CHAPTER 5

5. PROBABILITY& PROBABILITY DISTRIBUTION

5.1 PROBABILITY

- Probability theory is the foundation upon which the logic of inference is built.
- It helps us to cope up with uncertainty.
- In general, probability is the chance of an outcome of an experiment. It is the measure of how likely an outcome is to occur.

5.2 Definitions of some probability terms

1. **Experimen**t: Any process of observation or measurement or any process which generates well defined outcome.
2. **Probability Experiment**: It is an experiment that can be repeated any number of times under similar conditions and it is possible to enumerate the total number of outcomes without predicting an individual out come. It is also called random experiment.

Example: If a fair die is rolled once it is possible to list all the possible outcomes i.e.1, 2, 3, 4, 5, 6 but it is not possible to predict which outcome will occur.

3. **Outcome** :The result of a single trial of a random experiment
4. **Sample Space**: Set of all possible outcomes of a probability experiment
5. **Event**: It is a subset of sample space. It is a statement about one or more outcomes of a random experiment .They are denoted by capital letters.

   Example: Considering the above experiment let A be the event of odd numbers, B be the event of even numbers, and C be the event of number 8.

$$\Rightarrow A = \{1,3,5\}$$

$$B = \{2,4,6\}$$

$$C = \{\ \}\ or\ empty\ space\ or\ \ impossible\ event$$

Remark:

If S (sample space) has n members then there are exactly 2^n subsets or events.

6. **Equally Likely Events**: Events which have the same chance of occurring.
7. **Complement of an Event: the complement of an event A means non-**occurrence of A and is denoted by $A', or\ A^c, or\ \overline{A}$ contains those points of the sample space which don't belong to A.
8. **Elementary Event**: an event having only a single element or sample point.
9. **Mutually Exclusive Events**: Two events which cannot happen at the same time.

10. **Independent Events**: Two events are independent if the occurrence of one does not affect the probability of the other occurring.
11. **Dependent Events**: Two events are dependent if the first event affects the outcome or occurrence of the second event in a way the probability is changed.

Example: .What is the sample space for the following experiment

a) Toss a die one time.
b) Toss a coin two times.
c) A light bulb is manufactured. It is tested for its life length by time.

Solution

a) S={1,2,3,4,5,6}
b) S={(HH),(HT),(TH),(TT)}
c) S={t /t≥0}

- Sample space can be
  - Countable (finite or infinite)
  - Uncountable

5.3 Counting Rules

In order to calculate probabilities, we have to know

- The number of elements of an event
- The number of elements of the sample space.

That is in order to judge what is **probable**, we have to know what is **possible.**

- In order to determine the number of outcomes, one can use several rules of counting.
  - The addition rule
  - The multiplication rule
  - Permutation rule
  - Combination rule

- To list the outcomes of the sequence of events, a useful device called **tree diagram is** used.

The addition rule

Suppose that the 1st procedure designed by 1 can be performed in n1 ways. Assume that 2nd procedure designed by 2 can be performed in n2 ways. $(n_1 + n_2 + ... + n_k)\, ways.$ suppose further more that, it is not possible that both procedures 1 and 2 are performed together then the number of ways in which we can perform 1 or 2 procedure is n_1+n_2 ways, and also if we have another procedure that is designed by k with possible way of n_k we can conclude that there is $n_1+n_2+\ldots+n_k$ possible ways.

Example: suppose we planning a trip and are deciding by bus and train transportation. If there are 3 bus routes and 2 train routes to go from A to B. find the available routes for the trip.

Solution:

There are 3+2 =5 routes for someone to go from A to B.

The Multiplication Rule:
If a choice consists of k steps of which the first can be made in n_1 ways, the second can be made in n_2 ways… the k^{th} can be made in n_k ways, then the whole choice can be made in $(n_1 * n_2 * .... * n_k)$ *ways.*

Example 1
An air line has 6 flights from A to B, and 7 flights from B to C per day. If the flights are to be made on separate days, in how many different ways can the airline offer from A to C?

Solution: In operation 1 there are 6 flights from A to B, 7 flights are available to make flight from B to C. Altogether there are 6*7 = 42 possible flights from A to C.

Example 2
suppose that in a medical study patients are classified according to their blood type as A, B , AB, and O; according to their RH factors as + or - and according to their blood pressure as high, normal or low ,then in how many different ways can a patient be classified ?

Solution: The 1st classification done in 4 ways, the 2nd in 2 ways, and the 3rd in 3 ways. Thus patient can be classified in 4*2*3 = 24 different ways.

Example 3
The digits 0, 1, 2, 3, and 4 are to be used in 4 digit identification card. How many different cards are possible if

a) Repetitions are permitted.
b) Repetitions are not permitted.

