Groundwater Flow Modeling

Introduction

- A mathematical groundwater model is used to **simulate** and describe real-world groundwater flow.
- The mathematical model is developed by translating a conceptual model in the form of **governing equations**, with associated **boundary** and **initial conditions**.
- This model can then be solved using a **numerical model**, which is developed through the implementation of computer programs (codes).
- A groundwater simulation model is a **non-unique model** due to different sets of **assumptions** used for simplifying the mathematical description of groundwater flow.
  - ✓ assumptions such as homogeneity, isotropy, direction of flow, geometry of the aquifer, mechanisms of contaminant transport and its reaction.

Introduction

- Models can also simulate **more complicated problems** with higher accuracy, utilizing **more inputs**, **system parameters**, and **boundary conditions**.
- A successful model can result from a complete **site investigation** and **field data**.

- The model selection is a trade-off between the computational burdens including boundary conditions, grid discretization, time steps, the model accuracy, and ways to avoid truncation errors.
- The performance and efficiency of a model depends upon how accurate the mathematical equations **approximate the physical system being modeled**.

Introduction

- However, it should be noted that the developed model is an approximation and **not an exact simulation** of real-world groundwater flow.

- The first step in groundwater system modeling is the **conceptualization of the model**, which includes
  - ✓ a set of assumptions for the system's components,
  - ✓ the media properties, and
  - ✓ the transport processes in the system.

Process of Modeling

The process of aquifer modeling consists of the following activities:

- The parameters characterizing the physical framework of the aquifer and the system condition are identified.
- The hydrogeological parameters are estimated using field data in specific points.

- The spatial distribution of parameters is estimated utilizing interpolation/extrapolation methods.
- The entire estimated parameters and field data are utilized to make the conceptual model structure.

Process of Modeling

- A mathematical model is developed to describe the conceptual model by expressing the system condition using the **groundwater flow equations**.
- The mathematical model is transformed to a numerical model to find the **aquifer response** including hydraulic head or pollutants concentrations.

- The generated model is solved by numerical methods of solution.
- The model is **calibrated** for predicting the behavior of a considered system by simulating the **available field data**.
- The model is **verified** to eliminate errors resulting from the **numerical approximations**.

Process of Modeling

- The **sensitivity analysis** is done to select the estimates of model coefficients, which need to be estimated more accurately, and also to decipher the error bounds.
- The **management strategies** are suggested for **aquifer restoration** and **optimal utilization** of the groundwater resources.

- The model accuracy depends upon
  - ✓ the level of conceptualization and understanding of the groundwater system
  - ✓ the assumptions embedded in the derivation of the mathematical equations.

Mathematical Modeling

- Groundwater simulation models have been widely used in groundwater system analysis and management.
- These models generally require the **solution of partial differential equation**.
- Prediction of **subsurface flow**, **water table level**, solute transport, and simulation of natural or human-induced stresses are necessary for groundwater management.

- Mathematical models may be **deterministic**, **stochastic** (statistical), or a combination of both.
- In stochastic models, a range of predictions, based on probabilities of occurrence, is provided.

Mathematical Modeling

- Such predictions can be used in **planning** and **decision-making** processes for the groundwater resource.
- Stochastic models can also help to evaluate the **uncertainties of a system**.
- Deterministic models widely used for *solving regional groundwater problems* are based on **cause-and-effect relationship** of **known systems** and **processes**.

- Deterministic models can be further classified as **analytical** and **numerical**.
- Another class of mathematical models in solving groundwater flow is **analytical modeling**, which is an **easy method** to evaluate the **physical characteristics of an aquifer**.

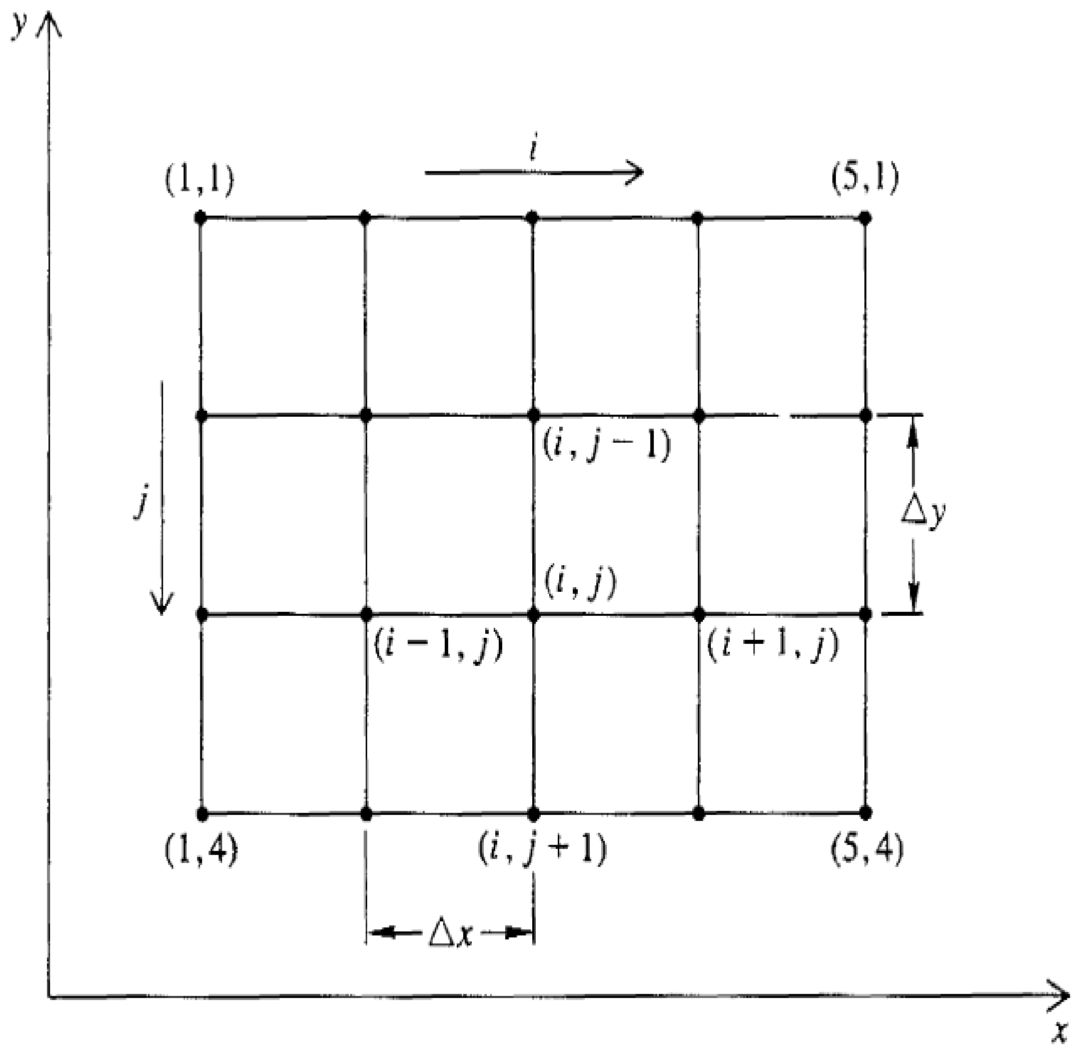
Mathematical Modeling

- This method of solution provides a **rapid preliminary analysis** of groundwater system utilizing a number of **simplifying assumptions**.
- These models cannot be used for **solving the problems** with the **irregularity** of the domain’s shape, the **heterogeneity** of the domain, and **complex boundary conditions**.

- The numerical models’ implementation is then carried out using computer programs for addressing more complicated problems.
- The exact solutions for some simple or idealized problems can be found by numerical models.
- These models can yield **approximate solutions** by **discretization of time** and **space**.
- Numerical models can be further classified as finite difference method (FDM), finite element method (FEM), and finite volume method (FVM)

Mathematical Modeling

- In FDM, the first derivative in partial differential equations.
- It is approximated by the difference between values of independent variables at **adjacent nodes** considering the **distance between the nodes**, and considering the **duration of time step increment** at two successive time levels.

