eq. (14.103) or (14.76) to determine the normal coordinates
$$(\mathbf{V} - \omega^2\mathbf{T})\mathbf{A} = 0$$

or
$$\sum_{m=1}^{2} |V_{lm} - \omega^2 T_{lm}| A_m = 0, \qquad l = 1, 2$$

That is, for $\omega^2 = \omega_1^2 = g/l$,
$$\begin{pmatrix} k + \frac{mg}{l} - \frac{mg}{l} & -k \\ -k & k + \frac{mg}{l} - \frac{mg}{l} \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = 0$$

or
$$\begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = 0$$

gives $\qquad \text{If } a_{11} = 1, \qquad a_{12} = 1 \qquad \textbf{(xx)}$

Similarly, for $\omega^2 = \omega_2^2 = (g/l) + (2k/m)$,

$$\begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = 0$$

That is, $\qquad \text{If } a_{21} = 1, \qquad a_{22} = -1 \qquad \textbf{(xxi)}$

Thus the normal modes are

$$\eta_1 = a_{11}x_1 + a_{12}x_2$$

$$\eta_2 = a_{21}x_1 + a_{22}x_2$$

Substituting these values of a_{11}, a_{12}, a_{21}, and a_{22} from Eqs. (xx) and (xxi) and x_1 and x_2 from Eqs. (xviii) and (xix), we get

$$\eta_1 = x_1 + x_2 = 2(A_1 e^{i\omega_1 t} + A_{-1} e^{-i\omega_1 t}) \qquad \textbf{(xxii)}$$

$$\eta_2 = x_1 - x_2 = 2(A_2 e^{i\omega_2 t} + A_{-2} e^{-i\omega_2 t}) \qquad \textbf{(xxiii)}$$

Thus each normal mode depends only on one frequency. Furthermore, we can see the physical meaning of these modes as before.

For the η_1 mode, we take $\eta_2 = 0$; therefore,

$$x_1 - x_2 = 0 \quad \text{or} \quad x_1 = x_2 \qquad \textbf{(xxiv)}$$

In order to really understand and illustrate the natural modes and normal modes of vibrations, we graph for arbitrary numerical values.

normal modes: X1 with frequency ω1 and X2 with frequency ω2

natural modes: x1 (= X1 + X2) and x2 (= X1 - X2)

We will first graph the normal modes and then the natural modes.

X1 and X2 (or η1 and η2) determine the normal coordinates with the characteristic frequencies ω1 and ω2.

$$X1 = A22\cdot\exp(-i\cdot\omega1\cdot t) + A11\cdot\exp(i\cdot\omega1\cdot t) \qquad X2 = A12\cdot\exp(-i\cdot\omega2\cdot t) - A21\cdot\exp(i\cdot\omega2\cdot t)$$

Let us now follow the reverse process, that is, find the values of natural displacements x1 and x2 from the relation X1 = x1 + x2 and X2 = x1 – x2 and then make the plots of x1 and x2. Note that we are going to use prime (') for the variables; otherwise we will get the numerical results because the values of the constants are already given.

Given

$$x1 + x2 = 2\cdot\left(A12'\cdot e^{i\cdot\omega1'\cdot t'} + A12'\cdot e^{-i\cdot\omega1'\cdot t'}\right) \qquad x1 - x2 = 2\cdot\left(A21'\cdot e^{i\cdot\omega2'\cdot t'} + A22'\cdot e^{-i\cdot\omega2'\cdot t'}\right)$$

$$\text{Find}(x1,x2) \rightarrow \begin{pmatrix} A22'\cdot\exp(-i\cdot\omega2'\cdot t') + A12'\cdot\exp(i\cdot\omega1'\cdot t') + A12'\cdot\exp(-i\cdot\omega1'\cdot t') + A21'\cdot\exp(i\cdot\omega2'\cdot t') \\ -A22'\cdot\exp(-i\cdot\omega2'\cdot t') + A12'\cdot\exp(i\cdot\omega1'\cdot t') + A12'\cdot\exp(-i\cdot\omega1'\cdot t') - A21'\cdot\exp(i\cdot\omega2'\cdot t') \end{pmatrix}$$

$$x1 = A22'\cdot\exp(-i\cdot\omega2'\cdot t') + A12'\cdot\exp(i\cdot\omega1'\cdot t') + A12'\cdot\exp(-i\cdot\omega1'\cdot t') + A21'\cdot\exp(i\cdot\omega2'\cdot t')$$

$$x2 = -A22'\cdot\exp(-i\cdot\omega2'\cdot t') + A12'\cdot\exp(i\cdot\omega1'\cdot t') + A12'\cdot\exp(-i\cdot\omega1'\cdot t') - A21'\cdot\exp(i\cdot\omega2'\cdot t')$$

This is the same result that we obtained earlier. We will use the original equations with the arbitrary numerical values and graph them.

Let us now make the graphs, as shown in Figure Ex.(14.2) (b) and (c), by using the arbitrary values given below.

$A11 := 4 \quad A12 := 2 \quad A21 := 2 \quad A22 := 4 \qquad g := 9.8 \qquad l := 2 \qquad k := 1 \qquad m := 1 \qquad i := \sqrt{-1}$

$N := 200 \quad n := 0..N \quad t_n := \dfrac{n}{10} \qquad \omega 1 := \sqrt{\dfrac{g}{l}} \qquad \omega 1 = 2.214 \qquad \omega 2 := \sqrt{\dfrac{g}{l} + 2\cdot\dfrac{k}{m}} \qquad \omega 2 = 2.627$

$X1_n := A11\cdot\exp(i\cdot\omega 1\cdot t_n) - A12\cdot\exp(-i\cdot\omega 1\cdot t_n) \qquad X2_n := (A21\cdot\exp(-i\cdot\omega 2\cdot t_n) + A22\cdot\exp(i\cdot\omega 2\cdot t_n))$

$x1_n := A22\cdot\exp(-i\cdot\omega 2\cdot t_n) + A11\cdot\exp(i\cdot\omega 1\cdot t_n) + A12\cdot\exp(-i\cdot\omega 1\cdot t_n) + A21\cdot\exp(i\cdot\omega 2\cdot t_n)$

$x2_n := -A22\cdot\exp(-i\cdot\omega 2\cdot t_n) + A11\cdot\exp(i\cdot\omega 1\cdot t_n) + A12\cdot\exp(-i\cdot\omega 1\cdot t_n) - A21\cdot\exp(i\cdot\omega 2\cdot t_n)$

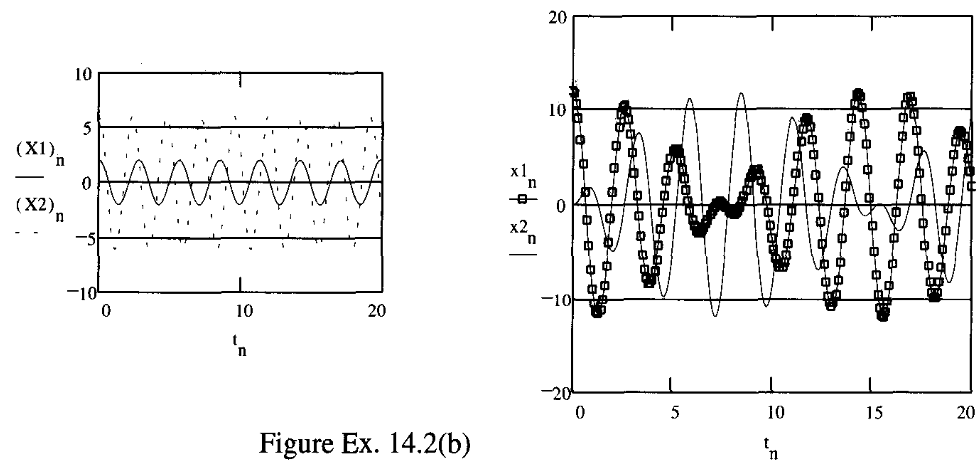


Figure Ex. 14.2(b)

Normal symmetric modes X1 and X2 and natural symmetric modes x1 and x2

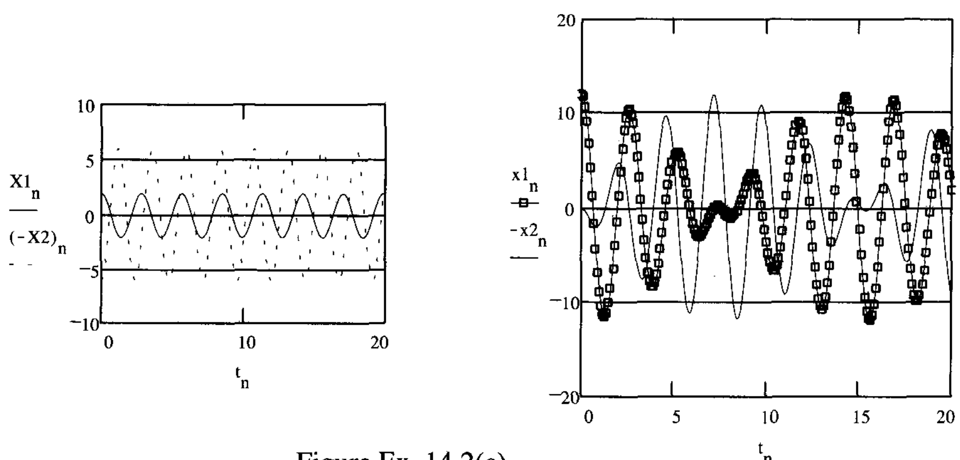


Figure Ex. 14.2(c)

