COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE
DEPARTMENT OF MATHEMATICS

Course Module On Transformation Geometry

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Contents

Chapter 1

Group of Transformation

Introduction

In This chapter we study the basic definition of Transformation and colineations and properties of transformation (Group transformation and involution).

By moving all the points a geometric figure according to certain rules, you can create an image of the original figure, each point on the original figure corresponds to a point on its image. The image of point A after a transformation of any type is point A'. We read A' as "A prime". This process is called **transformation.**

We consider objects as they moves as a result of a plane. objects refer to triangles, lines, points, circles etc. The concept "Transformation" is matched with the definition and properties of function.

Main Objectives of this Chapter: At the end of this Chapter students will be able to:

- Define transformation and collineation
- Verify group transformation
- Discuss on Involution

1.1 Definition of transformations and colineations

Question: What is one to one correspondence function?

Possible answer: One to one correspondence function is both one to one and on to function.

Definition 1.1.1. A transformation on the plane is a one to one correspondence from the set of points in the plane on to itself.

A transformation of the plane is a rule that assigns to each point in the plane a different point or the point itself. Note that each point in the plane is assigned to exactly one point. Transformations are used to scale, translate, rotate, reflect and shear shapes and objects. It is possible to affect this by changing their coordinate values.

For a given transformation α of the plane

1) α is one to one

   For every point R of the plane, there is a unique point Q such that $\alpha(R) = Q$.

2) α is on to

   For every point Q of the plane, there is a unique point S on the plane such that $\alpha(S) = Q$.

Example 1.1.2. Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\alpha(x, y) = (x^2, y)$, then show that α is a trans formation.

Solution: To show that α is a transformation, we shall check one to oneness and on toness of α.

Let (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$.

Suppose $\alpha((x_1, y_1)) = \alpha((x_2, y_2))$, then we want to show that $(x_1, y_1) = (x_2, y_2)$

$\alpha((x_1, y_1)) = \alpha((x_2, y_2))$ implies $(x_1^2, y_1) = (x_2^2, y_2)$

$\Rightarrow y_1 = y_2$ and $x_1^2 = x_2^2 \Rightarrow x_1 \neq x_2$. Thus $(x_1, y_1) \neq (x_2, y_2)$

Hence α is not one to one. So that α is not a transformation.

Example 1.1.3. *a*) Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined by $\alpha((x, y)) = (-x, y), \forall (x, y) \in \mathbb{R}^2$. Show that α is a transformation.

b) Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined by $\alpha((x, y)) = (x, y^3), \forall (x, y) \in \mathbb{R}^2$. Show that α is a transformation.

Solution:

a) To show that α is a transformation we will check one to oneness and ontoness of α

Let (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$

Suppose $\alpha(x_1, y_1) = \alpha(x_2, y_2)$, we need to show that $(x_1, y_1) = (x_2, y_2)$

$\Rightarrow (-x_1, y_1) = (-x_2, y_2)$

$\Rightarrow -x_1 = -x_2$ and $y_1 = y_2$

$\Rightarrow x_1 = x_2$ and $y_1 = y_2$

Therefore $(x_1, y_1) = (x_2, y_2)$ and hence α is one to one

Next to check one to oneness of α

For every $(u, v) \in \mathbb{R}^2$ (in image of α), there exist $(x, y) \in \mathbb{R}^2$ (in domain of α) such that $\alpha(x, y) = (u, v)$

$\Rightarrow (-x, y) = (u, v)$

$\Rightarrow -x = u$ and $y = v$

$\Rightarrow x = -u$ and $y = v$

$\Rightarrow (x, y) = (-u, v)$

$\Rightarrow$ Fir every $(u, v) \in \mathbb{R}^2$, there exist $(-u, v) \in \mathbb{R}^2$ such that $\alpha(-u, v) = (u, v)$

Therefore α is on to and hence a transformation.

b) Let (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$ Suppose $\alpha(x_1, y_1) = \alpha(x_2, y_2)$, we need to show that $(x_1, y_1) = (x_2, y_2)$

$\Rightarrow (-x_1, y_1^3) = (-x_2, y_2^3)$

$\Rightarrow -x_1 = -x_2$ and $y_1^3 = y_2^3$

$\Rightarrow x_1 = x_2$ and $y_1 = y_2$

Therefore $(x_1, y_1) = (x_2, y_2)$ and hence α is one to one.

To check on toness of α

For every $(u, v) \in \mathbb{R}^2$ (in image of α), there exist $(x, y) \in \mathbb{R}^2$ (in domain of α) such that $\alpha(x, y) = (u, v)$

$\Rightarrow \alpha(x, y) = (u, v)$

$\Rightarrow (x, y^3) = (u, v)$

$\Rightarrow x = u$ and $y^3 = v$

$\Rightarrow x = u$ and $y = v^{\frac{1}{3}}$

$\Rightarrow$ Fir every $(u, v) \in \mathbb{R}^2$, there exist $(u, v^{\frac{1}{3}}) \in \mathbb{R}^2$ such that $\alpha((u, v^{\frac{1}{3}}) = (u, v)$.

Therefore α is on to and hence a transformation.

Definition 1.1.4. A transformation f having the property that if L is a line then $f(L)$ is also a line. A transformation f with this property is called **a collineation.**

Example 1.1.5. 1) Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\alpha((x, y)) = (x - 1, \frac{1}{2}y)$, then show that α is a collineation.

2) Show that the mapping γ that sends each points (x, y) to$(-x + \frac{y}{2}, x + 2)$ is a collineation.

Solution: 1) First show that α is a transformation. Let (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$

1.2 Group of Transformation

Brainstorming

- Do you remember the word group from your abstract algebra course? Please define group of transformation by using your previous concept?
- What are the criteria of group of transformation?

Sample problem

Show that the following transformations form a group transformation.

1 Let $\alpha_b : \mathbb{R} \to \mathbb{R}$ be defined by $\alpha_b = x + b$. Let $G = \{\alpha_b : b \in \mathbb{R}\}$.

2 Let $\beta_a : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\beta_a(x, y) = (ax, y), \ \forall\, a \in \mathbb{R},\ a \neq 0$ such that $G = \{\beta_a : \forall\, a \in \mathbb{R},\ a \neq 0\}$.

Solution:

1)

Theorem 1.2.1. *i) The set of all transformations form a group.*

ii) The set of all collineations forms a group.

1.3 Composition Of Transformation

Definition 1.3.1. The **composition** $\beta \circ \alpha$ of transformation α and β is the mapping defined by $\beta \circ \alpha(p) = \beta(\alpha(p))$ for every point p. Note that α is applied first and then β is applied.

Theorem 1.3.2. *The composition $\beta \circ \alpha$ of transformations α and β is itself a transformation.*

Example 1.3.3. Suppose $\beta : \mathbb{R} \to \mathbb{R}$ given by $\beta(x) = 2x+1$ and $\alpha : \mathbb{R} \to \mathbb{R}$ given by $\alpha(x) = x^2 + 1$. Then find

a) $\beta \circ \alpha$

b) $\alpha \circ \beta$

Solution:

a) $\beta \circ \alpha(x) = \beta(\alpha(x)) = \beta(x^2 + 1) = 2x^2 + 3.$

b) $\alpha \circ \beta(x) = \alpha(\beta(x)) = \alpha(2x + 1) = (2x + 1)^2 + 1 = 4x^2 + 4x + 2.$

Question:

1 Is the composition of a transformation commutative?

2 Is the composition of a transformation associative?

1.4 Identity and inverse Transformation

Definition 1.4.1. The **identity** transformation I is defined by $I(p) = p$ for every point p. If I is in set G of transformations, then G is said to have the **identity property**.

Definition 1.4.2. Let γ^{-1} is the mapping defined by $\gamma^{-1}(A) = B$ if and only $A = \gamma(B)$. The transformation γ^{-1} is called the inverse of transformation γ.

Explain the steps to find the inverse of transformation

Example 1.4.3. 1 Let $\alpha : \mathbb{R} \to \mathbb{R}$ given by $\alpha_x = x + 1$, then find the inverse of a transformation α.

2 Let $\beta : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\beta(x, y) = (3x + 1, 2y - 1)$, then find the inverse of a transformation β.

Possible answer:

1 $\alpha^{-1}(x) = x - 1$.

2 $\beta^{-1}(x, y) = (\frac{x-1}{3}, \frac{y-1}{2})$

Theorem 1.4.4. *The inverse of a transformation is unique.*

1.5 Involution

Did you hear the word involution? Discus this concept with your friends.