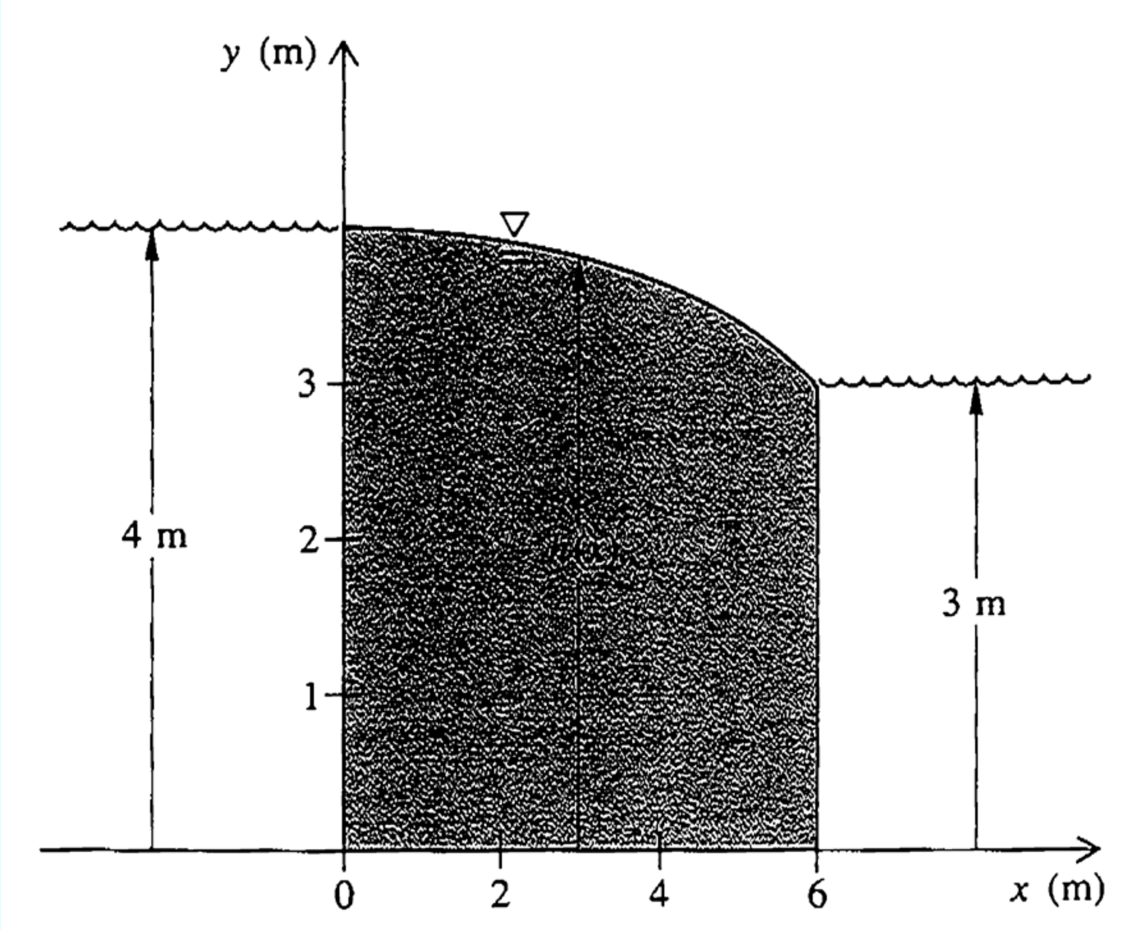


Finite difference grid showing index numbering convention.

Mathematical Modeling

Seepage through a dam:

A classical problem of unconfined flow systems is locating the top boundary of the saturated zone in an earthen dam. In the cross section shown in figure, the reservoir lake has an elevation of 4 m and the plung pool has an elevation of 3 m.



The bottom of the dam at y= 0 rests on impermeable bedrock. In this section, we solve the one-dimensional version of this problem using the Dupit approximation that flow is horizontal through the dam and that the upper saturated boundary intersects the reservoir and plunge pool levels at the two ends of the dam.

Mathematical Modeling

- The exact formulation of this seepage problem requires that the top surface be a no-flow boundary and that the head at each point on the boundary be equal to its elevation. In one dimension, the governing equation for this problem is $\frac{K}{2}\frac{d^2h^2}{dx^2} = -R$ with R = 0. The boundary conditions are that h=4 m at x= 0 and that h= 3 m at x = 6 m.

The analytical solution can be derived by integration of $d^2h^2/dx^2 = 0$

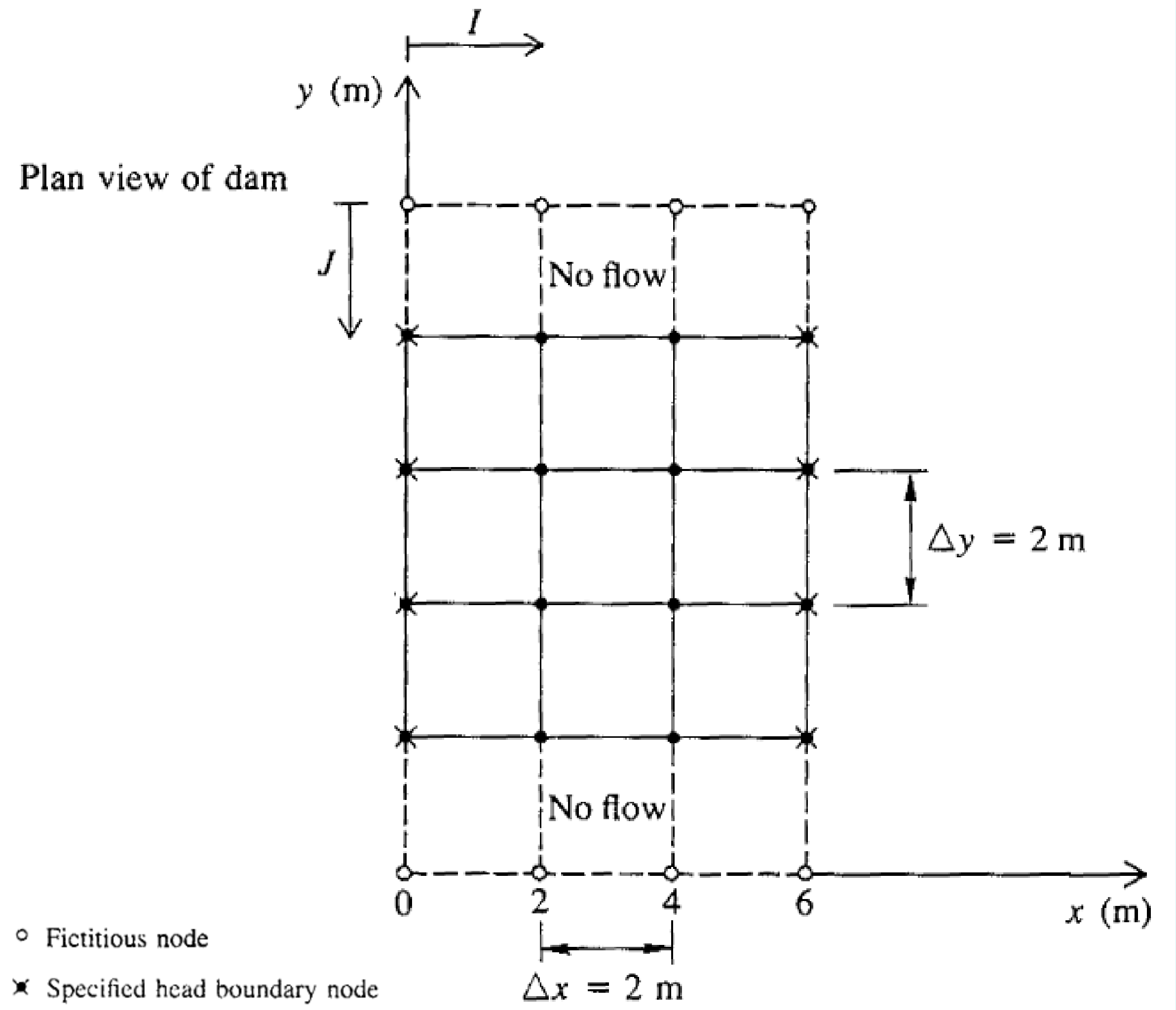
Leading to the general solution $h^2 = a_1x + a_2$, where a_1 and a_2 are constants determined by substitution of the boundary conditions.

The final analytical solution for the boundary conditions used in this problem is

$$h(x) = \sqrt{-1.17x + 16}$$

Mathematical Modeling

- We now present a finite difference solution.
- The boundaries at the top and bottom of figure shown are no-flow boundaries to force flow to be one-dimensional in the x-direction.
- Remember that use of the Dupit assumptions means that head does not vary vertically.
- The y-dimension of the grid is arbitrary in length, but we have chosen it to be 6 m simply for the sake of keeping the spacing between nodes such that $\Delta x = \Delta x = 2$ m.



Plan view of the finite difference grid used in the seepage through a dam problem. The two no-flow boundaries parallel to the x-axis keep the flow one-dimensional in the x-direction.

Mathematical Modeling

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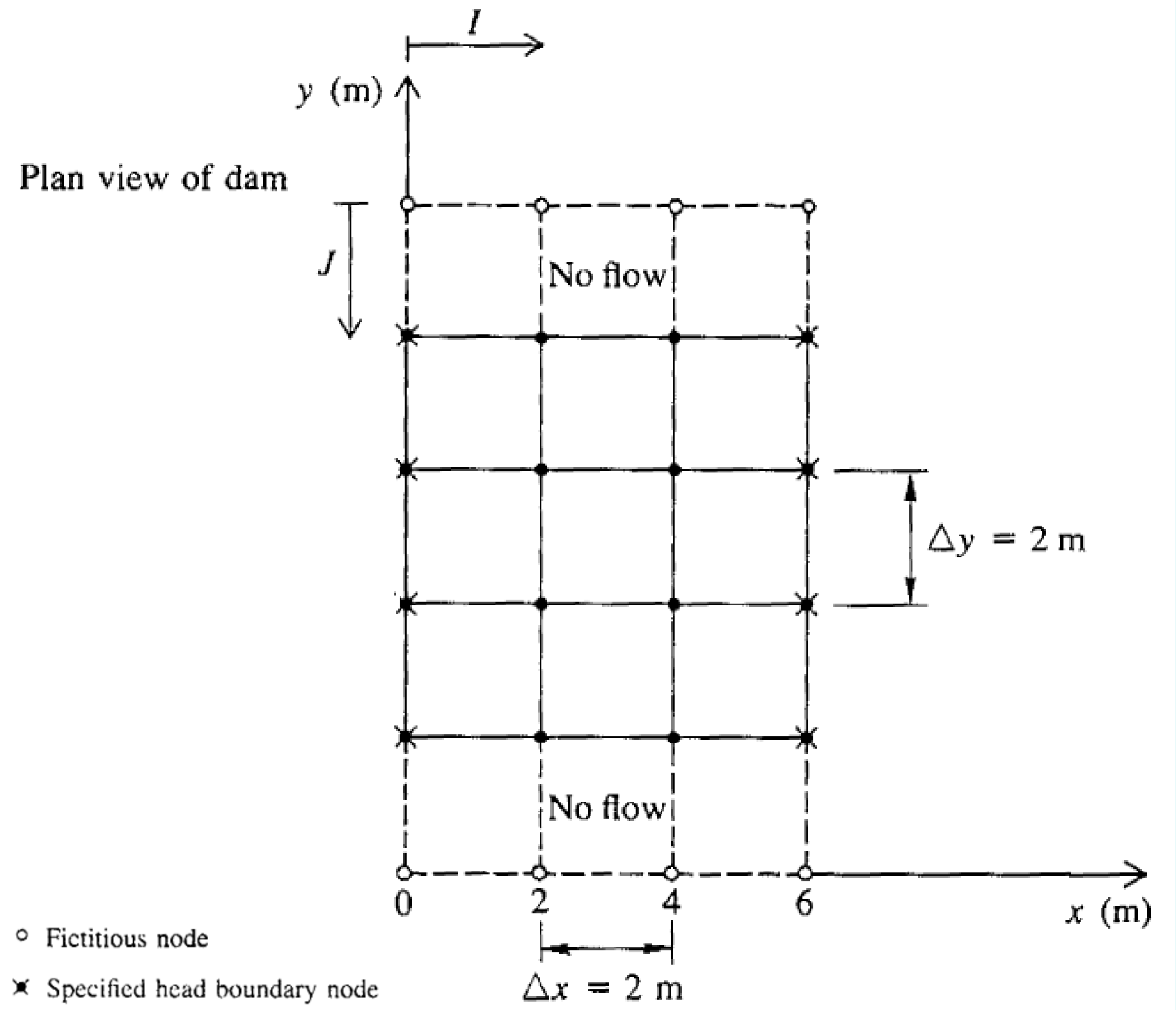


Plan view of the finite difference grid used in the seepage through a dam problem. The two no-flow boundaries parallel to the x-axis keep the flow one-dimensional in the x-direction.

