UNIT TWO

PROPERTIES OF TRANSFORMATIONS

1.3

2.1. Defn & Egs of Transformations.

Defn: A one-to-one mapping of a set onto itself, i.e., a 1-1 mapping of a set A onto precisely the set A, is called a Tranformation of A i.e. For any given transformation α & for a given pt P, $\exists!$ point Q $\ni \alpha(P)=Q$.

* In this course we mainly focus on various transformations of the plane, so, we re-state the defn as:

→ A transformation on the plane is a 1-1 correspondence from the set of pts in the plane onto itself.

Egs.

1. A mapping that sends (x,y) to (x,y) is a transformation. Such transformation is called Identity transformation & is denoted by i.

i.e., Identity transf. i is defined by $i(P)=P \quad \forall P$.

2. Let α be a mapping given by

$$\alpha((x,y)) = (x^3, y^3).$$ Show that α is a transformation.

<u>Soln</u>

To show that α is a transformation we need to show it is 1-1 & onto.

i. <u>1-1 ness</u>

Let $P_1=(x_1,y_1)$ & $P_2=(x_2,y_2)$ be pts in $\mathbb{R}^2 \ni$

$$\alpha(P_1) = \alpha(P_2)$$

$$\Rightarrow \alpha((x_1,y_1)) = \alpha((x_2,y_2))$$

$$\Rightarrow (x_1^3, y_1^3) = (x_2^3, y_2^3)$$

$$\Rightarrow x_1^3 = x_2^3 \quad \& \quad y_1^3 = y_2^3$$

$$\Rightarrow x_1 = x_2 \quad \& \quad y_1 = y_2$$

$$\Rightarrow (x_1,y_1) = (x_2,y_2) \quad \text{or} \quad P_1 = P_2$$

$\therefore \alpha$ is 1-1.

a. $\beta((x,y)) = (x, y^2)$

Sol^{n}

β is not a transformation. B/c

See that $\beta((1,-2)) = \beta((1,2))$ but $(1,-2) \neq (1,2)$

$\Rightarrow \beta$ is not 1-1.

& also

there is no pt (x,y) ∋ $\beta((x,y)) = (1,-2)$

$\Rightarrow \beta$ is not onto.

~~Hence~~

b. $\theta((x,y)) = (x^3, y)$ is a transfo.

c. $\tau((x,y)) = (x+2, y-3)$ is a trans

d. $r((x,y)) = (x^3-x, y)$ is not. B/c $r((1,1)) = r((-1,1))$ but

$(1,1) \neq (-1,1)$ $\Rightarrow r$ is not 1-1. Thus, r is not a transformation.

Theorem: The composite $\beta \circ \alpha$ of transformations α & β defined by $(\beta\circ\alpha)(P) = \beta(\alpha(P))$ is itself a transformation.

Proof

It is sufficient to show that $\beta\circ\alpha$ is both 1-1 & onto.

i. 1-1 ness. Let P_1 & P_2 be pts in $\mathbb{R}^2$ ∋

$(\beta\circ\alpha)(P_1) = (\beta\circ\alpha)(P_2)$

$\Rightarrow \beta(\alpha(P_1)) = \beta(\alpha(P_2))$ by def^{n}

$\Rightarrow$ ~~β~~ $\alpha(P_1) = \alpha(P_2)$ since β is 1-1.

$\Rightarrow$ $P_1 = P_2$ since α is 1-1

$\therefore$ $\beta \circ \alpha$ is 1-1.

ii. Ontoness.

Let Q be any pt in $\mathbb{R}^2$.

Since β is onto $\exists$ pt $C \in \mathbb{R}^2 \ni \beta(C) = Q$.

Since α is onto, there is a pt A in $\mathbb{R}^2 \ni$

$\alpha(A) = C$.

~~Since~~ Thus, α is onto.

Thus, $(\beta \circ \alpha)(A) = \beta(\alpha(A)) = \beta(C) = Q$

$\Rightarrow$ $\exists$ a pt A in $\mathbb{R}^2 \ni \beta\circ\alpha$ of $\mathbb{R}^2$.

$\Rightarrow$ $\beta\circ\alpha$ is onto.

$\therefore$ $\beta\circ\alpha$ is a transformation.

Note: From the defn, we observe that if γ is a transformation, γ^{-1} is a transformation where γ^{-1} is the inverse of γ, γ^{-1} is the mapping defined by $\gamma^{-1}(B) = A$ iff $B = \gamma(A)$

Theorem: The set of all transformations form a group.

proof:

Let G be a set of all transformations.

i. For every $\alpha, \beta \in G$, $\alpha \circ \beta \in G$. (composition of two transfor.)

ii. For $\forall\, \alpha, \beta, \gamma \in G$,

$[\gamma \circ (\beta \circ \alpha)](p) = \gamma((\beta\circ\alpha)(p)) = \gamma(\beta(\alpha(p))) = (\gamma\circ\beta)(\alpha(p))$

$= [(\gamma\circ\beta)\circ\alpha](p)$

$\therefore$ G have associative property.

iii. The identity transformation i is an idty elt for the operation o.

iv. For every elt $\alpha \in G$, there exist $\alpha^{-1} \in G$ ∋

$$\alpha^{-1} o \alpha = i = \alpha o \alpha^{-1}$$

∴ G is a grp.

Note:

i. If every elt of transf. grp G_2 is an elt of transf. grp G_1, then G_2 is a subgrp of G_1.

ii. Transf.s α & β may or may not satisfy the commutative law:

$$\alpha o \beta = \beta o \alpha.$$

Involutions

* If there is no ambiguity, we abreviate "$\beta o \alpha$" as "$\beta\alpha$" & read as "The product of α multiplied by β on the left" or "The product of β multiplied by α on the right" or simply "The product ~~beta~~ - alpha".

Thus, $(\varepsilon o (\delta o \gamma)) o (\beta o \alpha) = \varepsilon\delta\gamma\beta\alpha$ since the composition of trans. is associative.

Hence, in $\gamma = \alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1$, with n a +ve integer a transformation γ is expressed as a product of transformations.

- If $\alpha_i = \alpha$, $\forall 1 \le i \le n$, $\gamma = \alpha^n$

Further, if $\alpha = \beta^{-1}$, then $\gamma = \beta^{-n}$.

We define $\beta^0 = i$, for any transformation β.

$(\beta^{-1})^n = \beta^{-n}$, for any trans. β & any integer n.

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Theorem:

i. If α, β & γ are elts of a group, then:

a. $\beta\alpha = \gamma\alpha \Rightarrow \beta = \gamma$ (right Cancellation law)

b. $\beta\alpha = \beta\gamma \Rightarrow \alpha = \gamma$ (left ,,)

c. $\beta\alpha = \alpha \Rightarrow \beta = i$

d. $\beta\alpha = \beta \Rightarrow \alpha = i$

e. $\beta\alpha = i \Rightarrow \beta = \alpha^{-1}$ & $\alpha = \beta^{-1}$

ii. In a grp, the inverse of a product is the product of the inverses ~~in reverse~~ in reverse order. i.e.,

$$(\omega \cdots \gamma\beta\alpha)^{-1} = \alpha^{-1}\beta^{-1}\gamma^{-1} \cdots \omega^{-1}.$$

Proof: Ex.

Note: If there is a smallest +ve integer $n \ni \alpha^n = i$, then transf. α is said to have order n; otherwise it is infinite order

eg.

1. Let ρ be a rotation of $\frac{360^\circ}{n}$ about the origin with n a +ve integer. Then ρ has order n, the set $\{\rho, \rho^2, \rho^3, \cdots \rho^n\}$ forms a grp of order n.

Here $\rho^n = \rho\rho\rho\rho \cdots \rho = \rho(\rho(\rho(\cdots(\rho(p))\cdots) = p$

$\Rightarrow \rho^n = i$

2. Let $\tau((x,y)) = (x+1, y)$. Element τ has infinite order & the set of all transformations τ^k with k an integer forms an infinite grp.

Defn: If every elt of a grp containing α is a power of α, then we say that the grp is cyclic with generator α & denote the grp as $\langle\alpha\rangle$.

Eg

1. If ρ is a rotation of 60°, then $\langle \rho \rangle$ is a cyclic grp of order 6.

2. If ρ_1 is a rotation of 36°, then $\langle \rho_1 \rangle$ is a cyclic grp of order 10.

* Observe that: $\langle \rho_1 \rangle = \langle \rho_1^3 \rangle$. Thus, a cyclic grp may have more than one generator.

Def: A transformation with order 2 is called Involution.

i.e., transformation γ is an involution iff $\gamma^2 = i$ but $\gamma \neq i$.

* In other words, a non identity transf. γ is an involution iff $\gamma = \gamma^{-1}$.

$\gamma^2 = i \iff \gamma \cdot \gamma = i \iff \gamma^{-1}\gamma\gamma = \gamma^{-1} i \iff \gamma = \gamma^{-1}$

Eg 1 → at the back → A rotation of 180°

2. Give a rotation which is an involution.

3. Find a & b ∋ α is an involution if $\alpha((x,y)) = (ay, \frac{x}{b})$

4. At the back

Sol $\frac{n}{}$

3. α is involution $\iff \alpha^2 = i$, $\alpha \neq i$.

$\alpha^2((x,y)) = \alpha\alpha((x,y)) = \alpha(\alpha(x,y)) = \alpha((ay, \frac{x}{b})) = (\frac{a}{b}x, \frac{a}{b}y)$

Thus $\alpha^2((x,y)) = (x,y) \Rightarrow (\frac{a}{b}x, \frac{a}{b}y) = (x,y) \Rightarrow \frac{a}{b} = 1 \Rightarrow a = b$

Def: A transf. that sends a line to a line is called a Collineation

i.e., A transf. α is a collineation iff whenever l is a line, $\alpha(l)$ is also a line.

Eg. 1. A transf. α defined by $\alpha((x,y)) = (x, 2y)$ is a collineation.

Proof:

Eg

1. If ρ is a rotation of 60°, then $<\rho>$ is a cyclic grp of order 6.

2. If ρ_1 is a rotation of 36°, then $<\rho_1>$ is a cyclic grp of order 10.

* Observe that: $<\rho_1> = <\rho_1^3>$. Thus, a cyclic grp may have more than one generator.

Defn: A transformation with order 2 is called Involution.

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$\gamma^2 = i \Leftrightarrow \gamma.\gamma = i \Leftrightarrow \gamma^{-1}\gamma\gamma = \gamma^{-1} i \Leftrightarrow \gamma = \gamma^{-1}$

Eg 1 → at the back

2. Give a rotation which is an involution. → A rotation of 180°

3. Find a & b ∋ α is an involution if $\alpha((x,y)) = (ay, \frac{x}{b})$

4. At the back

Soln

3. α is involution $\Leftrightarrow \alpha^2 = i$, $\alpha \neq i$.

α $((x,y)) = \alpha\alpha((x,y)) = \alpha(\alpha(x,y)) = \alpha((ay, \frac{x}{b})) = (\frac{a}{b}x, \frac{a}{b}y)$

Thus $\alpha^2((x,y)) = (x,y) \Rightarrow (\frac{a}{b}x, \frac{a}{b}y) = (x,y) \Rightarrow \frac{a}{b} = 1 \Rightarrow \underline{a = b}$

Defn: A transf. that sends a line to a line is called a Collineation

i.e., A transf. α is a collineation iff whenever l is a line, $\alpha(l)$ is also a line.

Eg. 1. A transf. α defined by $\alpha((x,y)) = (x, 2y)$ is a collineation.

proof:

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Let l be a line with eqn $ax+by+c=0$ i.e.
$\forall (x,y) \in l \Rightarrow ax+by+c=0$ ---

Let $\alpha((x,y)) = (u,v)$
$\Rightarrow (u,v)$ is on the image of l.

$\alpha((x,y)) = (x, 2y)$
$\Rightarrow (x,2y) = (u,v)$
$\Rightarrow x = u$ & $2y = v$
$\Rightarrow u = x$ & $y = \frac{v}{2}$

$ax+by+c=0 \Rightarrow au + b(\frac{v}{2}) + c = 0$

$\Leftrightarrow 2au + bv + 2c = 0$ → w/c is again a st. line. (linear in terms of u & v)

$\Rightarrow (u,v)$ a soln of $2ax+by+2c=0$ i.e., (u,v) on the line

$2ax+by+2c=0$

∴ trans. α is a collineation.