Normal antisymmetric modes X1 and –X2 and natural antisymmetric modes x1 and –x2

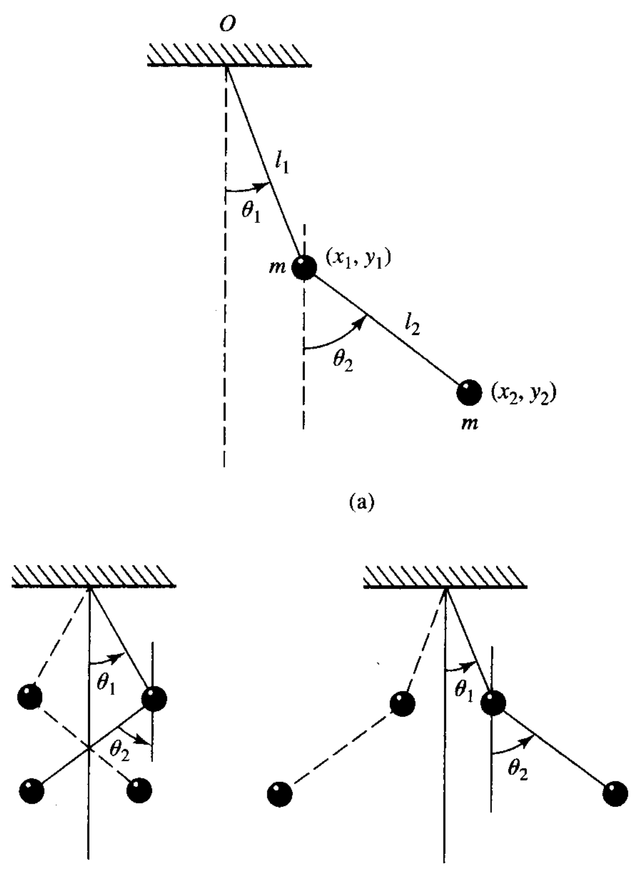
Answer the following by looking at the two graphs.
Do the symmetric modes repeat themselves for each mass?
Do the antisymmetric modes repeat themselves for each mass?
What is the difference between the two types of modes with respect to their frequencies and the amplitudes?

EXERCISE 14.2 For the system shown in Fig. 14.2 and discussed in Section 14.3, find the normal frequencies and normal modes using the matrix method discussed above.

Example 14.3

Find the frequencies of small oscillation for a double pendulum, as shown in Fig. Ex. 14.3(a). We may assume that

$$m_1 = m_2 = m \quad \text{and} \quad l_1 = l_2 = l$$



(b) Antisymmetric mode (c) Symmetric mode

Figure Ex. 14.3

Solution

Let (x_1, y_1) and (x_2, y_2) be the coordinates of the two masses of the pendulums such that the lengths of the pendulums make angles θ_1 and θ_2 as shown. From Fig. Ex. 14.3(a),

$$x_1 = l_1 \sin \theta_1$$

$$x_2 = l_1 \sin \theta_1 + l_2 \sin \theta_2$$

$$y_1 = l_1 \cos \theta_1$$

$$y_2 = l_1 \cos \theta_1 + l_2 \cos \theta_2$$

Thus the potential energy of the system is

$$V = -mgy_1 - mgy_2 = -mgl \cos \theta_1 - mgl(\cos \theta_1 + \cos \theta_2) \tag{i}$$

$$\left.\frac{\partial V}{\partial \theta_1}\right|_{\substack{\theta_1 = 0 \\ \theta_2 = 0}} = 0 \quad \text{and} \quad \left.\frac{\partial V}{\partial \theta_2}\right|_{\substack{\theta_1 = 0 \\ \theta_2 = 0}} = 0$$

The components V_{lm} are

$$V_{11} = \left.\frac{\partial^2 V}{\partial \theta_1^2}\right|_{\substack{\theta_1 = 0 \\ \theta_2 = 0}} = mgl + mgl = 2mgl$$

$$V_{22} = \left.\frac{\partial^2 V}{\partial \theta_2^2}\right|_{\substack{\theta_1 = 0 \\ \theta_2 = 0}} = mgl$$

and $$V_{12} = V_{21} = 0$$

Thus $$V = \begin{pmatrix} 2mgl & 0 \\ 0 & mgl \end{pmatrix} \tag{ii}$$

Since $$\begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} > 0 \tag{iii}$$

Therefore, the associated homogeneous quadratic form is positive.

The components T_{lm} of T are calculated as follows:

$$\begin{aligned} T &= \tfrac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2) \\ &= \tfrac{1}{2}m[l \cos \theta_1\, \dot{\theta}_1]^2 + \tfrac{1}{2}m[l(-\sin \theta_1)\dot{\theta}_1]^2 \\ &\quad + \tfrac{1}{2}m[l \cos \theta_1\, \dot{\theta}_1 + l \cos \theta_2\, \dot{\theta}_2]^2 + \tfrac{1}{2}m[l(-\sin \theta_1)\dot{\theta}_1]^2 + [l(-\sin \theta_2)\dot{\theta}_2]^2 \\ &= \tfrac{1}{2}ml^2\dot{\theta}_1^2 + \tfrac{1}{2}m[l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2 \cos(\theta_1 - \theta_2)\dot{\theta}_1\, \dot{\theta}_2] \end{aligned} \tag{iv}$$

At the equilibrium point, $\theta_1 = \theta_2 = 0$,

$$T = \tfrac{1}{2}(2ml^2)\dot{\theta}_1^2 + \tfrac{1}{2}ml^2\dot{\theta}_2^2 + ml^2\dot{\theta}_1\, \dot{\theta}_2 \tag{v}$$

The components T_{ll} and T_{lm} are the coefficients of $\frac{1}{2}\dot{\theta}_1^2$ and $\dot{\theta}_l\dot{\theta}_m$; that is,

$$T_{11} = 2ml^2, \qquad T_{22} = ml^2, \qquad T_{12} = T_{21} = ml^2$$

Therefore,

$$\mathbf{T} = \begin{pmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{pmatrix} \tag{vi}$$

The normal frequencies of the double pendulum are given by

$$|\mathbf{V} - \omega^2\mathbf{T}| = 0 \tag{vii}$$

$$\begin{vmatrix} 2mgl - \omega^2 2ml^2 & -\omega^2 ml^2 \\ -\omega^2 ml^2 & mgl - \omega^2 ml^2 \end{vmatrix} = 0$$

which gives

$$\omega_1^2 = (2 - \sqrt{2})\frac{g}{l} \quad \text{and} \quad \omega_2^2 = (2 + \sqrt{2})\frac{g}{l} \tag{viii}$$

The normal modes for a double pendulum for $\omega^2 = \omega_1^2$ are

$$\begin{pmatrix} 2mgl - (2 - \sqrt{2})\frac{g}{l}2ml^2 & -(2 - \sqrt{2})\frac{g}{l}ml^2 \\ -(2 - \sqrt{2})\frac{g}{l}ml^2 & mgl - (2 - \sqrt{2})\frac{g}{l}ml^2 \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 0 \tag{ix}$$

which reduces to

$$(2 - 2\sqrt{2})a_{11} + (2 - \sqrt{2})a_{21} = 0 \tag{x}$$

$$(2 - \sqrt{2})a_{11} + (1 - \sqrt{2})a_{21} = 0 \tag{xi}$$

and

$$\text{If } a_{11} = 1, \qquad a_{21} = \sqrt{2} \tag{xii}$$

Similarly, for $\omega^2 = \omega_2^2$, we get

$$\text{If } a_{12} = 1, \qquad a_{22} = -\sqrt{2} \tag{xiii}$$

a_{11} and a_{12} correspond to particle 1, and a_{21} and a_{22} correspond to particle 2. The two modes are

$$\eta_1 = a_{11}x_1 + a_{12}x_2 = x_1 + x_2 \tag{xiv}$$

$$\eta_2 = a_{21}x_1 + a_{22}x_2 = \sqrt{2}(x_1 - x_2) \tag{xv}$$

In mode η_1, the particles oscillate out of phase and it is an antisymmetric mode, as shown in Fig. Ex. 14.3(b). In mode η_2, they oscillate in phase and it is a symmetric mode, as shown in Fig. Ex. 14.3(c).

The above remarks are illustrated using numerical values. Below are graphed the natural modes x1, x2; normal modes X1, X2; and the sum of natural modes x1 + x2 and sum of the normal modes X1 + X2.