In the finite difference approximation, derivatives are replaced by differences taken between nodal points. A central approximation to $\partial^2 h/\partial x^2$ at (x_0, y_0) is obtained by approximating the first derivative at $(x_0 + \Delta x/2, y_0)$ and at $(x_0 - \Delta x/2, y_0)$, and then obtaining the second derivative by taking a difference between the first derivatives at those points. That is

$$\frac{\partial^2 h}{\partial x^2} \simeq \frac{\dfrac{h_{i+1,j} - h_{i,j}}{\Delta x} - \dfrac{h_{i,j} - h_{i-1,j}}{\Delta x}}{\Delta x}$$

which simplifies to

$$\frac{\partial^2 h}{\partial x^2} \simeq \frac{h_{i-1,j} - 2h_{i,j} + h_{i+1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 h}{\partial y^2} \simeq \frac{h_{i,j-1} - 2h_{i,j} + h_{i,j+1}}{(\Delta y)^2} \tag{2.4}$$

According to Laplace's equation, we must add the preceding two equations and set the result equal to zero. If we consider a square grid of points such that $\Delta x = \Delta y$, then the finite difference approximation for Laplace's equation at the point (i, j) simplifies to

$$h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1} - 4h_{i,j} = 0 \tag{2.5}$$

Equation 2.5, or a generalized form of it, is the most widely used equation in finite difference solutions of steady-state flow problems, and it will form the heart of a computer program. There will be one equation of the form of Equation 2.5 for each interior point (i, j) of the problem.

If the finite difference equation, Equation 2.5, were solved for $h_{i,j}$, then

$$h_{i,j} = \frac{h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1}}{4} \tag{2.7}$$

That is, the value of $h_{i,j}$ at any point is the average value of head computed from its four nearest neighbors in the nodal array.

Mathematical Modeling

- The solution is shown in Figure. The heads in each column of the solution are identical to the others, as they should be, since the top and bottom no-flow boundary conditions constrain flow to be one-dimensional. The values computed from the numerical solution for the height of the water table are h = 3.70 m at x = 2 m and h= 3.37 m at x = 4 m.

4.00	3.70	3.37	3.00
4.00	3.70	3.37	3.00
4.00	3.70	3.37	3.00
4.00	3.70	3.37	3.00

- **Although each approach has some advantages and disadvantages, the FDMs are generally easier to program because of their conceptual and mathematical simplicity.**

Analytical Modeling

- An analytical model provides a solution for a mathematical description of a physical process.
- In this order, groundwater flow equations require several simplifying assumptions including the domain's shape, the boundary and initial conditions.

- Because of the simplifications inherent with analytical models,
  - it is not possible to account for field conditions that change with time or space (Mandle, 2002).
- Thus, the system under study may vary from actual conditions.

Analytical Modeling

- Analytical models have mostly been in use for particular sets of conditions and involve manually solving equations, such as
  - ✓ Darcy's law, the Theis equation,
  - ✓ generating solutions utilizing curve-matching techniques,
  - ✓ inverse solutions for interpretation of flow tests and
  - ✓ verification of numerical models.
- In most of the analytical methods for solution of two-dimensional (2D) groundwater problems, a suitable function is first determined to transform the problem from a geometrical domain into a domain with a more straightforward solution algorithm (Karamouz et al., 2003).

Numerical Modeling

In groundwater hydrology, numerical modeling can be used for several purposes:

- For the aquifer considered, investigating groundwater system dynamics and understanding the flow patterns.
- As an assessment and planning tool for evaluating recharge, discharge, aquifer storage processes, transport of contaminants, and quantifying sustainable yield.
- As a predictive tool for simulating future scenarios and to recreate the impacts of various activities.
- For planning and designing practical solutions for different development and management scenarios.
- As a groundwater management tool for assessing alternative policies and regulatory guidelines.
- As visualization tools for communicating key messages to the public and decision makers.

Numerical Modeling

- For the groundwater simulation, in the last few decades, a variety of numerical methods such as finite difference method (FDM), finite volume method (FVM), finite element method (FEM), method of characteristics (MOC), boundary element method (BEM), analytic element method (AEM), meshless method, etc. have been developed by engineers and scientists.

- Depending on the groundwater problem to be solved, each of these methods has its own advantages and disadvantages and the choice depends on the complexity of the problem, data available, computational facilities, and investigator's familiarity with the method.
- Out of the many available numerical methods for groundwater simulation, FDM, FEM, and AEM are the most popular numerical modeling techniques among engineers and scientists.

Numerical Modeling

The main features of the various numerical models are:

1. The models are solved only at specified points in the space and time domains defined for the problem (discrete values).
2. The PDEs that describe the groundwater flow are transformed by a set of mathematical equations in certain points as discrete values of the state variables.
3. The solution is for a specified set of numerical values of the various model coefficients rather than a general relationship in terms of these coefficients.
4. A computer program is employed to solve the large number of equations that must be solved simultaneously.

Finite Difference Method

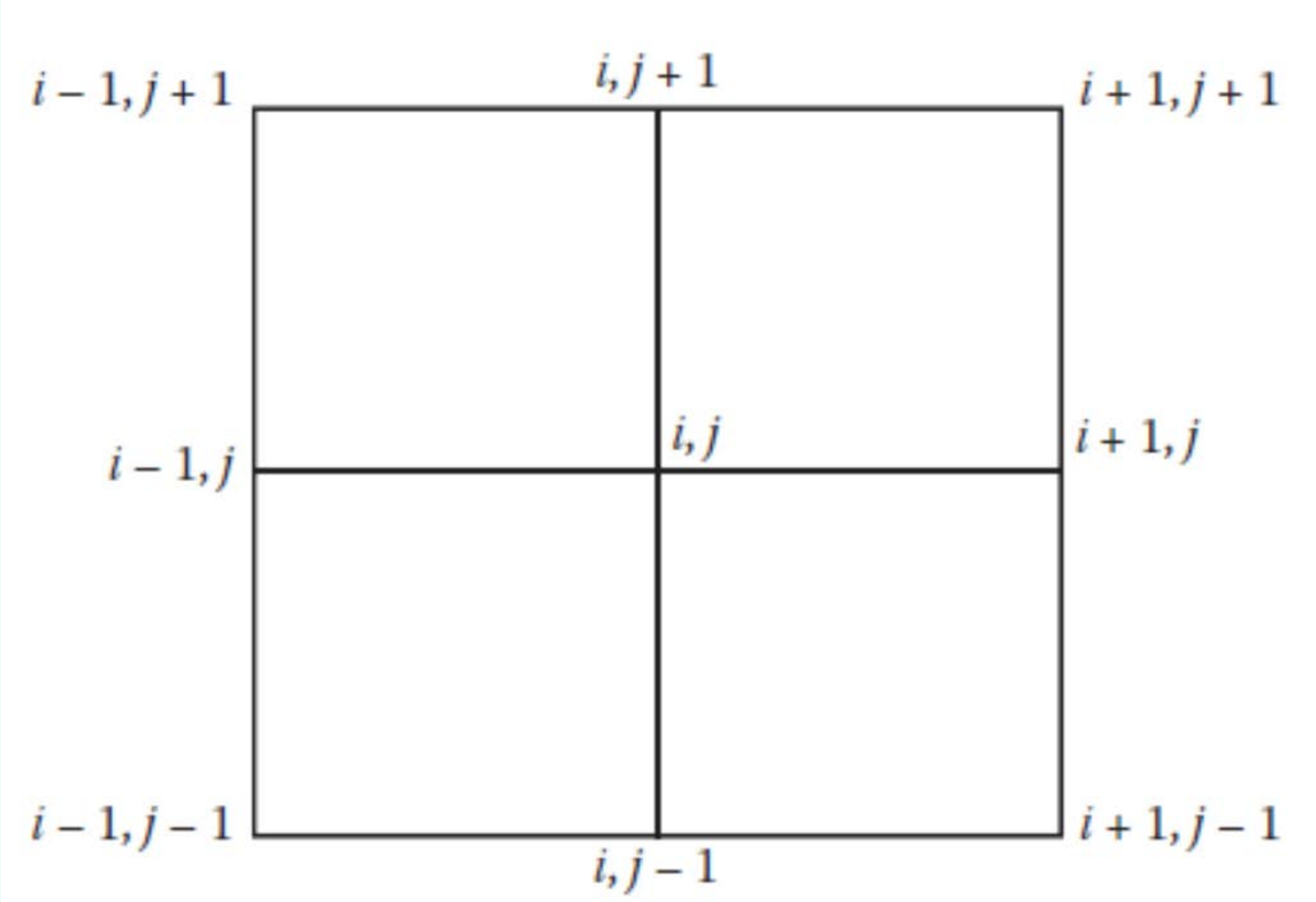
- Then the governing equation is approximated using a suitable scheme of difference at each node and a system of equations is formed.
- This system can be solved using direct or iterative solution techniques, after the appropriate application of the boundary conditions, to get the unknown groundwater head or concentration.
- Considering homogeneous isotropic confined aquifer in two dimensions, flow equation can written as

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T}\frac{\partial h}{\partial t} - \frac{R(x,y,t)}{T}$$

where *S* is the storage coefficient, *T* the transmissivity, and *R* the recharge or pumping rate.

