CHAPTER TWO :MATRICES AND ITS APPLICATIONS

SECTION ONE: MATRIX CONCEPTS

Section Objectives

Up on completing this section, you will be able to:

- Know the definition and meaning of a matrix.
- Know dimension of a matrix and basic types of matrices.
- Develop an insight towards basic operations in matrix and the techniques.
- Develop know-how towards inverse of a matrix
- Build an insight on matrix algebra principles and concepts.

Definition of a Matrix

A matrix is a rectangular array of numbers, parameters, or variables each of which has a carefully ordered place within the matrix. The numbers (parameters or variables) are referred to as elements of the matrix. The numbers in the horizontal like are called rows; the numbers in a vertical line are called columns. It is customary to enclose the elements of a matrix in parentheses, brackets, or braces to signify that they must be considered as a whole and not individually.

A matrix is often denoted by a single letter in bold face type. The first subscript in a matrix refers to the row and the second subscript refers to the column.

A general matrix of order *m x n* is written as:

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \vdots \\ X_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}_{(mxn)}$$

Matrix *X* above has m rows and n columns or it is said to be a matrix of order (size) m x n (read as m by n).

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{3x3}$$

Here *A* is a general matrix composed of 3x3 = 9 elements, arranged in three rows and three columns. The elements all have double subscripts which give the address or placement of the element in the matrix; the first subscript identifies the row in which the element appears and the second identifies the column. For instance, a_{23} is the element which appears in the second row and the third column and a_{32} is the element which appears in the third row and the second column.

2.2. Dimensions and Types of Matrices

Dimension of a matrix is defined as the number of rows and columns.

Based on their dimension (order), matrices are classified in to the following types:

A. **A row matrix:** is a matrix that has only one row and can have many columns.

E .g. $\boldsymbol{A} = \begin{pmatrix} 2 & 5 & 7 \end{pmatrix}$ is a row matrix of order 1x3.

B. **A column matrix:** is a matrix with one column and can have many rows.

E.g. $\boldsymbol{B} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ is a column matrix of dimensions 3x1.

C. **A square matrix*:*** is a matrix with equal number of rows and columns.

E.g. $\boldsymbol{C} = \begin{bmatrix} 6 \end{bmatrix}$; $\boldsymbol{D} = \begin{bmatrix} 2 & 6 \\ 3 & 8 \end{bmatrix}$; $\boldsymbol{E} = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 2 & 5 \\ 8 & 6 & 9 \end{pmatrix}$

D. **A diagonal matrix:** is a square matrix where its all non- diagonal elements are zero.

E.g. $\boldsymbol{x} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 11 \end{pmatrix}$ is a diagonal matrix of order 3x3.

E. **A scalar matrix:** a square matrix is called a scalar matrix if all its non- diagonal elements are zero and all diagonal elements are equal.

E.g. $\boldsymbol{Y} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\boldsymbol{Z} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

F. A unit matrix (Identity matrix)*:* is a type of diagonal matrix where its main diagonal elements are equal to one.

E.g. $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

G. **A null matrix (zero matrix):** a matrix is called a null matrix if all its elements are zero.

E.g. $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

H. ***A symmetric matrix:*** a matrix is said to be symmetric if $\boldsymbol{A} = \boldsymbol{A}^t$.

E.g. $\boldsymbol{A} = \begin{pmatrix} 8 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 5 \end{pmatrix}$

I. **<u>Idempotent matrix:</u>** this is a matrix having the property that $\boldsymbol{A^2 = A}$.

E.g. If $\boldsymbol{A} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$; then $\boldsymbol{AA}$= $\boldsymbol{A^2} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

Dear distance learner, what do you conclude about the relationship of scalar matrix and diagonal matrix? And about unit matrix and scalar matrix?

Remark:

- It is seen above that every scalar matrix is a diagonal matrix; whereas a diagonal matrix need not be a scalar matrix. Every unit matrix is a scalar matrix; whereas a scalar matrix need not be a unit matrix.

2.3. Matrix Operations and Properties

1. ***Matrix equality:*** two matrices are said to be equal if and only if they have the same dimension and corresponding elements of each matrix are equal.

E.g. $\boldsymbol{A} = \begin{Bmatrix} 3 & 0 \\ 1 & -4 \end{Bmatrix}$ $\boldsymbol{B} = \begin{Bmatrix} 3 & -4 \\ 1 & 0 \end{Bmatrix}$ $\boldsymbol{C} = \begin{Bmatrix} 3 & 0 \\ 1 & -4 \end{Bmatrix}$

- $A \neq B$; $A = C$; $B \neq C$.

2. ***Transpose of a matrix:*** If the rows and columns of a matrix are interchanged the new matrix is known as the transpose of the original matrix. If the original matrix is denoted by ***A***, the transpose is denoted by A' or $\boldsymbol{A^t}$**.** Transposition means interchanging the rows or columns of a given matrix. That is, the rows become columns and the columns become rows.

E.g $\quad \boldsymbol{B} = \begin{Bmatrix} 3 & 5 & 6 & 9 \\ 0 & 11 & 13 & 8 \\ 6 & 8 & 3 & 4 \end{Bmatrix}$

The transpose of matrix ***B***, denoted by B' or $\boldsymbol{B^t}$ is given as:

$$\boldsymbol{B^t} = \begin{Bmatrix} 3 & 0 & 6 \\ 5 & 11 & 8 \\ 6 & 13 & 3 \\ 9 & 8 & 4 \end{Bmatrix}$$

The dimension of ***B*** is changed from 3x4 to 4x3.

$$\boldsymbol{A} = \begin{Bmatrix} 1 & 3 \\ 0 & 4 \\ 2 & 8 \end{Bmatrix}_{(3x2)} \qquad A' = \begin{Bmatrix} 1 & 0 & 2 \\ 3 & 4 & 8 \end{Bmatrix}_{(2x3)}$$

<u>Properties of the transpose</u>

The following properties are held for the transpose of a matrix:

- *Property 1:* $(A^t)^t = A$
- *Property 2:* $(aA)^t = aA^t$, where (a) is a scalar ($a^t = a$)
- *Property 3:* $(A+B)^t = A^t + B^t$
- *Property 4:* $(AB)^t = B^tA^t$

3. ***Addition and subtraction of matrices*****:** Two matrices A and B can be added or subtracted if and only if they have the same order, which is the same number of rows and columns. That is, the number of columns of matrix A is equal to the number of columns of matrix B, and the number of rows of matrix A is equal to the number of rows of matrix B. Two matrices of the same order are said to be conformable for addition and subtraction. The sum and subtraction of two matrices of the same order is obtained by adding together or subtracting corresponding elements of the two matrices.

If A= (a_{ij}) and B= (b_{ij}), then C = A+B is the matrix having a general element of the form; $C_{ij} = a_{ij} + b_{ij}$. $D = A-B \rightarrow C_{ij} = a_{ij} - b_{ij}$.

Example:

$$A = \begin{Bmatrix} 2 & 0 \\ -5 & 6 \end{Bmatrix} \qquad B = \begin{Bmatrix} 3 & 6 \\ 4 & 1 \end{Bmatrix}$$

Then;

$$A+B = \begin{Bmatrix} 2+3 & 0+6 \\ -5+4 & 6+1 \end{Bmatrix} = \begin{Bmatrix} 5 & 6 \\ -1 & 7 \end{Bmatrix}$$

$$\text{If} \quad A = \begin{Bmatrix} 1 & 5 \\ 6 & 7 \\ 8 & 9 \end{Bmatrix} \qquad B = \begin{Bmatrix} 10 & 2 \\ 8 & 6 \end{Bmatrix}$$

- A+B is not defined, since orders of A and B are not the same.

Properties of matrix addition

a. *Commutative law:* A+B = B+A

$$\text{E.g.} \quad A = \begin{Bmatrix} 2 & 3 \end{Bmatrix} \quad B = \begin{Bmatrix} 4 & 3 \end{Bmatrix} \quad C = \begin{Bmatrix} 1 & 5 \end{Bmatrix}$$

1 0 2 1 3 6

$$A+B = \begin{Bmatrix} 2+4 & 3+3 \\ 1+2 & 0+1 \end{Bmatrix} = B + A = \begin{Bmatrix} 4+2 & 3+3 \\ 2+1 & 1+0 \end{Bmatrix} = \begin{Bmatrix} 6 & 6 \\ 3 & 1 \end{Bmatrix}$$

b. *Associative law:* (A + B) + C = A + (B + C)

$$(A+B)+C = \begin{Bmatrix} 6 & 6 \\ 3 & 1 \end{Bmatrix} + \begin{Bmatrix} 1 & 5 \\ 3 & 6 \end{Bmatrix} = \begin{Bmatrix} 7 & 11 \\ 6 & 7 \end{Bmatrix}$$

$$A+(B+C) = \begin{Bmatrix} 2 & 3 \\ 1 & 0 \end{Bmatrix} + \left\{ \begin{Bmatrix} 4 & 3 \\ 2 & 1 \end{Bmatrix} + \begin{Bmatrix} 1 & 5 \\ 3 & 6 \end{Bmatrix} \right\}$$

$$= \begin{Bmatrix} 2 & 3 \\ 1 & 0 \end{Bmatrix} + \begin{Bmatrix} 5 & 8 \\ 5 & 7 \end{Bmatrix} = \begin{Bmatrix} 7 & 11 \\ 6 & 7 \end{Bmatrix}$$

c. <u>Existence of identity:</u> A+ 0 = 0 + A = A.

<u>Note:</u> The subtraction (difference) of two matrices of the same order is obtained by subtracting corresponding elements.

Referring to the above matrices given in (a);

$$A - B = \begin{Bmatrix} 2 & 3 \\ 1 & 0 \end{Bmatrix} - \begin{Bmatrix} 4 & 3 \\ 2 & 1 \end{Bmatrix} = \begin{Bmatrix} 2-4 & 3-3 \\ 1-2 & 0-1 \end{Bmatrix} = \begin{Bmatrix} -2 & 0 \\ -1 & -1 \end{Bmatrix}$$

$$B-A = \begin{Bmatrix} 4 & 3 \\ 2 & 1 \end{Bmatrix} - \begin{Bmatrix} 2 & 3 \\ 1 & 0 \end{Bmatrix} = \begin{Bmatrix} 4-2 & 3-3 \\ 2-1 & 1-0 \end{Bmatrix} = \begin{Bmatrix} 2 & 0 \\ 1 & 1 \end{Bmatrix}$$

- A-B ≠ B-A, thus matrix subtraction is not commutative.

$$(A-B)-C = \left\{ \begin{Bmatrix} -2 & 0 \\ -1 & -1 \end{Bmatrix} - \begin{Bmatrix} 1 & 5 \\ 3 & 6 \end{Bmatrix} \right\} = \begin{Bmatrix} -3 & -5 \\ -4 & -7 \end{Bmatrix}$$

$$A-(B-C) = \begin{Bmatrix} 2 & 3 \\ 1 & 0 \end{Bmatrix} - \left\{ \begin{Bmatrix} 4 & 3 \\ 2 & 1 \end{Bmatrix} - \begin{Bmatrix} 1 & 5 \\ 3 & 6 \end{Bmatrix} \right\}$$

$$= \begin{Bmatrix} 2 & 3 \\ 1 & 0 \end{Bmatrix} - \begin{Bmatrix} 3 & -2 \\ -1 & -5 \end{Bmatrix} = \begin{Bmatrix} -1 & 5 \\ 2 & 5 \end{Bmatrix}$$

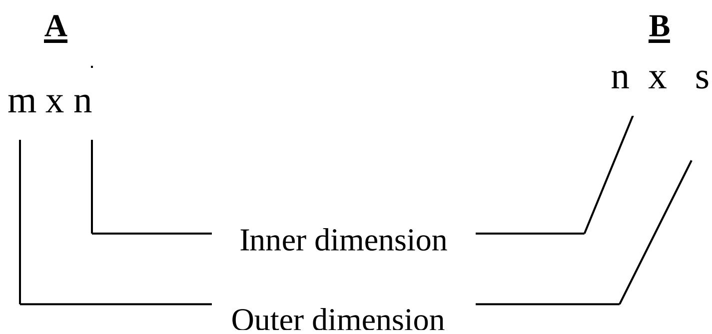
- (A-B) –C ≠ A- (B-C) ; matrix subtraction is not associative.

