

Lecture Notes on  
Classical Mechanics-II  
Phys-2042

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# Chapter One

## Dynamics of System of Particles

### 1.1 System of Particles and Centre of Mass

For a system composed of  $n$  particles, the total mass  $M$  is given by

$$M = \sum_{\alpha} m_{\alpha} \quad (1.1)$$

where  $m_{\alpha}$  is the mass of the  $i^{\text{th}}$  particle, with  $\alpha = 1, \dots, n$ . If each particle is (mathematically) connected to the origin of the system through a position vector  $r_i$ , then the centre of mass vector is defined as

$$\mathbf{R} = \frac{1}{M} \sum_{\alpha} m_{\alpha} \mathbf{r}_{\alpha} \quad (1.2)$$

For a continuous system, the summation over  $\alpha$  is replaced with an integral over an infinitesimal amount of mass  $dm$  such that

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r} dm \quad (1.3)$$

It is important to realize that the position vector  $\mathbf{R}$  of the centre of mass depends on the origin chosen for the coordinate systems.

### 1.2 The Conservation of Linear Momentum

The force acting on particle  $\alpha$  of a system of particles is composed of the resultant of all forces external to the system  $F_{\alpha}^{(e)}$ , and the resultant of the internal forces  $f_{\alpha}$  stemming from its interaction with the other particles that are part of the system. If we define these internal interaction forces as  $f_{\alpha\beta}$ , the resulting force  $f_{\alpha}$  acting on particle  $\alpha$  is

$$f_{\alpha} = \sum_{\beta \neq \alpha} f_{\alpha\beta} \quad (1.4)$$

The total force  $F_{\alpha}$  acting on the particle is,

$$F_{\alpha} = F_{\alpha}^{(e)} + f_{\alpha} \quad (1.5)$$

From Newton's Second law we can write

$$\dot{P}_\alpha = F_\alpha^{(e)} + f_\alpha \quad (1.6)$$

$$\begin{aligned} \frac{d^2}{dt^2}(m_\alpha r_\alpha) &= F_\alpha^{(e)} + f_\alpha \\ &= F_\alpha^{(e)} + \sum_{\beta \neq \alpha} f_{\alpha\beta} \end{aligned} \quad (1.7)$$

where no summation on repeated index is implied. Summing equation (1.7) over all particles we get

$$\begin{aligned} \frac{d^2}{dt^2}(\sum m_\alpha r_\alpha) &= \sum F_\alpha^{(e)} + \sum_\alpha \sum_{\beta \neq \alpha} f_{\alpha\beta} \\ &= F + \sum_{\alpha, \beta \text{ pairs}} (f_{\alpha\beta} + f_{\beta\alpha}), \end{aligned} \quad (1.8)$$

where we have defined the sum over all external forces as

$$F = \sum_\alpha F_\alpha^{(e)} \quad (1.9)$$

and the second term on the right of equation (1.8) was replaced by a single summation over every pair of internal interactions between the particles. However, we know from Newton's Third Law that  $f_{\alpha\beta} = -f_{\beta\alpha}$ . We can therefore write, from equation (1.8) that

$$M\ddot{R} = F \quad (1.10)$$

This last equation can also be used to express the conservation of momentum since

$$\begin{aligned} \mathbf{P} &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \\ &= \frac{d}{dt}(\sum_\alpha m_\alpha \mathbf{r}_\alpha) \\ \mathbf{P} &= M\dot{\mathbf{R}}, \end{aligned} \quad (1.11)$$

and then

$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \mathbf{F} \quad (1.12)$$

**We can summarize as follows**

- I. *The center of mass of a system moves as if it were a single particle of mass equal to the total mass of the system, acted upon by the total external force, and independent of the internal forces (as long as  $f_{\alpha\beta} = -f_{\beta\alpha}$  (Newton's Third Law) holds).*
- II. *The total linear momentum of a system is the same as that of a single particle of mass  $M$  located at the position of the center of mass and moving in the manner the center of mass moves.*
- III. *The total linear momentum for a system free of external forces is a constant and equal to the linear momentum of the center of mass (the law of conservation of linear momentum for a system).*

### 1.3 The Conservation of Angular Momentum

As we saw in the previous chapter on central force motion, it is often more convenient to define the positions of the particles composing a system by vectors  $\mathbf{r}'_{\alpha}$  originating at the centre of mass (see Figure 1-1). The position vector  $\mathbf{r}_{\alpha}$  in the inertial frame is

$$\mathbf{r}_{\alpha} = \mathbf{R} + \mathbf{r}'_{\alpha}. \quad (1.13)$$

The angular momentum of the  $\alpha$ th particle is given by

$$L_{\alpha} = \mathbf{r}_{\alpha} \times \mathbf{p}_{\alpha} \quad (1.14)$$

and summing over all particles,

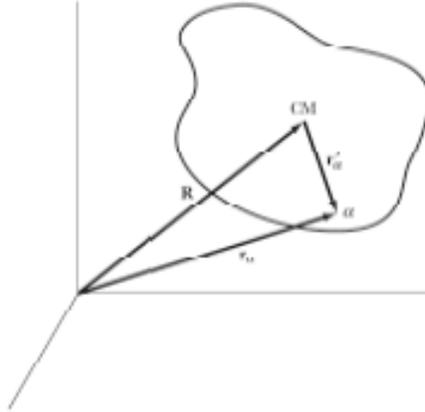


Figure 7.1 - Description of the position of a particle using its position vector from the centre of mass of the system.

$$\begin{aligned}
 \mathbf{L} &= \sum \mathbf{L}_a = \sum (\mathbf{r}_a \times \mathbf{p}_a) = \sum (\mathbf{r}_a \times m_a \mathbf{v}_a) \\
 &= \sum [(\mathbf{R} + \mathbf{r}'_a) \times m_a (\dot{\mathbf{R}} + \dot{\mathbf{r}}'_a)] \\
 &= \sum m_a [(\mathbf{R} \times \dot{\mathbf{R}}) + (\mathbf{R} \times \dot{\mathbf{r}}'_a) + (\mathbf{r}'_a \times \dot{\mathbf{R}}) + (\mathbf{r}'_a \times \dot{\mathbf{r}}'_a)]
 \end{aligned}
 \tag{1.15}$$

The second and third terms on the right hand side equal zero from

$$\begin{aligned}
 \sum m_a [(\mathbf{R} \times \dot{\mathbf{r}}'_a) + (\mathbf{r}'_a \times \dot{\mathbf{R}})] &= \sum m_a \left[ \frac{d}{dt} (\mathbf{R} \times \mathbf{r}'_a) - (\dot{\mathbf{R}} \times \mathbf{r}'_a) + (\mathbf{r}'_a \times \dot{\mathbf{R}}) \right] \\
 &= \frac{d}{dt} \left( \mathbf{R} \times \sum m_a \mathbf{r}'_a \right) + 2 \left( \sum m_a \mathbf{r}'_a \right) \times \dot{\mathbf{R}} = 0,
 \end{aligned}
 \tag{1.16}$$

since, from equations (1.2) and (1.13),

$$\sum m_a \mathbf{r}'_a = 0
 \tag{1.17}$$

Equation (1.15) now becomes

$$L = (\mathbf{R} \times \sum m_a \dot{\mathbf{R}}) + \sum m_a (\mathbf{r}'_a \times \dot{\mathbf{r}}'_a)$$

$$L = R \times P + \sum_{\alpha} r'_{\alpha} \times p'_{\alpha} \quad 1.18$$

We, therefore, have this important result

- IV. *The angular momentum about an origin is the sum of the angular momentum of the centre of mass about that origin and the angular momentum of the system about the position of the centre of mass.*

The time derivative of the total angular momentum is

$$\begin{aligned} \dot{\mathbf{L}} &= \sum_{\alpha} \dot{\mathbf{L}}_{\alpha} = \sum_{\alpha} (\mathbf{r}_{\alpha} \times \dot{\mathbf{p}}_{\alpha}) \\ &= \sum_{\alpha} (\mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha}^{(e)}) + \sum_{\alpha} \left( \mathbf{r}_{\alpha} \times \sum_{\beta \neq \alpha} \mathbf{f}_{\alpha\beta} \right) \\ &= \sum_{\alpha} (\mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha}^{(e)}) + \sum_{\alpha < \beta} [(\mathbf{r}_{\alpha} \times \mathbf{f}_{\alpha\beta}) + (\mathbf{r}_{\beta} \times \mathbf{f}_{\beta\alpha})] \end{aligned} \quad 1.19$$

Where  $\sum_{\alpha < \beta}$  means a sum over  $\alpha$  and  $\beta$  with  $\alpha < \beta$

We know, however, from Newton's Third Law that  $f_{\alpha\beta} = -f_{\beta\alpha}$  so that equation (1.19) can be re-written

$$\dot{\mathbf{L}} = \sum_{\alpha} (\mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha}^{(e)}) + \sum_{\alpha < \beta} [(\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}) \times f_{\alpha\beta}] \quad 1.20$$

If we further limit ourselves to internal forces  $f_{\alpha\beta}$  that are also directed along the straight line joining the two interacting particles (i.e., along  $\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}$ ), we must have the following

$$(\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}) \times f_{\alpha\beta} = \mathbf{0} \quad 1.21$$

The time derivative of the total angular momentum is then

$$\dot{\mathbf{L}} = \sum_{\alpha} (\mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha}^{(e)}) \quad 1.22$$

or if we express the right hand side as a sum of the external torque applied on the different particles  $\boldsymbol{\tau}_{\alpha}^{(e)}$ ,

$$\dot{\mathbf{L}} = \sum_{\alpha} \boldsymbol{\tau}_{\alpha}^{(e)} = \boldsymbol{\tau}^{(e)} \quad 1.23$$

We, therefore, have the following results

- V. *If the net resultant external torque about an axis vanishes, then the total angular momentum of the system about that axis remains a constant in time.*

Furthermore, since we found that the total internal torque also vanishes, i.e.,

$$\sum_{\alpha} (\mathbf{r}_{\alpha} \times \sum_{\beta \neq \alpha} \mathbf{f}_{\alpha\beta}) = \sum_{\alpha < \beta} [(\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}) \times \mathbf{f}_{\alpha\beta}] = 0 \quad 1.24$$

And we can state that

- VI. *The total internal torque must vanish if the internal forces are central (i.e.,  $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$  and the internal forces between two interacting particles are directed along the line joining them), and the angular momentum of an isolated system cannot be altered without the application of external forces.*

#### 1.4) Conservation of energy

Consider a system of particles that evolves from a starting configuration “1” to an ulterior configuration “2” where the positions  $\mathbf{r}_{\alpha}$  the particles may have changed in the process. We can write the total work done on the system as the sum of the work done on individual particles.

$$\begin{aligned} W_{12} &= \sum_{\alpha} \int_1^2 \mathbf{F}_{\alpha} \cdot d\mathbf{r}_{\alpha} \\ &= \sum_{\alpha} \int_1^2 m_{\alpha} \frac{d\mathbf{v}_{\alpha}}{dt} \cdot \frac{d\mathbf{r}_{\alpha}}{dt} dt = \sum_{\alpha} \int_1^2 m_{\alpha} \frac{d\mathbf{v}_{\alpha}}{dt} \cdot \mathbf{v}_{\alpha} dt \\ &= \sum_{\alpha} \int_1^2 \frac{1}{2} m_{\alpha} \frac{dv_{\alpha}^2}{dt} dt = \sum_{\alpha} \int_1^2 \frac{d}{dt} \left( \frac{1}{2} m_{\alpha} v_{\alpha}^2 \right) dt \\ &= \sum_{\alpha} \int_1^2 d \left( \frac{1}{2} m_{\alpha} v_{\alpha}^2 \right) = T_2 - T_1, \end{aligned} \quad 1.25$$

Where

$$T = \sum_{\alpha} T_{\alpha} = \sum_{\alpha} \frac{1}{2} m_{\alpha} v_{\alpha}^2 \quad 1.26$$

Using equation (1.13) we can write

$$\begin{aligned} v_{\alpha}^2 &= \dot{\mathbf{r}}_{\alpha} \cdot \dot{\mathbf{r}}_{\alpha} \\ &= (\dot{\mathbf{R}} + \dot{\mathbf{r}}'_{\alpha}) \cdot (\dot{\mathbf{R}} + \dot{\mathbf{r}}'_{\alpha}) \\ &= \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + 2(\dot{\mathbf{r}}'_{\alpha} \cdot \dot{\mathbf{R}}) + (\dot{\mathbf{r}}'_{\alpha} \cdot \dot{\mathbf{r}}'_{\alpha}) \\ &= V^2 + 2(\dot{\mathbf{r}}'_{\alpha} \cdot \dot{\mathbf{R}}) + (v'_{\alpha})^2 \end{aligned} \quad 1.27$$

Where  $v'_{\alpha} = |\dot{\mathbf{r}}'_{\alpha}|$  and  $V = |\dot{\mathbf{R}}|$ . Inserting equation (1.27) into equation (1.26), while using the earlier result that states that  $\sum_{\alpha} m_{\alpha} \dot{\mathbf{r}}'_{\alpha} = 0$ , we find that

$$T = \frac{1}{2} M V^2 + \sum_{\alpha} m_{\alpha} (v'_{\alpha})^2 \quad 1.28$$

In other words

VII. *The total kinetic energy of the system is equal to the sum of the kinetic energy of a particle of mass  $M$  moving with velocity of the center of mass and the kinetic energy of motion of the individual particles relative to the center of mass.*

Alternatively, we can rewrite first of equations (1.25) by separating the total force applied on each particle in its external and internal components

$$W_{12} = \sum_{\alpha} \int_1^2 F_{\alpha}^{(e)} \cdot d\mathbf{r}_{\alpha} + \sum_{\alpha, \beta \neq \alpha} \int_1^2 f_{\alpha\beta} \cdot d\mathbf{r}_{\alpha} \quad 1.29$$

If the forces involved are conservatives, we can then derive them from potentials such That

$$F_{\alpha}^{(e)} = -\nabla_{\alpha} U_{\alpha} \quad 1.30$$

$$f_{\alpha\beta} = -\nabla_{\alpha}U_{\alpha\beta},$$

Where  $U_{\alpha}$  and  $\bar{U}_{\alpha\beta}$  are independent potential functions. The gradient operator  $\nabla_{\alpha}$  is a vector operator meant to apply to the coordinate components of the  $\alpha$ th particle (i.e.,  $\alpha$  is the index that specifies a given particle, and does not represent a coordinate such as  $x$ ,  $y$ , or  $z$ ).

