UNIT ONE

LINEAR EQUATIONS AND THEIR INTERPRETATIVE APPLICATIONS

Unit Objectives

After a thorough study of this unit, you will be able to:

- Understand basic concepts of linear equations and functions.
- Develop acquaintance of graphic representation of linear equations.
- Compute and formulate slope and equation of a line.
- Systematically apply the linear equation algebra and geometry in solving real world situations.

Unit Introduction

In the face of changing business environment, organizations encounter diverse set of problems and challenges as well as prospects. Consequently, managers are expected to make appropriate decisions and take actions that enable the organization take advantages and overcome difficulties. In making such decisions and actions, one may require to apply mathematical tools and quantitative techniques. In other words, there are various subjects of decision of which relation to one another is at least approximated and explained by linear equations, for instance, sales volume can be linearly related to advertisement expense. The same holds true between output level and number of employees engaged on some activity and cost of production. Furthermore, demand for and supply of a given product can be well approximated and explained by a linear equation. In such real business instances, the concept and interpretative application of linear equations have a considerable importance.

Cognizant to the above fact, we need to be well acquainted with the fundamentals of linear equations algebra and geometry as related to its business application. This chapter, therefore, is dedicated to our study of linear equations. To this end, the unit is organized in to two sections. In the first section, you will learn about basic concepts of linear equations and their graphic representation and then you will proceed to the application of mathematical concepts of linear equations in solving business problems.

Section One: Linear Equations, Functions and Graphs

Section Overview:

1.1 Basic Concepts of Linear Equations and Functions
1.2 Graph of a Linear Equation
1.3 The Distance between Two Points
1.4 Developing Equation of a Line

1.1 Basic Concepts of Linear Equations and Functions

An equation is a statement of equality, which shows two mathematical expressions are equal. Equations always involve one or more unknown quantities that need to be solved. Among the different types of equations, linear equation is the one that we are going to deal with in some detail. *Linear equations* are equations whose terms[1] are a constant time a variable to the first power. Accordingly, equations that can be transposed to the form,

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = c$$

are said to be linear equations.

Where, $a_1, a_2, a_3, \ldots a_n$ and c are constants

$x_1, x_2, x_3, \ldots x_n$ are variables (unknown quantities)

$a_1 x_1, a_2 x_2, \ldots a_n x_n$ and c are the terms of the equation (terms of a linear equation represent the parts separated by plus, minus, and equal signs)

As it occurs in many business application cases, a linear equation may involve two variables, x and y, and constants a, b, and c in which case the equation relating x and y takes the form,

$$a x + b y = c$$

The following are all examples of linear equations.

$2x + 3y = 9$, $3x – 9y + z = 23$, $4y + 7.5x – 11 = 14$

On the other hand, $4xy + 7x = 8$ is not a linear equation because the tem $4x y$ is a product of a constant and two variables. Likewise $5x^2 + 3y = 25$ is not linear because of the term $5x^2$ which is a constant times one variable to the second – power.

Dear student, we observe many business situations in which linear relationships arise. Consider the following example.

Example 1.1

Assume that Ethiopian Electric Power Corporation Charges Birr 0.55 per kilowatt-hour consumed and a fixed monthly charge of Birr 7 for rent of electric meter. If y is the total monthly charge and x is the number of kilowatt-hours consumed in a given month, write the equation for y in terms of x.

Solution

The total monthly charge will be, 0.55 times the number of monthly KWh consumption plus Birr 7 for meter rent.

Thus, using the symbols given,

$$y = 0.55x + 7$$

The equation of this example is linear with two variable x and y. In such linear equations, we need to note that the constants can be positive or negative, and can be fractions when graphs of these equations is plotted it will be a straight line. This is the reason for the term equation.

Linear Functions: functional relationship refers to the case where there is one and only one corresponding value of the dependent variable for each value of the independent variable. The relationship between x and y as expressed by

$$y = 0.55x + 7.$$

is called a functional relationship since for each value of x (independent variable), there is a single corresponding value for y (dependent). Thus if we write y as expression involving x and constants x is called the independent variable, then the value of y depends upon what value we may assign to x and as a result it is called the dependent variable. Therefore, a linear function refers to a linear equation, which does have one corresponding value of dependent variable for each value of the independent variable.

Exercise 1.1

Suppose that a car rent company charges Birr 65 per hour a car is rented. In addition, Birr 150 for insurance premium. Write the equation for the total amount charged by the company in terms of the hours the car is rented.

1.2 Graph of a Linear Equation

Linear equations in two variables can be plotted on a coordinate plane with two dimensions. Such equations have graphs that are straight lines. This means that the graph of the relationship between the variables takes the form of a straight line. Any straight-line graph can be sketched by plotting just two points which satisfy the linear equation and then joining them with a straight line. Now let us further develop this approach by considering the following example.

Example 1.2

Sketch the graph of the equation $2y - 3x = 3$.

Solution

To plot the graph, you may arbitrarily select two values for x and obtain the corresponding values for y. Therefore, lets set $x = 0$. Then the equation becomes $2y – 3(0) = 3$.

That is, $2y = 3$

$y = 3/2$

This means that when $x = 0$, the value of y is 3/2. So, the point with coordinates (0, 3/2) lies on the line of $2y – 3x = 3$.

In the same way, let $y = 0$. Then the equation becomes $-3x = 3$.

That is, $x = 3/-3 = -1$.

This means, when $y = 0$, the value of x is -1. So, the point with coordinates (-1, 0) lies on the line of equation $2y – 3x = 3$. These two points are plotted on the coordinate plane with horizontal “x – axis” and vertical “y-axis” as follows.

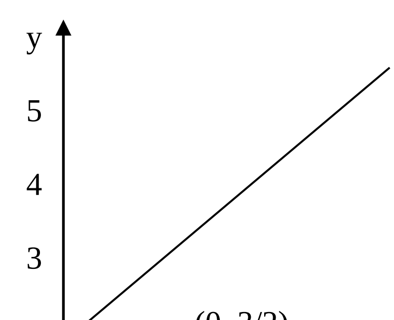

Fig 1.1 Linear Equation Graph

Exercise 1.2

Find two coordinate points that satisfy the equation $3x + 4y = 24$. Then, using the two coordinate points plot the graph of the given function.

1.3 The Distance between Two Points

The distance between two points is the length of a straight-line segment that joins the points. To determine the length of a given segment in coordinate geometry, algebraic procedures are applied to the x – and y coordinates of the end points of the segment. Distance on horizontal and vertical line segments are used in the computation of the distance. Distance on a vertical segment (also called vertical separation) is found by computing the positive difference of the y-coordinates of the end points of the segment. Distance on the horizontal segment (also called horizontal separation) is found by computing the positive difference of the x-coordinate of the end points of the segment.