Remark:

- A+B = B+A and (A+B) +C= A+ (B+C) → matrix addition is both commutative and associative.
- A-B ≠ B-A and (A-B) - C ≠ A- (B-C) → matrix subtraction is neither commutative nor associative.

4. ***Matrix Multiplication***

Two matrices A and B can be multiplied together to get AB if the number of columns in A is equal to the number of rows in B.



- If two matrices have the same inner dimension, then we can get the product of the matrices. The resulting matrix will have a dimension equal to the outer dimensions of the two matrices. There are two types of matrix multiplication: multiplication by a scalar and multiplication by a matrix.

i. Scalar multiplication: in this type of multiplication, we multiply the scalar by each element of the given matrix.

E.g. If $B = \left\{\begin{matrix} 3 & 4 & 0 \\ 1 & 2 & 5 \\ 3 & 4 & 1 \end{matrix}\right\}$

$$(5).\ B = (5) \left\{\begin{matrix} 3 & 4 & 0 \\ 1 & 2 & 5 \\ 3 & 4 & 1 \end{matrix}\right\} = \left\{\begin{matrix} 15 & 20 & 0 \\ 5 & 10 & 25 \\ 15 & 20 & 5 \end{matrix}\right\}$$

ii. Multiplication by a matrix: multiplication by a matrix can be performed if the number of columns in the first matrix is equal to the number of rows in the second matrix. In this type of multiplication, we always multiply each row of the first matrix by each column of the second matrix and sum the resulting outcome.

E.g.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{pmatrix}_{(3x2)} \qquad B = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 0 & 5 \end{pmatrix}_{(2x3)}$$

Then,

$$A \text{ x } B = \begin{pmatrix} (1x2) + (2x3) & (1x1) + (2x0) & (1x4) + (2x5) \\ (3x2) + (4x3) & (3x1) + (4x0) & (3x4) + (4x5) \\ (0x2) + (1x3) & (0x1) + (1x0) & (0x4) + (1x5) \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 1 & 14 \\ 15 & 3 & 32 \\ 3 & 0 & 5 \end{pmatrix} \quad (3x3)$$

Dear student, can you show whether matrix multiplication is both communicative and associative or not?

Properties of matrix multiplication

Property 1: The distributive law is valid in matrix multiplications.

$$A (B+C) = AB + AC$$

$$(B+C)A = BA + CA$$

Property 2: The associative law is valid in matrix multiplication.

$$(AB)C = A(BC) = ABC$$

Property 3: If I is an identity matrix, then;

$$AI = IA = A$$

In general, as long as the order of the matrix is maintained, matrix multiplication is associative, but matrix multiplication is not commutative ***except*** for:

a) The multiplication of a matrix with an identity matrix;

i.e. A.I = I. A =A

b) The multiplication of a matrix with its inverse;

i.e., $A.A^{-1} = A^{-1}.A = I$

Solved problems

1. Interest at the rates of 0.06, 0.07 and 0.08 is earned on respective investments of $3000, $2000 and $4000.

a) Express the total amount of interest earned as the product of a row vector by a column vector.

b) Compute the total interest by matrix multiplication.

Solution:

Given: Let the interest rate matrix be I and investment matrix be B.

a) $I = \begin{Bmatrix} 0.06 & 0.07 & 0.08 \end{Bmatrix}_{(1x3)}$; $B = \begin{Bmatrix} 3000 \\ 2000 \\ 4000 \end{Bmatrix}_{(3x1)}$

Total interest = (Interest rate matrix) (Investment matrix) = I.B.

$$I.B = \begin{Bmatrix} 0.06 & 0.07 & 0.08 \end{Bmatrix} \cdot \begin{Bmatrix} 3000 \\ 2000 \\ 4000 \end{Bmatrix}$$

$$= \begin{Bmatrix} (0.06x\ 3000) + (0.07x2000) + (0.08x4000) \end{Bmatrix}$$

Total interest $= \begin{Bmatrix} 640 \end{Bmatrix}$ *(1x1)*

= *$ 640*

2. Finfine Furniture Factory (3F) produces three types of executive chairs namely A, B and C. The following matrix shows the sale of executive chairs in two different cities.

$$\textbf{\textit{Cities}} \quad \begin{array}{c} \\ C_1 \\ \\ C_2 \end{array} \overset{\textbf{\textit{Executive chairs}}}{\begin{array}{ccc} A & B & C \\ \left\{ \begin{array}{c} 400 \\ \\ 300 \end{array} \right. & \begin{array}{c} 300 \\ \\ 200 \end{array} & \left. \begin{array}{c} 200 \\ \\ 100 \end{array} \right\} \end{array}} \textit{(2x3)}$$

If the cost of each chair (A, B and C) is Birr 1000, 2000 and 3000 respectively, and the selling price is Birr 2500, 3000 and 4000 respectively;

a) Find the total cost of the factory for the total sale made.

b) Find the total profit of the factory.

<u>*Solution:*</u>

Given: Let the quantity matrix be q

Let the price matrix be p

Let the unit cost matrix be v;

a) $q = \left\{ \begin{array}{ccc} 400 & 300 & 200 \\ \\ 300 & 200 & 100 \end{array} \right\}$ $\quad p = \left\{ \begin{array}{c} 1500 \\ 3000 \\ 4000 \end{array} \right\}$ $\quad V = \left\{ \begin{array}{c} 1000 \\ 2000 \\ 3000 \end{array} \right\}$

- *Total cost = (unit cost) (Quantity)*

$$= \left\{ \begin{array}{ccc} 400 & 300 & 200 \\ \\ 300 & 200 & 100 \end{array} \right\} \cdot \left\{ \begin{array}{c} 1000 \\ 2000 \\ 3000 \end{array} \right\}$$

$$\left\{ \begin{array}{c} 1,600,000 \\ \\ \\ \end{array} \right\}$$

$$= \quad 1{,}000{,}000$$

Total cost = Birr 1,600,000 + Birr 1,000,000 = *Birr 2,600,000*

b) *Total profit = Total Revenue - Total Cost*

Total Revenue = (price) (quantity)

$$= \begin{Bmatrix} 400 & 300 & 200 \\ 300 & 200 & 100 \end{Bmatrix} \cdot \begin{Bmatrix} 1500 \\ 3000 \\ 4000 \end{Bmatrix}$$

$$= \begin{Bmatrix} 2{,}300{,}000 \\ 1{,}450{,}000 \end{Bmatrix}$$

Total Revenue = Birr 2,300,000 + Birr 1,450,000 = *Birr 3,750,000*

Profit = Birr 3,750,000 – Birr 2, 600,000

= *Birr 1,150,000*

Exercise 2 - 1

1. Find a and b if $\begin{Bmatrix} a+b & a-b \\ 1 & 2 \end{Bmatrix} = \begin{Bmatrix} 7 & 3 \\ 1 & 2 \end{Bmatrix}$

2. Classify the following matrices:

a) $\begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}$ b) $\begin{Bmatrix} 1 & -1 & 3 \end{Bmatrix}$ c) $\begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$ d) $\begin{Bmatrix} -2 \\ 3 \\ 5 \end{Bmatrix}$

3. A = $\begin{Bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{Bmatrix}$ and B = $\begin{Bmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{Bmatrix}$; find A + B.

4. If $A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 6 & 3 \\ 1 & 4 & 3 \end{pmatrix}$; find the value of 2A + 3B.

5. If $A = \begin{pmatrix} 8 & 4 \\ 3 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix}$;

Find the matrix x such that 2A +4B – 3x = 0.

6. If $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$;

Find A-B and B-A.

7. Given the matrices:

$A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \\ 0 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & -1 \end{pmatrix}$; Determine where possible:

a) AB
b) BA
c) 2A

8. Verify whether AB = BA for matrices:

$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

9. Given:

$A = \begin{pmatrix} 8 & 1 & -2 \\ -9 & 9 & 9 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 3 \\ 5 & 6 & -4 \end{pmatrix}$ $C = \begin{pmatrix} 4 & -3 & 1 \\ 6 & 2 & -1 \end{pmatrix}$

6 -3 9 7 -9 8 0 -4 3

Show that; i) A(B+C) = AB + AC ii) (A+B)C = AC + BC.

10. Given: A = $\begin{pmatrix} 7 & 5 \\ 1 & 3 \\ 8 & 6 \end{pmatrix}$ B = $\begin{pmatrix} 4 & 9 & 10 \\ 2 & 6 & 5 \end{pmatrix}$ C = $\begin{pmatrix} 2 \\ 6 \\ 7 \end{pmatrix}$

Determinant of a matrix

Definition: the determinant is a single number or scalar and is found only for square matrices. If the determinant of a matrix is equal to zero, the determinant is said to vanish and the matrix is termed singular.

1.Let A = $\begin{pmatrix} a_{11} \end{pmatrix}_{(1x1)}$, then the determinant of A denoted by $|A|$ or det A is a11.

i.e det A= $|A|$ = $/a_{11}/$

2. Let A = $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{(2x2)}$

$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is known as a determinant of order two

and its value is given as: $|A| = a_{11}a_{22} - a_{12}a_{21}$.

E.g. A= $\begin{pmatrix} 6 & 4 \\ 7 & 9 \end{pmatrix}$; $|A| = \begin{vmatrix} 6 & 4 \\ 7 & 9 \end{vmatrix}$ = 6(9)-7(4)=*26*

3. L et A = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ is called a third order determinant

$$|A| = +a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

E.g. Let $A = \begin{Bmatrix} 1 & 2 & 4 \\ 0 & -1 & 0 \\ -2 & 0 & 3 \end{Bmatrix}$; Find $|A|$.

$$|A| = \begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & 0 \\ -2 & 0 & 3 \end{vmatrix} = +1\begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} -2\begin{vmatrix} 0 & 0 \\ -2 & 3 \end{vmatrix} + 4\begin{vmatrix} 0 & -1 \\ -2 & 0 \end{vmatrix}$$

$= 1(-1x3 – 0x0) -2(0x3- (-2x0)) + 4(0x0 – (-2x-1))$

$= -3 -0 \;\; -8$

$= \underline{\underline{-11}}$

Note: The value of determinant of 2nd order is equal to the product of the elements along the principal diagonal minus the product of the off diagonal elements.