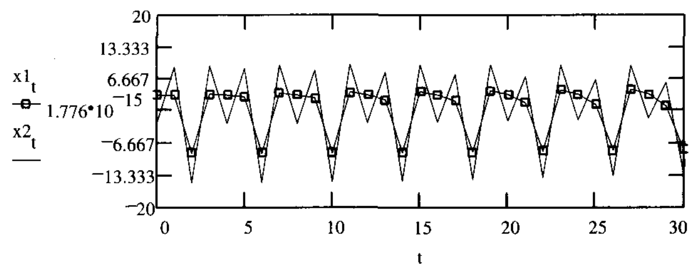
$A11 := 2 \quad A12 := 4 \quad A21 := 3 \quad A22 := 6 \quad k1 := 5 \quad k2 := 15 \quad m := 2$

$N := 50 \quad t := 0..N \quad \omega 1 := \sqrt{\frac{k1}{m}} \quad \omega 1 = 1.581 \quad \omega 2 := \sqrt{\frac{k1 + k2}{m}} \quad \omega 2 = 3.162 \quad i := \sqrt{-1}$

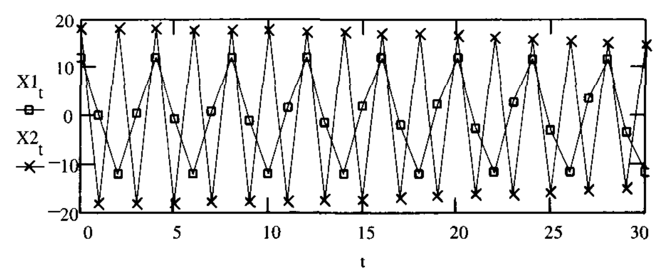
$x1_t := A11 \cdot e^{i \cdot \omega 1 \cdot t} + A12 \cdot (e)^{-i \cdot \omega 1 \cdot t} + A21 \cdot e^{i \cdot \omega 2 \cdot t} - A22 \cdot (e)^{-i \cdot \omega 2 \cdot t} \qquad X1_t := 2 \cdot A11 \cdot e^{i \cdot \omega 1 \cdot t} + 2 \cdot A12 \cdot e^{-i \cdot \omega 1 \cdot t}$

$x2_t := \left(A11 \cdot e^{i \cdot \omega 1 \cdot t} + A12 \cdot e^{-i \cdot \omega 1 \cdot t}\right) - A21 \cdot e^{i \cdot \omega 2 \cdot t} - A22 \cdot e^{-i \cdot \omega 2 \cdot t} \qquad X2_t := 2 \cdot A21 \cdot e^{i \cdot \omega 2 \cdot t} + 2 \cdot A22 \cdot e^{-i \cdot \omega 2 \cdot t}$

(a) In each of the graphs, explain the differences (in terms of frequencies, amplitudes, and phase differences) between:
x1 and x2
X1 and X2
x1 + x2 and X1 + X2

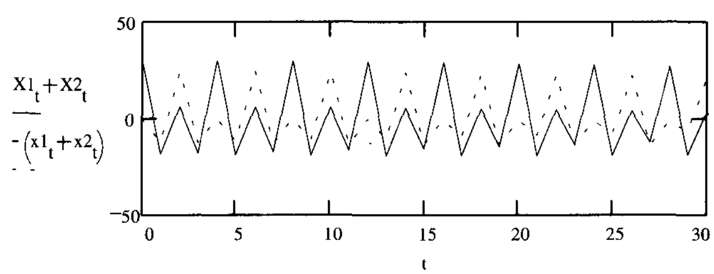


(b) What are the outstanding features of normal modes as shown by these graphs?



(c) What is the significance of the maximum and minimum values of the two graphs?

(d) What do you conclude from these graphs?



EXERCISE 14.3 Consider the situation shown in Fig. Exer. 14.3. Mass M is constrained to move on a smoother frictionless track AB. Another mass m is connected to M by a massless inextensible string of length l. Calculate the frequencies of small oscillations. Draw graphs similar to those in Exercise 14.3.

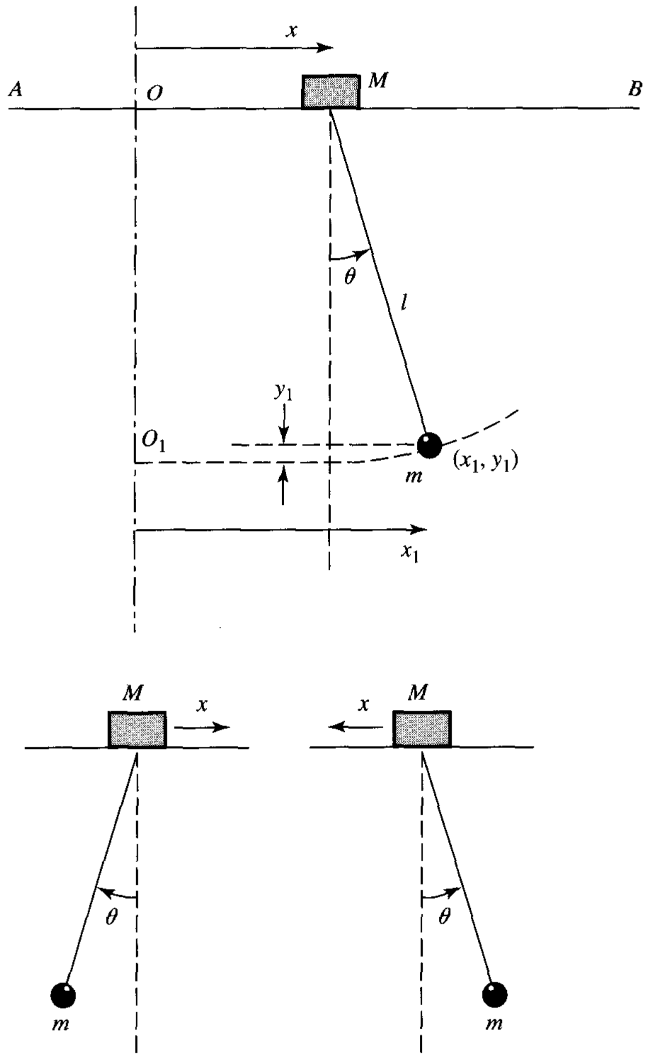


Figure Ex. 14.3

14.7 SYMPATHETIC VIBRATIONS AND BEATS

Let us consider two simple oscillators each of length l and mass m that are coupled by a spring constant k, as shown in Fig. Ex. 14.2. If the spring offers a small resistance to the relative motion of the two pendulums, we say that the system has *weak coupling*, whereas if the spring offers a greater resistance, the system is said to have *strong coupling*. If the pendulums are not exactly equal in length or in mass, we say that the two pendulums are *out of tune* or *detuned.*

For the present, let us assume that the two pendulums are exactly of equal length and mass, and they are weakly coupled by a spring. We assume that they oscillate in the same plane. Let us further assume that the one pendulum is excited by giving an initial displacement while the other pendulum is at rest. As time passes, the resulting oscillations of the two pendulums are as

Figure 14.4

Below the resonance between two weakly coupled oscillators such as pendulums is shown. We may use Eqs. (14.119) and (14.120) or (14.121).

$n := 200$ $\quad i := 0..n$ $\quad t_i := \frac{i}{20}$ $\quad A := 10$ $\quad A11 := 10$ $\quad A12 := 10$

$\omega 1 := 88$ $\quad \omega 2 := 90$ $\quad \omega 0 := \frac{\omega 2 - \omega 1}{2}$ $\quad \omega 0 = 1$ $\quad T := \frac{2 \cdot \pi}{\omega 0}$ $\quad T = 6.283$

$x1_i := A11 \cdot \cos(\omega 1 \cdot t_i) - A12 \cdot \cos(\omega 2 \cdot t_i)$ $\quad x2_i := (A11 \cdot \cos(\omega 1 \cdot t_i) + A12 \cdot \cos(\omega 2 \cdot t_i))$

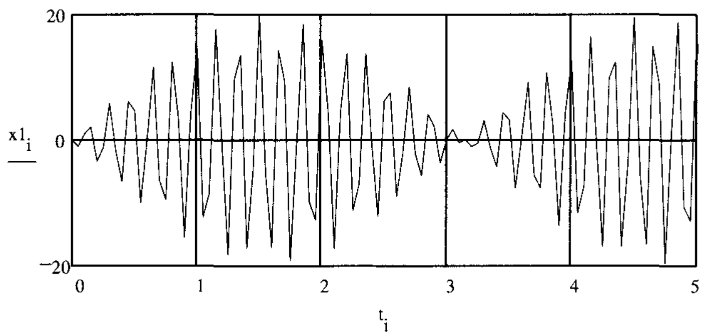
First oscillator:

$t \equiv 0$ $\quad x1 \equiv 0$ $\quad v1 \equiv 0$

$x1_0 = 0$

$\max(x1) = 19.947$

$\min(x1) = -19.604$



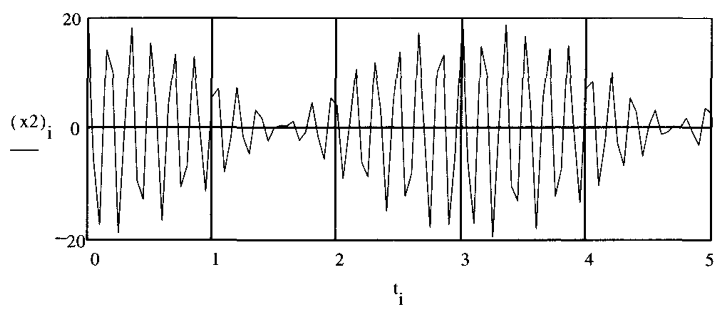
Second oscillator:

$t \equiv 0$ $\quad x2 \equiv A$ $\quad v2 \equiv 0$

$x2_0 = 20$

$\max(x2) = 20$

$\min(x2) = -19.636$



(a) What determines the amplitude of the oscillations in the two cases?
(b) In the two graphs draw the envelope of the oscillations.
(c) How will the increase or decrease in frequency affect the resonance?
(d) How will your increase or decrease in the amplitude affect the resonance?
(e) How do the above graphs illustrate the transfer of energy from one oscillator to the other and vice versa?

shown in Fig. 14.4. As is clear, the oscillations are modulated, and the energy is continuously being transferred from one pendulum to the other. When one pendulum is oscillating with maximum amplitude, the other pendulum is at rest, and vice versa. This is the phenomenon of *resonance* or sympathetic vibration between two systems. The alternation of energy between the two pendulums can be shown mathematically as explained next. This is the *theory of resonance,* as illustrated in Fig. 14.4. A slight detuning leads to the phenomenon of beats, as we shall see later.