Thus, given two points (x_1, y_1) and (x_2, y_2), the quantity $/ x_2 – x_1 /$, is called the horizontal separation of the two points. Further, the quantity $/ y_2 – y_1 /$ is the vertical separation of the two points.

Example 1.3

Given the points A (-5, 7), B (-3, -9), C (-5, 15), D (12, 6), find the horizontal and vertical distance of the segment,

a. AB b. AD c. BD

Solution

a. The horizontal distance (Separation) of the points A (-5, 7) and B (-3, -9) is given by

Horizontal distance $= / x_2 - x_1 /$

$= /-3-(-5)/$

$= /-3+5/ = 2$

Vertical distance $= / y_2 - y_1 /$

$= / -9 - 7 /$

$= /-16/ = 16$

b. Horizontal distance $= / x_2 - x_1/$

$= / 12 - (-5)/$

$= 12+ 5/ = 17$

Vertical distance AD $= / y_2 - y_1 / = / 6 – 7 /$

$= / -1 / = 1$

c. Horizontal distance BD $= / x_2 - x_1/ = /12- (-3)/$

$= / 12+3/ = 15$

Vertical distance BD $= / y_2 - y_1/ = / 6- (-9)/$

$= / 6 + 9/ =15$

Exercise 1.3

Find the vertical and horizontal separation of the following points.

a. (5, 7) and (-3, 2)

b. (5, - 3) and (-11, -7)

c. (6, 2) and (6, -4)

d. (3, 4) and (9, 4)

Dear student, as you recall all lines in a coordinate plane are not vertical and/or horizontal. Hence, in case the segment is slant to any direction the actual distance between (x_1, y_1) and (x_2, y_2) may be calculated from Pythagoras' Theorem, using their horizontal and vertical separations.

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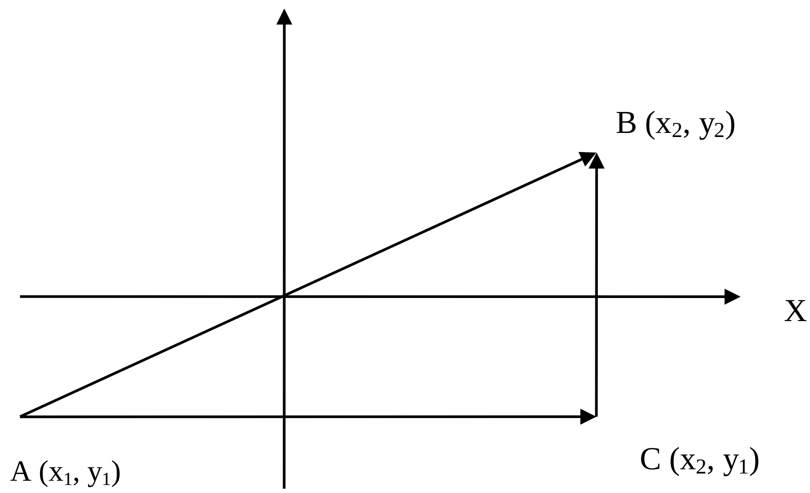


Fig 1.2 Places of Coordinates

In the above diagram, $AB^2 = AC^2 + BC^2$.

That is, the distance *d* between point A and B is given by:

$$d^2 = (\text{horizontal separation})^2 + (\text{vertical separation})^2$$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1.4

Calculate the distance *d* between the points (5, -3) and (-11, -7).

Solution

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(-11-5)^2 + (-7\ (-3))^2}$$

That is, $d = \sqrt{256 + 16} = \sqrt{272}$

$$d = 16.5$$

Exercise 1.4

Find the distance between the points given below.

a. (5, 10) and (11, 8)

b. (0, 0) and (9, 12)

1.4 Developing Equation of a Line

We have three alternative forms of developing the equation of a straight line. These are, slope-intercept form, slope-point form, and two-point form. Let us consider these approaches further.

The Slope – Intercept Form

Slope is a measure of steepness or inclination of a line and it is represented by the letter m. the slope of a non-vertical line is defined in several ways. It is the rise over the run. It is the change in y over the change in x. Thus, given coordinates of two points (x_1, y_1) and (x_2, y_2)

$$\text{Slope} = m = \frac{Rise}{Run} = \frac{y_2 - y_1}{x_2 - x_1}, \qquad \text{Where,} \qquad x_1 \neq x_2$$

If the value of the slope is positive, the line rises form left to right. If the slope is negative, the line falls from left to right. If the slope is zero, the line is horizontal. If the slope is undefined then the line is vertical.

Example 1.6

Obtain the slope of the straight-line segment joining the two points (8, -13) and (-2, 5).

Solution

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-13)}{-2-8} = \frac{18}{-10}$$

Therefore, the line that passes through these two points falls downwards from left to right. On the other hand, if the equation of a line is given, then the slope can be determined more simply. Thus, if a liner equation is written in the form y = m x + b, "m" is the slope and "b" is often referred as the intercept term and it is the value at which the straight line intercepts the Y-axis.

Example 1.7

An agent rents car for one day and charges Birr 22 plus 20 cents per mile driven.

a. Write the equation for one day's rental (y) in terms of the number of miles driven (x).
b. Interpret the slope and the y – intercept.
c. What is the renter's total cost if a car is driven 100 miles? What is the renter's cost per mile?

Solution

Given fixed (constant) cost of Birr 22 = b

Slope = m = 20 cents = Birr 0.2

y = Total cost for one day rental

x = Number of miles driven

a. The equation

$y = m x + b$

$y = 0.2x + 22$

b. Interpretation

The slope, m = 20 cents (Birr 0.2) means that each additional mile driven adds 20 cents to total cost. b = Birr 22 is the fixed cost (the amount to be paid irrespective of the mile driven). Hence, it will be the total cost when no mile is driven.

c. Total cost of driving 100 miles (x = 100)

$= 0.2\,(x) + 22$

Total cost of the renter $= 0.2\,(100) + 22$

$= 20+22 =$ Birr 42

Cost per mile when x = 100 miles is given by total cost of driving 100 miles divided by 100 miles. Putting it in equation form,

$$Cost\ \ per\ mile = \frac{Total\quad Cost}{Total\ Number\quad Miles\,Driven}$$

$= 42 \div 100 =$ Birr 0.42

Example 1.8

Write $8x – 2y – 6 = 0$ in slope intercept form and determine slope and y-intercept of the equation. In addition, find the coordinates of x- and y-intercept.

Solution

We proceed to isolate y with a coefficient of one on the left-hand side of the equation to obtain;

$y = m x + b$ form finally.

