UNIT FOUR
MATHEMATICS OF FINANCE

Unit Objectives

After a thorough study of this chapter, you will able to:

- Understand the concept of time – value of money.
- Identify the concept and computation of simple interest.
- Compute compound interest.
- Know the concept and business application of annuities.

Unit Introduction

Mathematics of finance is concerned with the analysis of time-value of money. The fundamental premise behind such analysis is the concept that entails the value of money changes overtime. Putting it in simple terms, the value of one birr today is not the same after a year. Suppose, if you deposit Birr 1000 at a bank for some period, you will find the sum grows to a higher sum at the time of withdrawal. The rationale behind is that the banks reinvest the deposits received at some other profitable ventures and hence, it is using depositors' money. As a result, it pays interest on depositors' money as compensation. Thus, in one way or another as value of money changes overtime, we find a difference between the present and future value of money. In sum, the difference arises because a rational being is assumed to invest/use money available on productive activity that result in a higher future sum and, the difference between the present and future value of money is referred to as time-value of money.

Mathematics of finance has an important implication in organizations as transactions and business dealings are mostly pecuniary. Such matters as lending and borrowing money for various purposes, leasing materials, accumulating funds for future use, sell of bonds are some of the cases that involves the concept of time value of money, Likewise, finance mathematics is equally important in our personal affairs. For example, we might be interested in owning a house, in financing our educational fees, having a car, having enough retirement funds etc. All these cases and others involve financial matter. Cognizant of this fact, we proceed to the study of mathematics of finance in this unit. In doing so, the unit is organized in to three main sections.

The first section advances to our study of simple interest and discounts. We further explore about compound interest and annuities in the second and third sections respectively.

Section One: Simple Interest and Discounts

Section Objectives:

After you study this section, you are expected to understand:

- How to compute simple interest
- Ordinary and exact simple interest
- Simple discounts
- Promissory notes and bank discounts

Section Overview:

5.1 Basic Concepts

5.2 Simple Interest and Its Computation

5.3 Ordinary and Exact Interest

5.4 Solving the Principal (P), Interest Rate (i) and Time

5.5 Simple Discount: Present Value

5.6 Promissory Notes and Bank Discount

5.1 Basic Concepts

In business, we usually pay some money for using services and goods. Such payments go by various names. For instance, the money we pay for hiring a taxi is known as fair. The amount we pay for education is called tuition fee. Likewise, the cost we pay for electric consumption commonly called electric charge. Back to our case, we also incur cost in using money for a certain period. This cost is referred to as *interest*. Thus, interest is the payment made for use of the principal (money) or a fee, which is paid for having the use of money. The amount of money that is borrowed or lent or invested or money available at hand at the beginning is called the *principal* and denoted by P. Interest is usually paid in proportion to the principal and the period of time over which the money is used. The percent of the principal that is charged for the use of the principal for a unit of time is called *the rate of interest* (interest rate, i). The length of time for which the principal is borrowed, lent or invested is called *the time or term of the loan* and commonly how symbolized by n. The *future or maturity value,* which is also denoted by F, is the sum of the principal and all the interest earned.

Based on computation of the respective interest, there are two types of interests. These are,

i. *Simple interest:* it is the return on a principal amount for one time period.

ii. *Compound interest:* it is the return on a principal amount for two or more time period, assuming that the interest for each time period is added to the principal amount at the end of each period and earns interest on all subsequent periods.

5.2 The Simple Interest

Interest that is paid solely on the amount of the principal P is called simple interest. Simple interest is usually associated with loans or investments that are short term in nature. In addition, it is always computed based on the original principal.

The Simple Interest Formula:

The computation of simple interest is based on the following formula.

$$I = p\,i\,n$$

Where, I = Simple interest (in dollars or birr)

P = Principal (in dollar, or birr) and it is the amount

i = Rate of interest per period (the annual simple interest rate)

n = Number of years or fraction of one year

In computing simple interest, any stated time period such as months, weeks or days should be expressed in terms of years. Accordingly, if the time period is given in terms of,

i. Months, then

$$n = \frac{\text{Number of months}}{12}$$

ii. Weeks, then

$$n = \frac{\text{Number of Weeks}}{52}$$

iii. Days, then

a. Exact interest

$$n = \frac{\text{Number of days}}{365}$$

b. Ordinary simple interest

$$n = \frac{\text{Number of days}}{360}$$

Maturity value (future value) represents the accumulated amount or value at the end of the time periods given. Thus,

Future value (F) = Principal (P) + Interest (I)

Example 5.1

A credit union has issued a 3 year loan of Birr 5000. Simple interest is charged at a rate of 10% per year. The principal plus interest is to be repaid at the end of the third year.

a. Compute the interest for the 3-year period.

b. What amount will be repaid at the end of the third year?

Solution

Given values in the problem

3 – Years loan = Principal = Birr 5000

Interest rate = i = 10% = 0.1

Number of years (n) = 3 years

a. I = p i n

I = 5000 x 0.1 x 3

I = Birr 1500

b. The amount to be repaid at the end of the third year is the maturity (future) value of the specified money (Birr 5000). Accordingly, F = P + I

F = 5000 + 1500

F = Birr 6500

Or, using alternative approach,

F = P + I

Then, substitute I = P i n in the expression to obtain

F = P + Pin

F = P (1 + in)

Consequently, using this formula we can obtain

F = 5000 (1+ (0.1x3)

F = 5000 x 1.3

F = Birr 1500

Example 5.2

A person "lends" Birr 10,000 to a corporation by purchasing a bond from the corporation. Simple interest is computed quarterly (four times a year) at a rate of 2% per quarter, and a check for the interest is mailed each quarter to all bondholders. The bonds expire at the end of 5 years, and the final check includes the original principal plus interest earned during the last quarter. Compute the interest earned each quarter and the total interest, which will be earned over the five-year life of the bonds.

Dear student, please try to solve the problem before going to the solution part.

Solution

Given values in the problem, P = Birr 10,000 i = 2% per quarter n = 5 years

Required:

Interest per quarter and interest over the five-year periods

Interest per quarter (one quarter) = Pin

= 10000 x 0.2 x 1

= Birr 200

There, at each quarter the interest earned on Birr 10,000 is Birr 200.