The first term on the right hand side of equation (7.29) can be written as

$$\begin{aligned} \sum_{\alpha} \int_1^2 \mathbf{F}_{\alpha}^{(e)} \cdot d\mathbf{r}_{\alpha} &= -\sum_{\alpha} \int_1^2 (\nabla_{\alpha}U_{\alpha}) \cdot d\mathbf{r}_{\alpha} \\ &= -\sum_{\alpha} U_{\alpha} \Big|_1^2. \end{aligned} \tag{1.31}$$

The last term of the same equation is transformed to

$$\begin{aligned} \sum_{\alpha, \beta \neq \alpha} \int_1^2 \mathbf{f}_{\alpha\beta} \cdot d\mathbf{r}_{\alpha} &= \sum_{\alpha < \beta} \int_1^2 (\mathbf{f}_{\alpha\beta} \cdot d\mathbf{r}_{\alpha} + \mathbf{f}_{\beta\alpha} \cdot d\mathbf{r}_{\beta}) \\ &= \sum_{\alpha < \beta} \int_1^2 \mathbf{f}_{\alpha\beta} \cdot (d\mathbf{r}_{\alpha} - d\mathbf{r}_{\beta}). \end{aligned} \tag{1.32}$$

Before we use the last of equations (1.30) to further transform equation (1.32), we consider the following differential

$$\begin{aligned} d\bar{U}_{\alpha\beta} &= (\nabla_{\alpha}\bar{U}_{\alpha\beta}) \cdot d\mathbf{r}_{\alpha} + (\nabla_{\beta}\bar{U}_{\alpha\beta}) \cdot d\mathbf{r}_{\beta} \\ &= (-\mathbf{f}_{\alpha\beta}) \cdot d\mathbf{r}_{\alpha} + (-\mathbf{f}_{\beta\alpha}) \cdot d\mathbf{r}_{\beta} \\ &= -\mathbf{f}_{\alpha\beta} \cdot (d\mathbf{r}_{\alpha} - d\mathbf{r}_{\beta}), \end{aligned} \tag{1.33}$$

Since  $\nabla_{\beta}\bar{U}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha} = \mathbf{f}_{\alpha\beta}$  (note also that  $\bar{U}_{\beta\alpha} = \bar{U}_{\alpha\beta}$ ). Combining this result with equations (1.29), (1.31), and (1.32), we get

$$W_{12} = -\sum_{\alpha} U_{\alpha} \Big|_1^2 - \sum_{\alpha < \beta} \bar{U}_{\alpha\beta} \Big|_1^2. \tag{1.34}$$

If we define the total potential energy as

$$U = U_{\alpha} + \bar{U}_{\alpha\beta} \quad 1.35$$

We get

$$W_{12} = -U|_1^2 = U_1 - U_2. \quad 1.36$$

Combining equation (1.36) and the last of equations (1.25), we find that

$$T_2 - T_1 = U_1 - U_2 \quad 1.37$$

Or,

$$T_1 + U_1 = T_2 + U_2, \quad 1.38$$

and finally

$$\mathbf{E}_1 = \mathbf{E}_2, \quad 1.39$$

We have therefore proved the conservation of energy for a system of particles where all the forces can be derived from a potential that are independent of time; such a system is called *conservative*.

*VIII. The total energy for a conservative system is constant.*

## 1.5) Motion of systems with variable mass

### 1.5.1 Introduction

So far we have considered DMSs and CMSs in which masses of particles  $m_n$  and their number have not changed. In nature and technology, however, phenomena are commonly known where the number of particles of a system or their mass change over time. If floating icebergs are heated by the Sun's rays, then the ice melts and their mass decreases. If the falling snow becomes frozen to the floating icebergs, then their mass increases. Earth's mass increases when meteorites fall on its surface. In turn, the mass of the meteorites before they reach Earth's surface decreases as a result of burning in Earth's atmosphere. The mass of rockets decreases as the fuel they contain burns. The mass of elements transported on a conveyor belt changes as a result of their loading and unloading.

## 1.5.2 Change in Quantity of Motion and Angular Momentum

Let the mass of a mechanical system  $m.t/$  be changing in time according to the equation

$$m(t) = m_o - m_1(t) + m_2(t) , \quad 1.5.1$$

Where  $m(t) = m(t_o)$  ,  $m_1(t) \geq 0$ ,  $(m_2(t) \geq 0)$  denotes the mass of particles leaving (entering) the system(Fig. 1.2)

Let us choose a time instant  $t$  during motion of the system, and let for this instant the momentum  $\mathbf{p}$  of the considered system of particles increase by  $\Delta \mathbf{p}$  during time  $\Delta t$ . Then, by  $\mathbf{p}^*$  let us denote the momentum of analogous system, but of a constant mass. At the instant  $t + \Delta t$  the quantity of motion of a system of variable mass is equal to

$$P + \Delta p = p^* + \Delta p^* - \Delta p_1 + \Delta p_2 \quad 1.5.2$$

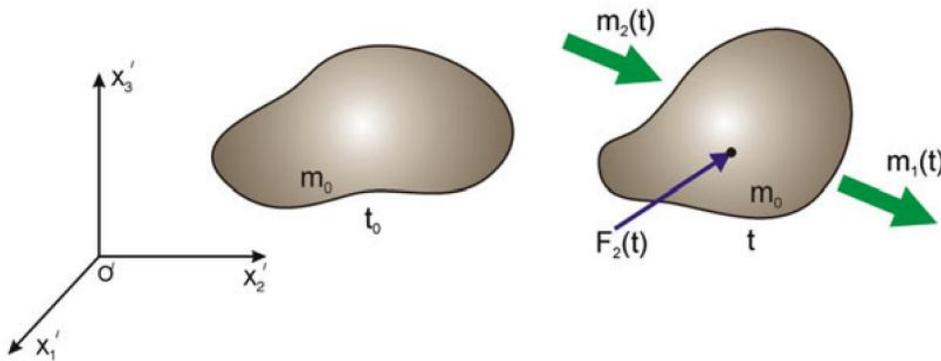


Fig. 1.5.2 Motion of a body of variable mass with respect to the inertial coordinate system  $O'X'_1X'_2X'_3$

This means that the increment of momentum of the investigated system follows from the increment of momentum of a system of constant mass and the additional quantity of motion delivered ( $\Delta \mathbf{p}_2$ ) and removed ( $\Delta \mathbf{p}_1$ ) to/from the system during time  $\Delta t$ .

From the preceding equation we obtain

$$\Delta p = \Delta p^* - \Delta p_1 + \Delta p_2 \quad 1.5.3$$

Because at the instant  $t$  we have

$$P = P^* \tag{1.5.4}$$

Dividing by  $\Delta t$  and on the assumption that  $\Delta t \rightarrow 0$  we get,

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta p}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta p^*}{\Delta t} - \lim_{\Delta t \rightarrow 0} \frac{\Delta p_1}{\Delta t} + \lim_{\Delta t \rightarrow 0} \frac{\Delta p_2}{\Delta t}, \tag{1.5.5}$$

Hence

$$\frac{dp}{dt} = F + F_1^R + F_2^R \tag{1.5.6}$$

where

$$F = \lim_{\Delta t \rightarrow 0} \frac{\Delta p^*}{\Delta t} = \frac{dp^*}{dt},$$

$$F_1^R = - \lim_{\Delta t \rightarrow 0} \frac{dp_1}{dt}, \quad F_2^R = - \lim_{\Delta t \rightarrow 0} \frac{dp_2}{dt} \tag{1.5.7}$$

and  $\mathbf{F}$  is a main vector of a system of external forces acting at the time instant  $t$ . Equation (1.5.6) extends the well-known theorem concerning the change in the quantity of motion (momentum) of a system. On its right-hand side additionally appear the so-called *thrust forces*,  $F_1^R$  and  $F_2^R$ .

In a similar way one can generalize the theorem regarding the change in angular momentum (moment of momentum) of a system. Applying an argument analogous to the previous one, we obtain

$$K + \Delta K = K^* + \Delta K^* - \Delta K_1 + \Delta K_2 \tag{1.5.8}$$

where  $\mathbf{K}$  is the moment of momentum of the system with respect to a certain arbitrary chosen fixed pole in the coordinate system  $O'X'_1X'_2X'_3$  and  $\Delta \mathbf{K}1(2)$  denotes the sum of moments of a quantity of motion for those particles that left (entered) the considered system of variable mass during the time interval  $\Delta t$ . Dividing the preceding equation by  $\Delta t$  and proceeding to the limit as  $\Delta t \rightarrow 0$  we have

$$\frac{dK}{dt} = M + M_1^R + M_2^R, \tag{1.5.9}$$

Where

$$M = \lim_{\Delta t \rightarrow 0} \frac{\Delta M^*}{\Delta t} = \frac{dK^*}{dt},$$

$$M_1^R = - \lim_{\Delta t \rightarrow 0} \frac{\Delta K_1}{\Delta t}, \quad M_2^R = \lim_{\Delta t \rightarrow 0} \frac{\Delta K_2}{\Delta t}.$$

Equation (1.5.9) is a generalization of a theorem concerning changes in the angular momentum of a mechanical system. On its right-hand side additionally appear *moments of a thrust force*,  $M_1^R$  and  $M_2^R$ .

## 1.6 Collision and conservation laws

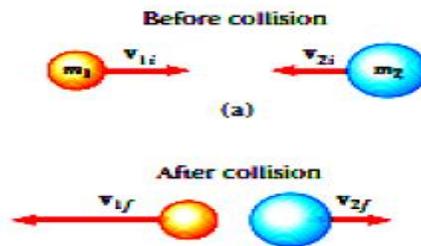
Consider the collision between two particles of masses  $m_1$  and  $m_2$  shown in the figure below. If the two particles form an isolated system, the momentum of the system must be conserved. Therefore, the total momentum of an isolated system just before a collision equals the total momentum of the system just after the collision. But, the total kinetic energy of the system of particles may or may not be conserved, depending on the type of collision. Whether or not kinetic energy is conserved is used to classify collisions as either *elastic* or *inelastic*.

- A. **Elastic collision:** - An elastic collision between two objects is one in which the total kinetic energy as well as total momentum of the system is conserved.

Consider two particles of masses  $m_1$  and  $m_2$  moving with initial velocities and along the same straight line, as shown in the figure below. The two particles collide head-on and then leave the collision site with different velocities,  $v_{1f}$  and  $v_{2f}$ . If the collision is elastic, both the momentum and kinetic energy of the system are conserved. Therefore, considering velocities along the horizontal direction we have:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad 1.6.1$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad 1.6.2$$



Because all velocities in the figure are either to the left or the right, they can be represented by the corresponding speeds along with algebraic signs indicating direction. We shall indicate  $v$  as positive if a particle moves to the right and negative if it moves to the left.

From conservation of kinetic energy (eq. 1.6.2), we can cancel the factor  $\frac{1}{2}$  as

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2i}^2 - v_{2f}^2)$$

Then factor both sides

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2i} - v_{2f})(v_{2i} + v_{2f}) \quad (a)$$

By separating the terms containing  $m_1$  and  $m_2$

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2i} - v_{2f}) \quad (b)$$

Dividing equation (a) by eq. (b)

$$v_{1i} + v_{1f} = v_{2i} + v_{2f}$$

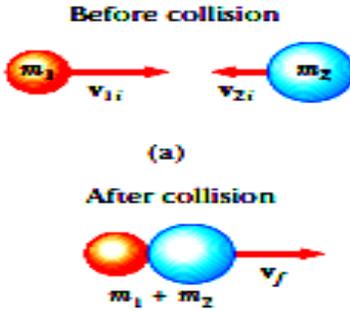
$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f}) \quad 1.6.3$$

**B. Inelastic Collision:** - An inelastic collision is one in which the total kinetic energy of the system is not conserved. But the momentum of the system is conserved. Therefore, for inelastic collision of two particles:

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 \neq \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

**C. Perfectly Inelastic Collision:** -When the colliding objects stick together after the collision, the collision is called perfectly inelastic. Consider two particles of masses  $m_1$  and  $m_2$  moving with initial velocities and along the same straight line, as shown in the figure below.