The value of determinant of 3rd order is equal to the summation of three products. To derive the three products:

Let $A = \begin{Bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Bmatrix}$

i) Take the first element of the first row $a_{11,}$ and mentally delete the row and column in which it appears. Then, multiply a_{11} by the determinant of the remaining elements.

ii) Take the second element of the first row a_{12} and mentally delete the row and column in which it appears. Then, multiply a_{12} by -1 times the determinant of the remaining elements.

iii) Take the third element of the first row a_{13}, and mentally delete the row and column in which it appears. Then, multiply a_{13} by the determinant of the remaining elements.

<u>Minors and Cofactors:</u> The element of a matrix remaining after the deletion process from a sub-determinant of the matrix is called a minor. Thus the minor M_{ii} is the determinant of the sub-matrix formed by deleting the i^{th} row and j^{th} column of the matrix.

Given : $A = \begin{Bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Bmatrix}$

Here, $/m_{11/}$ is the Minor of a_{11}, $/m_{12}/$ the minor of a_{12} and $/m_{13}/$ the minor of a_{13}.

$$/m_{11}/ = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\, a_{33} - a_{32}\, a_{23}$$

$$/m_{12}/ = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\, a_{33} - a_{31} a_{23}$$

$$/m_{13}/ = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\,a_{32} - a_{31}\,a_{22}$$

$$\text{Given : } A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$/m_{11}/ = /a_{22}/ = a_{22}$; $/m_{21}/ = /a_{12}/ = a_{12}$

$/m_{12}/ = /a_{21}/ = a_{21}$; $/m_{22}/ = /a_{11}/ = a_{11}$

Example:

$$\text{If } A = \begin{Bmatrix} 2 & 1 \\ 0 & 4 \end{Bmatrix} \text{ ; then,}$$

The minor of a_{11} ($/m_{11}/$), the element in the first row and first column is; $/m_{11}/ = /4/ = 4$

The minor of a_{12}, $/m_{12}/ = 0$

The minor of a_{21}, $/m_{21}/ = 1$

The minor of a_{22}, $/m_{22}/ = 2$

- The cofactor (C_{ij}) of the element a_{ij} of the matrix A is the minor of a_{ij} multiplied by $(-1)^{i+j}$; so that if i+j is even, the cofactor and the minor are equal, and if i+j is odd, the cofactor is the negative of the minor.
- The adjoint of A is the transpose of the cofactor matrix of A.

Taking the above example, the cofactors are computed as follows:

C_{11} = cofactor of a_{11} (the element in the first row and first column)

$= /m_{11}/. (-1)^{1+1}$

$= 4(-1)^2 = 4$

$C_{12} = /m_{12}/. (-1)^{1+2}$

$= 0(-1)^3 = 0$

$C_{21} = /m_{21}/. (-1)^{2+1}$

$= (1). (-1)3 = -1$

$C_{22} = /m_{22}/. (-1)^{2+2}$

$= 2(-1)^4 = 2$

The cofactor matrix denoted by C, is given by:

$$C = \begin{Bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{Bmatrix} = \begin{Bmatrix} 4 & 0 \\ -1 & 2 \end{Bmatrix}$$

- Adjoint of (A), which is the transpose of the cofactor matrix (c^t) is given by :

$$\text{Adjoint (A)} = c^t = \begin{Bmatrix} 4 & -1 \\ 0 & 2 \end{Bmatrix}$$

$$\text{Given:} \quad A = \begin{Bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & -1 \end{Bmatrix}$$

The minor of a_{11} $\longrightarrow$ $/m_{11}/ = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$

The minor of a_{12} $\longrightarrow$ $/m_{12}/ = \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} = -4$

The minor of a_{13} $\longrightarrow$ $/ m_{13}/ = \begin{vmatrix} 2 & -1 \\ & \end{vmatrix}$

$$1 \quad 0 \qquad = 1$$

The minor of a_{21} $\longrightarrow$ $/ m_{21} / = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = 1$

The minor of a_{22} $\longrightarrow$ $/ m_{22} / = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$

The minor of a_{23} $\longrightarrow$ $/ m_{23} / = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$

The minor of a_{31} $\longrightarrow$ $/ m_{31} / = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3$

The minor of a_{32} $\longrightarrow$ $/ m_{32} / = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$

The minor of a_{33} $\longrightarrow$ $/ m_{33} / = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$

The cofactor matrix is given by:

$$C = \left\{ \begin{matrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{matrix} \right\} = \left\{ \begin{matrix} 1 & 4 & 1 \\ 1 & -2 & 1 \\ 3 & 0 & -3 \end{matrix} \right\}$$

$$\text{Adjoint (A)} = c^t = \left\{ \begin{matrix} 1 & 1 & 3 \\ 4 & -2 & 0 \\ 1 & 1 & -3 \end{matrix} \right\}$$

2.4 Inverse of a Matrix

In scalar algebra, the inverse of a number is that number which, when multiplied by the original number, gives a product of 1. Hence, the inverse of x is simply 1/x; or in slightly different notation, x^{-1}. In matrix algebra, the inverse of a matrix is that which, when multiplied by the original matrix, gives an identity matrix. The inverse of a matrix is denoted by the superscript "-1". Hence, $AA^{-1} = A^{-1}A = I$.

Note that: A matrix must be square to have an inverse, but not all square matrices have an inverse. The necessary and sufficient condition for a square matrix to possess its inverse is that $/A/ \neq 0$.

Finding the inverse of a matrix requires the concept of row operations to be performed. The row operations are the following:

a. *Multiply or divide a row by a non- zero constant*;

If $A = \begin{Bmatrix} 2 & 3 \\ 6 & 9 \end{Bmatrix}$ multiply row one (R_1) by -2 to get matrix B.

Then, $B = \begin{Bmatrix} -4 & -6 \\ 6 & 9 \end{Bmatrix}$

Divide row two (R_2) by 3 to get matrix C. Then, matrix

$$C = \begin{Bmatrix} 2 & 3 \\ 2 & 3 \end{Bmatrix}$$

b. *Add a multiple of one row to another row*;

If $A = \begin{Bmatrix} 1 & 2 \\ 3 & 4 \end{Bmatrix}$ multiply R_1 by 2 and add to R_2 to get matrix x.

Matrix $X = \begin{Bmatrix} 1 & 2 \\ 5 & 8 \end{Bmatrix}$

C. *Interchanging of rows;*

If $A = \begin{Bmatrix} 1 & 0 \\ 2 & 4 \end{Bmatrix}$ Interchange R_1 and R_2 ($R_1 \leftrightarrow R_2$) ; to get matrix D.

$$D = \begin{Bmatrix} 2 & 4 \\ 1 & 0 \end{Bmatrix}$$

N.B: The first row elements in the original matrix become second row elements in the new matrix and vice versa.

Dear distance learner! Do you know the three most important methods to find inverse of a matrix?

The most important methods to find inverse of a given matrix include the following:

1. *Gauss- Jordan Inversion method*
2. *The zero-first method*
3. *The cofactor technique*

Dear student! Now let us look at each of the inversion methods one by one.

I. Gauss- Jordan Inversion Method

This method was developed by a mathematician called Gauss and it was named so by the founder. The Gauss- Jordan inversion method starts by writing the given matrix at the left and the corresponding identity matrix next to it, at the right. Then, select and carryout row operations that will convert the given matrix in to an identity matrix and apply the same operations to the matrix at the right simultaneously. When the left or the given matrix becomes an identity matrix, the matrix at the right will be the desired inverse matrix.

i.e. $\{A/I\} \xrightarrow{\text{Apply Elementary Row Operation (ERO)}} \{I/A^{-1}\}$

Example: Find the inverse of the following matrix using the Gauss- Jordan method.

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

<u>Solution</u>

Steps:

1st: write the given matrix at the left and the corresponding identity matrix at the right;

$$\{A/I\} = \left(\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)$$

N.B: corresponding identity matrix for 2x2 matrix is of dimension 2x2.

2nd : Interchange R_1 and R_2;

$$\left(\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{array}\right)$$

3rd: Multiply R_1 by -3 and add the result to R_2;

$$\begin{array}{lcrrrr} -3R_1 & = & -3 & -3 & 0 & -3 \\ + & & & & & \\ \underline{R_2} & = & 3 & 2 & 1 & 0 \\ \hline & & 0 & -1 & 1 & -3 \end{array}$$

The resulting matrix is given by:

$$\left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ & & & \end{array}\right)$$

$0 \quad -1 \quad 1 \quad -3$

*4^th^*: Simply add R_2 entries to R_1 entries;

$$\begin{array}{rrrrr} R_2 = & 0 & -1 & 1 & -3 \\ + & & & & \\ R_1 = & 1 & 1 & 0 & 1 \\ \hline & 1 & 0 & 1 & -2 \end{array}$$

The resulting matrix is given by:

$$\left\{\begin{array}{rr|rr} 1 & 0 & 1 & -2 \\ 0 & -1 & 1 & -3 \end{array}\right\}$$

*5^th^*: Multiply R_2 by -1;

$(-1)(R_2) = \ 0 \quad 1 \quad -1 \quad 3$

The resulting matrix is given by;

$$\left\{\begin{array}{rr|rr} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 3 \end{array}\right\}$$

Dear student! Have you noticed that the original matrix is converted in to identity matrix and the corresponding identity matrix to inverse matrix?

Thus; the inverse matrix A, denoted by A^{-1} is given as:

$$A^{-1} = \left\{\begin{array}{rr} 1 & -2 \\ -1 & 3 \end{array}\right\}$$

Check! $A.A^{-1} = A^{-1}. A = I$

$$= \left\{\begin{array}{rr} 3 & 2 \\ 1 & 1 \end{array}\right\} . \left\{\begin{array}{rr} 1 & -2 \\ -1 & 3 \end{array}\right\} = \left\{\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array}\right\}$$

Example: Find the inverse of B = $\begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 4 & 0 & 3 \end{pmatrix}$

Solution:

1st → write the given matrix at the left and the corresponding identity matrix at the right.

$$\{B/I\} = \left(\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 4 & 0 & 3 & 0 & 0 & 1 \end{array}\right)$$

N.B: The corresponding identity matrix for a 3x3 square matrix is of dimension 3x3.