Thus, $8x – 2y – 6 = 0$ is equivalent with

$$8x – 2y = 6$$

$$-2y = -8x + 6$$

$$\frac{-2y}{-2} = \frac{-8x}{-2} + \frac{6}{-2}$$

$$y = 4x – 3$$

Therefore, y = 4x – 3 is an equation in the y = m x + b form that is, 4 is the slope and -3 is the y-intercept of the line. Further, the x-intercept is found by setting y = 0. Thus, the value of x when y=0 is,

$$y = 4(x) - 3$$

$$0 = 4(x) - 3 = -1$$

The x intercept coordinate is (3/4, 0). In the same manner, we can obtain the y-intercept coordinate is (0, -3).

The Slope – Point Form

In this form, we will be provided with the slope and a point on the line, say (x_1, y_1). Then, we determine the intercept from the slope and the given point and develop the equation.

Accordingly, the expression we need further is the equation that is true not only for the point (x_1, y_1) but also for all other points say (x, y) on the line. Therefore, we have points (x_1, y_1) and (x, y) with slope m. The slope of the line is $y - y_1 / x - x_1$ and this is equal for all pair of points on the line. Thus, we have

$$\frac{y - y_1}{x - x_1} = m$$

the following formula for slope-point form:

Alternatively,

$$y - y_1 = m(x - x_1)$$

Example 1.9

Find the equation of a line that has a slope of 3 and that passes through the point (3, 4).

Dear student, please try to solve the example before you go to the solution part.

Solution

Given m = 3 and $(x_1, y_1) = (3, 4)$. By substituting these values in the formula

$$y - y_1 = m(x - x_1)$$

We will obtain $y - 4 = 3(x - 3)$.

Then, $y - 4 = 3x - 9$

$$y = 3x - 9 + 4$$

$$y = 3x - 5.$$

In another approach, $y = m x + b$ where $(x, y) = (3, 4)$

$$y = 3x + b$$

$$4 = 3(3) + b$$

$$4 = 9 + b$$

$$4 - 9 = b = -5 \text{ is the y-intercept.}$$

Thus, $y = 3x - 5$ is the equation of the line.

Exercise 1.6

If the relationship between total cost (y) and the number of units made (x) is linear and if cost increases by Birr 3 for each additional unit made and if the total cost of making 10 units is Birr 40, find the equation of the relationship between cost and number of units made.

The Two – Point Form

In this case, two points that are on the line are given and completely used to determine equation of a straight line. In doing so, we first compute the slope and then use this value with either points to generate the equation. Taking two points designated by $(x_1 - y_1)$ and $(x_2 - y_2)$ and another point (x, y), we can develop the expression for the equation of the line as follows.

Therefore, $(y - y_1)(x_2 - x_1) = (y_2 - y_1)(x - x_1)$

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

is the expression for the two-point form of generating equation of a straight line.

Example 1.10

A publisher asks a printer for quotations on the cost of printing 1000 and 2000 copies of a book. The printer quotes Birr 4500 for 1000 copies and Birr 7500 birr for 2000 copies. Assume that cost (y) is linearly related to the number of books printed (x).

a. Write the coordinates of the given points

b. Write the equation of the line

Solution

Given the values

x_1 = 1000 Books $\quad$ y_1= Birr 4500

x_2 = 2000 books $\quad$ y_2= Birr 7500

a. Coordinates of the points are:

(x_1, y_1) and (x_2, y_2)

Thus, (1000, 4500) and (2000, 7500)

b. To develop the equation of the line, first let's compute the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{7500 - 4500}{2000 - 1000}$$

$$= 3000 \div 1000 = 3$$

Then, consider the formula of two-point form of developing equation of a line as given by,

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

We have obtained the value for the slope m = 3 as it's expressed by

$$y_2 - y_1 \div x_2 - x_1.$$

Subsequently, by substitution this value in the above formula will result in;

$$\frac{y - y_1}{x - x_1} = 3$$

Then, $y - y_1 = 3(x - x_1)$

In continuation, substitute the value (1000, 4500) in place of x_1 and y_2 in the equation $y - y_1 = 3(x - x_1)$. As a result, you will obtain,

y – 4500 = 3 (x – 1000)

y – 4500 = 3x – 3000

y = 3x – 3000 + 4500

y = 3x + 1500 ……………………… is the equation of the line.

Exercise 1.7

As the number of units manufactured increases form 100 to 200, manufacturing cost increases form 350 birr to 650 birr. Assume that the given data establishes relationship between cost (y) and the number of units made (x), and assume that the relationship is linear. Find the equation of this relationship.

Section Two: Applications of Linear Equations

Section Overview:

1.5 Linear Cost – Output Relations Analysis

1.6 Market Equilibrium Analysis

1.7 Break – Even Analysis

1.5 Linear Cost – Output Relations Analysis

As of the very beginning, we aimed at developing our understanding on the interpretative application of linear equations in business. Consequently, our interest and purpose in this section is to learn how we can approximate and relate the mathematical terminology and technique of linear equations in addressing real world business issues. In dealing, we are going to consider three application areas of linear equations. These are the linear cost – output relations analysis, break – even analysis, and market equilibrium analysis. In this particular section, we will consider these application areas to some detail.

In order to grasp the concept of linear cost output relations, let us consider the relationship among different types of cost on the following a coordinate plane.