Suppose, for the case in Fig. Ex. 14.2, at $t = 0$, we have $x_1 = 0$, $\dot{x}_1 = 0$, $x_2 = A$, and $\dot{x}_2 = 0$. Applying these conditions to Eqs. (xviii) and (xix) in Example 14.2, that is, we get [or for the system shown in Fig. 14.2, resulting in Eqs. (14.39) and (14.40)]

$$x_1(t) = A_1 e^{i\omega_1 t} + A_{-1} e^{-i\omega_1 t} + A_2 e^{i\omega_2 t} + A_{-2} e^{-i\omega_2 t} \tag{14.114}$$

$$x_2(t) = A_1 e^{i\omega_1 t} + A_{-1} e^{-i\omega_1 t} - A_2 e^{i\omega_2 t} - A_{-2} e^{-i\omega_2 t} \tag{14.115}$$

we obtain, at $t = 0$,

$$A_1 + A_{-1} + A_2 + A_{-2} = 0 \tag{14.116a}$$

$$A_1 + A_{-1} - A_2 - A_{-2} = A \tag{14.116b}$$

$$i\omega_1(A_1 - A_{-1}) + i\omega_2(A_2 - A_{-2}) = 0 \tag{14.117a}$$

$$i\omega_1(A_1 - A_{-1}) - i\omega_2(A_2 - A_{-2}) = 0 \tag{14.117b}$$

Solving these equations yields

$$A_1 = A_{-1} = \frac{A}{4} \quad \text{and} \quad A_2 = A_{-2} = -\frac{A}{4} \tag{14.118}$$

Substituting these in Eqs. (14.114) and (14.115), we obtain

$$x_1(t) = \frac{A}{4}[(e^{i\omega_1 t} + e^{-i\omega_1 t}) - (e^{i\omega_2 t} + e^{-i\omega_2 t})]$$

$$x_2(t) = \frac{A}{4}[(e^{i\omega_1 t} + e^{-i\omega_1 t}) + (e^{i\omega_2 t} + e^{-i\omega_2 t})]$$

Since $2\cos\theta = e^{i\theta} + e^{-i\theta}$, we may write

$$x_1 = \frac{A}{2}(\cos\omega_1 t - \cos\omega_2 t) \tag{14.119}$$

$$x_2 = \frac{A}{2}(\cos\omega_1 t + \cos\omega_2 t) \tag{14.120}$$

Equations (14.119) and (14.120) may also be written as

$$x_1 = A\sin\left(\frac{\omega_2 - \omega_1}{2}t\right)\sin\left(\frac{\omega_1 + \omega_2}{2}t\right) \tag{14.121}$$

$$x_2 = A\cos\left(\frac{\omega_2 - \omega_1}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right) \tag{14.122}$$

Let $(\omega_1 + \omega_2)/2 = \omega_0$ and $\omega_2 \simeq \omega_1$; then we may write

$$x_1 = A \sin\left(\frac{\omega_2 - \omega_1}{2} t\right) \sin \omega_0 t \tag{14.123}$$

$$x_2 = A \cos\left(\frac{\omega_2 - \omega_1}{2} t\right) \cos \omega_0 t \tag{14.124}$$

Note that, at $t = 0$, $x_1 = 0$, and $x_2 = A$, as it should be. These equations state that x_1 and x_2 are executing oscillatory motions $\sin \omega_0 t$ and $\cos \omega_0 t$, with their slowly varying amplitudes given respectively by

$$A \sin\left(\frac{\omega_2 - \omega_1}{2} t\right) \tag{14.125}$$

and

$$A \cos\left(\frac{\omega_2 - \omega_1}{2} t\right) \tag{14.126}$$

This implies that, as the amplitude of x_1 becomes larger, that of x_2 becomes smaller and smaller, and vice versa. This is demonstrated in Fig. 14.4. This means that there is a transfer of energy back and forth. The period T of this energy transfer is

$$T = \frac{2\pi}{\omega} = \frac{4\pi}{\omega_2 - \omega_1} \tag{14.127}$$

If the two pendulums are slightly detuned (have slightly different frequencies), the energy exchange will still take place, but this exchange is not complete. The initially excited second, pendulum reaches a certain minimum amplitude, but not zero amplitude. The first pendulum initially at rest, does reach zero amplitude during its oscillations. This results in the phenomenon of *beats,* as shown in Fig. 14.5. Thus sympathetic vibration or resonance is upset by slight detuning. We can apply these considerations to another example, that of the double pendulum, as discussed in Example 14.3. If the two masses and the two lengths are equal, we still can have sympathetic resonance vibrations. But suppose the upper mass (and hence weight) is much larger than the lower mass. This leads to slight detuning and to the formation of beats. Suppose we set the pendulum in motion by pulling the upper mass slightly away from the vertical and releasing it. In the subsequent motion, at regular intervals, the lower mass will come to rest, while the upper mass will have a maximum amplitude, or the upper mass will have a minimum amplitude (different from zero) when the lower mass has maximum amplitude. This is the phenomenon of beats, as illustrated in Fig. 14.5. Once again, due to slight detuning, there is an incomplete transfer of energy.

If instead of looking at the normal modes, we look at the motion of the two separately, the resulting natural modes of the two are as was shown in Fig. 14.4. It is clear that when one has maximum displacement, the other has minimum and vice versa.

If in the preceding examples, both pendulums were set in motion simultaneously either (1) in the same direction or (2) in opposite directions, we would find that there would be no energy exchange between the two pendulums. We get the normal modes of vibrations as discussed in Section 14.3 and in Example 14.2.

Figure 14.5

Below the phenomenon of beats resulting from two slightly detuned, weakly coupled oscillators (pendulums in this case) are shown.

$n := 200 \quad i := 0..n \quad \omega 1 := 12 \quad \omega 2 := 13$

$t_i := \dfrac{i}{20} \quad A := 10 \quad \omega 0 := \dfrac{\omega 2 + \omega 1}{2} \quad \omega 0 = 12.5$

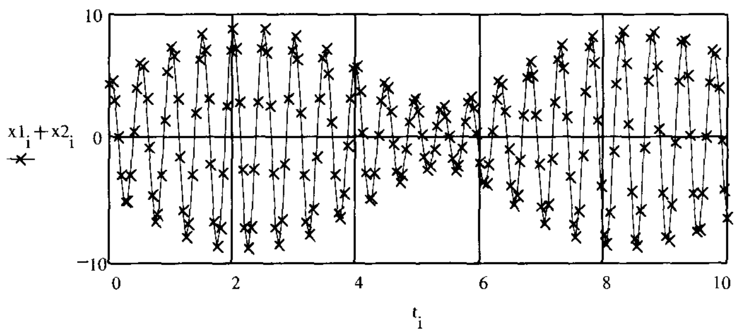
$T := \dfrac{2\cdot\pi}{\omega 0} \quad |T| = 0.503$

$x1_i := (A)\cdot\left[\sin\left[\dfrac{\omega 2 - \omega 1}{2}\cdot(t_i + 5)\right]\cdot\sin(\omega 0\cdot t_i + 5)\right] \qquad x2_i := A\cdot\cos\left(\dfrac{\omega 2 - \omega 1}{2}\cdot t_i\right)\cdot\cos(\omega 0\cdot t_i)$

$x1_0 = -5.739$

$\max(x1) = 9.992$

$\min(x1) = -9.962$

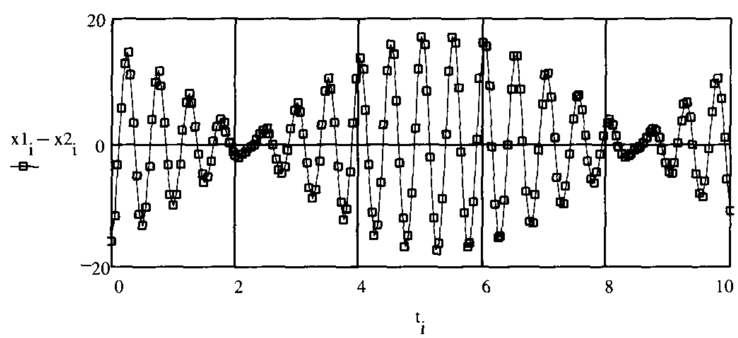


Upper mass displaced at t=0

$x2_0 = 10$

$\max(x2) = 10$

$\min(x2) = -9.921$



Lower mass not displaced at t=0

What is the significant difference between the two graphs?