Theorem 1.5.1. *Suppose α and β are elements of group of transformation. Then*

i) $\beta\alpha = \gamma\alpha \Rightarrow \beta = \gamma$ *(Right cancellation law).*

ii) $\beta\alpha = \beta\gamma \Rightarrow \alpha = \gamma$ *(Left cancellation law).*

iii) $\beta\alpha = \alpha \Rightarrow \beta = I$*, where* I *is identity.*

iv) $\beta\alpha = \beta \Rightarrow \alpha = I$*.*

v) $\beta\alpha = I \Rightarrow \beta = \alpha^{-1}$ *and* $\alpha = \beta^{-1}$*.*

Theorem 1.5.2. *In a group, the inverse of a product is the product of the inverses in reverse order.*

Review Exercise

1) Suppose α is a transformation on the plane. Write "True" or "False" for the following sentences.
   a) If $\alpha(P) = \alpha(Q)$, then $P = Q$.
   b) For any point P there is a point Q such that $\alpha(P) = Q$.
   c) For any point P there is a unique point Q such that $\alpha(Q) = P$.
   d) For any point P there is a point Q such that $\alpha(Q) = P$.
   e) A transformation is necessarily a collineation.
   f) A collineation is necessarily a transformation.

2) Which of the mappings defined on the Cartesian plane by the equations below are transformation?
   a) $\alpha(x, y) = (x^3, y^3)$
   b) $\beta(x, y) = (\cos x, \sin y)$
   c) $\lambda(x, y) = (x^3 - x, y)$
   d) $\alpha(x, y) = (4x, 6y)$
   e) $\alpha(x, y) = (-x, x + 6)$
   f) $\pi(x, y) = (3y, x + 4)$
   g) $\sigma(x, y) = \left(x^{\frac{1}{3}}, e^y\right)$
   h) $\pi(x, y) = (-x, -y)$

3 Which of the transformation in Exercise 2 are collineations? For each collination, find the image of the line with equation $2x + 3y + 6 = 0$.

4 Let $A = (a, b)$ in the $xy-$plane. Find the equations for x' and y', where $\alpha((x, y)) = (x', y')$ and α is a mapping such that for any point P the mid-point of P and $\alpha(P)$ is always A.

5 If $A(1, 3)$ and $B(-2, -1)$ are given points, write an equation for the line such that $\alpha_l(A) = B$.

6 Give three examples of transformations on the Cartesian plane that are not collineations.

7 Show that a collineation determines a one to one correspondence from the set of all lines onto itself.

8 Find the pre-image of the line $y = 3x + 2$ under the collineation α, where

$\alpha((x, y)) = (3y, x - y)$.

9 Show that the lines with equations $ax + by + c = 0$ and $dx + ey + f = 0$

$a)$ are parallel if and only if $ae - bd = 0$ and

$b)$ are perpendicular if and only if $ad + be = 0$.

10 Prove or disprove that a mapping on the Cartesian plane that preserves betweeness among the points is necessarily a collineation.

11 Verify that the set of all transformations has associative property.

12 Write "True" or "False" for the following statements

$a)$ If α and β are transformations, then $\alpha = \beta$ if and only if $\alpha(P) = \beta(P)$ for every point P.

b Transformation G is in every transformations

c $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$ for transformation α and β.

Chapter 2

Affine Geometry

In This chapter we study the basic definitions of affine space, affine geometry, lines and planes in affine geometry, concurrency and classical theorems.

Main Objectives of this Chapter: At the end of this Chapter students will be able to:

- Define an affine space
- Relate affine geometry and vectors
- Discuss the lines and planes in affine space.
- Explain the concurrency of lines in affine space
- State and prove the classical theorem.

Method of teaching: It will be delivered through brain storming, interactive lecture.

Assessment Techniques: The students achievement will be assessed through questioning and answering, class activity and home work.

2.1 Affine Space

Brainstorming:

1 What is vector space:

2 What is difference and similarity of affine space and vector space?

3 What is the difference between affine geometry and Euclidean geometry?

Definition 2.1.1. A non-empty set W is said to be an an affine space associated with vector space V if and only if the following conditions are satisfied.

i) For any two points A, B in W, there exist a unique vector $u = \overrightarrow{AB} = B - A$ in V and u is the zero vector if and only if $A = B$.

ii) For any point A in W and vector u in V, there exist a unique point B such that $u = \vec{AB}$.

iii For any points A, $B\,C$ in W, we have $\vec{AB} + \vec{BC} = \vec{AC}$.

Question: Verify affine space by giving some example.

Which conditions must be satisfied to say two vectors are perpendicular and parallel?

Sample problem:

1 Let $\vec{a} = (2, 3, k)$ and $\vec{b} = (1, -2, 2)$ are perpendicular vectors, then find the value of k. An: $k = 2$

2 Let $\vec{a} = (3, x, 2)$ and $\vec{b} = (x, 4, 7)$ are perpendicular vectors, then find the value of x. An: $x = -2$

3 Show that $\vec{a} = (1, 2, 3)$ and $\vec{b} = (2, 4, 6)$ are parallel.

Theorem 2.1.2. *1 Parallelism relation denoted by $\|$ on a vectors is an equivalence relation.*

2 If $\vec{a}$ and $\vec{b}$ are not parallel, then $\forall\, r, t \in \mathbb{R}$, then the equation $r\vec{a} = t\vec{b}$ has a unique solution $r = t = 0$.

2.2 Geometry in Affine Space

Brainstorming:

1 What are the element of affine space?

2 How can analyze geometric theorems in affine space?

3 Describe the co-linearity of the point in affine space?

Theorem 2.2.1. *Suppose A, B, C, D are distinct points such that any three of them are co-linear. Then $\vec{AB} = \vec{CD}$ if and only if $\vec{AB} \parallel \vec{CD}$ and $\vec{AC} \parallel \vec{BD}$.*

Theorem 2.2.2. *i) The line segment joining the midpoints of two sides of a triangle is parallel to the third sides and its length is half of that side.*

ii) The diagonal of a rhombus are perpendicular.

Students activities

1 Prove the above theorems.

2 Using vectors prove the Pythagorean theorem.

3 State and prove cosine and sine laws by using vectors.

2.3 Lines and planes in affine space

Brainstorming:

1 What are the lines and planes in Euclidean geometry?

2 Could you define lines and planes in affine geometry?

2.3.1 Lines in Affine Space

Definition 2.3.1. Let W be an affine space. Then any lines in W passes through two different points A and B is defined as the set $< A, B >= \left\{X \in W : \vec{AX} = r\vec{AB}, r \in \mathbb{R}\right\}$ or $\left\{L : X = A + rL : X = A + r\overrightarrow{AB}, r \in \mathbb{R}\right\}$. Here the vector $\vec{AB}$ is called directed vector of the line and the scalar r is called parameter.

Now letting $X = (x, y, z)$, $A = (x_0, y_0, y_0)$ and $\vec{d} = \vec{AB} = (a, b, c)$ we get $L : X = (x_0, y_0, y_0) + r\vec{d}$ (This is called vector equation of the line).
Equating the corresponding components from the vector equation
$L : (x, y, z) = (x_0, y_0, y_0) + r\vec{d}$, we have $\begin{cases} x = x_0 + ar \\ y = y_0 + br \\ z = z_0 + cr \end{cases}$ (This is parametric equation of the l

Sample Exercise:

Write the vector and parametric equation of the line passing through the following points.

a) $A = (1, 2, 3)$ and $B = (-2, 4, 5)$

b) $A = (-1, 3, 4)$ and $B = (2, 7, 8)$

c) $A = (-4, 5, 6)$ and $B = (-9, 3, 4)$

Theorem 2.3.2. *i There is a unique line through any two points in affine space.*

ii Any two different directed vectors of a line are parallel.

Activities

1 Prove the above theorems

2 Define parallel and perpendicular lines.

Theorem 2.3.3. *Two lines $L_1 : X = A + r\vec{u}$ and $L_2 : X = A + r\vec{v}$ passing through the same point A are identical or equal if and only if $\vec{u}$ and $\vec{v}$ are parallel.*

Theorem 2.3.4. *Given a line L a point Q not on L. Then there is exactly one line m parallel to l.*

Activities

1 State and prove the length of ratio Theorem.

2 If $A = (5, 0, 7)$ and $B = (2, -3, 6)$, then find the point P on the line $< A, B >$ which satisfies $\frac{AP}{PB} = 3$. Is P is between A and B? Or out side?