The two particles collide head-on, stick together, and then move with some common velocity  $v_f$  after the collision. The total momentum before the collision equals the total momentum of the composite system after the collision

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f$$

gives

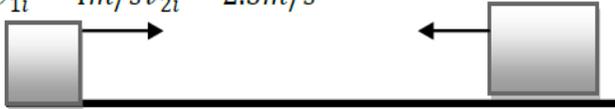
$$v_f = \frac{(m_1 v_{1i} + m_2 v_{2i})}{m_1 + m_2} \quad 1.6.4$$

### Example

A block of mass  $m_1=1.6\text{kg}$  initially moving to the right with a speed of  $4\text{m/s}$  on a horizontal frictionless track collides with a second block of mass  $m_2=2.1\text{kg}$  initially moving to the left with a speed of  $2.5\text{m/s}$ . If the collision is elastic, find the velocities of the two blocks after collision.

**Solution**

$$v_{1i} = 4\text{m/s} \quad v_{2i} = 2.5\text{m/s}$$



*Before Collision*

We are given that We are asked to find  $v_{1f}$  &  $v_{2f}$

$$m_1 = 1.6\text{kg}, v_{1i} = 4\text{m/s}$$

Since the collision is elastic both momentum and

$$m_2 = 2.1\text{kg}, v_{2i} = -\frac{2.5\text{m}}{\text{s}} \text{ kinetic energy are conserved}$$

From conservation of momentum

$$(1.6\text{kg})(4\text{m/s}) + (2.1\text{kg})(-2.5\text{m/s}) = (1.6\text{kg})v_{1f} + (2.1\text{kg})v_{2f}$$

$$1.6v_{1f} + 2.1v_{2f} = 1.15 \quad (\text{a})$$

From conservation of kinetic energy

$$v_{1i} + v_{1f} = v_{2i} + v_{2f}$$

$$v_{1f} - v_{2f} = v_{2i} - v_{1i}$$

$$v_{1f} - v_{2f} = -2.5\text{m/s} - 4\text{m/s}$$

$$v_{1f} - v_{2f} = -6.5\text{m/s} \quad (\text{b})$$

Solving equation (a) and (b) simultaneously

$$v_{1f} = -3.38\text{m/s}$$

$$v_{2f} = 3.12\text{m/s}$$

## 1.7) Two body problem in center of mass coordinate system

In the previous lecture, we discussed a variety of conclusions we could make about the motion of an arbitrary collection of particles, subject only to a few restrictions. Today, we will consider a much simpler, very well-known problem in physics - an isolated system of two particles which interact through a central potential. This model is often referred to simply as the two-body problem. In the case of only two particles, our equations of motion reduce simply to

$$\begin{aligned}m_1 \ddot{r}_1 &= F_{21} \\m_2 \ddot{r}_2 &= F_{12}\end{aligned}\tag{1.7.1}$$

A famous example of such a system is of course given by Newton's Law of Gravitation, where the two particles interact through a potential energy given by

$$\begin{aligned}U_{12}(|r_1 - r_2|) &= U_{21}(|r_2 - r_1|) \\&= G \frac{m_1 m_2}{|r_1 - r_2|^2},\end{aligned}\tag{1.7.2}$$

where  $G$  is Newton's constant,

$$G = 6.673 \times 10^{-11} \text{ Nm}^2/\text{kg}^2\tag{1.7.3}$$

How can we go about finding the most general solution to this set of equations? As with any physics problem, the first thing we should do is make maximal use of the symmetries or conservation laws of our problem. First, because the two particles interact via a central potential, these two forces should obey Newton's third law, as we discussed in the previous lecture. We know that as a result, the total momentum of our system will be conserved, and so we should consider the center of mass,

$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} = \frac{m_1 r_1 + m_2 r_2}{M}\tag{1.7.4}$$

the time derivative of which is given by the center of mass velocity,

$$V_{CM} = \frac{m_1 v_1 + m_2 v_2}{M}\tag{1.7.5}$$

Now, in the previous lecture, we found that the acceleration of the center of mass depended on the net external force,

$$F_{ext} = M a_{CM}\tag{1.7.6}$$

Since our system is isolated, the center of mass acceleration must be zero, and hence the center of mass velocity must be a constant,

$$V_{CM}^{(0)} = \frac{m_1 v_1^{(0)} + m_2 v_2^{(0)}}{M} \quad 1.7.7$$

Thus, the center of mass motion is given by

$$R(t) = V_{CM}^{(0)} t \quad 1.7.8$$

Since we already know the motion of the center of mass on general grounds, we can make use of this information to simplify our problem. To see how, let's define the vector of relative distance

$$r = r_1 - r_2 \quad 1.7.9$$

A simple algebraic rearrangement then yields

$$r_1 = R + \frac{m_2}{m_1+m_2} r ; \quad r_2 = R - \frac{m_1}{m_1+m_2} r \quad 1.7.10$$

From this expression, and the fact that we already know  $\mathbf{R}$ , our problem simply reduces to finding  $\mathbf{r}$ .

To find the equation of motion satisfied by  $\mathbf{r}$ , we return to our original equations of motion and multiply the first by  $m_2$  and the second by  $m_1$ , in order to find

$$\begin{aligned} m_1 m_2 \ddot{r}_1 &= m_2 F_{21} \\ m_1 m_2 \ddot{r}_2 &= m_1 F_{12} \end{aligned} \quad 1.7.11$$

If we then subtract the second equation from the first, we have

$$\begin{aligned} m_1 m_2 (\ddot{r}_1 - \ddot{r}_2) &= m_2 F_{21} - m_1 F_{12} \\ \frac{m_1 m_2}{(m_1+m_2)} \ddot{r} &= F_{21} \end{aligned} \quad 1.7.12$$

where we have made use of Newton's third law in the second equation. Because the forces are derived from a central potential which only depends on the distance between the two particles, we have

$$\mathbf{F}_{21} = -\frac{\partial}{\partial \mathbf{r}_1} U_{12}(|\mathbf{r}_1 - \mathbf{r}_2|) = \nabla_1 U_{12}(|\mathbf{r}_1 - \mathbf{r}_2|) \quad 1.7.13$$

Now, since the potential only depends on  $\mathbf{r}$ , and not the center of mass  $\mathbf{R}$ , we can use the chain rule to write for the x-component of the derivative, for example,

$$\frac{\partial}{\partial r_{1x}} U_{12}(|\mathbf{r}|) = \frac{\partial r_x}{\partial r_{1x}} \frac{\partial}{\partial r_x} U_{12}(|\mathbf{r}|) + \frac{\partial R_x}{\partial r_{1x}} \frac{\partial}{\partial R_x} U_{12}(|\mathbf{r}|) = \frac{\partial}{\partial r_x} U_{12}(|\mathbf{r}|), \quad 1.7.14$$

and so on for the other coordinates. Thus, I find that I can write

$$m_* \ddot{\mathbf{r}} = -\frac{\partial}{\partial \mathbf{r}} U(|\mathbf{r}|) \equiv \mathbf{F}(\mathbf{r}), \quad 1.7.15$$

Where

$$m_* = \frac{m_1 m_2}{(m_1 + m_2)} \quad 1.7.16$$

is the **reduced mass** of the system. Thus, our problem has effectively been reduced to a one-particle system - mathematically, it is no different than a single particle with position vector  $\mathbf{r}$  and mass  $m_*$ , subject to an external force  $\mathbf{F}$ . Therefore, conservation of momentum has dramatically simplified our system.

## Conservation of angular momentum of two body problem

Since our two particles interact with each other through a central potential, we know that the total angular momentum of the system is conserved. However, since we have reduced our problem to a one-particle system, it makes more sense to reformulate this statement in terms of the angular momentum of this fictitious particle,

$$L = m_* \mathbf{r} \times \mathbf{v}, \quad 1.7.17$$

Where

$$\mathbf{v} = \dot{\mathbf{r}} \quad 1.7.18$$

Now, a short exercise in the chain rule shows us that

$$\mathbf{F}(\mathbf{r}) = -\frac{\partial}{\partial \mathbf{r}} U(|\mathbf{r}|) = -\frac{\mathbf{r}}{r} \frac{dU(r)}{dr}; \quad r = |\mathbf{r}|. \quad 1.7.19$$

Therefore, the torque on the particle due to F is

$$\tau = r \times F \propto r \times r = 0 \quad 1.7.20$$

That is, the torque vanishes because the force is parallel to the displacement vector. Thus, in the absence of any torque, the angular momentum of the particle must be constant,

$$\frac{d}{dt}L = 0 \quad 1.7.21$$

This fact is a general result for the motion of a particle in an external central potential.

For our one-particle system, conservation of angular momentum allows us to make a further simplification. For any three vectors, we can form the scalar triple product,

$$a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b) \quad 1.7.22$$

The fact that all three of these expressions are equal is left as an exercise on your homework. If we use this identity, we can see that

$$r \cdot L = m_* r \cdot (r \times v) = m_* v \cdot (r \times r) = 0 \quad 1.7.23$$

Because this inner product is zero, it must be the case that r is always perpendicular to the angular momentum L,

$$r \perp L \quad 1.7.24$$

However, because the angular momentum is constant, there must be a fixed vector in space which the position vector r is always perpendicular to. Given that the position vector is always perpendicular to a certain orientation in space, it must be the case that the position vector always lies in a plane. As a result of this fact, not only has our problem been reduced to a one particle system, it has also been effectively reduced to two dimensions. Because our problem is described by a radial force in two dimensions, at this point it is most convenient to switch over to polar coordinates,

$$r_x = r \cos\theta ; r_y = r \sin\theta \quad 1.7.25$$

We have chosen the convention that the plane which the particle travels in is the x-y plane, and that the angular momentum is oriented along the z-axis. In this set of coordinates, we can write

$$\frac{d\theta}{dt} = \frac{l}{m_* r^2} ; l \equiv |L| \quad 1.7.26$$

which you'll show on your homework. This expression for the time derivative of the angular coordinate makes another fact clear - the sign of  $d\theta/dt$  is always positive, so that the particle always rotates around the center of our coordinate system in the same direction.

## 1.8 collisions in the Center of Mass reference frame

### i) Elastic collisions in the CM frame

In center of mass frame (zero momentum frame),



➤ Conservation of momentum is given by,

$$m_1 u_1' + m_2 u_2' = 0$$

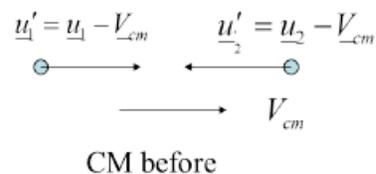
$$m_1 v_1' + m_2 v_2' = 0 \tag{1.8.1}$$

➤ Conservation of energy is given by,

$$\frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \tag{1.8.2}$$

### ii) Inelastic collision in CM frame

➤ Case of perfectly inelastic collision ( $e=0$ )



➤ After collision, total mass ( $m_1 + m_2$ ) is at rest in CM:

$$\begin{array}{c}
 m_1 + m_2 \\
 \text{●} \quad \mathbf{v}' = \mathbf{0} \\
 \text{CM after}
 \end{array}$$

- KE in CM:  $T_{\text{cm}} = T_{\text{LAB}} - \frac{1}{2}(m_1 + m_2) v_{\text{CM}}^2$
- The maximum energy that can be lost during the collision is then,

$$\mathbf{T}_{\text{CM}} = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 - \frac{1}{2}(m_1 + m_2)v_{\text{CM}}^2 \qquad 1.8.3$$

# Chapter Two

## Rigid Body Dynamics

### 2.1 Introduction

In this chapter, unless otherwise stated, the following notation conventions will be used:

1. Einstein's summation convention. Whenever an index appears twice (an only twice), then a summation over this index is implied. For example,

$$x_i x_i = \sum_i x_i x_i = \sum_i x_i^2 \quad 2.1$$

2. The index  $i$  is reserved for Cartesian coordinates. For example,  $x_i$ , for  $i = 1, 2, 3$ , represents either  $x$ ,  $y$ , or  $z$  depending on the value of  $i$ . Similarly,  $p_i$  can represent  $p_x$ ,  $p_y$ , or  $p_z$ . This does **not** mean that any other indices cannot be used for Cartesian coordinates, but that the index  $i$  will only be used for Cartesian coordinates.
3. When dealing with systems containing multiple particles, the index  $\alpha$  will be used to identify quantities associated with a given particle when using Cartesian coordinates. For example, if we are in the presence of  $n$  particles, the position vector for particle  $\alpha$  is given by  $r_\alpha$ , and its kinetic energy  $T_\alpha$  by

$$T_\alpha = \frac{1}{2} m_\alpha \dot{x}_{\alpha,i} \dot{x}_{\alpha,i}, \quad \alpha = 1, 2, \dots, n \text{ and } i = 1, 2, 3. \quad 2.2$$

Take note that, according to convention 1 above, there is an implied summation on the Cartesian velocity components (the index  $i$  is used), but **not** on the masses since the index  $\alpha$  appears more than twice. Correspondingly, the total kinetic energies is written as

$$\begin{aligned} T &= \frac{1}{2} \sum_{\alpha=1}^n m_\alpha \dot{x}_{\alpha,i} \dot{x}_{\alpha,i} \\ &= \frac{1}{2} \sum_{\alpha=1}^n m_\alpha (\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \end{aligned} \quad 2.3$$

### 2.2 The Inertia Tensor

Let's consider a rigid body composed of  $n$  particles of mass  $m_\alpha$ ,  $\alpha = 1, 2, \dots, n$ . If the body rotates with an angular velocity  $\boldsymbol{\omega}$  about some point fixed with respect to the body coordinates (this "body" coordinate system is what we used to refer to as "noninertial" or "rotating" coordinate system), and if this point moves linearly with a velocity  $\mathbf{V}$  with respect to a fixed (i.e., inertial) coordinate system, then the velocity of the  $\alpha$ th particle is given by

$$v_\alpha = V + \omega \times r_\alpha \quad 2.4$$

Where we omitted the term

$$v_{\alpha,r} = \left(\frac{dr_\alpha}{dt}\right)_{\text{rotating}} = 0 \quad 2.5$$

since we are dealing with a rigid body. We have also dropped the  $f$  subscript, denoting the fixed coordinate system, as it is understood that all the non-vanishing velocities will be measured in this system; again, we are dealing with a rigid body.

