CHAPTER 9

Rigid Body Motion: I

9.1 INTRODUCTION

In previous chapters we have dealt with the motion of a particle or a system of particles under the influence of external forces. In actual, everyday motions, we have to deal with rigid objects of different shapes and sizes which may or may not reduce to equivalent point masses. We will show now that to describe the motion of rigid bodies and apply conservation laws, we must understand the full meanings of center of mass, moment of inertia, and radius of gyration.

Discussion of angular motion is complex in such cases, so simple cases of rotation about a fixed axis will be discussed here, while rotation about an axis passing through a fixed point will be discussed in Chapter 13. Furthermore, in this chapter we will assume that the bodies are rigid and do not deform, which is true in ideal cases only. We will briefly discuss deformable continua in order to understand the elastic properties of objects, which, in turn, is necessary to an understanding of the equilibrium of flexible cables, strings, and solid beams.

9.2 DESCRIPTION OF A RIGID BODY

A *rigid body* is defined as a system consisting of a large number of point masses, called particles, such that the distances between the pairs of point masses remain constant even when the body is in motion or under the action of external forces. This is an idealized definition of a rigid body because (1) there is no such thing as true point masses or particles, and (2) no body of any physical size is strictly rigid; it becomes deformed under the action of applied forces. However, the concept of an idealized rigid body is useful in describing motion, and the resulting deviations are not that significant.

Forces that maintain constant distances between different pairs of point masses are internal forces and are called *forces of constraint.* Such forces come in pairs and obey Newton's third law in the strong form; that is, they are equal and opposite and act along the same line of action. Hence we can apply the laws of conservation of linear momentum and angular momentum to the description of the motion of rigid bodies. Furthermore, in any displacement, the relative distances and the orientations of different particles remain the same with respect to each other; hence no net work is done by the internal forces or the forces of constraint. This implies that for a perfectly rigid body the law of conservation of mechanical energy holds as well.

Our next step is to establish the number of independent coordinates needed to describe the position in space or configuration of a rigid body. Suppose a rigid body consists of N particles. Since the position of each particle is specified by three coordinates, we may be led to conclude that we need $3N$ coordinates to describe the position of the rigid body. This would be true only if the positions and the motions of all particles were independent. But this is not so. The distance r_{kl} between any pair of particles is constant, and there are many such pairs. We are going to show that only *six* independent coordinates are needed to describe the position of a rigid body. Let us consider the rigid body shown in Fig. 9.1. To describe the position of the point mass k we need not specify its distances from all other point masses in the body; we need its distances from three other noncollinear points, such as P_1, P_2, and P_3, as shown in Fig. 9.1. Thus, if the positions of these three points are known, the positions of the remaining points in the body are fixed by the constraints. But P_1, P_2, and P_3 need at the most *nine* coordinates to describe their positions in space. Even these nine coordinates are not all independent. The distances r_{12}, r_{13}, and r_{23} are all constants; that is,

$$r_{12} = d_1, \qquad r_{13} = d_2, \qquad r_{23} = d_3 \tag{9.1}$$

where d_1, d_2, and d_3 are constants. These three relations, called the *equations of constraints*, reduce the number of independent coordinates needed to describe the position of the rigid body to six.

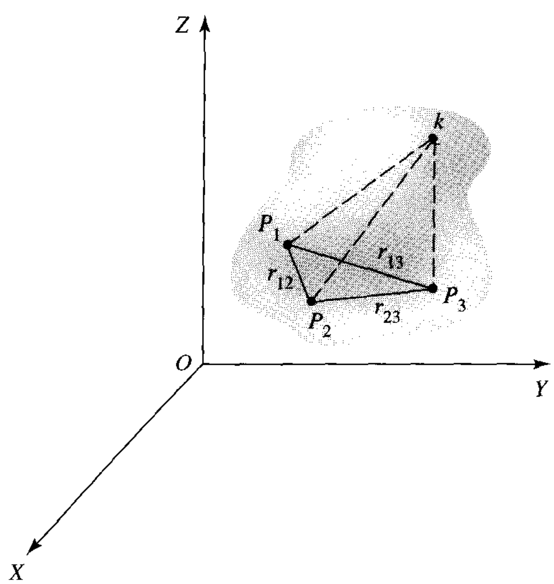


Figure 9.1 The position of any point mass at k in space may be determined by knowing the positions of three noncollinear points P_1, P_2, and P_3.

There is an alternative way of explaining that only six coordinates are needed to establish the positions of the three reference points. The reference point P_1 needs only three coordinates (x_1, y_1, z_1) to specify its position. Once P_1 is fixed, P_2 can be specified by only two coordinates, since it will be constrained to move on the surface of a sphere whose center is at P_1. These two coordinates are (θ_2, ϕ_2). With these two points fixed, point P_3 lies on a circle of radius a whose center lies on an axis joining points P_1 and P_2. Thus only *six* coordinates are needed to locate three noncollinear points P_1, P_2, and P_3 of a rigid body. Once these are fixed, the locations of all other points of the rigid body are fixed; that is, the configuration of a rigid body in space is fixed. If there are other constraints on the body, the number of coordinates needed to specify the position of a rigid body may be less than six.

There are several ways of choosing these six coordinates. One such way is shown in Fig. 9.2. The primed coordinates $X'Y'Z'$, the *body set of axes*, drawn in the rigid body can completely specify the rigid body relative to the external coordinates XYZ, the *space set of axes.* Thus three coordinates are needed to specify the origin of the body set of axes, while the remaining three must specify the orientation of the body set of axes (primed axes) relative to the coordinate axes parallel to the space axes (unprimed axes), as shown in Fig. 9.2. Thus we must know the coordinates of O' with respect to O and the orientation of $X'Y'Z'$ axes relative to the XYZ axes.

Let us consider the motion of a rigid body constrained to rotate about a fixed point. Since there is no translational motion, we are concerned only with the torques that produce rotational motion. But before doing this, we must choose three coordinates that describe the orientation of the body axes relative to the space axes. The choice is not so simple. No simple symmetric set of coordinates can be found that will describe the orientation of the rigid body. We shall postpone the discussion of the rotation of a body about a fixed point until Chapter 13. In this chapter, we limit ourselves to the discussion of simple problems involving rotation about a fixed axis (and not a fixed point).

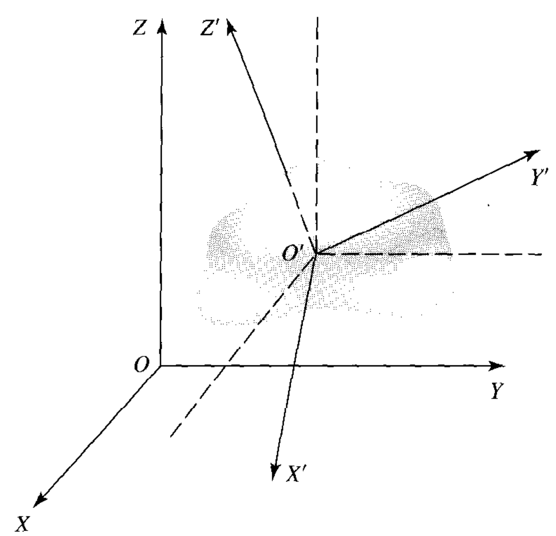


Figure 9.2 The configuration of a rigid body is specified by six independent coordinates: the coordinates of O' with respect to O and the orientation of the $X'Y'Z'$ axes relative to the XYZ axes.

9.3 CENTER OF MASS OF A RIGID BODY

Even a small-sized solid body contains a very large number of atoms and molecules. It is convenient to represent the structure of this body by its *average density*, ρ, defined as mass per unit volume, that is, $\rho = M/V$, where M is the mass and V is the volume, while the *local density* or simply *density* may be defined as

$$\rho = \frac{dM}{dV} \tag{9.2}$$

where dM is the mass of a volume element dV. Since the body is assumed to be continuous over the whole volume, the total mass being given by finite summation over mass particles m_k must now be replaced by an integral over a volume space of infinitesimal masses dM; that is,

$$\sum m_k \rightarrow M = \iiint dM = \iiint \rho \, dV \tag{9.3}$$

For a system containing a discrete number of particles of masses m_k at distances r_k, the center of mass $\mathbf{R}$ was defined in Chapter 8 as

$$\mathbf{R} = \frac{\Sigma m_k \mathbf{r}_k}{\Sigma m_k} \tag{9.4}$$

For an extended rigid body, the summation can be replaced by an integration over the whole volume of the body; that is, the center of mass $\mathbf{R}(X, Y, Z)$ is

$$\mathbf{R} = \frac{\iiint \mathbf{r} \, dM}{\iiint dM} = \frac{1}{M}\iiint \mathbf{r}\rho \, dV \tag{9.5}$$

where $dM = \rho \, dV$, and M is the total mass of the body. In component form, the center of mass may be written as

$$X = \frac{1}{M}\iiint x\rho \, dV, \qquad Y = \frac{1}{M}\iiint y\rho \, dV, \qquad Z = \frac{1}{M}\iiint z\rho \, dV \tag{9.6}$$

If a rigid body is in the form of a thin shell, the equation for the center of mass takes the form

$$\mathbf{R} = \frac{1}{M}\iint \mathbf{r}\sigma \, dA \tag{9.7}$$

where σ is the *surface density* defined as the mass per unit area, dA is a small element of area, and the total mass M is given by

$$M = \iint \sigma \, dA \tag{9.8}$$

Similarly, if the body is in the form of a thin wire, the center of mass is

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\lambda \, dL \tag{9.9}$$

where λ is the *linear density* defined as mass per unit length, dL is a small element of length. and the total mass M is given by

$$M = \int \lambda \, dL \tag{9.10}$$

If ρ, σ, and λ are constants, they can be taken out of the integration signs, thereby making the problem somewhat simpler.

Suppose a system consists of two or more discrete parts such that the center of mass of M_1 is at $\mathbf{r}_1$, that of M_2 is $\mathbf{r}_2, \ldots$, then the center of mass of the system is

$$\mathbf{R} = \frac{M_1\mathbf{r}_1 + M_2\mathbf{r}_2 + \cdots}{M_1 + M_2 + \cdots} \tag{9.11}$$

In component form,

$$X = \frac{M_1x_1 + M_2x_2 + \cdots}{M_1 + M_2 + \cdots} \tag{9.12}$$

with similar expressions for Y and Z. Note that (x_1, y_1, z_1), $(x_2, y_2, z_2), \ldots$, are the coordinates of the center of masses of $M_1, M_2, \ldots$, respectively.

In calculating the center of mass, we should be able to take advantage of symmetry considerations. Suppose a body has a plane of symmetry; that is, every mass m_k has a mirror image of itself m_k' relative to the same plane. Let us assume that the XY plane is the plane of symmetry. In this case,

$$Z = \frac{\Sigma(m_kz_k + m_k'z_k')}{\Sigma(m_k + m_k')} \tag{9.13}$$

But, due to symmetry, $m_k = m_k'$ and $z_k = -z_k'$; that is, $Z = 0$, which implies that the center of mass lies in the XY plane, the plane of symmetry. Similarly, if a rigid body has a line of symmetry, the center of mass lies on this line. Let us discuss some examples to explain the application of the preceding equations.

Center of Mass of a Solid Hemisphere, Hemispherical Shell, and Semicircle

Figure 9.3 shows a solid hemisphere of radius a and uniform density ρ so that its mass $M = (2\pi/3)a^3\rho$. From symmetry considerations, we know that its center of mass lies on the radius that is normal to the plane face. That is, as shown, it lies on the Z-axis. To calculate Z, the center of mass, we consider the volume element shown shaded, so that

$$dV = \pi(a^2 - z^2)\, dz \tag{9.14}$$

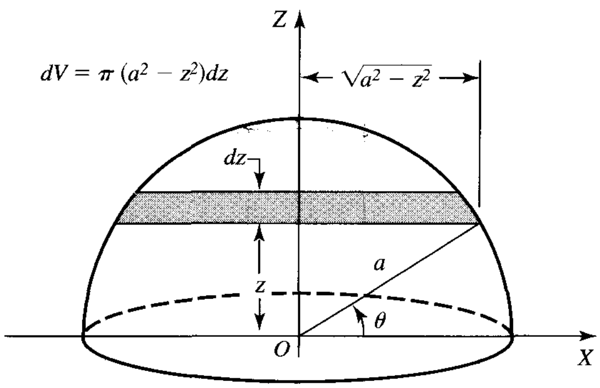


Figure 9.3 Center of mass for a solid hemisphere.

Therefore, the center-of-mass coordinate Z, according to Eq. (9.5), is

$$Z = \frac{\int_0^a z\rho\, dV}{\int_0^a \rho\, dV} = \frac{\int_0^a z\rho\pi(a^2 - z^2)\, dz}{\int_0^a \rho\pi(a^2 - z^2)\, dz} = \frac{3}{8}a \tag{9.15}$$

For a hemispherical shell, the situation is as shown in Fig. 9.4. Again from symmetry considerations, the center of mass is on the Z-axis. A small surface of length $2\pi(a^2 - z^2)^{1/2}$ and width $a\, d\theta$), as shown, has an area

$$dA = 2\pi(a^2 - z^2)^{1/2} a\, d\theta \tag{9.16}$$

According to Fig. 9.4,

$$z = a \sin\theta, \qquad dz = a\cos\theta\, d\theta$$

$$d\theta = \frac{dz}{a\cos\theta} = \frac{dz}{(a^2 - z^2)^{1/2}}$$

while the center of mass is $\mathbf{R}(X, Y, Z)$. The translational motion of the body is described by

$$\mathbf{F} = M\ddot{\mathbf{R}} \tag{9.17}$$

where $\mathbf{F}$ is the total external force acting on a body of mass M. The rotational motion of the body is described by the equation

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} \tag{9.18}$$

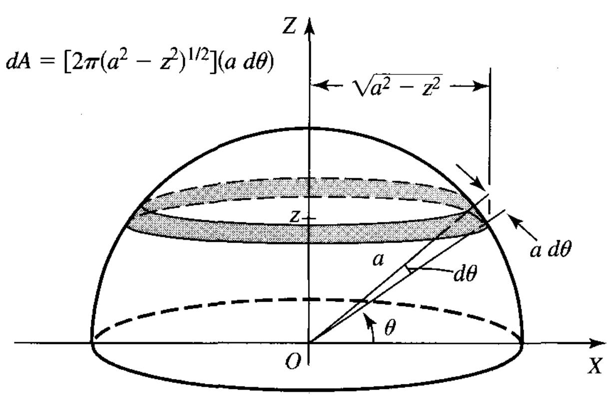


Figure 9.4 Center of mass for a hemispherical shell.

where **L** is the angular momentum and $\boldsymbol{\tau}$ is the total external torque acting about an axis passing through the center of mass. Thus Eqs. (9.17) and (9.18) represent six coupled equations to be solved simultaneously, a hard task to accomplish. But under certain constraints the number of equations can be reduced considerably, as we shall discuss later.

Example 9.1

Find the center of mass for (a) a solid cone and (b) a frustum of a cone. The two situations are shown in Fig. Ex. 9.1

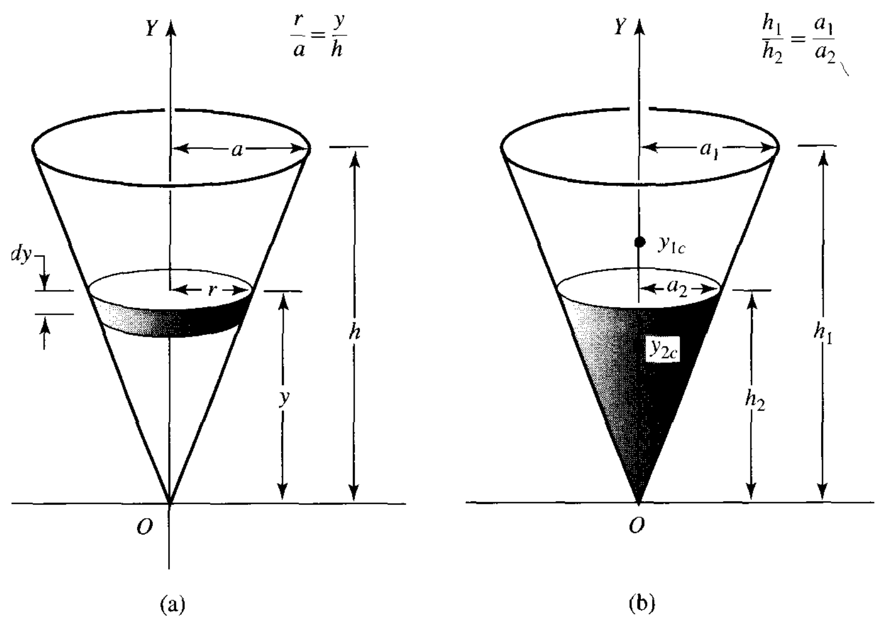


Figure Ex. 9.1

Solution

(a) Consider the cone, shown in Fig. Ex. 9.1(a), of radius a, height h, and density ρ. First consider a cone of radius r and thickness dy. Using the relation r/a = y/h, we can write the values of r and the mass dm of the cone. Eliminate r by substituting its value as shown.

$$r=\frac{a\cdot y}{h} \qquad dm=\rho\cdot\pi\cdot r^2\cdot dy \qquad dm=\rho\cdot\pi\cdot\left(\frac{a\cdot y}{h}\right)^2\cdot dy$$

Using Eq. (9.3), calculate the mass M of the cone by integration.

$$M=\int_0^h\left[\left(\frac{a\cdot y}{h}\right)^2\cdot\rho\cdot\pi\right]dy \qquad M=\frac{1}{3}\cdot a^2\cdot h\cdot\rho\cdot\pi$$

Using either of the relations in Eq. (9.5), calculate the center of mass Ycm of the cone as shown.

$$Ycm = \frac{\int_0^h y\cdot\left[\left(\frac{a\cdot y}{h}\right)^2\cdot\rho\cdot\pi\right]dy}{\int_0^h \left[\left(\frac{a\cdot y}{h}\right)^2\cdot\rho\cdot\pi\right]dy} \qquad Ycm = \frac{3}{4}\cdot h$$

(b) The frustum is a cone that has had the lower portion [shaded portion in Figure Ex. 9.1(b)] removed. Treat this as two cones, the whole cone and the lower shaded cone. Find the mass and the center of mass of the frustum by subtracting the values for the lower cone from those for the whole cone as shown.