2.3.2 Planes in Affine Space

Definition 2.3.5. Let W be an affine space. Then the plane π passing through three non-collinear points A, B, C is the set given by $< A, B, C >= \{X \in \pi : X = A + r\vec{AB} + t\vec{AC}\}$. From this equation, if we let $A = (x_0, y_0, z_0)$ and $\vec{AB} = (a, b, c)$, $\vec{Ac} = (d, e, f)$, then any arbitrary point $X = (x, y, z)$ on this plane is given by

$$< A, B, C >= \begin{cases} x = x_0 + ra + td \\ y = y_0 + rb + te \\ z = z_0 + rc + tf \end{cases}$$

Here r and t are parameters.

Activities

1 Find the vector and parametric equation of the plane passing through the points $A = (3, 4, 5)$, $B = (-3, 5, 6)$ and $C = (0, 3, 6)$.

2 How can we determine the distance from a point to a line in affine space?

3 Let L be a line through the points $A = (3, 4, 5)$, $B = (-3, 5, 6)$. Find point P on this line which is closest to the origin and calculate the shortest distance from the line to the origin.

4 State and prove intercept Theorem in an affine space.

2.4 Concurrency

Brainstorming

1 Did you hear the word concurrency?

2 What is point of concurrency?

2.5 The Classical Theorems

State and prove the following classical theorem

1 Menelaus Theorem

2 Ceva's Theorem

3 Desargue's Theorem

4 papu's Theorem

Chapter 3

Orthogonal Transformations

In this Chapter we study the basic properties of orthogonal transformation Fundamental types of orthogonal transformation, representation of orthogonal transformation as the product of reflection and Equations of orthogonal transformation in coordinates.

3.1 Definition of Orthogonal Transformation

Main Objectives of this Chapter: At the end of this Chapter Students will be able to:-

- Define an orthogonal transformation
- Describe fundamental types of orthogonal transformation
- Discus the representation of orthogonal transformation as the product of reflection
- Derive the equations of orthogonal transformation in coordinates.

Method of teaching: It will be delivered through brainstorming, interactive lecture.
Assessment Techniques: The student's achievement will be assessed through questioning and answering, class activity and home work.

Definition 3.1.1. A geometric transformation is said to an orthogonal(Isometry) when it preserves the distance between any pair of points in the plane. In other words α is an orthogonal transformation in Euclidean plane , when the quality $d(\alpha(P), \alpha(Q)) = d(p, q)$ holds for every pair of points P and Q in the plane.

Example 3.1.2. Show that the following maps are orthogonal transformation

a) $\alpha_b : \mathbb{R} \to \mathbb{R}$ given by $\alpha_b(x) = x + b$ for any constant b

b) $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\alpha(x, y) = (x + 1, y + 1)$

c) $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\gamma(x, y) = (x, -y)$

d) $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\alpha(x, y) = (x - 5, y + 3)$

Solution: a) Since the distance between two points x and y in $\mathbb{R}$ is

$$\begin{aligned} d(x, y) &= |x - y| \text{ and} \\ d(\alpha_b(x), \alpha_b(y)) &= |x + b - (y - b)| \\ &= |x - y| \\ &= d(x, y). \end{aligned}$$

Therefore α_b is an orthogonal transformation.

b) Let $P = (x, y)$ and $Q = (z, w)$ are two points in $\mathbb{R}^2$(in plane). Now distance between P and Q is give by

$$\begin{aligned} d(P, Q) &= |P - Q| \\ &= |(x, y) - (z, w)| \\ &= |(x - z), (y - w)| \text{ and} \\ d(\alpha(P), \alpha(Q)) &= |\alpha(P) - \alpha(Q)| \\ &= |\alpha(x, y) - \alpha(z, w)| \\ &= |(x + 1, y + 2) - (z + 1, w + 2)| \\ &= |x + 1 - (z + 1), y + 2 - (w + 2)| \\ &= |x - z, y - w| \\ &= d(P, Q). \end{aligned}$$

Therefore α is an orthogonal transformation.

Question: Is $\beta : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\beta(x(x,y)) = (ax+1, bx+2)$, for any constant a and b orthogonal transformation.

3.1.1 Properties of an orthogonal Transformation

Theorem 3.1.3. *The inverse of an orthogonal transformation is an orthogonal*

transformation.

Proof. Let β be an orthogonal transformation. Now let P and Q are any points, we need to show that $d(\beta^{-1}(P), \beta^{-1}(Q)) = d(P,Q)$.

$$\begin{aligned} d(\beta^{-1}(P), \beta^{-1}(Q)) &= \left|\beta^{-1}(P) - \beta^{-1}(Q)\right| \\ &= \left|\beta^{-1}\beta(P) - \beta^{-1}\beta(Q)\right|, \text{since } \beta \text{ is an orthoganal transformation.} \\ &= |I(P) - I(Q)| \\ &= |P - Q| \\ &= d(P,Q). \end{aligned}$$

Hence for any orthogonal transformation β, β^{-1} is also an orthogonal transformation. □

Example 3.1.4. If $\beta : \mathbb{R} \to \mathbb{R}$ defined by $\beta(x) = x+1$, then find the inverse of β and show that β^{-1} is also an orthogonal transformation.

Solution: Let $y = x+1 \Rightarrow x = y+1 \Rightarrow y = x-1$.

Therefore $\beta^{-1}(x) = x - 1$.

For any two points (x,y) then the distance between the two points is given by

$$\begin{aligned}
d(x,y) &= |x-y|, \text{ and} \\
d\left(\beta^{-1}(x), \beta^{-1}(y)\right) &= \left|\beta^{-1}(x) - \beta^{-1}(y)\right| \\
&= |x-1-(y-1)| \\
&= |x-y| \\
&= d(x,y)
\end{aligned}$$

Theorem 3.1.5. *The composition of any two orthogonal transformation is again an orthogonal transformation.*

Proof. Let α and β be any two orthogonal transformations.

we need to show that their compositions $\alpha \circ \beta$ is an orthogonal transformations.

Let P and Q be any two points, since α an orthogonal transformation

$|\alpha(P) - \alpha(Q)| = |P - Q|$ and β is also an orthogonal transformation, then we have

$|\beta(P) - \beta(Q)| = |P - Q|$

$$\begin{aligned}
d(\beta \circ \alpha(P), \beta \circ \alpha(Q)) &= |\beta \circ \alpha(P) - \beta \circ \alpha(Q)| \\
&= |\beta(\alpha(P)) - \beta(\alpha(Q))| \\
&= |\alpha(P) - \alpha(Q|\text{, since } \beta \text{ is an orthogonal transformation} \\
&= |P - Q|\text{, since } \alpha \text{ is an orthogonal transformation.}
\end{aligned}$$

Hence the composition $\beta \circ \alpha$ is an orthogonal transformation. □

Example 3.1.6. Let $\beta : \mathbb{R} \to \mathbb{R}$ given by $\beta(x) = x + 2$ and $\alpha : \mathbb{R} \to \mathbb{R}$ given by $\alpha(x) = x + 1$, show that $\beta \circ \alpha$ is an orthogonal transformation.

Solution: First find the composition function $\beta \circ \alpha$.

$\beta \circ \alpha(x) = \beta(\alpha(x)) = \beta(x+1) = x + 1 + 2 = x + 3.$

The distance between two point x and y is given by

$$\begin{aligned} d(x,y) &= |x-y| \text{ and} \\ d(\beta\circ\alpha(x),\beta\circ\alpha(y)) &= |\beta\circ\alpha(x)-\beta\circ\alpha(y)| \\ &= |\beta(\alpha(x))-\beta(\alpha(y))| \\ &= |x+3-(y+3)| \\ &= |x+3-y-3| \\ &= |x-y| \\ &= d(x,y). \end{aligned}$$

Hence the composition $\beta\circ\alpha(x)=x+3$ is an orthogonal transformation.

Theorem 3.1.7. *An orthogonal transformation maps distinct points in to distinct points.*

Proof. Let β be an orthogonal transformation. We need to show that

$A\neq B\Rightarrow\beta(A)\neq\beta(B)$.