The total kinetic energy of the body is given by

$$\begin{aligned} T &= \sum_{\alpha} T_{\alpha} = \frac{1}{2} \sum_{\alpha} m_{\alpha} v_{\alpha}^2 \\ &= \frac{1}{2} \sum_{\alpha} m_{\alpha} (\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_{\alpha})^2 \\ &= \frac{1}{2} \sum_{\alpha} m_{\alpha} V^2 + \sum_{\alpha} m_{\alpha} \mathbf{V} \cdot (\boldsymbol{\omega} \times \mathbf{r}_{\alpha}) + \frac{1}{2} \sum_{\alpha} m_{\alpha} (\boldsymbol{\omega} \times \mathbf{r}_{\alpha})^2. \end{aligned} \quad 2.6$$

Although this is an equation for the total kinetic energy is perfectly general, considerable simplification will result if we choose the origin of the body coordinate system to coincide with the centre of mass. With this choice, the second term on the right hand side of the last of equations (2.9.6) can be seen to vanish from

$$\sum_{\alpha} m_{\alpha} V \cdot (\boldsymbol{\omega} \times \mathbf{r}_{\alpha}) = V \cdot [\boldsymbol{\omega} \times (\sum_{\alpha} m_{\alpha} \mathbf{r}_{\alpha})] = 0, \quad 2.7$$

since the centre of mass  $\mathbf{R}$  of the body, of mass  $M$ , is defined such that

$$\sum_{\alpha} m_{\alpha} \mathbf{r}_{\alpha} = 0 \quad 2.8$$

The total kinetic energy can then be broken into two components: one for the translational kinetic energy and another for the rotational kinetic energy. That is,

$$T = T_{\text{trans}} + T_{\text{rot}} \quad 2.9$$

With

$$\boxed{\begin{aligned} T_{\text{trans}} &= \frac{1}{2} \sum_{\alpha} m_{\alpha} V^2 = \frac{1}{2} M V^2 \\ T_{\text{rot}} &= \frac{1}{2} \sum_{\alpha} m_{\alpha} (\boldsymbol{\omega} \times \mathbf{r}_{\alpha})^2 \end{aligned}}$$

2.10

The expression for  $T_{\text{rot}}$  can be further modified, but to do so we will now resort to tensor (or index) notation. So, let's consider the following vector equation

$$(\boldsymbol{\omega} \times \mathbf{r}_{\alpha})^2 = (\boldsymbol{\omega} \times \mathbf{r}_{\alpha}) \cdot (\boldsymbol{\omega} \times \mathbf{r}_{\alpha}), \quad 2.11$$

and rewrite it using the Levi-Civita and the Kronecker tensors,

$$\begin{aligned} (\boldsymbol{\varepsilon}_{ijk} \omega_j r_{\alpha,k}) (\boldsymbol{\varepsilon}_{lmn} \omega_l r_{\alpha,n}) &= \boldsymbol{\varepsilon}_{ijk} \boldsymbol{\varepsilon}_{lmn} \omega_j r_{\alpha,k} \omega_l r_{\alpha,n} \\ &= (\delta_{jn} \delta_{lm} - \delta_{jn} \delta_{lm}) \omega_j r_{\alpha,k} \omega_l r_{\alpha,n} \\ &= \omega_j \omega_l r_{\alpha,k} r_{\alpha,n} - \omega_j \omega_l r_{\alpha,k} r_{\alpha,n}. \end{aligned} \quad 2.12$$

Inserting this result in the equation for  $T_{\text{rot}}$  in equation (2.10) we get

$$T_{\text{rot}} = \frac{1}{2} \sum_{\alpha} m_{\alpha} [|\boldsymbol{\omega}|^2 |\mathbf{r}_{\alpha}|^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_{\alpha})^2]. \quad 2.13$$

Alternatively,

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2} \sum_{\alpha} m_{\alpha} [\omega_j \omega_l r_{\alpha,k} r_{\alpha,k} - \omega_l r_{\alpha,i} \omega_j r_{\alpha,i}] \\ &= \frac{1}{2} \sum_{\alpha} m_{\alpha} [(\omega_l \omega_j \delta_{ij}) r_{\alpha,k} r_{\alpha,k} - \omega_l r_{\alpha,i} \omega_j r_{\alpha,i}] \\ &= \frac{1}{2} (\omega_l \omega_j) \sum_{\alpha} m_{\alpha} [\delta_{ij} r_{\alpha,k} r_{\alpha,k} - r_{\alpha,i} r_{\alpha,i}]. \end{aligned} \quad 2.14$$

We now define the components  $I_{ij}$  of the so called inertia tensor  $\{I\}$  by

$$\boxed{I_{ij} = \sum_{\alpha} m_{\alpha} [\delta_{ij} r_{\alpha,k} r_{\alpha,k} - r_{\alpha,i} r_{\alpha,i}]} \quad 2.15$$

and the rotational kinetic energy becomes

$$T_{\text{rot}} = \frac{1}{2} I_{ij} \omega_i \omega_j \quad 2.16$$

Or in vector notation

$$T_{rot} = \frac{1}{2} \omega \cdot \{I\} \cdot \omega \quad 2.17$$

For our purposes it will be usually sufficient to treat the inertia tensor as a regular  $3 \times 3$  matrix. Indeed, we can explicitly write  $\{I\}$  using equation (2.15) as

$$\{I\} = \begin{pmatrix} \sum_{\alpha} m_{\alpha} (x_{\alpha,2}^2 + x_{\alpha,3}^2) & -\sum_{\alpha} m_{\alpha} x_{\alpha,1} x_{\alpha,2} & -\sum_{\alpha} m_{\alpha} x_{\alpha,1} x_{\alpha,3} \\ -\sum_{\alpha} m_{\alpha} x_{\alpha,2} x_{\alpha,1} & \sum_{\alpha} m_{\alpha} (x_{\alpha,1}^2 + x_{\alpha,3}^2) & -\sum_{\alpha} m_{\alpha} x_{\alpha,2} x_{\alpha,3} \\ -\sum_{\alpha} m_{\alpha} x_{\alpha,3} x_{\alpha,1} & -\sum_{\alpha} m_{\alpha} x_{\alpha,3} x_{\alpha,2} & \sum_{\alpha} m_{\alpha} (x_{\alpha,1}^2 + x_{\alpha,2}^2) \end{pmatrix} \quad 2.18$$

It is easy to see from either equation (2.15) or equation (2.18) that the inertia tensor is symmetric, that is,

$$I_{ij} = I_{ji} \quad 2.19$$

The diagonal elements  $I_{11}$ ,  $I_{22}$ , and  $I_{33}$  are called the **moments of inertia** about the  $x_1$ -,  $x_2$ -, and  $x_3$ -axes, respectively. The negatives of the off-diagonal elements are the **products of inertia**. Finally, in most cases the rigid body is continuous and not made up of discrete particles as was assumed so far, but the results are easily generalized by replacing the summation by a corresponding integral in the expression for the components of the inertia tensor

$$I_{ij} = \int_V \rho(\mathbf{r}) (\delta_{ij} x_k x_k - x_i x_j) dx_1 dx_2 dx_3, \quad 2.20$$

where  $\rho(\mathbf{r})$  is the mass density at the position  $\mathbf{r}$ , and the integral is to be performed over the whole volume  $V$  of the rigid body.

### Example

Calculate the inertia tensor for a homogeneous cube of density  $\rho$ , mass  $M$ , and side length  $b$ . Let one corner be at the origin, and three adjacent edges lie along the coordinate axes (see Figure 2-1).

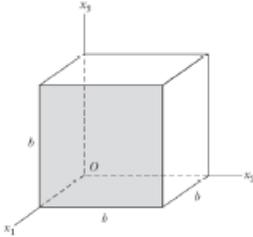


Fig. 2.1 A homogeneous cube of sides  $b$  with the origin at one corner.

### Solution.

We use equation (2.20) to calculate the components of the inertia tensor. Because of the symmetry of the problem, it is easy to see that the three moments of inertia  $I_{11}$ ,  $I_{22}$ , and  $I_{33}$  are equal and that same holds for all of the products of inertia. So,

$$\begin{aligned} I_{11} &= \int_0^b \int_0^b \int_0^b \rho(x_2^2 + x_3^2) dx_1 dx_2 dx_3 \\ &= \rho \int_0^b dx_3 \int_0^b dx_2 (x_2^2 + x_3^2) \int_0^b dx_1 \\ &= \rho b \int_0^b dx_3 \left( \frac{b^3}{3} + bx_3^2 \right) = \rho b \left( \frac{b^4}{3} + \frac{b^4}{3} \right) \\ &= \frac{2}{3} \rho b^5 = \frac{2}{3} Mb^2. \end{aligned}$$

And for the negative of the products of inertia

$$\begin{aligned} I_{12} &= - \int_0^b \int_0^b \int_0^b \rho x_1 x_2 dx_1 dx_2 dx_3 \\ &= -\rho \left( \frac{b^2}{2} \right) \left( \frac{b^2}{2} \right) (b) \\ &= -\frac{1}{4} \rho b^5 = -\frac{1}{4} Mb^2. \end{aligned}$$

It should be noted that in this example the origin of the coordinate system is not located at the centre of mass of the cube.

## 2.3 Angular Momentum

Going back to the case of a rigid body composed of a discrete number of particles; we can calculate the angular momentum with respect to some point  $O$  fixed in the body coordinate system with

$$L = \sum_{\alpha} r_{\alpha} \times p_{\alpha}. \quad 2.23$$

Relative to the body coordinate system the linear momentum of the  $\alpha$ th particle is

$$p_{\alpha} = m_{\alpha} v_{\alpha} = m_{\alpha} \omega \times r_{\alpha}, \quad 2.24$$

and the total angular momentum becomes

$$L = \sum_{\alpha} m_{\alpha} r_{\alpha} \times (\omega \times r_{\alpha}) \quad 2.25$$

Resorting one more time to tensor notation we can calculate  $r_{\alpha} \times (\omega \times r_{\alpha})$  as

$$\begin{aligned} \varepsilon_{ijk} x_{\alpha,j} \varepsilon_{ilm} \omega_l x_{\alpha,m} &= \varepsilon_{kij} \varepsilon_{ilm} x_{\alpha,j} \omega_l x_{\alpha,m} \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) x_{\alpha,j} \omega_l x_{\alpha,m} \\ &= x_{\alpha,j} x_{\alpha,i} \omega_j - x_{\alpha,j} \omega_j x_{\alpha,i}, \end{aligned} \quad 2.26$$

or alternatively in vector notation

$$\mathbf{r}_{\alpha} \times (\boldsymbol{\omega} \times \mathbf{r}_{\alpha}) = r_{\alpha}^2 \boldsymbol{\omega} - \mathbf{r}_{\alpha} (\mathbf{r}_{\alpha} \cdot \boldsymbol{\omega}). \quad 2.27$$

Then, the total angular momentum is given by

$$\boxed{\mathbf{L} = \sum_{\alpha} m_{\alpha} [r_{\alpha}^2 \boldsymbol{\omega} - \mathbf{r}_{\alpha} (\mathbf{r}_{\alpha} \cdot \boldsymbol{\omega})]} \quad 2.28$$

Using the tensor notation the component of the angular momentum is

$$\begin{aligned} L_i &= \sum_{\alpha} m_{\alpha} (x_{\alpha,k} x_{\alpha,k} \omega_i - x_{\alpha,i} \omega_j x_{\alpha,j}) \\ &= \omega_j \sum_{\alpha} m_{\alpha} (\delta_{ij} x_{\alpha,k} x_{\alpha,k} - x_{\alpha,i} x_{\alpha,j}), \end{aligned} \quad 2.29$$

and upon using equation (2.15) for the inertia tensor

$$L_i = I_{ij} \omega_j \quad 2.30$$

Or in tensor notation

$$\mathbf{L} = \{\mathbf{I}\} \cdot \boldsymbol{\omega} \quad 2.31$$

Finally, we can insert equation (2.31) for the angular momentum vector into equation (2.17) for the rotational kinetic energy to obtain

$$T_{rot} = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega} \quad 2.32$$

## 2.4 The Principal Axes of Inertia

We now set on finding a set of body axes that will render the inertia tensor diagonal in form. That is, given equation (9.18) for  $\{\mathbf{I}\}$ , we want to make a change in the body basis vectors (i.e., a change of variables) that will change the form of the inertia tensor to

$$\{\mathbf{I}\} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad 2.33$$

We will then require that all the products of inertia be zero. Carrying this program will provide a significant simplification for the expressions of the angular momentum and the kinetic energy, as measured in the inertial reference frame. That is, these two quantities will be given by

$$L_1 = I_1 \omega_1$$

$$L_2 = I_2 \omega_2$$

$$L_3 = I_3 \omega_3 \quad 2.34$$

and

$$T_{rot} = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2), \quad 2.35$$

The set of axes that allow this transformation is called the **principal axes of inertia**. When the equations for the components of the angular momentum can be expressed as in equation (2.33), then  $\mathbf{L}$ , and  $\omega$  are directed along the same axis.