The preceding discussion for coupled *mechanical oscillating systems* can be extended to electrical systems. Sympathetic oscillations are of great importance in electrical circuits. In electrical systems, we have a primary and a secondary circuit that are usually inductively coupled with each other. Thus, if the primary circuit is excited, the secondary circuit will also oscillate

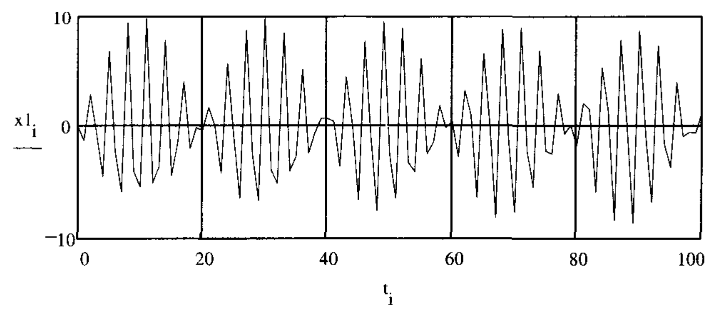
Figure 14.5 (continued)

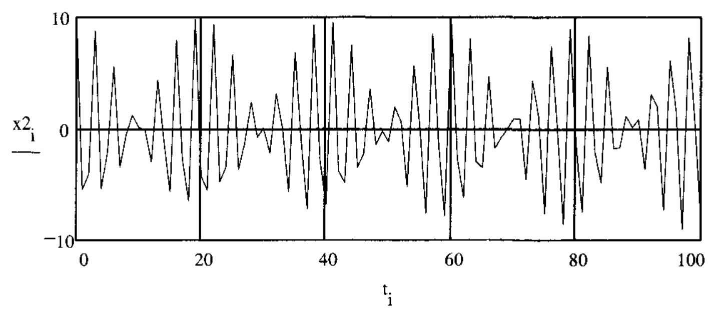
The transfer of displacement is equivalent to transfer of energy, between two lightly coupled oscillators. Thus if we graph x1 and x2 separately, it illustrates the trasfer of energy between the two lightly coupled oscillators as shown below.

$n := 100$ $i := 0..n$ $t_i := i$ $A := 10$ $\omega1 := 40$ $\omega2 := 42$

$\omega0 := \frac{\omega2 + \omega1}{2}$ $\omega0 = 41$

$x1_i := A \cdot \sin\left(\frac{\omega2 - \omega1}{2} \cdot t_i\right) \cdot \sin(\omega0 \cdot t_i)$ $x2_i := A \cdot \cos\left(\frac{\omega2 - \omega1}{2} \cdot t_i\right) \cdot \cos(\omega0 \cdot t_i)$





(a) How do you explain that when x1 is minimum x2 is maximum and vice versa?
(b) What is the phase relation between x1 and x2 and how do you explain it?

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strongly if there is a resonance. Unlike the coupled pendulums considered previously, in electrical circuits damping must be included. As discussed in Chapter 4, damping is equivalent to ohmic resistance, mass corresponds to the self-inductance, and restoring force to the capacitance effects. Furthermore, in electrical oscillations, we deal not only with "position coupling" but also with "velocity and acceleration coupling."

14.8 VIBRATION OF MOLECULES

We shall consider possible modes of vibrations for diatomic and triatomic molecules. A typical diatomic molecule may be regarded as equivalent to two masses m_1 and m_2 connected by a massless spring of spring constant k and of unstretched length a, vibrating along the line joining the two masses, as shown in Fig. 14.6. Let x_1 and x_2 be the coordinates of m_1 and m_2 measured from a fixed point O. The potential energy and kinetic energy of the system are

$$V = \tfrac{1}{2}k(x_2 - x_1 - a)^2 \tag{14.128}$$

$$T = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 \tag{14.129}$$

The expression for the potential energy is not a homogeneous quadratic function; hence a linear transformation to normal coordinates is not possible. But this difficulty can be overcome by making the substitution

$$u = x_2 - a \quad \text{and} \quad \dot{u} = \dot{x}_2 \tag{14.130}$$

Substituting these in Eqs. (14.128) and (14.129),

$$V = \tfrac{1}{2}k(u - x_1)^2 \tag{14.131}$$

$$T = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{u}^2 \tag{14.132}$$

By using x_1 and u as generalized coordinates, we can solve the Lagrangian equation for x_1 and u. By using proper linear combinations of x_1 and u, we can find the normal coordinates X_1 and X_2 corresponding to ω_1 and ω_2 respectively. Thus

$$X_1 = \frac{m_1}{m_2}x_1 + u \qquad \text{and} \qquad X_2 = x_1 - u \tag{14.133}$$

If mode X_1 is excited, then X_2 must be suppressed; that is,

$$\text{For mode } X_1: \qquad X_2 = x_1 - u = 0$$

or

$$x_1 = u = x_2 - a \tag{14.134}$$

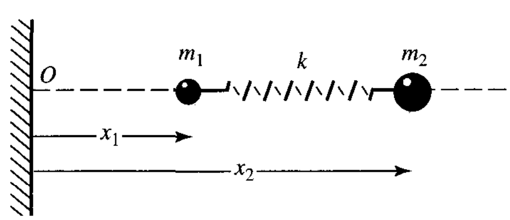


Figure 14.6 Schematic of a system equivalent to a diatomic molecule.

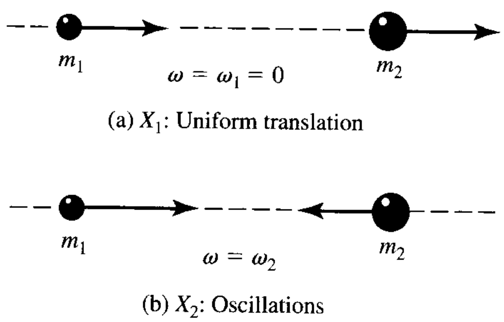


Figure 14.7 Two possible normal modes of vibration of the system of Fig. 14.6.

which corresponds to uniform translation motion of the system, as shown in Fig. 14.7(a). Similarly, if mode X_2 is excited, then X_1 must be suppressed; that is,

$$\text{For mode } X_2: \qquad X_1 = \frac{m_1}{m_2} x_1 + u = 0$$

or

$$x_1 = -\frac{m_2}{m_1} u = -\frac{m_2}{m_1}(x_2 - a) \tag{14.135}$$

which indicates that the two masses oscillate relative to the center of mass, as shown in Fig. 14.7(b).

The results obtained can be arrived at by an inspection of the situation and recognizing the basic physical problem. Let us demonstrate this in the case of a triatomic molecule such as CO_2, as shown in Fig. 14.8. CO_2 is a linear molecule, and if the motion is constrained along a line, it will have three degrees of freedom and hence three normal coordinates.

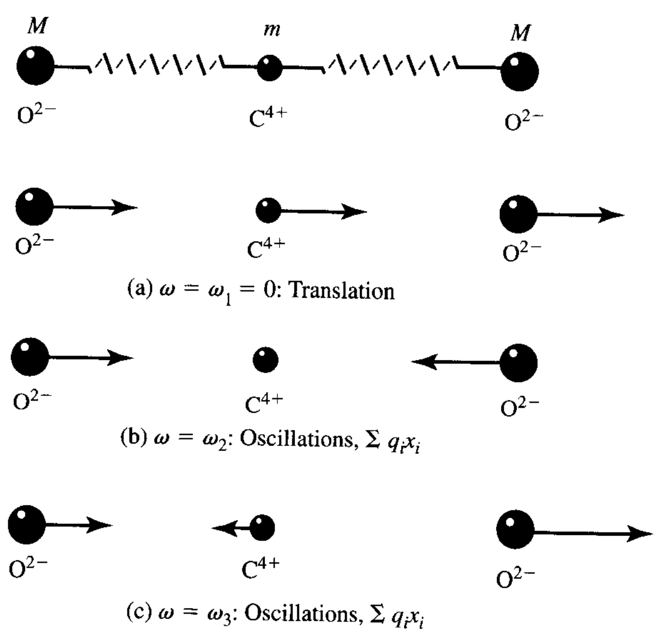


Figure 14.8 A triatomic molecule and its three possible normal modes of vibration.

14.9 DISSIPATIVE SYSTEMS AND FORCED OSCILLATIONS

So far in the discussion of small oscillations, we neglected the effects of viscous or frictional forces. A common situation is one in which the viscous damping forces are proportional to the first power of the velocity. In such situations, the motion of the ith particle may be described by Newton's second law as

$$m_i\ddot{\mathbf{r}} = \mathbf{F}_i - c_i\dot{\mathbf{r}}_i \tag{14.136}$$

which in component form may by written as

$$m_i\ddot{x}_i = F_{ix} - c_i\dot{x}_i \tag{14.137a}$$

$$m_i\ddot{y}_i = F_{iy} - c_i\dot{y}_i \tag{14.137b}$$

$$m_i\ddot{z}_i = F_{iz} - c_i\dot{z}_i \tag{14.137c}$$

where c_i are constants and F_{ix}, F_{iy}, and F_{iz} are the components of a resultant force $\mathbf{F}_i$ that are derivable from a potential, and the potential is a homogeneous quadratic function of the coordinates.