Solutions

a)

1st digit	2nd digit	3rd digit	4th digit
5	5	5	5

There are four steps

1. Selecting the 1st digit, this can be made in 5 ways.
2. Selecting the 2nd digit, this can be made in 5 ways.
3. Selecting the 3rd digit, this can be made in 5 ways.
4. Selecting the 4th digit, this can be made in 5 ways.

$$\Rightarrow 5*5*5*5 = 625 \text{ } \textit{different cards are possible}.$$

b)

1^{st} digit	2^{nd} digit	3^{rd} digit	4^{th} digit
5	4	3	2

There are four steps

5. Selecting the 1^{st} digit, this can be made in 5 ways.
6. Selecting the 2^{nd} digit, this can be made in 4 ways.
7. Selecting the 3^{rd} digit, this can be made in 3 ways.
8. Selecting the 4^{th} digit, this can be made in 2 ways.

$$\Rightarrow 5*4*3*2 = 120 \textit{ different cards are possible.}$$

<u>Permutation</u>

An arrangement of n objects in a specified order is called permutation of the objects.

Permutation Rules:

1. The number of permutations of n distinct objects taken all together is n!

   Where $n! = n*(n-1)*(n-2)*.....*3*2*1$

   ${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$ =n!. In definition 0! = 1! = 1

2. The arrangement of n objects in a specified order using r objects at a time is called the permutation of n objects taken r objects at a time. It is written as ${}_nP_r$ and the formula is

   $${}_nP_r = \frac{n!}{(n-r)!}$$

3. The number of permutations of n objects in which k_1 are unlike k_2 are unlike ---- etc is

   $${}_nP_r = \frac{n!}{k_1!*k_2*...*k_n}$$

Example:

1. Suppose we have a letters A, B, C, D
   a) How many permutations are there taking all the four?

b) How many permutations are there two letters at a time?

2. How many different permutations can be made from the letters in the word “CORRECTION”?

Solutions:

1.

a)

$$Here \quad n = 4, \quad there\, are\ four\, disnict\ object$$

$$\Rightarrow There\ are\ 4! = 24\ permutations.$$

b)

$$Here \quad n = 4, \quad r = 2$$

$$\Rightarrow There\ are\ {}_4P_2 = \frac{4!}{(4-2)!} = \frac{24}{2} = 12\ permutations.$$

2.

$$Here\, n = 10$$

$$Of\ which\ 2\, are\, C,\ 2\, are\, O,\ 2\, are\ R\ ,1E,1T,1I,1N$$

$$\Rightarrow K_1 = 2,\ k_2 = 2,\ k_3 = 2,\ k_4 = k_5 = k_6 = k_7 = 1$$

$$U\sin\ g\, the\, 3^{rd}\ rule\, of\ permutation\, ,there\, are$$

$$\frac{10!}{2!*2!*2!*1!*1!*1!*1!} = 453600\ permutations.$$

Exercises:

1. Six different statistics books, seven different physics books, and 3 different Economics books are arranged on a shelf. How many different arrangements are possible if;
   i. The books in each particular subject must all stand together
   ii. Only the statistics books must stand together

Combination

A selection of objects without regard to order is called combination.
Example: Given the letters A, B, C, and D list the permutation and combination for selecting two letters.

Solutions:

Permutation				Combination	
AB	BA	CA	DA	AB	BC
AC	BC	CB	DB	AC	BD
AD	BD	CD	DC	AD	DC

Note that in permutation AB is different from BA. But in combination AB is the same as BA.

Combination Rule

The number of combinations of r objects selected from n objects is denoted by $_{n}C_{r}$ or $\binom{n}{r}$ and is given by the formula:

$$\binom{n}{r} = \frac{n!}{(n-r)!*r!}$$

Examples:

1. In how many ways a committee of 5 employees be chosen out of 9 employees?

Solutions:

$$n = 9 \ , \quad r = 5$$

$$\binom{n}{r} = \frac{n!}{(n-r)!*r!} = \frac{9!}{4!*5!} = 126\, ways$$

Example: Out of 5 male workers and 7 female workers of a factory, a task force consisting of 5 workers is to be formed. In how many ways can this be done if the task force will consist of

(a) 2 male and 3 female workers?
(b) All female workers?
(c) At least 3 male workers?

Solution:

a) $\binom{5}{2}\binom{7}{3}=\frac{5!}{2!3!}\times\frac{7!}{3!4!}=350$

$b)$ $\binom{5}{0}\binom{7}{5}=\frac{5!}{0!5!}\times\frac{7!}{5!2!}=21$

$c)$ $\binom{5}{3}\binom{7}{2}+\binom{5}{4}\binom{7}{1}+\binom{5}{5}\binom{7}{0}=210+35+1=246$

Exercises:

1. Out of 5 accountants and 7 economists a committee consisting of 2 accountants and 3 economists is to be formed. In how many ways this can be done if
   a) There is no restriction
   b) One particular economists should be included
   c) Two particular accountants cannot be included on the committee.
2. If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and a dictionary, in how many ways this can be don if
   a) There is no restriction.
   b) The dictionary is selected?
   c) 2 novels and 1 book of poems are selected?

5.4 Approaches to measuring Probability

There are four different conceptual approaches to the study of probability theory. These are:

- The classical approach.
- The relative frequency approach.
- The axiomatic approach.
- The subjective approach.

The classical approach

This approach is used when:

- All outcomes are equally likely.

- Total number of outcome is finite, say N.