M1 and M2 are the masses of the two cones, and Y1cm and Y2cm are their center of masses. Using the geometry of the figure, replace the values of h2 by (a2/a1)h1.

$$\frac{h1}{h2} = \frac{a1}{a2} \qquad h2 = \frac{a2}{a1}\cdot h1 \qquad M1 = \frac{1}{3}\cdot a1^2\cdot h1\cdot\rho\cdot\pi$$

$$M2 = \frac{1}{3}\cdot a2^2\cdot h2\cdot\rho\cdot\pi \qquad M2 = \frac{1}{3}\cdot\frac{a2^3}{a1}\cdot h1\cdot\rho\cdot\pi$$

$$Y2cm = \frac{3}{4}\cdot h2 \qquad Y2cm = \frac{3}{4}\cdot\frac{a2}{a1}\cdot h1$$

Substituting the values of M1, M2, Y1cm, and Y2cm, and simplifying, we find the center of mass Ycm of the frustum of a cone.

$$Ycm = \frac{M1\cdot Y1cm - M2\cdot Y2cm}{M1 - M2}$$

$$Ycm = \frac{3}{4}\cdot(a2 + a1)\cdot\left(a2^2 + a1^2\right)\cdot\frac{h1}{\left[a1\cdot\left(a2^2 + a1\cdot a2 + a1^2\right)\right]}$$

9.4 ROTATION ABOUT AN AXIS

After pure translation motion, the next simplest motion of a rigid body is its rotational motion about a fixed axis. When a body is free to rotate about a fixed axis, it needs only the coordinate to specify its orientation. Let us consider a rigid body that rotates about a fixed Z-axis, as shown in Fig. 9.5(a). The position of the body may be specified by an angle θ, which is between the line OA drawn on the body and the X-axis. Let us consider a particle of mass m_k to be a representative particle located at a distance $\mathbf{R}_k(X_k, Y_k, Z_k)$ from the origin, moving with a velocity $\mathbf{v}_k$ and angular velocity ω. The path of such a particle is a circle of radius $r_k = (x_k^2 + y_k^2)^{1/2}$ with its center on the Z-axis. Let ψ be the angle between the direction of the line OA in the body and the radius r_k from the Z-axis to the mass m_k. Since for a rigid body ψ is constant, as shown in the figure, $\phi = \theta = \psi$ and hence

$$\dot{\phi} = \dot{\theta} = \omega \tag{9.19}$$

while

$$v_k = r_k\omega \tag{9.20}$$

or in vector notation

$$\mathbf{v}_k = \boldsymbol{\omega} \times \mathbf{r}_k \tag{9.21}$$

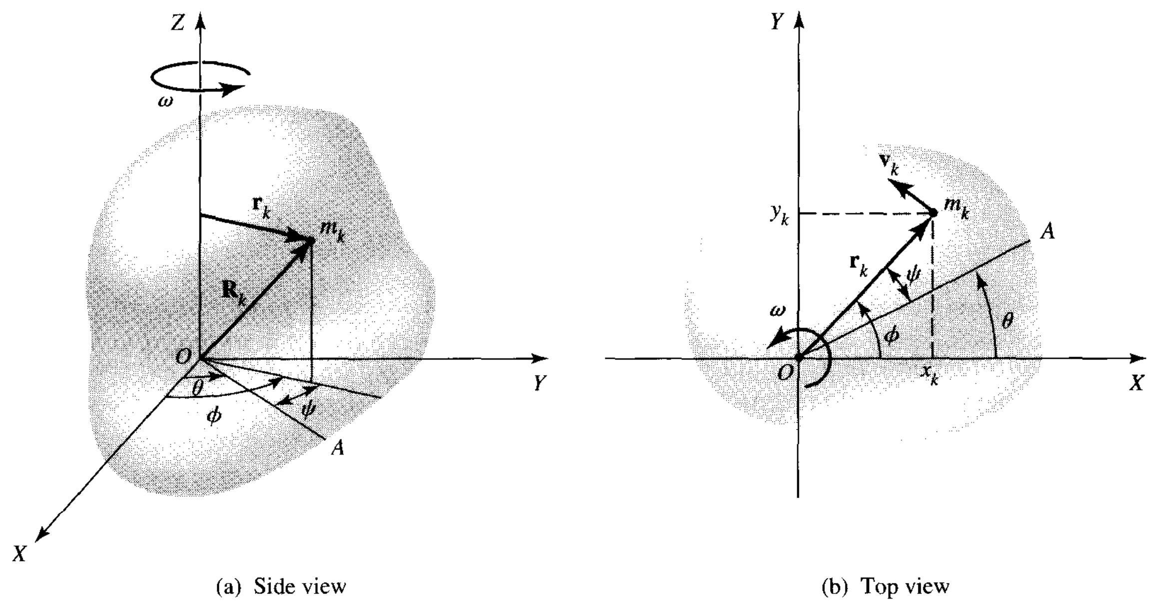


Figure 9.5 Rotation of a rigid body about a fixed axis: (a) side view, and (b) top view.

From Eq. (9.21) or from Fig. 9.5(b),

$$\dot{x}_k = -v_k \sin\phi = -\omega y_k \tag{9.22a}$$

$$\dot{y}_k = v_k \cos\phi = \omega x_k \tag{9.22b}$$

$$\dot{z}_k = 0 \tag{9.22c}$$

and

$$v_k = r_k\omega = (\dot{x}_k^2 + \dot{y}_k^2)^{1/2} \tag{9.23}$$

For further calculations, we can either use the rectangular coordinates (x, y, z) or cylindrical coordinates (r, θ, z). The kinetic energy K of the rotating body about the Z-axis is

$$K = \sum_k \tfrac{1}{2} m_k v_k^2 = \tfrac{1}{2}\left[\sum_k m_k r_k^2\right]\omega^2$$

or

$$K = \tfrac{1}{2} I_z \omega^2 = \tfrac{1}{2} I_z \dot{\theta}^2 \tag{9.24}$$

where

$$I_z = \sum_k m_k r_k^2 = \sum_k m_k(x_k^2 + y_k^2) \tag{9.25}$$

The quantity I_z is constant for a given rigid body rotating about a given axis (Z-axis in this case) and is called the *moment of inertia* about that axis. Since the body is continuous, we may replace the summation by the integration, and express I_z as

$$I_z = \iiint r^2\, dm = \iiint r^2 \rho\, dV \tag{9.26}$$

Let us now calculate the angular momentum of the body about the Z-axis. By definition, the angular momentum of the body about the Z-axis is

$$L = \sum_k r_k(m_k v_k) = \left(\sum_k m_k r_k^2\right)\omega \tag{9.27}$$

or
$$L = I\omega = I\dot{\theta} \tag{9.28}$$

The rate of change of angular momentum for any system is equal to the total external torque (or total moment of force) τ (also written as N). Thus, for a rigid body rotating about the Z-axis, since I is constant,

$$\tau_z = \frac{dL}{dt} = I\frac{d\omega}{dt} = I\ddot{\theta} \tag{9.29}$$

This is an equation of motion for rotation of a rigid body about a fixed axis and is analogous to the equation for the translational motion of a particle along a straight line, that is, Newton's second law.

Similarly, the moment of inertia I is analogous to mass m; that is, I is a measure of the rotational inertia of a body relative to some fixed axis of rotation, just as m is a measure of the translational inertia of a body. Remember, the difference is that the moment of inertia depends on the axis of rotation, while the mass does not depend on its position. Such analogy may be shown between translational and rotational quantities, as well (see Table 9.1).

Furthermore, as an analogy with the translational motion, we may define the rotational potential energy as

$$V(\theta) = -\int_{\theta_s}^{\theta} \tau_z(\theta)\, d\theta \tag{9.30}$$

Table 9.1. Analogy between Rectilinear Motion and Rotational Motion about an Axis

Rectilinear	Rotational
Position: x	Angular position: θ
Velocity: $v = dx/dt$	Angular velocity: $\omega = d\theta/dt$
Acceleration:	Angular acceleration:
$a = dv/dt = d^2x/dt^2$	$\alpha = d\omega/dt = d^2\theta/dt$
$v = v_0 + at$	$\omega = \omega_0 = \alpha t$
$x = v_0t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
Mass: M	Moment of inertia: $I = \Sigma m_k r_k^2$
Linear momentum: $p = mv$	Angular momentum: $L = I\omega$
Force: F	Torque: $\tau = rF \sin\theta$
$F = ma$	$\tau = I\alpha$
$F = dp/dt$	$\tau = dL/dt$
Translational kinetic energy:	Rotational kinetic energy:
$K = \frac{1}{2}mv^2$	$K = \frac{1}{2}I\omega^2$
Potential energy:	Potential energy:
$V(x) = -\int_{x_s}^{x} F(x)\, dx$	$V(\theta) = -\int_{\theta_s}^{\theta} \tau(\theta)\, d\theta$
$F(x) = -\frac{dV(x)}{dx}$	$\tau(\theta) = -\frac{dV(\theta)}{d\theta}$

and $$\tau_z = -\frac{dV}{d\theta} \tag{9.31}$$

Thus the rotational potential energy is the work done against the forces that produce the torque τ_z when the body is rotated from a standard angular position θ_s to a new position θ.

Example 9.2

A stick of mass M and length L is initially at rest in a vertical position on a frictionless table, as shown in Fig. Ex. 9.2. If the stick starts falling, find the speed of the center of mass as a function of the angle that the stick makes with the vertical.

Solution

The situation is as shown in the figure. We can find the speed of the center of mass by using the conservation of energy method. The only force acting on the stick is the gravitational force Mg in the vertical downward direction. Since there is no horizontal force acting on the rod, the center of mass falls vertically downward as shown. Let $\dot{y} = vy$ be the speed of the center of mass.

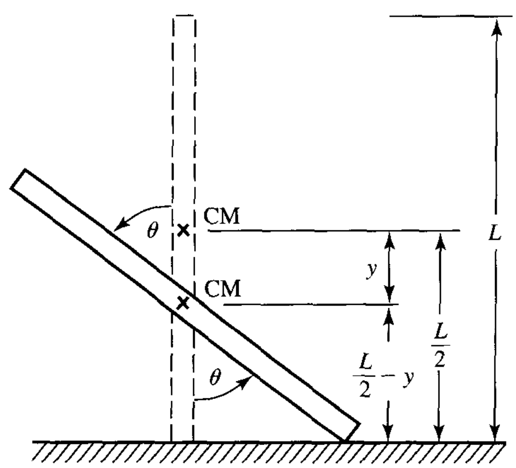


Figure Ex. 9.2

Ki = 0
Ei = initial total energy
Ef = final total energy
Vi = initial potential energy.
V = potential energy at different y
E = total energy, which is the sum of the translational kinetic energy (Kt), rotational kinetic energy (Kr), and potential energy V. The energy equation for any position y is

$$Ei = K + V = Ef \qquad Ei = Vi = M\cdot g\cdot\frac{L}{2} \qquad V = M\cdot g\cdot\left(\frac{L}{2} - y\right)$$

$$Ei = Kt + Kr + V$$

$$Ei = E = \frac{M\cdot vy^2}{2} + \frac{Io\cdot\omega\theta^2}{2} + M\cdot g\cdot\left(\frac{L}{2} - y\right) \qquad \omega\theta = \frac{2\cdot vy}{L\cdot\sin(\theta)}$$

$$M\cdot g\cdot\frac{L}{2} = \frac{M\cdot vy^2}{2} + \frac{Io\cdot\omega\theta^2}{2} + M\cdot g\cdot\left(\frac{L}{2} - y\right)$$

λ is the linear mass density equal to M/L. Io is the moment of inertia of the rod about the center of mass. Solve for Io and then substitute for λ.

$$Io = \int_{\left(\frac{-L}{2}\right)}^{\frac{L}{2}} x^2 \cdot \lambda \, dx \qquad Io = \frac{1}{12} \cdot L^3 \cdot \lambda \qquad \lambda = \frac{M}{L} \qquad Io = \frac{1}{12} \cdot L^2 \cdot M$$

Substitute the value of Io and ωθ in the energy equation and solve the equation for the value of velocity vy.

$$\frac{1}{2} \cdot M \cdot g \cdot L = \frac{1}{2} \cdot M \cdot vy^2 + \frac{1}{2} \cdot Io \cdot \left(\frac{2 \cdot vy}{L \cdot \sin(\theta)}\right)^2 + M \cdot g \cdot \left(\frac{1}{2} \cdot L - y\right)$$

$$\frac{1}{2} \cdot M \cdot g \cdot L = \frac{1}{2} \cdot M \cdot vy^2 + \frac{1}{6} \cdot M \cdot \frac{vy^2}{\sin(\theta)^2} + M \cdot g \cdot \left(\frac{1}{2} \cdot L - y\right)$$

The two resulting solutions for vy are

$$\left[\begin{array}{c} \dfrac{1}{\left[6 \cdot \left(\dfrac{-1}{2} \cdot M - \dfrac{1}{6} \cdot \dfrac{M}{\sin(\theta)^2}\right)\right]} \cdot \sqrt{2} \cdot M \cdot \sqrt{3 \cdot \sin(\theta)^2 + 1} \cdot \sqrt{g} \cdot \sqrt{y} \cdot \dfrac{\sqrt{3}}{\sin(\theta)} \\ \dfrac{-1}{\left[6 \cdot \left(\dfrac{-1}{2} \cdot M - \dfrac{1}{6} \cdot \dfrac{M}{\sin(\theta)^2}\right)\right]} \cdot \sqrt{2} \cdot M \cdot \sqrt{3 \cdot \sin(\theta)^2 + 1} \cdot \sqrt{g} \cdot \sqrt{y} \cdot \dfrac{\sqrt{3}}{\sin(\theta)} \end{array}\right]$$

Using one of the roots and simplifying, we obtain the final solution for vy. The value of vy may be further simplified as shown.

$$vy = \frac{-1}{\left[6 \cdot \left(\dfrac{-1}{2} \cdot M - \dfrac{1}{6} \cdot \dfrac{M}{\sin(\theta)^2}\right)\right]} \cdot \sqrt{2} \cdot M \cdot \sqrt{3 \cdot \sin(\theta)^2 + 1} \cdot \sqrt{g} \cdot \sqrt{y} \cdot \frac{\sqrt{3}}{\sin(\theta)}$$

$$vy = -i \cdot \sin(\theta) \cdot \frac{\sqrt{2}}{\sqrt{-4 + 3 \cdot \cos(\theta)^2}} \cdot \sqrt{g} \cdot \sqrt{y} \cdot \sqrt{3}$$

(a) What factors determine the velocity of the center of mass?

(b) Explain the changes in the magnitude of V, Kr, and Kt as the stick is falling and just before it hits the floor.