The problem of finding the principal axes is mathematically equivalent to solving a set of linear equations. More precisely, we have from equation (2.31) that

$$L = \{\mathbf{I}\} \cdot \omega \quad 2.36$$

but we are actually looking for a way to reduce this equation to the following form

$$L = \{I\} \cdot \omega = I\omega \quad 2.37$$

Mathematically speaking,  $I$ , which is called a **principal moment of inertia**, is an eigenvalue of the inertia tensor, and  $\omega$ , which will give us the corresponding principal axis of inertia, is an eigenvector. The system of equations (2.37) can be written as

$$\begin{aligned} L_1 &= I\omega_1 = I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ L_2 &= I\omega_2 = I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ L_3 &= I\omega_3 = I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3, \end{aligned} \quad 2.38$$

or, after some rearranging

$$\begin{aligned} (I_{11} - I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 &= 0 \\ I_{21}\omega_1 + (I_{22} - I)\omega_2 + I_{23}\omega_3 &= 0 \\ I_{31}\omega_1 + I_{32}\omega_2 + (I_{33} - I)\omega_3 &= 0. \end{aligned} \quad 2.39$$

The mathematical condition necessary for this set of equation to have a nontrivial solution is that the determinant of the coefficient vanishes

$$\begin{vmatrix} (I_{11} - I) + I_{12} + I_{13} \\ I_{21} + (I_{22} - I) + I_{23} \\ I_{31} + I_{32} + (I_{33} - I) \end{vmatrix} = 0. \quad 2.40$$

The expansion of this determinant leads to the so-called **secular** or **characteristic equation** for the eigenvalues  $I$  (i.e.,  $I_1$ ,  $I_2$ , and  $I_3$  in equation (2.34)); it is a third order polynomial. Once the characteristic equation has been solved, the principal axes can be determined by inserting the

eigenvalues back in equation (2.39) and evaluating the ratios of the angular velocity components  $(\omega_1, \omega_2, \omega_3)$ , therefore, determining the corresponding eigenvectors.

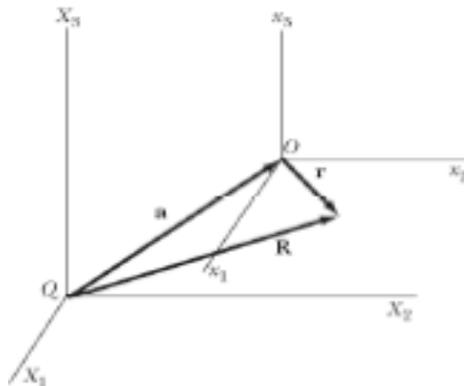
It is important to realize that in many cases, the rigid body under study will exhibit some symmetry that will allow one to guess what the principal axes are. For example, a cylinder will have one of its principal axes directed along the centre axis of the cylinder. The two remaining axes will be directed at right angle to this axis (and to each other).

Finally, here are a few definitions: a body that has

- i)  $I_1 = I_2 = I_3$  is called a **spherical top**,
- ii)  $I_1 = I_2 \neq I_3$  is a **symmetric top**,
- iii)  $I_1 \neq I_2 \neq I_3$  is an **asymmetric top**, and finally, if
- iv)  $I_1 = 0, I_2 = I_3$  the body is a **rotor**.

## 2.5 Moments of Inertia for Different Body Coordinate Systems

We consider two sets of coordinate axes that are oriented in the same direction, but have different origins. The  $xi$ -axes have their origin  $O$  located at the centre of mass of the rigid body, and the  $Xi$ -axes have their origin  $Q$  located somewhere else inside, or outside, of the body (Figure 2.2)



**Figure 2-2** – The  $Xi$ -axes are fixed in the rigid body and have the same orientation as the  $xi$ -axes, but its origin  $Q$  is not located at the same point  $O$ , which is the centre of mass of the body.

The elements of the inertia tensor  $\{ \mathbf{J} \}$  relative to the  $Xi$ -axes are

$$J_{ij} = \sum_{\alpha} m_{\alpha} (\delta_{ij} X_{\alpha,k} X_{\alpha,k} - X_{\alpha,i} X_{\alpha,j}) \quad 2.41$$

If the vector  $\mathbf{a}$  connects the origin Q to the centre of mass (and origin) O, then the *general vector*  $\mathbf{R}$  for the position of a point within the rigid body is written as  $R_\alpha = \mathbf{a} + r_\alpha$ , or using components

$$X_{\alpha,i} = a_i + x_{\alpha,i}. \quad 2.42$$

Inserting equation (2.42) into equation (2.41) we get

$$\begin{aligned} J_{ij} &= \sum_{\alpha} m_{\alpha} \left[ \delta_{ij} (a_k + x_{\alpha,k})(a_k + x_{\alpha,k}) - (a_i + x_{\alpha,i})(a_j + x_{\alpha,j}) \right] \\ &= \sum_{\alpha} m_{\alpha} (\delta_{ij} x_{\alpha,k} x_{\alpha,k} - x_{\alpha,i} x_{\alpha,j}) \\ &\quad + \sum_{\alpha} m_{\alpha} \left[ \delta_{ij} (2x_{\alpha,k} a_k + a_k a_k) - (a_i x_{\alpha,j} + a_j x_{\alpha,i} + a_i a_j) \right] \\ &= I_{ij} + \sum_{\alpha} m_{\alpha} (\delta_{ij} a_k a_k - a_i a_j) + \sum_{\alpha} m_{\alpha} (\delta_{ij} 2x_{\alpha,k} a_k - a_i x_{\alpha,j} - a_j x_{\alpha,i}). \end{aligned} \quad 2.43$$

But from the definition of the centre of mass itself, the last term on the right hand side of the last of equations (2.43) equals zero since

$$\sum_{\alpha} m_{\alpha} x_{\alpha,i} = 0 \quad 2.44$$

We then find the final result that

$$\boxed{J_{ij} = I_{ij} + M (\delta_{ij} a^2 - a_i a_j)} \quad 2.45$$

with  $M = \sum_{\alpha} m_{\alpha}$  and  $a^2 = a_k a_k$ .

We see from equation (2.45) that *the inertia tensor components are minimum when measured relative to the centre of mass.*

## 2.6 The Euler Angles

We stated in previous sections that of the six degrees of freedom of a rigid body, three are rotational in nature (the other three are for the translation motion of the centre of mass). In this section, we set to determine the set of angles that can be used to specify the rotation of a rigid body.

We know that the transformation from one coordinate system to another can be represented by a matrix equation such as

$$\mathbf{x} = \lambda \mathbf{x}' \quad 2.46$$

If we identify the inertial (or fixed) system with  $\mathbf{x}'$  and the rigid body coordinate system with  $\mathbf{x}$ , then the rotation matrix  $\lambda$  describes the relative orientation of the body in relation to the fixed system. Since there are three rotational degrees of freedom,  $\lambda$  is actually a product from three individual rotation matrices; one for each independent angles. Although there are many possible choices for the selection of these angles, we will use the so-called **Euler angles  $\phi, \theta$  and  $\psi$** .

The Euler angles are generated in the following series of rotation that takes the fixed  $\mathbf{x}'$  system to the rigid body  $\mathbf{x}$  system( see Fig. 2.3)

1. The first rotation is counterclockwise through an angle  $\phi$  about the  $x'_3$ -axis. It transforms the transforms the inertial system into an intermediate set of  $x''_i$ -axis. The transformation matrix is

$$\lambda_\phi = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 2.47$$

With  $0 \leq \phi \leq 2\pi$ , and

$$x'' = \lambda_\phi x' \quad 2.48$$

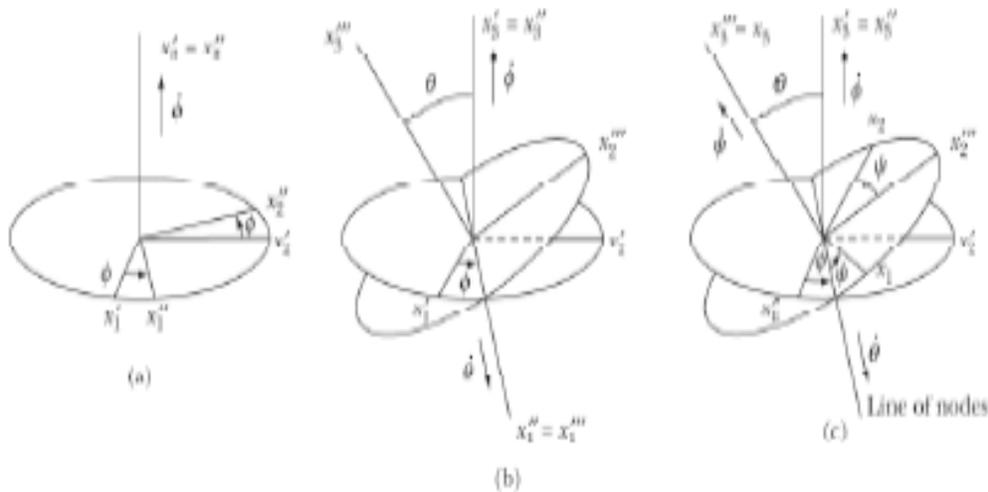


Fig. 2.4 The Euler angles are used to rotate the fixed  $\mathbf{x}'$  system to the rigid body  $\mathbf{x}$  system. (a) The first rotation is counterclockwise through an angle  $\phi$  about the  $x'_3$ -axis. (b) The second rotation is counterclockwise through an angle  $\theta$  about the  $x''_1$ -axis. (c) The third rotation is counterclockwise through an angle  $\psi$  about the  $x'''_3$ -axis.

2. The second rotation is counterclockwise through an angle  $\theta$  about the  $x_1''$ -axis. (also called the **line of nodes**). It transforms the inertial system into an intermediate set of  $x_i'''$  - axis. The transformation matrix is

$$\lambda_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}, \quad 2.49$$

with  $0 \leq \theta \leq \pi$ , and

$$x''' = \lambda_\theta x''. \quad 2.50$$

3. The third rotation is counterclockwise through an angle  $\psi$  about the  $x_3'''$ -axis. It transforms the inertial system into the final set rigid body  $x_i$  - axis. The transformation matrix is

$$\lambda_\psi = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad 2.51$$

With  $0 \leq \psi \leq \pi$

$$\mathbf{x} = \lambda_\psi \mathbf{x}''' \quad 2.52$$

Combining the three rotations using equations (2.48), (2.50), and (2.52) we find that the complete transformation is given by

$$\mathbf{x} = \lambda_\psi \lambda_\theta \lambda_\phi \mathbf{x}', \quad 2.53$$

and the rotation matrix is

$$\lambda = \lambda_\psi \lambda_\theta \lambda_\phi \quad 2.54$$

Upon calculating this matrix, we find that its components are

$$\begin{aligned}
 \lambda_{11} &= \cos(\psi)\cos(\phi) - \cos(\theta)\sin(\phi)\sin(\psi) \\
 \lambda_{21} &= -\sin(\psi)\cos(\phi) - \cos(\theta)\sin(\phi)\cos(\psi) \\
 \lambda_{31} &= \sin(\theta)\sin(\phi) \\
 \lambda_{12} &= \cos(\psi)\sin(\phi) + \cos(\theta)\cos(\phi)\sin(\psi) \\
 \lambda_{22} &= -\sin(\psi)\sin(\phi) + \cos(\theta)\cos(\phi)\cos(\psi) \\
 \lambda_{32} &= -\sin(\theta)\cos(\phi) \\
 \lambda_{13} &= \sin(\psi)\sin(\theta) \\
 \lambda_{23} &= \cos(\psi)\sin(\theta) \\
 \lambda_{33} &= \cos(\theta).
 \end{aligned}$$

with  $0 \leq \psi \leq 2\pi$  ,  $0 \leq \theta \leq \pi$  ,  $0 \leq \phi \leq 2\pi$