Definition: If a random experiment with N equally likely outcomes is conducted and out of these N_A outcomes are favorable to the event A, then the probability that event A occur denoted $P(A)$ is defined as:

$$P(A) = \frac{N_A}{N} = \frac{No.of\ outcomes\ favourable to\ A}{Total\ number\ of\ \text{outcomes}} = \frac{n(A)}{n(S)}$$

Examples:

1. A fair die is tossed once. What is the probability of getting
   a) Number 4?
   b) An odd number?
   c) An even number?
   d) Number 8?

Solutions:

First identify the sample space, say S

$S = \{1, 2, 3, 4, 5, 6\}$

$\Rightarrow N = n(S) = 6$

a) Let A be the event of number 4

$A = \{4\}$

$\Rightarrow N_A = n(A) = 1$

$$P(A) = \frac{n(A)}{n(S)} = 1/6$$

b) Let A be the event of odd numbers

$A = \{1,3,5\}$

$\Rightarrow N_A = n(A) = 3$

$$P(A) = \frac{n(A)}{n(S)} = 3/6 = 0.5$$

c) Let A be the event of even numbers

$A = \{2,4,6\}$

$\Rightarrow N_A = n(A) = 3$

$$P(A) = \frac{n(A)}{n(S)} = 3/6 = 0.5$$

d) Let A be the event of number 8

$$A = \emptyset$$

$$\Rightarrow N_A = n(A) = 0$$

$$P(A) = \frac{n(A)}{n(S)} = 0/6 = 0$$

Example: Out of 5 male workers and 7 female workers of a factory, a task force consisting of 5 workers is to be formed. What is the probability that the task force will consist of

(a) 2 male and 3 female workers?
(b) all female workers?
(c) at least 3 male workers?

Solution: Total possible committee = n(S) = $\binom{12}{5} = 792$

a) Let A = 2 male and 3 female workers , n(A) = $\binom{5}{2}\binom{7}{3} = 350$

Hence, P(A) = $\frac{n(A)}{n(S)} = \frac{350}{792} = 0.442$

$$b)\ P(all\ \ female) = \frac{\binom{5}{0}\binom{7}{5}}{\binom{12}{5}} = \frac{21}{792} = 0.0265$$

$$c)\ P(at\ least\ 3\ male) == \frac{246}{792} = 0.312$$

Example: If 3 light bulbs are chosen at random from a dozen of bulbs of a company which 4 are defective, what is the probability that

a) none is defective b) all defective

c) 1 defective and 2 non defective d) 2 defective and 1 non defective

Solution: there are c (12, 3) ways of choosing 3 bulbs from 12 i.e. 220

a) $\frac{c(8,3)}{220} = \frac{56}{220} = \frac{14}{55}$ b) $\frac{c(4,3)}{220} = \frac{4}{220} = \frac{1}{55}$

c) $\frac{c(4,1) \times c(8,2)}{220} = \frac{4 \times 28}{220} = \frac{28}{55}$ d) $\frac{c(4,2) \times c(8,1)}{220} = \frac{12}{220} = \frac{12}{55}$

Exercises:

1. If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and a dictionary, what is the probability that
   a) The dictionary is selected?
   b) 2 novels and 1 book of poems are selected?

Short coming of the classical approach:

This approach is not applicable when:

- The total number of outcomes is infinite.
- Outcomes are not equally likely.

The Frequentist Approach

This is based on the relative frequencies of outcomes belonging to an event.

Definition: The probability of an event A is the proportion of outcomes favorable to A in the long run when the experiment is repeated under same condition.

$$P(A) = \lim_{N \to \infty} \frac{N_A}{N}$$

Example: If records show that 60 out of 100,000 bulbs produced are defective. What is the probability of a newly produced bulb to be defective?

Solution:

Let A be the event that the newly produced bulb is defective.

$$P(A) = \lim_{N \to \infty} \frac{N_A}{N} = \frac{60}{100,000} = 0.0006$$

Axiomatic Approach:

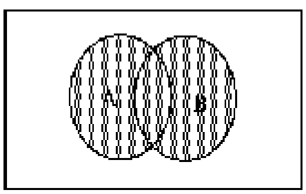
Let E be a random experiment and S be a sample space associated with E. With each event A a real number called the probability of A satisfies the following properties called axioms of probability or postulates of probability.

1. $P(A) \geq 0$
2. $P(S) = 1, \ S \text{ is the sure event.}$
3. If A and B are mutually exclusive events, the probability that one or the other occur equals the sum of the two probabilities. i. e.

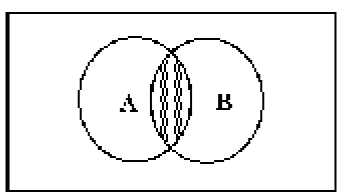
$$P(A \cup B) = P(A) + P(B)$$

4. $P(A') = 1 - P(A)$
5. $0 \leq P(A) \leq 1$
6. P (∅) = 0, ∅ is the impossible event.

Remark: Venn-diagrams can be used to solve probability problems.