2nd → Divide R_1 by 2 (or multiply R_1 by ½);

$R_2/2$ = 2/2 2/2 3/2 1/2 0/2 0/2

= 1 1 3/2 ½ 0 0

The resultant matrix is: $\left(\begin{array}{ccc|ccc} 1 & 1 & 3/2 & ½ & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 4 & 0 & 3 & 0 & 0 & 1 \end{array}\right)$

3rd → Multiply R_1 by -4 and add to R_3 ($-4R_1 + R_3$);

$-4R_1$ = -4 -4 -6 -2 0 0

\+

R_3 = 4 0 3 0 0 1

---

0 -4 -3 -2 0 1

The resultant matrix is: $\left(\begin{array}{ccc|ccc} 1 & 1 & 3/2 & ½ & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -4 & -3 & -2 & 0 & 1 \end{array}\right)$

4th → Multiply R_2 by -1 and add to R_1 ($-1R_2+R_1$);

$(-1)R_2$ = 0 -1 -1 0 -1 0

+

R_1 = 1 1 3/2 ½ 0 0

1	0	½	½	-1	0

The resultant matrix is given by:

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & ½ & ½ & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -4 & -3 & -2 & 0 & 1 \end{array}\right\}$$

5^{th} → Multiply R_2 by 4 and add to R_3 ($4R_2$ +R3);

$4R_2$ =	0	4	4	0	4	0
+	0	-4	-3	-2	0	1
R_3	0	0	1	-2	4	1

The resultant matrix is given by:

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & ½ & ½ & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 4 & 1 \end{array}\right\}$$

6^{th} → Multiply R_3 by -1/2 and add to R_1 (-1/2 R_{3+} R_1);

-1/2 R_3 =	0	0	-1/2	1	-2	-1/2
+						
R_1 =	1	0	½	½	-1	0
	1	0	0	3/2	-3	-1/2

5f

The resultant matrix is green by:

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -3 & -1/2 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 4 & 1 \end{array}\right\}$$

$7^{th} \rightarrow$ Multiply R_3 by -1 and add to R_2 $(-1R_3+R_2)$;

$(-1)\ R_3$ =		0	0	-1	2	-4	-1
+							
R_2	=	0	1	1	0	1	0
		0	1	0	2	-3	-1

The resultant matrix is green by:

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -3 & -1/2 \\ 0 & 1 & 0 & 2 & -3 & -1 \\ 0 & 0 & 1 & -2 & 4 & 1 \end{array}\right\}$$

Thus; $B^{-1} = \left\{\begin{array}{ccc} 3/2 & -3 & -1/2 \\ 2 & -3 & -1 \\ -2 & 4 & 1 \end{array}\right\}$

Dear student! Have you understand that the elementary row operations converted the original matrix (B) in to an identity matrix (I) and the corresponding identity matrix to the desired inverse (B^{-1})? Dear student! Reanalyze the above two examples to justify the situation.

Exercise 2 - 2

1. Find the inverse of the following matrices using the Gauss – Jordan method.

$$A = \begin{Bmatrix} 3 & 3 \\ 2 & 2 \end{Bmatrix}$$

2. Write the expanded matrix form of a 2 by 3 general matrix.
3. If matrix p is 1 by 2 and we have py = q, where q is a 1 by 1 matrix:

a. What are the dimensions of matrix y?

b. Write the expanded vector form of the equation.

c. Write the usual algebraic form.

4. Find the inverse of $A = \begin{Bmatrix} 3 & 4 \\ 2 & 1 \end{Bmatrix}$

The Zero – first method

In using this method, first find zeros in the off-diagonal followed by ones in the main diagonal.

Example;

If $C = \begin{Bmatrix} 2 & 3 \\ 4 & 7 \end{Bmatrix}$; Find C^{-1} using zero –first method.

Solution:

1st → Write the augmented matrix; (C/I)

$$\left\{\begin{array}{cccc} 2 & 3 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right\}$$

2nd

2^{nd} → To translate the off- diagonal element in the send row and first column (i.e., 4) in to zero; the elementary row operation is; $-2R_1 + R_2$;

$$\begin{array}{rrrrr} -2R_1 = & -4 & -6 & -2 & 0 \\ + & & & & \\ \underline{R_2 =} & \underline{4} & \underline{7} & \underline{0} & \underline{1} \\ & 0 & 1 & -2 & 1 \end{array}$$

The resultant matrix is; $\left\{\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right\}$

3^{rd} →To translate the remaining off – diagonal element (i.e., 3) in to zero; the elementary row operation is; $-3R_2 + R_1$;

$$\begin{array}{rrrrr} -3R_2 = & 0 & -3 & 6 & -3 \\ + & & & & \\ R_1 = & 2 & 3 & 1 & 0 \\ \hline & 2 & 0 & 7 & -3 \end{array}$$

The resultant matrix is; $\left\{\begin{array}{cc|cc} 2 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right\}$

4^{th} → To translate the main- diagonal entry (i.e., 2) in to one; the elementary row operation is; $R_1/2$;

$R_1/2 = 1 \quad 0 \quad 7/2 \quad -3/2$

The final resultant matrix is given by:

$$\left\{\begin{array}{cc|cc} 1 & 0 & 7/2 & -3/2 \\ 0 & 1 & -2 & 1 \end{array}\right\}$$

Thus; $C^{-1} = \left\{\begin{array}{cc} 7/2 & -3/2 \\ -2 & 1 \end{array}\right\}$

Check: $C.C^{-1} = C^{-1} = I$

$$\left\{\begin{array}{cc} 2 & 3 \\ 4 & 7 \end{array}\right\} \quad \left\{\begin{array}{cc} 7/2 & -3/2 \\ -2 & 1 \end{array}\right\} \left\{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right\}$$

✓

Example: Find the inverse of the following matrix by using the zero – first method.

$$D = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & -2 \end{pmatrix}$$

<u>Solution:</u>

*1*1^{st} →Write the augmented matrix;

$$\left\{\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \end{array}\right\}$$

N.B: the corresponding identity matrix for a 3x3 matrix is of dimension 3x3.

2nd →Interchange R_1 and R_3;

The resultant matrix is given by;

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & 1 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{array}\right\}$$

3rd → Add R_1 to R_2;

$$\begin{array}{lcccccc} R_1 = & 1 & 0 & -2 & 0 & 0 & 1 \\ + & & & & & & \\ R_2 = & -1 & 1 & 2 & 0 & 1 & 0 \\ \hline & 0 & 1 & 0 & 0 & 1 & 1 \end{array}$$

The resultant matrix is given by:

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{array}\right\}$$

4th → Add R_2 to R_3;

$$\begin{array}{lcccccc} R_2 = & 0 & 1 & 0 & 0 & 1 & 1 \\ + & & & & & & \\ R_3 = & 0 & -1 & 1 & 1 & 0 & 0 \\ \hline & 0 & 0 & 1 & 1 & 1 & 1 \end{array}$$

The resultant matrix is given by:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 0 & 1 \\ & & & & & \\ & & & & & \end{array}\right)$$

$$\begin{array}{cccccc} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}$$

$5^{th} \rightarrow 2R_3 + R_1$;

$$\begin{array}{lcccccc} 2R_3 = & 0 & 0 & 2 & 2 & 2 & 2 \\ \quad + & & & & & & \\ R_1 = & 1 & 0 & -2 & 0 & 0 & 1 \\ \hline & 1 & 0 & 0 & 2 & 2 & 3 \end{array}$$

The resultant matrix is given by: $\left\{\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 3 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right\}$

Thus; $D^{-1} = \left\{\begin{array}{ccc} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right\}$

Check: $D.D^{-1} = D^{-1}.D = I$

$$\left\{\begin{array}{ccc} 0 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & -2 \end{array}\right\} . \left\{\begin{array}{ccc} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right\} = \left\{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right\}$$

✓

<u>Exercise 2 - 3</u>

Find the inverse of the following using the zero –first method;

a) $A = \left\{\begin{array}{ccc} 2 & 1 & 3 \\ 3 & 1 & 2 \end{array}\right\}$

1 2 3

b) $C = \begin{Bmatrix} 3 & 2 \\ 1 & 5 \end{Bmatrix}$ c) $D = \begin{Bmatrix} 3 & 3 \\ 4 & 4 \end{Bmatrix}$

III. The Cofactor Technique

This method involves computation of minors, cofactors and the adjoint matrix in order to find the inverse of a given matrix.

Formula wise; the inverse of a given matrix A is given by:

$A^{-1} = 1/\left|A\right|(\text{Adj. } A)$

Dear student! Remember the computation of a determinant; minors; cofactors and adjoint of a matrix in the previous sections of the chapter.

N.B:

- The minor of a_{ij} is the determinant of a given matrix; say A with row i and column j eliminated.
- The cofactor C_{ij} of the element a_{ij} of matrix A is the minor of a_{ij} multiplied by $(-1)^{i+j}$.
- The adjoint of A is the transpose of the cofactor matrix of A.

Example; If $A = \begin{Bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{Bmatrix}$; Find its inverse using the above formulae.

Solution:

$a_{11} = 1$ and its cofactor $c_{11} = (-1)^{1+1} \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 7$

a_{12}= -2 and its cofactor $c_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix} = -1$

$a_{13} = 3$ and its cofactor $c_{13} = (-1)^{1+4} \begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} = 11$

Similarly; C_{21}= 7; C_{22}= 11; C_{23}= 5, C_{31}= -7; C_{32}= 7 and C_{33}=7.

Adjoint of matrix A, Adj. (A) = the transpose of the cofactor matrix (C^t).

Cofactor matrix;

$$C = \begin{Bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{Bmatrix} = \begin{Bmatrix} 7 & -1 & 11 \\ 7 & 11 & 5 \\ -7 & 7 & 7 \end{Bmatrix}$$

$$\text{Adj. (A)} = C^t = \begin{Bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{Bmatrix}$$

Determinant of matrix A; det (A) = 1x7 – 2x1 +3x11

= 38

Then, $A^{-1} = \dfrac{(\text{Adj (A)})}{|A|}$

$$= \quad 1/38 \begin{Bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{Bmatrix} = \begin{Bmatrix} 7/38 & 7/38 & -7/38 \\ -1/38 & 11/38 & 7/38 \\ 11/38 & 5/38 & 7/38 \end{Bmatrix}$$

Exercise 2 - 4

Find the inverse of the following using the cofactor technique;

a)

$$C = \begin{Bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & -1 \end{Bmatrix}$$

b)

$$x = \begin{Bmatrix} 1 & -1 \\ 1 & 1 \end{Bmatrix}$$

Dear Student! What do you conclude about inverse of matrix?

SECTION TWO: MATRIX APPLICATIONS

Section Objectives

Up on completing this section, you will be able to:

- Develop an insight towards application areas of matrix algebra.
- Handle large linear systems using matrix algebra.
- Undertake markov chain analysis with the help of matrix algebra.

2.5 Solving System of Linear Equations

A system of linear equations can be solved by the following three methods using matrix algebra:

a) Cramer's rule (the determinant method)
b) The inverse method
c) The Gauss- Jordan method

Let us see each of the three methods one by one.

a) Cramer's Rule:

This method sometimes called the determinant method; works according to this formula:

$X_i = \frac{/A_i/}{/A/}$; where x_i = indicates the variables we want to solve for.

$/A_i/$ = is the determinant obtained by putting the right-hand side of the system in place of the column of coefficients of the variable whose solution is needed; and

$/A/$ = is the determinant of the system.