Interest over the five year period = pin

In the case, n represents all quarters with in 5 years. That is,

n = Number of years x 4

= 5x4 = 20

Then, Interest (I) = Pin

= 10000 x 0.02 x 20 = Birr 4000

Remark

The interest rate of 2% is given as at a quarterly rate. Hence, in computation of the interest we shall not change it in to annual rate. As long as the interest rate is provided in the desired time interval, we shall not make adjustment on the rate given as well as the period. Yet, if conversion from one time interval to another is demanded, we have to be consistent. For example, in the above example the i = 2%

is for a quarter. The yearly rate will be 2x4 = 8%. By using this annual rate, we can compute the total interest as follows.

$I = Pin$

$= 10000 \times 0.08 \times 5 = $ Birr 4000

Exercise 5.1

1. Suppose, a small handicraft enterprise has requested a two year loan of Birr 6500 from the commercial Bank of Ethiopia. If the bank approves the loan at an annual interest rate of 7.5%,
    b. What is the simple interest on the loan?
    c. What is the maturity value of the loan?
2. If the above loan (exercise 5.1) is offered at a rate of 21% and is due in 3 months, what is the maturity value of the loan?

5.1.2 Solution for P, i and n

Dear students, in the computation of simple interest we might be required to find out the value of the principal, interest rate and the time period in some cases. Such computation for P, i and n is simply made by driving the formula for the unknown values from the formula we have used for simple interest.

Example 5.3

1. How long must one leave Birr 300 invested in order to learn Birr 28 interest at 3% per year?
2. At what rate will Birr 150 produce interest of Birr 20.25 in 4.5 years?
3. What principal is required to produce interest of Birr 38.50 in two year at 3.5 % per year?

Solution

1. The question involves determining the time period which is enough to earn an interest of Birr 28 on Birr 300.

   The given values in the problem are P = Birr 300 I = Birr 28 i = 3% and n = ?

   $I = Pin$, now solve for n in this formula.

   $$n = \frac{I}{Pi} = \frac{28}{300 \times 0.30} = \frac{28}{9} = 4 \text{ years}$$

2. Given values in the problem

   P=Birr 150

   i=Birr 20.25

n=4.5 years

The required value is the rate of interest.

I = pin, Solve for i

$$i = \frac{I}{Pn} = \frac{20.25}{150x4.5} = \frac{20.25}{675} = 0.03 \quad \text{or} \quad 3\%$$

3. Given values in the problem

I = Birr 38.50

n =2 years

I = 3.5% per year

Required: Principal (P)

We can find out the value of P In the same manner with the above examples a follows.

I = pin, solve for P

$$P = \frac{I}{i \quad n} = \frac{38.5}{0.035 \times 2} = \frac{38.5}{0.07}$$

P = Birr 550

Thus, Birr 550 is required to produce interest of Birr 38.5 in 2 years at 3.5% rate.

5.4 Ordinary and Exact Interest

In computing simple interest, the number of years or time, n, can be measured in days. In such case, there are two ways of computing the interest.

i. *The Exact Method:* if a year is considered as 365 days, the interest is called exact simple interest. If the exact method is used to calculate interest, then the time is

n = number of days / 365

ii. *The Ordinary Method (Banker's Rule):* if a year is considered as 360 days, the interest is called ordinary simple interest. The time n, is calculated as

n = number of days/ 360

Example 5.4

Find the interest on Birr 1460 for 72 days at 10% interest using,

a. The exact method
b. The ordinary method

Solution

Given	a) P=1460	b) P=1460
P = Birr 1460	n=72/*365*	n= 72/*360*
n = 72 days	i= 0.1	i=0.1
i = 10% = 0.1	I=Pin =1460*72/365*0.1=28.8	I=Pin= 1460*72/365*0.1=29.2

5.5 Simple Discount: Present Value

The principal that must be invested at a given rate for a given time in order to produce a definite amount or accumulated value is called *present value*. The present value is analogous to a principal P. It involves discounting the maturity or future value of a sum of money to a present time. Hence, the simple present value formula is derived from the future value (F) formula as follows.

Future Value = Principal + Interest

F = P + I   but I = Pin

Thus,   F = P + Pin

F = P (1+ in)

Then from this, solve for P.

$$P = F/_{1+in}$$

If P is found by the above formula, we say that F has been discounted. The difference between F and P is called the simple discount and is the same as the simple interest on P.

Example 5.5

1. 90 days after borrowing money a person repaid exactly Birr 870.19. How much money was borrowed if the payment includes principal and arch nary simple interest at 9 ½ %?
2. What is the present value of Birr 645 due in 2 ½ years if the interest rate is 3%? What is the simple discount?

Solution

1. Given values in the problem,

   n in ordinary method = Number of days / 360

   = 90 /360

   n = 0.25

   F = the amount repaid = Birr 870.19

   i = 9 ½% = 9.5% = 0.095

Required:

The amount borrowed which is the same as simple present value, P.

$$P = F/_{1+in}$$

$= 870.19 / (1+ (0.095 \times 0.25))$

$P = 870.19 \div 1.024$

P = Birr 849.795

2. Given values in the problem,

F = Birr 645

n = 2.5 years

i = 3% = 0.03

$P = F/ 1+ in$

$P = 645/ 1+ (0.03 \times 2.5)$

$P = 645/1.075$

P = Birr 600

Exercise 5.4

Solve for the missing quantities.

Question	Present Value (P)	Simple discount (I)	Future value (F)	Rate (i)	Time (n)
1	Birr 400	Birr 18	?	?	½ years
2	Birr 600	Birr 60	?	4%	?
3	?	Birr 126	Birr 1026	?	3½ years
4	Birr 474.81	?	Birr 481,93	?	¼ years
5	Birr 2510.14	?	Birr 2566.62	4 ½%	?

5.6 Promissory Notes and Bank Discount

Definitions

A *promissory note* is a promise to pay a certain sum of money on a specified date. It is also considered as a written contract containing an unconditional promise by the debtor called the maker of the note to pay a certain sum of money to the creditor called the payee of the note, under terms clearly specified in the contract. Promissory note is unconditional in a sense that it gives the maker of the note an exclusive right either to sell, borrow, or discount it against the value of the note.