A U B



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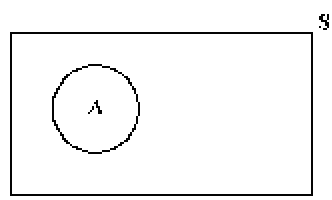

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In general $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

5.5. Conditional Probability and Independence

5.5.1 Conditional Probability

Definition: The conditional probability of an event A, given that event B has occurred with P(B)>0, denoted by P(A|B), is defined as P(A|B) = $\frac{P(A \cap B)}{P(B)}$.

P(B)≠0

Note:

- $P(S|B) = 1$, for any event B and S = sample space
- $P(A^c|B) = 1 - P(A|B)$

Example: A fair die is tossed once. What is the probability that the die shows a 4 given that the die shows even number?

Let A = $\{4\}$, B= $\{2,4,6\}$, then $A \cap B = \{4\}$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1/6}{3/6} = \frac{1}{3}$$

Example: A random sample of 200 adults are classified below by sex and their employed status.

Employee	Male	Female
Accountant	38	45
Casher	28	50
Manager	22	17

If a person is picked at random from this group, find the probability that

(a) The person is a male given that the person is casher

(b) The person is not manager given that the person is a female.

Solution: Let C= the person is casher, M = the person is male, F = the person is female,N = the person is manager and N^c =the person is not manager.

a) $$P(M|C) = \frac{P(M\cap C)}{P(C)} = \frac{28/200}{78/200} = \frac{28}{78}$$

$$P(N^c|F) = \frac{P(N^c\cap F)}{P(F)} = \frac{95/200}{112/200} = \frac{95}{112}$$

Exercise

1. A lot consists of 20 defective and 80 non-defective items from which two items are chosen without replacement. Events A & B are defined as A = {the first item chosen is defective}, B = {the second item chosen is defective}
   a. What is the probability that both items are defective?
   b. What is the probability that the second item is defective?

5.5.2 Independent Event

Definition: Two events A and B are said to be independent (in the probability sense) if $P(A\cap B) = P(A)\,P(B)$.

In other words, two events A and B are independent means the occurrence of one event A is not affected by the occurrence or non-occurrence of B and vice versa.

Remark: If two events A and B are independent, then P(B\A) = P(B), for P(A) > 0 and P(A|B) = P(A) where P(B) > 0.

NB: If at least one of the relations violates the above equation, the events are said to be dependent

Example; A company contains four female and six male workers. What is the probability of getting two female workers in choosing one after the other under the following conditions?

a. The first worker selected is replaced
b. The first worker selected is not replaced

Solution; Let A= first selected worker is female
B= second selected is female

Required $p(A\cap B)$

a. $p(A\cap B)= p(A).p(B)=(4/10)(4/10)=4/25$

b. $p(A\cap B)= p(B/A).p(A)=(4/10)(3/9)=2/15$

5.6. RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

5.6.1 Definitions of random variable and probability distributions

Definition: *A random variable* is a numerical description of the outcomes of the experiment or a numerical valued function defined on sample space, usually denoted by capital letters.

Example: If X is a random variable, then it is a function from the elements of the sample space to the set of real numbers. i.e.

X is a function X: S $\rightarrow$ R

➔A random variable takes a possible outcome and assigns a number to it.

Example: Flip a coin three times, let X be the number of heads in three tosses.

$$\Rightarrow S=\{(HHH),(HHT),(HTH),(HTT),(THH),(THT),(TTH),(TTT)\}$$

$$\Rightarrow \quad X(HHH)=3, \quad X(HHT)=X(HTH)=X(THH)=2,$$

$$X(HTT)=X(THT)=X(TTH)=1$$

$$X(TTT)=0$$

X = {0, 1, 2, 3}

➔X assumes a specific number of values with some probabilities.

Random variables are of two types:

1. *Discrete random variable*: are variables which can assume only a specific number of values. They have values that can be counted

Examples:

- Toss coin n times and count the number of heads.
- Number of children in a family.
- Number of car accidents per week.
- Number of defective items in a given company.
- Number of customers visit commercial bank per day.

2. *Continuous random variable*: are variables that can assume all values between any two given values.

Examples:

- Height of students at certain college.
- Mark of a student.
- Life time of items that bought at the market.
- Length of time required to complete a given training.
- Price of goods.

Definition: *a probability distribution* consists of a value a random variable can assume and the corresponding probabilities of the values.

Example: Consider the experiment of tossing a coin three times. Let X is the number of heads. Construct the probability distribution of X.

Solution:

- First identify the possible value that X can assume.
- Calculate the probability of each possible distinct value of X and express X in the form of frequency distribution.

$X = x$	0	1	2	3
$P(X = x)$	$1/8$	$3/8$	$3/8$	$1/8$

Probability distribution is denoted by P for discrete and by f for continuous random variable.