Given a system of equations:

i) $a_{11}x+a_{12}y = b_1 \rightarrow$ algebraic form

$a_{21}x+a_{22}y = b_2$

The above system of equations can also be rewritten in expanded matrix form as follows:

$$\begin{Bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Bmatrix} \cdot \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}$$

Matrix of coefficients denoted by A — *column vector of variables (X)* — *column vector of constants (B)*

Using Cramer's rule, the solution is given by:

$$X = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{|A|} \quad ; \quad \text{and} \quad Y = \frac{\begin{vmatrix} a11 & b1 \\ a21 & b2 \end{vmatrix}}{|A|}$$

Example; $x - y = 1$, $x + y = 2$

$$= \begin{Bmatrix} 1 & -1 \\ 1 & 1 \end{Bmatrix} \cdot \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \rightarrow \text{matrix expression}$$

Then, $x = \frac{\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}}{|A|}$

$$= \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$\frac{}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}} = 3/2$$

$$y = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}{|A|} \qquad \frac{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}} = 1/2$$

ii) Given a system of equations:

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Expanded form:

$$\underbrace{\begin{Bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}}_{\boldsymbol{X}} = \underbrace{\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}}_{\boldsymbol{B}}$$

Then; the value of x is given by:

$$\frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} ; \quad y = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{}$$

$$\begin{vmatrix} a_{11} & a_{22} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example; Solve using Cramer's rule:

$$2x + y - z = 0$$
$$x + y + z = 0$$
$$y - z = 1$$

Expanded form of the above system is:

$$\underset{\boldsymbol{A}}{\begin{Bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{Bmatrix}} . \underset{\boldsymbol{X}}{\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}} = \underset{\boldsymbol{B}}{\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}}$$

$$|A| = +2\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} -1\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} +(-1)\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -4$$

Thus; $x = \dfrac{\begin{vmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}}{|A|} = 2/-4 = -1/2$

$$Y = \dfrac{\begin{vmatrix} 2 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}}{|A|} = -3/-4 = 3/4$$

$$Z = \begin{vmatrix} 2 & 1 & 0 \\ & & \\ & & \end{vmatrix}$$

$$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix} \quad = \quad -1/4$$

$$|A|$$

b) The Inverse Method:

It is used to find the solution of linear equations when the number of equations is equal to the number of variables (i.e. for square matrices only).

Consider the following system of linear equations:

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

These equations can be expanded as:

$$\begin{Bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Bmatrix} \cdot \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

Matrix of coefficient (A) *column vector of Variables (x)* *column vector of constants (B)*

- Therefore; $AX = B$

If we multiply the above equation by A^{-1}, we get $(A^{-1}A)(x) = A^{-1}B$

$IX = A^{-1}B$, but $A^{-1}A = I$; therefore; $IX = X$

Thus; $\underline{\underline{X = A^{-1}B}}$

Because we know matrix B, we need to find A^{-1}, which we know how to obtain. If we find the inverse, we multiply it by vector B, and the outcome will be the solution. In total, in using the inverse method to find the solution of linear equations:

1st Find the inverse of the coefficient matrix.

2nd Multiply the inverse with the column vector of constants.

Example:

Solve the following equations using the inverse method.

a) $2x_1 + 3x_2 = 17$

$x_1 + 2x_2 = 10$

b) x+2y-3Z=11

3x+2y+Z=1

2x+y - 5Z=11

Dear student, please try to solve these examples before going to the solution part.

Solution :

a) 1st Write the expanded form;

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 17 \\ 10 \end{pmatrix}$$

2nd Find the inverse of the coefficient matrix;

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right)$$

⇨ Interchange R_1 and R_2;

$$\left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right)$$

⇨ $-2R_1 + R_2$;

-2 R = -2 -4 0 -2

$$
\begin{array}{rrrrr}
+ & & & & \\
R_2 = & 2 & 3 & 1 & 0 \\
\hline
 & 0 & -1 & 1 & -2
\end{array}
$$

The resultant matrix is given by:

$$
\left(\begin{array}{rr|rr} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right)
$$

⇨Multiply R_2 by -1;

$$-1(R_2) = 0 \quad 1 \quad -1 \quad 2$$

The resultant matrix is given by:

$$
\left(\begin{array}{rr|rr} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right)
$$

Thus, the inverse of the coefficient matrix is:

$$
\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}
$$

3rd Multiply the inverse by the constant matrix;

$X = A^{-1}B$

$$
\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} . \begin{pmatrix} 17 \\ 10 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}
$$

Therefore, $X_1 = 4$ and $X_2 = 3$.

b) *1st* ⇨ write the expanded form;

$$
\begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & 1 \\ & & \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 11 \\ 1 \end{pmatrix}
$$

2 1 -5 Z 11

2nd ⇨Find the inverse of the coefficient matrix;

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & -5 & 0 & 0 & 1 \end{array}\right)$$

⇨ $-3R_1 + R_2$;

$-3R_1$ =	-3	-6	9	-3	0	0
+						
R_2	3	2	1	0	1	0
	0	-4	10	-3	1	0

The resultant matrix is given by:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 10 & -3 & 1 & 0 \\ 2 & 1 & -5 & 0 & 0 & 1 \end{array}\right)$$

⇨ $-2R_1 + R_3$;

$-2R_1$=	-2	-4	6	-2	0	0
+						
R_3=	2	1	-5	0	0	1
	0	-3	1	-2	0	1

The resultant matrix is given by:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 10 & -3 & 1 & 0 \\ 0 & -3 & 1 & -2 & 0 & 1 \end{array}\right)$$

⇨ $R_2/2 + R_1$;

$R_2/_2$ =	0	-2	5	-3/2	½	0
+						
R_1 =	1	2	-3	1	0	0

1 0 2 -½ ½ 0

The resultant matrix is:

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & 2 & -½ & ½ & 0 \\ 0 & -4 & 10 & -3 & 1 & 0 \\ 0 & -3 & 1 & -2 & 0 & 1 \end{array}\right\}$$

⇨ $-R_2/4$;

$-R_2/_4$= 0 1 -5/2 ¾ -¼ 0

The resultant matrix is given by:

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & 2 & -½ & ½ & 0 \\ 0 & 1 & -5/2 & ¾ & -¼ & 0 \\ 0 & -3 & 1 & -2 & 0 & 1 \end{array}\right\}$$

⇨ $3R_2+R_3$;

$3R_2$ = 0 3 -15/2 9/4 -¾ 0

+

R_3= 0 -3 1 -2 0 1

0 0 -13/2 ¼ -3/4 1

The resultant matrix is:

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & 2 & -½ & ½ & 0 \\ 0 & 1 & -5/2 & ¾ & -¼ & 0 \\ 0 & 0 & -13/2 & ¼ & -¾ & 1 \end{array}\right\}$$

⇨ $-2/13R_3$

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & 2 & -½ & ½ & 0 \\ 0 & 1 & -5/2 & ¾ & -¼ & 0 \\ 0 & 0 & 1 & -1/26 & 3/26 & -2/13 \end{array}\right\}$$

⇨ $5/2\ R_3+R_2$

$5/2\ R_3$ = 0 0 5/2 -5/52 15/52 -10/26

+

$$R_2 = \frac{0 \quad 1 \quad -5/2 \quad 3/4 \quad -1/4 \quad 0}{0 \quad 1 \quad 0 \quad 17/26 \quad -1/26 \quad -5/13}$$

The resultant matrix is given by:

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & 2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 17/26 & 1/26 & -5/13 \\ 0 & 0 & 1 & -1/26 & 3/26 & -2/13 \end{array}\right\}$$

⇨ $-2R_3+R_1$;

$$\begin{array}{rcccccc} -2R_3= & 0 & 0 & -2 & 1/13 & -3/13 & 4/13 \\ + & & & & & & \\ R_1= & 1 & 0 & 2 & -1/2 & 1/2 & 0 \\ \hline & 1 & 0 & 0 & -11/26 & 7/26 & 4/13 \end{array}$$

The final resultant matrix is given by:

$$\left\{\begin{array}{ccc|ccc} 1 & 0 & 0 & -11/26 & 7/26 & 4/13 \\ 0 & 1 & 0 & 17/26 & 1/26 & -5/13 \\ 0 & 0 & 1 & -1/26 & 3/26 & -2/13 \end{array}\right\}$$

3rd ⇨Multiply the inverse (A^{-1}) with the constants matrix (B); i.e. $X=A^{-1}B$;

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} -11/26 & 7/26 & 4/13 \\ 17/26 & -7/13 & -5/13 \\ -1/26 & 3/26 & -2/13 \end{pmatrix} * \begin{pmatrix} 11 \\ 1 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ \end{pmatrix}$$ So, x = -1, y= 3, and z = -2.

z -2

c) The Gauss-Jordan Method:

This method works through several operations to reduce a given matrix of coefficients of a system equation in to an identity matrix. It is used to find the solution of linear equations:

i) When the number of equations (m) is equal to the number of variables (n); i.e. m = n.

ii) When the number of equations (m) is greater than the number of variables (n); i.e., m>n.

iii) When the number of equations (m) is less than the number of variables (n); i.e., m<n.

- It applies the concept of row operations both on the coefficient matrix (A) and the column vector of constants (B) in order to convert them to an identity matrix (I) and the solution matrix (S) respectively.

That is, ***(A/B)*** $\xrightarrow{\textbf{\textit{row operations}}}$ ***(I/S)***

i. ***Number of equations (m) equals number of variables (n)***

The intention to convert (A/B) to (I/S) when the number of equations (m) equals the number of variables (m) will result in:

1. Unique solution if the coefficient matrix has an inverse.
2. An infinite solution if the elements in the last row are all zeros including the constants' column.
3. No solution if there is a row that is all zeros except in the constants' column.

Example-1: Solve the following system of two equations:

$$2x_1 + 3x_2 = 5$$
$$x_1 + x_2 = 3$$

Solution:

Form the augmented matrix A consisting of the coefficients of x_1 and x_2 and the column of the right-hand side of the above system. That is,

$$(A/B) = \left(\begin{array}{cc|c} 2 & 3 & 5 \\ 1 & 1 & 3 \end{array}\right)$$

As mentioned, we want to reduce the coefficient matrix in to an identity matrix of dimension 2 x 2; this can be done through the following operations:

Multiply the first row by (½), keeping the second row intact, and yields:

$$\left(\begin{array}{cc|c} 1 & 3/2 & 5/2 \\ 1 & 1 & 3 \end{array}\right)$$

The first element of the first row is (1); accordingly, the first row is the pivotal row to perform the operations with. We need now to make the first element of the second row (0). To do so, we keep the first row intact, multiply it by (-1), and add the result to the second row, which yields:

$$\left(\begin{array}{cc|c} 1 & 3/2 & 5/2 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

Now, we need to convert the (-½) in to (1) to create a pivotal row. This can be done by multiplying the second row by (-2); keeping the first row intact yields:

$$\left(\begin{array}{cc|c} 1 & 3/2 & 5/2 \\ 0 & 1 & -1 \end{array}\right)$$

Again, we want to convert the (3/2) in to (0). To do so, we keep the second row intact (the pivot), multiply it by (-3/2), and add the result to the first row, which yields:

$$\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array}\right)$$

As can be seen through several operations, we reduce the coefficient matrix in to an identity matrix of dimension 2x2. The last column of the above matrix is the solution for x_1 and x_2, respectively.