A *bank discount* is the amount of money received or collected after discounting a note before its due date. It is not unusual when borrowing money from a bank that one is required to pay a charge based on the total amount that is to be repaid (maturity value), instead of the principal used. If the maturity value is used in determining the charge for use of money, we say that the *promissory note (or simply the note) is discounted*. Consequently, a charge of loan computed in this manner is called *'Bank Discount'* and it is always computed based on the maturity value. Bank discount is the amount that is charged on maturity value.

Hence, the amount of money payable to the debtor or the amount that the borrower receives is called *'Proceed.'* The amount that the borrower is going to pay to the creditor (lender) is called *'maturity value.'* To further our understanding of this concept, let's develop mathematical expressions (formula) for computation of the variables at stake.

Proceed = Maturity Value – Bank Discount

Symbolically,

$$P = F - D, \text{ and } \quad D = Fdt$$

Where,

P = Proceed

F = Maturity value

D = Bank discount

d = Rate of discount

t = Time of discount

Now we can further elaborate the above formula for proceed. To begin with,

$$P = F - D, \quad \text{but} \quad D = Fdt$$

Therefore,

$$P = F - Fdt = F(1 - dt)$$

In sum, proceeds can be calculated by

$$P = F(1 - dt)$$

For example, if Birr 1000 is borrowed at 12% for 6 months, the borrower receives the proceeds, P, and pays back F = Birr 1000. The proceeds will be Birr 1000 minus the interest on Birr 1000. This will be:

$$P = 1000 - (1000 \times 0.12 \times 6/12) = \text{Birr } 940$$

Or, $P = 1000 (1 - (0.12 \times 6/12)$

$P = \text{Birr } 940$

Remark

i. Proceeds are an amount received now for payment in the future. Therefore, they are analogous to present value. Yet, proceeds are not equal to present value because the proceeds from a futures obligation to pay are always less than the present value of that obligation if, of course, the same rate of interest is used in both adulations.

ii. Proceeds should be completed when the interest rate is stated by the qualifier word as discount rate or a bank discount or interest deducted–in–advance, and present value should be computed where the interest is given without such qualifiers, discount.

iii. The computation of simple interest and bank discount is the same except in the former case principal and in the later case the maturity values are used for between trimmings the amount discount.

Having the idea of promissory notes and bank discounts, we may now progress to consider some illustrative problems.

Example 5.6

1. Find the bank discount and proceeds on a note whose maturity value is Birr 480 which is discounted at 4% ninety days before it is due.
2. A borrower signed a note promising to pay a bank Birr 5000 ten months from now.
   a. How much will the borrower receive if the discount rate is 6%?
   b. How much would the borrower have to repay in order to receive Birr 5000 now?

Dear student, please try to solve the above problem before going to the next part.

Solution

1. Given values in the problem

$F = \text{Birr } 480$

$d = 4\% = 0.04$

$t = 90 \text{ days or } 3 \text{ months} = 3/12 = 90/360 = 0.25$

$D = ?$ and $P = ?$

To find the value of the bank discount, we use the formula $D = Fdt$. Accordingly,

$$D = 480 \times 0.04 \times 3/12$$

$D =$ Birr 4.8 is the amount of bank discount.

In the same manner, the proceed can be obtained as follows.

$$P = F - D \quad \text{or} \quad P = F(1 - d\,t)$$

$$P = 480 - 4.8 \quad \text{or} \quad P = 480\,(1 - (0.04 \times 0.25)$$

$$P = \text{Birr } 475.2 \quad \text{or} \quad P = 480(0.99) = \text{Birr } 475.2$$

2. Given values:

$F =$ Birr 5000

$t = 10$ months $= 10/12$ year $= 0.83$ year

a. $d = 6\% = 0.06$

Proceeds $= P = ?$

$$P = F(1 - dt)$$

$$= 5000\,(1 - (0.06 \times 10/12)$$

$$= 5000(1 - 0.05)$$

$$P = 5000\,(0.95)$$

$P =$ Birr 4750 is the amount that the borrower receive now.

b. Given $d = 0.04$ or 4%

$t = 10$ months or $10/12 = 0.83$ years

$P =$ Birr 5000

$F = ?$

Now to find the value of F, simply we need to solve for F in the formula $P = F(1- dt)$.

Thus, $F = P \div (1 - d\,t)$

$$F = \frac{5000}{1 - (0.04 \times 0.83)} = 5000 / 09668$$

$F =$ Birr 5171.7

Exercise 5.5

1. A person signs a note promising to pay a bank Birr 1500 eight months from now and receives Birr 1350. Find the discount rate.
2. Find the bank discount and proceeds on a 120-day note for Birr 720 bearing 5% interest if discounted at 4% 90 days before it is due.

Section Two: Compound Interest

Section Objectives:

Upon completion of this section, you will be able to:

- Identify procedures of computing compound interest.
- Determine the compound amount of a sum of money and discounting of the same.

Section Overview:

- The Compound Interest and Its Formula
- Present Value of a Compound Amount

5.7 The Compound Interest and Its Formula

As it has been highlighted earlier, compound interest involves the case where interest earned during the earlier periods also earns interest during the later period. If, instead of being paid when due, the interest as investment is added to the principal and the sum is used as new principal, we say that compound interest is being used. Under this procedure, the interest for each period is added to the principal for purpose of computing interest for the next period. The sum to which the principal and interest on it grow during the period is called the *maturity or accumulated value of the principal*. The difference between the compound amount and the principal is called *compound interest*. The sum that is invested is called the *present value or the principal*. The time interval between the date on which the principal was invested and the date on which it is repaid is called the *term of the investment (loan)*.