Properties of Probability Distribution:

1. $P(x) \geq 0, \quad if \quad X \quad is \ discrete.$
   $f(x) \geq 0, \ if \ X \ is \ continuous.$

2. $\sum_{x} P(X = x) = 1 , \quad if \quad X \quad is \quad discrete.$
   $\int_{x} f(x)dx = 1 , \quad if \ is \ continuous.$

Note:

1. If X is a continuous random variable then

$$P(a < X < b) = \int_{a}^{b} f(x)dx$$

2. Probability of a fixed value of a continuous random variable is zero.

$$\Rightarrow P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b)$$

3. If X is discrete random variable the

$$P(a < X < b) = \sum_{x=a+1}^{b-1} P(x)$$
$$P(a \leq X < b) = \sum_{x=a}^{b-1} p(x)$$
$$P(a < X \leq b) = \sum_{x=a+1}^{b} P(x)$$
$$P(a \leq X \leq b) = \sum_{x=a}^{b} P(x)$$

4. Probability means *area* for continuous random variable.

5.6.2 Introduction to expectation

Definition:

1. Let a discrete random variable X assume the values X_1, X_2, …, X_n with the probabilities $P(X_1)$, $P(X_2)$, ….,$P(X_n)$ respectively. Then the expected value of X ,denoted as E(X) is defined as:

$$E(X) = X_1P(X_1) + X_2P(X_2) + ... + X_nP(X_n)$$
$$= \sum_{i=1}^{n} X_i P(X_i)$$

2. Let X be a continuous random variable assuming the values in the interval (a, b) such that $\int_a^b f(x)dx = 1$, then

$$E(X) = \int_a^b x\, f(x)dx$$

Examples:

1. What is the expected value of a random variable X obtained by tossing a coin three times where is the number of heads

Solution:

First construct the probability distribution of X

$X = x$	0	1	2	3
$P(X = x)$	$1/8$	$3/8$	$3/8$	$1/8$

$$\Rightarrow E(X) = X_1P(X_1) + X_2P(X_2) + ... + X_nP(X_n)$$
$$= 0*1/8 + 1*3/8 + ... + 2*1/8$$
$$= 1.5$$

2. Suppose a charity organization is mailing printed return-address stickers to over one million homes in the Ethiopia. Each recipient is asked to donate\$1, \$2, \$5, \$10, \$15, or \$20. Based on past experience, the amount a person donates is believed to follow the following probability distribution:

$X = x$	\$1	\$2	\$5	\$10	\$15	\$20
$P(X = x)$	0.1	0.2	0.3	0.2	0.15	0.05

What is expected that an average donor to contribute?

Solution:

$X = x$	\$1	\$2	\$5	\$10	\$15	\$20	Total
$P(X = x)$	0.1	0.2	0.3	0.2	0.15	0.05	1
$xP(X = x)$	0.1	0.4	1.5	2	2.25	1	7.25

$$\Rightarrow E(X) = \sum_{i=1}^{6} x_i P(X = x_i) = \$7.25$$

Mean and Variance of a random variable

Let X is given random variable.

1. The expected value of X is its mean $\Rightarrow Mean\ of\ X = E(X)$
2. The variance of X is given by:

$$Variance\ of\ X = \text{var}(X) = E(X^2) - [E(X)]^2$$

Where:

$$E(X^2) = \sum_{i=1}^{n} x_i^{\,2} P(X = x_i) \ , \quad if\ \ X\ \ is\ discrete$$
$$= \int_x x^2 f(x)dx \ , \qquad if\ \ X\ is\ continuous.$$

Examples:

1. Find the mean and the variance of a random variable X in example 2 above.

Solutions:

$X = x$	\$1	\$2	\$5	\$10	\$15	\$20	Total
$P(X = x)$	0.1	0.2	0.3	0.2	0.15	0.05	1
$xP(X = x)$	0.1	0.4	1.5	2	2.25	1	7.25
$x^2P(X = x)$	0.1	0.8	7.5	20	33.75	20	82.15

$$\Rightarrow E(X) = 7.25$$

$$Var(X) = E(X^2) - [E(X)]^2 = 82.15 - 7.25^2 = 29.59$$

2. Two dice are rolled. Let X is a random variable denoting the sum of the numbers on the two dice.
   i) Give the probability distribution of X
   ii) Compute the expected value of X and its variance

There are some general rules for mathematical expectation.

Let X and Y are random variables and *k* is a constant.

RULE 1 $E(k) = k$

RULE 2 $Var(k) = 0$

RULE 3 $E(kX) = kE(X)$

RULE 4 $Var(kX) = k^2 Var(X)$

RULE 5 $E(X+Y) = E(X) + E(Y)$

5.6.3 Common Discrete Probability Distributions

1. Binomial Distribution

A binomial experiment is a probability experiment that satisfies the following four requirements called assumptions of a binomial distribution.

1. The experiment consists of n identical trials.
2. Each trial has only one of the two possible mutually exclusive outcomes, success or a failure.
3. The probability of each outcome does not change from trial to trial, and
4. The trials are independent, thus we must sample with replacement.

Examples of binomial experiments

- Tossing a coin 20 times to see how many tails occur.
- Asking 200 people if they watch BBC news.
- Registering a newly produced product as defective or non defective.
- Asking 100 people if they favor the ruling party.

- Rolling a die to see if a 5 appears.