9.5 CALCULATION OF MOMENT OF INERTIA

For a system consisting of masses m_k located at distances r_k from an axis of rotation, the moment of inertia is given by

$$I = \sum_{k=1}^{N} m_k r_k^2 \tag{9.32}$$

It is important to remember that r_k is the perpendicular distance of m_k from the rotation axis. For an extended, continuous rigid body, the moment of inertia about an axis of rotation is given by

$$I = \int r^2 \, dm \tag{9.33}$$

where r is the perpendicular distance of the mass element dm from the rotation axis. For a one-dimensional body with a linear mass density λ (mass per unit length), for a two-dimensional body with an area mass density σ (mass per unit area), and for a three-dimensional body with volume mass density ρ (mass per unit volume), the moment of inertia in each case may be written as

$$I = \int r^2\lambda\, dt \tag{9.34}$$

$$I = \iint r^2\sigma\, dA \tag{9.35}$$

$$I = \iiint r^2\rho\, dV \tag{9.36}$$

where dl is the length element, dA is the area element, and dV is the volume element.

The definition of moment of inertia may be extended to the case of a composite body. Thus, if $I_1, I_2, \ldots,$ are the moments of inertia of the various parts of the body about a particular axis, then the moment of inertia of the whole body about the same axis is

$$I = I_1 + I_2 + \cdots \tag{9.37}$$

We now calculate the moment of inertia of rigid bodies of different shapes.

Thin Rod

Let us consider a thin rod of length L and mass M, so that the linear mass density will be $\lambda = M/L$. Suppose we want to find the moment of inertia about an axis perpendicular to the rod at one end [Fig. 9.6(a)]. According to Eq. (9.34),

$$I = \int_0^L x^2\lambda\, dx = \tfrac{1}{3}L^3\lambda = \tfrac{1}{3}ML^2 \tag{9.38}$$

where we have substituted $\lambda = M/L$. If the axis of rotation were at the center of the rod, as shown in Fig. 9.6(b), the moment of inertia would be

$$I = \int_{-L/2}^{+L/2} x^2\lambda\, dx = \tfrac{1}{12}L^3\lambda = \tfrac{1}{12}ML^2 \tag{9.39}$$

Before proceeding further, it is important at this point to introduce and prove two most important theorems: The parallel axis theorem and the perpendicular axis theorem.

Parallel Axis Theorem

Consider a body rotating about an axis passing through O. There is no loss in generality by assuming this to be a Z-axis. By definition, the moment of inertia about an axis passing through O is

$$I_0 = \sum_k m_k(x_k^2 + y_k^2) = \iiint (x^2 + y^2)\rho\, dV \tag{9.40}$$

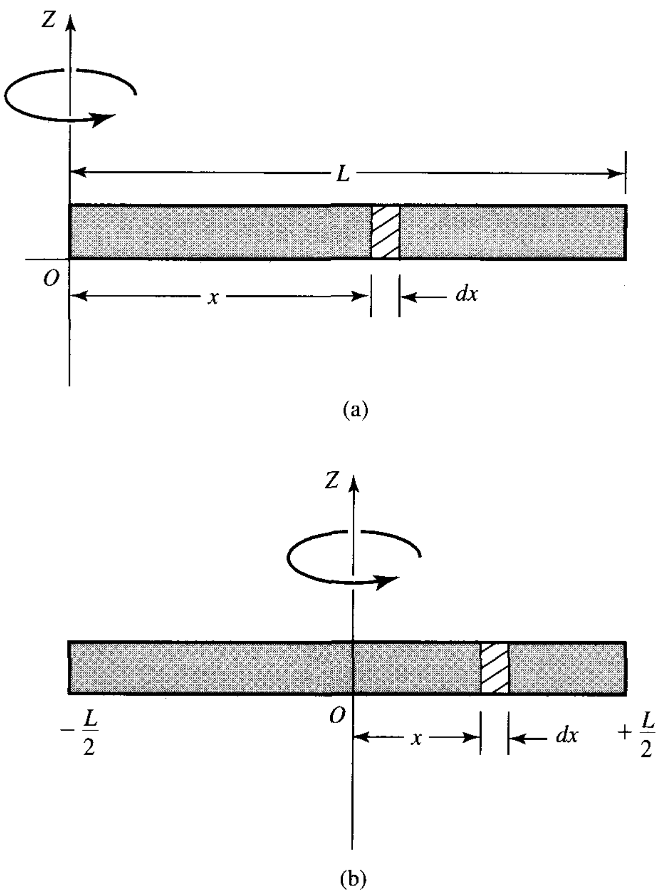


Figure 9.6 The moment of inertia for a thin rod (a) about an axis perpendicular to the rod at one end, and (b) about an axis perpendicular to the rod at the center.

where mass m_k is at a distance $\mathbf{r}_k$ from the origin and $(x_k^2 + y_x^2)^{1/2}$ from the Z-axis. According to Fig. 9.7,

$$\mathbf{r}_k = \mathbf{r}_c + \mathbf{r}'_k \tag{9.41}$$

where $\mathbf{r}_c$ is the distance of the center of mass from the origin O, and $\mathbf{r}'_k$ is the relative coordinate of m_k with respect to the CM. Using Eq. (9.41) (and dropping k),

$$x^2 + y^2 = (x_c + x')^2 + (y_c + y')^2 = x_c^2 + y_c^2 + x'^2 + y'^2 + 2x_cx' + 2y_cy' \tag{9.42}$$

Substituting this in Eq. (9.40), we obtain

$$\begin{aligned} I_0 = \iiint (x'^2 + y'^2)\rho\, dV + (x_c^2 + y_c^2)\iiint \rho\, dV \\ + 2x_c \iiint x'\rho\, dV + 2y_c\iiint y'\rho\, dV \end{aligned} \tag{9.43}$$

where the first term on the right is the moment of inertia about an axis parallel to the Z-axis and passing through the center of mass; that is,

$$I_c = \iiint (x'^2 + y'^2)\rho\, dV \tag{9.44}$$

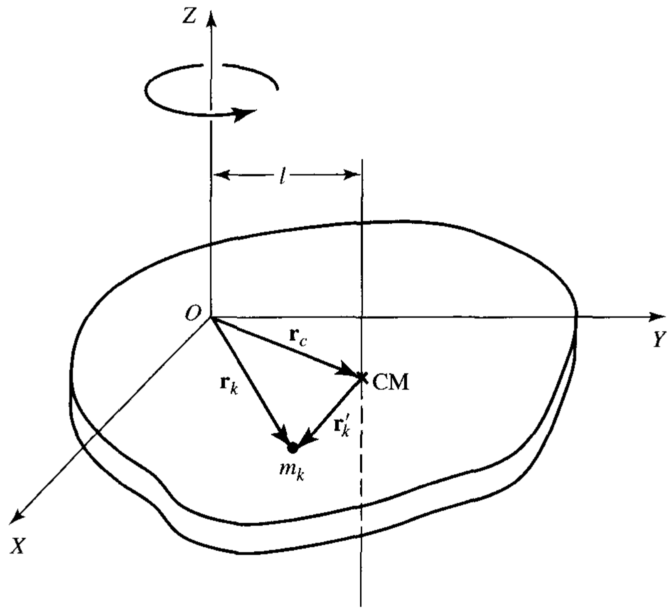


Figure 9.7 Parallel axis theorem.

The second term on the right side of Eq. (9.43) is equal to the mass M of the body multiplied by the square of the distance l between the center of mass and the Z-axis; that is,

$$(x_c^2 + y_c^2)\iiint \rho\, dV = (x_c^2 + y_c^2)M = Ml^2 \tag{9.45}$$

The last two terms in Eq. (9.43) are zero by definition of the center of mass; that is, they simply locate the center of mass relative to itself.

$$\iiint x'\rho\, dV = \iiint y'\rho\, dV = 0 \tag{9.46}$$

Thus, combining Eqs. (9.44), (9.45), and (9.46) with Eq. (9.43), we obtain

$$I_0 = I_c + Ml^2 \tag{9.47}$$

which is the parallel axis theorem and may be stated as follows:

> ***Parallel Axis Theorem.*** *The moment of inertia of a body about any axis is equal to the sum of the moment of inertia about a parallel axis through the center of mass and the moment of inertia about the given axis for the total mass of the body located at the center of mass.*

Thus, if we know the center of mass of a body and the moment of inertia about the center of mass, then the moment of inertia about any parallel axis can be calculated by using this theorem. This theorem can be applied to composite bodies as well.

Perpendicular Axis Theorem

A body whose mass is concentrated in a single plane is called a *plane lamina*. The perpendicular axis theorem is applicable to a plane lamina of any shape. Let us consider a rigid body in the form of a lamina in the XY-plane, as shown in Fig. 9.8. For rotation about the Z-axis, the moment of inertia about the Z-axis is given by

$$I_Z = \sum m_k(x_k^2 + y_k^2) = \iiint (x^2 + y^2)\rho\, dV \tag{9.48}$$

If the body were rotating about the X-axis, its moment of inertia about the X-axis would be (for a thin lamina, $z = 0$; hence no z^2 term)

$$I_X = \iiint y^2 \rho\, dV \tag{9.49}$$

and, similarly, the moment of inertia about the Y-axis would be

$$I_Y = \iiint x^2 \rho\, dV \tag{9.50}$$

Combining Eqs. (9.49) and (9.50) with Eq. (9.48),

$$I_Z = I_X + I_Y \tag{9.51}$$

which is the perpendicular axis theorem and may be stated as follows:

> ***Perpendicular Axis Theorem.*** *The sum of the moments of inertia of a plane lamina about any two perpendicular axes in the plane of the lamina is equal to the moment of inertia about an axis that passes through the point of intersection and perpendicular to the plane of the lamina.*

Let us apply these theorems to different situations.

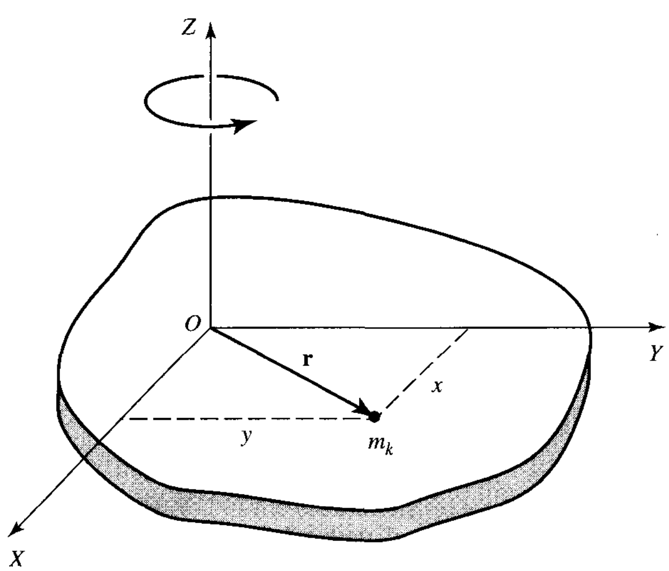


Figure 9.8 Perpendicular axis theorem as applied to a lamina in the XY plane.

Hoop or Cylindrical Shell

Consider a hoop or ring of mass M and radius a, as shown in Fig. 9.9. All the mass M is concentrated at a distance a from the axis. Hence the moment of inertia about the Z-axis is

$$I_Z = Ma^2 \tag{9.52}$$

Now suppose that we want to calculate the moment of inertia about an axis AA' that is parallel to the Z-axis and perpendicular to the plane of the ring, passing through the edge of the ring as shown. The situation is no longer symmetrical, and the direct calculation of the moment of inertia about the axis AA' is no longer trivial. But the application of the parallel axis theorem [Eq. (9.47)] makes such calculations simple; that is,

$$I_0 = I_c + Ml^2$$

When applied to the situation in Fig. 9.9, it gives

$$\begin{aligned} I_{AA'} &= I_Z + Ma^2 = Ma^2 + Ma^2 \\ &= 2Ma^2 \end{aligned} \tag{9.53}$$

Next we proceed to calculate the moment of inertia of the ring about an axis in the plane of the ring, such as about an X- or Y-axis. From the symmetry of the situation,

$$I_X = I_Y \tag{9.54}$$

and applying the perpendicular axis theorem, Eq. (9.51), $I_Z = I_X + I_Y$ gives

$$Ma^2 = I_X + I_X = I_Y + I_Y$$

or

$$I_X = I_Y = \tfrac{1}{2}Ma^2 \tag{9.55}$$

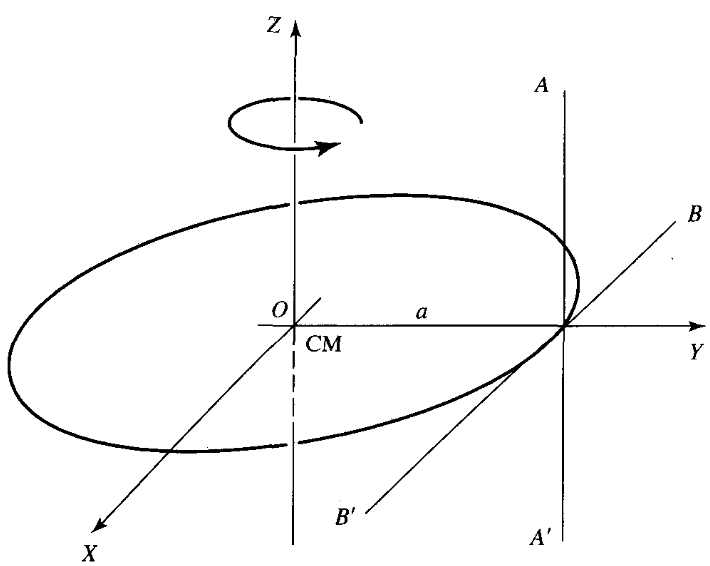


Figure 9.9 Moment of inertia for a hoop or ring about three different axes: Z-axis, AA'-axis, and BB'-axis.

We can now apply the parallel axis theorem to find the moment of inertia about the BB' axis, which is in the plane of the ring and tangent to the edge, as shown in Fig. 9.9. Thus

$$I_{BB'} = I_X + Ma^2 = \tfrac{1}{2}Ma^2 + Ma^2$$
$$= \tfrac{3}{2} Ma^2 \tag{9.56}$$

A cylindrical shell is simply a large number of rings piled one upon another. Thus the moment of inertia of a cylindrical shell or hollow cylinder of mass M, radius a, and length l may be calculated in a manner similar to the preceding ring.

Radius of Gyration

It is convenient to express the moment of inertia of a rigid body in terms of a distance k, called the radius of gyration, defined as

$$I = Mk^2, \qquad k = \sqrt{\frac{I}{M}} \tag{9.57}$$

That is, the *radius of gyration* is that distance from the axis of rotation where we may assume all of the mass of the body to be concentrated. Thus, for example, the radius gyration k of a thin rod with the axis of rotation passing through the center is

$$k = \sqrt{\frac{I}{M}} = \sqrt{\frac{(1/12)Ma^2}{M}} = \frac{a}{\sqrt{12}} \tag{9.58}$$

Once we know k for a rigid body rotating about a given axis, the moment of inertia is simply calculated from $I = Mk^2$.

Circular Disk, Solid Cylinder

Let us consider a solid disk of mass M and radius a, rotating about an axis through its center and perpendicular to the plane of the disk, as shown in Fig. 9.10. Let us divide the disk into several concentric rings, such as the one shown shaded in the figure. Thus the moment of inertia of this ring about the given axis is

$$dI = r^2\,dm$$

where r is the radius of the ring. The density per unit area is $\sigma = M/\pi a^2$; hence the mass dm of the ring is

$$dm = \sigma\,dA = \frac{M}{\pi a^2}\,2\pi r\,dr = \frac{2M}{a^2}\,r\,dr$$

Thus the moment of inertia of the disk may be written as

$$I = \int_0^a dI = \int_0^a r^2\,dm = \int_0^a r^2\frac{2M}{a^2}\,r\,dr = \frac{2M}{a^2}\int_0^a r^3\,dr = \frac{1}{2}Ma^2 \tag{9.59}$$

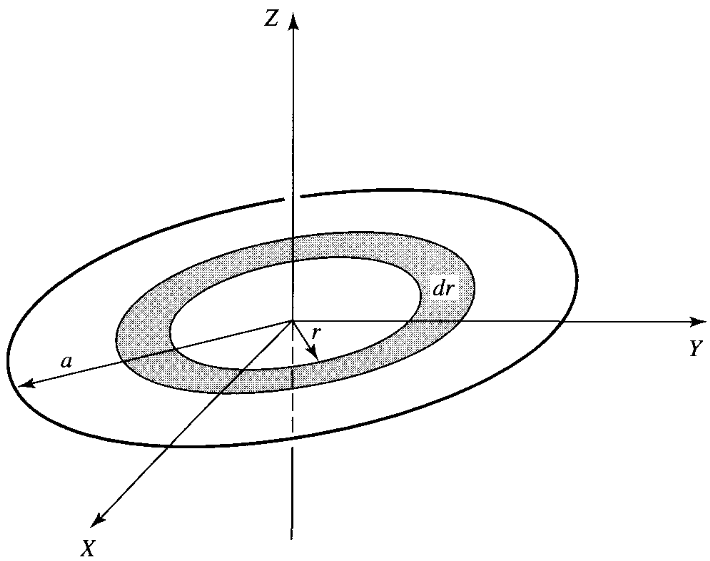


Figure 9.10 Moment of inertia for a disk about an axis perpendicular to the plane of the disk.