Suppose the system has l degrees of freedom and is described by l independent coordinates:

$$q_1', q_2', \ldots, q_l' \tag{14.138}$$

The relations between these and the x, y, and z coordinates are given by the following $3n$ equations for n particles.

$$x_i = x_i(q_1', q_2', \ldots, q_l')$$

$$y_i = y_i(q_1', q_2', \ldots, q_l')$$

$$z_i = z_i(q_1', q_2', \ldots, q_l') \tag{14.139}$$

Note that there is no explicit dependence on time t because kinetic energy T is a homogeneous quadratic function of time. Multiply each of Eqs. (14.137), respectively, by the quantities $\partial x_i/\partial q_j'$, $\partial y_i/\partial q_j'$, and $\partial z_i/\partial q_j'$; adding all three and summing over all the n particles yields

$$\sum_{i=1}^{n} m_i\left(\ddot{x}_i\frac{\partial x_i}{\partial q_j'} + \ddot{y}_i\frac{\partial y_i}{\partial q_j'} + \ddot{z}_i\frac{\partial z_i}{\partial q_j'}\right)$$

$$= \sum_{i=1}^{n}\left(F_{ix}\frac{\partial x_i}{\partial q_j'} + F_{iy}\frac{\partial y_i}{\partial q_j'} + F_{iz}\frac{\partial z_i}{\partial q_j'}\right) - \sum_{i=1}^{n} c_i\left(\dot{x}_i\frac{\partial x_i}{\partial q_j'} + \dot{y}_i\frac{\partial y_i}{\partial q_j'} + \dot{z}_i\frac{\partial z_i}{\partial q_j'}\right) \tag{14.140}$$

where

$$\text{First term on the left} \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j'}\right) - \frac{\partial T}{\partial q_j'}$$

$$\text{First term on the right} \equiv -\frac{\partial V}{\partial q_j'} \equiv Q_i,$$ the generalized force, excluding the dissipative forces

$$\text{Second term on the right} \equiv -\frac{\partial}{\partial \dot{q}_j'}\left[\frac{1}{2}\sum_{i=1}^{n} c_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)\right] = -\frac{\partial F_r}{\partial \dot{q}_j'}$$

and $F_r = \frac{1}{2}\Sigma c_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$ is the dissipative function named by Rayleigh and represents one-half the rate at which the energy is being dissipated through the action of frictional forces. Thus Eq. (14.140) may be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}'_j}\right) - \frac{\partial T}{\partial q'_j} = -\frac{\partial V}{\partial q'_j} = -\frac{\partial F_r}{\partial \dot{q}'_j} \tag{14.141}$$

Since $L = T - V$, Eq. (14.137) or (14.141) takes form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}'_j}\right) = \frac{\partial L}{\partial q'_j} + Q_{rj} \tag{14.142}$$

where Q_{rj} is the generalized damping force

$$Q_{rj} = -\frac{\partial F_r}{\partial \dot{q}'_j} \tag{14.143}$$

For sufficiently small motions, the expressions for V, T, and F_r may be written as

$$V = a_{11}q_1'^2 + \cdots + a_{ll}q_1'^2 + 2a_{12}q_1'q_2' + \cdots \tag{14.144a}$$

$$T = b_{11}\dot{q}_1'^2 + \cdots + b_{ll}\dot{q}_1'^2 + 2b_{12}\dot{q}_1'\dot{q}_2' + \cdots \tag{14.144b}$$

$$F_r = c_{11}\dot{q}_1'^2 + \cdots + c_{ll}\dot{q}_1'^2 + 2c_{12}\dot{q}_1'\dot{q}_2' + \cdots \tag{14.144c}$$

where $a_{ll}, \ldots, b_{ll}, \ldots$, and $c_{ll}, \ldots$, are constants.

The resulting differential equations of motion obtained from Eq. (14.141) or (14.142) are similar to the undamped case, except that terms of the form $\dot{q}$ are present. To calculate normal modes, we must find new coordinates that are linear combinations of $q'_1, q'_2, \ldots, q'_l$ so that V, T, and F_r, when expressed in terms of coordinates $\eta_1, \eta_2, \ldots, \eta_j$, do not contain cross terms; that is, they contain the sum of the squares of the new coordinates and their time derivatives. Because of the presence of F_r, it is not always possible to find such new coordinates. In some situations it is possible to find a normal coordinate transformation, and the resulting differential equations are of the form

$$m_j\ddot{\eta}_j + c_j\dot{\eta}_j + k_j\eta_j = 0 \tag{14.145}$$

which have solutions of the form

$$\eta_j = A_j e^{-\lambda_j t}\cos(\omega_j t + \phi_j) \tag{14.146}$$

Thus, unlike the case of undamped motion in which one observes oscillations, in the present case the motion may be underdamped, critically damped, or overdamped, as the case may be; hence the motion may be nonoscillatory. The normal coordinates and their phases are the same as in the corresponding problem of undamped motion. The amplitude decreases exponentially with time, while the frequencies are different from the ones in the undamped case.

First, we must assume that the driving forces are small enough so that the squares of the displacements and velocities will be such that the equations of motion are still linear. If the forces are constant, such as a system under gravitational force, the only change is in the equilibrium position about which the oscillations take place. If the driving force is periodic, it is possible to discuss motion in terms of normal coordinates. For convenience, let us assume that a single harmonic force of the type $Q_{jext}\cos\omega t$ or $Q_{jext}e^{i\omega t}$ is applied. The resulting equation of motion in normal coordinates is of the form (in the presence of a linear restoring force, dissipative force, and driving force)

$$m_j\ddot{\eta}_j + c_j\dot{\eta}_j + k_j\eta_j = Q_{jext}e^{i\omega t} \tag{14.147}$$

If the driving frequency is equal to one of the normal frequencies of the system, the corresponding normal mode will assume the largest amplitude in the steady state. Furthermore, if the damping constants are small, not all normal modes are excited to any appreciable extent; only one normal mode that has the same frequency as the driving force will be excited.

Example 14.4

Let us consider once again the situation of two coupled pendula, as discussed in Example 14.2. Let us assume that the driving force is $F\cos\omega t$, and the frictional force proportional to velocity is $c\dot{x}$, where c is a constant. Discuss the solution of this problem.

Solution

The equations describing the system are

$$m\ddot{x}_1 + \frac{mg}{l}x_1 + k(x_1 - x_2) = -c\dot{x}_1 + F\cos\omega t$$

$$m\ddot{x}_2 + \frac{mg}{l}x_2 - k(x_1 - x_2) = -c\dot{x}_2 + F\cos\omega t$$

Equations involving normal coordinates X_1 and X_2 are ($\eta_1 = X_1 = x_1 + x_2$ and $\eta_2 = X_2 = x_1 - x_2$)

$$\ddot{X}_1 + \frac{c}{m}\dot{X}_1 + \frac{g}{l}X_1 = \frac{2F}{m}\cos\omega t$$

$$\ddot{X}_2 + \frac{c}{m}\dot{X}_2 + \left(\frac{g}{l} + \frac{2k}{m}\right)X_2 = 0$$

We should be able to recognize these differential equations, which have the following solutions:

$$X_1 = e^{-(c/2m)t}(A_1e^{i\omega_1' t} + A_{-1}e^{-i\omega_1' t}) + \frac{2F\cos(\omega t - \phi)}{[m^2(\omega_0^2 - \omega^2)^2 + \omega^2c^2]^{1/2}}$$

and

$$X_2 = e^{-(c/2m)t}(A_2e^{i\omega_2' t} + A_{-2}e^{-i\omega_2' t})$$

where

$$\omega_0 = \left(\frac{g}{l}\right)^{1/2}, \qquad \omega_1' = \left(\frac{g}{l} - \frac{c^2}{4m^2}\right)^{1/2}, \qquad \omega_2' = \left[\left(\frac{g}{l} + \frac{2k}{m}\right) - \frac{c^2}{4m^2}\right]^{1/2}$$

$$\tan\phi = \frac{\omega c}{[m(\omega_0^2 - \omega^2)]}, \qquad \text{for } g/l > c^2/4$$

Both X_1 and X_2 contain transient terms. Only X_1 possesses a steady-state term, and only X_1 will remain excited (for any initial conditions) with the same frequency as the driving frequency, which is similar to a system having one degree of freedom, X_2 will decay in a short interval. These points are illustrated in the following graphs.