If an amount of money, P, earns interest compounded at a rate of I percent per period it will grow after n periods to the compound amount F, and it is computed by the formula:

Compound amount formula: $F_n = P(1 + i)^n$

Where, P = Principal

i = Interest rate per compounding periods

n = Number of compounding periods (number of periods in which the principal earn interest)

F = Compound amount

A period, for this purpose, can be any unit of time. If interest is compounded annually, a year is the appropriate compounding or conversion or interest period. If it is compounded monthly, a month is the appropriate period. It is important to know that the number of compounding period/s within a year is/are used in order to find the interest rate per compounding periods and it is denoted by i in the above formula. Consequently, when the interest rate is stated as annual interest rate and is compounded more than once a year, the interest rate per compounding period is computed by the formula:

$i = j / m$, where j is annual quoted or nominal interest rate

m number of conversation periods per year or the compounding periods per year

$n = m \times t$, where t is the number of years

Example 5.7

Assume that we have deposited Birr 6000 at commercial Bank of Ethiopia which pays interest of 6% per year compounded yearly. Assume that we want to determine the amount of money we will have on deposit (our account) at the end of 2 years (the first and second year) if all interest is left in the savings account.

Solution

Give values in the problem, P = Birr 6000, j = 6% = 0.06, t = 2 years

m = compounded annually = i.e. only once

$n = m \times t = 1 \times 2 = 2$

$i = j / m = 0.06 / 1 = 0.06$

Then, the required value is the maturity or future value

$F = P(1 + i)^n$

$= 6000 (1 + 0.006)^2$

$=$ Birr $6000 (1.06)^2$

$=$ Birr 6741.6

Exercise 5.8

An individual accumulated Birr 30,000 ten years before his retirement in order to buy a house after he is retried. If the person invests this money at 12% compounded monthly, how much will be the balance immediately after his retirement?

Solution

Given values, P = Birr 30,000, t = 10 years, i = 12% = 0.12

m = compounded monthly = 12

$i = j / m = 0.12 / 12 = 0.01$

$n = m \times t = 12 \times 10 = 120$ and what is required is the Future Value F.

Then, $F = P (1 + i)^n$

$= 30,000 (1.01)^{120}$

$= 30,000 (1.01)^{120}$

$F =$ Birr 99,011.61

Having the understanding of how compound interest works and computation of future value, in subsequent example, we will consider how to determine the number of periods it will take for P birr deposited now at i percent to grow to an amount of F birr.

Example 5.9

A newly married couple has Birr 15000 to the purchase a house. For the type of house they are interested in buying they estimate a birr 20,000 down payment will be necessary. How long will the money have to be inserted at 10 % compounded quarterly to grow to Birr 20,000?

Solution

Given the values, P = Birr 15000, F = Birr 20,000

m = quarterly = 4 times a year

$j = 10\% = 0.1$

$i = j \div m = 0.1 \div 4 = 0.025$

a. We are required to determine the period within which Birr 15000 could grow to Birr 20,000. In accordance, given the formula $F = p (1 + i)^n$ we solve out for n.

$20,000 = 15,000 (1 + 0.1)^n$

$20,000 = 15000 (1.025)^n$, then divide both sides by 15,000.

$20,000 \div 15,000 = 1.025^n$

$1.33 = 1.025^n$

Now, we shall apply the rule of logarithm we considered in the preceding chapter. Hence, we can take the logarithm of both sides.

$\log 1.33 = \log 1.025^n$

Then, applying the rule of common logarithm (logarithm in base 10 = $\log_{10}$), we will find that

$\log 1,025^n = n \log 1.025$

Thus,

$$\frac{\log 1.33}{\log 1.025} = n$$

log 1.33 = n log 1.025

$$n = \frac{0.124938736}{0.1010723865}$$

$$n = 11.65 \quad or \quad 12 \quad days$$

Using the logarithm table or scientific calculators, we obtain

Since we have considered the interest rate per quarter, the resulting period will be in terms of quarters. However, if we use the annual interest rate of 10% the resulting answer will be in terms of years.

Students, in the example above we have seen how to compute the time 'n' for a given sum of money to grow to some specified amount at a specified interest rate. Further, let us consider a problem that involves computation of unknown interest rate.

Example 5.10

Find the semi-annual interest rate at which Birr 1760 will accumulate to Birr 3800 in 26 years.

Solution

Given values in the problem are, P = Birr 1760 , F = Birr 3800 , t = 26 years

m = number of conversion periods in a year = twice (semi-annual)

n = m x t = 2 x 26 = 52 semi-annuals

i =? (We are required to compute the semi-annual interest rate)

First, we have the compound amount formula $F_n = p(1 + i)^n$. Then,

$3800 = 1760 (1 + i)^{52}$

Now, let us divide both sides by 1760 to obtain

$3800 \div 1760 = (1 + i)^{52}$

$2.16 = (1+i)^{52}$

At this point, we shall apply the logarithmic function rule. Hence, taking common logarithm of both sides, we obtain.

$\log 2.16 = \log(1 + i)^{52}$

$\log 2.16 = 52 \log (1 + i)$

In further approach, let us divide both sides by 52 to obtain

$\log 2.16 \div 52 = \log (1 + i)$

$0.334454 \div 52 = \log (1 + i)$

$$0.006432 = \log (1 + i)$$

To obtain (1 + i), we must convert the given expression to exponential function form or take the antilogarithm of both sides. The first approach is simpler and involves few steps than the later method. Thus, as we have seen in the fourth chapter, we know that:

$$\log_a x = y \quad \text{is equal to} \quad a^y = x .$$

Accordingly, we obtain

$$10^{0.0064432} = 1 + i$$

$$1.0149 = 1 + i$$

$$i = 0.0149 \approx 1.5\% .$$

The rate of interest per conversion period (semi annual rate) is 1.5%. In the same way, the annual nominal rate j will be 3%, which can be obtained by multiplying i by the number of conversion periods per year.

Having considered the above examples, now you can further check your understanding by working out the following exercises.

Exercise 5.6

1. If Birr 6500 is invested at 8 ½ % compound
   a. Annually b. Semi annually
   c. Quarterly d. Monthly,
   what is the amount after 7 years?
2. At what interest rate compounded quarterly will a sum of money double in 4 years?
3. How long will it take for Birr 4750 to accumulate to Birr 7500 at $5^1/3\%$ compounded semi-annually?
4. If, in 11 years, Birr 1200 accumulates to Birr 1482, what is the compound interest rate provided it is converted annually?