Now $A\neq B\Rightarrow A-B\neq 0\Rightarrow d(A,B)\neq 0$ and also β is an orthogonal transformation. Then we have

$d(A,B)=d\left(\beta(A),\beta(B)\right)$, so $d(A,B)\neq 0\Rightarrow d\left(\beta(A),\beta(B)\right)\neq 0\Rightarrow\beta(A)\neq\beta(B)$ □

Theorem 3.1.8. *An orthogonal transformation maps*

i) *any three non-collinear points in to non-collinear points*

ii) *any collinear points in to collinear points.*

Proof. *i)* Let A,B and C be non-collinear points, Then by triangle inequality the non-collinearity means

$d(A,B)+d(B,C)>d(A,C)$. Now let A',B' and C' be respectively the image of A,B and C. Since the orthogonal transformation preserves the distance

$d(A', B') + d(B', C') > d(A', C')$. This show that A', B' and C' can not be collinear.

ii) Suppose A, B and C are any three points and β is an orthogonal transformation. Let $A' = \beta(A)$, $B' = \beta(B)$ and $C' = \beta C$.

Since β preserves distance, if $AB + BC = AC$, then by collinearity of these points we have

$A'B' + B'C' = A'C'$, as $AB = A'B'$, $B'C' = BC$ and $AC = A'C'$.

Hence $A - B - C \Rightarrow A' - B' - C'$ in other words B is between A and C, then B' is between A' and C'.

□

Theorem 3.1.9. *An orthogonal transformation maps lines in to lines and parallel, lines in to parallel lines.*

Proof. Let L is a line determined by two distinct points A and B and α is an orthogonal transformation such that $A' = \alpha(A)$ and $B' = \alpha(B)$, but $A \neq B$.

$\alpha(A) \neq \alpha(B) \Rightarrow A' \neq B'$. Hence A and B are two distinct points and determine a unique line L'. Besides α is an orthogonal transformation which preserves collinearity for any point P on L, $\alpha(P) = P'$ is on L'. Moreover if $L \parallel M$, then $\alpha(L) \parallel \alpha(M)$. □

Theorem 3.1.10. *(Three point Theorem): Two orthogonal transformation with the same image at three non-collinear points are equal.*

Proof. Exercise □

3.2 Orientation preserving and Orientation reversing Orthogonal Transformation

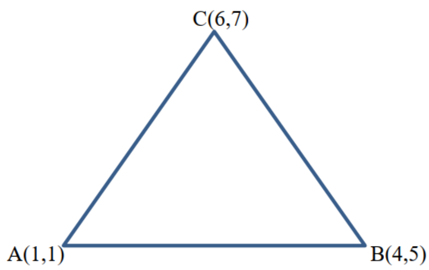
Definition 3.2.1. Let A, B and C be vertices of $\triangle ABC$. Then the orientation of $\triangle ABC$ is determined by using vectors $\vec{AB}$ and $\vec{AC}$ as follows:

a) If det $\left(\vec{AB}, \vec{Ac}\right) > 0$, then the pair $\left(\vec{AB}, \vec{Ac}\right)$ has positive orientation and so $\triangle ABC$.

b) If det $\left(\vec{AB}, \vec{Ac}\right) < 0$, then the pair $\left(\vec{AB}, \vec{Ac}\right)$ has negative orientation and so $\triangle ABC$.

Example 3.2.2. Determine the orientation of $\triangle ABC$ with vertices $A = (1,1)$, $B = (4,5)$ and $C = (6,7)$.

Solution:

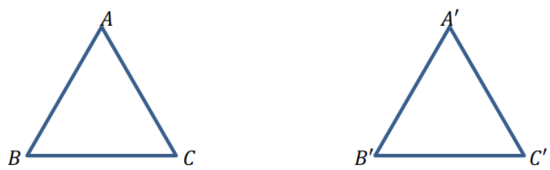


$\vec{AB} = B - A = (4,5) - (1,1) = (3,4)$ and $\vec{AC} = C - A = (6,7) - (1,1) = (5,6)$. Then

$$\det\left(\vec{AB}, \vec{Ac}\right) = det \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix} = 18 - 20 = -2 < 0$$

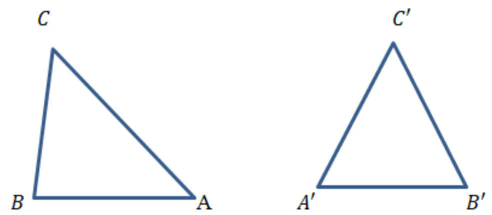
Therefore the pair $\left(\vec{AB}, \vec{Ac}\right)$ has negative orientation and so $\triangle ABC$.

Definition 3.2.3. 1) An orthogonal transformation said to be orientation preserving if for each ordered triples of non-collinear points (A, B, C) the orientation of the image triple (A', B', C') is the same as that of the given triple.



The two triangles $\triangle ABC$ and $\triangle A'B'C'$ have the same orientation.

2) An orthogonal transformation is said to orientation reserving if for each ordered triple of non-collinear points, the orientation of the image triple is opposite that of the given triples.



The two triangles $\triangle ABC$ and $\triangle A'B'C'$ have opposite orientation.

Example 3.2.4. Suppose β an orthogonal transformation that maps $\triangle ABC$ to $\triangle A'B'C'$, where the vertices of the triangles are $A = (4, 2)$, $B = (1, -2)$, $C = (12, -4)$ and $A' = (-1, 3)$, $B' = (3, 0)$, $C' = (5, 11)$ Determine whether β is orientation preserving or orientation reserving.

Solution: $\vec{AB} = B - A = (1, -2) - (4, 2) = (-3, -4)$ and
$\vec{AC} = C - A = (12, -4) - (4, 2) = (8, -6)$, then

$$\det\left(\vec{AB}, \vec{AC}\right) = \det\begin{pmatrix} -3 & 8 \\ -4 & -6 \end{pmatrix} = 18 + 32 = 50 > 0$$

Similarly $\vec{A'B'} = B' - A' = (3, 0) - (-1, 3) = (4, -3)$ and
$\vec{A'C'} = C' - A' = (5, 11) - (-1, 3) = (6, 8)$, then

$$\det\left(\vec{A'B'}, \vec{A'C'}\right) = \det\begin{pmatrix} 4 & 6 \\ -3 & 8 \end{pmatrix} = 32 + 18 = 50 > 0$$

Therefore $\triangle ABC$ and $\triangle A'B'C'$ both have positive orientation and hence β is orientation preserving orthogonal transformation.

Definition 3.2.5. Let α be an orthogonal transformation given by $\alpha(X) = AX + \vec{b}$, where A is standard matrix. Then *a)* α preserves orientation if and only if $\det(A) > 0$

b) α reserve orientation if and only if $\det(A) < 0$.

Example 3.2.6. $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal transformation given by $\alpha((x, y)) = (x + 1, y + 2)$. Then show that α is orientation preserving orthogonal transformation.

Solution: $\alpha(X) = AX + \vec{b}$, where $X = (x, y)$, then $\alpha(X) = \alpha((x, y)) = (x + 1, y + 2) = \begin{pmatrix} x+1 \\ y+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Here $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 - 0 = 1 > 0$.

Hence α is orientation preserving orthogonal transformation.

3.3 Fundamental Orthogonal Transformation and Half turns

Definition 3.3.1. A mapping $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ is called a translation if it has the equation of the form $\tau(x, y) = (x', y') = (x + a, y + b)$, where a and b are unique.

A translation from P to Q is denoted by $\tau_{P,Q}$ defined by $\tau_{P,Q} : P \to Q \Rightarrow \tau_{P,Q}(x, y) = (x', y') = (x + a, y + b)$.

Note: $\vec{PQ} = Q - P = (x+a, y+b) - (x, y) = (x+a-x, y+b-y) = (a, b)$, then $\vec{PQ} = (a, b)$ is a translation vector. Thus we can write a translations as $\tau_{P,Q}(P) = P + \vec{PQ}$.

Example 3.3.2.

1) Let $\tau_{A,B}(-1, 5) \to B(7, 2)$, then find the translation vector.

2) Construct the equation of the translation taking points $P(-3, 1)$ to $Q(1, 5)$ and compute the image of $R(3, 2)$.

3) Show that a translation is a transformation.

Solution:

1) $\tau_{A,B}(-1,5) = (7,8)$, $A = (-1,5)$, $B = (7,2)$ and

$$\begin{aligned}
\tau_{A,B}(A) &= A + \vec{AB} \\
\Rightarrow (7,2) &= (-1,5) + (-1+a, 5+b) - (-1,5) \\
&= (-1,5) + (a,b) \\
&= (a-1, b+5) \\
\Rightarrow &= a-1 = 7 \text{ and } b+5 = 2 \\
\Rightarrow a &= 8 \text{ and } b = -3
\end{aligned}$$

Therefore the translation vector is $(a,b) = (8,-3)$. Or simply the translation vector is given by $(a,b) = \vec{AB} = B - A = (7,2) - (-1,5) = (8,-3)$.