The same result can be obtained by using polar coordinates involving double integration. In the expression for the moment of inertia,

$$I = \iint r^2 \sigma \, dA$$

$dA = r\,d\theta\,dr$ is the area shown in Fig. 9.11 and the mass per unit area is $\sigma = M/\pi a^2$. Substituting this in the preceding expression for I, we get

$$I = \frac{M}{\pi a^2} \int_0^a \int_0^{2\pi} r^3 \, dr \, d\theta = \frac{M}{\pi a^2} 2\pi \int_0^a r^3 \, dr$$

That is,

$$I = \tfrac{1}{2} Ma^2$$

the result obtained in Eq. (9.59). To obtain the moment of inertia about different axes we can make use of the parallel and perpendicular axis theorems.

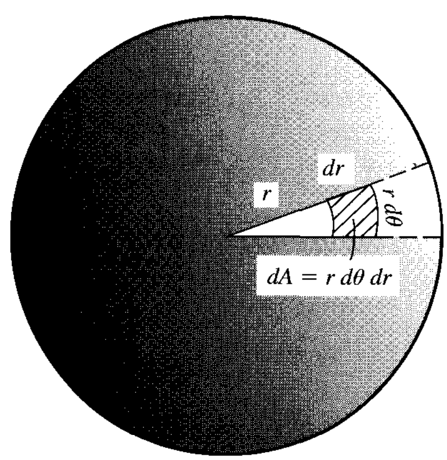


Figure 9.11 Moment of inertia for a disk using plane polar coordinates.

Sphere and Spherical Shell

Let us calculate the moment of inertia of a uniform solid sphere of radius a and mass M about an axis, say the Z-axis, passing through the center, as shown in Fig. 9.12. We can regard the sphere as made of disks, as shown in the figure. Let dI be the moment of inertia of this disk about the Z-axis so that the moment of inertia of the whole sphere will be $I = \int dI$. We calculate dI first. The disk shown has a radius $r = a \sin\theta$ while the density, mass per unit volume, of the material of the disk is

$$\rho = \frac{M}{4\pi a^3/3}$$

and the volume of the disk, with $z = a\cos\theta$, is

$$dV = \pi r^2\, dz = \pi(a\sin\theta)^2\, d(a\cos\theta) \tag{9.60}$$

Thus the moment of inertia of the disk, using Eq. (9.59), is

$$dI = \frac{1}{2}r^2\, dm = \frac{1}{2}r^2\rho\, dV = \frac{1}{2}(a\sin\theta)^2 \frac{M}{4\pi a^3/3}\pi(a\sin\theta)^2\, d(a\cos\theta) = \frac{3}{8}Ma^2\sin^5\theta\, d\theta$$

Hence the moment of inertia of the sphere about the Z-axis is given by

$$I = \int dI = -\tfrac{3}{8}Ma^2\int_0^{\pi}\sin^5\theta\, d\theta = \tfrac{2}{5}Ma^2 \tag{9.61}$$

We can obtain the same result by using the rectangular coordinates.

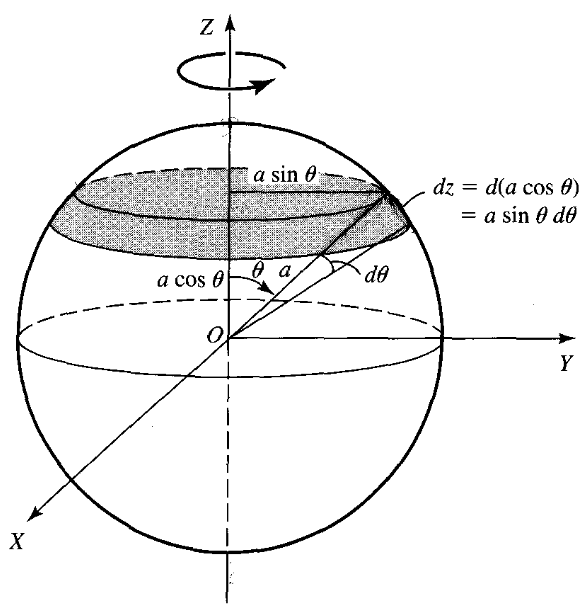


Figure 9.12 Moment of inertia for a solid sphere about the Z-axis passing through its center using spherical coordinates.

Finally, we can calculate the moment of inertia of a thin spherical shell. This can be done either by direct integration, as we have been doing, or alternatively by the application of Eq. (9.56). The final result is

$$I = \tfrac{2}{3}Ma^2 \tag{9.62}$$

9.6 SIMPLE PENDULUM

This is the first of many examples of the treatment of rotational motion. A simple pendulum consists of a mass m suspended from a fixed point O by a massless taut string (or a massless rod) of length l, as shown in Fig. 9.13. The system is treated as a rigid one. When the mass m is displaced from the vertical equilibrium position, it moves back and forth in an arc of a circle as shown. Thus the motion of a pendulum is equivalent to a rotational motion in a vertical plane and about the Z-axis through O, the axis being perpendicular to the plane. Let us apply Eq. (9.29) to this situation:

$$\tau_z = I_z\ddot{\theta} \tag{9.29}$$

where

$$I_z = ml^2 \tag{9.63}$$

and the torque about the Z-axis produced by the force mg is

$$\tau_z = -(mg \sin \theta)l \tag{9.64}$$

The negative sign is taken because the torque acts in such a way as to decrease the angle θ. Substituting Eqs. (9.63) and (9.64) in Eq. (9.31), we obtain

$$\ddot{\theta} + \frac{g}{l}\sin \theta = 0 \tag{9.65}$$

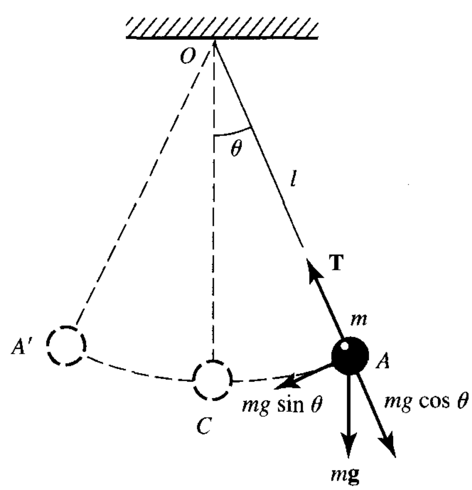


Figure 9.13 Simple pendulum.

This equation is not so easy to solve. But if we assume the angular displacement θ to be very small, that is, $\theta \ll \pi/2$, then $\sin\theta \simeq \theta$ and Eq. (9.65) takes the form

$$\ddot{\theta} + \frac{g}{l}\theta \simeq 0 \tag{9.66}$$

which is the equation for simple harmonic motion and has the solution

$$\theta = \theta_0 \cos(\omega t + \phi) \tag{9.67}$$

where
$$\omega = 2\pi f = \frac{2\pi}{T} = \sqrt{\frac{g}{l}} \tag{9.68}$$

θ_0 and ϕ being two arbitrary constants that determine the amplitude and the phase of the oscillations from the initial conditions. Notice that the frequency f and time period T are independent of the amplitude of the oscillations, provided the amplitude is small enough for Eq. (9.66) to hold good. That T is independent of the amplitude for small displacements makes the pendulum well suited for use in clocks to regulate the rate. The exact solution of the pendulum motion, as we shall show, indicates that the time period of the pendulum increases with a slight increase in amplitude.

We now discuss the motion of the pendulum without the restriction that the amplitude be small. Since the motion of the pendulum is under a conservative force, we can solve the pendulum motion problem by the energy integral. The rotational potential energy associated with the torque given by Eq. (9.64) is

$$V(\theta) = -\int_{\theta_s}^{\theta} \tau_z(\theta)\, d\theta = -\int_{\pi/2}^{\theta} (-mgl\sin\theta)\, d\theta = -mgl\cos\theta \tag{9.69}$$

where we have taken the standard reference angle θ_s to be $\pi/2$. The kinetic energy of mass m is

$$K = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}ml^2\dot{\theta}^2 \tag{9.70}$$

Thus the energy integral describing the motion is

$$K + V = E = \text{constant}$$

$$\tfrac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = E \tag{9.71}$$

Before solving this equation, we discuss the general features of the motion by drawing an energy diagram. Figure 9.14 shows the graph of $V(\theta)$, $K(\theta)$, and $E(\theta)$ versus θ. The graph of $V(\theta)$ versus θ has the maximum value mgl and the minimum value $-mgl$. For a mass m with energy slightly greater than $-mgl$, the motion will be simple harmonic. For E between $-mgl$ and $+mgl$, the motion is oscillatory and not harmonic. For $E > mgl$, the motion becomes nonoscillatory and the pendulum has enough energy to swing around in a complete circle. But the motion is still periodic, the period being equal to the time it takes to make one revolution, that is, for θ to increase or decrease by 2π.

Figure 9.14

Below is the graph of potential energy V versus θ for a pendulum.

The length L, mass m, amplitude θ0 and the frequency ω of the simple pendulum are as given here. The initial phase ϕ is assumed to be zero.

$N := 100 \quad i := 0..N \quad t_i := \frac{i}{10} \quad g := 9.8$

$L := 1.5 \quad m := .5 \quad \theta 0 := 1 \quad \omega := \sqrt{\frac{g}{L}} \quad \theta_i := \omega \cdot t_i$

Using Eqs.(9.67) through (9.70), we can obtain the expressions for the potential energy V, velocity v, kinetic energy K, and the limits of the potential energy mgL and −mgL as shown.

$V_i := -m \cdot g \cdot L \cdot \cos\left(\left(\omega \cdot t_i\right)\right) \quad v_i := -\omega \cdot \theta 0 \cdot L \cdot \sin\left(\omega \cdot t_i\right)$

$K_i := \frac{1}{2} \cdot m \cdot \left(v_i\right)^2$

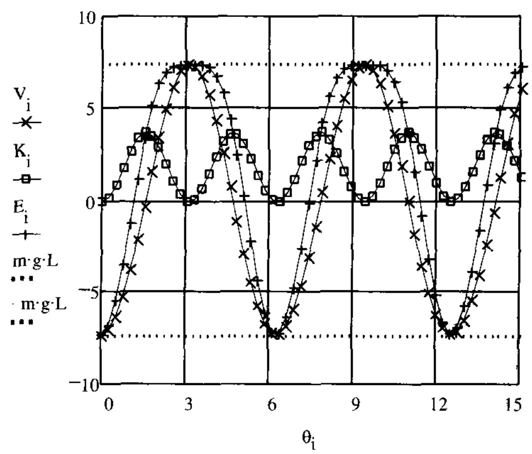
$E_i := K_i + V_i \quad m \cdot g \cdot L = 7.35$

$\max(V) = 7.35 \quad \min(V) = -7.35$

$\max(K) = 3.675 \quad \min(K) = 0$

$m \cdot g \cdot L = 7.35 \quad -m \cdot g \cdot L = -7.35$

$E := \max(V) - \min(K) \quad E = 7.35$



Looking at the graphs explain the variation in the values of K, V, and E at different angles.

From Eq. (9.71), when $\theta = \theta_0$, $\dot{\theta} = 0$, the total energy is

$$E = -mgl \cos \theta_0 \tag{9.72}$$

Substituting this in Eq. (9.71) and rearranging, we obtain

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \sqrt{\frac{2g}{l}} \int_0^t dt \tag{9.73}$$

By assuming θ to be small and using $\cos \theta \simeq 1 - \frac{1}{2}\theta^2$, after integrating, we obtain the same result as obtained for small amplitudes. But we can proceed to solve Eq. (9.73) without a restriction on the amplitude. To obtain an accurate solution, we must transform Eq. (9.73) into a proper form of an elliptic integral. Using the identity $\cos\theta = 1 - 2\sin^2\theta/2$ to write

$$\cos\theta - \cos\theta_0 = 2\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)$$

Eq. (9.73) becomes

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}} = 2\sqrt{\frac{g}{l}}\int_0^t dt \qquad \textbf{(9.74)}$$

where θ changes between $\pm\,\theta_0$. Now let us change the variables by substituting

$$\sin\phi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)} = \frac{\sin(\theta/2)}{K} \qquad \textbf{(9.75)}$$

where

$$K = \sin\frac{\theta_0}{2} \qquad \textbf{(9.76)}$$

As the pendulum swings through a cycle, θ varies between $-\theta_0$ and $+\theta_0$; hence ϕ changes between $-\pi$ and $+\pi$. That is, ϕ runs from 0 to 2π for each cycle. Using Eqs. (9.75) and (9.76), Eq. (9.74) takes the form

$$\int \frac{d\phi}{\sqrt{1 - K^2\sin^2\phi}} = \sqrt{\frac{g}{l}}\int dt \qquad \textbf{(9.77)}$$

Let us integrate this equation over one cycle; that is, as ϕ changes from 0 to 2π, t changes from 0 to T. Thus

$$\int_0^{2\pi} \frac{d\phi}{\sqrt{1 - K^2\sin^2\phi p}} = \sqrt{\frac{g}{l}}\,T \qquad \textbf{(9.78)}$$

This is an *elliptic integral* of the *first kind*, and its value can be obtained from standard tables. However, it is more demonstrative to expand the integrand and then integrate; that is,

$$T = \sqrt{\frac{l}{g}}\int_0^{2\pi}\left(1 + \frac{1}{2}K^2\sin^2\phi + \cdots\right)d\phi = \sqrt{\frac{l}{g}}\left(2\pi + \frac{2\pi}{4}K^2 + \cdots\right) \qquad \textbf{(9.79)}$$

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \cdots\right) \qquad \textbf{(9.80)}$$

which clearly states that, as the amplitude θ_0 of the oscillations becomes large, the period becomes slightly larger than for small oscillations. Even for oscillations of small amplitude, the expression $T = 2\pi\sqrt{(l/g)}$, can be improved by assuming

$$\theta_0 \ll 1, \qquad \sin^2(\theta_0/2) \simeq \frac{\theta_0^2}{4}$$

Hence for small oscillations, Eq. (9.80) takes the form

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\,\theta_0^2 + \cdots\right) \tag{9.81}$$

which is a more accurate expression because of the presence of the second term on the right. Expression (9.81) can be experimentally verified by measuring the time periods of two pendulums of the same lengths but with different amplitudes of oscillation.

9.7 PHYSICAL PENDULUM

A rigid body suspended and free to swing under its own weight about a fixed horizontal axis of rotation is known as a *physical pendulum* or *compound pendulum*. The rigid body can be of any shape as long as the horizontal axis does not pass through the center of mass. As shown in Fig. 9.15, the pendulum swings in an arc of a circle about an axis of rotation passing through O, the point O being the point of suspension. The point C is the center of mass of the physical pendulum. The distance between O and C is l. The position of the pendulum is specified by an angle θ between the line OC and the vertical line OA.