Example -2 :

$$X+Y = 2$$
$$2X+2Y = 4$$

<u>Solution:</u>

1st → write the expanded form;

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ Y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\left(A/B \right) \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 4 \end{array}\right)$$

2nd → $-2R_1 + R_2$;

$-2R_1 =$ -2 -2 -4

+

$R_2 =$ $\underline{2 \quad 2 \quad 4}$

$0 \quad 0 \quad 0$

The resultant matrix is given by:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right) \leftrightarrow$$ Infinite solution since the last row entries are all zeros.

Dear Student! Notice that, no row operations can convert matrix A in to an identity matrix. So, no further operation is required. The bottom row entries (all being zero) indicate the case of an infinite solution.

Example-3 :

X+Y+Z = 4

5X-Y+7Z = 25

2X-Y+3Z = 8

Solution:

$1^{st} \rightarrow$ Expanded form;

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & -1 & 7 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 4 \\ 25 \\ 8 \end{pmatrix}$$

$$2^{nd} \rightarrow [A/B] = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 5 & -1 & 7 & 25 \\ 2 & -1 & 3 & 8 \end{array}\right)$$

$3^{rd} \rightarrow$ Apply row operations to convert $[A/B] \rightarrow [I/S]$

⇨ $-5R_1 + R_2$;

$-5R_1$ =	-5	-5	-5	-20
+				
R_2 =	5	-1	7	25
	0	-6	2	5

The resultant matrix is:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -6 & 2 & 5 \\ 2 & -1 & 3 & 8 \end{array}\right)$$

⇨ $-2R_1+R_3$;

$-2R_1$=	-2	-2	-2	-8
+				
R_3 =	2	-1	3	8
	0	-3	1	0

The resultant matrix is:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -6 & 2 & 5 \\ 0 & -3 & 1 & 0 \end{array}\right)$$

⇨ $R_2/-6$;

$$= \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1/3 & -5/6 \\ 0 & -3 & 1 & 0 \end{array}\right)$$

⇨ $-R_2+ R_1$;

$-R_2$ =	0	-1	1/3	5/6
+				
R_1 =	1	1	1	4
	1	0	4/3	29/4

The resultant matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 4/3 & 29/4 \\ 0 & 1 & -1/3 & -5/6 \\ 0 & -3 & 1 & 0 \end{array}\right)$$

⇨ $3R_2 + R_3$:

$3R_2 =$	0	3	-1	-5/2
+				
$R_3 =$	0	-3	1	0
	0	0	0	-5/2

The resultant matrix is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 4/3 & 29/4 \\ 0 & 1 & -1/3 & -5/6 \\ 0 & 0 & 0 & -5/2 \end{array}\right) \leftrightarrow \textit{No solution}$$

Dear student! You have noticed that bottom row entries are all zeros except the constants' column (which is -5/2) resulting in no solution case.

ii. Number of equations (m) greater than number of variables (n)

The intention to convert $[A/B] \rightarrow [I/S]$ will result either in:

1. An n by n identity matrix above m-n bottom rows that are all zeros, giving the unique solution.
2. A row that is all zeros except in the constants' column indicating that there are no solutions.
3. A matrix in a form different from (1) and (2) indicating an unlimited solution.

Example-1

$2x_1 - 3x_2 = 6$

$x_1 + 5x_2 = 29$

$3x_1 - 4x_2 = 11$

Solution:

1st → write the expanded form;

$$= \begin{Bmatrix} 2 & -3 \\ 1 & 5 \\ 3 & -4 \end{Bmatrix} . \begin{Bmatrix} x_1 \\ \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 29 \\ 11 \end{Bmatrix}$$

2nd → Apply row operations;

$$\left\{ \begin{array}{cc|c} 2 & -3 & 6 \\ 1 & 5 & 29 \\ 3 & -4 & 11 \end{array} \right\}$$

⇨ Interchange R_1 and R_2;

$$\left\{ \begin{array}{cc|c} 1 & 5 & 29 \\ 2 & -3 & 6 \\ 3 & -4 & 11 \end{array} \right\}$$

⇨ $2R_1 + R_2$;

$$\left\{ \begin{array}{cc|c} 1 & 5 & 29 \\ 0 & -13 & -52 \\ 3 & -4 & 11 \end{array} \right\}$$

⇨ $R_2/-13$;

$$\left\{ \begin{array}{cc|c} 1 & 5 & 29 \\ 0 & 1 & 4 \\ 3 & -4 & 11 \end{array} \right\}$$

⇨ $-5R_2 + R_1$;

$$\left\{ \begin{array}{c|c} \quad & \quad \end{array} \right\}$$

$$\begin{matrix} 1 & 0 & 9 \\ 0 & 1 & 4 \\ 3 & -4 & 11 \end{matrix}$$

⇨ $-3R_1 + R_3$;

$$\begin{matrix} 1 & 0 & 9 \end{matrix}$$
$$\left\{\begin{array}{cc|c} 0 & 1 & 4 \\ 0 & -4 & -16 \\ & & \end{array}\right\}$$

⇨ $4R_2+R_3$;

$$\left\{\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right\}$$

↔ Unique solution

Thus; $x_1 = 9$, and $x_2 = 4$

Example-2;

$2x_1+x_2 = 30$

$x_1+ 2x_2 = 24$

$4x_1+5x_2 = 72$

Solution:

1st → Expanded form;

$$\left\{\begin{matrix} 2 & 1 \\ 1 & 2 \\ 4 & 5 \end{matrix}\right\} \left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\} = \left\{\begin{matrix} 30 \\ 24 \\ 72 \end{matrix}\right\}$$

2nd → Apply row operations;

$$\left\{ A/B \right\} ⇨ \left\{\begin{matrix} 2 & 1 \\ 1 & 2 \\ 4 & 5 \end{matrix}\right\} = \left\{\begin{matrix} 30 \\ 24 \\ 72 \end{matrix}\right\}$$

⇨ Interchange R_1 and R_2;

$$\left\{\begin{array}{cc|c} 1 & 2 & 24 \\ 2 & 1 & 30 \\ 4 & 5 & 72 \end{array}\right\}$$

⇨ $-2R_1 + R_2$;

$$\left\{\begin{array}{cc|c} 1 & 2 & 24 \\ 0 & -3 & -18 \\ 4 & 5 & 72 \end{array}\right\}$$

⇨ $-4R_1 + R_3$;

$$\left\{\begin{array}{cc|c} 1 & 2 & 24 \\ 0 & -3 & -18 \\ 0 & -3 & -24 \end{array}\right\}$$

⇨ $R_2/-3$;

$$\left\{\begin{array}{cc|c} 1 & 2 & 24 \\ 0 & 1 & 6 \\ 0 & -3 & -24 \end{array}\right\}$$

⇨ $-2R_2 + R_1$;

$$\left\{\begin{array}{cc|c} 1 & 0 & 12 \\ 0 & 1 & 6 \\ 0 & -3 & -24 \end{array}\right\}$$

⇨ $3R_2 + R_3$;

$$\left\{\begin{array}{cc|c} 1 & 0 & 12 \\ & & \\ & & \end{array}\right\}$$

$$\begin{matrix} 0 & 1 & 6 \\ 0 & 0 & -6 \end{matrix} \quad \leftrightarrow \textit{No solution}$$

iii. Number of equations (m) less than number of variables (n)

The attempt to convert matrix (A/B) → (I/S), when m <n, will result either in:

1. A row which is all zeros except in the constant column, indicating that there are no solutions.
2. A matrix in a form different from (1), indicating that there are unlimited number of solutions.

Example;

$4x_1 + 6x_2 - 3x_3 = 12$

$6x_1 + 9x_2 - 9/2x_3 = 20$

Solution:

$$1^{st} \rightarrow \left\{\begin{array}{ccc|c} 4 & 6 & -3 & 12 \\ 6 & 9 & -9/2 & 20 \end{array}\right\}$$

*2^nd^* $\rightarrow R_1/4$;

$$\left\{\begin{array}{ccc|c} 1 & 3/2 & -3/4 & 3 \\ 6 & 9 & -9/2 & 20 \end{array}\right\}$$

$3^{rd} \rightarrow -6R_1+R_2$;

$$\left\{\begin{array}{ccc|c} 1 & 3/2 & -3/4 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right\} \quad \leftrightarrow \text{No solution case!}$$

Exercise 2-5

1. Solve the following using the Cramer's rule;

a. $5x_1+3x_2 =13$
$4x_2 =11$

b. $x_1+x_2+3x_3 =3$
$x_1+x_3 = 2$

c. $2x_1+4x_2 = 7$
$4x_1+3x_2 =1$
$4x_1+2x_2+3x_3 =5$

2. Solve the following using the inverse method;

a. $2x+z = 3$
$3x+5y+7z = 1$
$3z = 7$

b. $3x_1 + 2x_2 = 6$
$x_1+3x_2 = 5$
$4x_1+ 2x_2+3x_3 =5$

c. $x_1+x_2+3x_3 = 3$
$x_1+x_3 = 2$

3. Solve the following using the Gauss-Jordan method;

a. $x_1+x_2+3x_3 =3$
$x_1+2x_2+x_3 =2$
$4x_1+2x_2+3x_3=5$

b. $2x_2 = 4$
$x_1+3x_2 = 5$

Solving Word Problems

Steps:

1. Represent the unknown quantities by letters.
2. Translate the quantities from the statement of the problem and form an algebraic expression; then, set up an equation.
3. Solve the equations for the unknowns.
4. Check the findings as per the statement in the problem.

Example-1

A manufacturer produces two products p aid q. Each unit of product p requires in its production 20 units of raw material A and 10 units of raw material B whereas each unit of product q requires 30 units of raw material A and 50 units of raw material B. There is a limited supply of only 1200 units of raw material A and 950 units of raw material B. How many units of P and q can be produced if the manufacturer is to exhaust the supply of raw materials (to operate at full capacity).

Solution:

Given: Raw materials	Type of Products		Availability
	P	q	
A	20	30	1200
B	10	50	950

Step-1: Let x and y represent the number of units of product P and q to be produced respectively at full capacity.