2) The translation vector $\vec{PQ} = Q - P = (1,5) - (-3,1) = (4,4) = (a,b)$. Then

$\tau(x,y) = (x',y') = (x+a, y+b) = (x+4, y+4)$.

Therefore the equation of translation $\tau(x,y) = (x+4, y+4)$ and hence the image of $R = (3,7) = \tau_{P,Q}(R) = R + \vec{PQ}$

$\Rightarrow \tau_{P,Q}(3,7) = (3+4, 7+4) = (3,11)$.

3) Let $\tau(x,y) = (x',y') = (x+a, y+b)$.

Let $(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$(in the domain) and suppose $\tau(x_1,y_1) = \tau(x_2,y_2)$ we need to show that $(x_1,y_1) = (x_2,y_2)$.

$\Rightarrow \tau(x_1,y_1) = \tau(x_2,y_2)$

$\Rightarrow (x_1+a, y_1+b) = (x_2+a, y_2+b)$

$\Rightarrow x_1 + a = x_2 + a$ and $y_1 + b = y_2 + b$

$\Rightarrow x_1 = x_2$ and $y_1 = y_2$

$\Rightarrow (x_1, y_1) = (x_2, y_2)$.

Therefore $\tau_{P,Q}$ is one to one.

For any $(u, v) \in \mathbb{R}^2$, there is $(x, y) \in \mathbb{R}^2$ such that

$\tau(x, y) = (u, v) \Rightarrow (x + a.y + b) = ()u, v$

$\Rightarrow x + a = u$ and $y + b = v$

$\Rightarrow x = u - a$ and $y = v - b$.

For every $(u, v) \in \mathbb{R}^2$, there exist $(u - a, v - b) \in \mathbb{R}^2$ such that

$\tau_{P,Q}(u - a, v - b) = (u, v)$.

Therefore $\tau_{P,Q}$ is on to and hence a translation is a transformation.

Theorem 3.3.3. *Given point Q, there is a unique translation taking P and Q, namely $\tau_{P,Q}$.*

Proof. Let $P = (c, d)$ and $Q = (e, f)$

i) **Existence:** Taking $\vec{PQ} = Q - P = (e, f) - (c, d) = (e - c, f - d)$. By definition we have

$\tau_{P,Q}(x, y) = (x, y) + \vec{PQ} = (x, y) + (e - c, f - d) = (x + (e - c), y + (f - d)$.
There fore $\tau_{P,Q}$ is exist (a translation)

ii) **Uniqueness:** Suppose there exist $\tau_{R,S}$ which takes

$\tau_{R,S}(P) = Q \Rightarrow P + \vec{RS} = Q \Rightarrow P - Q = \vec{RS} \Rightarrow \vec{PQ} = \vec{RS}$.

Therefore $\tau_{P,Q} = \tau_{R,S}$ and hence $\tau_{P,Q}$ is unique.

□

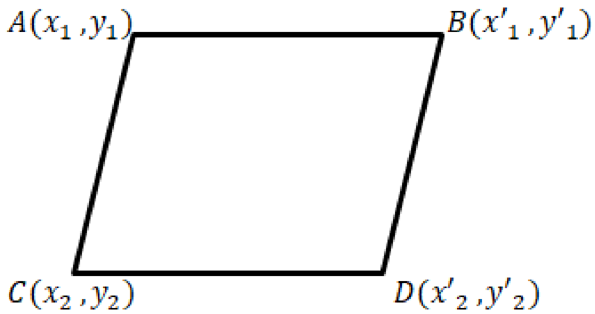
From this we have $\tau_{P,Q}(R) = S$ and $\tau_{P,Q} = \tau_{R,S}$ for points P, Q, R and S. Note that identity is a special case of a translation as $I = \tau_{P,Q}$ for each point P and also if $\tau_{P,Q}(R) = R$ for point R, then $\tau_{P,Q} = R \Rightarrow R + \vec{PQ} = R \Rightarrow \vec{PQ} = 0$

$\Rightarrow P - Q = 0 \Rightarrow P = Q$ as $\tau_{P,Q} = \tau_{R,R} = I$.

Theorem 3.3.4. *Suppose A, B and C are non-collinear points then $\tau_{A,B} = \tau_{C,D}$ if and only if $\Box ABCD$ is a parallelogram.*

Proof. ($\Rightarrow$)Suppose $\tau_{A,B} = \tau_{C,D}$, then we need to show that $\Box ABCD$ is a parallelogram.

Let $A = (x_1, y_1)$ and $C = (x_2, y_2)$



Since $\tau_{A,B}(A) = B \Rightarrow A + \vec{AB} = B \Rightarrow A + B - A = B \Rightarrow \tau_{A,B} = B$ and
$\tau_{C,D}(C) = C + \vec{CD} \Rightarrow C + D - C = D$ and
$\tau_{A,B}(A) = B \Rightarrow B = \tau_{A,B}(x_1, y_1) = (x_1, y_1) + (a, b) = (x_1 + a, y_1 + b)$
$\Rightarrow B = (x_1 + a, y_1 + b)$
$\Rightarrow \vec{AB} = (a, b)$ and $\vec{CD} = (a', b') \Rightarrow \vec{AB} = \vec{CD}$
Since $\tau_{A,B} = \tau_{C,D} \Rightarrow (a, b) = (a', b')$, by using distance formula we have
$AB = \sqrt{(x_1 + a - x_1)^2 + (y_1 + b - y_1)^2} = \sqrt{a^2 + b^2}$ and
$CD = \sqrt{(x_2 + a' - x_2)^2 + (y_2 + b' - y_2)^2} = \sqrt{a'^2 + b'^2} = \sqrt{a^2 + b^2}$,
since $(a, b) = (a', b')$, then we have $AB = CD \Rightarrow AB \parallel \vec{CD}$ and similarly we have
$AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and
$BD = \sqrt{((x_2 + a') - (x_1 + a'))^2 + ((y_2 + b') - (y_1 + b'))^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = AC$
$AC = BD \Rightarrow AC \parallel BD$
Therefore by definition of two pairs of opposite sides are congruent and

parallel the quadrilateral is a parallelogram and hence $\square ABCD$ is a parallelogram.

($\Leftarrow$) Suppose $\square ABCD$ is a parallelogram, we need to show that $\tau_{A,B} = \tau_{C,D}$

$AB = CD \Rightarrow \vec{AB} = \vec{CD} \Rightarrow \tau_{A,B} = \tau_{C,D}$ $\square$

Theorem 3.3.5. *A translation is a collineation.*

Proof. BY Example 3.3.2 $\tau_{P,Q}$ is a transformation.

Let L is a line with equation $ax + by + c = 0$ and $\tau_{P,Q}(x, y) = (x', y') = (x + h, y + k)$

$\Rightarrow x' = x + h$ and $y' = y + k$

$\Rightarrow x = x' - h$ and $y = y' - k$

$\Rightarrow$ Now substitute $x = x' - h$ and $y = y' - k$ from $ax + by + c = 0$

$\Rightarrow a(x' - h) + b(y' - k) + c = 0$

$\Rightarrow ax' - ah + by' - bk + c = 0$

$\Rightarrow ax' + by' + c - ah - bk = 0$

Hence $\tau_{P,Q}(L)$ is a line.

Therefore a translation is a collineation. $\square$

Definition 3.3.6. A collineation α is said to a dilatation if L is parallel to $\alpha(L)$ ($L \parallel \alpha(L)$) for every line L.

Note: While any collineation sends a pair of parallel lines to a pair of parallel lines, a dilatation sends each given line to a line parallel to the given line. For example, we shall see that a rotation of 90^o is a collineation but not a dilatation.

Let α be a transformation and S a set of points.

Definition 3.3.7. A translation α α fixes line L if and only if $\alpha(L) = L$ and a transformation α that fixes a point $\alpha(P) = P$ in general α fixes a set S if and only if $\alpha(S) = S$.

Remark 3.3.8. Show that L and $\alpha(L)$ are the same if and only if $ah+bk = 0$.