2. Determine whether the ff. transf. is collineation. For each collineation find the image of the line with eqn $ax+by+c=0$.

a. $\beta((x,y)) = (x, y^3)$

Soln

If (x,y) is on a line l with eqn $ax+by+c=0$, then
$ax+by+c=0$ ... ⊛

Let $\beta((x,y)) = (u,v)$
$\Rightarrow (x,y^3) = (u,v)$
$\Rightarrow x = u$ & $y = v^{\frac{1}{3}}$

putting this in ⊛ $au + bv^{\frac{1}{3}} + c = 0$ w/c is not a st. line
$\Rightarrow (u,v)$ is a pt on $ax + by^{\frac{1}{3}} + c = 0$
∴ β is not a collineation.

Eg 1. On the cartesian plane define:

$i((x,y)) = (x,y)$, $\alpha((x,y)) = (x,-y)$ → ref about x-axis.

$\beta((x,y)) = (-x,-y)$ & $\gamma((x,y)) = (-x,y)$ → ref about y-axis

→ ref about the origin

i, α, β & γ are transformations.

$(\alpha \circ \alpha)(x,y) = \alpha(\alpha((x,y))) = \alpha((x,-y)) = (x,y)$

$\Rightarrow \alpha \circ \alpha = i$ i.e. $\alpha^2 = i$ $\Rightarrow$ α is an involution.

$(\alpha \circ \beta)(x,y) = \alpha(\beta(x,y)) = \alpha(-x,-y) = (-x,y) = \gamma(x,y)$

$\Rightarrow \alpha \circ \beta = \gamma$ i.e. $\alpha\beta = \gamma$.

In general

$\circ$	i	α	β	γ
i	i	α	β	γ
α	α	i	γ	β
β	β	γ	i	α
γ	γ	β	α	i

- From the table. $G = \{i, \alpha, \beta, \gamma\}$ is a grp of transformations, w/c is a finite grp of order 4. i.e., $O(G) = 4$.
- G is not a cyclic grp b/c it has no generator.
- α, β & γ are all involutions.

Eg 4. Construct the grp of rotations generated by a rotation of the plane about the origin by 90°.

a. What is the order of the grp? $O(G) = 4$. $G = \{\alpha^0, \alpha^1, \alpha^2, \alpha^3\}$

b. Is the grp cyclic? yes. $= \{i, \rho_{0,90}, \rho_{0,180} = \rho^2_{0,90}, \rho^3_{0,90}\}$

b. $\alpha((x,y)) = (-x + y/2, x+2)$

<u>Soln</u>

Set $u = -x + y/2$ & $V = x+2$ is [illegible]

we have unique solns $x = V-2$ & $y = 2u + 2V - 4$

$\exists$ $\alpha((x,y)) = (u,v)$ for any no u & v.

Hence α is a transf.

Let l be a line with eqn $ax+by+c=0$

If (x,y) is on l, then $ax+by+c=0$

$$a(V-2) + b(2u+2V-4) + c = 0$$

$$2bu + (a+2b)V + (c-4b-2a) = 0$$

$\Rightarrow$ (u,v) is on the line with eqn

$$(2b)x + (a+2b)y + (c-4b-2a) = 0$$

$\Rightarrow$ $\alpha((x,y))$ is on the line with eqn

$$(2b)x + (a+2b)y + (c-4b-2a) = 0$$

So, the line with eqn $ax+by+c=0$ goes to the line with eqn $a'x + b'y + c' = 0$ where

$$a' = 2b, \quad b' = a+2b \quad \& \quad c' = c-4b-2a$$

Hence α is a collineation.

2. Find the image of the line $y = 3x+2$ under collineation α if $\alpha((x,y))$ is

simultaneous

a. $(2x, y)$ b. $(x, -y)$ c. $(2y-x, x-2)$ d. $(x, y+1)$ e. $(x+y, 2x-...)$

(3) Find the preimage of the line with eqn $y = 3x+2$ under collineation α where $\alpha((x,y)) = (3y, x-y)$

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Sol:

2 a. $\alpha((x,y)) = (2x, y)$

If (x,y) lie on l, then $y = 3x+2$

Let $u = 2x$ & $v = y$

$\Rightarrow x = \frac{u}{2}$ & $v = y$

Thus, $\alpha((x,y)) = (2x,y) = (u,v)$ lie in the line

$v = 3(\frac{u}{2}) + 2$

$\Rightarrow v = \frac{3}{2}u + 2$

Hence $(2x,y)$ lie on the eqn $y = \frac{3}{2}x + 2$

Thus, the image of the line $y = 3x+2$ is

$$y = \frac{3}{2}x + 2$$

Check: Two pts form of eqn of a line.

3. Let a line l is the preimage of the line with eqn $y = 3x+2$.

Let (x,y) be a pt on the image of l, then

~~Since~~ $\cancel{\alpha((x,y)) = (3y, x-y)}$

$y = 3x+2$

Let (a,b) be the preimage of (x,y) under α, then

(a,b) is a pt on l &

$\alpha((a,b)) = (x,y)$ $\Rightarrow 3b = x$ & $a-b = y$

$\Rightarrow (3b, a-b) = (x,y)$ $\Rightarrow$ $\cancel{y = \frac{a}{3}}$ & $\cancel{x = y+b}$ $\cancel{\Rightarrow x = \frac{a}{3}+b}$

$y = 3x+2$ $\quad$ $a-b = 3(3b)+2$

$\Rightarrow \cancel{b = 3(\frac{a}{3}+b)+2}$ $\quad$ $a-b = 9b+2$

$\cancel{\frac{a}{3} = a+3b+2}$ $\quad$ $a-2 = 10b$

$\Rightarrow a-2 = 10b$

$\Rightarrow b = \frac{1}{10}a - \frac{1}{5}$

Thus, (a,b) is a pt on the line $y = \frac{1}{10}x - \frac{1}{5}$

$\Rightarrow$ The eqn of line l is $y = \frac{1}{10}x - \frac{1}{5}$

Check!

Find the image of line l given by $y = \frac{1}{10}x - \frac{1}{5}$ under a Collineation $\alpha((x,y)) = (3y, x-y)$. Let $(u,v) = \alpha((x,y))$

$\Rightarrow$ ~~let~~ $u = 3y$, & $v = x - y$

$y = \frac{u}{3}$ & $x = v + y = v + \frac{u}{3}$

$\frac{u}{3} = \frac{v + \frac{u}{3}}{10} - \frac{1}{5} \Rightarrow v = 3u + 2$

$\therefore (u,v)$ is a pt on $y = 3x + 2$

Hence the pre-image of a line with eqn $y = 3x+2$ under α is $y = \frac{1}{10}x - \frac{1}{5}$

Theorem: The set of all collineations form a grp.

Proof:

Let S be the set of all collineations.

i. Closure Property:

Let α & β ~~be~~ $\in S$

Let l is a line

Sqse

Then $\alpha(l)$ is a line since α is a collineation.

.. β " " " .

& $\beta(\alpha(l))$ " " "

Hence $(\beta \circ \alpha)(l)$ is a line, & ~~$\beta\circ\alpha$~~

$\therefore \beta \circ \alpha$ is a collineation.

ii. ~~Composi~~ If $\alpha, \beta, \gamma \in S$

$\alpha(\beta\gamma) = (\alpha\beta)\gamma$ since collineations are transformations

$\Rightarrow$ Composition of ~~tra~~ Collineations satisfies associative property.

iii. The idty trans is an idty elt of $\mathcal{S}$.

iv. Existence of inverse.

Let l be a line & $\alpha \in \mathcal{S}$.

There is a line $m \ni \alpha(m) = l$ since α is a collineation (or a transf. it is onto).

So, $\alpha^{-1}(l) = \alpha^{-1}(\alpha(m)) = \alpha^{-1} \circ \alpha(m) = i(m) = m$

Hence, α^{-1} is a collineation.

i.e. $\alpha^{-1} \in \mathcal{S}$, $\forall \alpha \in \mathcal{S}$.

$\therefore$ $\mathcal{S}$ forms a grp.

Defn: A collineation α is a dilatation if m & $\alpha(m)$ are parallel for every line m.

Eg. A collineation α defined by $\alpha((x,y)) = (x+2, y+3)$ is a dilatation.

proof.

Let l be arb. line with eqn $y = ax+b$.

Let (x,y) be any pt on l, then

$y = ax+b$ & $\alpha((x,y)) = (x+2, y+3)$

Set $u = x+2$ & $v = y+3$

$\Rightarrow$ $x = u-2$ & $y = v-3$

Now $y = ax+b$ $\Rightarrow$ $v-3 = a(u-2)+b$

$\Rightarrow$ $v = au - 2a + b + 3$

$\Rightarrow$ (u,v) is on the line m with eqn:

$y = ax + (-2a+b+3)$

$\Rightarrow$ l & m are $//$.

$\therefore$ α is a dilatation.

Note: The dilatations form a grp called the dilatation grp.

Prf

Let D be the set of all dilatations.

i. closure property.

Let $\alpha, \beta \in D$ & l be any line $\ni$

$\alpha(l) = l'$ & $\beta(l') = l''$

$\Rightarrow$ $l \parallel l'$ & $l' \parallel l''$ since $\alpha, \beta \in D$

So, by transitivity of parallelness $l \parallel l''$.

Thus, $\beta\alpha(l) = \beta(\alpha(l)) = \beta(l') = l''$

$\Rightarrow \beta\alpha \in D$.

$\Rightarrow$ The composite of two dillatations is a dilatation.

ii. Existence of inverse.

Holds by the symmetry of llness for lines i.e., $l \parallel l' \Rightarrow l' \parallel l$.

for $\forall$ $\alpha \in D$

if $\alpha(l) = l'$ ~~the~~ ~~and~~

$\exists \alpha^{-1} \ni \alpha^{-1}(l') = l$, where $l' \parallel l$ by symmetry.

Hence the inverse of a dilatation is a dilatation.

The remaining are trivial. (show?)

Hence, the dilatations form a grp.

unit 3

Discuss the 3 rigid motions trans, rotn, & reflc. These trans change only positions of objects (plane figs) not their shape & size.

UNIT THREE

ISOMETRIC TYPE MOTIONS

Defn: The mapping α of a plane Π onto itself is said to be isometry (orthogonal mapping) if for any two pts M & N of Π, the distance b/n M & N is equal to the distance b/n $\alpha(M)$ & $\alpha(N)$.

- Remark: An isometry is the name given for any transformation that preserves distance. It comes from the Greek isos (equal) & metron (measure).

Eg. w/c of the ff. mapping is an isometry.

a. $\alpha((x,y)) = (x,y)$ b) $\alpha((x,y)) = (2x,-2y)$

b. $\gamma((x,y)) = (-x+1, -y)$

c. $\beta((x,y)) = (y,x)$

d. $\theta((x,y)) = (x+4, y-3)$

Theorem: An isometry is a collineation that preserves b/nness, mid pts, segments, rays, triangles, angles, angle measure & perpendicularity.

Proof:

a. To show ~~b/n~~ betweenness, mid pts & segments preservance:
Let α be an isometry and Spse A, B & C be any three distinct pts $\ni$

$\alpha(A) = A'$, $\alpha(B) = B'$ & $\alpha(C) = C'$.

i. since α preserves distance
$AB = A'B'$, $BC = B'C'$ & $AC = A'C'$

Thus, if $AB + BC = AC$; then $A'B' + B'C' = A'C'$

Hence, B is b/n A & C $\Rightarrow$ B' is b/n A' & C'

$\therefore$ α preserves b/nness.

ii. if $AB = BC$, then $A'B' = B'C'$.

Thus, if B is the mid pt of A & C, then
B' is " " " " A' & C'.

$\Rightarrow \alpha$ preserves mid pt.

iii. Since $\overline{AB}$ is the union of A, B & all pts b/n A & B,
$\alpha(\overline{AB})$ is the union of A', B' & all pts b/n A' & B'.

So $\alpha(\overline{AB}) = \overline{A'B'}$

Hence α preserves line segments

b. Preserving rays, lines, angles & $\perp$rity.

(Ex.)

i. Preserving rays

Let α be an isometry

Let pts $A, B, C, D, E \ldots$ lie on a ray $\overrightarrow{AB}$ &
$\alpha(A) = A'$, $\alpha(B) = B'$, $\alpha(C) = C'$, $\alpha(D) = D'$ ....