Correspondingly, the rate of change with time (i.e., the angular speed) associated with each of the three Euler angles are defined as  $\dot{\theta}$ ,  $\dot{\phi}$ ,  $\dot{\psi}$ . The vectors associated with  $\dot{\theta}$ ,  $\dot{\phi}$  and  $\dot{\psi}$  can be written as

$$\begin{aligned}
 \dot{\phi} &= \dot{\phi}\mathbf{e}_3' \\
 \dot{\theta} &= \dot{\theta}\mathbf{e}_1'' \\
 \dot{\psi} &= \dot{\psi}\mathbf{e}_3''' = \dot{\psi}\mathbf{e}_3.
 \end{aligned}
 \tag{2.55}$$

Taking the projections of the unit bases vectors appearing in equation (2.55) on the rigid body bases vectors, we find

$$\begin{aligned}
 \dot{\phi} &= \dot{\phi}[\sin(\theta)\sin(\psi)\mathbf{e}_1 + \sin(\theta)\cos(\psi)\mathbf{e}_2 + \cos(\theta)\mathbf{e}_3] \\
 \dot{\theta} &= \dot{\theta}[\cos(\psi)\mathbf{e}_1 - \sin(\psi)\mathbf{e}_2] \\
 \dot{\psi} &= \dot{\psi}\mathbf{e}_3.
 \end{aligned}
 \tag{2.56}$$

Combining the three equations (2.56), we can express the components of the total angular velocity vector  $\omega$  as a function of  $\dot{\theta}$ ,  $\dot{\phi}$  and  $\dot{\psi}$

$$\boxed{
 \begin{aligned}
 \omega_1 &= \dot{\phi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi) \\
 \omega_2 &= \dot{\phi}\sin(\theta)\cos(\psi) - \dot{\theta}\sin(\psi) \\
 \omega_3 &= \dot{\phi}\cos(\theta) + \dot{\psi}
 \end{aligned}
 }
 \tag{2.57}$$

## 2.8 Euler's Equations of motion for a rigid body

To obtain the equations of motion of a rigid body, we can always start with the fundamental equation

$$\left(\frac{d\mathbf{L}}{dt}\right)_{fixed} = \mathbf{N}, \quad 2.58$$

where  $\mathbf{N}$  is the torque, and designation "fixed" is used since this equation can only be applied in an inertial frame of reference. We also know that

$$\left(\frac{d\mathbf{L}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L}, \quad 2.59$$

Or

$$\left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \mathbf{N} \quad 2.60$$

Using tensor notation we can write the components of equation (2.60) as

$$\dot{L}_i + \varepsilon_{ijk}\omega_j L_k = N_i. \quad 2.61$$

Now, if we choose the coordinate axes for the body frame of reference to coincide with the principal axes of the rigid body, then we have from equations (2.34)

$$\begin{aligned} L_1 &= I_1 \omega_1 \\ L_2 &= I_2 \omega_2 \\ L_3 &= I_3 \omega_3 \end{aligned} \quad 2.62$$

Since the principal moments of inertia  $I_1$ ,  $I_2$ , and  $I_3$  are constant with time, we can combine equations (2.61) and (2.62) to get

$$\begin{aligned} I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2 \omega_3 &= N_1 \\ I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3 \omega_1 &= N_2 \\ I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1 \omega_2 &= N_3. \end{aligned} \quad 2.63$$

Alternatively, we can combine these three equations into one using indices

$$\boxed{(I_i - I_j)\omega_i\omega_j - \sum_k \varepsilon_{ijk}(I_k\dot{\omega}_k - N_k) = 0} \quad 2.64$$

where *no summation* is implied on the  $i$  and  $j$  indices. Equations (2.64) are the so-called **Euler equations of motion** for a rigid body.

## 2.9 Principle of virtual work

### 2.9.1 Constraints and degrees of freedom

The number of degrees of freedom of a system is equal to the number of variables required to describe the state of the system. For instance:

- A particle constrained to move along the  $x$  axis has one degree of freedom, the position  $x$  on this axis.
- A particle constrained to the surface of the earth has two degrees of freedom, longitude and latitude.
- A wheel rotating on a fixed axle has one degree of freedom, the angle of rotation.
- A solid body in free space has six degrees of freedom: a particular atom in the body can move in three dimensions, which accounts for three degrees of freedom; another atom can move on a sphere with the first particle at its center for two additional degrees of freedom; any other atom can move in a circle about the line passing through the first two atoms. No other independent motion of the body is possible.
- $N$  atoms moving freely in three-dimensional space collectively have  $3N$  degrees of freedom.

#### i) Holonomic constraints

Suppose a mass is constrained to move in a circle of radius  $R$  in the  $x$ - $y$  plane. Without this constraint it could move freely over this plane. Such a constraint could be expressed by the equation for a circle,  $x^2 + y^2 = R^2$ . A better way to represent this constraint is

$$F(x, y) = x^2 + y^2 - R^2 = 0 \quad 2.65$$

As we shall see, this constraint may be useful when expressed in differential form:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 2x dx + 2y dy = 0 \quad 2.66$$

A constraint that can be represented by setting to zero a function of the variables representing the configuration of a system (e.g., the x and y locations of a mass moving in a plane) is called *holonomic*.

In a more complex system, there may be more than one constraint. For instance, if the mass in the above case is moving in x-y-z space, but in addition is constrained to remain on a horizontal surface at elevation a, the additional constraint

$$G(z) = z - a = 0 \quad 2.67$$

would apply.

## ii) Non-holonomic constraints

Sometimes a constraint on the motion of an object cannot be represented in holonomic form. For instance, imagine a car moving on a horizontal plane. The car would normally have three degrees of freedom, two translational represented by the car's position (x; y) and a rotational degree of freedom about the vertical axis, represented by an angle  $\phi$  counterclockwise from the x axis. (We neglect the possibility of the car overturning!) However, if the car is not skidding, it is constrained at a particular instant to move in the direction it is pointing, which can be represented by the differential relation

$$\sin\phi dx - \cos\phi dy = 0 \quad 2.68$$

This constraint cannot be integrated to the form  $F(x; y) = 0$ , because  $\phi$  can change as the car moves due to the driver turning the steering wheel. It thus depends on more than x and y. Thus, for infinitesimal motions, the car can only move along a particular line in the x-y plane as represented by equation (2.68), whereas with driver input, the car can reach any point in this plane with any rotational orientation, but only through finite motions. (This is what makes parallel parking so complicated!) Thus a car exhibits only one degree of freedom in infinitesimal motion, but three degrees of freedom in finite motion.

A constraint of this type is called non-holonomic. In general, non-holonomic constraints are more difficult to deal with than holonomic constraints.

## 2.9.2 Internal and external forces on a system

In mechanics, the definition of what constitutes a system is arbitrary; the choice is completely

up to us, and is based on what we are trying to accomplish. A system consisting of many atoms is in principle very complicated, because one must consider not only external forces acting on the system, but internal forces acting between each pair of atoms within the system. However, Newton's third law says that the force of atom A on atom B is equal and opposite to the force of atom B on atom A. Thus, the net force on the system due to atoms within the system acting on each other is zero. This result is related to the conservation of linear momentum in isolated systems. Thus, in considering the overall motion of a system, only external forces need be considered.

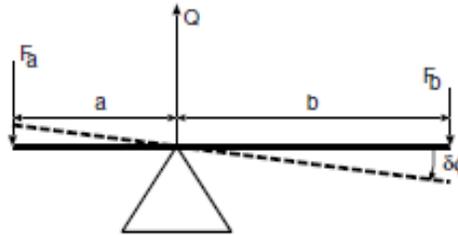


Figure 2.5 The balance beam.

## Principle of virtual work

The modern approach to a statics problem is to apply the two conditions that the total force and the total torque acting the system of interest each sum to zero. Sommerfeld invokes an older method of handling such problems called the principle of virtual work. This method has the advantage that forces of constraint, i.e., forces that keep the system from moving, may be neglected, thus potentially simplifying the analysis. Only forces, not torques, need to be considered, as the locations at which the forces are applied are used in the analysis. Since the action of a torque is really the action of a force applied at a particular location, the consideration of torques becomes less important.

### Uneven balance beam

A simple example is the uneven balance beam, illustrated in figure 2.5. The modern approach sets the total force and torque on the beam to zero:

$$Q - F_a - F_b = 0 \quad 2.69$$

$$aF_a - bF_b = 0 \quad 2.70$$

Solving the first equation tells us that the upward force of the pivot on the beam just balances the two downward forces at the opposite ends of the beam:

$$Q = F_a + F_b \quad 2.71$$

The second equation gives us the ratio of the two end forces:

$$\frac{F_a}{F_b} = \frac{b}{a}. \quad 2.72$$

These forces can also be obtained in terms of the beam dimensions and Q:

$$F_a = \frac{bQ}{a+b}$$

$$F_b = \frac{aQ}{a+b} \quad 2.73$$

The principle of virtual work can be used to obtain the same results. The idea is that if the beam tilts by a small angle  $\delta\phi$  in the clockwise direction, then the forces  $F_a$  and  $F_b$  respectively do work  $-F_a a \delta\phi$  and  $F_b b \delta\phi$ . This is because the left end of the beam moves a distance  $a \delta\phi$  in a direction opposite that of the force whereas the right end of the beam moves a distance  $b \delta\phi$  in the same direction as the force. The force of the pivot on the beam does no work, as the pivot is assumed not to move, so the total work done is

$$\delta W = (-F_a a + F_b b) \delta\phi \quad 2.74$$

This work increment is zero as a result of the torque balance expressed by equation (2.70).

Thus, assuming that the work increment is zero in a small displacement of the system is equivalent to the condition of zero net torque.

If instead of tilting the beam, the pivot and beam are lifted vertically by a small distance  $\delta z$ , the work done by the three forces in this case would be

$$\delta W = (Q - F_a - F_b) \delta z, \quad 2.75$$

which according to equation (2.69) is also zero. Setting  $\delta W = 0$  in this equation allows us to obtain the pivot support force Q. Thus, setting the work done in small displacements of the system to zero allows us to determine all of the relevant forces via computing the work done by these forces. The work is called virtual, because no real motion of the system is envisioned, with the corresponding complications resulting from generation of kinetic energy. In this example, there is little computational advantage in using the principle of virtual work over the method of zero forces and torques. However, its virtues become more evident as the problem becomes more complex.

## Problems

1. Use the principle of virtual work to determine the ratio  $M=m$  that results in static equilibrium in figure 2.6.

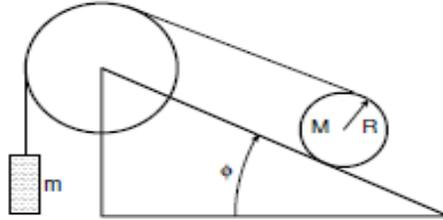


Figure 1.4.1: The pulley on the left is frictionless and the wheel on the right rolls up or down the ramp. The string wraps around the wheel on the ramp.

2. Use the principle of virtual work to determine the tensions in the clothes line from which a mass  $M$  is hung, as illustrated in figure 2.7. Note that you will need to apply the principle twice, once for small virtual displacements of the point  $P$  horizontally and again for small vertical displacements. The tensions  $T_a$  and  $T_b$  are kept constant in these displacements.

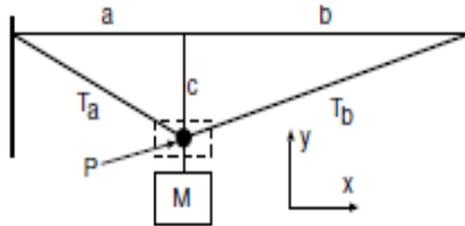


Figure 2.6: Mass hanging from clothes line.

# Chapter Three

## Theory of Small Oscillations

### 3.1 Introduction

When a conservative system is displaced slightly from its "stable" equilibrium position, it undergoes oscillation. The cause of oscillation is the restoring forces which are called into play. Restoring forces can do both positive and negative work. When the work done is positive, the restoring forces change the potential energy into kinetic energy and when the work done is negative, they change kinetic energy back into potential energy. For most mechanical systems, when the system is not too far from the equilibrium, the restoring force is proportional to the displacement ( $F = -kx$ ). Such oscillators are called linear oscillators. For linear oscillators, the oscillation frequencies are independent of the amplitude of oscillation. Oscillator motion can be damped in the presence of resistive forces. Resistive forces extract energy from the oscillator. For low velocities, the resistive forces are proportional to velocity. Oscillators, whether damped or undamped, can be driven by external agencies which continuously supply energy to the oscillator to keep it oscillating. Such oscillators are known as "forced" or "driven" oscillators. Driven oscillators can cause amplitude of oscillation to become very large when the driving frequency matches the natural frequency of oscillation. This is known as the phenomenon of resonance.