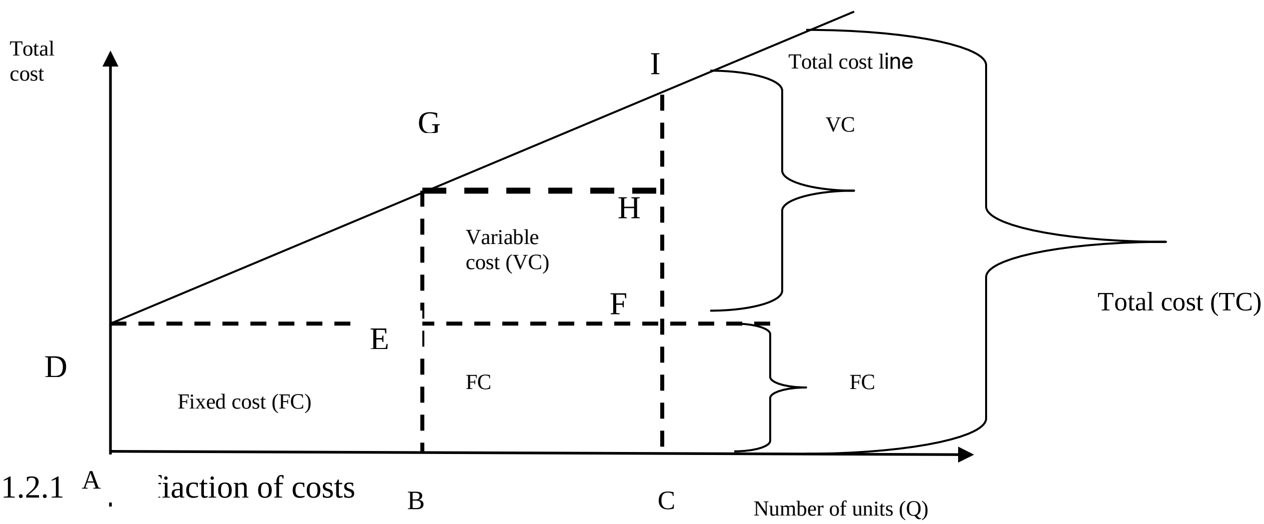


Fig. 1.2.1 iaction of costs

Definitions

Fixed cost is a cost component that does not change with the number of units produced. The variable cost is a cost component that varies with the number of units produced. Then at each level of production, total cost is the summation of fixed cost and variable cost. Marginal cost is the additional cost incurred in producing one more unit of output.

Observations

Assume that total manufacturing cost and the number of units produced are linearly related. The total cost originates from the fixed cost line because of zero level of production the total cost will be equal to the fixed cost (see the above figure (Fig 1.2.1)). Accordingly,

- Fixed costs (FC) = AD = BE = CF
- The segment BG is the Total Cost (TC) of producing AB units of outputs.
- The segment CI is the TC of producing AC units of outputs.
- The segment AD is the TC of producing zero units of outputs.
- The ratio

$$FI \div DF = \frac{\text{Variable Cost}}{\text{Number of Units}} = \text{Variable Cost Per Unit } \left({}^{VC}\!\big/_{unit} \text{ or VC per Unit}\right)$$

- Marginal cost is given by change in TC divided by change in Quantity (Q). Thus,

  Marginal cost (MC) = $\frac{\Delta TC}{\Delta Q}$ = VC per unit.

- Therefore, marginal cost and VC/unit are the considered as the slope of TC line and they are constant as long as total cost and quantity produced are linear.
- When TC ÷ Q = Average cost per unit (AC).
- Unlike MC and VC per units, AC per unit is not constant although cost and quantity, produced are linearly related.

Example 1.11

Given the total cost function C = 5Q + 10 and if 10 units are produced, find TC, AC, MC, FC and VC per unit.

Solution

i. TC = 5(10) + 10 = 60

ii. AC = $\frac{\text{Total cost of producing 10 units}}{\text{Number of units produced}}$ = 60 ÷ 10 = 6

   =>Suppose, if 5 units were produced,

   TC = 5(5) + 10 = 35.

   In this case, AC = 35 ÷ 5 = 7. Therefore, the AC is not constant.

iii. MC = is the same as the slope of the equation. Thus MC = 5

iv. FC = the fixed cost remains constant at any level of production. Thus, 10 is the fixed cost.

v. VC/unit = Variable cost ÷ Number of units

$$= \frac{5(Q)}{Q} = \frac{5(10)}{10} = 5$$

Example 1.12

If the total factory cost (y) of making x units of a product is given by y = 3x + 20, and if 50 units are made,

a. What is the variable cost (VC)?
b. What is the total cost (TC)?
c. What is the variable cost per unit (VC/unit)?
d. What is the average cost per unit (AC/ unit)?
e. What is the marginal cost of the 50th unit?

Solution

Given, total cost = y = 3x + 20

x = Units produced

x = 50 Units

a. Variable cost is the cost that varies with the level of production and it can be obtained by multiplying the slope or the marginal cost with the number units produced (x).

   That is, VC = m x

   = 3x 50 = Birr 150

b. TC is the sum of fixed cost and variable cost. In the equation given y =3x + 20, the term 3x represent the VC and the constant 20 is the fixed cost. Thus

   TC = y = 3x+20

   = 3(50) + 20 + 150 + 20

   = Birr 170

c. Variable cost per unit = vc / unit = vc / x = 150 / 50 = Birr 3
d. AC is given by total cost divided by Number of units produced. Thus,

   AC/ unit = TC/ x = (VC+FC)/ x = (150 + 20) / 50 = 170 / 50 = Birr 3.40

e. The slope of a linear equation is equal to the marginal cost of any given level of production thus, MC = 3.

   In alternative approach, MC is the extra (additional) cost of producing one more unit of output. Thus, the marginal cost of producing the 50th unit is equal to the additional cost in producing the 50th unit.

$$\text{Therefore, } MC = \Delta TC = \frac{\text{TC of producing 50 units} - \text{TC of producing 49 units}}{50 - 49}$$

$$= \frac{(3(50)+20) - (3(49)+20)}{50 - 49}$$

$$= \frac{170 - 167}{1} = \text{Birr } 3$$

Exercise 1.8

If total factory cost, y, of making x units of a product is y = 10x + 500 and if 1,000 units are made:

a. What is the variable cost?
b. What is the total cost?
c. What is the variable cost per unit?
d. What is the marginal cost of the last unit made?

1.6 Break – Even Analysis

Break-even analysis is an economic theory that states that profits arise from the excess of total revenue over total cost.

This is, Profit = Total Revenue – Total Cost

In the application of break-even analysis, there are two important concepts we need to distinguish. These are the break-even point and break-even chart. The break-even point (BEP) is the volume or level of output resulting in neither a loss nor a profit. It is a point at which revenue and cost are equal. The break-even chart is a convenient means of graphically describing the relationship between cost and revenues at different levels of output.

In business concept, we have two cases of applying break-even analysis based on the type of business activity under consideration. The first case is break-even analysis for manufacturing companies. The other is for retail businesses. The distinction between the two cases is that manufacturing companies usually state their cost equation in terms of quantity (output) as their business involves producing and selling. In such a case, the break-even point is commonly computed in terms of output level. On the other hand, as retail businesses are concerned with purchase and sell of merchandise, they state their cost equation in terms of revenue, which is the same as the Birr (dollar) volume of sales. Thus, the break-even point is commonly computed in terms of break-even level of sales (dollar sales volume).

Case 1: Break-Even Analysis for Manufacturing Businesses

In this case, we shall consider a manufacturer who produces q units of a product and sells the product at a price of p per unit. In proceeding, let us specify the symbols to be used in our study of the case before hand.

C = total cost of producing and selling q units

q = number of units produced and sold

v = variable cost per unit made (assumed to be constant)

FC = fixed cost (constant amount)

P = Selling price per unit

R = total revenue received, which is equal to sales volume in terms of dollar or birr.