By part (a)

$AB + BC + CD + DE + \cdots = A'B' + B'C' + C'D' + \cdots$

Thus, α preserves rays.

[OR $\overrightarrow{AB}$ is the union of $\overline{AB}$ & all pts $C \ni A-B-C$, then $\alpha(\overrightarrow{AB})$ is the union of $\overline{A'B'}$ & all pts $C' \ni A'-B'-C'$ so, we say α preserves rays.]
$\alpha(\overrightarrow{AB}) = \overrightarrow{A'B'}$

ii Preserving lines.

By (bi) $\overrightarrow{AB} = \overrightarrow{A'B'}$ & $\overrightarrow{BA} = \overrightarrow{B'A'}$

$\overleftrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{BA}$ ~~and~~ $\overrightarrow{A'B'} \cup \overrightarrow{B'A'} = \overleftrightarrow{A'B'}$

$\alpha(\overleftrightarrow{AB}) = \alpha(\overrightarrow{AB}) \cup \alpha(\overrightarrow{BA}) = \overrightarrow{A'B'} \cup \overrightarrow{B'A'} = \overleftrightarrow{A'B'}$

$\Rightarrow \alpha$ preserves lines

Hence α is a collineation.

ii. If $AB = BC$, then $A'B' = B'C'$

Thus, if B is the mid pt of A & C, then
B' is " " " A' & C'.

$\Rightarrow \alpha$ preserves mid pt.

iii. Since $\overline{AB}$ is the union of A, B & all pts b/w A & B,
$\alpha(\overline{AB})$ is the union of A', B' & all pts b/w A' & B'.

So $\alpha(\overline{AB}) = \overline{A'B'}$

Hence α preserves line segments

b. Preserving rays, lines, angles & ⊥rity.

(Ex.)

i. Preserving rays

Let α be an isometry

Let pts A, B, C, D, E --- lie on a ray $\overrightarrow{AB}$ &
$\alpha(A) = A'$, $\alpha(B) = B'$, $\alpha(C) = C'$, $\alpha(D) = D'$ ----

By part (a)

$AB + BC + CD + DE + \cdots = A'B' + B'C' + C'D' + \cdots$

Thus, α preserves rays.

[OR $\overrightarrow{AB}$ is the union of $\overline{AB}$ & all pts $C \ni A-B-C$, then
$\alpha(\overrightarrow{AB})$ is the union of $\overline{A'B'}$ & all pts $C' \ni A'-B'-C'$ so,
$\alpha(\overrightarrow{AB}) = \overrightarrow{A'B'}$ & we say α preserves rays.]

ii Preserving lines

By (bi) $\overrightarrow{AB} = \overrightarrow{A'B'}$ & $\overrightarrow{BA} = \overrightarrow{B'A'}$

$\overleftrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{BA}$ ~~$\alpha(\overrightarrow{AB})$~~ $\cup \overrightarrow{B'A'} = \overleftrightarrow{A'B'}$

$\alpha(\overleftrightarrow{AB}) = \alpha(\overrightarrow{AB}) \cup \alpha(\overrightarrow{BA}) = \overrightarrow{A'B'} \cup \overrightarrow{B'A'} = \overleftrightarrow{A'B'}$

$\Rightarrow \alpha$ preserves lines

Hence α is a collineation.

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iii. Preserving $\angle$s

Let $\angle ACB$ be formed by the two rays $\overrightarrow{OA}$ & $\overrightarrow{OB}$.

Since α preserves rays $\overrightarrow{OA}$ & $\overrightarrow{OB}$ are carried to rays $\overrightarrow{O'A'}$ & $\overrightarrow{O'B'}$

Hence $\angle AOB$ is carried into $\angle A'O'B'$.

We need also to show that

$\angle AOB \cong \angle A'O'B'$

By the proof in part ⓐ $\overline{OA} = O'A'$, $\overline{OB} = O'B'$ & $AB = A'B'$

$\Rightarrow \Delta AOB \cong \Delta A'O'B'$ by SSS.

$\Rightarrow \angle AOB \cong \angle A'O'B'$

$\therefore \alpha$ preserves angle measure

Hence, preserving $\perp$larity follows.

iv. Preserving Δs is immediate from the above arguments.

Ex.

Prove that the set of all isometries form a grp.

Proof:

Let G be the set of all isometries.

i. closurity

Let α & $\beta \in G$.

Let also M & N be any two pts on a plane π $\ni$

$\alpha(M) = M'$ & $\alpha(N) = N'$

$\beta(M') = M''$ & $\beta(N') = N''$

Thus, since α is iso, $MN = M'N'$ &

since β is iso, $M'N' = M''N''$

$\Rightarrow MN = M''N''$

$\Rightarrow$ The ~~distance b/n~~ distance b/n M & N is equal to the distance b/n $\beta\circ\alpha(M)$ & $\beta\circ\alpha(N)$

$\Rightarrow \beta\circ\alpha$ preserves distance

$\Rightarrow \beta\circ\alpha \in G$.

ii. Associativity : obvious

iii. Idty mapping is an idty elt of G.

iv. Let $\alpha \in G$. Since α is an isometry it is a transf. & for any line segment $A'B'$, there exist segment AB ∋

$$\alpha(\overline{AB}) = \overline{A'B'} \qquad \Rightarrow \quad \overline{AB} = \alpha^{-1}(\overline{A'B'})$$

$\Rightarrow \quad \alpha^{-1}\alpha(\overline{AB}) = \alpha^{-1}(\overline{A'B'})$

$\Rightarrow \quad \alpha^{-1}$ preserves distance.

$\Rightarrow \quad \alpha^{-1} \in G$.

$\therefore$ G is a grp.

★ Translation, rotation & reflection are isometric type motions which preserve shape & size.

3.1. Translations.

Defn: A translation is a transformation τ defined by

$$\tau((x,y)) = (x+a,\ y+b) \quad \text{where } a, b \in \mathbb{R}.$$

Ex. Show that a translation is an isometry.

proof:

Let the translation τ be defined by

$$\tau((x,y)) = (x+a,\ y+b), \quad a, b \in \mathbb{R}.$$

Let $P(x,y)$ & $Q(s,t)$ be any two pts

The distance b/n P & Q,

$$PQ = \sqrt{(x-s)^2 + (y-t)^2}$$

$\tau(P) = (x+a,\ y+b)$

$\tau(Q) = (s+a,\ t+b)$

Distance b/n $\tau(P)$ & $\tau(Q)$ is

$$= \sqrt{(x-s)^2 + (y-t)^2}$$

$\therefore PQ = \tau(P)\tau(Q)$

$\Rightarrow \tau$ is an isometry.

* For any two distinct pts $P(a,b)$ & $Q(c,d)$ there is a unique translation τ that takes P to Q & is given by

$$\tau((x,y)) = (x+(c-a),\ y+(d-b))$$

Such translation is ~~given~~ denoted by $\tau_{P,Q}$

Ex. Write the unique translation that takes $(2,3)$ to $(5,1)$.

Note: For any pts P, Q, R & S:

i. if $\tau_{P,Q}(R) = S$, then $\tau_{P,Q} = \tau_{R,S}$

ii. The idty is a special case of a translation as

$$i = \tau_{P,P}$$

iii. If $\tau_{P,Q}(R) = R$, then $P = Q$ as

$$\tau_{P,Q} = \tau_{R,R} = i$$

Theorem: Spse A, B & C are noncollinear pts. Then $\tau_{A,B} = \tau_{C,D}$ iff $\square CABD$ is a llgm.

proof:

Let A, B, C be non collinear pts.

$(\Rightarrow)$ spse $\tau_{A,B} = \tau_{C,D}$

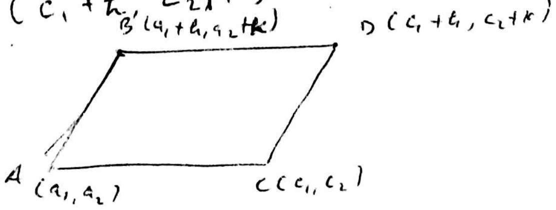
Let the translation τ is given by

$$\tau((x,y)) = (x+h,\ y+k)$$

Let $A = (a_1, a_2)$ & $C = (c_1, c_2)$

& $B = \tau(A) = (a_1+h, a_2+k)$

$D = \tau(C) = (c_1+h, c_2+k)$



$\Rightarrow$ $AC = BD$ since τ is an isometry.

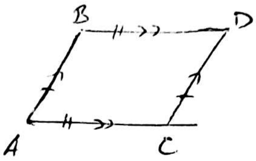
&

$AB = CD = \sqrt{h^2+k^2}$

Hence, $\square CABD$ is a //gm.

($\Leftarrow$) Ex

Spse $\square CABD$ is a //gm



WTS: $\tau_{A,B} = \tau_{C,D}$.

Let $\tau_{A,B}((x,y)) = (x+h, y+k)$ &

$\tau_{C,D}((x,y)) = (x+s, y+t)$

Now we need to show $h = s$ & $k = t$

If $A = (a_1, a_2)$, then $B = (a_1+h, a_2+k)$. &

" $C = (c_1, c_2)$, then $D = (c_1+s, c_2+t)$.

By properties of a //gm we have:

i. $AB = CD \Rightarrow \boxed{h^2+k^2 = s^2+t^2}$ — ①

ii. $AB \parallel CD \Rightarrow \boxed{\frac{k}{h} = \frac{t}{s}}$

$\Rightarrow \boxed{ks = ht}$ ②

From ① we have $hk\left(\frac{h}{k}+\frac{k}{h}\right) = st\left(\frac{s}{t}+\frac{t}{s}\right)$

$\Rightarrow hk\left(\frac{h}{k}+\frac{k}{h}\right) = st\left(\frac{h}{k}+\frac{k}{h}\right)$ $\quad \frac{t}{s} = \frac{k}{h}$ & $s/t = h/k$.

$\Rightarrow \boxed{hk = st}$ ... ③

Adding ② & ③ we get

$hk + ks = st + ht$

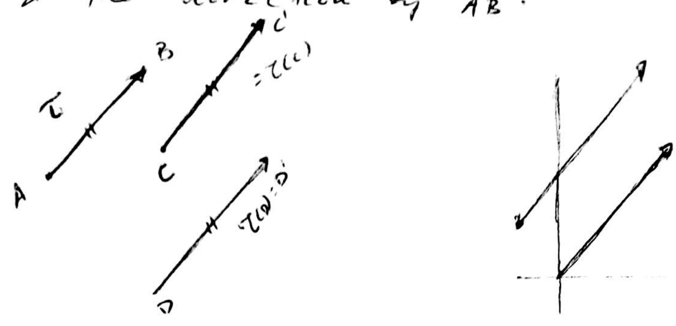
$\Rightarrow k(s+h) = t(s+h)$

$\Rightarrow k = t$

$\Rightarrow h = s$

$\therefore \tau_{A,B} = \tau_{C,D}$.

Note: For a non identity translation $\tau_{A,B}$, the distance is given by AB & the direction by $\overrightarrow{AB}$.



* We say a transformation α fixes pt P iff $\alpha(P) = P$.

Transformation α fixes a line l iff $\alpha(l) = l$ & in general, α fixes a set S of pts iff $\alpha(S) = S$

Theorem:

a. A translation is a dilatation.

b. If $P \neq Q$, then $\tau_{P,Q}$ fixes no pts & fixes exactly those lines that are $\parallel$ to $\overleftrightarrow{PQ}$.

Proof:

Let $\tau((x,y)) = (x+h, y+k)$ be a translation.

a. Let l be a line with eqn.

$ax + by + c = 0$

Let (x,y) be a pt on l.

Then $\tau((x,y)) = (x+h, y+k)$ & $ax + by + c = 0$

Set $(x+h, y+k) = (x', y')$

Then $x = x' - h$ & $y = y' - k$

& $ax + by + c = 0$

$\Rightarrow a(x'-h) + b(y'-k) + c = 0$

$\Rightarrow ax' + by' + (c - ah - bk) = 0$

$\Rightarrow (x', y')$ is on the line m with eqn

$ax + by + (c - ah - bk) = 0$

Since $m \parallel l$,

τ is a dilatation.

b. Spse $P \neq Q$.