### 3.2 Equilibrium and potential energy

Consider a system with  $\{q_j\}$  as the generalised coordinates. Since the system is conservative, the forces acting on the system are derivable from a potential energy function  $V(q_1, q_2, \dots, q_N)$ . Lagrange defined equilibrium as a configuration in which all generalized forces vanish, i.e.

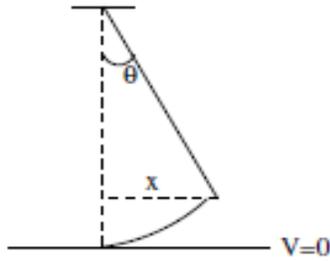
$$\frac{\partial V}{\partial q_j} = 0. \tag{3.1}$$

Clearly, in such a situation, the system will not change its configuration. However, even when  $Q_i = 0$ , the system may not be stable in the sense that if it is slightly disturbed from a position of equilibrium, it may not return to the position of equilibrium. If it does, such a configuration is called one of stable equilibrium - otherwise the equilibrium is unstable.

Example: Simple Pendulum

The potential energy is given by  $V(\theta) = mgl(1 - \cos\theta)$ , so that

$$F(\theta) = -\frac{\partial V}{\partial \theta} = -mgl\sin\theta = -mgx$$



The “generalised force” corresponding to  $\theta$  in this case is actually the restraining torque. Equilibrium occurs when the restoring torque is zero. There are two such positions,  $\theta = 0$  and  $\theta = \pi = 0$ . Let us look at the form of the Lagrangian near these two positions.

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

Near  $\theta = 0$ ,  $\cos\theta \approx 1 - \frac{1}{2}\theta^2$  so that

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}mgl\theta^2$$

so that the potential energy is  $V(\theta) = \frac{1}{2}mgl\theta^2$  and the corresponding generalised force is  $-mgl\theta$  which is of restoring nature. On the other hand, near the second position of equilibrium  $\theta = \pi$ ,  $\cos\theta = \cos(\pi + \delta\theta) \approx -\cos\delta\theta = -1 + \frac{1}{2}\delta\theta^2$ . In this situation,

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}mgl(\delta\theta)^2$$

the corresponding force is “anti-restoring”, making the equilibrium unstable. For one dimensional holonomic systems, equilibrium can be either stable or unstable (leaving out a trivial case of neutral equilibrium where the potential energy function is spatially flat) for which the potential energy has an extremum

$$\frac{\partial V}{\partial q_i} = 0$$

for every generalised coordinate  $q_i$ . Let the position of equilibrium be  $q_{i0}$ . If the position is one of stable equilibrium, the potential energy has to be minimum. This is because, the system being conservative, the total energy is constant. If we go away from the position of minimum potential energy, it leads to an increase in the potential energy and a consequent decrease in the kinetic energy. Thus the system returns back to the equilibrium position.

For stable equilibrium, we, therefore, have

$$\frac{\partial^2 V}{\partial q_i \partial q_j} > 0$$

The converse would be true for an unstable equilibrium.

### 3.3) Two coupled oscillators and normal coordinates

We consider the problem of two particles of similar mass  $M$  connected by a spring of Constant  $k_{12}$ , and further each particle connected to fixed points with springs of constant  $\kappa$ . The motion of particles is restricted to direction along the  $x$ -axis, so the system has two degrees of freedom  $x_1$  and  $x_2$  that give the displacement of the masses from their respective equilibrium position (see Figure 3-1).

The kinetic and potential energies of the system is given by

$$T = \frac{1}{2} M (\dot{x}_1^2 + \dot{x}_2^2), \quad 3.1$$

And

$$U = \frac{1}{2} \kappa (x_1^2 + x_2^2) + \frac{1}{2} \kappa_{12} (x_2 - x_1)^2, \quad 3.2$$

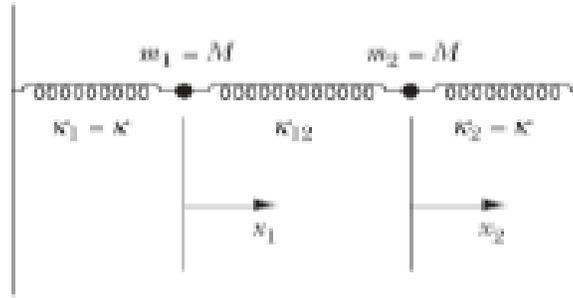
respectively. Using  $L = T - U$  for the Lagrangian, we can easily calculate the equations of motion to be

$$\begin{aligned} M\ddot{x}_1 + (\kappa + \kappa_{12})x_1 - \kappa_{12}x_2 &= 0 \\ M\ddot{x}_2 + (\kappa + \kappa_{12})x_2 - \kappa_{12}x_1 &= 0. \end{aligned} \quad 3.3$$

Because we expect oscillatory motions for the systems response, we attempt a solution of the form

$$x_k(t) = B_k e^{i\omega t}, \quad k=1,2 \quad 3.4$$

with  $B_k$  the complex amplitudes and  $\omega$  a frequency of oscillation. As we will see,  $B_k$  and  $\omega$  can take different values depending on the mode of oscillation.



**Figure 3-2** Two masses connected by a spring to each other and by other springs to fixed points.

Using equations (3.4) along with  $\ddot{x}_k = -\omega^2 x_k$  we can transform equations (2.3) to

$$\begin{aligned} -M\omega^2 B_1 e^{i\omega t} + (\kappa + \kappa_{12}) B_1 e^{i\omega t} - \kappa_{12} B_2 e^{i\omega t} &= 0 \\ -M\omega^2 B_2 e^{i\omega t} + (\kappa + \kappa_{12}) B_2 e^{i\omega t} - \kappa_{12} B_1 e^{i\omega t} &= 0. \end{aligned} \tag{3.5}$$

Regrouping terms and simplifying (by dropping the common exponential term), this equation can be written in a matrix form as

$$\begin{bmatrix} (\kappa + \kappa_{12} - \omega^2 M) & -\kappa_{12} \\ -\kappa_{12} & (\kappa + \kappa_{12} - \omega^2 M) \end{bmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0. \tag{3.6}$$

As usual, for this system of equations to have a non-trivial solution the determinant of the matrix on the left side of equation (3.6) must vanish. That is,

$$\begin{vmatrix} (\kappa + \kappa_{12} - \omega^2 M) & -\kappa_{12} \\ -\kappa_{12} & (\kappa + \kappa_{12} - \omega^2 M) \end{vmatrix} = 0. \tag{3.7}$$

The expansion of this determinant yields the so-called **characteristic equation** of the system

$$(\kappa + \kappa_{12} - \omega^2 M)^2 - \kappa_{12}^2 = 0, \tag{3.8}$$

or, if we take the square root,

$$\kappa + \kappa_{12} - \omega^2 M = \pm \kappa_{12}. \tag{3.9}$$

Solving for  $\omega$ , we find the **characteristic frequencies** (or **eigenfrequencies**, or **eigenvalues**) of the system. In this case, there are four frequencies:  $\pm\omega_2$  and  $\pm\omega_1$ , with

$$\omega_1 = \sqrt{\frac{k+2k_{12}}{M}} \quad , \quad \omega_2 = \sqrt{\frac{k}{M}} \quad 3.10$$

If we set  $\omega = \pm\omega_1$  in equations (3.4) and insert it in equation (3.6), we find that  $B_1 = -B_2$ . Similarly, if we set  $\omega = \pm\omega_2$  in equations (3.4) and insert it in equation (3.6), we find that  $B_1 = B_2$ . If we associate one amplitude constant for each eigenfrequency, i.e.,  $B_i^\pm$  for  $\pm\omega_i$ , we can write the complete solution to the system of equations (3.6) as

$$\begin{aligned} x_1(t) &= B_1^+ e^{i\omega_1 t} + B_1^- e^{-i\omega_1 t} + B_2^+ e^{i\omega_2 t} + B_2^- e^{-i\omega_2 t} \\ x_2(t) &= -B_1^+ e^{i\omega_1 t} - B_1^- e^{-i\omega_1 t} + B_2^+ e^{i\omega_2 t} + B_2^- e^{-i\omega_2 t} . \end{aligned} \quad 3.11$$

We see from this last set of equations that the position of the particles are both functions of the two frequencies  $\omega_1$  and  $\omega_2$ . The two degrees of freedom  $x_1(t)$  and  $x_2(t)$  are not, therefore, independent of each other. We would like to find out if there exists a transformation that will lead to a new set of coordinates that would be *decoupled* along the different modes of oscillation. Inspection of equations (3.11) suggests an obvious candidate. That is, if we introduce the following new coordinates

$$\begin{aligned} \eta_1 &= x_1 - x_2 \\ \eta_2 &= x_1 + x_2 \end{aligned} \quad 3.12$$

Or

$$\begin{aligned} x_1 &= \frac{1}{2}(\eta_1 + \eta_2) \\ x_2 &= \frac{1}{2}(\eta_2 - \eta_1) . \end{aligned} \quad 3.13$$

and we substitute this last set of equations into equations (10.5) we find

$$\begin{aligned} M(\ddot{\eta}_1 + \ddot{\eta}_2) + (\kappa + 2\kappa_{12})\eta_1 + \kappa\eta_2 &= 0 \\ M(\ddot{\eta}_1 - \ddot{\eta}_2) + (\kappa + 2\kappa_{12})\eta_1 - \kappa\eta_2 &= 0 . \end{aligned} \quad 3.14$$

By adding and subtracting the last two equations, we easily solve this system to obtain

$$\begin{aligned} M\ddot{\eta}_1 + (\kappa + 2\kappa_{12})\eta_1 &= 0 \\ M\ddot{\eta}_2 + \kappa\eta_2 &= 0 . \end{aligned} \quad 3.15$$

We can proceed as was done for  $x_1(t)$  and  $x_2(t)$  to find that

$$\eta_1(t) = C_1^+ e^{i\omega_1 t} + C_1^- e^{-i\omega_1 t}$$

$$\eta_2(t) = C_2^+ e^{i\omega_2 t} + C_2^- e^{-i\omega_2 t},$$

3.16

where the frequencies  $\omega_1$  and  $\omega_2$  are defined by equations (3.10). We see from equations (3.15) and (3.16) that  $\eta_1(t)$  and  $\eta_2(t)$  are *decoupled* and independent.

The constants  $C_i^\pm$  are to be determined from the initial conditions. For example, if we have  $x_1(0) = -x_2(0)$  and  $\dot{x}_1(0) = -\dot{x}_2(0)$ , then  $\eta_2(0) = \dot{\eta}_2(0) = 0$  and  $C_2^+ = C_2^- = 0$ ; that is,  $\eta_2(t) = 0$  at all times. We find that in this case the particles oscillate out of phase with each other at frequency  $\omega_1$ ; this is the *anti-symmetrical mode of oscillation*. Conversely, if we set  $x_1(0) = x_2(0)$  and  $\dot{x}_1(0) = \dot{x}_2(0)$ , we find that  $\eta_1(t) = 0$  at all times. The particles then oscillate in phase with each other at frequency  $\omega_2$ ; this is the *Symmetrical mode of oscillation*.

### 3.4 Theory of small oscillations

Any mechanical system can perform oscillations in the neighbourhood of a position of stable equilibrium. These oscillations are an extremely important feature of the system whether they are intended to occur (as in a pendulum clock), or whether they are undesirable (as in a suspension bridge!). Analogous oscillations occur in continuum mechanics and in quantum mechanics. Here we present the theory of such oscillations for **conservative systems** under the assumption that the amplitude of the oscillations is small enough so that the **linear approximation** is adequate. This treatment is restricted to systems with two degrees of freedom and does not make use of Lagrange's equations. Although the material in the present chapter is self-contained, it is helpful to have solved a few simple normal mode problems before.

The best way to develop the theory of small oscillations is to use Lagrange's equations. We will show that it is possible to approximate the expressions for  $T$  and  $V$  from the start so that the linearized equations of motion are obtained immediately. The theory is presented in an elegant matrix form which enables us to make use of concepts from linear algebra, such as eigenvalues and eigenvectors. We prove that fundamental result that a system with  $n$  degrees of freedom always has  $n$  harmonic motions known as **normal modes**, whose frequencies are generally different. These **normal frequencies** are the most important characteristic of the oscillating system. One important application of the theory is to the internal vibrations of molecules. Although this should

really be treated by quantum mechanics, the classical model is extremely valuable in making qualitative predictions and classifying the vibrational modes of the molecule.