The cost function then is given by:

$$C = v\,q + FC \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

and,

$$\text{Revenue} = \text{Price per unit} \times \text{Number of units sold}$$

$$R = p\,q \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

Thus, if the manufacture is to break – even on operations, which is to neither incur loss nor earn profit, revenue (2) must equal cost (1).

That is, at break – even

$$p\,q = v\,q + FC \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3)$$

You may now solve equation (3) for the production volume q;

$$p\,q = v\,q + FC$$

$$p\,q - v\,q = FC$$

$$q\,(p - v) = FC$$

$$q = \frac{FC}{p-v}$$

Thus, the ***Break – Even Quantity*** denoted by q_{e}, is given by;

$$q_e = \frac{FC}{p-v}$$

To further our understanding of break-even analysis, let us consider the following break-even chart.

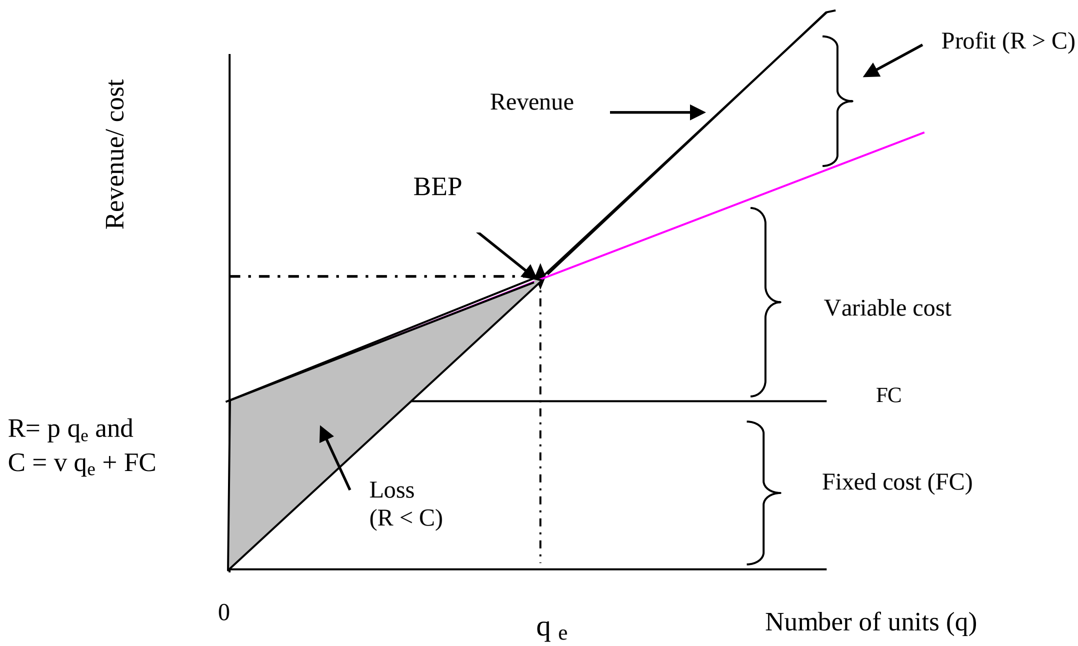


Fig 1.2.2 Break Even Point (BEP) and other points

Observations: From the above break – even chart, we observe certain important points.

i. As such, the total revenue line passes through the origin and hence has a y-intercept of zero while the total cost line has a y – intercept which is equal to the amount of the fixed cost.

ii. The fixed cost line which is parallel to the quantity axis (x – axis) is constant at all levels of output.

iii. To the left of the break – even point the revenue line is found below the cost line and hence any vertical separation indicates a loss while to the right the opposite is true.

iv. The total variable cost, which is the gap between the total cost and the fixed cost line increases as more units are produced.

v. Important linear cost – output expressions (equations):

- $C = v\,q + FC$
- $R = p\,q$
- Average Revenue (AR) = $R \div q = p\,q \div q = p$
- Average Variable Cost (AVC) = $v\,q \div q = v$ = Slope (m)
- Average Fixed Cost (AFC) = $FC \div q$
- Average Cost = $C \div q$ = AVC + AFC

- Profit (π) = R – C

Example 1.13

A book company produces children's books. One time fixed costs for Little Home are $12,838 that includes fees to the author, the printer, and for the building. Variable costs amount to $14.50 per book the books are then sold to bookstores around the country at $39.00 each. How many books must be printed and sold to break-even?

Solution

Given, v = $14.50

FC = $12,838

p = $39

Let q = the number of books printed and sold

Thus, C = v q + FC

C = 14.5q + 12,838 is the cost equation.

The revenue (R) is also given by,

$$R = p\,q$$

$$= 39q$$

Then to obtain the quantity of books to be printed and sold to break-even, you need to equate the R and C equations.

$$39q = 14.5q + 12{,}838$$

$$39q – 14.5q = 12838$$

$$24.5q = 12838$$

$$q = 12838/24.5$$

q = 524 books must be printed and sold to break – even.

Example 1.14

A manufacture has a fixed cost of Birr 60,000 and a variable cost of Birr 2 per unit made and sold at selling price of Birr 5 per unit. Required:

a. Write the revenue and cost equations
b. Computer the profit, if 25,000 units are made and sold
c. Compute the profit, if 10,000 units are made and sold
d. Find the breakeven quantity
e. Find the break-even birr volume of sales

f. Construct the break-even chart

Solution

Given the values,

FC = Birr 60,000 , v = Birr 2 , p = Birr 5

a. Revenue equation = p q

R = 5q

Cost equation = v q + FC

C = 2q + 60,000

b. Level of production = q = 25,000 units

Thus, R= 5q = 5 x 25,000

R = Birr 125,000

Likewise, C = v q + FC

C = (2 x 25,000) + 60,000

C = 50,000 + 60,000 = Birr 110,000

Then, Profit = R – C

π = 125,000 – 110,000

π = Birr 15,000

c. Level of production = q = 10,000 units

Thus, R = 5q = 5 x 10,000

R = Birr 50,000

Likewise, C = v q + FC

= 2 (10,000) + 60,000

= Birr 80,000

Then, Profit = R – C

π = 50,000 – 80.000

π = (Birr 30,000)

The manufacturer losses Birr 30,000 at 10,000 units level of production.

d. The break-even quantity is given by;

$q_e = \frac{FC}{p-v} = \frac{60{,}000}{5-2} = \frac{60{,}000}{3} = 20{,}000$ Units are required to be produced to break-even on operation.

e. The break – even birr volume of sales can be simply obtained by substituting the break – even quantity in the revenue (R) equation. This is,

$$R = 5q$$

Substituting q_e in place of q will result in,

$$R = p\ q_e = 5 \times 20{,}000$$

$$R = \text{Birr } 100{,}000$$

f. In sketching break – even chart, first we need to find x – and y – interests for both the revenue and cost equations. To this end, considering the cost equation, $C = 2q + 60{,}000$ we can obtain; the y – intercept (at q = 0) = FC = Birr 60,000.