Step 2: Formulate the equations:

Raw material A: $20x+30y=1200 \Rightarrow 2x+3y=120$

Raw material B: $10x+50y = 950 \Rightarrow x+5y = 95$

Step 3: Solve the equations:

Let us apply the Gaussian method;

Expanded form:

$$\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} . \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 120 \\ 95 \end{pmatrix}$$

$(A/B) \rightarrow (I/S)$

$$\left(\begin{array}{cc|c} 2 & 3 & 120 \\ & & \end{array}\right)$$

1 5 95

→ Interchange $R_1$and R_2;

$$\left(\begin{array}{cc|c} 1 & 5 & 95 \\ 2 & 3 & 120 \end{array}\right)$$

→ $-2R_1+R_2$;

$$\left(\begin{array}{cc|c} 1 & 5 & 95 \\ 0 & -7 & -70 \end{array}\right)$$

→ $R_2/-7$;

$$\left(\begin{array}{cc|c} 1 & 5 & 95 \\ 0 & 1 & 10 \end{array}\right)$$

→ $-5R_2+R_1$;

$$\left(\begin{array}{cc|c} 1 & 0 & 45 \\ 0 & 1 & 10 \end{array}\right)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 45 \\ 10 \end{pmatrix}$$

⇨x = 45 units

y = 10 units

Step -4: Cross- Checking

- 20x+30y =1200

  20(45)+30(10) =1200

  1200 =1200 ✓

- 10x+50y = 950

  10(45)+50(10)=950

  950 = 950 ✓

Example -2

Attendance records indicated that 1600 people attended a football game and the total ticket receipt was 2800 birr.

The admission price was 1.50 birr for students and 2.50 birr for others.

Determine the number of students and non-students who attended the game.

Dear student, please try to solve the above example before going to the next solution.

Solution:

Given:	*Attendants*		*Total*
	Students	*Non-students*	*receipt*
Admission price …….	1.50 birr	2.50 birr	2800 birr
Number of People	-	-	1600

Step -1: Let S and N represent the number of students and non – students who attended the same respectively.

Step- 2: Develop the equations;

Number of people: S+N = 1600

Receipt : 1.50 S+ 2.50 N = 2800

Step- 3: Apply the Gaussian method;

$$\begin{pmatrix} 1 & 1 \\ 1.5 & 2.5 \end{pmatrix} \cdot \begin{pmatrix} S \\ N \end{pmatrix} = \begin{pmatrix} 1600 \\ 2800 \end{pmatrix}$$

$$\Rightarrow \ [A/B] \rightarrow [I/S]$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1600 \\ 1.5 & 2.5 & 2800 \end{array}\right)$$

$\Rightarrow$ $-1.5\ R_1 + R_2$;

$$\left(\begin{array}{cc|c} 1 & 1 & 1600 \\ 0 & 1 & 2800 \end{array}\right)$$

$\Rightarrow$ $-R_2 + R_1$;

$$\left(\begin{array}{cc|c} 1 & 0 & 1200 \\ 0 & 1 & 400 \end{array}\right)$$

$$\begin{pmatrix} S \\ N \end{pmatrix} = \begin{pmatrix} 1200 \\ 400 \end{pmatrix}$$

⇨ Number of students(S) = 1200.

Number of non-students (N) = <u>400</u>

Total = *<u>1600</u>*

Example -3

A mixture containing X – pounds of ingredient A, Y- pounds of ingredient B and Z – pounds of ingredient C is to be made. The mixture is expected to have a weight of 5 pounds and contains

1500 units of vitamin and 2500 units of calories. The vitamin and calorie content of the three ingredients is shown below.

Ingredients	*Number of pounds*	*Units of Vitamin Per pound*	*Units of Calories per pound*
A	X	500	300
B	Y	200	600
C	Z	100	700

Determine how many pounds of each ingredient should be in the 5 pound mixture.

Solution:

Let pounds of ingredient A be X

″ ″ B ″ Y

″ ″ C ″ Z

Equations: x+y+Z = 5 …..Weight of the mixture

5x+2y+Z =15…………… Vitamine

3x+6y+7Z = 25………….. Calories

Solve using matrix algebra**;**

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & 2 & 1 \\ 3 & 6 & 7 \end{pmatrix} . \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ 25 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 5 & 2 & 1 & 15 \\ 3 & 6 & 7 & 25 \end{array}\right)$$

⇨ $-5R_1+R_2$;

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -3 & -4 & -10 \\ 3 & 6 & 7 & 25 \end{array}\right)$$

⇨ $-3R_1+R_3$;

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -3 & -4 & -10 \\ 0 & 3 & 4 & 10 \end{array}\right)$$

⇨ $R_{2/}-3$;

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 4/3 & 10/3 \\ 0 & 3 & 4 & 10 \end{array}\right)$$

⇨ $-3R_2+R_3$;

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 4/3 & 10/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

⇨ $-R_2+R_1$;

$$\left(\begin{array}{ccc|c} 1 & 0 & -1/3 & 5/3 \\ 0 & 1 & 4/3 & 10/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

⇨ Unlimited solution case!

2.6 Markov Chain Analysis

Definition: Markov analysis is the process of determining future occurrences depending on present occurrences. It is a continuous process by which an outcome at each stage is determined from the previous outcome by applying a certain fixed proportion.

There are two important data of the model:

a) The set of transition probabilities, and

b) The current or initial state

Based on these inputs, the model makes two predictions:

i. The probability of the system being in any state at any given time (i.e., state probability).

ii. The long run or steady state or equilibrium probability.

The markov model is based on the transitional probability describing a situation that changes between two stages. The first occurs at time (t-1), and the second stage takes place at time (t). In other words, the current probability of a certain state depends on the probability of the immediate preceding state only.

The markov model has various applications in business and economics. For example, it can be applied to a model where customers are buying products from two different stores and where managers are interested in knowing how these customers change stores or continue buying from the same store over the long run. Also, the markov model can be applied to a situation where people migrate from region to region between time (t-1) and (t).

Dear student! What do you know about transition probabilities, stages, states, regular transition matrix, steady state matrix, absorbing and non- absorbing chains?

States, Stages and Transitional Probabilities

Imagine the following table of probabilities:

		State at the end of stage "t"				
		1	2	3	4…………….	n
State at the end of stage "t-1"	1	P_{11}	P_{12}	P_{13}	P_{14}…………………	P_{1n}
	2	P_{21}	P_{22}	P_{23}	P_{24}…………………	P_{2n}
	3	P_{31}	P_{32}	P_{33}	P_{34}…………………	P_{3n}
	⋮	⋮	⋮	⋮	⋮	⋮

n	P_{n1}	P_{n2}	P_{n3}	P_{n4}……………..........P_{nn}

The table reveals some technical concepts. The first concept is the state concept such as state 1, 2, and so forth, which means that the system is in a certain category or condition. For example; it is sunny today (stage t-1) and will be cloudy to morrow (Stage t). The second concept is the stage concept. The table shows that there are two stages; stage t-1 and stage t. This means that if stage t-1 represents yesterday, stage t represents today (or a future day). Similarly, if stage t-1 represents the month of May, stage t indicates the month of June (or a future month). The third concept is the transition probabilities, P_{ij}. P_{ij}, i =1, 2….n and j =1, 2….n, which represent the probabilities that the system will change from state i to state j during stage t. For example, P_{12} reflects the probability that the system will change from state 1 to state 2 during stage t. Similarly p_{23} reflects the probability that the system will change form state 2 to state 3 during stage t. In fact, this is the reason why the probability matrix P, where

$$P = \begin{pmatrix} P_{11} & p_{12} & p_{13} & p_{14} \ldots\ldots\ldots p_{1n} \\ P_{21} & P_{22} & P_{23} & P_{24} \ldots\ldots\ldots P_{2n} \\ P_{31} & P_{32} & P_{33} & P_{34} \ldots\ldots\ldots P_{3n} \\ - \cdot - \cdot - & - \cdot - \cdot - & - \cdot - \cdot - & - \cdot - \cdot - \\ p_{n1}\ p_{n2} & p_{n3} & p_{n4} & P_{nn} \end{pmatrix}$$

is called the transition matrix.

The transition matrix, which is a stochastic matrix, is a square matrix of dimension nxn with the following properties:

i. All the elements P_{ij} must be between 0 and 1 (i.e. $p_{ij} \geqq 0$), and
ii. The sum of each row must equal 1.

The fourth technical concept that the table of probabilities provides is the concept of a system. In markov analysis, the system consists of all the states along with their transition probabilities, provides is the concept of a system. In markov analysis, the system consists of all the states along with their transition probabilities, P_{ij}.

Example:

Assume a weather record showing that it will be sunny to tomorrow with a probability of 70 percent if it is sunny today; otherwise it will be cloudy tomorrow if it is sunny today. Also, the record shows that the probability that it will be cloudy tomorrow 50 percent of the time if it is cloudy today; otherwise it will be sunny tomorrow if it is cloudy today. The task is to formulate the transition probability matrix, taking in to consideration the two states of sunny and cloudy. The transition probability is shown below. The table reveals two stages: today (t-1) and tomorrow (t). The transition probabilities reflect two states: Cloudy and sunny.

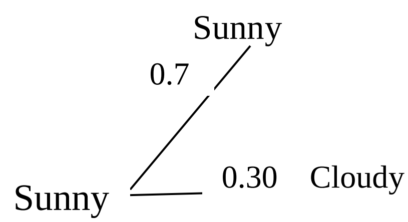
		Tomorrow	
		Sunny	***Cloudy***
Today	***Sunny***	0.70	0.30
	Cloudy	0.50	0.50

As can be seen, each entry of the transition matrix is between zero and one, and each entry can be described as follows.

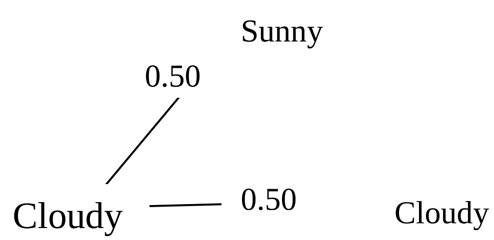
The probability 0.70 represents the probability that it will be sunny tomorrow if it is sunny today. The probability of 0.30, which is (1-0.70), reflects the probability that it will be cloudy tomorrow if it is sunny today. The probability of 0.50 represents the probability that it will be sunny tomorrow if it is cloudy today. The entry, 0.50, represents the probability that it will be cloudy tomorrow if it is cloudy today.

Stated differently, these transition probabilities are actually conditional probabilities. If S represents the event sunny, and C represents the event cloudy, then given that the state cloudy has occurred today, the probability of being sunny tomorrow (or P(S/C) is equal to 0.70; this is called a conditional probability. Similarly, P(C/C) is the conditional probability of cloudy tomorrow given it is cloudy to day, which is equal to 0.50.

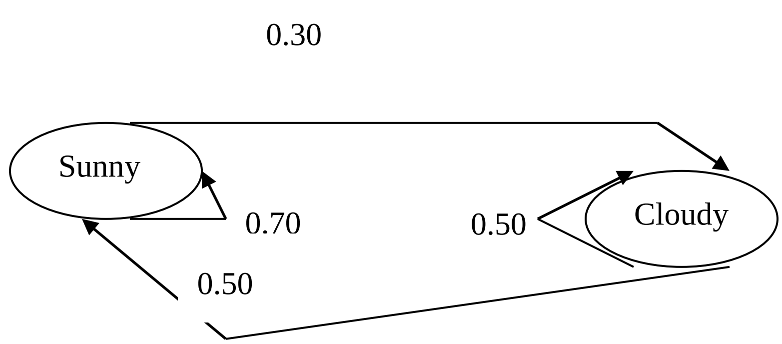
The transition (conditional) probabilities can be described by probability trees. Given that it is sunny today, the probabilities that it will be sunny or cloudy tomorrow are 0.70 and 0.30, respectively, as shown in the following:



Given the state that it is cloudy today, the probabilities that it will be sunny or cloudy tomorrow are 0.50 and 0.50 respectively, and this is shown below:



The transition probabilities for the above problem are shown in the following transition diagram:



Example:

Assume that a market is shared by two cigarette brands namely Rothmans and Nyala. The Ethiopian Tobacco Corporation conducted a market survey and has concluded the following information about the proportion of smokers who stay with the same brand or change brands in consecutive months.