Then either $h \neq 0$ or $k \neq 0$.

Then for any pt (x,y), $\tau((x,y)) \neq (x,y)$

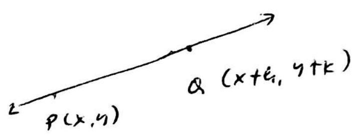
$\Rightarrow$ $\tau_{P,Q}$ fixes no pt if $P \neq Q$.

* To show $\tau_{P,Q}$ fixes those lines $\parallel$ to $\overleftrightarrow{PQ}$.

Let τ be a translation that sends P to Q & is given by

$$\tau((x,y)) = (x+h, y+k)$$

Then the slope of $\overleftrightarrow{PQ}$ is $\frac{k}{h}$.



$\Rightarrow$ The eqn of any line $l \parallel$ to $\overleftrightarrow{PQ}$ is given by

$$y = \frac{k}{h}x + c$$

Claim: $\tau(l) = l$

Since $\tau((x,y)) = (x+h, y+k)$, let

$u = x+h$ & $v = y+k$

$\Rightarrow$ $x = u-h$ & $y = v-k$

Then $y = \frac{k}{h}x + c \Rightarrow (v-k) = \frac{k}{h}(u-h) + c$

$\Rightarrow$ $v = \frac{k}{h}u + c$

$\Rightarrow$ (u,v) is also on $y = \frac{k}{h}x + c$

$\Rightarrow$ $\tau(l) = l$.

$\therefore$ τ fixes those lines w/c are $\parallel$ to $\overleftrightarrow{PQ}$.

Theorem: The translations form an abelian grp T.

Proof: (Ex.)

Let τ be a translation defined by

$\tau((x,y)) = (x+h, y+k)$

Let $S = (a,b)$, $T = (c,d)$ & $R = (e,f)$

Since there is a unique translation from one pt to another, the unique translation from pt S to T is:

$\tau((x,y)) = (x + (c-a), y + (d-b))$ &

the unique translation from T to R is:

$\tau((x,y)) = (x + (e-c), y + (f-d))$

Thus,

i. closure

$$\tau_{S,T}\tau_{T,R}((x,y)) = \tau_{S,T}((x+(e-c), y+(f-d)))$$

$$= (x+(e-a), y+(f-b))$$

$$= \tau_{S,R}$$

~~One can show~~

OR, Let $S = (a,b)$, $T = (c,d)$ & $R = (a+c, b+d)$

Then, $\tau_{0,T}\tau_{0,S}((x,y)) = \tau_{0,T}((x+a, y+b)) = (x+a+c, y+b+d)$

$$= \tau_{0,R}((x,y)).$$

$\Rightarrow$ a product of two translations is a translation.

By taking $R=0$, we see that the inverse of the translation $\tau_{0,S}$ with $S = (a,b)$ is $\tau_{0,T}$ with $T = (-a,-b)$.

Hence, the set of all translations form a grp.

Further, since $a+c = c+a$ & $b+d = d+b$ it follows $\tau_{0,T}\tau_{0,S} = \tau_{0,S}\tau_{0,T}$ $\Rightarrow$ Translations Commute.

$\therefore$ The translations form an abelian grp.

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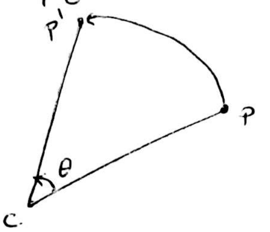
3.2. Rotations

Defn: A rotation about pt C through directed angle $\theta°$ is the transformation $\rho_{C,\theta}$ that fixes pt C only & otherwise sends a pt P to P' with $CP = CP'$ & θ is the directed angle measure of the directed angle from $\vec{CP}$ to $\vec{CP'}$



Remark:

i. We agree that $\rho_{C,0}$ is the idty ι.

ii. Rotation $\rho_{C,\theta}$ is said to have center C & directed angle θ.

Eg. Find $\rho_{C,\theta}(P)$ if $C = (0,0)$ & $\theta = \pi/2$, $P = (0,4)$

$C = (0,0)$ & " , $P = (3,4)$

$P' = (-4,3)$

Sol^n

$\rho_{C,\theta}(P) = (\ ,\ )$

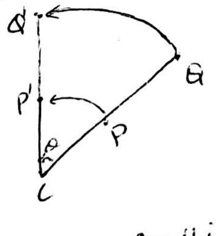
Theorem: Rotation is an isometry.

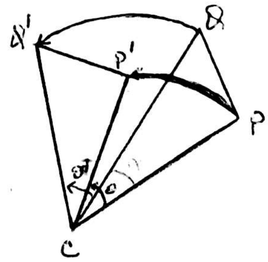
Proof:

Spse $\rho_{C,\theta}$ sends pts P & Q to pt P' & Q' resp.

* If C, P, Q are collinear, ~~then~~ $CP = CP'$ & $CQ = CQ'$ by defn.

$\Rightarrow PQ = P'Q'$.





* If C, P, Q are non collinear, then

$\triangle PCQ \cong \triangle P'CQ'$ by SAS.

$\Rightarrow PQ = P'Q'$

So, $\rho_{c,\theta}$ is a transformation that preserves distance & hence it is an isometry.

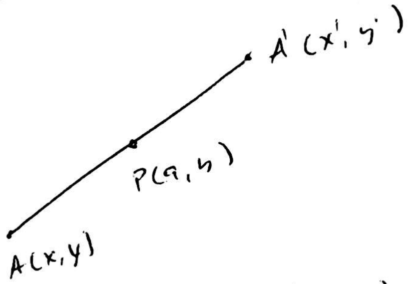
Note: A rotation of 90° is a collineation but not a dilatation.

(if -90° $(x,y) \to (y,-x)$).

$$\rho_{c,90^\circ} = \tau((x,y)) = (-y,x).$$

What can you say about a rotation of 180°?

Defn: A rotation of 180° about some fixed pt P is called a half turn & is denoted by σ_p.



We observe that if pt $A(x,y)$ is rotated 180° about pt $P(a,b)$ to pt $A'(x',y')$, then P is the mid pt of A & A'.

So, using mid pt formula,

$$\frac{x+x'}{2} = a \quad , \quad \& \quad \frac{y+y'}{2} = b$$

$$\Rightarrow \quad x' = -x+2a \quad \& \quad y' = -y+2b$$

Thus σ_p is a mapping given by

$$\sigma_p((x,y)) = (-x+2a, -y+2b)$$

Remark:

i. σ_p is an involutory transformation.

ii. σ_p fixes exactly the one pt P.

9. Show that:

1. $\sigma_p \sigma_p = \iota$ (From this we have $\sigma_p = \sigma_p^{-1}$ $\rightarrow$ involution)

2. σ_p is a dilatation.

3. σ_p fixes line l iff P is on l.

<u>Proof:</u>

Let $P(a,b)$ be a pt.

1. $\sigma_p \sigma_p ((x,y)) = \sigma_p ((-x+2a, -y+2b)) = (-(-x+2a)+2a, -(-y+2b)+2b)$

$= (x-2a+2a,\ y-2b+2b) = (x,y)$

$\Rightarrow \sigma_p \sigma_p = \iota$

2. Spse line l has eqn $ax+by+c=0$

Let $P = (h,k)$

Then $\sigma_p((x,y)) = (-x+2h, -y+2k)$

Let $u = -x+2h$ & $v = -y+2k$ $\Rightarrow$ $x = -u+2h$ & $y = -v+2k$

Then $ax+by+c=0$ iff

$a(-u+2h) + b(-v+2k) + c = 0$

$-au - bv + 2ah + 2bk + c = 0$

$\Leftrightarrow$ $au + bv - 2ah - 2bk - c = 0$

$\Leftrightarrow$ $au + bv + (c - c) - 2ah - 2bk - c = 0$

$\Leftrightarrow$ $au + bv + c - 2(ah + bk + c) = 0$

Thus, (u,v) lie on a line m with eqn:

$ax+by+c - 2(ah+bk+c) = 0$

Hence, σ_p is a collineation

Since $l \parallel m$, σ_p is a dilatation.

l & m in ~~…~~ #2 are the same iff

$ah + bk + c = 0$

& this holds iff (h, k) is on l.

Theorem: If Q is the mid pt of pts P & R, then

$$\sigma_Q \sigma_P = \tau_{P,R} = \sigma_R \sigma_Q$$

~~$\sigma_Q\sigma_P$ - The product of half turn of pt P, Q is equal to the product of half turn of point R, Q~~ $= \sigma_R \sigma_Q$

Proof:

Let $P = (a, b)$ & $Q = (c, d)$

Then $\sigma_Q \sigma_P((x,y)) = \sigma_Q((-x+2a, -y+2b)) = (-(-x+2a)+2c, -(-y+2b)+2d)$

$= (x + 2(c-a), y + 2(d-b))$

$\Rightarrow \sigma_Q \sigma_P$ is a translation.

Spse R is a pt $\ni$ Q is the mid pt of P & R.

Then $\sigma_Q \sigma_P(P) = \sigma_Q(P) = R$ &

$\sigma_R \sigma_Q(P) = \sigma_R(R) = R$

Since $\sigma_Q \sigma_P$ is a translation & there is a unique translation taking P to R, $\tau_{P,R}$ (as $\sigma_Q \sigma_P(P) = R$),

$$\sigma_Q \sigma_P = \tau_{P,R} = \sigma_R \sigma_Q.$$

* From the above theorem we see that a product of two half turns is a translation &, conversely, a translation is a product of two half turns.

Note: $\sigma_Q \sigma_P$ moves each pt twice the directed distance from P to Q.

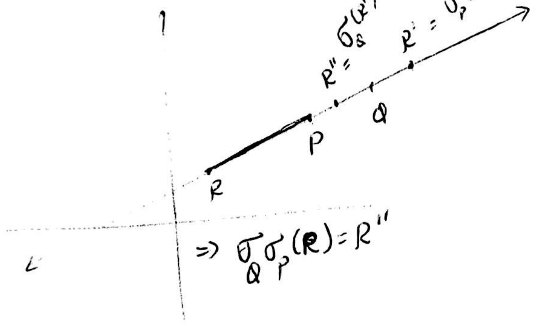
Proof:

Let $R(x,y)$ be any pt

i. Let R, P & Q be collinear (in any order R-P-Q, P-R-Q or P-Q-R).

Let's consider R-P-Q.

Spse $RP > PQ$ (of course we may have $RP = PQ$ or $RP < PQ$)



$RR' = 2RP = 2PR'$ by defn

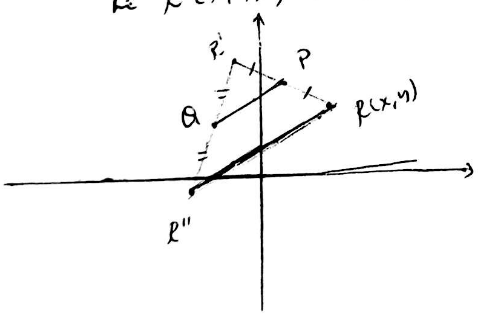
$R''R' = 2QR'$,,

$$RR'' = RR' - R'R''$$
$$= 2RP - 2QR'$$
$$= 2[PR' - QR']$$
$$RR'' = 2PQ$$

Show the rest possible cases.

ii. Let R, P & Q be non collinear.

Le $R(x,y)$, $P(a,b)$, $Q(c,d)$.



Theorem: A product of three half turns is a halfturn.

Proof:

Let $P = (a,b)$; $Q = (c,d)$; $R = (e,f)$.

Then $\sigma_R \sigma_Q \sigma_P ((x,y)) = \sigma_R \sigma_Q ((-x+2a, -y+2b)) = \sigma_R ((x+2(c-a), y+2(d-b)))$

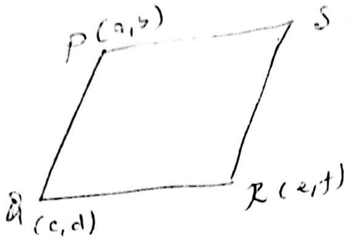
$= (-[x+2(c-a)]+2e, -[y+2(d-b)]+2f) = (-x+2(a-c+e), -y+2(b-d+f))$

$= (-x+2(a-c+e), -y+2(b-d+f))$

$= \tau_S((x,y))$, where $S = (a-c+e, b-d+f)$

Note: / Corollary /

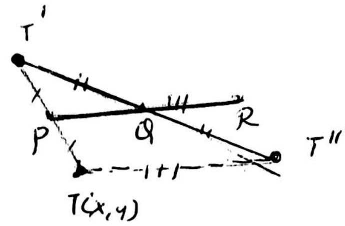
- In the above thm if P, Q & R are non collinear, then $\square PQRS$ is a $\parallel$gm.