5.8 Present Value of a Compound Amount

As we have considered in the simple interest case and as extended in the compound amount as well, future (maturity) value is the value of the present sum of money at some future date (time). Conversely, present value (or simply principal) is the current birr or dollar value equivalent of the future amount. It is the sum of money that is invested initially and that is expected to grow to some amount in the future at a specified rate. If we put the present and future (maturity) values on a continuum as shown below, we can observe that they are inverse to one another. And, future value is always greater than the present value or the principal since it adds/earns interest over specified time-period.

0 **1** **2** **3 . . .** **n**

Present Value (P) Future Value (Compound Amount)

$$P = \frac{F_n}{(1+i)^n} = F_n(1+i)^n \qquad\qquad F_n = P(1+i)^n$$

Future value is obtained by compounding technique and the expression $(1 + i)^n$ is called *compound factor*. On the other hand, present value is obtained by discounting techniques and the expression $(1 + i)^{-n}$ is referred to as the *compound discount factor*. The formula for present value of compound amount is simply derived from compound amount formula by solving for P.

Examples 5.11

1. What is the present value of
   a. Birr 5000 in 3 years at 12% compounded annually?
   b. Birr 8000 in 10 years at 10% compounded quarterly?
2. Suppose that a person can invest money in a saving account at a rate of 6% per year compounded quarterly. Assume that the person wishes to deposit a lump sum at the beginning of the year and have that some grow to Birr 20,000 over the next 10 years. How much money should be deposited at the beginning of the year?
3. A young man has recently received an inheritance of birr 200,000. He wants to make a portion of his inheritance and invest it for his late years. His goal is to accumulate Birr 300,000 in 15 years. How much of the inheritance should be invested if the money will earn 8% per year compounded semi-annually? How much interest will be earned over the 15 years?

Solution

1. (a) Given the values, Fn = F_3 = Birr 5000, t = 3 years m = 1 (compounded annually)

   n = t x m = 3 x 1 = 3

   j = 12 % = 0.12

   i = j / m = 0.12/1 = 0.12 and we are required to find Present Value P.

   Thus, $P = F_n (1 + i)^{-n}$

   $= 5000 (1 + 0.12)^{-3}$

$= 5000 (1.12^{-3})$

$= 5000 (0.7118)$

$P =$ Birr 3559

(b) $F_n = F_{40} =$ Birr 8000, $t =$ 10 years , $m =$ quarterly $= 4$

$n = t \times m = 10 \times 4 = 40$, $i = 10\% = 0.1$, $i = j / m = 0.1 / 4 = 0.025$

$p = ?$ but $P = F_n (1 + i)^{-n}$

$= 8000 (1 + 0.025)^{-40} = 8000 (1.025)^{-40}$

$P =$ Birr 2979.5

2. Given the values, $i = 6\% = 0.06$, $m =$ quarter $= 4$ times a year

$i = j \div m = 0.06 \div 4 = 0.015$

$F =$ Birr 20,000 shall be accumulated

$t = 10$ years

$n = m \times t = 10 \times 4 = 40$ interest periods

$P =$ how much should be deposited now?

$P = F_n(1 + i)^{-n}$

$= 20,000 (1+0.015)^{-40} = 20,000(1.015^{-40})$

$P =$ Birr 11,025.25

3. Inheritance = Birr 200,000

$F_n =$ Birr 300,000 (the person's goal of deposit) , $t =$ 15 years , $j = 8\% = 0.08$

$m =$ semi-annual $= 2$ times a year

$i = j \div m = 0.08 \div 2 = 0.04$

$n = t \times m = 15 \times 2 = 30$ interest periods/semi-annuals

$P =$ how much of the inheritance should be invested now? $P = F_n(1 + i)^{-n}$

$I =$ Amount of interest?

$= 300,000 (1+0.04)^{-30} = 300,000(1.04)^{-30} = 300,000(0.3083)$

$=$ Birr 92,490

The present value of Birr 300,000 after 15 years at 4% semi-annual interest rate is equal to Birr 92,490. Therefore, from the total inheritances received Birr 92,490 needs to be deposited now.

Amount of compound interest = Future Value – Preset Value = 300,000 – 92,490

Amount of compound interest = Birr 207,510

Dear student, now it is the time for you to practice and solve out the following problems.

Section Three: Annuities

Section Objectives:

After going through this section, you will able to:

- Define and explain annuity and its types.
- Identify and compute maturity value of an ordinary annuity.
- Develop the comprehension of sinking funds payments.
- Develop acquaintance of the mathematical application of present value and mortgage payments.

5.9 Introductory Concepts

Annuity refers to a sequence or series of equal periodic payments, deposits, withdrawals, or receipts made at equal intervals for a specified number of periods. For instance, regular deposits to a saving account, monthly expenditures for car rent, insurance, house rent expenses, and periodic payments to a person from a retirement plan fund are some of particular examples of annuity.

Payments of any type are considered as annuities if all of the following conditions are present:

i. The periodic payments are equal in amount
ii. The time between payments is constant such as a year, half a year, a quarter of a year, a month etc.
iii. The interest rate per period remains constant.
iv. The interest is compounded at the end of every time.

Annuities are classified according to the time the payment is made. Accordingly, we have two basic types of annuities.

i. ***Ordinary annuity:*** is a series of equal periodic payment is made at the end of each interval or period. In this case, the last payment does not earn interest.
ii. ***Annuity due:*** is a type of annuity for which a payment is made of the beginning of each interval or period.

It is only for ordinary annuity that we have a formula to compute the present as well as future values. Yet, for annuity due case, we may drive it from the ordinary annuity formula. To proceed, let us first consider some important terminologies that we are going to use in dealing with annuities.