Definition: The outcomes of the binomial experiment and the corresponding probabilities of these outcomes are called **Binomial Distribution**.

$$Let\ P = the\ probability\ of\ success$$

$$q = 1 - p = the\ probability\ of\ \ failure\ on\ any\ given\ trial$$

Then the probability of getting x successes in n trials becomes:

$$P(X = x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0,1,2,....,n$$

And this is some times written as:

$$X \sim Bin(n, p)$$

When using the binomial formula to solve problems, we have to identify three things:

- The number of trials (n)
- The probability of a success on any one trial (p) and
- The number of successes desired (X).

Examples:

1. What is the probability of getting three heads by tossing a fair coin four times?

   Solution:

   Let X be the number of heads in tossing a fair coin four times

   $$X \sim Bin(n = 4, p = 0.50)$$

   $$\Rightarrow P(X = x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0,1,2,3,4.$$

   $$\Rightarrow P(X = 3) = \binom{4}{3} 0.5^4 = 0.25$$

2. **S**uppose that a company wants to produce six different types of products in the given time,the probability that the product will be defective is 30%. then,
   a) What is the probability of producing more than three defective products?
   b) What is the probability of producing at least two defective products?
   c) What is the probability of producing at most three defective products?

d) What is the probability of producing less than five defective products?

Solution

a) Let X = the number of defective products that the company produces.

$$X \sim Bin(n=6, p=0.30)$$

a) $P(X>3)=?$

$$\Rightarrow P(X=x)=\binom{n}{x}p^x q^{n-x}, \quad x=0,1,2,..6$$

$$=\binom{6}{x}0.3^x 0.7^{6-x}$$

$$\Rightarrow P(X>3)=P(X=4)+P(X=5)+P(X=6)$$

$$=0.060+0.010+0.001$$

$$=0.071$$

Thus, we may conclude that if 30% of the products are defective products, the probability is 0.071 (or 7.1%) that more than four of the defective products are produced by the company.

b) $P(X \geq 2)=?$

$$P(X \geq 2)=P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)$$

$$=0.324+0.185+0.060+0.010+0.001$$

$$=0.58$$

c) $P(X \leq 3)=?$

$$P(X \leq 3)=P(X=0)+P(X=1)+P(X=2)+P(X=3)$$

$$=0.118+0.303+0.324+0.185$$

$$=0.93$$

d) $P(X<5)=?$

$$P(X<5)=1-P(X \geq 5)$$

$$=1-\{P(X=5)+P(X=6)\}$$

$$=1-(0.010+0.001)$$

$$=0.989$$

Exercises:

1. A merchant claims that her item has 45% to be sold per day. What is the probability that

a) Exactly 3 of her next 4 items to be sold?

b) None of her next 4 items to be sold?

2. Explain why the following experiments are not Binomial

- Rolling a die until a 6 appears.
- Asking 20 people how old they are.
- Drawing 5 cards from a deck for a poker hand.

Remark: If X is a binomial random variable with parameters n and p then

$$E(X) = np \quad , \qquad Var(X) = npq$$

2. Poisson Distribution

- A random variable X is said to have a Poisson distribution if its probability distribution is given by:

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0,1,2,......$$

$$Where \;\; \lambda = the\ average\, number.$$

- The Poisson distribution depends *only* on the average number of occurrences per unit time of space.
- The Poisson distribution is used as a distribution of rare events, such as:
    - Number of misprints.
    - Natural disasters like earth quake.
    - Accidents.
    - Hereditary.
    - Arrivals
- The process that gives rise to such events are called Poisson process.

Examples:

1. If 1.6 accidents can be expected an intersection on any given day, what is the probability that there will be 3 accidents on any given day?

Solution; Let X =the number of accidents, $\lambda = 1.6$

$$X = poisson(1.6) \Rightarrow p(X = x) = \frac{1.6^x e^{-1.6}}{x!}$$

$$p(X = 3) = \frac{1.6^3 e^{-1.6}}{3!} = 0.1380$$

Exercise

2. On the average, five customers pass a certain street corners every ten minutes, what is the probability that during a given 10 minutes the number of customers passing will be
    a. 6 or fewer
    b. 7 or more
    c. Exactly 8…….

If X is a Poisson random variable with parameters λ then

$$E(X) = \lambda \quad , \qquad Var(X) = \lambda$$

Note:

The Poisson probability distribution provides a close approximation to the binomial probability distribution when n is large and p is quite small or quite large with $\lambda = np$.

$$P(X = x) = \frac{(np)^x e^{-(np)}}{x!}, \quad x = 0,1,2,......$$

$$Where \ \ \lambda = np = the\ average\ number.$$

Usually we use this approximation if $np \le 5$. In other words, if $n > 20$ and $np \le 5$ [or $n(1-p) \le 5$], then we may use Poisson distribution as an approximation to binomial distribution.