Finite Difference Method

Corresponding FDM form in explicit form (refer to Figure below), the above flow Equation can be written as (Wang and Anderson, 1982)



$$\frac{h^n_{I+1,J} - 2h^n_{I,J} + h^n_{I-1,J}}{(\Delta x)^2} + \frac{h^n_{I,J+1} - 2h^n_{I,J} + h^n_{I,J+1}}{(\Delta y)^2}$$

$$= \left(\frac{S}{T}\right) \frac{h^{n+1}_{I,J} - h^n_{I,J}}{(\Delta t)} - \frac{R^n_{I,J}}{T} \qquad A$$

Figure. A typical finite difference grid

Finite Difference Method

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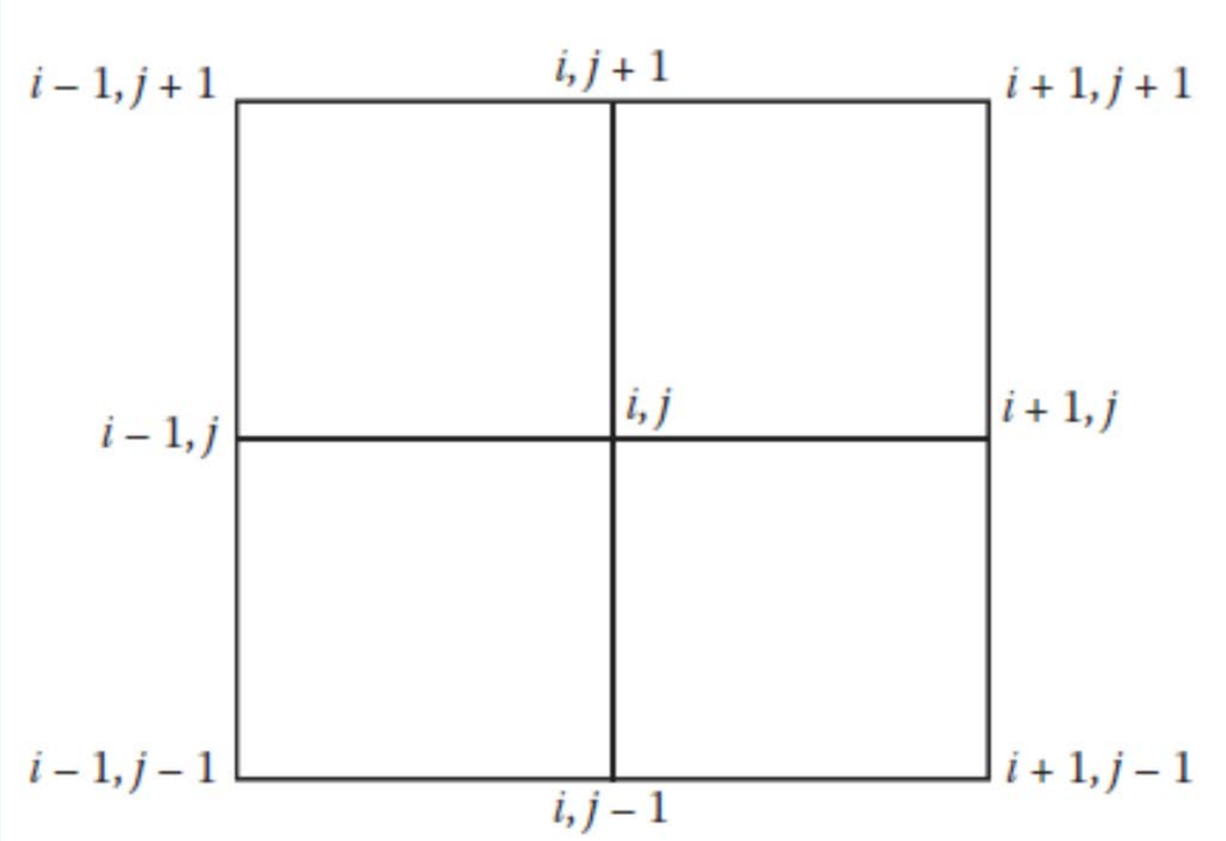


Figure. A typical finite difference grid

$$\frac{h_{I+1,J}^{n+1} - 2h_{I,J}^{n+1} + h_{I-1,J}^{n+1}}{\Delta x^2} + \frac{h_{I,J+1}^{n+1} - 2h_{I,J}^{n+1} + h_{I,J-1}^{n+1}}{\Delta y^2}$$

$$= \frac{S}{T}\frac{h_{I,J}^{n+1} - h_{I,J}^{n}}{\Delta t} - \frac{R_{I,J}^{n+1}}{T} \qquad \text{B}$$

Here $n + 1$ is the current time step and n the previous time step. In Equation A, as there is only one unknown, we can explicitly get the unknown head. In Equation B, as there are more unknowns, the equations are to be formed for all grids and simultaneously solved for all unknowns.

Finite Difference Method

- The continuous media is expressed in Equation 1.
- It is replaced by a finite set of discrete points in space, and also time is discretized:

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = S_s \frac{\partial h}{\partial t} \qquad (1)$$

- The partial derivatives are replaced by terms calculated from the differences in head values at each node.
- Simultaneous linear algebraic difference equations are achieved through the derivatives of variables.
- Their solutions lead to the head values at nodes and time by solving the linear algebraic equations.

Finite Difference Method

- The solutions obtained by this method are just approximations.
- Figure 1 shows a three-dimensional (3D) discretized aquifer. In this aquifer, terms of (*i*, *j*, *k*) address the location of each node in the mesh, where *i*, *j*, and *k* represent rows, columns, and layers, respectively.

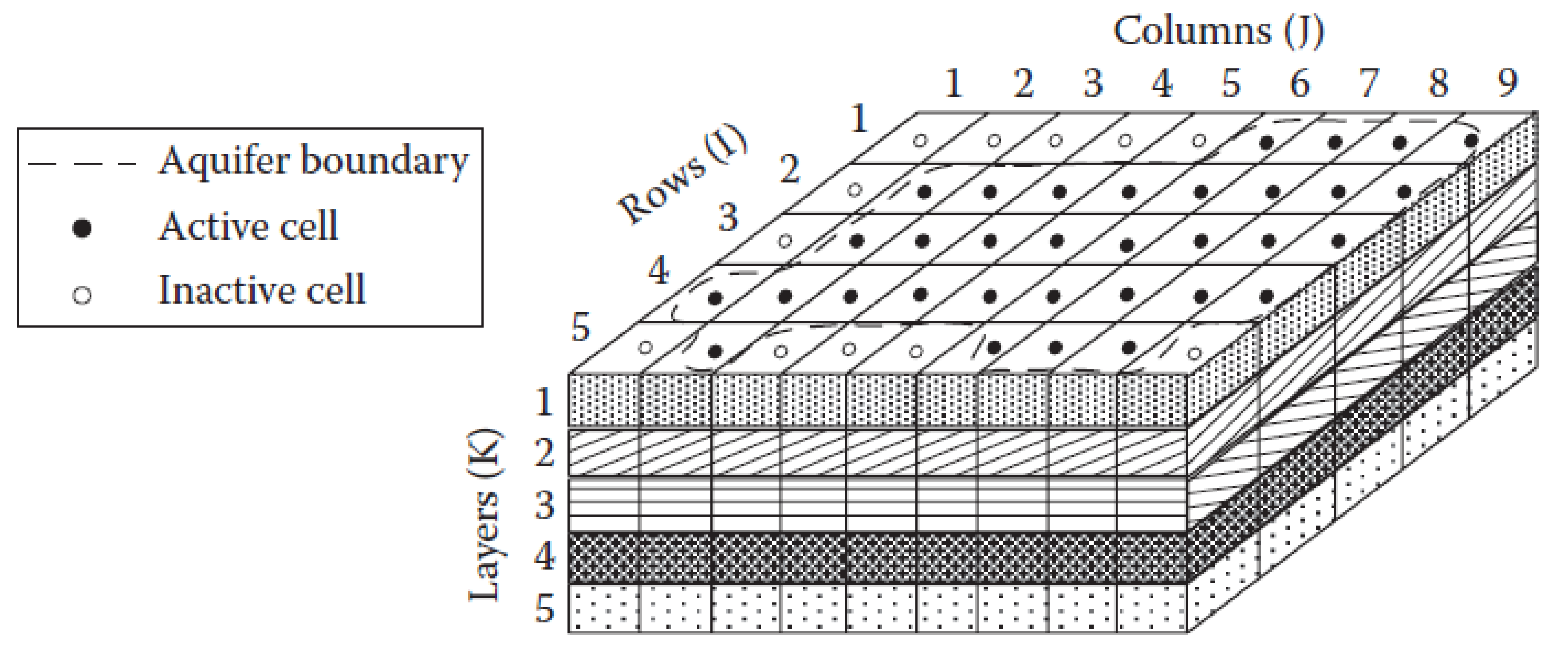


Figure 1. Illustration of discretization of continuous media into finite difference cells.

Finite Difference Method

- The head in each node is a function of both space and time and it is required to discretize the continuous media into discrete nodes and time.
- The calculation is solved for the nodes that are located within each cell.
- There are many schemes for locating nodes in cells such as a mesh-centered node or block-centered node as shown in Figure 2.