- Of the smokers who bought Rothmans last month, 80% buy it again and 20 percent change to Nyala this month.
- Of the smokers who bought Nyala last month, 40% buy it again and 60% change to Rothmans this month. If for a sample of 1000 smokers, 400 buy Rothmans and 600 buy Nyala in the first month of observations, what figure can we expect for the second, third and fourth months?

Solution:

This month

Last month	*Rothmans*	*Nyala*
Rothmans	0.80	0.20
Nyala	0.60	0.4

⇨ Transition probabilities

Rothmans	*Nyala*
400	600

⇨ Current or initial state vector

- ***Remarks:***
  - ✓ The elements in the main-diagonal indicate brand loyalty.
  - ✓ The elements in the off-diagonal indicate shifting of customers.
  - ✓ The rows indicate the proportion of those retained and lost by a given brand.
  - ✓ The column elements indicate the proportion of customers retained and gained by a given brand.

Let R ⇨ represents Rothmans

N ⇨ represents Nyala

⇨ The expected figure in the second month:

(Current state) (*Transition probabilities*)

$$= \begin{array}{cc} \underline{R} & \underline{N} \\ \end{array} \begin{pmatrix} 400 & 600 \end{pmatrix} \begin{array}{c} R \\ N \end{array}\!\!\begin{pmatrix} \underline{R} & \underline{N} \\ 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix} = \begin{array}{cc} \underline{R} & \underline{N} \end{array} \begin{pmatrix} 680 & 320 \end{pmatrix}$$

⇨ The expected figure in the third month:

$$= \begin{pmatrix} \text{State of second month} \end{pmatrix} \cdot \begin{pmatrix} \text{Transition probabilities} \end{pmatrix}$$

$$= \begin{array}{cc} \underline{R} & \underline{N} \end{array} \begin{pmatrix} 680 & 320 \end{pmatrix} \begin{array}{c} R \\ N \end{array}\!\!\begin{pmatrix} \underline{R} & \underline{N} \\ 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix} = \begin{array}{cc} \underline{R} & \underline{N} \end{array} \begin{pmatrix} 736 & 264 \end{pmatrix}$$

⇨ The expected figure in the four month:

$$= \begin{pmatrix} \text{Third month state probability} \end{pmatrix} = \begin{pmatrix} \text{Transition probabilities} \end{pmatrix}$$

$$= \begin{pmatrix} \underline{R} & \underline{N} \\ 736 & 264 \end{pmatrix} \begin{matrix} \\ R \\ N \end{matrix} \begin{pmatrix} \underline{R} & \underline{N} \\ 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} \underline{R} & \underline{N} \\ 747 & 253 \end{pmatrix}$$

Markov Chain Formula:

$$V_{ij(n)} = V_{ij(n-1)}\ p$$

Where;

$V_{ij(n)}$ = the state probability at the n^{th} period

$V_{ij(n-1)}$ = the state probability at the $(n-1)^{th}$ period

P = the transition probability

Taking the above example, if we continue the process; the state probabilities for the following consecutive months will be:

Month	***Rothmans***	***Nyala***
5……………………………..…749.44		250.56
6………………………… ….749.888		250.112
7……………………………..749.9776		250.0224
8……………………………..749.99552		250. 00448
9……………………………..749.999104		250.0008964
.	.	.
	.	.
		≈ 250

.

.

n ≈ *750*

Pre-multiplying P with (750 250) gives us:

$$\begin{pmatrix} 750 & 250 \end{pmatrix} \cdot \begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 750 & 250 \end{pmatrix}$$

⇨ Steady state probability

Note: Finding the steady state probability using the above pre-multiplication of the rounded figure is cumbersome and time taking.

In reality, it is very difficult to find the steady state transition matrix by multiplying the P matrix by itself many times. The alternative approach for finding the steady state matrix is to do the following:

$$S = SP$$

For the steady state matrix (S) of two states only, we have:

$$\begin{pmatrix} S_1 & S_2 \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \end{pmatrix}$$

For the above example, $P = \begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$;

Then;

$$\begin{pmatrix} S_1 & S_2 \end{pmatrix} . \begin{pmatrix} 0.6 & 0.4 \\ 0.8 & 0.2 \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \end{pmatrix}$$

After performing the required multiplication, we obtain;

$-0.2S_1 + 0.6\ S_2 = 0$ ……………..e_1

$0.2\ S_1 + (-0.6\ S_2) = 0$…………..$e_2$

$S_1 + S_2 = 1$ …………………..e_3 The sum of the rows in the state vector equals 1.

This is a system of three equations but two unknowns. As the first two equations are linearly dependent, one of the equations has to be dropped. Assume, the second equation is dropped, the system becomes;

$$-0.2S_1 + 0.6\ S_2 = 0$$
$$S_1 + S_2 = 1$$

Solve simultaneously!

Multiply the second equation by 0.2;

$$-0.2S_1 + 0.6\ S_2 = 0$$
$$\underline{0.2S_1 + 0.2S_2 = 0.2}$$
$$0 + 0.80S_2 = 0.2$$

Solve simultaneously!

$$S_2 = \frac{0.20}{0.80} = \underline{\underline{0.25}}$$

Then, substituting $S_2 = 0.25$ in to any one of the equations,

$S_1 = 0.75$.

Therefore; $(S_1\ \ S_2) = (0.75\text{x}100 \quad 0.25\text{x}100) = (750\ 250)$, which is the steady state vector.

Example:

The market for a particular product is shared by three department stores: X, Y and Z. A market survey has produced the market transition table below which describes the proportion of customers who buy at the same store again or change stores in consecutive months.

From/To	**X**	**Y**	**Z**
X	0.6	0.2	0.2
Y	0.1	0.6	0.3
Z	0.2	0.6	0.2

Required:

a. Find the share of the market which each store would command at the steady state.

b. If the sample of 4000 people is assumed to be customers of the three stores, calculate the number of customers who used each store at the steady state.

Solution:

a) Let V_1, V_2 and V_3 represent the steady state vector;

At the steady state: $(V_1 \quad V_2 \quad V_3).(P) = (V1 \quad V_2 \quad V_3)$

Then,

$$(V_1 \quad V_2 \quad V_3) \cdot \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.2 & 0.6 & 0.2 \end{pmatrix} = (V_1 \quad V_2 \quad V_3)$$

After matrix multiplication, we obtain the following;

$0.6V_1 + 0.1V_2 + 0.2V_3 = V_1 \Rightarrow -0.4V_1 + 0.1V_2 + 0.2V_3 = 0$………..$e_1$

$0.2V_1 + 0.6V_2 + 0.6V_3 = V_2 \Rightarrow 0.2V_1 - 0.4V_2 + 0.6V_3 = 0$ ………..…e_2

$0.2V_1 + 0.3\, V_2 + 0.2V_3 = V_3 \Rightarrow 0.\, 2V_1 + 0.3V_2 - 0.8V_3 = 0$……….$e_3$

$V_1 + V_2 + V_3 = 1$..e_4

Equate e_1 and e_4

X (0.4)

$-0.4V_1 + 0.1V_2 + 0.2V_3 = 0$

$\underline{V_1+V_2+V_3 = 1}$

$-\, 0.4V_1 + 0.1V_2 + 0.2V_3 = 0$

$\underline{0.\, 4V_1 + 0.4V_2 + 0.4V_3 = 0}$

$0.5V_2 + 0.6V_3 = 0.4$.................................e_5

Equate e_2 and e_4

X (-0.2)

$0.2V_1 - 0.4V_2 + 0.6V_3 = 0$

$\underline{V_1 + V_2 + V_3 = 1}$

$0.2V_1 - 0.4V_2 + 0.6V_3 = 0$

$\underline{-0.2V_1 - 0.2V_2 - 2V_3 = -0.2}$

$-0.6V_2 + 0.4V_3 = -0.2$.................................e_6

Equate e_5 and e_6

X(0.6) $0.5V_2 + 0.6V_3 = \quad 0.4$

X (0.5) $\underline{-0.6V_2 + 0.4V3 = -0.2}$

$0.3V_2 + 0.36V_3 + = 0.24$

$\underline{-0.3V_2 + 0.2V_3 = -0.1}$

$0.56V_3 = 0.14$

$$V_3 = \frac{0.14}{0.56} = \underline{\underline{0.25}}$$

Substitute $V_3 = 0.25$ in e_6;

Then, $-0.6\, V_2 + 0.4V_3 = -0.2 \Rightarrow -0.6V_2 + 0.4\, (0.25) = -0.2$

$-0.6V_2 = -0.3$

$V_2 = \frac{-0.3}{-0.6} = \underline{\underline{0.50}}$

Substitute $V_2 = 0.5$ and $V_3 = 0.25$ in e_4;

Then; $V_1 + V_2 + V_3 = 1 \Rightarrow V_1 + 0.50 + 0.25 = 1$

$V_1 = 1 - 0.75 = \underline{\underline{0.25}}$

$(V_1 \ \ V_2 \ \ V_3) = (0.25 \ \ 0.5 \ \ 0.25) \Rightarrow$ Steady state vectors

b) Total number of people = $\underline{\underline{4000}}$

- $\Rightarrow$ Customers of store x at the steady state = 0.25x 4000 = $\underline{\underline{1000}}$
- $\Rightarrow$ Customers of store y at the steady state = 0.5x 4000 = $\underline{\underline{2000}}$
- $\Rightarrow$ Customers of store z at the steady state = 0.25x 4000 = $\underline{\underline{1000}}$

Exercise 2- 6

1. At a point in time, 1 percent of the population uses a drug and 99 percent do not. In a year, 1/10 of one percent of non – users become users, but all users remain users.

(a) What will be the percent of users and non- users after one transition?

(b) What is the steady state vector?

2. A division of the ministry of public health has conducted a sample survey on public attitude towards the use of condom. For the survey, the division concluded that currently only 20 percent of the population uses condom and every month 10 percent of non–users become users whereas 5 percent of the users discontinue using. Based on this information;

(a) What will be the percentage of users from the total population after just two months?

(b) What will be the proportion of users and non- users at the steady state?

3. A college of business has three departments: Economics; Accountancy and Finance, with the following transition probabilities:

	Economics	*Accounting*	*Finance*
Economics	0.30	0.30	0.40
Accountancy	0.01	0.89	0.10
Finance	0.25	0.20	0.55

If the initial distributions of students are (200 500 600), respectively,

(a) Find the distribution after two years.

(b) Find the long – run probabilities for each department and the long –run distribution of students.