Proof. Let L is a line with $ax+by+c=0$ and $\tau_{P,Q}(L)$ is a line with equation $ax'+by'+c-ah-bk=0$.
The slope of L is $\dfrac{-a}{b}$ and slope $\vec{PQ}=\dfrac{k}{h}$ because the point
$\vec{PQ}=(h,k)=\dfrac{k-0}{h-0}=\dfrac{k}{h}$ Slope of L is $\dfrac{-a}{b}$ then $L \parallel \vec{PQ}$
$\Rightarrow \dfrac{-a}{b}=\dfrac{k}{h}\Rightarrow -ah=bk$
Therefore $ah+bk=0$. □

Theorem 3.3.9. *i) A translation is a dilatation.*

ii) If $P \neq Q$, then $\tau_{P,Q}$ fixes no point.

iii) If $P \neq Q$, then $\tau_{P,Q}$ fixes exactly the lines that are parallel to $\vec{PQ}$.

Proof.

i) Let $\tau_{P,Q}(x,y)=(x',y')=(x+a,y+b)\Rightarrow(a,b)=\vec{PQ}$

$\Rightarrow L:\ ax+by+c=0$ and $\tau_{P,Q}(L):\ ax'+by'+c-ah-bk=0$

$\Rightarrow$ Slope of L is $\dfrac{-a}{b}$ and slope of $\tau_{P,Q}(L)$ is $\dfrac{-a}{b}$

Therefore $L \parallel \tau_{P,Q}(L)$ and hence a translation is a dilatation.

ii) Suppose $\tau_{P,Q}(P)=P\Rightarrow P+\vec{PQ}=P\Rightarrow\vec{PQ}=0\Rightarrow P-Q=0\Rightarrow P=Q$. Which contradict the hypothesis $P\neq Q$.

iii) Suppose $L\parallel\vec{PQ}$ we need to show that $\tau_{P,Q}(L)=L$.

Since $\tau_{P,Q}$ is a dilatation, then the slope of L and $\tau_{P,Q}(L)$ are the same.

$\Rightarrow L:\ ax+by+c=c$ and $\tau_{P,Q}(L):\ ax'+by'+c-ah-bk=0$, we need to show that $ah+bk=0$.

$\Rightarrow L\parallel\vec{PQ}$, then slope of $L=\dfrac{-a}{b}$ and and slope of $\tau_{P,Q}(L)=\dfrac{h}{k}$

$\Rightarrow\dfrac{-a}{b}=\dfrac{h}{k}\Rightarrow -ah=bk\Rightarrow ah+bk=0$

Therefore $\tau_{P,Q}(L)$ and L have the same slope and hence $\tau_{P,Q}(L)=L$. □

Theorem 3.3.10. *i) A set of translation forms an abelian group, called the translation group.*

ii) A set of dilatation form a group, called the dilatation group.

Proof.

$i)$ Let τ_O be a set of transformations and translations are collineations. Let $S = (a, b), T = (c, d)$ and $R = (a + c, b + d)$. Then

$\Rightarrow \tau_{O,S}(x, y) = (x', y') = (x + a, y + b)$ and $\tau_{O,T}(x, y) = (x', y') = (x + c, y + d)$

$\Rightarrow \tau_{O,S} \circ \tau_{O,T}((x, y)) = \tau_{O,S}(x + c, y + d) = (x + c + a, y + d + b)$ $\tau_{O,S} \circ \tau_{O,T} = \tau_{O,K}$, where $\tau_{O,K} = (x + c + a, y + d + b)$, the τ is closed.

Let $S = (a, b)$ and $T = (-a, -b)$

$\Rightarrow \tau_{O,S} \circ \tau_{O,T}((x, y)) = \tau_{O,S}(x - a, y - b) = (x - a + a, y - b + b) = (x, y)$. This the inverse is exist.

Moreover: For $S = (a, b)$ and $T = (c, d)$

$$\begin{aligned}
\tau_{O,S} \circ \tau_{O,T}((x, y)) &= \tau_{O,S}(x + c, y + d) \\
&= (x + c + a, y + d + b) \\
&= (x + (c + a), y + (d + b)) \\
&= (x + (a + c), y + (b + d)) \\
&= \tau_{O,T} \circ \tau_{O,S}.
\end{aligned}$$

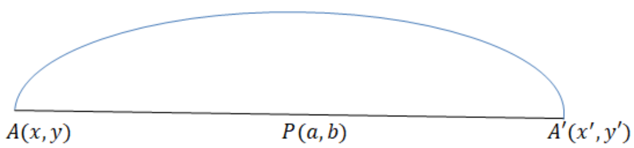
Hence a set of translation τ forms an abelian (commutative) group.

$ii)$ Dilatations are collineations. By the symmetry of parallelness for lines ($i.e., L \parallel L' \Rightarrow L' \parallel L$), the inverse of a dilatation is a dilatation. By the transitivity of parallelness for lines ($i.e., L \parallel L'$ and $L' \parallel L" \Rightarrow L \parallel L"$), the product of two dilatations is a dilatation. So the dilatations form a group (of transformations). □

3.4 Half-turns

A half turn-turns out to be an involutory rotation; that is, a rotation of 180^o. So, a half-turn is just a special case of a rotation. Although we have not formally introduced rotations yet, we look at this special case now because half-turns are nicely related to translations and have such easy equations. Informally, we observe that if point A is rotated 180^o about point P to point A', then P is the midpoint of A and A'. Hence, we need only the midpoint formulas to obtain the desired equations. From equations

$$\begin{cases} \dfrac{x+x'}{2} = a \\ \\ \dfrac{y+y'}{2} = b \end{cases}$$



$a = \dfrac{x+x'}{2}$ and $b = \dfrac{y+y'}{2} \Rightarrow x' = -x = 2a$ and $y' = -y + 2b$,
$\Rightarrow \sigma_P((x,y)) = (x',y') = (-x = 2a, -y+2b)$. we can make our definition as follows.

Definition 3.4.1. If $P = (a,b)$, then the half-turn σ_P about point P is the mapping $\sigma_P : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\sigma_P(x,y) = (x',y') = (-x+2a, -y+2b)$.

Example 3.4.2. Find the image of points $(5,-1)$ under half-turn centered at $(3,2)$.

Solution: $\sigma_P(x,y) = (x',y') = (-x+2a, -y+2b)$
$\Rightarrow \sigma_{(3,2)}(5,-1) = (x',y') = (-5+2(3), -(-1)+2(2)) = (1,5)$.
Hence the image of $(5,-1)$ under half-turn at $(3,2)$ is $(1,5)$.

Note: For the half-turn about the origin we have $\sigma_O((x,y)) = (-x,-y)$

Lemma 3.4.3. *Half-turn is a collineation.*

Proof. Let $L: ax+by+c=0$ and $P=(h,k)$ be a point on the line

$\sigma_P((x,y)) = (x',y') = (-x+2h, -y+2k)$

$\Rightarrow x' = -x+2h$ and $y' = -y+2k$

$\Rightarrow x = -x'+2h$ and $y = -y'+2k$. Now substitute $\Rightarrow x = -x'+2h$ and $y = -y'+2k$. from $ax+by+c=0$ we get

$\Rightarrow a(-x'+2h)+b(-y'+2k)+c=0$

$\Rightarrow -ax'+2ah-by'+2bk+c=0$

$\Rightarrow ax'+by'-2ah-2bk-c=0$

$\Rightarrow ax'+by'-(2ah+2bk+c)=0$, which is equation of line

Hence $ax'+by'+c'=0$, where $c'=-2ah-2bk-c$ is equation of line L'.

Therefore Half-turn is a collineation. □

Generally L and L' $(\sigma(L))$ have the same slope $m=\frac{a}{b}$ which implies that $L \parallel L'$.

Therefore Half-turn is dilatation.

Theorem 3.4.4. *i) A half-turn is an involutory dilatation.*

ii) The midpoint of points A and $\sigma_P(A)$ is P.

iii) Half-turn σ_P fixes point A if and only if $A=P$.

iv) Half-turn σ_P fixes line L if and only if P is on L.

Proof.

i) If $\sigma_P(A)=A'$, then $\sigma_P(A')=A$

$\Rightarrow \sigma_P^2(A) = \sigma_P(\sigma_P(A)) = \sigma_P(A') = A \Rightarrow \sigma_P^2(A) = i(A)$ with order 2

Therefore Half-turn is involutory dilatation.

ii) $\sigma_P(A) = A' = (-x+2a, -y+2b)$, $P=(a,b)$ and $A=(x,y)$, then the midpoint of A and A' is $\left(\frac{x-x+2a}{2}, \frac{y-y+2b}{2}\right) = (a,b) = P$.