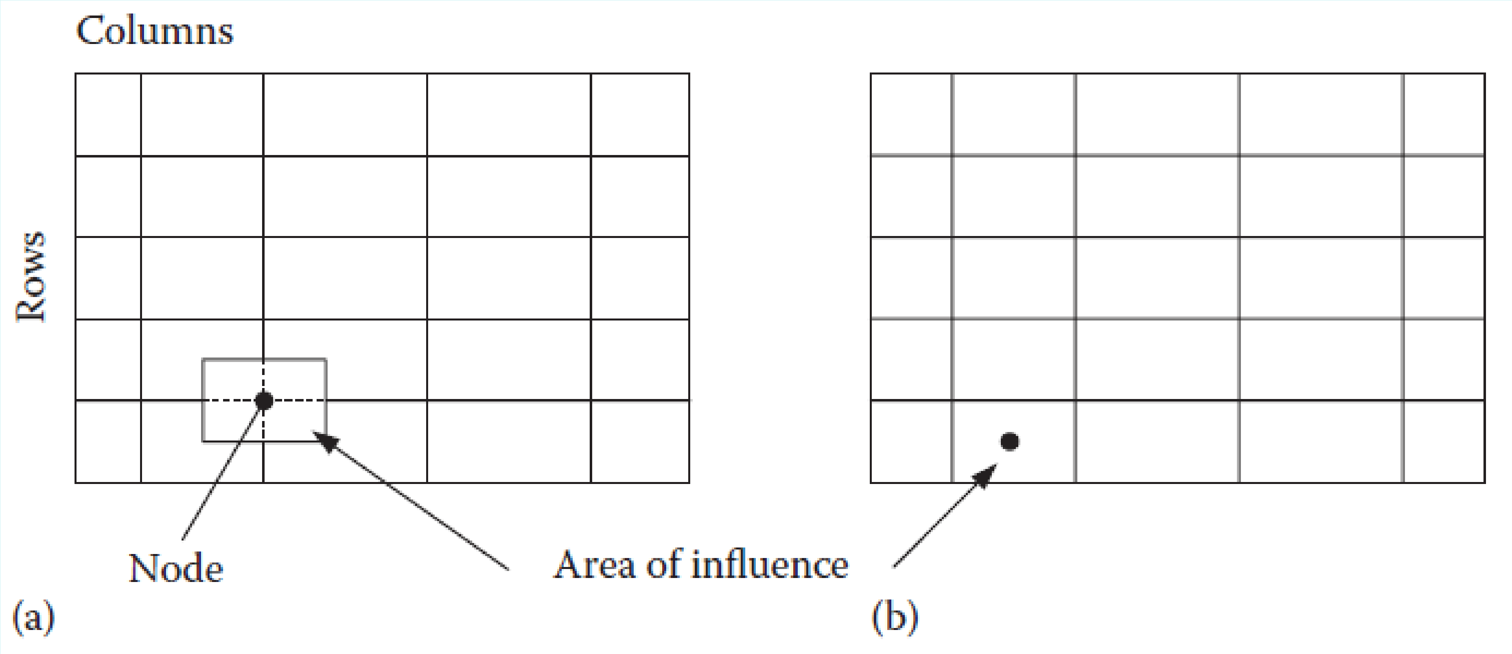


Figure 2. Illustration of 2D space discretization methods: (a) mesh-centered and (b) block-centered.

Finite Difference Method

- Finite-difference approximations for the first- and second-order derivatives of the groundwater flow and mass transport equations can be developed directly from the Taylor series approximations of the temporal and spatial derivatives.
- In the discrete groundwater model, the head or mass concentration at the nodal points $x + \Delta x$ or $x - \Delta x$ is expressed in terms of the state variables and all the derivatives are evaluated at an adjacent nodal point, x,

$$h(x - \Delta x) = h(x) - \left.\frac{\partial h}{\partial x}\right|_x \Delta x + \left.\frac{\partial^2 h}{\partial x^2}\right|_x \frac{\Delta x^2}{2!} - \left.\frac{\partial^3 h}{\partial x^3}\right|_x \frac{\Delta x^3}{3!} + \cdots \qquad (2)$$

By rearranging Equation 2,

Finite Difference Method

$$\left.\frac{\partial h}{\partial x}\right|_x = \frac{h(x+\Delta x)-h(x)}{\Delta x} - \frac{\Delta x}{2!}\left.\frac{\partial^2 h}{\partial x^2}\right|_x - \frac{\Delta x^2}{3!}\left.\frac{\partial^3 h}{\partial x^3}\right|_x - \cdots \qquad (3)$$

- The forward and backward first differences are obtained as

$$\left.\frac{\partial h}{\partial x}\right|_x = \frac{h(x+\Delta x)-h(x)}{\Delta x} + O(\Delta x) \qquad (4)$$

$$\left.\frac{\partial h}{\partial x}\right|_x = \frac{h(x)-h(x-\Delta x)}{\Delta x} + O(\Delta x) \qquad (5)$$

respectively.

Finite Difference Method

- Differencing Equations 2 and 3 and isolating the first-order derivative gives a central difference approximation to the first derivative:

$$\left.\frac{\partial h}{\partial x}\right|_x = \frac{h(x+\Delta x)-h(x)}{\Delta x} + O(\Delta x^2) \qquad (6)$$

- The error, $\bar{E}$, is on the order of Δx, which means a positive constant δ exists, which is independent of Δx, and $|\bar{E}| \leq \delta|\Delta x|$ for all sufficiently small Δxs.
- The errors of these approximations are called truncation errors.
- The truncation error of the central difference approximation is

$$\bar{E} = -\frac{\Delta x^2}{3!}\left.\frac{\partial^3 h}{\partial x^3}\right| = O(\Delta x^2) \qquad (7)$$

Finite Difference Method

- It is decreased by decreasing the values of Δx and Δt.
- An approximation to the second derivative can be obtained by Equation 2:

$$\left.\frac{\partial^2 h}{\partial x^2}\right|_x \cong \frac{h(x+\Delta x)-2h(x)+h(x-\Delta x)}{\Delta x^2} \qquad (8)$$

- The truncation error of the second-order approximation is

$$\bar{E} = \frac{\Delta x^2}{12}\left.\frac{\partial^4 h}{\partial x^4}\right|_{\xi} = O(\Delta x^2) \qquad (9)$$

- Higher-order derivative approximations can be developed using linear difference operators.

Finite Difference Method

- Similar discretization could be used for time intervals.
- Therefore, the approximations of $\partial h/\partial t$ using forward and backward differences are as follows:

$$\left.\frac{\partial h}{\partial t}\right|_t = \frac{h^{t+1} - h^t}{\Delta t} \qquad (10)$$

$$\left.\frac{\partial h}{\partial t}\right|_t = \frac{h^t - h^{t-1}}{\Delta t} \qquad (11)$$

respectively.

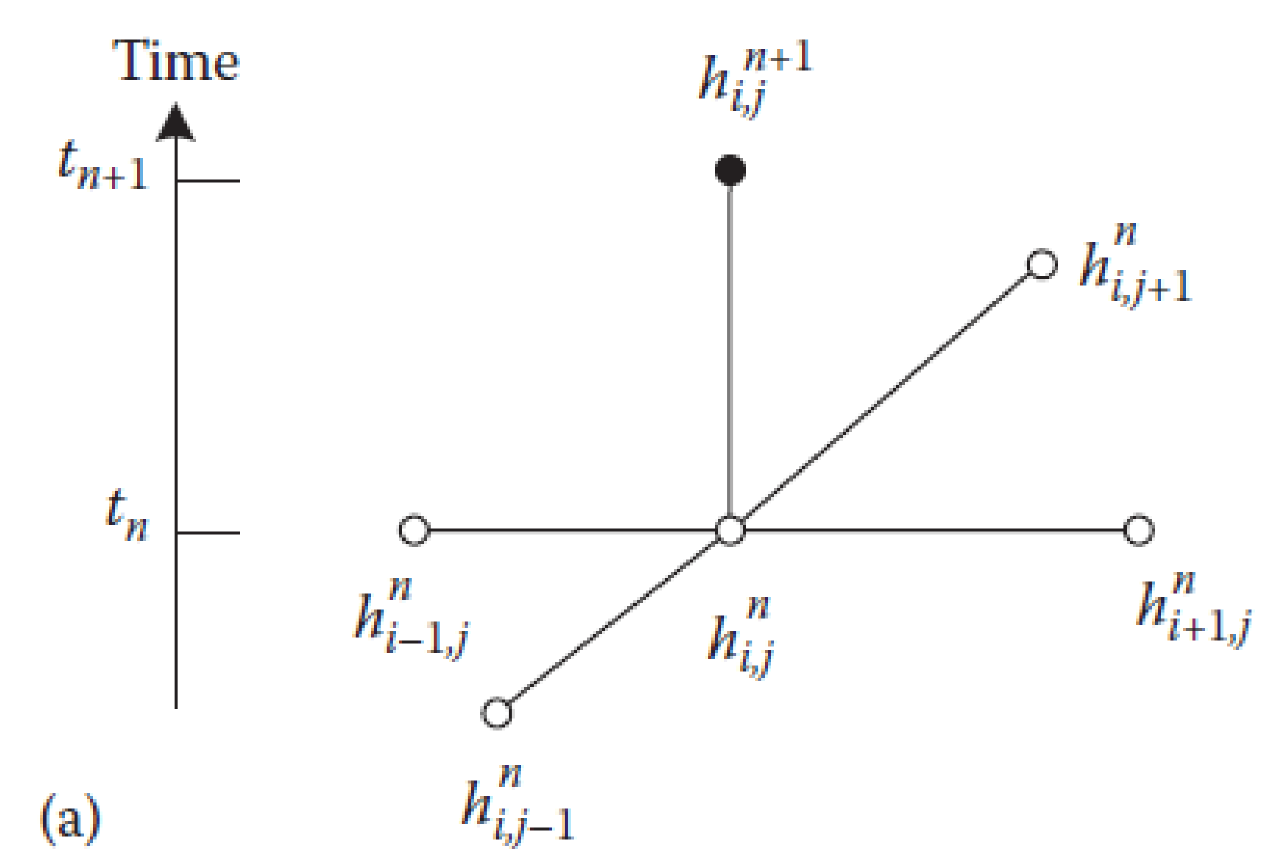
- Figure 3 shows time and space discretization at node (i, j) in a 2D finite difference grid, and illustrates the application of some simple finite-difference methods for solving groundwater flow equation.