Example:

1. Find the binomial probability P(X=3) by using the Poisson distribution if $p = 0.01$ and $n = 200$

Solution:

$$U\sin g\ Poisson\ ,\lambda = np = 0.01*200 = 2$$

$$\Rightarrow P(X=3) = \frac{2^3 e^{-2}}{3!} = 0.1804$$

$$U\sin g\ \ Binomial\ ,\ n = 200,\ \ p = 0.01$$

$$\Rightarrow P(X=3) = \binom{200}{3}(0.01)^3(0.99)^{99} = 0.1814$$

5.6.4 Common Continuous Probability Distributions

1. **Normal Distribution**

A random variable X is said to have a normal distribution if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \qquad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$$

$$Where\ \ \mu = E(X), \quad \sigma^2 = Var(X)$$

$$\mu\ \ and\ \ \sigma^2\ \ are\ the\ \ Parameters\ of\ the\ Normal\ \ Distribution.$$

Properties of Normal Distribution:

1. It is bell shaped and is symmetrical about its mean and it is mesokurtic. The maximum ordinate is at $x = \mu$ and is given by

   $$f(x) = \frac{1}{\sigma\sqrt{2\pi}}$$
2. It is asymptotic to the axis, i.e., it extends indefinitely in either direction from the mean.
3. It is a continuous distribution.
4. It is a family of curves, i.e., every unique pair of mean and standard deviation defines a different normal distribution. Thus, the normal distribution is completely described by two parameters: mean and standard deviation.
5. Total area under the curve sums to 1, i.e., the area of the distribution on each side of the mean is 0.5. $\Rightarrow \int_{-\infty}^{\infty} f(x)dx = 1$
6. It is unimodal, i.e., values mound up only in the center of the curve.

7. $Mean = Median = \mathrm{mod}\, e = \mu$
8. The probability that a random variable will have a value between any two points is equal to the area under the curve between those points.

Note: To facilitate the use of normal distribution, the following distribution known as the standard normal distribution was derived by using the transformation

$$Z = \frac{X - \mu}{\sigma}$$

$$\Rightarrow f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

Properties of the Standard Normal Distribution:

- Same as a normal distribution, but also...

- Mean is zero
- Variance is one
- Standard Deviation is one

- Areas under the standard normal distribution curve have been tabulated in various ways. The most common ones are the areas between

$$Z = 0 \ and \ a \ positive \ value of \ Z.$$

- Given a normal distributed random variable X with

Mean μ $and\ s\tan dard\ deviation\ \ \sigma$

$$P(a < X < b) = P(\frac{a - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{b - \mu}{\sigma})$$

$$\Rightarrow P(a < X < b) = P(\frac{a - \mu}{\sigma} < Z < \frac{b - \mu}{\sigma})$$

Note:

$$\begin{aligned} P(a < X < b) &= P(a \le X < b) \\ &= P(a < X \le b) \\ &= P(a \le X \le b) \end{aligned}$$

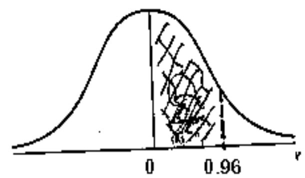
Examples:

1. Find the area under the standard normal distribution which lies

a) Between $Z = 0 \ \ and \ \ Z = 0.96$

Solution:

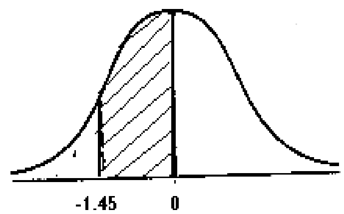
$$Area = P(0 < Z < 0.96) = 0.3315$$



b) Between $Z = -1.45 \ \ and \ \ Z = 0$

Solution:

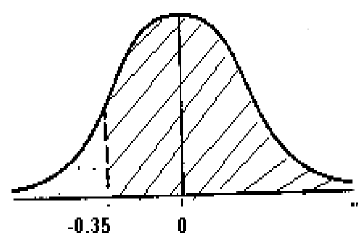
$$Area = P(-1.45 < Z < 0)$$
$$= P(0 < Z < 1.45)$$
$$= 0.4265$$



c) To the right of $Z = -0.35$

Solution:

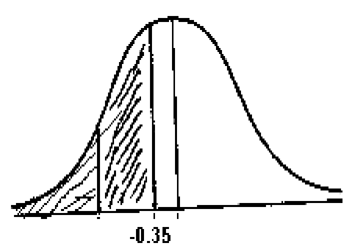
$$Area = P(Z > -0.35)$$
$$= P(-0.35 < Z < 0) + P(Z > 0)$$
$$= P(0 < Z < 0.35) + P(Z > 0)$$
$$= 0.1368 + 0.50 = 0.6368$$



d) To the left of $Z = -0.35$

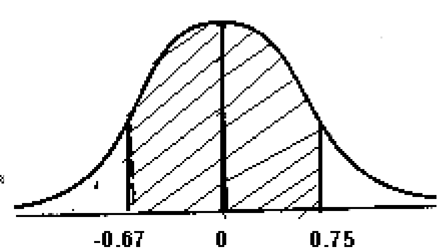
Solution:

$$Area = P(Z < -0.35)$$
$$= 1 - P(Z > -0.35)$$
$$= 1 - 0.6368 = 0.3632$$



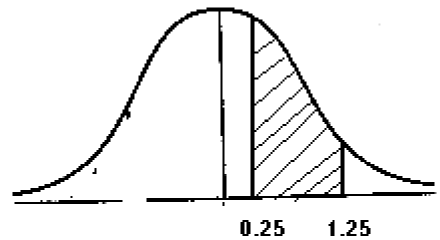
e) Between
$Z = -0.67 \ \ and \ Z = 0.75$