The torque τ_0 about the axis of rotation through O produced by the force Mg acting at C is

$$\tau_0 = -Mgl\sin\theta \tag{9.82}$$

If I is the moment of inertia of the body about the axis of rotation through O, the equation of motion

$$\tau_0 = I\ddot{\theta}$$

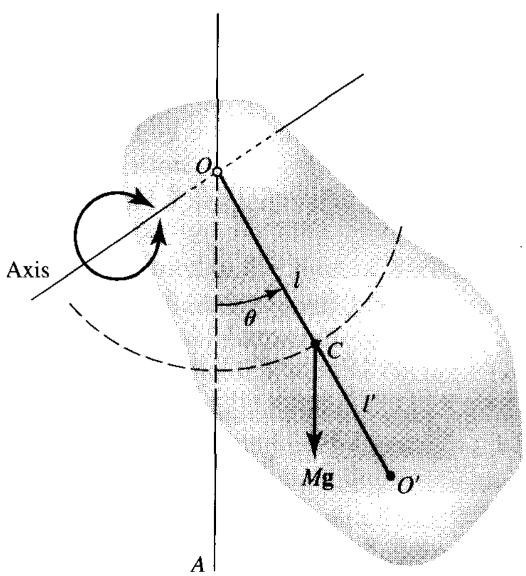


Figure 9.15 Physical pendulum.

takes the form

$$-Mgl \sin \theta = I\ddot{\theta}$$

or

$$\ddot{\theta} + \frac{Mgl}{I} \sin \theta = 0 \tag{9.83}$$

Once again, as in the case of a simple pendulum, for small oscillations we may assume that $\sin \theta \simeq \theta$, and hence

$$\ddot{\theta} + \frac{Mgl}{I} \theta \simeq 0 \tag{9.84}$$

This is the equation of a simple harmonic oscillator and has the solution

$$\theta = \theta_0 \cos(\omega t + \phi) \tag{9.85}$$

where the amplitude θ_0 and the phase angle ϕ are the two arbitrary constants to be determined from the initial conditions. The angular frequency ω is

$$\omega = \sqrt{\frac{Mgl}{I}} \tag{9.86}$$

while the time period T and frequency f are

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{I}{Mgl}} \tag{9.87}$$

If k is the radius of gyration for the moment of inertia about the axis of rotation through O, then

$$I = Mk^2 \tag{9.88}$$

Substituting Eq. (9.88) in Eq. (9.87) gives

$$T = 2\pi\sqrt{\frac{k^2}{gl}} \tag{9.89}$$

which states that *a simple pendulum of length k^2/l will have the same time period as that of a physical pendulum given by Eq. (9.87).*

Let us say that the moment of inertia of the rigid body about an axis passing through the center of mass C and parallel to the axis through O is I_C, and that the corresponding radius of gyration k_c is given by

$$I_C = Mk_c^2 \tag{9.90a}$$

Using the parallel axis theorem, we get the following relation between I and I_c:

$$I = I_C + Ml^2$$

$$Mk^2 = Mk_c^2 + Ml^2$$

or

$$k^2 = k_c^2 = l^2 \tag{9.90b}$$

Thus the time period T given by Eq. (9.89) may be written as

$$T = 2\pi\sqrt{\frac{k_c^2 + l^2}{gl}} \tag{9.91}$$

Let us now change the axis of rotation of this physical pendulum to a different position O' at a distance l' from the center of mass C, as shown in Fig. 9.15. The time period T' of oscillations about this new axis of rotation is

$$T' = 2\pi\sqrt{\frac{k_c^2 + l'^2}{gl'}} \tag{9.92}$$

Furthermore, suppose we assume that O' and l' are adjusted so that the two time periods T and T' are equal; that is,

$$T = T'$$

$$\frac{k_c^2 + l^2}{l} = \frac{k_c^2 + l'^2}{l'} \tag{9.93}$$

which simplifies to

$$k_c^2 = ll' \tag{9.94}$$

The point O' related to O by this relation is called the *center of oscillation* for the point O. Similarly, O is also the center of oscillation for O'.

Substituting Eq. (9.94) into Eq. (9.91) or (9.92) yields

$$T = 2\pi\sqrt{\frac{l + l'}{g}} \tag{9.95}$$

or

$$g = 4\pi^2\frac{l + l'}{T^2} \tag{9.96}$$

Thus, if we know the distance between O and O'—that is, if we know $l + l'$ and measure the time period T—the value of g can be measured very precisely, without knowing the position of the center of mass. Henry Kater used this method for an accurate determination of g. Kater's pendulum, shown in Fig. 9.16, has two knife edges. The pendulum can be suspended from either edge. The position of the edges can be adjusted so that the two time periods are equal. Once this is done, $l + l'$ is measured accurately and, knowing T, the value of g can be calculated from Eq. (9.96).

9.8 CENTER OF PERCUSSION

We shall now discuss some everyday applications of physical pendulum types of problems. Consider the body shown in Fig. 9.17, which is free to rotate about an axis passing through O. Suppose we strike a blow at point O', which is at a distance D from the axis of rotation through O.

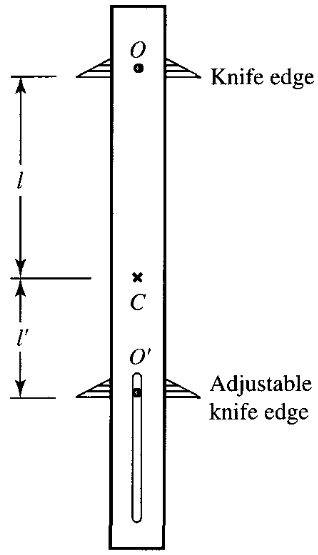


Figure 9.16 Kater's pendulum.

The blow applied is perpendicular to the line OCO', where C is the center of mass of the body. The forces acting on the body during the impact are force F' at the point of impact and another force F that is applied so as to keep the body fixed during the impact. If the body starts rotating with angular frequency ω, a radial force F_r, at O along the line $O'CO$ provides the necessary centripetal force. We want to find the condition under which F will be either zero or minimum. This can be done by the application of the laws of conservation of linear momentum and angular momentum.

By application of the laws of conservation of linear momentum and angular momentum it can be shown that if k_c is the radius of gyration for the momentum of inertia about C and l and l' are, respectively, distances of O and O' from C, then the following relation must be satisfied:

$$ll' = k_c^2 \tag{9.97}$$

Thus, if this relation is satisfied, when a blow is struck at O', no impulse will be felt at O. Such a point O' is called the center of percussion relative to point O. That is, *the point of application of an impulse (or blow) for which there is no reaction at the axis of rotation is known as the center of percussion*. The relation of Eq. (9.97) is exactly the same as for the physical pendulum

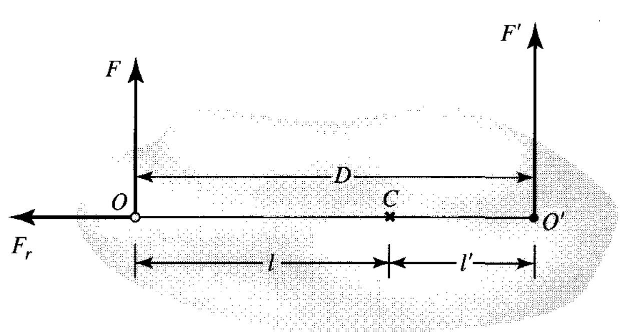


Figure 9.17 Relative positions of the center of oscillation O, the center of percussion O', and the center of mass C for a rigid body.

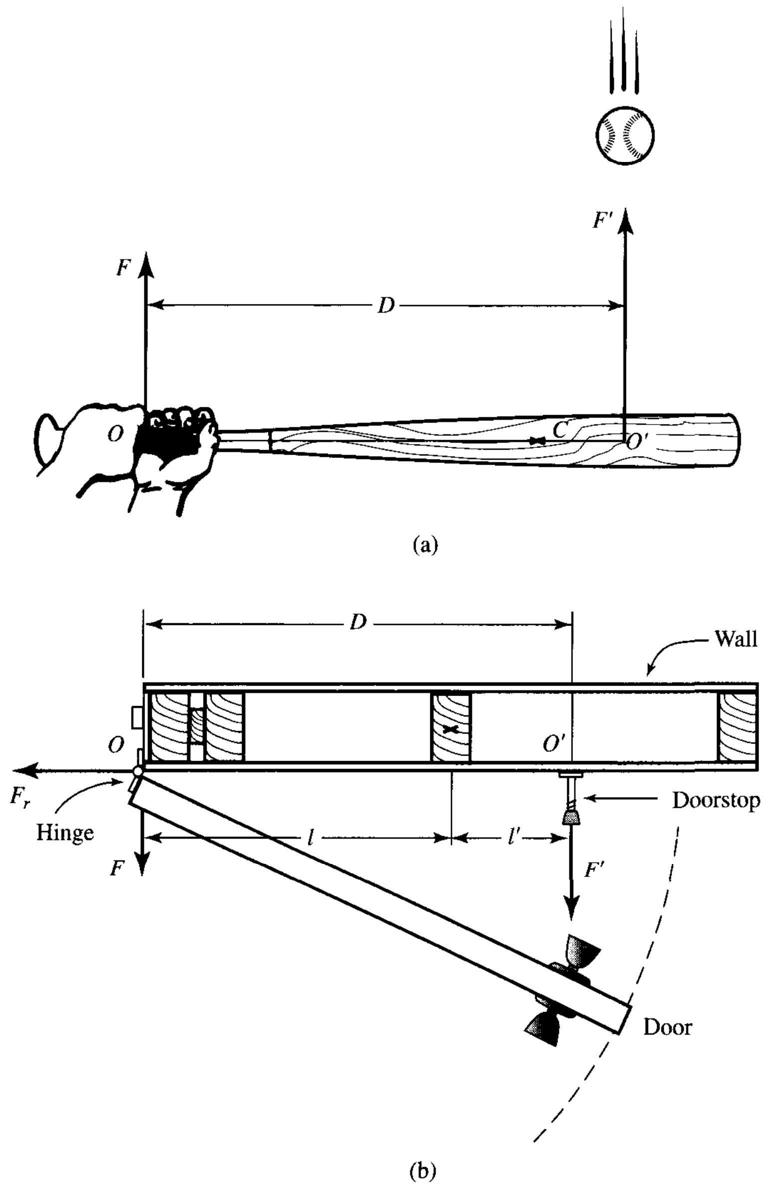


Figure 9.18 Relative position of O, O', and C in the case of (a) a batter hitting a ball, and (b) a door hitting a door stop.

given by Eq. (9.94). Thus the center of oscillation O and the center of percussion O' are identical. That is, the points O and O' are interchangeable.

Let us mention two important everyday applications. For instance, a batter hits a ball with a bat. The batter should try to hit at the center of the percussion at O' relative to his hand on the bat at O. This will minimize the blow to the hand; that is, it will avoid a reaction on the batter's hand, as shown in Fig. 9.18(a). As a second instance, consider a door stop used to prevent tearing door hinges loose. The door stop must be installed in such a way that, as shown in

Fig. 9.18(b), it is at a point of percussion at O' at a distance D from the hinges which are on the axis of rotation of the door.

9.9 DEFORMABLE CONTINUA

In most solids, atoms and molecules are arranged in some order. How rigidly these atoms and molecules are held about their equilibrium positions depends on the relative strength of the short-range electrical forces between them. Even though systems such as a vibrating string consist of a large number of discrete particles, it is advantageous to replace a system of discrete particles with a continuous distribution of matter. So far we have treated such continuous matter as rigid systems. In actual practice, matter is deformable. When under the action of internal and external forces, a change in the size and shape of the body may result. Thus, in this section, we shall be dealing with matter that we assume to be continuous and also deformable, that is, a *deformable continuum*. When external forces are applied to such a system, a distortion results because of the displacement of the atoms from their equilibrium positions and the body is said to be in a state of stress. After the external force is removed the body returns to its equilibrium position, providing the applied force was not too great. The ability of a body to retun to its original shape is called *elasticity*. To reach a quantitative definition of elasticity we must understand the definitions of stress and strain.

Suppose a body with surface area A is acted on by external forces having a resultant $\mathbf{F}$, where $\mathbf{F}$ is neither normal nor tangent to the surface. The *average stress* $\bar{\mathbf{S}}$ acting on an area A is defined as the *force per unit area*

$$\bar{\mathbf{S}} = \frac{\mathbf{F}}{A} \tag{9.98}$$

Let us now consider a small area ΔA, as shown in Fig. 9.19(a), that is acted on by a force $\Delta\mathbf{F}$. Thus the stress at point P is defined as

$$\mathbf{S} = \underset{\Delta A \to 0}{\text{limit}} \frac{\Delta\mathbf{F}}{\Delta A} = \frac{d\mathbf{F}}{dA} \tag{9.99}$$

The magnitude of $\mathbf{S}$ depends on the orientation of plane P in which area ΔA is located. We may resolve stress $\mathbf{S}$ into normal and tangential components by resolving $\Delta\mathbf{F}$ into two components—the normal component force ΔF_n and the tangential component force ΔF_t—which are normal and tangent to plane P. The *normal stress* σ is defined as

$$\sigma = \underset{\Delta A \to 0}{\text{limit}} \frac{\Delta\mathbf{F}_n}{\Delta A} = \frac{d\mathbf{F}_n}{dA} \tag{9.100}$$

while the *shear stress* τ is defined as

$$\tau = \underset{\Delta A \to 0}{\text{limit}} \frac{\Delta F_t}{\Delta A} = \frac{dF_t}{dA} \tag{9.101}$$

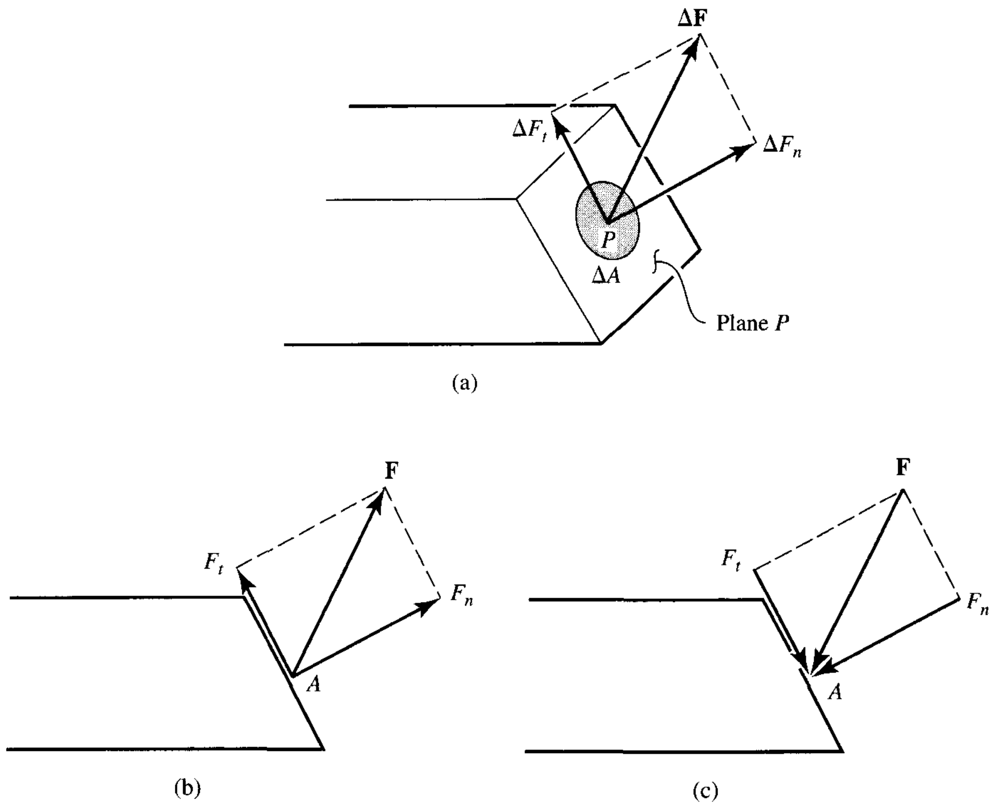


Figure 9.19 (a) Force $\Delta\mathbf{F}$ acting on an area ΔA. (b) Force $\mathbf{F}$ produces a tension and a shear stress. (c) Force $\mathbf{F}$ produces a compression and a shear stress.

If the normal stress is a pull, it is called *tension,* and if the normal stress is a push, it is called *compression,* while the tangential stress is called *shearing stress.* Thus, in Fig. 9.19(a) and (b), the normal force results in tension, while in Fig. 9.19(c) it results in compression, as shown; the tangential components in all three cases result in shear stress. The magnitude of tension and compression in parts (b) and (c) is F_n/A, while the shear stress is F_t/A. (Note: Stresses can result from internal forces as well, but according to Newton's third law, they cancel each other.)

The effect of stress is to cause distortion or a change in size and shape. *A quantity called strain refers to the relative change in size or shape of the body when under the applied stress.* We shall limit our discussion to three types of strain: (1) change in length, (2) change in shape, and (3) change in volume.

Consider a wire or rod of length L_0 and cross-sectional area A subject to a normal a tensile force F. The applied force increases the length. If the final length is L, then the change is $\Delta L = L - L_0$. We define the normal or tensile stress to be

$$\sigma = \frac{F_n}{A} \qquad \textbf{(9.102)}$$

while the fractional change in length ϵ, which is called the *longitudinal* or *tensile strain*, is given by

$$\epsilon = \frac{L - L_0}{L_0} = \frac{\Delta L}{L_0} \tag{9.103}$$

It is found experimentally that the ratio of stress to strain is a constant for a given material. This is called the *elastic modulus*. The ratio of the longitudinal stress to the longitudinal strain is called *Young's* (or *stretch*) *modulus*, Y; that is,

$$Y = \frac{\sigma}{\epsilon} = \frac{F_n/A}{\Delta L/L_0} \tag{9.104}$$

Since strain is a dimensionless quantity, the units of Young's modulus are the same as those of stress, that is, N/m^2, lb/in^2, and so on. [The transverse strain (change in length perpendicular to the force) is small and will be considered shortly.]

If the material of the wire obeys Hooke's law, the change in length is proportional to the applied force,

$$F_n = k\Delta L \tag{9.105}$$

where k is the stiffness or spring constant. We may write Eq. (9.104) as

$$F_n = \frac{YA}{L_0}\Delta L \tag{9.106}$$

Comparing these two equations, we get

$$k = \frac{YA}{L_0} \tag{9.107}$$

Since, for a given material Y, A and L_0 are constant, so is k.