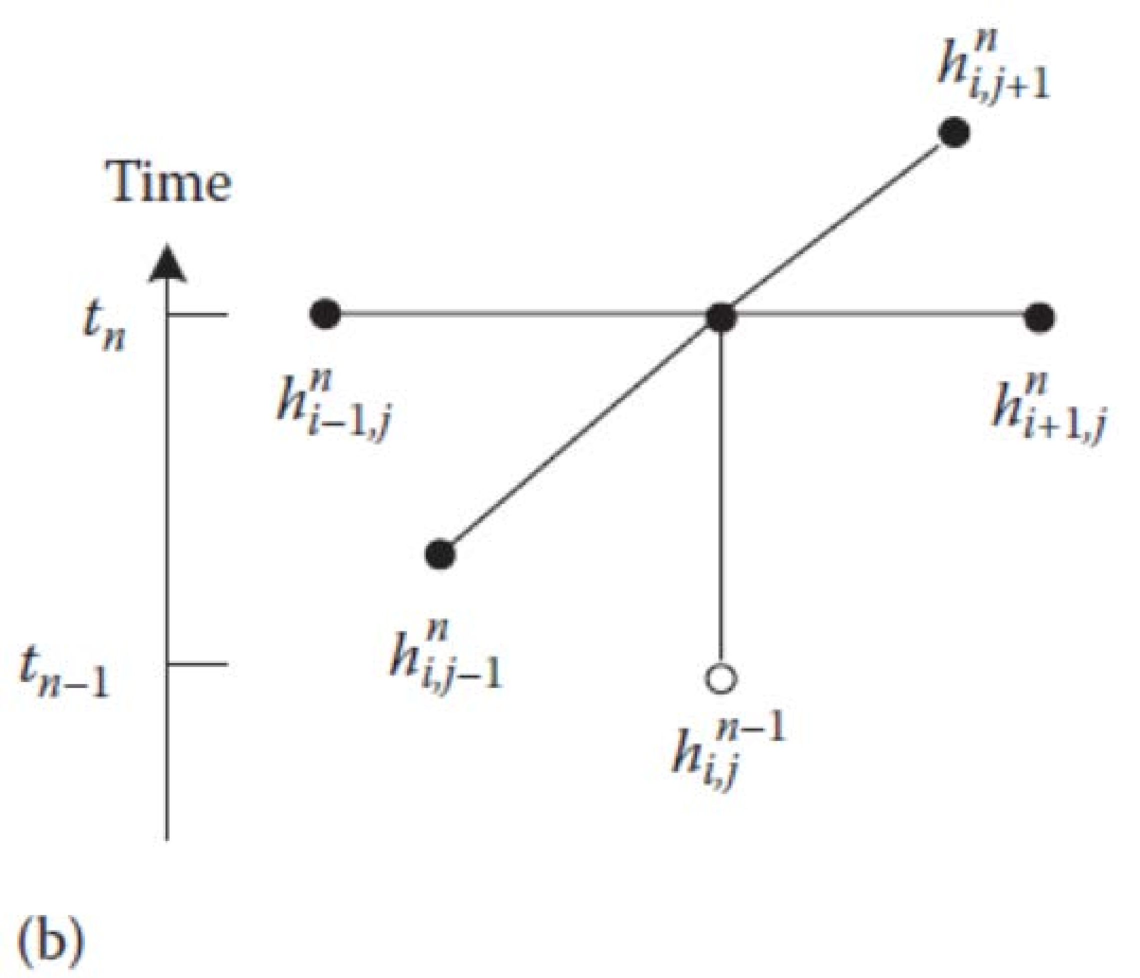
Finite Difference Method

Forward Difference Equation

- The head at point (i, j) at time step (n + 1) can be obtained utilizing the value of h at time step n and forward difference time derivative.
- Therefore, there is a finite difference equation for each node at time step n + 1 with only one unknown variable.
- As shown in Figure 6.3a, all values of head h are known at all spatial nodes at time n.
- This method is called forward difference or explicit method.

Finite Difference Method





Finite Difference Method

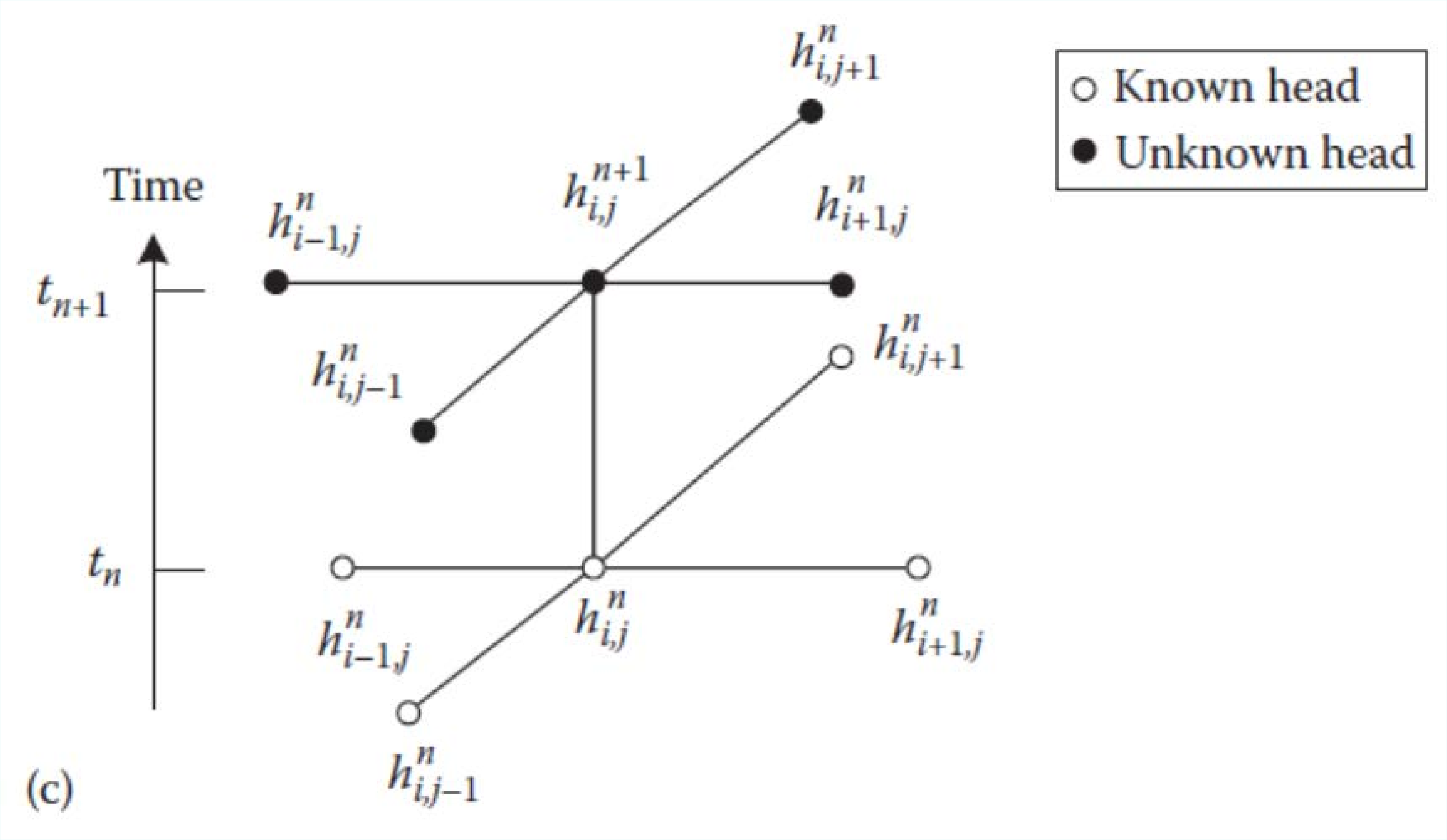


Figure 3 (a) The forward difference, (b) backward difference, and (c) Crank–Nicholson method in a 2D finite-difference grid.

Finite Difference Method

- In a 2D groundwater flow equation for a heterogeneous, anisotropic aquifer, Equation (1) can be written as

$$S_s\left(\frac{h_{i,j}^{n+1}-h_{i,j}^{n}}{\Delta t}\right)=K_{x(i-1/2,j)}\left(\frac{h_{i-1,j}^{n}-h_{i,j}^{n}}{(\Delta x)^2}\right)+K_{x(i+1/2,j)}\left(\frac{h_{i+1,j}^{n}-h_{i,j}^{n}}{(\Delta x)^2}\right)$$
$$+K_{y(i,j-1/2)}\left(\frac{h_{i,j-1}^{n}-h_{i,j}^{n}}{(\Delta y)^2}\right)+K_{y(i,j+1/2)}\left(\frac{h_{i,j+1}^{n}-h_{i,j}^{n}}{(\Delta y)^2}\right) \qquad 12$$

- In this equation, only h_{ij}^{n+1} is unknown. Equation (12) can be solved explicitly at each grid for the head at the new time level.
- Since the solution is dependent only upon known values of heads in the adjacent grids at the beginning of the time periods, the computation for h_{ij}^{n+1} in any grid can be made in any order without regard to values of h_{ij}^{n+1} for any other grid (Karamouz et al., 2003).