Therefore, the cost equation arises from the point 60,000 on the y- axis (cost and revenue axis). Alternatively, in a simple way, the total cost line starts at the point of fixed cost.

On the other hand, the graph for the revenue originates from the origin because the revenue at zero production level is zero. Further, the two lines crosses each other at the BEP which has a coordinate of (20,000, 100,000), the 20,000 units on the quantity axis (x – axis) and Birr 100,000 on cost revenue axis (y – axis).

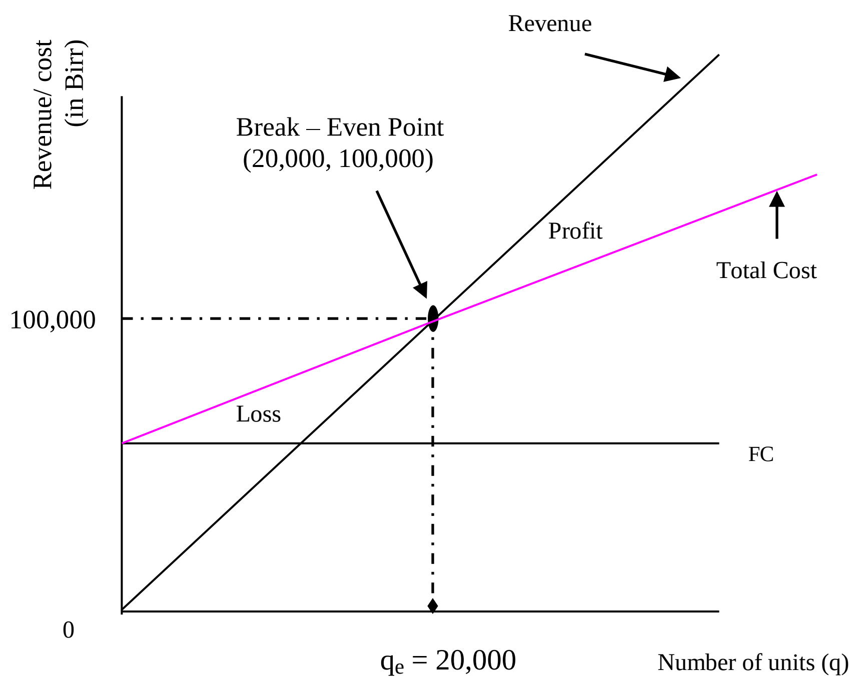


Fig 1.2.3 Solved example of BEP

Note: The BEP coordinate is given by (q_e, p q_e)

Exercise 1.9

Suppose a company has a fixed cost of Birr 35,000 and a variable cost of Birr 1.75 per unit for its products. Let us further consider that selling price is Birr 2.70 per unit.

a. Write the revenue and cost equations of the company.
b. At what level of output is the company break-even?
c. What is the amount of revenue when the company produces 300,000 units?
d. Plot the break-even chart and show the break-even point.

Case 2: Break – Even Analysis for Retail Businesses

Retail businesses and other financial managers are more likely think of break-even analysis in accounting terms. In this case, we shall consider a retailer or company that purchase products and sell them at a price above the cost. The difference between the purchase cost and retail price is known as *'markup'*. Markup is the component of profit but not exactly the same as profit. Thus, certain costs must be deducted to obtain the profit.

Example 1.15

An item that costs Birr 860 is priced to sell at Birr 940. Calculate the markup.

Solution

Given the values, Cost = Birr 860 and Retail price = Birr 940

Markup = Retail Price – Cost

= 940 – 860

Markup = Birr 80

Markup is viewed in one of the following two ways: as a percentage of retail price or as a percentage of cost.

a. Markup as a Percentage of Retail Price

The markup percentage on retail price is also called 'margin' in financial statements is given by

$$Margin = \frac{Markup}{Retail\ Price}$$

$$Margin\ \ (Markup)\ \ Percentage = \frac{Markup}{Retail\ \ Price} \times 100$$

b. Markup as a Percentage of Cost

Markup as a percentage of cost is given by

$$Markup \quad Percentage = {}^{Markup}\!/\!_{Cost} \times 100\%$$

Example 1.16

Assume an item that cost Birr 40 is made available for sale at a price of Birr 48. Find the markup percentage on cost.

Solution

$$Markup \quad Percentage = {}^{Markup}\!/\!_{Cost} \times 100\%$$

$$= {}^{48-40}\!/\!_{40} \times 100\%$$

$$= 8/40 \text{ x } 100\%$$

$$= 20\%$$

Exercise 1.10

Naïf supermarket purchases an item at Birr 115 and resells it at a price of Birr 128.

a. Compute the markup.
b. Compute the margin or markup as a percentage of retail price.
c. Compute markup as a percentage on cost.

Dear student considering the above concept about markup let us proceed to determining the break-even level of sales.

Driving the Break-Even Level of Sales

Let us symbolize some important components of the formula. Thus, consider the equation

y = m x + b, where y = represent the total cost

m = represent the variable cost per dollar of sales

x = represent the sales volume (Revenue)

m x = total variable cost

b = the fixed cost

As we have considered in the former case, at break even revenue is equal to cost. That is, y = x. Further, at break – even, the amount of dollar sale is equal to the cost, thus the break – even level of sales (x_e) is equal to y and x.

Therefore, $y = x = x_e$

Then at the break-even point, y and x can be substituted by x_e in the equation of

$y = mx + b$.

Accordingly, $x_e - m(x_e) = b$. Now let us solve for x_e.

$x_e = m(x_e) + b$

$x_e - m(x_e) = b$

$x_e(1 - m) = b$

$$\frac{xe\,(1-m)}{1-m} = \frac{b}{1-m}$$

Thus, $x_e = \dfrac{b}{1-m}$ is the expression for the break – even level of sales.

Or,

$$x_e = \frac{Fixed \quad Cost}{1 - Variable \quad Cost \quad Per \quad Dollar \quad of \quad Sales}$$

Example 1.17

Suppose that in making a budget for next year's operations top management of Hirmata Business Group has set a sales goal of Birr 200,000 per week. Margin is to be 45% of retail price and other variable cost is estimated at Birr 0.05 per birr of sales. Fixed cost is projected at Birr 56,000.

a. What is the linear sales-cost equation?
b. What is the breakeven volume of sales in birr per week?
c. What is the company's profit if sales goal is attained?
d. What is the company's profit if it sells merchandise that worth Birr 100,000?
e. Plot the company's cost-sales model.