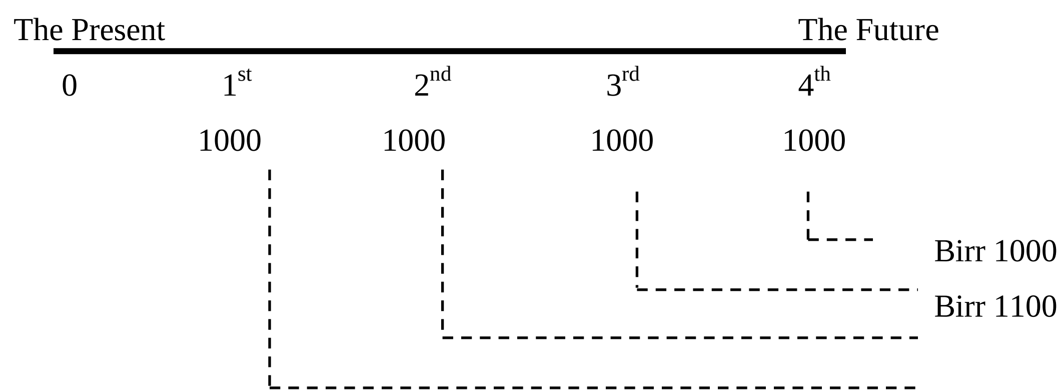
i. *Payment interval or period:* it is the time between successive payments of an annuity.
ii. *Term of annuity:* it is the period or time interval between the beginning of the first payment period and the end of the last one.
iii. *Conversion or interest period:* it is the interval between consecutive conversions of interest.
iv. *Periodic payment/rent:* it is the amount paid at the end or the beginning or each payment period.
v. *Simple annuity:* is the one in which the payment period and the conversion periods coincides each other.

Following the above basic overview about annuities, we shall progress to deal with practical business problems, which relate with determining the maturity and present values of annuities with specific application cases.

5.10 Sum of Ordinary Annuity: Maturity Value

Maturity value of ordinary annuity is the sum of all payments made and all the interest earned therefrom. It is an accumulated value of a series of equal payments at some point of time in the future. Suppose you started to deposit Birr 1000 in to a saving account at the end of every year for four years. How much will be in the account immediately after the last deposit if interest is 10% compounded annually?

In attempting this problem, we should understand that the phrase at the end of every year implies an ordinary annuity case. Likewise, we are required to find out the accumulated money immediately after the last deposit which also indicate the type of annuity. Further, the term of the annuity is four years with annual interest rate of 10%. Thus, we can show the pattern of deposits diagrammatically as follows.



Birr 1210

Birr 1331

Total Future Value = Birr 4641

The first payment earns interest for the remaining 3 periods. Therefore, the compound amount of it at the end of the term of annuity is given by,

$$F = P(1+i)^n = 1000(1+0.1)^3 = \text{Birr } 1331$$

In the same manner, the second payment earns interest for two periods (years). So,

$$F = 1000(1+0.1)^2 = 1210$$

The 3rd payment earns interest for only one period. So,

$$F = 1000(1+0.1)^1 = 1100$$

No interest for the fourth payment since it is made at the end of the term. Thus, its value is 1000 itself. In total, the maturity value amounts to Birr 4641.

This approach of computing future value of ordinary annuity is complex and may be tiresome in case the term is somewhat longer. Thus, in simple approach we can use the following formula for sum of ordinary annuity (Future Value).

$$Fn = R\left[\frac{(1+i)^n - 1}{i}\right]$$

Where, n = the number of payment periods

i = interest rate per period

R = payment per period

F_n = future value of the Annuity or sum of the annuity after n periods

Now, let us consider the above example. That is,

R = Birr 1000

i = 0.1 and n = 4

$$F_4 = 1000\left[\frac{(1+0.1)^4 - 1}{0.1}\right]$$

Future Value = Birr 4641

Example 5.12

1. A person plans to deposit 1000 birr in a savings account at the end of this year and an equal sum at the end of each following year. If interest is expected to be earned at the rate of 6% per year compound semi-annually, to what sum will the deposit (investment) grow at the time of the fourth deposit?
2. A 12-year-old student wants to begin saving for college. She plans to deposit Birr 50 in a saving account at the end of each quarter for the next 6 years. Interest is earned at a rate of 6% per year compounded quarterly. What should be her account balance 6 years from now? How much interest will she earn?

Solution

1. The known values in the problem are,

R = 1000, j = 6% = 0.06 , m = semi-annual = twice a year

i = 0.06 ÷ 2 = 0.03

n = 4

F_4 = ? $F_4 = R\left[\frac{(1+i)^n - 1}{i}\right]$

$$= 1000\left[\frac{(1+0.03)^4 - 1}{0.03}\right]$$

$$= 1000\ \left[(1.03)^4 - 1/0.03\right]$$

= 1000 x 4. 183627

F = Birr 4183.63

2. R = Birr 50

t = 6 years

m = quarterly = 4 times a year

n = t x m = 6 x 4 = 24 quarters

j = 6% = 0.06

i = j ÷ m = 0.06 ÷ 4 = 0.015

$$F_{24} = 50\left[\frac{(1+0.015)^{24} - 1}{0.015}\right]$$

$= 50\ \left[(1.015^{24} - 1) \div 0.015\right]$

F_{24} = Birr 1431.68

Interest = Maturity Value - Sum of Deposits

= Maturity value - (24 (50))

= F^{24} -1200 = 1431.68 - 1200

= Birr 231.68

5.11 Ordinary Annuities: Sinking Fund Payments

A sinking fund is a fund into which periodic payments or deposits are made at regular interval to accumulate a specified amount (sum) of money in the future to meet financial goals and/or obligations. The equal periodic payment to be made constitute an ordinary annuity and our interest is to determine the equal periodic payments that should be made to meet future obligations. Accordingly, we will be given the Future Amount, F, in n period and our interest is to determine the periodic payment, R. Then we can drive the formula for R as follows.

$$F_n = R\left[\frac{(1+i)^n - 1}{i}\right]$$

Multiply both sides by

$$\frac{i}{(1+i)^n - 1}$$

That is,

$$F_n \times \frac{i}{(1+i)^n - 1} = R\left[\frac{(1+i)^n - 1}{i}\right] \times \frac{i}{(1+i)^n - 1}$$

Then,

$$R = F_n\left[\frac{i}{(1+i)^n - 1}\right]$$

is the sinking found formula.

Where, R = Periodic payment amount of an annuity

i = Interest per period which is given by j ÷ m

j = Annual nominal interest rate

m = Number of conversion periods per year

n = Number of annuity payment or deposits (also, the number of compounding periods)

F = Future value of ordinary annuity

In general, a sinking fund can be established for expanding business, buying a new building, vehicles, settling mortgage payment, financing educational expenses etc.