Finite Difference Method

For example, in a one-dimensional (1D) groundwater flow equation for a heterogeneous, isotropic, and confined aquifer, Equation (12) can be written as

$$S\left(\frac{h_i^{n+1}-h_i^n}{\Delta t}\right)=Kb\left(\frac{h_{i-1}^n-h_i^n}{(\Delta x)^2}\right)+Kb\left(\frac{h_{i+1}^n-h_i^n}{(\Delta x)^2}\right) \quad 13$$

or

$$h_i^{n+1}=\frac{T\Delta t}{S(\Delta x)^2}(h_{i-1}^n+h_{i+1}^n)+h_i^n\left(1-\frac{2T\Delta t}{S(\Delta x)^2}\right) \quad 14$$

- Explicit finite-difference equations are simple to solve but when time increments are too large, small numerical errors can propagate into larger errors in the next computational stages.
- A stable solution is ensured in 1D heterogeneous case if

Finite Difference Method

- A stable solution is ensured in 1D heterogeneous case if

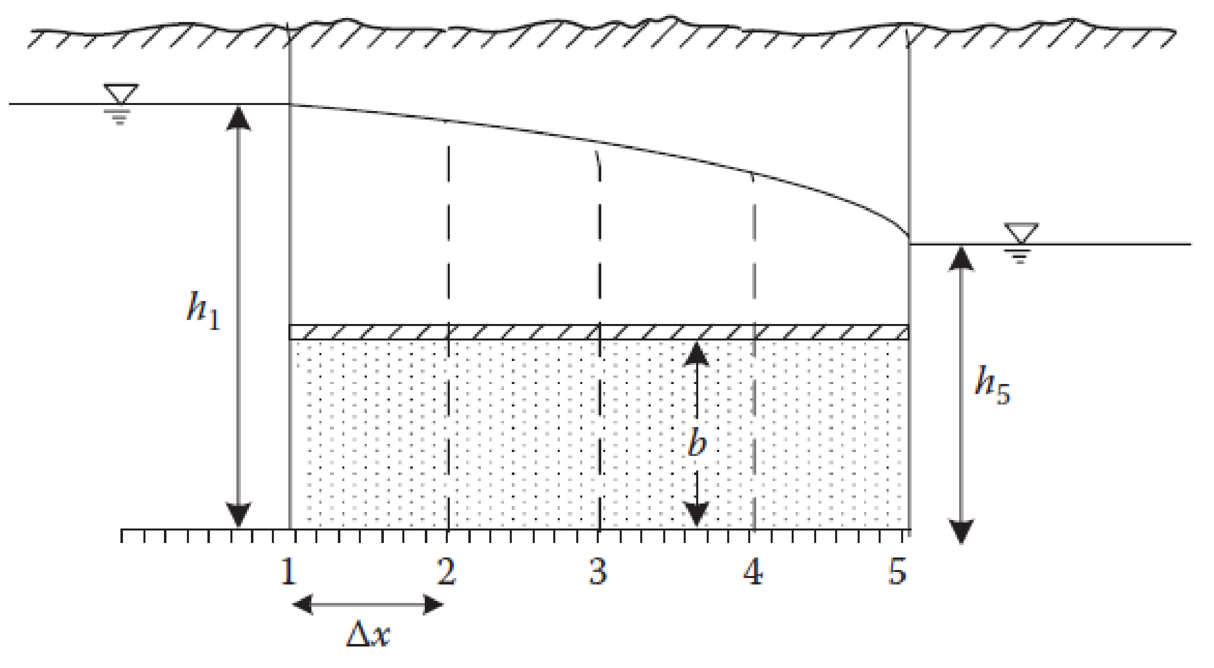
$$\frac{T\Delta t}{S(\Delta x)^2} < \frac{1}{2} \qquad 15$$

- Consequently, the time increment cannot be selected independently of the space increment.

Finite Difference Method

Example :

Consider a non-steady, 1D flow in a confined aquifer shown in Figure below. Let Δx = 3 m, b = 3 m, h_1 = 5 m, h_5 = 1 m for $t > 0$, K = 0.5 m/day, S = 0.03. The initial conditions are $h_1 = h_2 = h_3 = h_4 = h_5$ = 5 m. Determine the spatial variation of piezometric head.



Solution

To satisfy the stability requirement of Equation (15), the maximum time step Δt is computed as

$$h_i^{n+1} = \frac{T\Delta t}{S(\Delta x)^2}(h_{i-1}^n + h_{i+1}^n) + h_i^n\left(1 - \frac{2T\Delta t}{S(\Delta x)^2}\right)$$

Finite Difference Method

$$\Delta t < \frac{S(\Delta x)^2}{2T} = \frac{1}{2}\frac{(0.03)(3)^2}{0.5\times 3} = 0.09\,\text{day}$$

Therefore, the time increment is selected as 0.08 days. With assumption of $h_1 = h_2 = h_3 = h_4 = h_5 = 5\,\text{m}$, $h_1^{0.08} = h_2^{0.08} = h_3^{0.08} = h_4^{0.08} = 5\,\text{m}$, $h_5^{0.08} = 1\,\text{m}$. For the first time, step grid (4) is affected and Equation 6.14 becomes

$$h_4^{2\times 0.08} = \frac{T\Delta t}{S(\Delta x)^2}\left(h_3^{0.08} + h_5^{0.08}\right) + h_4^{0.08}\left(1 - \frac{2T\Delta t}{S(\Delta x)^2}\right)$$

$$= \frac{1.5\times 0.08}{0.03(3)^2}(5+1) + 5\times\left(1 - \frac{2\times 1.5\times 0.08}{0.03(3)^2}\right) = 3.22\,\text{m}$$

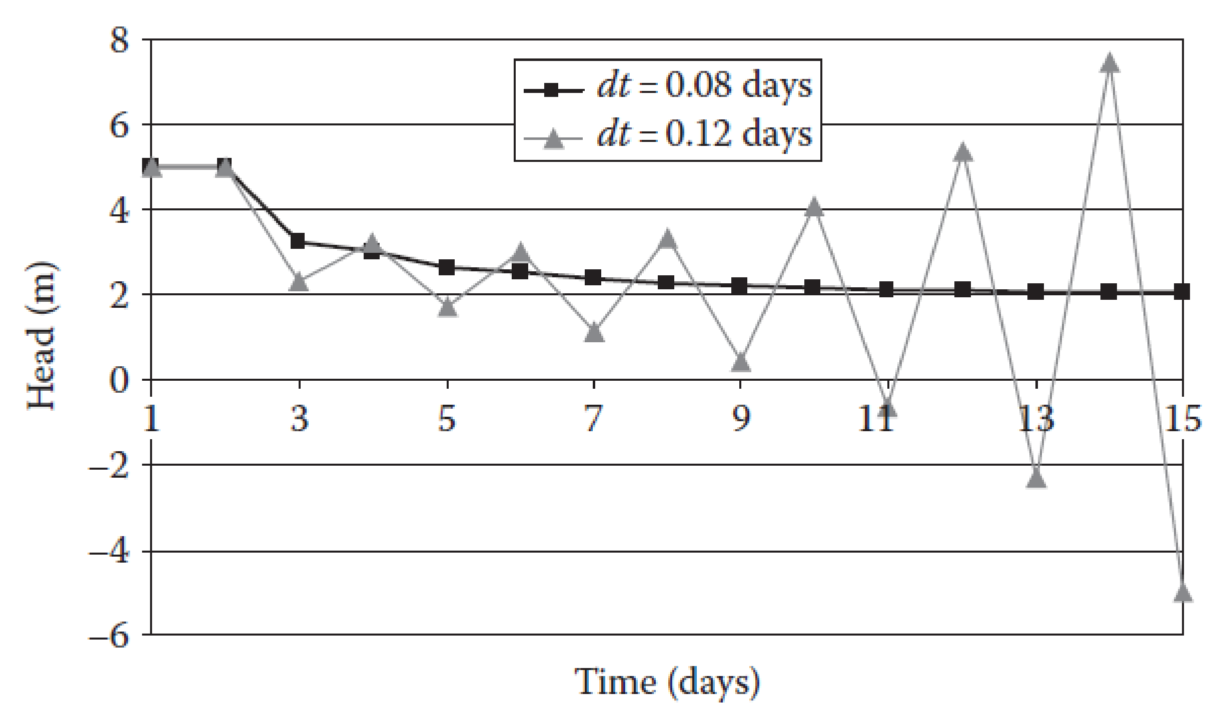
$$h_2^{2\times 0.08} = h_3^{2\times 0.08} = 5\,\text{m}$$

For the second time step $t = 3 \times 0.08$ for grid (4):

$$h_2^{3\times 0.08} = 5\,\text{m},\quad h_3^{3\times 0.08} = 4.21\,\text{m},\quad h_4^{3\times 0.08} = 3.02\,\text{m}$$

Finite Difference Method

The above process is repeated until the head at each grid is calculated at the desired time. To illustrate the stability problem, a set of calculations was made in which Δt = 0.12, was selected to be 0.12 days so that the expression for stability results in



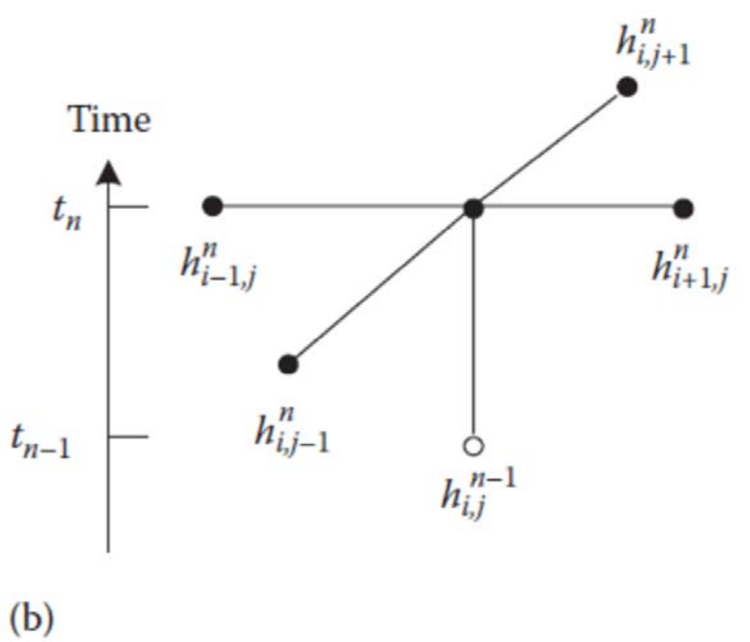
$$\frac{T\Delta t}{S(\Delta x)^2} = \frac{1.5 \times 0.12}{0.03(3)^2} > \frac{1}{2}$$

The calculated head in grid (4) as a function of time is shown in Figure. The computed values fluctuate with each time step for Δt = 0.12, giving completely erroneous results. Also, the amplitude of the fluctuation increases with increasing time.