Solution

Given values, Margin = 45% = 0.45

Other variable cost = 0.05 per birr of sales

Fixed cost (b) = birr 56,000

Sales goal (Revenue R) = Birr 200,000

x = the monetary (dollar) amount of sales (sales volume)

In addition, if margin is given as 45% the remaining 55% or 0.55 represent the cost. Thus, the variable cost per birr of sales is equal to

$$m = (100\ \% - \text{margin percentage}) + \text{other variable cost}$$
$$= (100\% - 45\%) + 0.05$$
$$= 0.55 + 0.05 = 0.60$$

Taking these values, we can solve out the problem

a. The equation

$$y = m\,x + b$$
$$y = 0.6\,x + 56.000$$

b. Break-even volume of sales

$$x_e = \frac{b}{1-m}$$

$$x_e = \frac{56{,}000}{1-0.60} = \frac{56{,}000}{0.4}$$

$$= \text{Birr } 140{,}000$$

c. For any amount of sales volume (Revenue) greater than birr 140,000 profit will be attained. At the targeted level of sale, the profit will be obtained as follows.

Profit = Revenue – Cost – R-C

= 200,000 – (0.06x200 + 56,000)

= Birr 24,000

d. Profit if the sales volume (revenue) is birr 100,000.

Profit (A) = R – C

= 100,000 – (m x + b), since cost (c) or y = mx + b

= 100,000 – (0.6 (100,000) + 65,000)

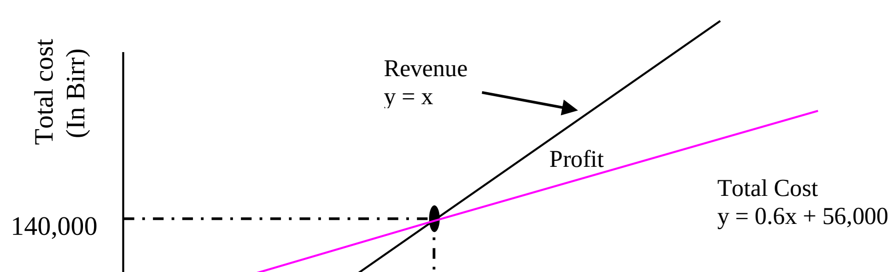
= 100,000 – (60,000 + 56,000)

= 100,000 – 116,000

= (Birr 16,000)

Hence, at sales volume of birr 100,000 the company incurs a loss of Birr 16,000.

e. Graph of cost – sales model or break – even chart



i. The break-even chart for sales volume is plotted in the same manner with chat of loran-even quantity except the x-axis in the former case represents dollar amount of sales volume (revenue).

ii. In the above diagram we note that the break-even point coordinates are equal for both x and y axis since at this point the cost and the revenue amount are equal and both are expressed in terms of monetary (dollar) sales.

iii. In the break even sales volume computation, the cost equation, $y = m x + b$, is given in terms of sales volume (Revenue) which is represented by x but, in case of competing the break-even quantity, in the cost equation $y = m q + b$, q represents the number of items or units produced. Thus, do not confuse x and q in the former and later cases respectively.

Exercise 1.11

A company expects fixed cost of Birr 23,400. Margin is to be 52% of retail, and variable cost in addition to cost of goods is estimated at birr 0.07 per dollar of sales.

a. Write the equation relating sales and costs.
b. Find the break-even birr of sale.
c. What will be profit on sales of Birr 62,500?
d. Make the break-even chart.

1.7 Market Equilibrium Analysis

The third major area in to which the concept of linear algebra and geometry applied is in the analysis of market equilibrium. Market equilibrium analysis is concerned with the supply and demand of a product in a case they are linearly related.

- ***Demand of a product:*** is the amount of a product consumers are willing and able to buy at a given price per unit. The linear demand function has a negative slope (falls downward from left to right as shown in the figure below) since demand for a product decreases as price increases.
- ***Supply of a product:*** is the amount q, of a product the producer is willing and able to supply (make available for sell) at a given price per unit, p. A linear supply curve or function has a

positive slope (rises upward from left to right as shown in the figure below) and the price and the amount of product supplied are directly related. This is because of the fact that suppliers are more interested to supply their product when the selling price increases.

- ***Market equilibrium:*** shows a market price that will equate the quantity consumers are willing and able to buy with the quantity suppliers are willing and able to supply. Thus, at the equilibrium,

Graphically,

Demand (DD) = Supply (SS)

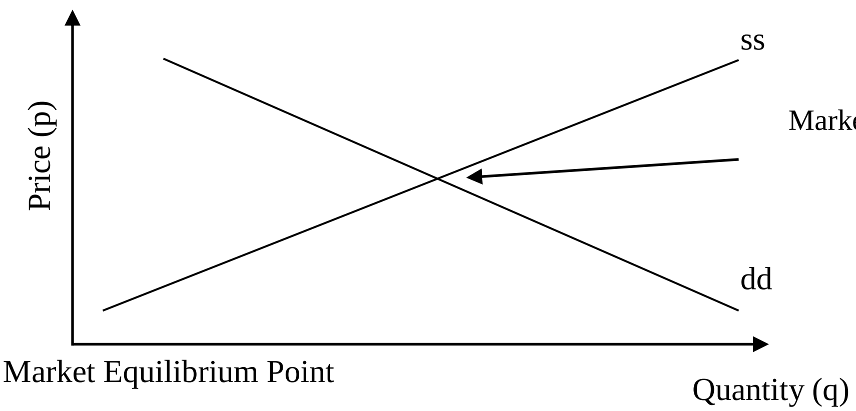


Fig 1.2.5 Market Equilibrium Point

Example 1.18

Suppose the supply and demand equation for a given product on a given day reveal the following.

Demand (DD): $P = 300 - 15q$

Supply (SS): $P = 500 + 5q$

a. Find the market equilibrium price and quantity.
b. Plot the demand and supply equation on a graph.

Solution

a. First, let us compute the *equilibrium quantity* for the given supply and demand functions.

Hence, at equilibrium: DD = SS

$$3000 - 15q = 500 + 5q$$

$$-15q - 5q = 500 - 3000$$

$$-20q = -2500$$

$$q = -2500 \div -20$$

$$q = 125 \text{ units}$$

The market equilibrium quantity is 125 units.