Thus elongation increases with incrasing force. When the force is removed, the wire returns to its original length. The plot of tensile stress σ versus tensile strain ϵ is shown in Fig. 9.20. The proportionality relation between σ and ϵ holds only if stress is less than a certain maximum value. As shown in Fig. 9.20, this is reached at point A, the *proportional limit* or *yield point*. If the value of the applied stress is between A and B, there is no proportionality; but when the stress is removed, the body does return to its original value. If the applied stress is beyond point C, permanent deformation occurs in the body and eventually, if the applied stress is high, it will break (*fracture point*).

The plot in Fig. 9.20 assumes that, in the expression for stress, the area A remains constant and is equal to the original area before applying any force. But in practical situations there will be lateral contraction. Thus the stress is indicative of the force and the ratio of the force and *true area*. Hence we may define the true strain as follows. Let dL be the infinitesimal amount of elongation when the instantaneous length is L. If L_0 is the initial length and L_f the final length, then the *true strain* is defined as

$$\epsilon_{\text{true}} = \int_{L_0}^{L_f} \frac{dL}{L} = \ln \frac{L_f}{L_0} \tag{9.108}$$

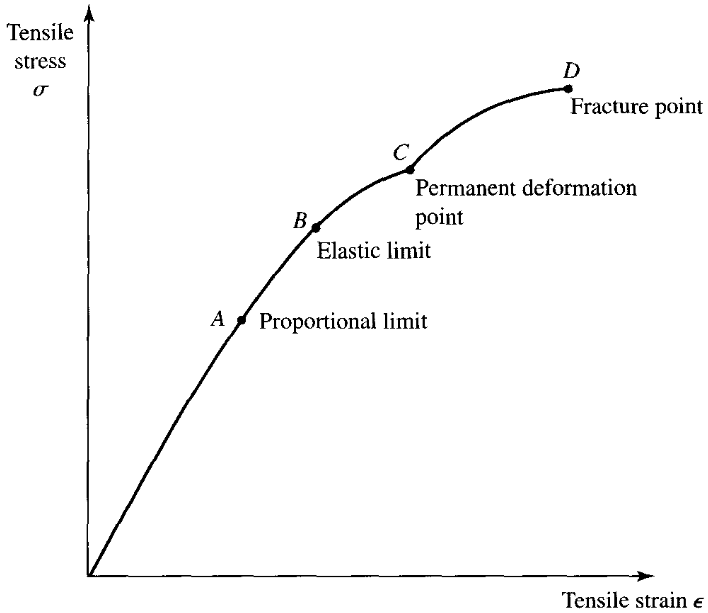


Figure 9.20 Elastic properties of a typical solid under normal stress.

For small elongations, if we expand ln (L_f/L_0) and neglect higher-order terms, we get ϵ as given by Eq. (9.103); that is, $\epsilon = \Delta L/L_0$.

Let us now consider the rigidity modulus resulting from the application of shear stress. When a pair of equal and opposite forces not acting along the same line of action is applied [Fig. 9.21(a)], the resulting shear stress produces a change in the shape of the body (but no change in length). The resulting strain is called a *shear strain.* It appears that the material consists of layers, and when stress is applied, the layers try to slide over one another. As shown in Fig. 9.21(a), the layer *ABCD*, under the action of a shear stress, has moved to *A'B'C'D'*, while the layer *PQRS* is not displaced. The shear stress τ as defined in Eq. (9.101) is

$$\tau = \frac{F_t}{A} \tag{9.109}$$

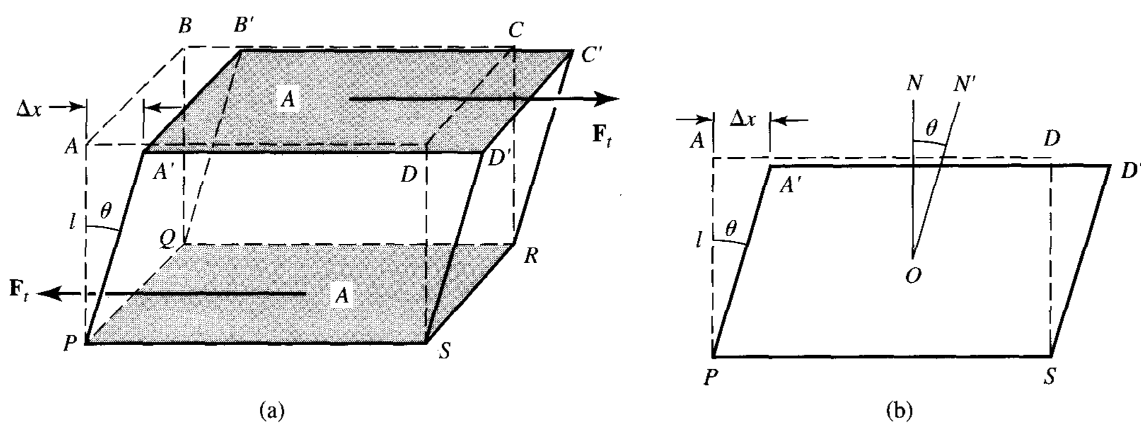


Figure 9.21 (a) A body is under the action of a pair of tangential forces, which results in a shearing stress. (b) Side view of part (a).

The shearing strain γ is defined as the ratio of the displacement Δx and length l, as shown in Fig. 9.21. For small values of Δx, this ratio is equal to the tangent of the angle θ. That is, shearing strain is

$$\gamma = \frac{\Delta x}{l} \simeq \tan\theta \tag{9.110}$$

Thus the *shear modulus*, or *modulus of rigidity*, or *torsion modulus*, η, is defined as

$$\eta = \frac{\tau}{\gamma} = \frac{F_t/A}{\tan\theta} \tag{9.111}$$

In the case of fluids, forces must be applied normal to the surface. Suppose that a fluid of volume V is acted on by a force F_n acting normal to an area A, resulting in a change in volume ΔV. The normal force applied to a fluid is called pressure P. Thus the stress and strain are given by

$$\text{Volume stress} = \sigma = \frac{F_n}{A} = \Delta P \tag{9.112}$$

and

$$\text{Volume strain} = \frac{\Delta V}{V} \tag{9.113}$$

Thus the *volume elasticity* or *bulk modulus*, B, defined as the ratio of the volume stress to volume strain, is given by

$$B = \frac{\Delta P}{-\Delta V/V} = -V\frac{\Delta P}{\Delta V} \tag{9.114}$$

The negative sign indicates that as pressure increases, volume decreases. The reciprocal of the bulk modulus is called *compressibility* β ($\beta = 1/B$).

Noting that $\Delta P = \sigma =$ normal stress, we may write Eq. (9.114) as

$$\sigma = -B\frac{\Delta V}{V} \tag{9.115}$$

When a stress is applied in one direction, it results in a longitudinal strain as well as a transverse (lateral) strain. In the case of simple tension, the ratio of the lateral strain ϵ_t to the longitudinal strain ϵ_l is called *Poisson's ratio* ν:

$$\nu = \frac{\epsilon_t}{\epsilon_l} \tag{9.116}$$

ν is small for glass ($\simeq 0.25$), while for rubber it is 0.5. There is a simple relationship among B, Y, and ν which we state here without proof:

$$B = \frac{Y}{3(1 - 2\nu)} \tag{9.117}$$

Note that this relation assumes that the material is uniform, that is, homogeneous and isotropic.

9.10 EQUILIBRIUM OF RIGID BODIES

To start, we shall discuss conditions of equilibrium. We shall apply these conditions to the investigation of equilibrium of flexible strings and cables and then to the equilibrium of solid beams.

Let us consider a body of mass M whose center of mass is at a distance R from a given point O, which is acted on by forces $\mathbf{F}_i$, and has angular momentum L_O about an axis passing through O. The equations of motion describing a rigid body are

$$\sum_i \mathbf{F}_i = M\ddot{\mathbf{R}} \tag{9.118}$$

and

$$\sum_i \boldsymbol{\tau}_{iO} = \frac{d\mathbf{L}_O}{dt} = I\ddot{\boldsymbol{\theta}} \tag{9.119}$$

where $\boldsymbol{\tau}_{iO}$ are the torques about an axis passing through O, I is the moment of intertia, and $\ddot{\boldsymbol{\theta}}$ is the angular acceleration. Once the torques about any one point O are known, the torques about any other point O' may be calculated from the following relation:

$$\sum_i \boldsymbol{\tau}_{iO'} = \sum_i \boldsymbol{\tau}_{iO} + (\mathbf{r}_O - \mathbf{r}_{O'}) \times \sum_i \mathbf{F}_i \tag{9.120}$$

where $\mathbf{r}_O$ and $\mathbf{r}_{O'}$ are the vector distances of points O and O' from the origin. Equation (9.120) states that *the total torque about O' is equal to the sum of two terms: the total torque about O and the total torque taken about O', assuming the total force is acting at O.* The proof is straightforward. Let $\mathbf{F}_i$ be the force acting at a point i, which is at a distance $\mathbf{r}_i$ from the origin. Then, according to the definition, the torque about O' is

$$\begin{aligned}
\sum_i \boldsymbol{\tau}_{iO'} &= \sum_i (\mathbf{r}_i - \mathbf{r}_{O'}) \times \mathbf{F}_i \\
&= \sum_i (\mathbf{r}_i - \mathbf{r}_O + \mathbf{r}_O - \mathbf{r}_{O'}) \times \mathbf{F}_i \\
&= \sum_i (\mathbf{r}_i - \mathbf{r}_O) \times \mathbf{F}_i + \sum_i (\mathbf{r}_O - \mathbf{r}_{O'}) \times \mathbf{F}_i \\
&= \sum_i \boldsymbol{\tau}_{iO} + (\mathbf{r}_O - \mathbf{r}_{O'}) \times \sum_i \mathbf{F}_i
\end{aligned}$$

which is the result stated previously.

For a rigid body to be in translational equilibrium—that is, at rest or moving with uniform velocity—the sum of the forces must be zero. For a body to be in rotational equilibrium—at rest or rotating with uniform velocity, the sum of the external torques must be zero. Thus, from

Eqs. (9.118) and (9.119), with $\ddot{\mathbf{R}} = 0$ and $\ddot{\boldsymbol{\theta}} = \mathbf{0}$, we get

$$\sum_i \mathbf{F}_i = 0 \tag{9.121}$$

and

$$\sum_i \boldsymbol{\tau}_{iO} = 0 \tag{9.122}$$

Note that if the sum of the torques about any point is zero, then [from Eq. (9.120)] it will be zero about any other point.

Since we notice that the motion of a rigid body is determined by the total forces and total torques, we can make the following statement, which we shall find useful in discussing the equilibrium of rigid bodies. Two systems of forces acting on a rigid body are *equivalent* if they produce the same resultant force and the same total torque about any point.

Here we state the definition of a couple. A *couple* is a system of forces whose sum is zero; that is,

$$\sum_i \mathbf{F}_i = 0 \tag{9.123}$$

The total torque resulting from a couple is the same about every point and is given by

$$\sum_i \boldsymbol{\tau}_{iO'} = \sum_i \boldsymbol{\tau}_{iO} = \sum_i \mathbf{r}_{iO} \times \mathbf{F}_i \tag{9.124}$$

Thus a couple may be characterized by a vector that is the total torque. This leads to a further statement: *All couples are said to be equivalent if they have the same total torque.*

From this discussion, we can deduce the following very useful result, referred to as the *rigid body theorem*:

> *Every system of forces acting on a rigid body can be reduced to a single force through an arbitrary point and a couple.*

Depending on the type of equilibrium, the resulting force and/or couple, may be zero.

9.11 EQUILIBRIUM OF FLEXIBLE CABLES AND STRINGS

An *ideal flexible cable* or *string* is one that will not support any compression or shearing stress, but there can be tension directed along the tangent to the string at any point. Cables, chains, and light strings used in different structures can be treated as ideal flexible strings. Furthermore, we shall assume that the weight of the cable is negligible compared to the external load acting on it, or that there is no external load and the weight of the cable is the only load. We can divide our discussion into two parts: (1) cables of negligible weight, and (2) cables with loads (or forces) distributed continuously along the length of the cable.

Cable with Concentrated Load

Consider an ideal flexible cable of negligible weight that is suspended between points P_1 and P_2 and is under the action of an external force **F** acting at point P_3, as shown in Fig. 9.22. The force **F** keeps the cable taut, as shown. Let l_1 be the length of the segment of the string between P_1P_3, l_2 the length between P_2P_3, while l_{12} is the distance between the points P_1 and P_2. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be the tensions in the two segments of the strings, as shown. By using the law of cosines, the angles α and β are given in terms of l_1, l_2, and l_3 by the following relations:

$$\cos\alpha = \frac{l_1^2 + l_{12}^2 - l_2^2}{2l_1l_{12}} \quad \text{and} \quad \cos\beta = \frac{l_2^2 + l_{12}^2 - l_1^2}{2l_2l_{12}} \tag{9.125}$$

It is assumed that the string does not stretch so that the position of point P_3 is independent of the force **F**. Since point P_3 is in equilibrium, we may write

$$\mathbf{F} + \mathbf{T}_1 + \mathbf{T}_2 = 0 \tag{9.126}$$

as shown by the triangular relation in Fig. 9.22(b). Using the law of sines, from Fig. 9.22(b) we obtain expressions for $\mathbf{T}_1$ and $\mathbf{T}_2$ in terms of **F**; that is,

$$\mathbf{T}_1 = \mathbf{F}\frac{\sin(\beta + \gamma)}{\sin(\alpha + \beta)} \quad \text{and} \quad \mathbf{T}_2 = \mathbf{F}\frac{\sin(\gamma - \alpha)}{\sin(\alpha + \beta)} \tag{9.127}$$

This means that we can find the angles in terms of distances from Eq. (9.125) and then use them in Eq. (9.127) to evaluate $\mathbf{T}_1$ and $\mathbf{T}_2$. But this is not the true answer because we have assumed that the string does not stretch Actually, tension determines the lengths of the segments of the strings, and we must take this into account to evaluate $\mathbf{T}_1$ and $\mathbf{T}_2$. These can be evaluated sufficiently by the method of *iterative approximations* or the *relaxation* method, explained next.

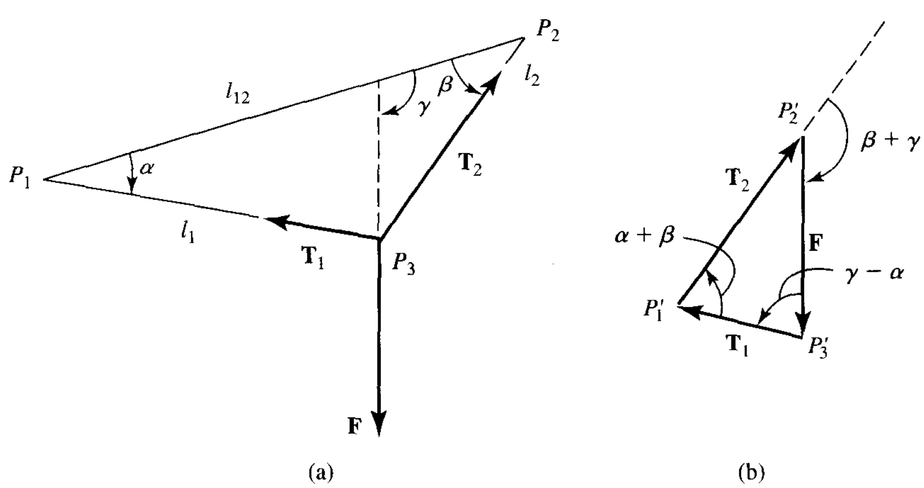


Figure 9.22 (a) Ideal flexible cable under the action of an external force **F**. (b) **F**, $\mathbf{T}_1$, and $\mathbf{T}_2$ form a closed triangle.

According to Hooke's law, the unstretched lengths l_{10} and l_{20} under tensions T_1 and T_2 become l_1 and l_2, given by

$$l_1 = l_{10}(1 + kT_1) \quad \text{and} \quad l_2 = l_{20}(1 + kT_2) \tag{9.128}$$

where k is a proportionality constant. We now use successive approximation to evaluate different quantities: α, β, T_1, and T_2. For a first approximation, assume that the string does not stretch, so that $l_1 = l_{10}$ and $l_2 = l_{20}$. Using these values in Eqs. (9.125) and (9.127), we evaluate α, β, T_1, and T_2. Now use these values for T_1 and T_2 in Eq. (9.128) to obtain new values for l_1 and l_2. Use these values for l_1 and l_2 in Eq. (9.125) to get new values for α and β, and use these in Eq. (9.127) to get better values for T_1 and T_2. These values for T_1 and T_2 can be used in Eq. (9.128) to get still better values for l_1 and l_2, and then Eqs. (9.125) and (9.127) are used to get better values for α, β, T_1, and T_2. This procedure can be repeated over and over until the values for l_1, l_2, α, β, T_1, and T_2 converge to the correct values. In most situations the amount of stretching is small, and the first few iterations lead to correct values. As stated earlier, this is the method of successive approximation.