Hence the midpoint of A and $\sigma_P(A)$ is P.

$iii)$ $(\Rightarrow)$ σ_P fixes A. then we need to show that $A = P$

Suppose $\sigma_P(A) = \sigma_P(x, y) = (x', y') = (x, y)$

$\Rightarrow (x', y') = (x, y)$

$\Rightarrow (-x + 2a, -y + 2b) = (x, y)$

$\Rightarrow -x + 2a = x$ and $-y + 2b = y$

$\Rightarrow 2a = 2x$ and $2b = 2y$

$\Rightarrow x = a$ and $y = b$

$\Rightarrow (x, y) = (a, b) \Rightarrow A = P$

$(\Leftarrow)$ Suppose $A = P$ we need to show that $\sigma_P(A) = A$

$\Rightarrow \sigma_P(A) = (-x + 2a. - y + 2b)$ but $A = P$

$\Rightarrow (-x + 2a. - y + 2b) = (x, y) = A$

$\Rightarrow -x + 2a = x$ and $-y + 2b = y$

$\Rightarrow (x, y) = (a, b) \Rightarrow \sigma_P(A) = A$ $\square$

Theorem 3.4.5. *i) The product of two half-turns is a translation*

ii) If Q is the midpoint of P and R then $\sigma_Q\sigma_P = \tau_{P,R} = \sigma_R\sigma_Q$.

iii) A product of three half-turns is a half-turn.

iv) $\sigma_R\sigma_Q\sigma_P = \sigma_P\sigma_Q\sigma_R$ for any points P, Q and R.

Proof. $i)$ Let σ_P and σ_Q are half-turns about $P(a, b)$ and $Q(c, d)$. We need to show that $\sigma_P\sigma_Q = \tau_{PQ}$

$$\begin{aligned}\sigma_P\sigma_Q(x, y) &= \sigma_P(-x + 2c, -y + 2d)\\ &= (-(-x + 2c) + 2a, -(-y + 2d) + 2b)\\ &= (x - 2c + 2a, y - 2d + 2b)\\ &= (x + 2(a - c), y + 2(b - d))\\ &= (x + a', y + b'), \text{where } a' = 2(a - c) \text{ and } b' = 2(b - d)\\ &= \tau_{P,Q}\end{aligned}$$

Hence the product of two half-turns is a translation.

$ii)$ Let Q is the midpoint of R and P

Let $P = (a, b)$ and $R = (c, d)$ and $Q = \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$

$$\begin{aligned}
\sigma_P(x, y) &= (x', y') = (-x + 2a, -y + 2b) \text{ and} \\
\sigma_R(x, y) &= (x', y') = (-x + 2c, -y + 2d) \\
\sigma_Q(x, y) &= (x', y') = (-x + a + c, -y + b + d) \\
\sigma_Q\sigma_R(x, y) &= \sigma_Q(-x + 2c, -y + 2d) \\
&= \left(-(-x + 2c) + 2(\frac{a+c}{2}), -(-y + 2d) + 2(\frac{b+d}{2})\right) \\
&= (x - 2c + a + c, y - 2d + b + d) \\
&= (x + a - c, y + b - d) \\
&= \tau_{P,R}
\end{aligned}$$

$iii)$ Let $P = (a, b), Q = (c, d)$ and $R = (e, f)$

$\Rightarrow \sigma_P(x, y) = (x', y') = (-x + 2a, -y + 2b)$

$\Rightarrow \sigma_Q(x, y) = (x', y') = (-x + 2c, -y + 2d)$

$$\Rightarrow \sigma_R(x, y) = (x', y') = (-x + 2e, -y + 2f)$$

$$\begin{aligned}
\sigma_P\sigma_Q\sigma_R(x, y) &= \sigma_P(\sigma_Q\sigma_R(x, y)) \\
&= \sigma_P(\sigma_Q(\sigma_R(x, y))) \\
&= \sigma_P(\sigma_Q(-x + 2e, -y + 2f)) \\
&= \sigma_P(-(-x + 2e) + 2c, -(-y + 2f) + 2d) \\
&= \sigma_P(x - 2e + 2c, y - 2f + 2d) \\
&= (-(x - 2e + 2c) + 2a, -(y - 2f + 2d) + 2b) \\
&= (-x + 2e - 2c + 2a, -y + 2f - 2d + 2b) \\
&= (-x + 2(a - c + e), -y + 2(b - d + f)) \\
&= (-x + 2a', -y + 2b'), \text{ where } a' = a - c + e \text{ and } b' = b - d + f.
\end{aligned}$$

$\therefore$ The product of three half-turns is also half-turn.

□

3.5 Reflections

A reflection will be defined as a transformation leaving invariant every point of a fixed line m and no other points. (An optical reflection along m in a mirror having both sides silvered, would yield the same result.) We make the following definition.

Definition 3.5.1. A reflection σ_m in a line m is a mapping defined by

$$\sigma_m(P) = \begin{cases} P, if\, point\, P\, is\, on\, m \\ \\ Q, \text{ if point P is off L and L is the perpendicular bisector of } \vec{PQ}. \end{cases}$$

The line m is usually referred to as the mirror of the reflection.

Note: We do not use the word reflection to denote the image of a point or of a set of points. A reflection is a transformation and never a set of points. Point $\sigma_m(P)$ is the image of point P under the reflection σ_m.

Proposition 3.5.2. *Properties of reflection which follows immediately from the definition:*

i) $\sigma_m \neq i$, *(non-identity)*

ii) $\sigma_m^2 = i$, *(involution)*

iii) *If* $\sigma_m(P) = Q$, *then* $m \perp \vec{PQ}$ *and is bisector*

iv) $\sigma_m(P) = P$ *if and only if* $P \in m$

v) σ_m *is a transformation.*

Proof.

ii) Let P such that $\sigma_m(P) = Q$

$\Rightarrow m \perp \vec{PQ}$ and bisector

$\Rightarrow \sigma_m(Q) = P$

$\Rightarrow \sigma_m^2(P) = \sigma_m(\sigma_m(P)) = \sigma_m(Q) = P.$

$\Rightarrow \sigma_m^2(P) = i(P)$

$\Rightarrow \sigma_m^2(P) = i$

v) First show that one one

Let $\sigma_m(P) = \sigma_m(Q)$, we need to show that $P = Q$

$\Rightarrow \sigma_m(\sigma_m(P)) = \sigma_m(\sigma_m(Q))$

$\Rightarrow i(P) = i(Q)$

$\Rightarrow P = Q$, since σ_m is involution

Hence σ_m is one to one.

Next to show that σ_m is on to

For every point Q there is a point P such that $\sigma_m(P) = Q$, for every P there is Q such that $\sigma_m(Q) = P$

$\Rightarrow \sigma_m(\sigma_m(P)) = \sigma_m^2(P) = P$

For every $P \in Q$ such that $\sigma_m(Q) = P$

Hence σ_m is on to

$\therefore$ The reflection is a translation. □

Theorem 3.5.3. *i) Reflection σ_m fixes line L point wise if and only if $L = m$*

ii) Reflection σ_m fixes point P if and only if P is on m

iii) Reflection σ_m fixes line L if and only if $L = m$.

Proof.

i) ($\Rightarrow$) Suppose σ_m fixes line L point wise, we need to show that $L = m$

Let $P \in L$, $\sigma_m(P) = P \Rightarrow P \in m$ (by definition)

we have that L and m have more than one common points $L = m$

($\Leftarrow$) Suppose $L = m$, we need to show that $P \in L$ and $P \in m$

$\Rightarrow \sigma_m(P) = P$ clearly elements of P is fixed.