Finite Difference Method

Backward Difference Equation

Figure shows the time derivative as a backward difference from the heads at time level, *n* – 1, which are the known heads. Therefore, the difference equation of each node will have five unknown variables. For a grid, which has *N* nodes, there is a system of *N* equations containing *N* unknown variables.



This system of equations can be solved simultaneously considering the boundary conditions. This method is called forward difference or implicit method. The implicit finite differential form of 2D groundwater equation, (Equation a) can be expressed as follows:

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = S_s \frac{\partial h}{\partial t} \quad \text{a}$$

Finite Difference Method

$$S_s\left(\frac{h_{i,j}^{n}-h_{i,j}^{n-1}}{\Delta t}\right)=k_{x(i-1/2,j)}\left(\frac{h_{i-1,j}^{n}-h_{i,j}^{n}}{(\Delta x)^2}\right)+k_{x(i+1/2,j)}\left(\frac{h_{i+1,j}^{n}-h_{i,j}^{n}}{(\Delta x)^2}\right)$$
$$+k_{y(i,j-1/2)}\left(\frac{h_{i,j-1}^{n}-h_{i,j}^{n}}{(\Delta y)^2}\right)+k_{y(i,j+1/2)}\left(\frac{h_{i,j+1}^{n}-h_{i,j}^{n}}{(\Delta y)^2}\right) \quad \text{b}$$

For example, in a 1D groundwater flow equation for a heterogeneous, isotropic, and confined aquifer, Equation (b) can be written as

$$S\left(\frac{h_i^{n}-h_i^{n-1}}{\Delta t}\right)=Kb\left(\frac{h_{i-1}^{n}-h_i^{n}}{(\Delta x)^2}\right)+Kb\left(\frac{h_{i+1}^{n}-h_i^{n}}{(\Delta x)^2}\right) \quad \text{c}$$

Rearranging Equation (c) so that all of the known values are on the right-hand side of the equal sign results in

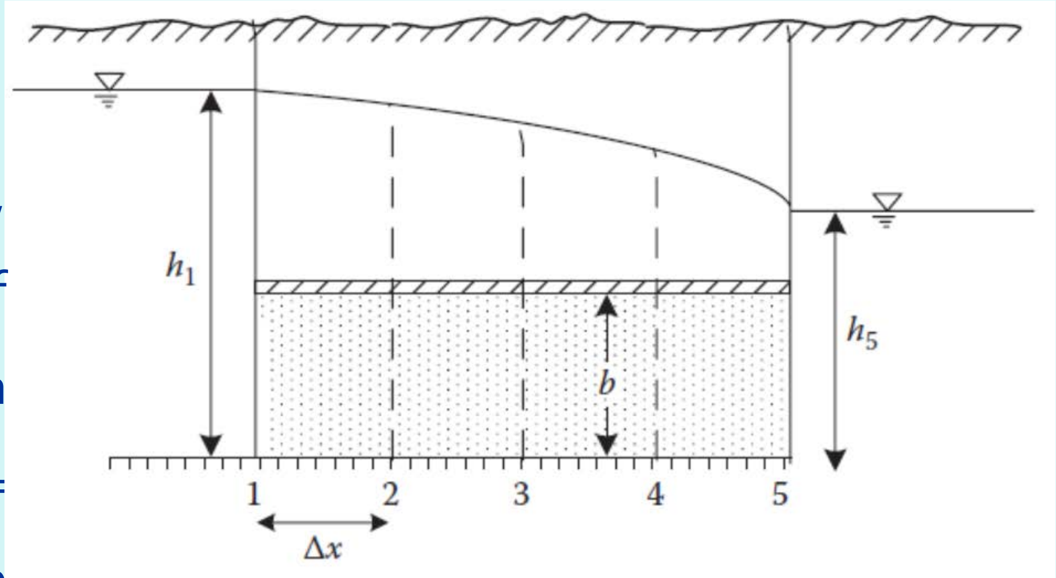
$$h_{i-1}^{n}-h_i^{n}\left(2+\frac{S(\Delta x)^2}{T\Delta t}\right)+h_{i+1}^{n}=-\frac{S(\Delta x)^2}{T\Delta t}h_i^{n-1} \quad \text{d}$$

Finite Difference Method

The head in grid (i) depends upon the value of head at time n in the adjacent grids, (i + 1) and (i − 1). Thus, Equation (d) represents a set of algebraic equations that must be solved simultaneously.

Example : Solve the previous example using the backward difference equation.

Consider a non-steady, 1D flow in a confined aquifer shown in Figure below. Let Δx = 3 m, b = 3 m, h_1 = 5 m, h_5 = 1 m for t > 0, K = 0.5 m/day, S = 0.03. The initial conditions are $h_1 = h_2 = h_3 = h_4 = h_5$ = 5 m. Determine the spatial variation of piezometric head.



Solution

Equation (d) is used for determining the three interior grids (2), (3), and (4). Grids (1) and (5) are boundary grids and values of head at these grids are specified as 5 and 1 m, respectively. With assumption of $h1 = h2 = h3 = h4 = h5$ = 5 m, and $h_1^{0.08} = h_2^{0.08} = h_3^{0.08} = h_4^{0.08}$ =1 = m, for the second time step Δt = 0.08 days, the following equations for grids (2), (3), (4) are obtained:

Finite Difference Method

$$5 - 4.25h_2^{0.16} + h_3^{0.16} = -11.25$$

$$h_2^{0.16} - 4.25h_3^{0.16} + h_4^{0.16} = -11.25$$

$$h_3^{0.16} - 4.25h_4^{0.16} + 1 = -11.25$$

Rearranging the above equations so that all known values are placed on the right-hand side and summing them up, we get

$$h_2^{0.16} = 4.9451, \quad h_3^{0.16} = 4.7665, \quad h_4^{0.16} = 4.0627\,\text{m}$$

MODFLOW

MODFLOW is a 3D, finite-difference groundwater model that was first released in 1984. MODFLOW is a modular 3D, finite-difference groundwater flow model, developed by the U.S. Geological Survey (McDonald and Harbaugh, 1988). This program is designed to simplify model development and data input for groundwater modeling to develop maps, diagrams, and text files. It is one of the most widely used groundwater simulation models. This model can be found and downloaded from USGS Web site. Many new capabilities have been added to the original model in recent versions.

MODFLOW

MODFLOW-2000 simulates steady and unsteady flow in an irregularly shaped flow system in which aquifer layers can be confined, unconfined, or a combination of confined and unconfined. Flow from external stresses, such as flow to wells, surface recharge, evapotranspiration, flow to drains, and flow through river beds, can be simulated. Hydraulic conductivities or transmissivities for any layer may differ spatially and be anisotropic (restricted to having the principal directions aligned with the grid axes), and the storage coefficient may be heterogeneous. Specified head and flux boundaries can be simulated by this model. It is also capable of modeling head-dependent flux across the outer boundary of the model, which allows water to be supplied to a boundary block in the modeled area at a rate proportional to the current head difference between a source of water outside the modeled area and the boundary block.

MODFLOW

In addition to simulating groundwater flow, the scope of MODFLOW-2000 has been expanded to incorporate related capabilities such as solute transport and parameter estimation. In this model, the groundwater flow equation is solved using the FDA. The flow region is subdivided into blocks in which the medium properties are assumed to be uniform. In plan view, the blocks are made from a grid of mutually perpendicular lines that may be variably spaced. Model layers can have varying thickness. A flow equation is written for each block, or cell. Several methods are provided for solving the resulting matrix problem; the user can choose the best one for a particular problem. Flow rate and cumulative volume, which are balanced from each type of inflow and outflow, are computed for each time step.

MODFLOW

The first step toward creating a model is the development of a grid system, then layers are added and parameters of hydraulic conductivity, initial head, storage coefficient, top elevation, and bottom elevation are specified for each cell. The number of stress periods over which the model will run and the time of their operation should be set.

Results could be analyzed through contour plots, if desired. Superimposing a grid over a graphic image or digitized line drawing is one of the most powerful features of graphic groundwater. These images are generally 2D plan views representing the area that is going to be modeled.

Data Requirement for a Groundwater Flow

In general, the data requirements for a groundwater flow model can be listed as below (Moore 1979):

- Geologic map and cross sections revealing the geometry, extent and boundary of the system
- Topographic map showing surface water bodies and divides
- Digital Elevation Model
- Contour maps and cross sections of the aquifer layers
- Isopach maps showing the thickness of the aquifers
- Water table and potentiometric maps for all aquifers
- Hydrographs for groundwater head and surface water levels and discharge rates
- Data for the hydraulic conductivity and transmissivity
- Storage data for the aquifers and confining beds
- Information about the colmation beds
- Spatial and temporal distribution of rates of evapotranspiration, groundwater recharge, surface water-groundwater interaction, groundwater extraction, and natural groundwater discharge (seepage, etc.)