Proof!

It is true since $\tau_{QP} = \tau_{R,S}$ by the thm pp. 23.

* If Q is the mid pt of P & R, then $\sigma_Q \sigma_P = \tau_{P,R}$



$PR = TT''$

$\overleftrightarrow{PR} \parallel \overleftrightarrow{TT''}$

Thus, $\sigma_Q \sigma_P = \tau_{P,R}$

$\Rightarrow \tau_{P,R}^{-1} = (\sigma_Q \sigma_P)^{-1} = \sigma_P^{-1} \sigma_Q^{-1} = \sigma_P \sigma_Q$ (σ_P - involution)

So $\sigma_Q \sigma_P = \sigma_P \sigma_Q \Leftrightarrow \tau_{P,R} = \tau_{P,R}^{-1}$ (i.e. iff $P = Q$)

Hence, half turns donot commute in general.

Skip

already discussed before.

Theorem: $\sigma_R \sigma_Q \sigma_P = \sigma_P \sigma_Q \sigma_R$ for any pts P, Q, R.

Proof!

For any pts P, Q, R there is a pt S $\ni$

$\sigma_R \sigma_Q \sigma_P = \sigma_S = \sigma_S^{-1} = (\sigma_R \sigma_Q \sigma_P)^{-1} = \sigma_P^{-1} \sigma_Q^{-1} \sigma_R^{-1} = \sigma_P \sigma_Q \sigma_R$

[illegible] since the product of two half turns is a translation, the half turns donot form a grp by themselves. However, the union of the translations & the half turns forms a grp.

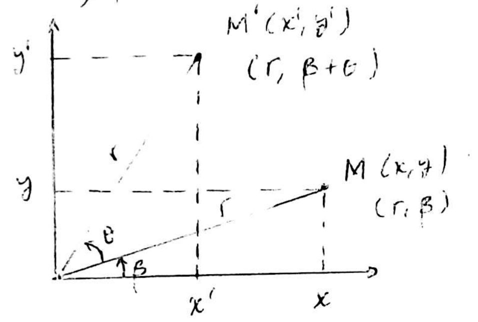
Equation of Rotation

A. First ~~we~~ take the center of rotation to be the origin, O.

Let ρ be a rotation about O through the directed angle θ.

Let $M(x,y)$ be any pt of the plane other than O.

Let (r,β) be the polar coordinates of M



If $\rho(M) = M'$, then clearly the polar coordinates of M' is $(r, \theta+\beta)$.

If the ~~pola~~ rectangular coordinate of M' is (x', y'), then

$$x' = r\cos(\theta+\beta) = r(\cos\theta\cos\beta - \sin\theta\sin\beta)$$

$$= r\cos\beta\cos\theta - r\sin\beta\sin\theta$$

$$= \underline{x\cos\theta - y\sin\theta}, \quad \text{since } x = r\cos\beta,\ y = r\sin\beta.$$

$$y' = r\sin(\theta+\beta) = r(\sin\theta\cos\beta + \sin\beta\cos\theta)$$

$$= r\cos\beta\sin\theta + r\sin\beta\cos\theta$$

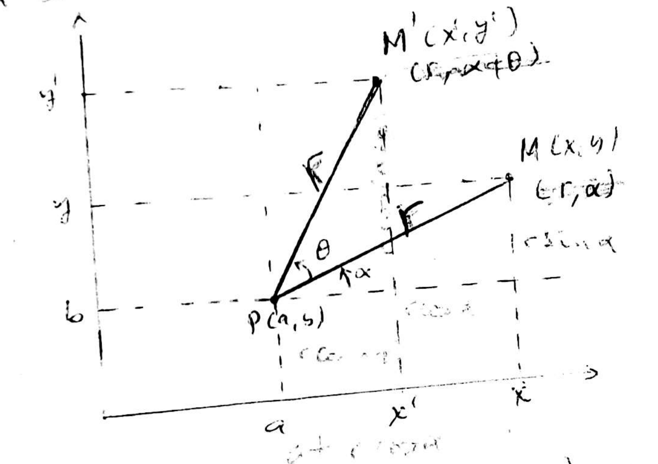
$$= \underline{x\sin\theta + y\cos\theta}$$

Theorem: In the Cartesian plane, rotating a pt (x, y) about the origin O through directed angle θ, $\rho_{O,\theta}$, has eqns:

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

B. Let the center of rotation be any pt $P(a,b)$.

Let $M(x,y)$ be any pt of the plane other than P. Let α be an angle b/n the +ve x-axis & line $\overleftrightarrow{PM}$.



Then $M = (a + r\cos\alpha,\ b + r\sin\alpha) = (x, y)$

If $M'(x', y') = \rho_P(M)$; then

$M' = (a + r\cos(\theta + \alpha),\ b + r\sin(\theta + \alpha))$

Then, $x' = a + r\cos(\alpha + \theta) = a + r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$

$y' = b + r\sin(\alpha + \theta) = b + r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$

But $r\cos\alpha = x - a$ & $r\sin\alpha = y - b$

Then, $x' = (x - a)\cos\theta - (y - b)\sin\theta + a$ &

$y' = (x - a)\sin\theta + (y - b)\cos\theta + b$

Theorem: In the Cartesian plane, rotation of (x, y) about the pt $P=(a, b)$ [illegible] $\rho_{P,\theta}$ then [illegible]: (note [illegible])

$$x' = (x-a)\cos\theta - (y-b)\sin\theta + a$$
$$y' = (x-a)\sin\theta + (y-b)\cos\theta + b$$

Theorem:

i. A non-idty rotation fixes exactly one pt, its center.

ii. A rotation with center C fixes every circle with center C.

iii. If C is a pt & θ & α are angles, then

$$\rho_{C,\theta}\,\rho_{C,\alpha} = \rho_{C,\theta+\alpha}$$

iv. $\rho^{-1}_{C,\theta} = \rho_{C,-\theta}$

v. The rotations with center C form an abelian grp.

Proof:

(i) - closure (iii)

(ii) for any $\rho_{C,\theta}$, $\rho_{C,\theta}\,\rho_{C,-\theta} = \rho_{C,0}$ (idty).

iii. Invertibility: $\forall \rho_{C,\theta}$, $\exists \rho_{C,-\theta}$ $\ni$ $\rho_{C,\theta}\,\rho_{C,-\theta} = \rho_{C,0} = i$.

iv. $\rho_{C,\alpha}\,\rho_{C,\theta} = \rho_{C,\alpha+\theta} = \rho_{C,\theta+\alpha} = \rho_{C,\theta}\,\rho_{C,\alpha}$ Commutative.

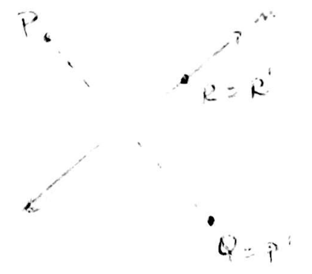
vi. The involutary rotations are halfturns.

vii. $\rho_{C,180} = \sigma_C$, for any pt C.

2.3 Reflections

Defn: Reflection σ_m in line m is the mapping defined by

$$\sigma_m(P) = \begin{cases} P, & \text{if } P \text{ is on } m \\ Q, & \text{if } P \text{ is off } m \text{ \& } m \text{ is } \perp \text{ bisector of } \overline{PQ} \end{cases}$$



(a) m is called the axis of σ_m

Remarks.

i. σ_m interchanges the half planes of ~~mmmt~~ m.

ii. $\sigma_m \neq i$ but $\sigma_m^2 = i$ as the $\perp$ bisector of $\overline{PQ}$ is the $\perp$ bisector of $\overline{QP}$. Therefore, σ_m is an involutory ~~mmmmy~~ mapping.

iii. σ_m is onto as $\sigma_m(P)$ is the pt mapped onto the given pt P since $\sigma_m(\sigma_m(P)) = P \quad \forall P$.

iv. σ_m is 1-1 as;

$$\sigma_m(A) = \sigma_m(B) \Rightarrow \sigma_m(\sigma_m(A)) = \sigma_m(\sigma_m(B))$$
$$\Rightarrow A = B.$$

v. σ_m fixes every pt P iff P is on m.

Let us derive a general formula for $\sigma_m(P)$ where m has eqn $ax + by + c = 0$.

Let ~~P~~ $P = (x, y)$ & $\sigma_m(P) = (x', y') = Q$

For the moment, Spse P is off m.

Now the line through pt P & Q is $\perp$ to line m.

i.e., a line $\overleftrightarrow{PQ}$ has slope $\frac{b}{a}$, (as slope of m $\times$ slope of $\overleftrightarrow{PQ} = -1$

$$\Rightarrow \frac{y'-y}{x'-x} = \frac{b}{a} \Rightarrow a(y'-y) = b(x'-x) \quad \text{---} (1)$$

Also $\left(\frac{x+x'}{2}, \frac{y+y'}{2}\right)$ is the mid pt ~~of~~ $\overline{PQ}$ & is on line m.

$$\Rightarrow a\left(\frac{x+x'}{2}\right) + b\left(\frac{y+y'}{2}\right) + c = 0 \quad \text{---} (2)$$

From eqns (1) & (2) we solve the Simultaneous eqns.

[illegible]

[illegible]

Solving [illegible]

$$x' = x - \frac{2a(ax+by+c)}{a^2+b^2}$$

$$y' = y - \frac{2b(ax+by+c)}{a^2+b^2}$$

<u>Theorem</u>: If line m has eqn $ax+by+c=0$, then T_m has eqn

$$x' = x - \frac{2a(ax+by+c)}{a^2+b^2}$$

$$y' = y - \frac{2b(ax+by+c)}{a^2+b^2}, \quad \text{where } T_m((x,y)) = (x',y')$$

Eg. Let m be a line with eqn $y = 5x+3$. Find $T_m((3,2))$.

<u>Soln</u>

Let $T_m((3,2)) = (x',y')$, $\quad 5x - y + 3 = 0$

$$\Rightarrow x' = x - \frac{2a(ax+by+c)}{a^2+b^2} = 3 - \frac{2(5)[5(3) - 1(2) + 3]}{5^2+(-1)^2}$$

$$= 3 - \frac{10(\cancel{16}^{8})}{\cancel{26}_{13}} = \frac{39-80}{13} = \frac{-41}{13}$$

$$y' = y - \frac{2b(ax+by+c)}{a^2+b^2} = 2 - \frac{2(-1)[5(3) - 1(2) + 3]}{26}$$

$$= 2 + \frac{2(16)}{26} = \frac{52+32}{26} = \frac{84}{26} = \frac{42}{13}$$

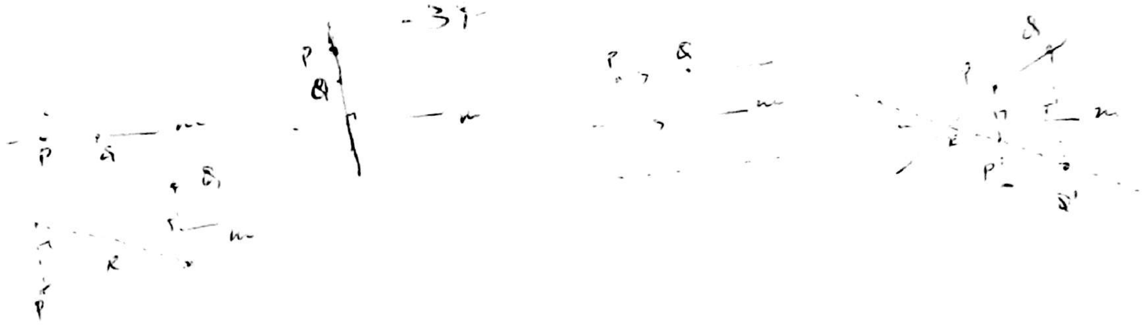
Theorem. Reflection T_m is an isometry.

proof:

Method 1. very simple algebraic procedure using the above Thm.