Example 5.13

1. A corporation wants to establish a sinking fund beginning at the end of this year. Annual deposits will be made at the end of this year and for the following 9 years. If deposits earn interest at the rate of 8% per year compounded annually, how much money must be deposited each year in order to have 12 million Birr at the time of the tenth deposit? How much interest will be earned?
2. Assume in the previous example that the corporation is going to make quarterly deposits and that interest is earned at the rate of 8% per year compounded quarterly. How much money should be deposited each quarter? How much less will the company have to deposit over the 10-year period as compared with annual deposits and annual compounding.
3. A firm wishes to establish a sinking found for the purpose of expanding the production facilities at one of its plants. The company needs to accumulate 500,000 birr over the next five years that earn interest at 6% compounded semi-annually.
   a. How much should the firm contribute to the found at the end of each semi-annual period in order to achieve the goal?
   b. Calculate the compound interest.
   c. Prepare the fund accumulation schedule.

Solution

1. Future level of deposit desired = F_n = Birr 12 million

   Term of the annuity = t = 10 years

Conversion periods = m = annual = 1

n = t x m = 10 x 1 = 10 annuals

j = 0.08

i = j ÷ m = 0.08 ÷ 1= 0.08

R = the amount to be deposited each year to have 12 million at the end of the 10th year = ? Then to obtain the value of R, we shall use the formula for sinking fund.

$$R = F_n \left[\frac{i}{(1+i)^n - 1} \right]$$

$$R = 12{,}000{,}000 \left[\frac{0.08}{(1+0.08)^{10} - 1} \right]$$

$$R = 12{,}000{,}000 \left[\frac{0.08}{(1.08)^{10} - 1} \right]$$

$$R = 12{,}000{,}000 \left[\frac{0.08}{1.158925} \right]$$

R = Birr 828,353.86

On the other hand, the amount of interest, I, is obtained by computing the difference between the maturity value (F_n = 12,000,000) and the sum of all periodic payments made. Thus,

I = F_n - R (10)

= 12,000,000 – 823,353.86 (10)

= 12,000,000 – 8,283,538.6

= Birr 3,716,461.4

2. This is the continuation of the previous example. Thus,

F_n = Birr 12,000,000

t = 10 years

m = Quarterly = 4 times a year

n = m x t = 4 x 10 = 40

j = 8% = 0.08

i = j ÷ m = 0.08 ÷ 4 = 0.02

R = Periodic payment at adjusted conversation or interest period?

$$R = F_n\left[\frac{i}{(1+i)^n - 1}\right]$$

$$R = 12{,}000{,}000 \left[\frac{0.02}{(1+0.02)^{40} - 1}\right]$$

$$R = 12{,}000{,}000 \left[\frac{0.02}{(1.02)^{40} - 1}\right]$$

= 12,000,000 (0.016555747)

R = Birr 198,668.94

In further computation to determine the difference in amount of deposit by changing the length of the conversion period, we see that as compared with the first case of annual conversion period, in the quarterly conversation scheme, the corporation will deposit R (10) minus R (40) less over the term of the annuity.

Thus, R (10) – R (40)

= Birr 828,353.86 (10) – 198,668.96 (40)

= 8,283,538.6 – 7,946,758.4

= Birr 336,780.2

3. Future financial goal of the firm = Fn = Birr 500.00

t = term of the annuity = 5 years

j = annual interest rate

m = Conversion period = Semi-annually = 2 times a year

i = j ÷ m = 0.06 ÷ 2 = 0.03

n = t x m = 5 x 2 = 10 semi-annuals

a. R=?

$$R = 500,000 \left[\frac{0.03}{(1+0.03)^{30} - 1} \right]$$

$= 500,000\ (0.08723050506)$

R = Birr 43,615.25

b. Compound Interest = Fn – (n x R)

= Fn - (R (10)

= 5000,000 - 43615.25(10)

= 500,000 - 436,152.5

Interest = Birr 63,847.5

c. Fund Amortization Schedule

Period	***Balance (Beginning)***	***Interest (Beginning Bal.* x *0.03)***	***Periodic Payment***	***Ending Balance (Beg. Bal.+ Interest + R)***
1st	0	0	43615.25	43615.25
2nd	43,615.25	1308.46	43615.25	88,538.95
3rd	88,538.95	2656.17	43615.25	134,810.37
4th	134,810.37	4044. 31	43615.25	182,469.93
5th	182,469.93	5474.01	43615.25	231,559.93
6th	231,559.28	6946.78	43615.25	282,121.31
7th	282,121.31	8463.64	43615.25	334,121.31
8th	334,121.31	10,026.01	43615.25	387,841.46
9th	387,841.46	11,635.24	43615.25	443,091.95
10th	443,091.95	13,292.76	43615.25	500,000

Now, you may practice the following exercise.

5.12 Present Value of Ordinary Annuity

The present value of annuity is an amount of money today, which is equivalent to a series of equal payments in the future. It is the value at the beginning of the term of the annuity. The present value of annuity calculation arise when we wish to determine what lump sum must be deposited in an account now if this sum and the interest it earns will provide equal periodic payment over a defined period of time, with the last payment making the balance in account zero.

Present value of ordinary annuity is given by the formula:

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Where, R= Periodic amount of an annuity

i = Interest per period which is given by j ÷ m

j = Annual nominal interest rate

m = Interest/ conversion periods per year

n = Number of annuity payments / deposits (also, the number of compounding periods)

P = Present value of ordinary annuity

Dear students, lets consider the following examples to make our understanding of business and financial application of present value of ordinary annuity clear.

Example 5.14

A person recently won a state lottery. The term of the lottery is that the winner will receive annual payments of birr 18,000 at the end of this year and each of the following 4 years. If the winner could invest money today at the rate of 6% per year compounded annually, what is the present value of the five payments?

Solution

R = Annual payments of Birr 18,000

Term of the annuity = t = this year and the following 4 years = 5 years

i = 6% = 0.06 (since the conversion period per year is annual)

n = 5

P

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$P = 18{,}000\left[\frac{1-(1+0.06)^{-5}}{0.06}\right] = 18{,}000\left[\frac{1-(1.06)^{-5}}{0.06}\right]$$

resent value of payments = P = ?