Now, we progress to find the *equilibrium price* for the supply and demand function of the given product. In the same manner with the above one, we can obtain the market equilibrium price by simply substituting the market equilibrium quantity in either of the supply or demand equations. Thus, let us take the supply function of $P = 500 + 5q$. Then substitute market equilibrium quantity of 125 units in place of q.

$$P = 500 + 5(125)$$

$$P = 500 + 625$$

$$P = \text{Birr } 1125$$

You will obtain the same result (i.e. Birr 1125), if you take the demand function of

$P = 3000 – 15q$ and substitute $q = 125$.

b. *Graph of demand and supply function:* In plotting the graph, first we need to get the x and y intercept for both the supply and demand equations. The y – intercept for demand equation is obtained by setting $q = 0$ in the equation $P = 3000 – 15q$. Thus, $P = 3000 – 15(0) = 3000$

Therefore, (0, 3000) is the y-intercept. Likewise, the x intercept is obtained by setting $P = 0$ in the equation $P = 3000 – 15q$. Consequently,

$$0 = 3000 – 15q$$

$$15q = 3000 \quad = \quad 3000 \div 15$$

$$q = 200$$

The point (200, 0) is the x-intercept of the demand function. The same procedure is to be followed in computing the x and y intercept for the supply function of $P = 500 + 5q$. The y – intercept is the value of P when $q = 0$.

Therefore, $P = 500 + 5(0)$

$$P = 500$$

The y – intercept is the point with coordinate of (0, 500). In the same manner, the x – intercept is the value of q at $P = 0$.

Thus, $0 = 500 + 5q$

$$-5q = 500 \quad = \quad 500 \div -5$$

$$q = -100$$

Hence, the x – intercept is given by the point (-100, 0). Off course, the graph (the straight line) of the supply function is also passes through the equilibrium point of (125 units, 1125 birr). Thus, we do not need to extend the line to the negative direction.

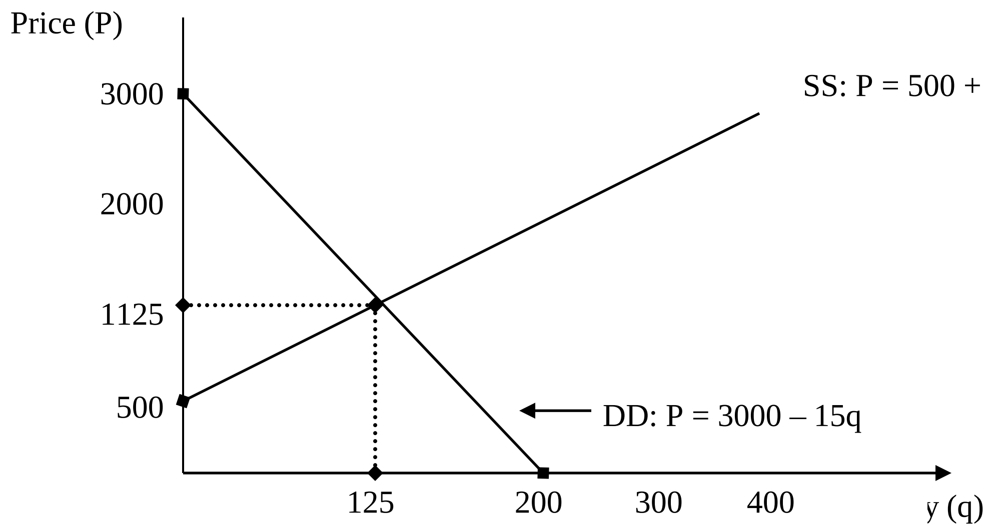


Fig 1.2.6 Solved Example of Equilibrium Point

Example 1.19

The market demand curve for a product is given by:

DD: $P = 200 - q$

SS: $p = 50 + 0.5q$

a. Find the excess demand at a price of Birr 80.
b. Find the excess supply at a price of Birr 130.
c. Find the equilibrium price and quantity.

Solution

a. Given DD: $P = 200 - q$ and P = Birr 80

Thus, at a price of Birr 80 the quantity q, demanded is computed by substituting 80 in place of p in the given function, $P = 200 - q$.

$$80 = 200 - q$$
$$80 - 200 = -q$$
$$-120 = -q$$
$$q = 120 \text{ units}$$

This implies, at the price of Birr 80, the supply will be 120 units. Now we progress to computing the amount of supply at a price of p = 80.

Thus, SS: $P = 50 + 0.5q$

$80 = 50 + 0.5q$

$80 - 50 = 0.5q$

$q = 30 \div 0.5 = 60$ units.

In this case, also if the price is Birr 80, the supply will be only 60 units. Therefore, in computing the excess demand, we take the difference between the demand and supply for the product.

Accordingly, ***Excess Demand = Demand – Supply***

= 120 units – 60 units

= 60 units.

Thus, if the price is Birr 80, there will be an excess demand for 60 units of the product.

b. Following the same procedure with the above case (i.e. item a), we can find out the excess supply for the product at a price of Birr 130. Accordingly, given demand function DD: $P = 200 - q$, we substitute Birr 130 in place of p to get the quantity demanded at the said price.

Thus, DD: $p = 200 - q$

$130 = 200 - q$

$130 - 200 = -q$

$q = 70$ units.

Therefore, at selling price of Birr 130, the demand is 70 units. On the other hand, the supply at a price of Birr 130 is obtained by,

SS: $P = 50 + 0.5q$

$130 = 50 + 0.5q$

$130 - 50 = 0.5q$

$0.5q = 80$

$q = 160$ units

Then, the *Excess Supply* is attained by:

Excess Supply = Supply – Demand

= 160 units – 70 units

= 90 units.

In general, at selling price of $p = 130$ the excess supply is equal to 90 units.

c. As we have seen earlier, equilibrium is attained when DD = SS.

That is, $$200 - q = 50 + 0.5\,q$$
$$200 - 50 = 0.5\,q + q$$
$$150 = 1.5\,q$$
$$q = 150 \div 1.5 = 100 \text{ units.}$$

Hence, at equilibrium the quantity demanded will be 100 units. In parallel, by substituting this value (100 units) in either of the supply or demand functions, we can obtain the equilibrium price. Let us take the demand function, DD: $P = 200 - q$ and then substitute 100 in place of q, to obtain

$$P = 200 - 100$$
$$P = 100 \text{ Birr.}$$

Therefore, the market equilibrium price is Birr 100.

Exercise 1.13

Jemal Plc. is a national distributor of Dell Computers. The selling price and quantity of computers distributed are linearly related. Further, the company's market analyst found out the following demand and supply functions for a particular year.

Demand (DD): $P = 3500 - 2q$

Supply (SS): $-q = 950 - p$

a. Find the excess demand for computers at a price of Birr 1400.
b. Find the excess supply of computers at a price of Birr 2100.
c. Find the market equilibrium quantity.
d. Find the market equilibrium price.
e. Sketch the demand and supply functions.