Cables under Distributed Loads

Parabolic cables. Consider a cable AB that is supporting a load that is uniformly distributed horizontally, as shown in Fig. 9.23. The load is denoted by vertical arrows pointing downward. Let this load be $\mathbf{w}$ per unit length, the length taken to be horizontal. O is the lowest point while $P(x, y)$ is any other point on the cable.

Let us consider the equilibrium of the portion OP of the cable that is horizontally loaded as shown in Fig. 9.24(a). Let $\mathbf{T}_0$ be the tension at the lowest point O that is horizontal, while the tension at P is $\mathbf{T}$, which makes an angle θ with the horizontal. The origin of the coordinate axes is taken to be at O, and XY is the vertical plane. $\mathbf{W}$ is the load acting on a portion of cable OP and is equal to $\mathbf{w}x$. Thus, for static equilibrium [from Fig. 9.24(b)],

$$\mathbf{T} + \mathbf{T}_0 + \mathbf{W} = 0 \tag{9.129}$$

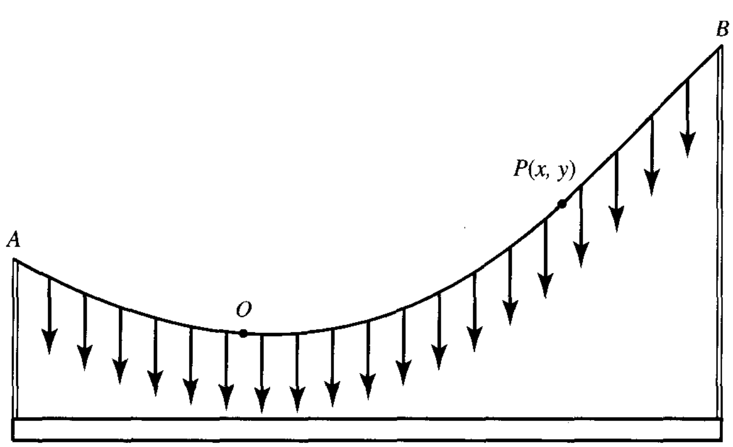


Figure 9.23 Cable AB supports a uniformly distributed load along a horizontal distance.

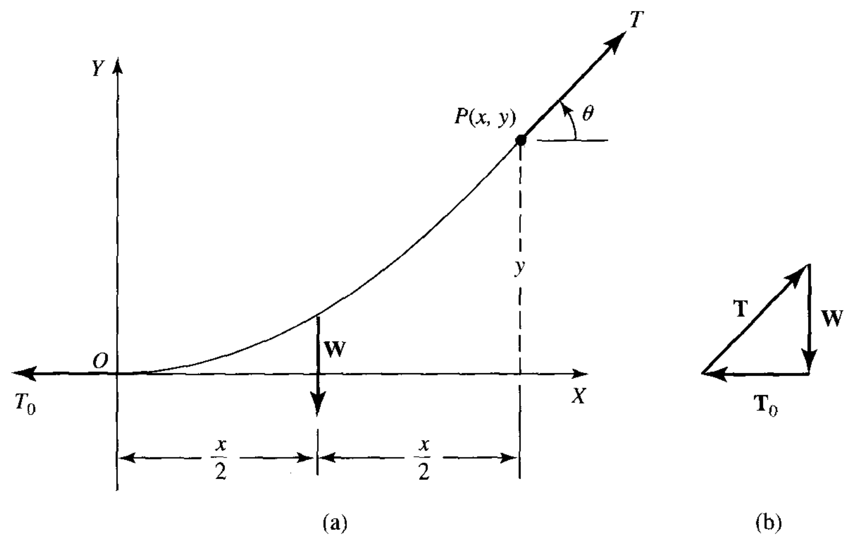


Figure 9.24 (a) Small segment *OP* of a uniformly loaded string. (b) For static equilibrium, $\mathbf{T}_0$, $\mathbf{T}$, and $\mathbf{W}$ form a closed triangle.

or

$$T\cos\theta = T_0 \tag{9.130}$$

$$T\sin\theta = W = wx \tag{9.131}$$

Solving for T and θ, we get

$$T = \sqrt{T_0^2 + w^2x^2} \tag{9.132}$$

and

$$\tan\theta = \frac{w}{T_0}x \tag{9.133}$$

Because the load is uniform, W is located at a distance $x/2$, as shown in Fig. 9.24(a). Taking the torque about P, we get

$$W\frac{x}{2} = T_0 y \quad \text{or} \quad wx\frac{x}{2} = T_0 y \tag{9.134}$$

That is,

$$y = \frac{w}{2T_0}x^2 \tag{9.135}$$

which is an equation of a *parabola*; that is, a cable under a uniform horizontal load has a parabolic shape. A cable in a suspension bridge is a typical example.

In Fig. 9.23, if A and B are at the same height, the horizontal distance L between A and B is called a *span*, while the vertical distance h of the lowest point O from A or B is called a *sag*. Substituting $y = h$ and $x = L/2$ in Eq. (9.135), the tension T_0 at the lowest point is

$$T_0 = \frac{wL^2}{8h} \tag{9.136}$$

Suppose point A is (x_A, y_A) and point P is (x_B, y_B). Using these values in Eq. (9.136), we can calculate the span $L = x_B - x_A$ (see Problem 9.46).

Catenary cables. Let us now consider a cable supporting a load that is distributed uniformly along its length (not along the horizontal distance as in parabolic cables). A typical example is that of a cable supporting its own weight, as shown in Fig. 9.25. Let point P be at a distance s from a point $s = 0$, where the tension T_0 is the supporting force at the end of $s = 0$ and is constant, while the tension at P is $\mathbf{T}(s)$. Let $\mathbf{w}(s)$ be the force per unit length at point s [note that $w(s) \neq w(x)$]. $\mathbf{w}\,ds$ represents the force on a small segment of length ds. Thus, for the portion of the cable AP that is in equilibrium, we must have

$$\mathbf{T}_0 + \mathbf{T}(s) + \int_0^s \mathbf{w}(s)\,ds = 0 \qquad \textbf{(9.137)}$$

We can obtain $\mathbf{T}(s)$ by differentiating Eq. (9.137) with respect to s; that is,

$$\frac{d\mathbf{T}}{ds} = -\mathbf{w}(s) \qquad \textbf{(9.138)}$$

From Fig. 9.25, θ is the angle that $\mathbf{T}$ makes with the X-axis. The vertical and horizontal components of Eq. (9.138) are

$$\frac{d}{ds}(T \sin\theta) = w \qquad \textbf{(9.139)}$$

and

$$\frac{d}{ds}(T \cos\theta) = 0 \qquad \textbf{(9.140)}$$

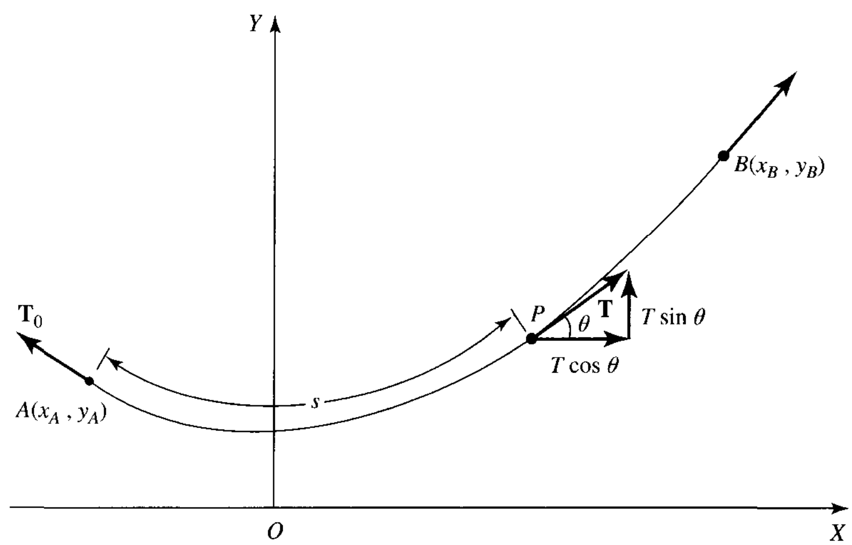


Figure 9.25 Cable supporting a load distributed uniformly along its length (and not its horizontal distance).

Keeping in mind that

$$ds = (dx^2 + dy^2)^{1/2} \tag{9.141}$$

the above equations, on solving, yield

$$y = \frac{C}{w}\cosh\left(\frac{w}{C}x + A\right) + B \tag{9.142}$$

This is the equation of a curve called a *catenary*. A, B, and C are the constants chosen so that y has proper values at the end points. Also, if we choose a coordinate system such that $y = 0$ at $x = 0$, then the constant $A = 0$.

9.12 EQUILIBRIUM OF SOLID BEAMS

General Treatment: Bending Moments

Let us consider a horizontal beam that is subject to vertical forces only—that is, the problem of a cantilever. Such a beam is under no compression or tension and there is no torsion about the axis of the beam. In these conditions, the beam bends only in a vertical plane. This is an example of a simple structure under shear forces and bending moments. We can calculate these quantities and the resulting bending as shown text.

Let the vertical forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ be the forces acting on a horizontal beam at distance $x_1, x_2, \ldots, x_n$, as shown in Fig. 9.26. The forces acting vertically upward are taken positive, and those acting downward are taken negative. Draw a plane AA' perpendicular to the beam and at a distance x from the left end of the beam. All the forces acting on the plane AA' due to the portion of the beam to the right of the plane can be reduced to an equivalent single force $\mathbf{F}_t$ through any point in the plane and a couple of torque $\boldsymbol{\tau}$. Since in the present case there are no compression or tension forces, $\mathbf{F}_t$ must be vertical; hence it is a *shearing force* acting from right to left across the plane AA', as shown. We assumed that there was no torsion in the beam. Since all the forces are vertical, the *bending moment* $\boldsymbol{\tau}$ (or torque $\boldsymbol{\tau}$) must be exerted from right to left

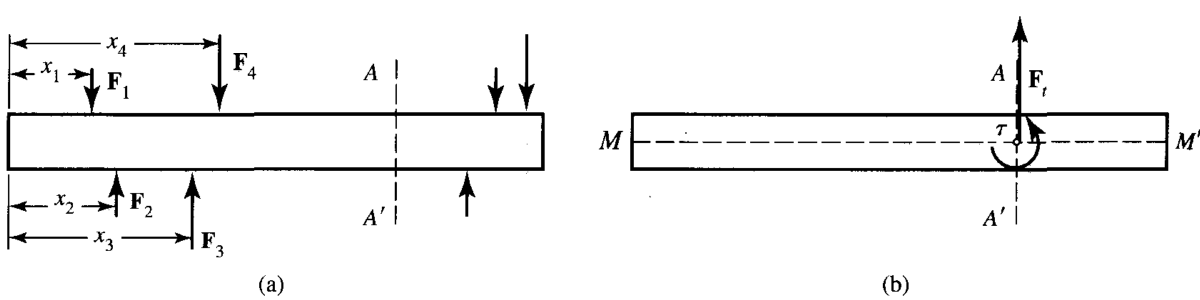


Figure 9.26 (a) Horizontal beam under the action of vertical forces. (b) The forces acting on the beam to the right of AA' are equivalent to a shear force $\mathbf{F}_t$ and torque $\boldsymbol{\tau}$.

along a horizontal axis perpendicular to the beam and in the plane AA'. The counterclockwise rotations of the plane about this horizontal axis are taken as positive. (From Newton's third law, force and torque equal and opposite to $\mathbf{F}_t$ and $\boldsymbol{\tau}$ are acting on the plane AA' from the beam on the left.)

We consider the equilibrium of the beam on the left of the plane AA'. Let $\mathbf{w}$ be the weight per unit length of the beam. Thus, from the two conditions of equilibrium, from Fig. 9.26, we get

$$\mathbf{F}_t + \sum_{x_i<x} \mathbf{F}_i - \int_0^x \mathbf{w}\, dx = 0 \tag{9.143}$$

and

$$\boldsymbol{\tau} - \sum_{x_i<x} (x - x_i)\mathbf{F}_i + \int_0^x (x - x')\mathbf{w}\, dx' - \boldsymbol{\tau}_0 = 0 \tag{9.144}$$

where $\mathbf{F}_t$ is the shearing force and $\boldsymbol{\tau}$ is the bending moment acting at a distance x from the left end. The second term on the left of Eq. (9.143) is the sum of the external forces acting on the beam from the left end up to the plane AA', and the third term is the weight of the same portion of the beam and is acting downward. $\boldsymbol{\tau}_0$ is the bending moment exerted by the left end of the beam on its support, providing the beam is fastened or clamped or supported at that end. If all the forces are known, $\mathbf{F}_t$ and $\boldsymbol{\tau}$ acting at x on the beam can be calculated from Eqs. (9.143) and (9.144). If the right end of the beam is free, then $\mathbf{F}_t = 0$ and $\boldsymbol{\tau} = 0$, and the equations can be used to calculate two other forces. Depending on whether the ends are free or supported, we can use these conditions to help solve the preceding equations.

The shearing force $\mathbf{F}_t$ and the bending torque $\boldsymbol{\tau}$ depend on the value of x and may be calculated as a function of x by differentiating Eqs. (9.143) and (9.144):

$$\frac{d\mathbf{F}_t}{dx} = \mathbf{w} \tag{9.145}$$

and

$$\frac{d\boldsymbol{\tau}}{dx} = \sum_{x_i<x} x_i\mathbf{F}_i - \int_0^x \mathbf{w}\, dx' = -\mathbf{F}_t \tag{9.146}$$

PROBLEMS

9.1. Find the center of the mass of the following:

(a) A thin uniform wire of linear mass density λ bent into an L-shape with both horizontal and vertical lengths equal.

(b) A thin uniform wire of linear mass density λ bent into a quadrant of a circle of radius R.

9.2. Find the center of mass of the following:

(a) A thin uniform sheet of metal of surface density σ cut into a semicircle of radius R.

(b) A thin uniform sheet of metal of surface mass density σ cut into a triangular piece with sides a, a, and b.

(c) A thin uniform sheet of metal of surface density σ cut into an octant of a thin spherical shell of radius R.

9.3. Find the center of mass of an octant of a solid sphere of radius R and uniform density ρ.

9.4. Find the center of mass of a sphere of radius R that is made up of layers of thin spherical shells centered about the center. The variation in density of these shells is **(a)** $\rho = \rho(z) = \rho_0(1 + z/R)$ and **(b)** $\rho = \rho(r) = \rho_0(1 + r/R)$.

9.5. Find the center of mass of a thin sheet in the XY-plane in the form of a parabola $y = ax^2$ and bounded between $y = 0$ and a straight line $y = b$. Calculate for the case when $b = 20$ cm and the surface density is 10 kg/m^2.

9.6. Find the center of mass of a paraboloid $z = a(x^2 + y^2)$ between $z = 0$ and $z = b$, as shown in Fig. P9.6. Calculate for the case when $b = 20$ cm, and the density is $\rho = 8000(1 - 0.5z)$ kg/m^3.

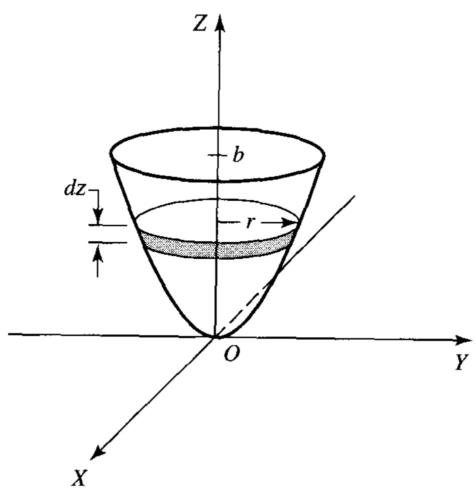


Figure P9.6

9.7. Consider a circular sheet of radius $2R$ having a uniform surface density σ. A circular hole of radius R is made at a distance R from the center of the first circle, as shown in Fig. P9.7. Find the center of mass of the remaining piece.