2. Geometrically, considering all the cases: for two pts P & Q

[illegible] are on m, ~~on the~~ PQ $\perp$ m, PQ $\parallel$ m, one of P or Q [illegible] on the same side



Defs:

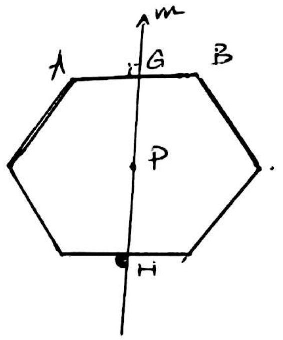
i. Line m is called a line of Symmetry for the set S of pts if $\sigma_m(S) = S$; i.e., σ_m fixes S.

ii. Pt P is a pt of Symmetry for the set S if $\sigma_P(S) = S$.

iii. Isometry α is a Symmetry for set S of pts if $\alpha(S) = S$.

Eg.

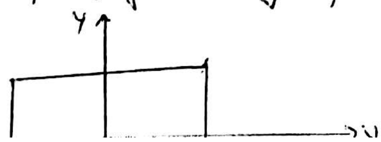
1. Consider a regular hexagon shown below & assume that m is $\perp$ bisector of $\overline{AB}$ & P is the mid pt of $\overline{GH}$.



Then,
- m is the line of Symmetry of the hexagon.
- P is pt of Symmetry " " "

(*) It is also clear that a rotation of $60°$ about P is also a Symmetry for the regular hexagon.

2 Let us consider the Symmetry of a rectangle (non square) given below



Denoting the reflection in the x-axis by σ_h & the reflection in the y-axis by σ_v, we then have $\sigma_h, \sigma_v, \sigma_c$ & i are all symmetries for the rectangle.

Note:
i. i is a symmetry for any set of pts.
ii The set $\{\sigma_h, \sigma_v, \sigma_c, i\}$ form a grp.

Theorem: The set of all symmetries of a set of pts forms a grp.

proof:

Let S be an non empty set of pts.
Let G be the set of all symmetries for S.
Then $G \neq \phi$ since $i \in G$.

* Closure property:
Spse $\alpha, \beta \in G$ (are symm. for S)
Then $\beta\alpha(S) = \beta(\alpha(S)) = \beta(S) = S$
$\Rightarrow \beta\alpha \in G$.

- Associativity & Existence of idty are obvious.

* Existence of Inverses
If $\alpha \in G$, then α & α^{-1} are transformations &
$\alpha^{-1}(S) = \alpha^{-1}(\alpha(S)) = \alpha^{-1}\alpha(S) = i(S) = S$
$\Rightarrow \alpha^{-1} \in G$.
$\therefore$ G is a grp.

Corollary: The set of all isometries forms a grp.

(*) The grp of all symmetries for a set S of pts is called the symmetry grp or the full grp of symmetries for S.

then the isometry fixes the line ...

proof:

Let α be an isometry with A & B fixed by α.

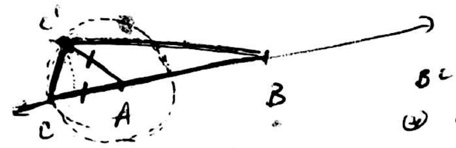
Let C be a pt on $\overleftrightarrow{AB}$ other than A & B.

Let $\alpha(C) = C'$.

Since α is an isometry,

$AC = AC'$ & $BC = BC'$

$\Rightarrow C = C'$

① 

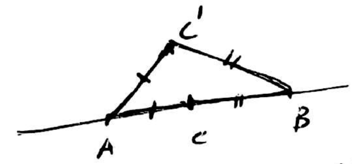
$BC = BC' = BC'$

⊕ CC'

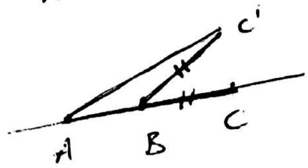
$BC = AB + CA = AB + AC'$

$\Rightarrow BC' = AB + AC'$ $\rightarrow\leftarrow$

sum of two sides of a Δ.

② 

$AC' + BC' = AB$ $\rightarrow\leftarrow$

③. 

$AC = AC'$

$AC = AB + BC = AC'$

$\rightarrow AB + BC' = AC'$ $\rightarrow\leftarrow$

Eg. Let α be an isometry with $\alpha(0,100) = (0,100)$ & $\alpha(0,-3) = (0,-3)$. Then, find $\alpha(0,33)$. Ans. $\alpha((0,33))$ $(0,33)$

Theorem: If an isometry fixes three non collinear pts, then the isometry must be the idty.

then the isometry must be the idty.

proof;

Spse an isometry fixes three non collinear pts A, B & C.

Then it fixes every pt on each line $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ & $\overleftrightarrow{AC}$ by the above theorem.

Hence, it fixes ΔABC.

Consider arb. pt Q on the plane, necessarily it lies on the line that intersects ΔABC in two distinct pts.

Thus, Q is on a line containing two fixed pts of the given isometry & therefore Q must also be fixed. --(by the above thm).

Hence, the isometry is an idty.

Eg. Let α be an isometry with $\alpha((2,1)) = (2,1)$, $\alpha((0,3)) = (0,3)$ & $\alpha((0,0)) = (0,0)$. Then $\alpha((3,4)) = (3,4)$

Theorem: If α & β are isometries $\ni$ $\alpha(P) = \beta(P)$, $\alpha(Q) = \beta(Q)$ & $\alpha(R) = \beta(R)$ for three non-collinear pts P, Q & R, then $\alpha = \beta$.

Proof:

Spse α & β are isometries $\ni$ $\alpha(P) = \beta(P)$, $\alpha(Q) = \beta(Q)$ & $\alpha(R) = \beta(R)$ for non collinear pts P, Q & R.

Apply β^{-1} to both sides of the above three eqns

Thus, $\beta^{-1}\alpha$ fixes each of the three non collinear pt P, Q & R

$\therefore \beta^{-1}\alpha = \iota \Rightarrow \alpha = \beta$

Theorem: An isometry that fixes two pts is a reflection or the idty.

Proof:

Spse isometry α fixes distinct pts P & Q on line m.

Spse $\alpha \neq i$.

Claim: $\alpha = \sigma_m$

If $\alpha \neq i$ there is a pt $R \ni \alpha(R) \neq R$ (not fixed by i)

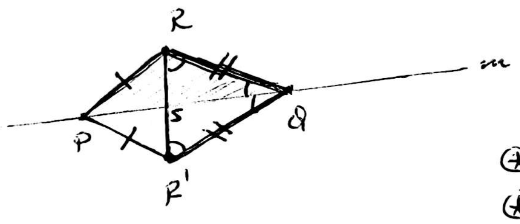
So R is off m otherwise $\alpha(R) = R$. (by thm ⊛ pp. 41)

& P, Q, R are three non-collinear pts.

Let $\alpha(R) = R'$

So $PR = PR'$ & $QR = QR'$ as α is an isometry.

~~Thus, m is ⊥ bisector of~~ $\cancel{\overline{RR'}}$



$\triangle PQR \cong \triangle PQR'$ SSS.

$\angle PQR \cong \angle PQR'$

$\triangle QRS \cong \triangle QR'S$ ASA

⊛ $\Rightarrow \angle QSR \cong QSR' \to 90°$

⊛ & $RS \cong R'S$

Thus, m is ⊥ bisector of $\overline{RR'}$.

Hence, $\alpha(R) = R' = \sigma_m(R)$

$\alpha(P) = P = \sigma_m(P)$

$\alpha(Q) = Q = \sigma_m(Q)$

$\Rightarrow \underline{\alpha = \sigma_m}$, by the above thm.

Eg Let α be an isometry with $\alpha((0,0)) = (0,0)$ & $\alpha((3,3)) = (3,3)$. What can you say about:

a. $\alpha((-1,-1))$ &

b. $\alpha((3,2))$?

Soln

a. α fixes distinct pts $(0,0)$ & $(3,3)$ on line m with eqn $y = x$.

Since $(-1,-1)$ lie on m

$\alpha((-1,-1)) = \underline{\underline{(-1,-1)}}$ (by the thm: An isometry that fixes distinct pts on a line fixes the line pt-wise.)

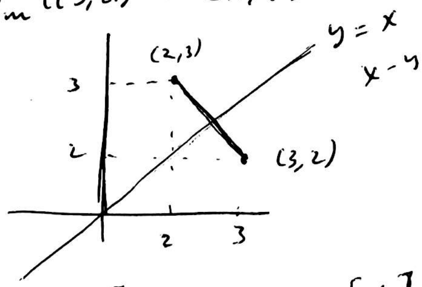
b. Since α fixes $(0,0)$ & $(3,3)$ & $(0,0)$, $(3,3)$ & $(3,2)$ are non collinear then by the above thm either:

i. α is an idty i.e.,

$\alpha((3,2)) = (3,2)$ OR

ii. α is a reflection; thus,

$\alpha((3,2)) = \sigma_m((3,2)) = (x', y')$



$x - y = 0$

$$x' = x - \frac{2a(ax+by+c)}{a^2+b^2}$$

$$y' = y - \frac{2b(ax+by+c)}{a^2+b^2}$$

$$\Rightarrow x' = 3 - \frac{2(1)[(1)(3)+(-1)(2)+0]}{2} = 3 - \frac{2[1]}{2} = 3 - 1 = \underline{\underline{2}}$$

$$y' = 2 - \frac{2(-1)[1]}{2} = 2 + 1 = \underline{\underline{3}}$$

$\Rightarrow (x', y') = \underline{\underline{(2,3)}}$

$\therefore \alpha((3,2)) = (2,3)$ OR

$\underline{\underline{\alpha((3,2)) = (3,2)}}$

Theorem: An isometry that fixes exactly one pt is a product of two reflections.

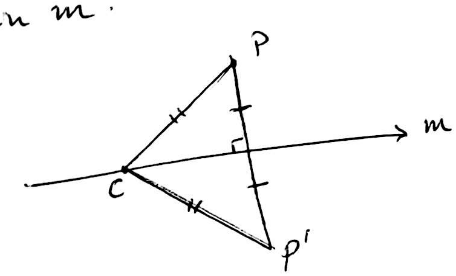
Proof:

Spse isometry α fixes exactly one pt C.

Let P be a pt d/t from C.

Let $\alpha(P) = P'$ & let m be the $\perp$ bisector of $\overline{PP'}$

Since $CP = CP'$ (since α is isometry), then C is on m.



So, $\sigma_m(C) = C$ & $\sigma_m(P') = P$

Then $\sigma_m \alpha(C) = \sigma_m(C) = C$

$\sigma_m \alpha(P) = \sigma_m(P') = P$

By the previous thm, $\sigma_m \alpha = i$ OR

$\sigma_m \alpha = \sigma_\ell$, where $\ell = \overleftrightarrow{CP}$

(*) However, $\sigma_m \alpha \neq i$ as otherwise $\alpha = \sigma_m$ & fixes more pts than C.

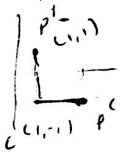
Thus, $\sigma_m \alpha = \sigma_\ell$ for some line ℓ (a $\overleftrightarrow{CP}$).

multiplying both sides of this eqn by σ_m on the left, we get

$$\underline{\underline{\alpha = \sigma_m \sigma_\ell}}$$

Q. Let α be an isometry that fixes only $(1,-1)$ & $\alpha((3,-1)) = (1,1)$.

Find the eq^n of α.



Sol^n

From the proof of the above thm, where m is the $\perp$ bisector of $\overline{PP'}$,

$\alpha = \sigma_m \sigma_l$

with $P = (3,-1)$, $P' = (1,1)$ & $C = (1,-1)$ &

$l = \overleftrightarrow{CP}$; $l: y = -1$ ~~$y = \frac{-1-1}{3-1}(x-1)+1$~~

$m: y = 3x-3$

$l: y = \frac{1}{2}x+2$

$\Rightarrow$ $l:$

The slope of $\overleftrightarrow{PP'} = -1$

" " $m = +1$

$m: y = (1)(x-1)-1 = x-2$

$\therefore$ $m: y = +x-2$

Thus, $\alpha = \sigma_m \sigma_l$, where $m: y = x-2$ & $l: y = -1$

Corollary: An isometry that fixes a pt is a product of at most two reflections.

Proof: (Ex.)

i. If the isometry fixes exactly one pt: It is a product of two reflections.