P = Birr 75,822.55

$$P = 18{,}000[4.212363785]$$

Example 5.15

A woman would like to borrow money from local microfinance institution which charges interest at 4% compounded quarterly. If the woman is able to pay Birr 100 at the end of each quarter for one year,

a. How much should she receive from the institution at the time of borrowing?
b. How much interest will the woman be charged?
c. Prepare the debt repayment schedule (Amortization schedule).

Dear student, please solve the above example before reading the solution part.

Solution

Interest charge rate = j = 4% = 0.04

Periodic payment by the woman = Birr 100

Term of the annuity (debt) = t = 1 year

Conversion period per year = quarterly = m = 4

Number of periods = n = t x m = 1 x 4 = 4 periods

Interest rate per conversion periods = j ÷ m = 0.04 ÷ 4 = 0.01

a. How much to receive now? That is, the present value of the annuities, p.

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$P = 100\left[\frac{1-(1+0.01)^{-4}}{0.01}\right] = 100\left[\frac{1-(1.01)^{-4}}{0.01}\right]$$

b.

= 100 (3.902)

Present Value = Birr (390.2)

Given the woman's potential to pay Birr 100 at the end of each quarter for one year, she can borrow Birr 390.2 at the beginning.

c. Interest charge = Total amount paid – Present value

= (R x n) – P

= (100 x 4) 390.2   =   400 – 390.2

= Birr 9.8

d. *The debt repayment schedule* is a table that shows a periodic status of payments that gradually make the debt account balance zero. This table is also called *amortization schedule*. Now let us proceed with preparing the schedule.

Period	***Beginning Balance (Debt)***	***Interest (I) (Debt*** x ***0.01)***	***Periodic Payment***	***Ending Balance (Debt + I – R)***
1st	Birr 390.2	Birr 3.902	Birr 100	Birr 294.102
2nd	Birr 294.102	Birr 2.941	Birr 100	Birr 197.043
3rd	Birr 197.043	Birr 1.97043	Birr 100	Birr 99.013
4th	Birr 99.013	Birr 0.99013	Birr 100	Birr 0

As you observe in the above amortization schedule, in ordinary annuity periodic payment the last balance becomes zero.

5.13 Mortgage Payments and Amortization

Dear student, would you define mortgage and amortization?

Another main area of application of annuities in to real world business situations in general and financial management practices in particular is mortgage amortization or payment. *Mortgage payment* is an arrangement where by regular payments are made in order to settle an initial sum of money borrowed from any source of finance. Such payments are made until the outstanding debt gets down to zero. An individual or a firm, for instance, may borrow a given sum of money from a bank to construct a building or undertake something else. Then the borrower (debtor) may repay the loan by effecting (making) a monthly payment to the lender (creditor) with the last payment settling the debt totally.

In mortgage payment, initial sum of money borrowed and regular payments made to settle the respective debt relate to the idea of present value of an ordinary annuity. Along this line, the expression for mortgage payment computation is derived from the present value of ordinary annuity formula. Our intention in this case is to determine the periodic payments to be made in order to settle the debt over a specified time – period.

Hence, we know that

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Now, we progress to isolate R on one side. It involves solving for R in the above present value of ordinary annuity formula. Hence, multiply both sides by the interest rate i to obtain:

$$P\,i = R\,[1 - (1 + i)^{-n}]$$

$$R = P\left[\frac{i}{1-(1+i)^{-n}}\right]$$

Further, we divide both sides by $[1 - (1 + i)^{-n}]$ and the result will be the mathematical expression or formula for computing mortgage periodic payments as follows.

Where, R = Periodic amount of an annuity

i = Interest per conversion period which is given by j ÷ m

j = Annual nominal interest rate

m = Interest or conversion periods per year

n = the number of annuity payments/deposits (number of compounding periods)

P = Present value of an ordinary annuity

Example 5.16

1. Emmanuel purchased a house for Birr 115,000. He made a 20% down payment with the remaining balance amortized in 30 years mortgage at annual interest rate of 11% compounded monthly.
    a. Find the monthly mortgage payment?
    b. Compute the total interest.
2. Assume you borrowed Birr 11,500 from a bank to finance construction of a swimming pool and agreed to repay the loan in 60 monthly equal installments. If the interest is 1.5% per month on the unpaid balance,
    a. How much is the monthly payment?
    b. How much interest will be paid over the term of the loan?

Solution

1. Total cost of purchase = Birr 115,000

Amount paid at the beginning (Amount of down payment) = 20% of the total cost

= 0.2 x 115,000 = Birr 23,000

Amount Unpaid or Mortgage or Outstanding Debt = 115,000 – 23,000

= Birr 92,000

t = 30 years

j = 11% = 0.11 , m = 12 , i = 0.11 ÷ 12 = 0.00916

n = t x m = 30 x 12 = 360 months

a. The periodic payment R = ?

$$R = P\left[\frac{i}{1-(1+i)^{-n}}\right]$$

$$R = 92{,}000\left[\frac{0.00916}{1-(1+0.00916)^{-360}}\right]$$

= 92,000 (0.009523233)

R = Birr 876.14

b. Total Interest = (R x n) – P

= 876.14 x 360 – 92,000

= Birr 223,409.49

Over the 30 years period Emmanuel is going to pay a total interest of Birr 223,409.49, which is well more than double of the initial amount of loan. Nonetheless, the high interest can be justified by the fact that value of a real estate is usually tend to increase overtime. Therefore, by the end of the term of the loan the value of the real estate (house) could be well higher than its purchase cost in addition to owning a house to live in for the 30 years and more.

2. Amount borrowed = Birr 11,500 = P

n = 60 months

i = 1.5% = 0.015

The interest rate is already given as monthly rate and it is equal to the interval of compounding and payment periods. Thus, we are not required to divide it by the number of conversion periods.

a. Monthly Installment Payment R = ?

$$R = P\left[\frac{i}{1-(1+i)^{-n}}\right]$$

$$R = 11{,}500\left[\frac{0.015}{1-(1+0.015)^{-60}}\right] = 11{,}500\,(0.025393)$$

$R = $ Birr 292.02 per month

b. Total interest paid (I) = (R x n) – P

$I = (292.02 \times 60) - 11,500$

$I = $ Birr 6021.405