Assuming the following values and graphing with and without the driving force, X1 and X2, respectively, gives

$g := 9.8 \quad l := 1 \quad c := 0.5 \quad m := 1 \quad k := 2 \quad n := 0..60 \quad t_n := n \quad i := \sqrt{-1}$

$A1 := 4 \quad A12 := 2 \quad A2 := 15 \quad A21 := 10 \quad F := 5$

$$\omega 0 := \sqrt{\frac{g}{l}} \qquad \omega 1 := \left(\frac{g}{l} - \frac{c^2}{4 \cdot m^2}\right)^{\frac{1}{2}} \qquad \omega 2 := \left[\left(\frac{g}{l} + \frac{2 \cdot k}{m}\right) - \frac{c^2}{4 \cdot m^2}\right]^{\frac{1}{2}}$$

$\omega := 3 \quad \phi := \frac{\pi}{2}$

$\omega 0 = 3.13$

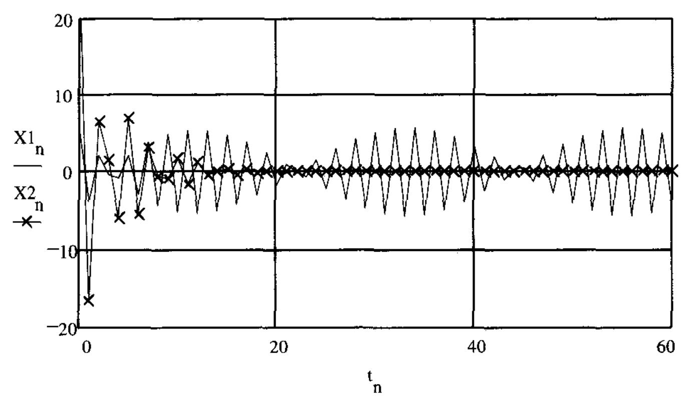
$\omega 1 = 3.12$

$\omega 2 = 3.706$

$$X1_n := e^{-\left(\frac{c}{2 \cdot m}\right) \cdot t_n} \cdot \left(A1 \cdot e^{i \cdot \omega 1 \cdot t_n} + A12 \cdot e^{-i \cdot \omega 1 \cdot t_n}\right) + \frac{2 \cdot F \cdot \cos\left(\omega \cdot t_n - \phi\right)}{\sqrt{m^2 \cdot \left(\omega 0^2 - \omega^2\right) + \omega^2 \cdot c^2}}$$

$$X2_n := e^{-\left(\frac{c}{2 \cdot m}\right) \cdot t_n} \cdot \left(A2 \cdot e^{i \cdot \omega 2 \cdot t_n} + A21 \cdot e^{-i \cdot \omega 2 \cdot t_n}\right)$$

Both X1 and X2 contain transient terms. Only X1 possesses a steady-state term and remains excited for any initial condition with the same frequency as the driving frequency. X2 will decay away in a short time.



EXERCISE 14.4 Repeat the above example with the applied force equal to F sin(ωt). What are the similarities and differences between the two?

PROBLEMS

14.1. A cube of side $2a$ is balanced on top of a rough spherical surface of radius R. Show that the equilibrium is stable if $R > a$ and unstable if $R = a$. What happens if $R = a$? Find the frequency of small oscillations.

14.2. In Problem 14.1, if the cube is replaced by a homogeneous solid hemisphere of radius r, show that for stable equilibrium $r < \frac{3}{5}R$.

14.3. A homogeneous rectangular slab of thickness d is placed atop and at right angles to a fixed cylinder of radius R with its axis horizontal. Assuming no slipping, show that the condition for stable equilibrium is $R < d/2$. Draw a potential energy function versus the angular displacement θ, and show that there is a minimum at $\theta = 0$ for $R > d/2$ but not for $R < d/2$. Find the frequency of small oscillations about equilibrium.

14.4. A homogeneous disk of mass M and radius R rolls without slipping on a horizontal surface and is attracted toward a point that lies at a distance d below the surface. The attractive force is proportional to the distance between the center of mass and the force center. Is the disk in stable equilibrium? If so, find the frequency of small oscillations.

14.5. Two identical springs each of natural length l_0 and stiffness constant k have their upper ends tied at two points A and B, which are a distance $2a$ apart. The two lower ends are tied together at C, and a mass m hangs it, as shown in Fig. P14.5. Find the position of equilibrium. Is it a position of stable equilibrium? Find the frequency of small oscillations.

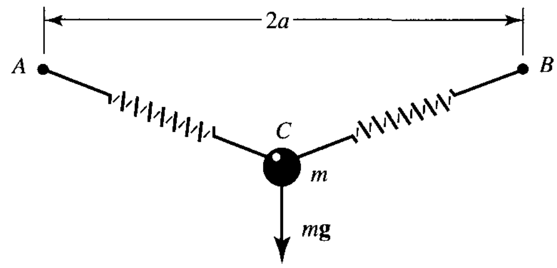


Figure P14.5

14.6. A mass m is subject to a force whose potential energy function is

$$V = V_0 \exp[(5x^2 + 5y^2 + 8z^2 - 8yz - 26ya - 8za)/a^2]$$

where V_0 and a are constants. Find, if any, positions of stable or unstable equilibrium. Find the normal frequencies of vibration about the minimum.

14.7. A particle of mass m moves along the X-axis under the influence of a potential energy given by $V(x) = -Axe^{-kx}$, where A and k are constants. Make a plot of $V(x)$ versus x. Find the position of equilibrium. Also calculate the frequency of small oscillations.

14.8. Consider a rod of length L and mass m supported by two springs, as shown in Fig. P14.8. Assuming that the rod remains in the vertical plane, calculate the normal frequencies of oscillation. Graph the normal modes.

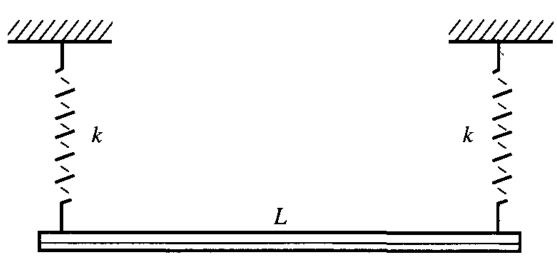


Figure P14.8

14.9. For the configuration of two masses and two springs as shown in Fig. P14.9, calculate the normal frequencies and normal coordinates, assuming that the motion is restricted to the vertical plane. Graph the natural as well as normal modes.

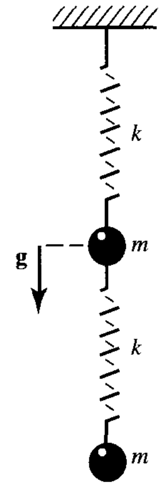


Figure P14.9

14.10. Three identical masses and four identical springs are connected as shown in Fig. P14.10. If the system is displaced from its equilibrium position along the line joining the masses, calculate the normal frequencies and normal coordinates for small oscillations. The unstretched length of each spring is a and k is its spring constant. Graph the natural as well as normal modes.

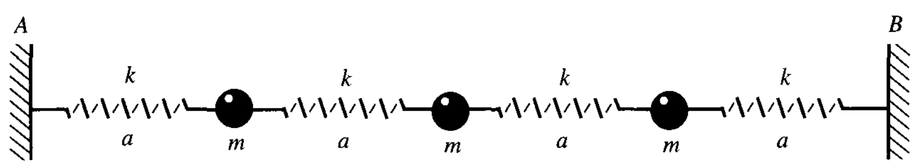


Figure P14.10

14.11. In Problem 14.10, there is a tension T in the spring at points A and B. Calculate the normal frequencies and normal coordinates for small transverse oscillations. Graph the tension versus displacement.

14.12. A light rod OA of length r is fixed at O, and a mass M is attached to the other end, as shown in Fig. P14.12. It is forced to move in the XY-plane. A pendulum of length l and mass m attached at A can oscillate in the YZ-plane. Find the normal frequencies and normal modes of vibration. Graph the normal modes of vibrations.

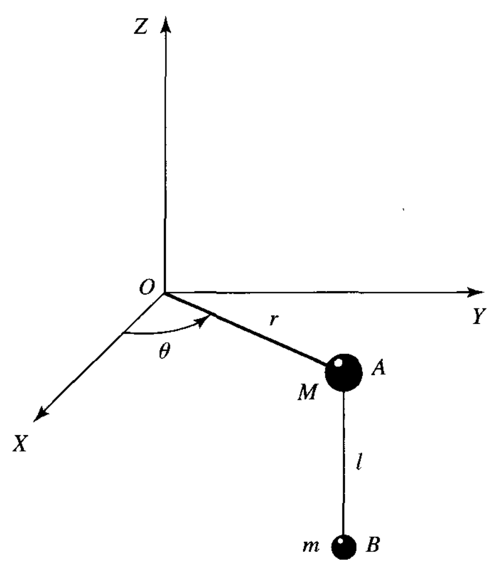


Figure P14.12

14.13. Three oscillators of mass m each are coupled in such a way that the force between them is given by the potential energy function

$$V = \frac{1}{2}[k_1(x_1^2 + x_3^2) + k_2x_2^2 + k_3(x_1x_2 + x_2x_3)]$$

where $k_3 = (2k_1k_2)^{1/2}$. Find the points of equilibrium and their stability. Find the normal frequencies of the system and normal modes of vibration. Graph the normal modes. Is there any physical significance of the null mode?

14.14. Three masses M, m, and m are connected by identical springs of stiffness constant k and placed on a fixed circular loop in space, as shown in Fig. P14.14. Calculate the normal frequencies and normal coordinates. What happens if $M = m$? Also describe the type of motion of these masses. Draw the polar graphs of the motion of the mass m and M.

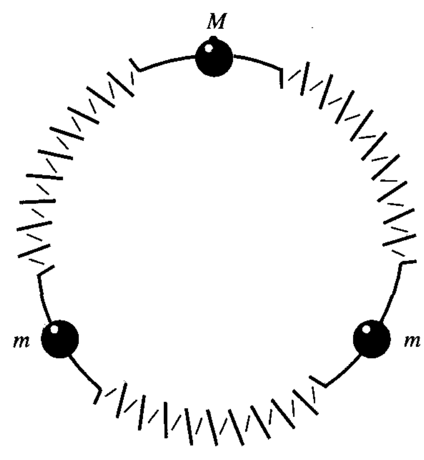


Figure P14.14

14.15. A particle of mass m is moving in a force field that is represented by the potential energy given by

$$V(x) = (1 - \alpha x)e^{-\alpha x}, \quad x \geqslant 0$$

where α is a positive constant. Find **(a)** the equilibrium points, **(b)** the nature of the equilibrium points, and **(c)** the frequency for small oscillations about equilibrium. Graph V and F versus x and displacement versus time.