By the above thm it is a product of two reflections & hence the result.

ii. If it fixes two pts: It is a reflection or an idty.

It is a reflection or an idty. It is a product of two reflections.

So, since an idty i is a product of two reflection (i.e. $i = \sigma_m \sigma_m$ for any line m);

the result is clear.

iii. If the isometry fixes at least three pts:

a. if the pts are noncollinear, it is an idty.

Hence the result.

b. if the pts are collinear it reduces to case (ii).

Theorem: Every isometry is a product of at most three ~~pts~~ reflections.

proof:

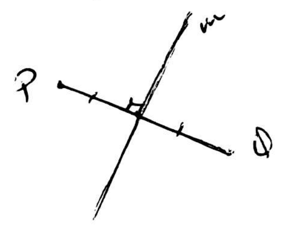
The idty is a product of two reflections.

spse a non-idty isometry α sends P to a d/t pt Q.

Claim: α is a product of at most three reflections.

Let m be the $\perp$ bisector of $\overline{PQ}$.

Then, $\sigma_m \alpha(P) = P$



Thus, by the above thm,

$\sigma_m \alpha$ must be a product ~~…~~ of at most two reflections. (since it fixes a pt P.)

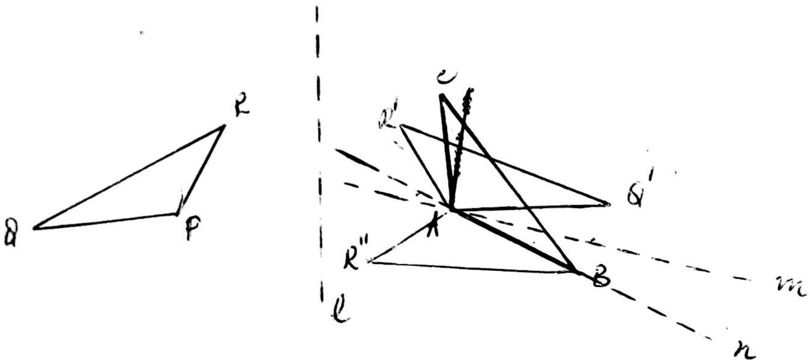
Hence, $\sigma_m \alpha = \beta$, β is a product of at most two refle.

$\Rightarrow \alpha = \sigma_m \beta$ since σ_m is involutory.

$\therefore$ α is a product of at most three reflections.

Theorem: If $\Delta PQR \cong \Delta ABC$, then there is a unique isometry $\alpha \ni$ $\alpha(P) = A$, $\alpha(Q) = B$ & $\alpha(R) = C$.

Proof:



Spse $\Delta PQR \cong \Delta ABC$. So, $AB = PQ$, $QR = BC$ & $PR = AC$.

If $P \neq A$, then let $\alpha_1 = \sigma_\ell$ where ℓ is $\perp$ bisector of $\overline{PA}$.

If $P = A$, then let $\alpha_1 = i$.

In either case, then $\alpha_1(P) = A$.

Let $\alpha_1(Q) = Q'$ & $\alpha_1(R) = R'$.

If $Q_1 \neq B$, then let $\alpha_2 = \sigma_m$ where m is $\perp$ bisector of $\overline{Q_1B}$.

$\to Q'$

In this case, A is on m as $AB = PQ = AQ'$.

If $Q_1 = B$, then let $\alpha_2 = i$.

$\to Q'$

In either case, we have $\alpha_2(A) = A$, & $\alpha_2(Q_1) = B$.

Let $\alpha_2(R') = R''$.

If $R_2 \neq C$, then let $\alpha_3 = \sigma_n$ where n is $\perp$ bisector of $\overline{R_2C}$.

$\to R''$

In this case, $n = \overleftrightarrow{AB}$ as $AC = PR = AR' = AR''$, & $BC = QR = Q'R' = BR''$.

If $R_2 = C$, then let $\alpha_3 = i$.

In any case, we have

$$\alpha_3(A) = A, \quad \alpha_3(B) = B \quad \& \quad \alpha_3(R'') = C.$$

Let $\alpha = \alpha_3\alpha_2\alpha_1$. Then

$$\alpha(P) = \alpha_3\alpha_2\alpha_1(P) = \alpha_3\alpha_2(A) = \alpha_3(A) = A$$

$$\alpha(Q) = \alpha_3\alpha_2\alpha_1(Q) = \alpha_3\alpha_2(Q') = \alpha_3(B) = B$$

$$\alpha(R) = \text{"} (R) = \quad i(R') = \alpha_3(R'') = C$$

As desired.

Note: If certain pts coincide, we may not need all three reflections.

Eg. Given $\Delta ABC \cong \Delta DEF$ where $A = (0,0)$, $B = (5,0)$, $C = (0,10)$, $D = (4,2)$; $E = (1,-2)$ & $F(12,-4)$

Find eqns of lines $\ni$ the product of the reflections in these lines takes ΔABC to ΔDEF.

Soln

Let α sends ΔABC to ΔDEF.

Now we need to find the eqns of lines l, m & n $\ni$

$\alpha = \sigma_n \sigma_m \sigma_l$

Let l be the $\perp$ bisector of $\overleftrightarrow{AD}$.

The slope of $\overleftrightarrow{AD} = \frac{1}{2}$

Thus, slope of line $l = -2$ & the mid pt of $\overline{AD} = (2,1)$ lies on l.

Hence the eqn of line l becomes:

$$y = -2(x-2) + 1$$

$$\Rightarrow y = -2x + 5 \iff 2x + y - 5 = 0$$

Let $\sigma_l(B) = B'(x', y')$ & $\sigma_l(C) = C'(x'', y'')$

Thus,

$$x' = 5 - \frac{2(2)[2(5) + 1(0) - 5]}{5} = 1$$

$$y' = 0 - \frac{2(1)[2(5) + 1(0) - 5]}{5} = -2$$

$$x'' = 0 - \frac{2(2)[2(0) + 1(10) - 5]}{5} = -4$$

$$y'' = 10 - \frac{2(1)[2(0) + 1(10) - 5]}{5} = 8$$

Thus $B' = (1,-2)$ & $C' = (-4,8)$

* Line m is the $\perp$ bisector of $\overline{B'C}$

Since $B' = C$, set $\sigma_m = 1$

$\Rightarrow \sigma_m(c') = c'' = c'$

Since $c'' = c' \neq F$, line n is the $\perp$ bisector of $\overline{C''F}$ i.e. $\overline{C'F}$.

& $n = \overleftrightarrow{DE}$

Thus, the eq^n of n becomes

$$y = \frac{4}{3}(x-4) + 2$$

$$\Leftrightarrow \quad -4x + 3y + 10 = 0$$

Hence, $\alpha = \sigma_n \sigma_m \sigma_\ell = \sigma_n \sigma_\ell$ where

n : $-4x + 3y + 10 = 0$ &

ℓ : $2x + y - 5 = 0$

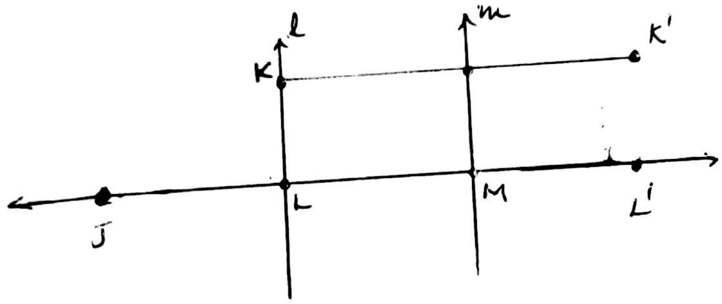
2.2.1. Product of Two Reflections

Theorem: If lines l & m are parallel, then $\sigma_m \sigma_l$ is a translation through twice the directed distance from l to m.

Proof:

Let l & m be distinct // lines.
Spse $\overleftrightarrow{LM}$ is commonly $\perp$ to l & m with L on l & M on m.

The directed distance from l to m is the directed distance from L to M.



Let K be a pt on l distinct from L.
Let $L' = \sigma_m(L)$ & $K' = \tau_{L,L'}(K)$

Then, $\tau_{K,K'} = \tau_{L,L'}$ &

$\square LKK'L'$ is a rectangle with m $\perp$ bisector of both $\overline{KK'}$ & $\overline{LL'}$.

So, $\sigma_m(K) = K'$.
Now, let $J = \sigma_l(M)$

Then, since L is the mid pt of $\overline{JM}$ &
M " " " " $\overline{LL'}$, we have

$\tau_{J,M} = \tau_{L,L'}$, where $\tau_{L,L'}$ is the translation through

Since an isometry is determined by any three non collinear pts.

$$\sigma_m \sigma_l = \tau_{L,L'} = \tau^2_{L,M}$$

Corollary: If line b is $\perp$ to line l at L & to line m at M, then

$$\sigma_m \sigma_l = \tau^2_{L,M} = \sigma_M \sigma_L$$

Proof:

From the above thm, we have

$$\sigma_m \sigma_l = \tau_{L,L'} = \tau^2_{L,M} \quad , \quad M \text{ is mid pt of } \overline{LL'}.$$

But $\tau_{L,L'} = \sigma_M \sigma_L$ where M is the mid pt of $\overline{LL'}$.

($\because$ we have a thm: If Q is mid pt of P & R, then $\sigma_Q \sigma_P = \tau_{P,R} = \sigma_R \sigma_Q$)

Then, $\sigma_m \sigma_l = \tau^2_{L,M} = \sigma_M \sigma_L$.

Corollary: Every translation is a product of two reflections in $\parallel$ lines & conversely, a product of two reflections in $\parallel$ lines is a translation.

Proof: (Ex. It is restatement of the thm.)

Eg. Let τ be a translation given by:

$$\tau((x,y)) = (x-2,\ y-4)$$

Find the eqns of n & m $\ni$

$$\tau = \sigma_m \sigma_n .$$

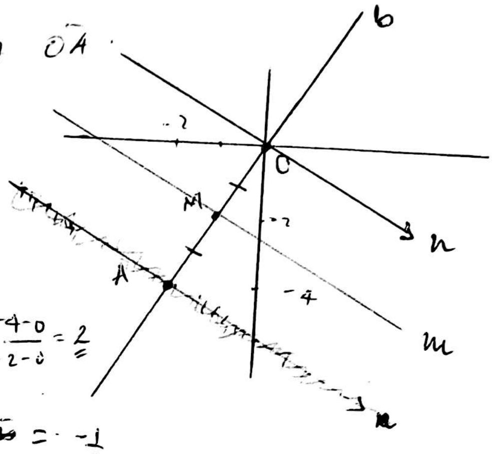
Sol'n

$\tau((x,y)) = (x-2,\ y-4)$

$\Leftrightarrow \quad \tau = \tau_{O,A}$, where $A = (-2,-4)$ & O is the origin.

Let M be the mid pt of $\overline{OA}$.

Then $M = (-1,-2)$.



Then slope of line $b = \frac{-4-0}{-2-0} = \underline{\underline{2}}$

Thus, the slope of n & m is $= -\frac{1}{2}$

Hence, the eqns of n & m are :

$n: \ y = \frac{-1}{2}x$ &

$m: y = \frac{-1}{2}(x-(-1)) - 2 = \frac{-1}{2}x - 5/2$

$\therefore \quad \tau = \tau_{O,A} = \sigma_M \sigma_O = \sigma_m \sigma_n \ \ni$

$m: \ y = \frac{-1}{2}x - \frac{5}{2}$ & $\Leftrightarrow x + 2y + 5 = \underline{\underline{0}}$

$n: \ y = -\frac{1}{2}x$

$\Leftrightarrow \quad x + 2y = 0$

⊛ General hint: If translation τ takes A to B, i.e.,

$\tau(A) = B$.

Find mid pt M of $\overline{AB}$.

Then find $n \perp$ to $\overleftrightarrow{AB}$ at A &

$m \perp$ " $\overleftrightarrow{AB}$ at M.

Thus, $\tau = \sigma_m \sigma_n$.

Quiz: If τ sends
$(-2,3) \to (2,5)$
$(-2,1) \to (4,5)$

Find l & m $\ni$

$\tau = \sigma_m \sigma_l$.