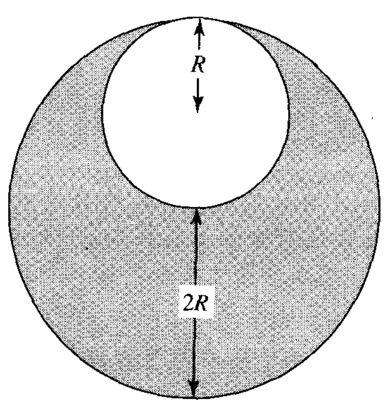


Figure P9.7

9.8. Find the center of the circle plate shown in Fig. P9.8 made up of two semicircular pieces of surface densities densities σ_1 and σ_2.

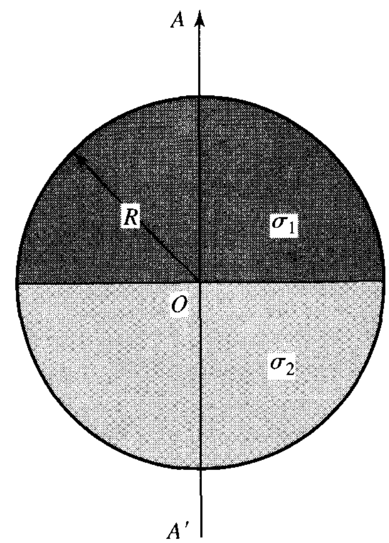


Figure P9.8

9.9. Find the center of mass of a solid hemisphere of radius R whose density varies linearly with distance from the center; that is, $\rho = \rho_0 r/R$.

9.10. Consider a solid sphere of uniform density ρ and radius R and a spherical cavity of radius $R/2$ centered at a distance of $R/2$ from the center. Find the center of mass.

9.11. Find the moment of inertia for a square lamina of mass M and side L, as shown in Fig P9.11, rotating about the following axes:
(a) Axis AA' passing through the center of mass and perpendicular to the lamina.
(b) Axis BB' parallel to AA' and at a distance $L/2$.
(c) Axis CC' parallel to one side of the lamina.

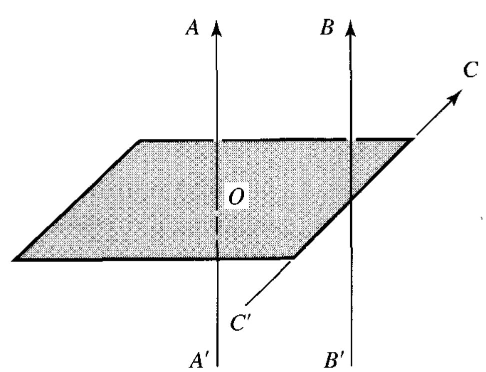


Figure P9.11

9.12. Consider a cube of mass M and side L, as shown in Fig. P9.12. Find the moment of inertia: **(a)** about an axis AA' perpendicular to a face and passing through the center of mass; **(b)** about an axis BB' parallel to the axis in part (a) and parallel to one edge; **(c)** about an axis CC'.

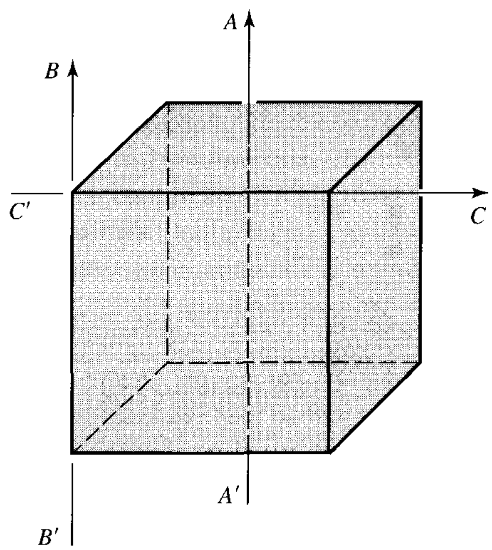


Figure P9.12

9.13. Find the moment of inertia for the following:

(a) A cylinder of mass M, radius R, and height H rotating about an axis of symmetry.

(b) Same as part (a), except rotating about an axis parallel to the symmetry axis and tangent to the surface.

9.14. Find the moment of inertia of a solid cone about its symmetry axis.

9.15. Find the moment of inertia for a frustum of a cone of mass M and radii R_1 and R_2 rotating about the symmetry axis.

9.16. Find the moment of inertia about an axis passing through the center O and perpendicular to the plane of a circular disk, as shown in Fig. P9.16. Also calculate I_X and I_Y. (The solid disk was removed from the hollow portion.)

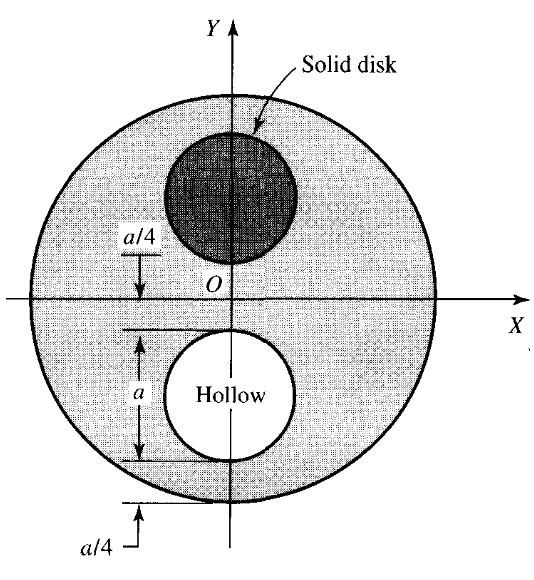


Figure P9.16

9.17. Consider a thin uniform square of side L with its diagonal along the X-axis, as shown in Fig. P9.17. The upper half of the square has a density σ_1 and the lower half σ_2. Find the moment of inertia about an axis passing through the center and perpendicular to the plane of the square. Also calculate I_X and I_Y.

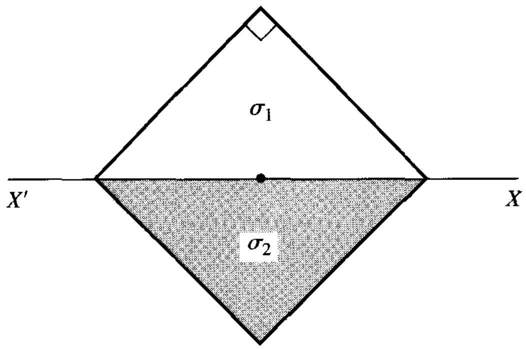


Figure P9.17

9.18. Find the moment of inertia for a sphere of radius R rotating about its axis of symmetry and having a density:
(a) $\rho_0(r) = \rho_0(kr/R)$:]
(b) $\rho_0(r) = \rho_0 e^{-kr/R}$
(c) $\rho_0(r) = \rho_0(1 - kr/R)$
Discuss the case in which $k \ll 1$.

9.19. Consider a thin disk of radius R and mass M. A small piece of maximum width $R/2$ has been cut off, as shown in Fig. P9.19. Calculate the center of mass and the moment of inertia for rotation about an axis perpendicular to the disk and passing through the center.

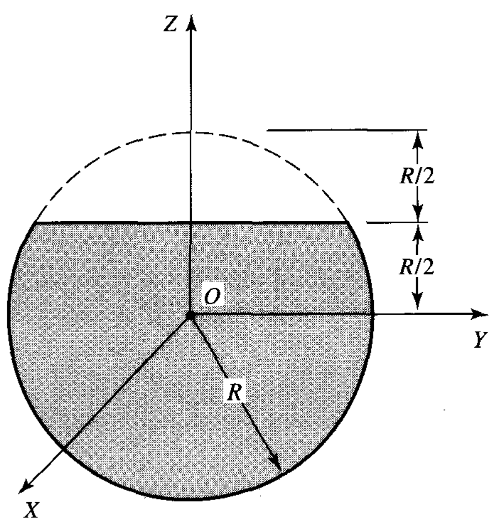


Figure P9.19

9.20. Consider a sphere of radius R that has a portion cut off, simialr to the disk in Fig. P9.19. Calculate the center of mass and the moment of inertia about the symmetry axis of the sphere.

9.21. Find the moment of inertia and the radius of gyration for a uniform rod of mass M and length L that is rotating about an axis through one end, making an angle θ with the rod, as shown in Fig. P9.21.

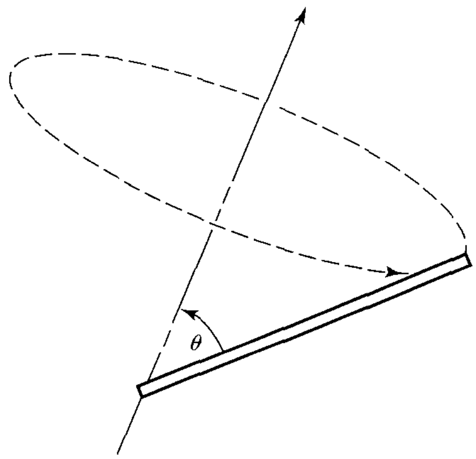


Figure P9.21

9.22. Show that the moment of inertia for a uniform octant of a sphere of mass m and radius a about an axis along one of the straight edges is $(3/2)ma^2$.

9.23. Calculate the moment of inertia for a parallelepiped about a symmetry axis.

9.24. Show that the moment of inertia for an ellipsoid of principal axes $2a$, $2b$, and $2c$ about the major axis is $(M/5)(b^2 + c^2)$.

9.25. A system consisting of a wheel attached to a fixed shaft is free to rotate without friction. A tape of negligible mass wrapped around the shaft is pulled with a steady constant force F. After a tape of length L has been pulled, the wheel acquires an angular velocity of ω. From these data, calculate the moment of inertia for the wheel.

9.26. If the total force acting on a system of particles is zero, show that the torque on the system is the same about all origins of different coordinate systems.

9.27. If the total linear momentum of a system of particles is zero, show that the angular momentum of the system is the same about all origins of different coordinate systems.

9.28. If τ_z is a funciton of θ alone, then, starting from the equation $dL/dt = I_z\ddot{\theta} = \tau_z$, show that the sum of the kinetic and potential energies is constant.

9.29. As in the case of translational motion, suppose the frictional torque is proportional to the angular velocity, that is, $\tau_f = -k\dot{\theta}$, while the driving torque is $\tau = \tau_0(1 + \alpha \cos \omega_0 t)$. Find the steady-state motion.

9.30. Consider a motor with an armature of 2-kg mass and radius of gyration of 8 cm. Its no-load full speed is when it draws a current of 2 A at 110 V at 1600 rpm. If the frictional torque is proportional to the angular velocity and the electrical efficiency is 75%, calculate the time required to reach a speed of 1200 rpm with no load.

9.31. A homogeneous circular disk of mass M and radius R has a light string wrapped around its circumference. One end of the string is attached to a fixed point. The disk is allowed to fall under gravity with the string unwinding. Find the acceleration of the center of mass.

9.32. A uniform rod of mass M and length L is placed like a ladder against a frictionless wall and frictionless horizontal floor. It is released from rest, making an angle θ with the vertical. Show that the initial reaction of the wall and the floor are (use only one variable to describe motion)

$$R_W = \tfrac{3}{4}\, mg \cos \alpha \sin \alpha, \qquad R_F = mg(1 - \tfrac{3}{4} \sin^2 \alpha)$$

and that the angle at which the rod will leave the wall is $\cos^{-1}(\tfrac{2}{3} \cos \alpha)$.

9.33. In Problem 9.32, if the coefficient of friction between the rod and the floor is μ, calculate **(a)** the horizontal and vertical components of the reaction as a function of angle θ, **(b)** the angle at which the rod begins to slip, and **(c)** the angular velocity when it hits the ground.

9.34. A uniform rod of length L and mass M is held horizontally with two hands at A and B. If A is suddenly released, at that instant what are **(a)** the torque about B, **(b)** the angular acceleration about B, **(c)** the vertical acceleration of the center of mass, and **(d)** the vertical force at B?

9.35. In the case of a simple pendulum, suppose we carry out correction to the fourth order instead of only the second order (as done in the text). Show that

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{K^2}{4} + \frac{9K^4}{64} + \cdots\right)$$

and that if θ is small, say θ_0, then

$$T \simeq 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \cdots\right)$$

9.36. Consider a simple pendulum of 1-m length ($T_0 \simeq 2$s) that has an amplitude of 5 cm. Show that

$$\frac{dT}{T_0} = -\theta_0\, d\theta_0$$

If θ_0 changes by 10%, that is, $d\theta = \theta_0/10$, calculate dT/T_0 and the resulting error per day.

9.37. In the case of a compound pendulum, we showed that when $T = T'$ the expression for g is given by Eq. (9.96). Suppose $T' = T(1 + \delta)$, where $\delta \ll 1$; find a new expression for g.

9.38. Consider a rod of mass M and length L. A mass m is attached at one end the rod is suspended from the other. If it behaves like a compound pendulum, calculate the time period of oscillations.

9.39. Consider a thin sheet of mass M in the shape of an equilateral triangle with each side of length L. Find the moment of inertia about an axis passing through the vertex and perpendicular to the sheet. If this behaves as a physical pendulum, find the period for small oscillations.

9.40. Consider a homogeneous hemisphere of mass M and radius R. With its flat face up, it rests on a perfectly rough horizontal surface. Find the expression for the length of an equivalent simple pendulum for small oscillations about the equilibrium position. Let the radius of gyration about the horizontal axis passing through the center of mass be k_g.

9.41. Consider a disk of mass m and radius r attached to a rod of length L and mass M. The system is suspended from the other end of the rod and allowed to oscillate. Find the time period of the oscillations. If the disk is mounted in such a way as to be free to spin (mounted on a frictionless bearing), what will be the period of oscillations?

9.42. Suppose a batter lets go of the bat after the ball hits the bat and that the bat starts rotating. For a quarter of a rotation of the bat, describe and sketch the motion of **(a)** the center of mass, and **(b)** the center of percussion; that is, plot $x(t)$ and $y(t)$ for both. Neglect the effect of gravity.

9.43. Consider a square plate of mass M and side L. The plate swings as a compound pendulum with an axis passing through one corner and perpendicular to the plane of the plate. Find the center of percussion and the period of oscillations.

9.44. A physical pendulum is made from a uniform disk of mass M and radius R suspended from a rod of negligible mass. The distance from the center of the disk and the point of oscillations is L. Find the time period of the oscillations. For what value of L will the time period be minimum? Locate the center of percussion.

9.45. Show that for a cube of side a when under a volume stress ΔP the volume strain is given by $\Delta V/V \simeq 3\Delta a/a$.

9.46. If two end points of a cable are at (x_A, y_A) and (x_B, y_B), show that the span L is given by $L = x_B - y_A = 8dT_0/w$, where d is a function of $y_A - y_B$.

9.47. Consider a wire of 1.5-m length and 2-mm diameter. It is clamped at the upper end and a 5-kg mass hangs at the lower end. Young's modulus is 9×10^{10} N/m^2, and its Poisson's ratio is 0.25. Calculate **(a)** the extension of the wire, **(b)** the decrease in the cross-sectional area due to the lateral strain, **(c)** the work done by the stretching force, and **(d)** the potential energy of the stretched wire.

9.48. Calculate the work done per unit volume in the following cases: **(a)** a shearing stress shearing the body through an angle θ, and **(b)** a uniform stress P producing a volume strain V.

9.49. A weightless rod of length L is clamped at two ends in a horizontal position. A weight W is placed at its center. Show that the equation for the shape of the rod is

$$y = \frac{W}{YK^2A}\left(\frac{x^3}{12} - \frac{Lx^2}{16}\right)$$

Calculate the deflection of the center.

SUGGESTIONS FOR FURTHER READING

ARTHUR, W., and FENSTER, S. K., *Mechanics,* Chapter 13. New York: Holt, Rinehart and Winston, Inc., 1969.

BECKER, R. A., *Introduction to Theoretical Mechanics*, Chapter 9. New York: McGraw-Hill Book Co., 1954.

FOWLES, G. R., *Analytical Mechanics*, Chapter 8. New York: Holt, Rinehart and Winston, Inc., 1962.

FRENCH, A. P., *Newtonian Mechanics*, Chapter 14. New York: W. W. Norton and Co., Inc., 1971.

HAUSER, W., *Introduction to the Principles of Mechanics*, Chapter 9. Reading, Mass.: Addison-Wesley Publishing Co., 1965.

KLEPPNER, D., and KOLENKOW, R. J., *An Introduction of Mechanics*, Chapter 6. New York: McGraw-Hill Book Co., 1973.

SLATER, J. C., *Mechanics*, Chapter 12. New York: McGraw-Hill Book Co., 1947.

STEPHENSON, R. J., *Mechanics and Properties of Matter*, Chapter 6. New York: John Wiley & Sons, Inc., 1962.

SYMON, K. R., *Mechanics*, 3rd ed., Chapter 5. Reading, Mass: Addison-Wesley Publishing Co., 1971.

TAYLOR, E. F., *Introductory Mechanics*, Chapter 8. New York: John Wiley & Sons, 1963.