$\therefore P \in L$ and $P \in m$. □

3.5.1 Equations of reflections

Theorem 3.5.4. *If line m has equation $ax + by + c = 0$, then reflection σ_m has equations:*

$$\begin{cases} x' = x - \dfrac{2a(ax+by+c)}{a^2+b^2} \\ \\ y' = y - \dfrac{2b(ax+by+c)}{a^2+b^2}. \end{cases}$$

Proof. Let $P = (x, y)$ and $\sigma_m(P) = (x', y') = Q$. For the moment, suppose that P is off m. Now, the line through points P and Q is perpendicular to line m. This geometric fact is expressed algebraically by the equation

$$b(x' - x) = a(y' - y).$$

Also $\left(\dfrac{x+x'}{2}, \dfrac{y+y'}{2}\right)$ is the midpoint of $\vec{PQ}$ and is on m. This geometric fact is expressed algebraically by the equation

$$a\left(\frac{x+x'}{2}\right)+b\left(\frac{y+y'}{2}\right)+c=0.$$

Rewriting these two equations as

$$\begin{cases} bx'-ay'=bx-ay \\ ax'+by'=-2c-ax-by \end{cases}$$

we see we have two linear equations in two unknowns x' and y'. Solving these equations for x' and y' (by using Cramers rule, for instance), we get

$$\begin{cases} x'=\dfrac{a^2x+b^2x-2a^2x-2aby-2ac}{a^2+b^2} \\ \\ y'=\dfrac{a^2y+b^2y-2b^2y-2abx-2bc}{a^2+b^2}. \end{cases}$$

With these equations in the form

$$\begin{cases} x'=x-\dfrac{2a(ax+by+c)}{a^2+b^2} \\ \\ y'=y-\dfrac{2b(ax+by+c)}{a^2+b^2} \end{cases}$$

it is easy to check that the equations also hold when P is on m. This proves the result. $\square$

Note: Suppose we had defined a reflection as a transformation having equations given by Theorem 3.5.4. Not only would this have seemed artificial, since these equations are not something you would think of examining in the first place, but just imagine trying to prove Theorem 3.5.3 from these equations. Although this is conceptually easy, the actual computation involves a considerable amount of algebra.

Example 3.5.5. 1) Find the image of the point $(4,5)$ by reflection on the line $L:\ 2x+3y+3=0$

2) Given $\sigma_m(a,b)=(3,4)$, where $L:\ x+y-1=0$. Then find the values of the point (a,b).

Solution:

1) Given $L:\ 2x+3y+3=0, a=2, b=3, c=3$ and $(x,y)=(4,5)$, then

$$\begin{aligned}
x' &= x-\frac{2a(ax+by+c)}{a^2+b^2}\\
&= 4-\frac{2(2)(2(4)-3(5)+3)}{4^2+3^2}\\
&= 4-\frac{4(-4)}{13}\\
&= \frac{52+16}{13}\\
&= \frac{68}{13}\\
y' &= y-\frac{2b(ax+by+c)}{a^2+b^2}\\
&= 5-\frac{2(3)(2(4)+3(5)+3)}{2^2+3^2}\\
&= 5-\frac{6(8+15+3)}{13}\\
&= \frac{65-156}{13}\\
&= \frac{-91}{13}.
\end{aligned}$$

Therefore the image $(x',y')=\left(\frac{68}{13},\frac{-91}{13}\right)$

3.5.2 Properties of a Reflection

We have already mentioned those properties of a reflection that follow immediately from the definition. Another important property is that a reflection preserves distance, which means the distance from $\sigma_m(P)$ to $\sigma_m(Q)$ is equal to the distance from P to Q, for all points P and Q. The following definition is fundamental.

Definition 3.5.6. A transformation is an isometry (or congruent transformation) if $P'Q' = PQ$ for all points P and Q, where $P' = \alpha(P)$ and $Q' = \alpha(Q)$.

In other words, an isometry is a distance-preserving transformation.
Note:

1) In fact, any distance-preserving mapping is an isometry. Such a mapping is one-to-one because points at nonzero distance cannot have images at zero distance but it is not clear that such a mapping is onto.

2) The name isometry comes from the Greek isos (equal) and metron (measure). An isometry is also called a rigid motion.

The set of all isometries form a group. This group is denoted by Isom.

Proposition 3.5.7. *Reflection σ_m is an isometry.*

Proof. We shall consider several cases. Suppose P and Q are two points, $P' = \sigma_m(P)$ and $Q' = \sigma_m(Q)$. We must show $P'Q' = PQ$.

a) If $\overleftrightarrow{PQ} = m$ or if $\overleftrightarrow{PQ} \perp m$, then the desired result follows immediately from the definition of σ_m.

b) Also, if $\overleftrightarrow{PQ}$ is parallel to L but distinct from m, the result follows easily as $\square PQP'Q'$ is a rectangle and so opposite sides $\overline{PQ}$ and $\overline{P'Q'}$ are congruent.

c) Further, if one of P or Q, say P, is on m and Q is off m, then $P'Q' = PQ$ follows from the fact that $P' = P$ and that m is the locus of all points equidistant from Q and Q'.

d) Finally, suppose P and Q are both off m and that $\overleftrightarrow{PQ}$ intersects m at point R, but is not perpendicular to m. So $RP = RP'$ and $RQ = RQ'$. The desired result, $P'Q' = PQ$, then follows provided R, P', Q' are shown to be collinear.

□

Theorem 3.5.8. *An isometry is a collineation that preserves betweenness, midpoints, segments, rays, triangles, angles, angle measure, and perpendicularity.*

Proof. Since these properties are shared by all isometries, we shall consider a general isometry α.

a) Suppose A, B, C are any three points and let $A' = \alpha(A), B' = \alpha(B), C' = \alpha(C)$. Since preserves distance, if $AB+BC = AC$ then $A'B'+B'C' = A'C'$ as $A'B' = AB, B'C' = BC$, and $A'C' = AC$. Hence, $A - B - C$ implies $A' - B' - C'$; in other words, if B is between A and C, then B' is between A' and C'. We describe this by saying that α preserves betweenness.

b) The special case $AB = BC$ in the argument above implies $A'B' = B'C'$. In other words, if B is the midpoint of A and C, then B' is the midpoint of A' and C'. Thus we say preserves midpoints.

c) More generally, since $\overline{AB}$ is the union of A, B, and all points between A and B, then $\alpha(\overline{AB}) = \overline{A'B'}$ is the union of A', B', and all points between A' and B'. So $\alpha(\overline{AB}) = \overline{A'B'}$ and we say preserves segments.

d) Likewise, since α is onto by definition and $\overrightarrow{AB}$ is the union of $\overline{AB}$ and all points C such that $A-B-C$, then $\alpha(\overrightarrow{AB})$ is the union of $\overline{A'B'}$ and all points C' such that $A' - B' - C'$. So $\alpha(\overrightarrow{AB}) = \overrightarrow{A'B'}$ and we say α preserves rays.

e) Since $\overleftrightarrow{AB}$ is the union $\overrightarrow{AB}$ and $\overrightarrow{BA}$, then $\alpha(\overleftrightarrow{AB})$ is the union of $\overrightarrow{A'B'}$ and $\overrightarrow{BA}$, which is $\overleftrightarrow{A'B'}$. So α is a transformation that preserves lines ; in other words, α is a collineation.

f) If A, B, C, are not collinear, then $AB+BC > AC$ and so $A'B'+B'C' > A'C'$ and A', B', C' are not collinear. Then, since $\triangle ABC$ is a union of the three segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ then we conclude that $\alpha(\triangle ABC)$ is just $\triangle A'B'C'$. So an isometry preserves triangles.

g) It follows that preserves angles as $\alpha(\angle ABC) = \angle A'B'C'$.

h) Not only does preserve angles, but also preserves angle measure. That is,$m(\angle ABC) = m(\angle A'B'C')$, since $\triangle ABC \cong \triangle A'B'C'$ by SSS.

Finally, if $\overrightarrow{BA} \perp \overrightarrow{AB}$ then $\overrightarrow{B'A'} \perp \overrightarrow{A'B'}$ since $m(\angle ABC) = 90$ implies $m(\angle A'B'C') = 90$ So α preserves perpendicularity. $\square$

3.6 Rotations

Definition 3.6.1. A rotation about point C through directed angle of θ is the transformation $\rho_{C,\theta}$ that fixes C and otherwise sends a point P to the point P', where $CP' = CP$ and θ is the directed angle measure of the directed angle from $\overrightarrow{CP}$ to $\overrightarrow{CP'}$.

Remark 3.6.2. *i)* We agree that $\rho_{C,0}$ is the identity i.

ii) Rotation $\rho_{C,\theta}$ is said to have centre C and directed angle θ.

Theorem 3.6.3. *A rotation is an isometry (preserve distance).*

Proof. Suppose $\rho_{C,\theta}$ sends points P and Q to P' and Q', respectively. If C, P, Q are collinear, then $PQ = P'Q'$ by the definition 3.6.1. If C, P, Q are not collinear, then $\triangle PCQ \cong \triangle P'C'Q'$ by SAS and $PQ = P'Q'$. So $\rho_{C,\theta}$ is a transformation that preserves distance. $\square$

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