14.16. A mass m is attached to a mass M by a light string of length l. The mass M slides without friction on a table, while the other mass hangs vertically through a hole in the table, as shown in Fig. P14.16. Find the steady-state motion, normal frequencies, and normal modes for small oscillations. Make appropriate polar graphs to describe the motion of masses m and M. What happens when M touches the whole in the table?

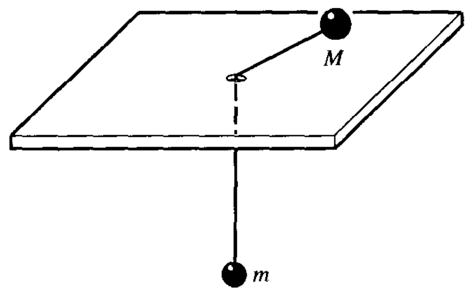


Figure P14.16

14.17. Suppose two identical harmonic oscillations are coupled via a force that is proportional to the relative velocity of the two masses (instead of a force proportional to distance). Find the normal frequencies and normal modes of vibrations.

14.18. A thin wire of mass M bent in the form of circle of radius R is suspended from a point on its circumference. A bead of mass m is attached to the wire and constrained to move on it (frictionless). Find the normal frequencies and normal modes of vibrations if the wire is free to swing in its own plane. If $M = m$, show that

$$\omega_1 = \sqrt{2}\sqrt{\frac{g}{R}} \quad \text{and} \quad \omega_2 = \frac{1}{\sqrt{2}}\sqrt{\frac{g}{R}}$$

Do the normal modes describe any physical situation?

14.19. Consider a double pendulum that consists of one pendulum of length l_1 and mass m_1, and the other of length l_2 and mass m_2. Calculate the normal frequencies. Also find the normal modes. For what initial conditions will the system oscillate in its normal modes? Draw the appropriate graphs to describe the motion.

14.20. Find the normal frequencies and normal modes for the system shown in Fig. P14.20, which consists of three springs and two masses forming a right-angled triangle. What type of motion is expected if one of the masses is displaced in the XY-plane?

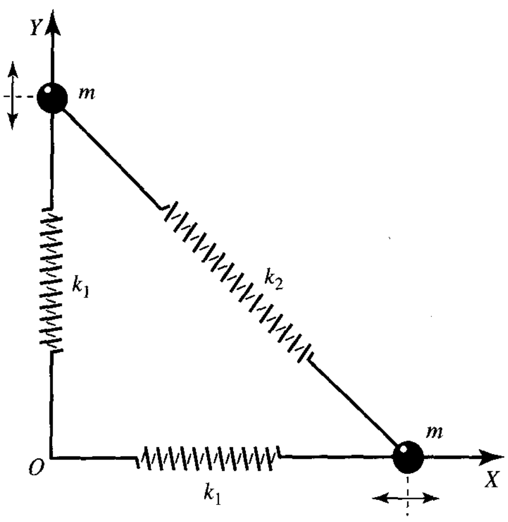


Figure P14.20

14.21. Consider a symmetrical rigid body mounted in weightless, frictionless gimbal rings. One ring exerts a torque $-k\phi$ (ϕ is the Euler angle) about the Z-axis. (This is done by attaching a hair spring.) Investigate the steady-state motion for small vibrations.

14.22. Derive expressions for the normal frequencies and the normal modes of vibration for the triatomic molecule discussed in Section 14.8, that is, CO_2. The mass of carbon is m_1 and that of oxygen is m_2, and assume that the force between adjacent atoms can be represented by a spring of spring constant k and that there is no interaction between the end atoms.

14.23. Consider a plane triatomic molecule consisting of equal masses at the vertices of an equilateral triangle, as shown in Fig. P14.23 (stretched and unstreched). The unstretched length of each spring is a, and the spring constant is k. Consider small oscillations in the plane of the triangle. How many normal modes do you expect and how many of these have zero normal frequencies? Find the frequency of small oscillations for a mode in which all three springs stretch symmetrically, as shown.

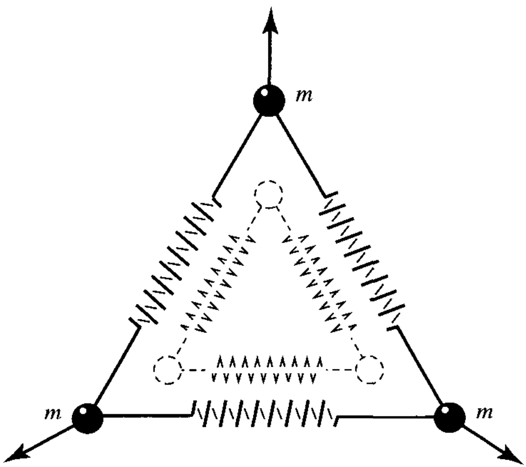


Figure P14.23

14.24. Two simple pendulums are coupled by a weak attractive force given by K/r^2, where r is the distance between the two particle masses. Show that, for a small displacement from equilibrium, the Lagrangian has the same form as that for two coupled oscillators. Furthermore, if one pendulum is set into oscillations while the other is at rest, then eventually the second pendulum will oscillate, and the first will come to rest. The process will repeat with time. Draw graphs that describe this motion.

14.25. As in Problem 14.24, let us once again consider the problem of two linearly coupled pendula, except that the lengths of the two are not equal. Find the normal frequencies and normal modes of vibrations. Show that, unlike the situation discussed in Problem 14.24, the energy of the system is never completely transferred to either of the pendula. Draw graphs to demonstrate this.

14.26. Two identical pendula coupled by a spring as shown in Fig. Ex. 14.2 are moving in a viscous medium that produces a retarding force proportional to its velocity. Find the normal frequencies and normal coordinates. Draw graphs to show the motion.

14.27. Three equal masses m are joined by two identical springs of spring constant k. The system is free to oscillate and move along the line joining the masses. The system is placed in a viscous medium that exerts a retarding force proportional to its velocity. Find the normal frequencies and the normal modes of oscillations. Draw graphs describing the motion.

14.28. Two equal masses and three identical springs are connected as shown in Fig. P14.28 and surrounded by a viscous medium which exerts a retarding force proportional to velocity. There is a tension of T in the springs at A and B. One mass is held and the second is displaced a distance d vertically, and then both are released. Find the normal frequencies and the normal modes of vibration. Draw graphs that describe the motion of the masses.

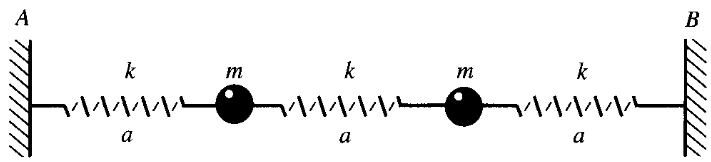


Figure P14.28

14.29. In Problem 14.24, the system is surrounded by a viscous medium that produces a retarding force proportional to its velocity. Find the normal frequencies and the normal modes of vibrations. Assume proper initial conditions.

14.30. Consider the system shown in Fig. P14.28. The unstretched length for each spring is a. **(a)** Find the normal frequencies and normal modes of vibration. **(b)** Suppose each mass m is subjected to a force $F = F_0 \sin \omega t$ at time $t = 0$ when the system is at rest. Discuss the motion using normal coordinates and draw graphs of this motion.

14.31. In Problem 14.30, each mass is subject to a frictional force $-bm\dot{x}_1$. Discuss the motion of the system and draw a graph of it.

SUGGESTIONS FOR FURTHER READING

Becker, R. A., *Introduction to Theoretical Mechanics,* Chapter 14. New York: McGraw-Hill Book Co., 1954.

Corben, H . C., and Stehle, P., *Classical Mechanics,* Chapter 8. New York: John Wiley & Sons, Inc., 1960.

Fowles, G. R., *Analytical Mechanics,* Chapter 11. New York: Holt, Rinehart and Winston, Inc., 1962.

French, A. P., *Vibrations and Waves,* Chapter 5. New York: W. W. Norton and Co., Inc., 1971.

*Goldstein, H., *Classical Mechanics,* 2nd ed., Chapter 6. Reading, Mass.: Addison-Wesley Publishing Co., 1980.

Hauser, W., *Introduction to the Principles of Mechanics,* Chapter 11. Reading, Mass.: Addison-Wesley Publishing Co., 1965.

*Landu, L. D., and Lifshitz, E. M., *Mechanics,* Chapter 5. Reading, Mass.: Addison-Wesley Publishing Co., 1960.

Marion, J. B., *Classical Dynamics,* 2nd ed., Chapters 6 and 7. New York: Academic Press, Inc., 1970.

*Moore, E. N., *Theoretical Mechanics,* Chapter 7. New York: John Wiley & Sons, Inc., 1983.

Rossberg, K., *Analytical Mechanics,* Chapter 10. New York: John Wiley & Sons. Inc., 1983.

Slater, J. C., *Mechanics,* Chapter 7. New York: McGraw-Hill Book Co., 1947.

Stephenson, R. J., *Mechanics and Properties of Matter,* Chapter 5. New York: John Wiley & Sons, Inc., 1962.

Symon, K. R., *Mechanics,* 3rd ed., Chapter 12. Reading, Mass.: Addison-Wesley Publishing Co., 1971.

*The asterisk indicates works of an advanced nature.