Solution:



$$Area = P(-0.67 < Z < 0.75)$$
$$= P(-0.67 < Z < 0) + P(0 < Z < 0.75)$$
$$= P(0 < Z < 0.67) + P(0 < Z < 0.75)$$
$$= 0.2486 + 0.2734 = 0.5220$$

f) Between $Z = 0.25$ *and* $Z = 1.25$



Solution:

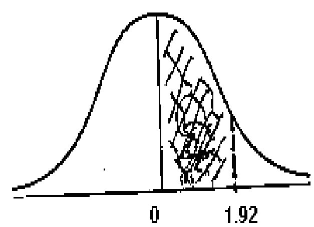
$$Area = P(0.25 < Z < 1.25)$$
$$= P(0 < Z < 1.25) - P(0 < Z < 0.25)$$
$$= 0.3934 - 0.0987 = 0.2957$$

2. Find the value of Z if

a) The normal curve area between 0 and z(positive) is 0.4726

Solution

$$P(0 < Z < z) = 0.4726 \text{ and from table}$$
$$P(0 < Z < 1.92) = 0.4726$$
$$\Leftrightarrow z = 1.92 \text{.....uniqueness of Areea.}$$



b) The area to the left of z is 0.9868

Solution

$$P(Z < z) = 0.9868$$
$$= P(Z < 0) + P(0 < Z < z)$$
$$= 0.50 + P(0 < Z < z)$$
$$\Rightarrow P(0 < Z < z) = 0.9868 - 0.50 = 0.4868$$
and from table
$$P(0 < Z < 2.2) = 0.4868$$
$$\Leftrightarrow z = 2.2$$

3) An income of a worker per day has a normal distribution with mean 80 birr and standard deviation 4.8. What is the probability that daily income of a worker is

a) Less than 87.2 birr
b) Greater than 76.4 birr
c) Between 81.2 and 86.0 birr

Solution

$$X \text{ is normal with mean}, \mu = 80, \text{ standard deviation}, \sigma = 4.8$$

a)

$$\begin{aligned} P(X < 87.2) &= P(\frac{X-\mu}{\sigma} < \frac{87.2-\mu}{\sigma}) \\ &= P(Z < \frac{87.2-80}{4.8}) \\ &= P(Z < 1.5) \\ &= P(Z < 0) + P(0 < Z < 1.5) \\ &= 0.50 + 0.4332 = \underline{\underline{0.9332}} \end{aligned}$$

b)

$$\begin{aligned} P(X > 76.4) &= P(\frac{X-\mu}{\sigma} > \frac{76.4-\mu}{\sigma}) \\ &= P(Z > \frac{76.4-80}{4.8}) \\ &= P(Z > -0.75) \\ &= P(Z > 0) + P(0 < Z < 0.75) \\ &= 0.50 + 0.2734 = \underline{\underline{0.7734}} \end{aligned}$$

c)

$$\begin{aligned} P(81.2 < X < 86.0) &= P(\frac{81.2-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{86.0-\mu}{\sigma}) \\ &= P(\frac{81.2-80}{4.8} < Z < \frac{86.0-80}{4.8}) \\ &= P(0.25 < Z < 1.25) \\ &= P(0 < Z < 1.25) - P(0 < Z < 1.25) \\ &= 0.3934 - 0.0987 = \underline{\underline{0.2957}} \end{aligned}$$

3. A normal distribution has mean 62.4.Find its standard deviation if 20.0% of the area under the normal curve lies to the right of 72.9

Solution

$$P(X > 72.9) = 0.2005 \Rightarrow P(\frac{X-\mu}{\sigma} > \frac{72.9-\mu}{\sigma}) = 0.2005$$

$$\Rightarrow P(Z > \frac{72.9-62.4}{\sigma}) = 0.2005$$

$$\Rightarrow P(Z > \frac{10.5}{\sigma}) = 0.2005$$

$$\Rightarrow P(0 < Z < \frac{10.5}{\sigma}) = 0.50 - 0.2005 = 0.2995$$

$$And \;\; from \; table \;\; P(0 < Z < 0.84) = 0.2995$$

$$\Leftrightarrow \frac{10.5}{\sigma} = 0.84$$

$$\Rightarrow \sigma = \underline{\underline{12.5}}$$

4. An expenditure of a person per day has a normal distribution with $\sigma = 5$.Find his / her mean expenditure if the probability that an expenditure of a person will be less than 52.5 birr per day is 0.6915.

Solution

$$P(Z < z) = P(Z < \frac{52.5-\mu}{5}) = 0.6915$$

$$\Rightarrow P(0 < Z < z) = 0.6915 - 0.50 = 0.1915.$$

$$But \;\; from \; the \, table$$

$$\Rightarrow P(0 < Z < 0.5) = 0.1915$$

$$\Leftrightarrow z = \frac{52.5-\mu}{5} = 0.5$$

$$\Rightarrow \mu = \underline{\underline{50}}$$