Theorem: If lines l, m & n are $\perp$ to line b, then there are unique lines p & q $\ni$

$$\sigma_m \sigma_l = \sigma_n \sigma_p = \sigma_q \sigma_n$$

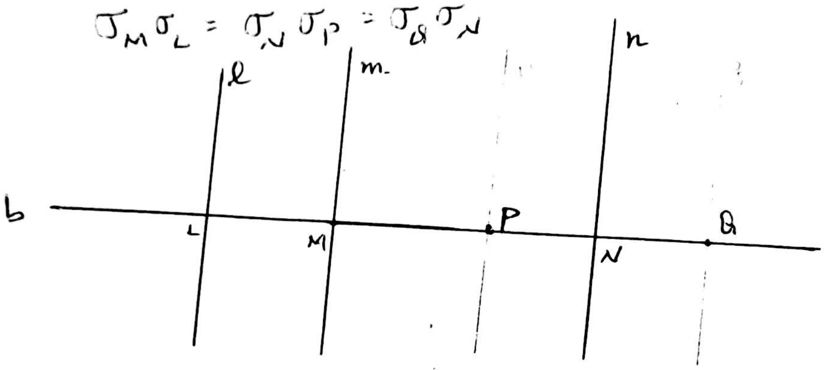
Further, the lines p & q are $\perp$ to b.

Proof:

Let b be $\perp$ to $\parallel$ lines ℓ, m & n at pts L, M & N resp.

Let P & Q be the unique pts on b $\ni$

$$\sigma_M \sigma_L = \sigma_N \sigma_P = \sigma_Q \sigma_N$$



Let line p be $\perp$ to b at P, & let line q be $\perp$ to b at Q.

Then, $\sigma_m \sigma_\ell = \sigma_M \sigma_L = \sigma_N \sigma_P = \sigma_n \sigma_p$ &

$$\sigma_m \sigma_\ell = \sigma_M \sigma_L = \sigma_Q \sigma_N = \sigma_q \sigma_n$$

Thus,

$$\sigma_m \sigma_\ell = \sigma_n \sigma_p = \sigma_q \sigma_n .$$

Corollary: If lines ℓ, m & n are $\perp$ to line b, then $\sigma_n \sigma_m \sigma_\ell$ is a reflection in a line $\perp$ to b.

Proof:

By the above thm:

$\exists$ some line $p \perp b$ $\ni$

$$\sigma_m \sigma_\ell = \sigma_n \sigma_p$$

$$\Rightarrow \quad \sigma_n \sigma_m \sigma_\ell = \sigma_p$$

(see eg ① next page.)

Eg 2. Let $\ell: y = 2x - 3$, $m: y = 2x - 7$ & $n: y = 2x$.

Find line p $\ni$ $\sigma_m \sigma_\ell = \sigma_n \sigma_p = \sigma_q \sigma_n$.

Q1. Let $l: y = x-2$, $m = y = x$ & $n: y = x+1$.

Find line p ∋

$$\sigma_m \sigma_l = \sigma_n \sigma_p = \sigma_q \sigma_n$$

Soln

Let's first get line $b \perp$ to all l, m & n, since they are ∥.

⇒ b has slope -1

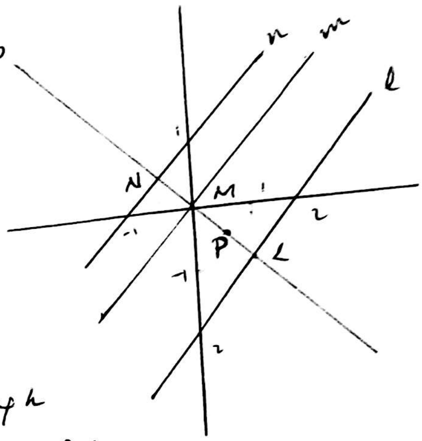
⇒ Any line whose slope is -1 is $\perp$ to all.

So, take $b: y = -x$

Then, $l \perp b$ at $L = (1,-1)$

$m \perp b$ at $M = (0,0)$

$n \perp b$ at $N = (-\frac{1}{2}, \frac{1}{2})$



Since $\sigma_m \sigma_l$ is a translation through twice the directed distance from l to m.

$\sigma_m \sigma_l$ translates any pt (x,y) by a distance of $2(LM)$ in the direction of $\overrightarrow{LM}$.

So, find pt P on b ∋ $\overrightarrow{LM}$ &

① $\overrightarrow{PN}$ is in the direction of $\overrightarrow{LM}$

② $PN = LM$.

⇒ $P = (\frac{1}{2}, \frac{-1}{2})$

∴ The required line p is the line $\perp$ to b at P.

Thus, $p: y = 1(x - \frac{1}{2}) - \frac{1}{2} = x - \frac{1}{2} - \frac{1}{2}$

⇒ $p: y = x - 1$

∴ $\sigma_m \sigma_l = \sigma_n \sigma_p$

check $(1,2) \longrightarrow (-1,4)$ in both cases

Theorem: If lines l & m intersect at pt C & directed angle measure of a directed angle from l to m is $\frac{\theta}{2}$, then

$$\sigma_m \sigma_l = \rho_{C,\theta}$$

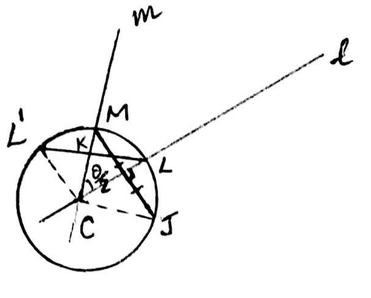
Proof;

Spse $\theta/2$ is the directed angle measure of one of the two directed angles from l to m.

Spse $-90° < \frac{\theta}{2} < 90°$ $[-\pi/2 < \frac{\theta}{2} < \frac{\pi}{2}]$.

Let L be a pt on l d/t from C.

Let pt M be the intersection of line m with a circle C_L with center C & radius $\overline{CL}$.



Let $L' = \rho_{C,\theta}(L)$.

Then L' is on the circle C_L (as $CL = CL'$ def of rotation).

m is $\perp$ bisector of $\overline{LL'}$, as $\Delta CL'K \cong \Delta CLK$.

So, $L' = \sigma_m(L)$

Let $J = \sigma_l(M)$

Then, l is the $\perp$ bisector of $\overline{JM}$.

So, J is on the circle C_L, & the directed angle measure from $\overline{CJ}$ to $\overline{CM}$ is θ.

Hence, $M = \rho_{C,\theta}(J)$

$\therefore$ $\sigma_m \sigma_l (C) = \sigma_m(C) = C = \rho_{C,\theta}(C)$

$\sigma_m \sigma_l (J) = \sigma_m(M) = M = \rho_{C,\theta}(J)$

$\sigma_m \sigma_l (L) = \sigma_m(L') = L' = \rho_{C,\theta}(L)$

Since, the non collinear pts L, J & C determine a unique isometry, we have

$$\sigma_m \sigma_l = \rho_{C,\theta}$$

So, $\sigma_m \sigma_l$ is the rotation about C through twice a directed angle from l to m.

Theorem ~~Note~~ From the above thm, we conclude that, every rotation is a product of two reflections in intersecting lines & conversely a product of two reflections in intersecting lines is a rotation.

i.e., spse $\rho_{C,\theta}$ is given.

Let l be any line through C & let m be a line through C $\ni$ the directed angle from l to m has directed angle measure $\theta/2$. Then

$$\rho_{C,\theta} = \sigma_m \sigma_l$$

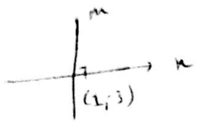
Eg Let n be a line with eqn $y = 2x - 5$. Find the eqn of the line m such that $\rho_{(1,-3),180°} = \sigma_m \sigma_n$

Soln

Since $\theta = 180°$, the directed angle from n to m is $90°$.

So line m is $\perp$ to n at $(1,-3)$

$\therefore$ the eq^n of line m is $y = \frac{-1}{2}(x-1) - 3 = \frac{-1}{2}x - 5/2$



eg. Let $\ell: y=0$ & $m: y=x$

If $\sigma_m\sigma_\ell = \rho_{C,\theta}$, where $\{C\} = \ell \cap m$, the $\rho_{C,\theta}((1,0)) = \_?$ Ans. $(0,1)$

Theorem: If lines ℓ, m & n are concurrent at pt C, then there are unique lines p & q $\ni$

$$\sigma_m\sigma_\ell = \sigma_n\sigma_p = \sigma_q\sigma_n$$

Furthermore, the lines p & q are concurrent at C.

Proof:

Given rays $\overrightarrow{CL}$, $\overrightarrow{CM}$ & $\overrightarrow{CN}$.

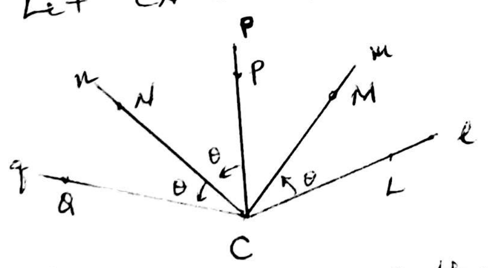
WOLG: Assume $m(\angle MCN) > m(\angle LCM)$.

Then there are unique rays $\overrightarrow{CP}$ & $\overrightarrow{CQ}$ $\ni$ the directed angles from $\overrightarrow{CL}$ to $\overrightarrow{CM}$,

the directed angle from $\overrightarrow{CP}$ to $\overrightarrow{CN}$ &

" " " " $\overrightarrow{CN}$ to $\overrightarrow{CQ}$ have the same angle measure θ.

Let $\overrightarrow{CN} = n$, $\overrightarrow{CP} = p$, $\overrightarrow{CL} = \ell$, $q = \overrightarrow{CQ}$, $\overrightarrow{CM} = m$



Then by the above thm

$\sigma_m\sigma_\ell = \rho_{C,2\theta}$, $\sigma_n\sigma_p = \rho_{C,2\theta}$, $\sigma_q\sigma_n = \rho_{C,2\theta}$

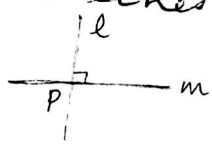
Thus, $\sigma_m\sigma_\ell = \sigma_n\sigma_p = \sigma_q\sigma_n$

Corollary 1. If lines l, m & n are concurrent at pt C, then $\sigma_n \sigma_m \sigma_l$ is a reflection in a line through C.

Proof: (Ex.)

Corollary 2. Halfturn σ_P is the product (in either order) of two reflections in any two lines $\perp$ at P.

Proof: (Ex.)



$\sigma_l \sigma_m = \sigma_P = \sigma_m \sigma_l$

Corollary 3. A product of two reflections is a translation or a rotation; only the idty is both a translation & a rotation.

Proof: (Ex.)

Defn: An isometry that is a product of an even no of reflections is said to be <u>even</u>; an isometry that is a product of an odd no of reflections is said to be odd.

* Note: Since an isometry is a product of reflections, then an isometry is even or odd.

Theorem: A product of four reflections is a product of two reflections

Proof: (Ex.)

Theorem: An even isometry is a product of two reflections. An odd isometry is a reflection or a product of three reflections. No isometry is both even and odd.

* The even isometry is a translation or a rotation
* An even involuto[illegible]

Case 1. proof: Let ℓ, m, n be concurrent at P. Then by the above thm, $\exists!$ line p $\ni$ $\sigma_m \sigma_\ell = \sigma_n \sigma_p$ $\Rightarrow$ $\sigma_n \sigma_m \sigma_\ell = \sigma_p$.

Case 2. proof: $\rho_{P,180} = \sigma_P$ $\forall P$ & $\rho_{P,180^\circ} = \sigma_m \sigma_\ell = \sigma_\ell \sigma_m$ for $\forall \ell, m \ni \ell \perp m$ by thm pp 50

Thus $\sigma_m \sigma_\ell = \sigma_P = \sigma_\ell \sigma_m$.

Case 3. (P79 #3) Any two lines may be $\parallel$ or intersect at a pt.

Thus, by the thms in pp 51 & 57 the product of two reflections is a translation or a rotation.

Since two distinct lines can not be $\parallel$ & intersect at a pt at the same time

so can't be both

& $\sigma_\ell \sigma_\ell = \tau_{P,P} = \rho_{P,0} = \iota$ $\forall$ line ℓ & $\forall$ point P.

$2b - a$

$b = \frac{a}{2}$

$a = \frac{b}{2}$

$\pi((\frac{a}{2}, \frac{b}{2})) = (2($