### 3.4.1 THE APPROXIMATE FORMS OF $T$ AND $V$

Now that we know small oscillations can take place about any minimum point of  $V$ , we can go on to find approximate equations that govern such motions. The obvious (but not the best!) way of doing this is as follows: Take the example of the double pendulum. In this case,  $T$  and  $V$  are given(see the following figure) by

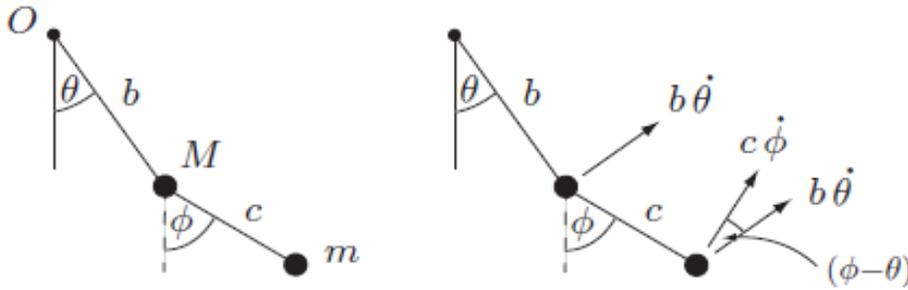


FIGURE 3.3 The double pendulum. **Left:** The generalised coordinates  $\theta, \phi$ . **Right:** The velocity diagram.

$$T = \frac{1}{2}M(b\dot{\theta})^2 + \frac{1}{2}m \left( (b\dot{\theta})^2 + (c\dot{\phi})^2 + 2(b\dot{\theta})(c\dot{\phi}) \cos(\theta - \phi) \right),$$

$$V = (M + m)gb(1 - \cos \theta) + mgc(1 - \cos \phi).$$

3.16 and 3.17

If these expressions are substituted into the Lagrange's equations, we obtain (after some simplification) the **exact equations of motion**

$$\begin{aligned} (M + m)b\ddot{\theta} + mc \cos(\theta - \phi)\ddot{\phi} + mc \sin(\theta - \phi)\dot{\phi}^2 + (M + m)g \sin \theta &= 0, \\ b \cos(\theta - \phi)\ddot{\theta} + c\ddot{\phi} - b \sin(\theta - \phi)\dot{\theta}^2 + g \sin \phi &= 0. \end{aligned}$$

This formidable pair of *coupled, second order, non-linear ODEs* govern the large oscillations of the double pendulum. However, for small oscillations about  $\theta = \phi = 0$ , these equations can be approximated by neglecting everything except linear terms in  $\theta, \phi$  and their time derivatives. On carrying out this approximation, the equations simplify dramatically to give

$$\begin{aligned} (M + m)b\ddot{\theta} + mc\ddot{\phi} + (M + m)g\theta &= 0, \\ b\ddot{\theta} + c\ddot{\phi} + g\phi &= 0. \end{aligned}$$

3.18 and 3.19

These are the **linearised equations** governing small oscillations of the double pendulum about the downward vertical. They are a pair of *coupled, second order, linear ODEs with constant coefficients*. An explicit solution is therefore possible. While the above method of finding the linearised equations of motion is perfectly correct, it is wasteful of effort and is also unsuitable

when presenting the general theory. What we did was to obtain the *exact* expressions for  $T$  and  $V$ , derive the *exact* equations of motion, and then linearise. In the linearisation process, many of the terms we took pains to find were discarded. It makes far better sense to approximate the expressions for  $T$  and  $V$  from the start so that, when these approximations are used in Lagrange's equations, the linearised equations of motion are produced immediately. The saving in labour is considerable and this is also a nice way to present the general theory.

Consider the double pendulum for example. The exact expression for  $V$  is given by equation (3.17) and when  $\theta, \phi$  are small, this is given approximately by

$$V = \frac{1}{2}(M + m)gb\theta^2 + \frac{1}{2}mgc\phi^2 + \dots,$$

where the neglected terms have power four or higher. Similarly, when  $\theta, \phi$  and their time derivatives are small,  $T$  is given approximately by

$$\begin{aligned} T &= \frac{1}{2}M(b\dot{\theta})^2 + \frac{1}{2}m\left((b\dot{\theta})^2 + (c\dot{\phi})^2 + 2(b\dot{\theta})(c\dot{\phi})(1 + \dots)\right), \\ &= \frac{1}{2}(M + m)b^2\dot{\theta}^2 + mbc\dot{\theta}\dot{\phi} + \frac{1}{2}mc^2\dot{\phi}^2 + \dots, \end{aligned}$$

where the neglected terms have power four (or higher) in small quantities. If these approximate forms for  $T$  and  $V$  are now substituted into Lagrange's equations, the linearized equations of motion (3.18), (3.19) are obtained immediately. This is clearly superior to our original method.

### The general approximate form of $V$

In the general case, suppose that the potential energy  $V(\mathbf{q})$  of the system  $S$  has a minimum at  $\mathbf{q} = \mathbf{0}$  and that  $V(\mathbf{0}) = 0$ . (If the minimum point of  $V$  is not at  $\mathbf{q} = \mathbf{0}$ , it can always be made so by a simple change of coordinates.) Then, for  $\mathbf{q}$  near  $\mathbf{0}$ ,  $V(\mathbf{q})$  can be expanded as an ( $n$ -dimensional) Taylor series in the variables  $q_1, q_2, \dots, q_n$ . For the special case when  $S$  has two degrees of freedom, this series has the form

$$\begin{aligned} V(q_1, q_2) &= V(0, 0) + \left(\frac{\partial V}{\partial q_1} q_1 + \frac{\partial V}{\partial q_2} q_2\right) \\ &+ \left(\frac{\partial^2 V}{\partial q_1^2} q_1^2 + 2\frac{\partial^2 V}{\partial q_1 \partial q_2} q_1 q_2 + \frac{\partial^2 V}{\partial q_2^2} q_2^2\right) + \dots, \end{aligned}$$

where all partial derivatives of  $V$  are evaluated at the expansion point  $q_1 = q_2 = 0$ . Now  $V$  has been selected so that  $V(0, 0) = 0$ . Also, since  $(0, 0)$  is a stationary point of  $V(q_1, q_2)$ , it follows that  $\partial V/\partial q_1 = \partial V/\partial q_2 = 0$  there. Thus the constant and linear terms are absent from the Taylor expansion of  $V$ . It follows that  $V$  can be approximated by

$$V^{\text{app}}(q_1, q_2) = v_{11}q_1^2 + 2v_{12}q_1q_2 + v_{22}q_2^2,$$

where  $v_{11}$ ,  $v_{12}$ ,  $v_{22}$  are constants given by

$$v_{11} = \frac{\partial^2 V}{\partial q_1^2}(\mathbf{0}, \mathbf{0}) \quad v_{12} = \frac{\partial^2 V}{\partial q_1 \partial q_2}(\mathbf{0}, \mathbf{0}) \quad v_{22} = \frac{\partial^2 V}{\partial q_2^2}(\mathbf{0}, \mathbf{0})$$

and the neglected terms have power three (or higher) in the small quantities  $q_1$ ,  $q_2$ . The corresponding approximation to  $V(\mathbf{q})$  in the case when  $S$  has  $n$ -degrees of freedom is

$$V^{\text{app}}(\mathbf{q}) = \sum_{j=1}^n \sum_{k=1}^n v_{jk} q_j q_k$$

3.20

where the  $\{v_{jk}\}$  are constants given by

$$v_{jk} = v_{kj} = \left. \frac{\partial^2 V}{\partial q_j \partial q_k} \right|_{\mathbf{q}=\mathbf{0}}$$

and the neglected terms have power three (or higher) in the small quantities  $q_1, q_2, \dots, q_n$ . This is the general form of the **approximate potential energy**  $V^{\text{app}}(\mathbf{q})$ . It is a homogeneous quadratic form in the variables  $q_1, q_2, \dots, q_n$ . In the theory that follows, we will always assume that  $\mathbf{q} = \mathbf{0}$  is also a minimum point of the *approximate* potential energy  $V^{\text{app}}(\mathbf{q})$ .\* This condition is equivalent to requiring that the quadratic form (15.8) should be **positive definite**. This simply means that it takes positive values except when  $\mathbf{q} = \mathbf{0}$ .

### The general approximate form of $T$

For any standard mechanical system with generalised coordinates  $\mathbf{q}$ , the kinetic energy  $T$  has the form

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{j=1}^n \sum_{k=1}^n t_{jk}(\mathbf{q}) \dot{q}_j \dot{q}_k,$$

a quadratic form in the variables  $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$  with coefficients that depend on  $\mathbf{q}$ . If we expand each of these coefficients as a Taylor series about  $\mathbf{q} = \mathbf{0}$ , the constant term is simply  $t_{jk}(\mathbf{0})$  and

$$T = \sum_{j=1}^n \sum_{k=1}^n t_{jk}(\mathbf{0}) \dot{q}_j \dot{q}_k + \dots$$

It follows that  $T$  can be approximated by

$$T^{\text{app}} = \sum_{j=1}^n \sum_{k=1}^n t_{jk} \dot{q}_j \dot{q}_k$$

3.21

where the constants  $\{t_{jk}\}$  are what we previously called  $\{t_{jk}(\mathbf{0})\}$ , and the neglected terms have power three (or higher) in the small quantities  $q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$ . This is the general form of the **approximate kinetic energy**  $V^{\text{app}}(\mathbf{q})$ . It is a homogeneous quadratic form in the variables  $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$ . Since  $T(\mathbf{q}, \dot{\mathbf{q}}) > 0$  except when  $\dot{\mathbf{q}} = \mathbf{0}$ , it follows that the quadratic form (15.9) must also be **positive definite**.

### 3.5 Small oscillations in normal coordinates

The preceding theory applies for any choice of the generalised coordinates  $\{q_j\}$ . Changing the generalised coordinates will change the  $V$ - and  $T$ -matrices, but the normal frequencies and the physical forms of the normal modes will be the same. This suggests that it might be possible to make a clever choice of coordinates so that the  $V$ - and  $T$ -matrices have a simple form leading to a much simplified theory. In particular, it would be very advantageous if  $\mathbf{T}$  and  $\mathbf{V}$  had **diagonal** form.

**Normal coordinates;** *A set of generalised coordinates in terms of which the  $T$ - and  $V$ -matrices have diagonal form are called **normal coordinates***

Actually, every oscillating system has normal coordinates, as we will now show. Let  $\mathbf{q}$  be the original choice of coordinates with corresponding matrices  $\mathbf{V}$  and  $\mathbf{T}$ . Then

$$T^{\text{app}} = \dot{\mathbf{q}}' \cdot \mathbf{T} \cdot \dot{\mathbf{q}}, \quad V^{\text{app}} = \mathbf{q}' \cdot \mathbf{V} \cdot \mathbf{q} \tag{3.22}$$

Now consider a change of coordinates from  $\mathbf{q}$  to  $\boldsymbol{\eta}$  defined by the linear transformation

$$\mathbf{q} = \mathbf{P} \cdot \boldsymbol{\eta} \iff \boldsymbol{\eta} = \mathbf{P}^{-1} \cdot \mathbf{q} \tag{3.23}$$

where  $\mathbf{P}$  can be any non-singular matrix. On substituting the transformation (15.28) into the expressions (3.22), we obtain

$$T^{\text{app}} = (\mathbf{P} \cdot \dot{\eta})' \cdot \mathbf{T} \cdot (\mathbf{P} \cdot \dot{\eta}) = \dot{\eta}' \cdot (\mathbf{P}' \cdot \mathbf{T} \cdot \mathbf{P}) \cdot \dot{\eta},$$

$$V^{\text{app}} = (\mathbf{P} \cdot \eta)' \cdot \mathbf{V} \cdot (\mathbf{P} \cdot \eta) = \eta' \cdot (\mathbf{P}' \cdot \mathbf{V} \cdot \mathbf{P}) \cdot \eta,$$

from which we see that this transformation of coordinates causes  $\mathbf{V}$  and  $\mathbf{T}$  to be transformed as

$$\mathbf{T} \rightarrow \mathbf{P}' \cdot \mathbf{T} \cdot \mathbf{P}, \quad \mathbf{V} \rightarrow \mathbf{P}' \cdot \mathbf{V} \cdot \mathbf{P}. \quad 3.24$$

Can we now choose the transformation matrix  $\mathbf{P}$  so that the new  $T$ - and  $V$ -matrices are diagonal?

Let  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$  be the amplitude vectors of the normal modes when they are expressed in terms of the coordinates  $q$  and let  $\omega_1, \omega_2, \dots, \omega_n$  be the corresponding normal frequencies. We will suppose that these amplitude vectors have been chosen so that they satisfy the orthonormality relations, that is

$$\mathbf{a}'_j \cdot \mathbf{T} \cdot \mathbf{a}_k = \begin{cases} 0 & (j \neq k), \\ 1 & (j = k). \end{cases} \quad 3.25$$

Now consider the matrix  $\mathbf{P}$  whose columns are the amplitude vectors  $\{\mathbf{a}_j\}$ , that is,

$$\mathbf{P} = (\mathbf{a}_1 | \mathbf{a}_2 | \dots | \mathbf{a}_n). \quad 3.26$$

Since the amplitude vectors are known to be linearly independent,  $\mathbf{P}$  has linearly independent columns and is therefore a non-singular matrix. Let us now try this  $\mathbf{P}$  as the transformation matrix. Then

$$\mathbf{P}' \cdot \mathbf{T} \cdot \mathbf{P} = \begin{pmatrix} \mathbf{a}'_1 \\ \mathbf{a}'_2 \\ \vdots \\ \mathbf{a}'_n \end{pmatrix} \cdot \mathbf{T} \cdot (\mathbf{a}_1 | \mathbf{a}_2 | \dots | \mathbf{a}_n).$$

The  $jk$ -th element of this matrix is given by

$$\mathbf{a}'_j \cdot \mathbf{T} \cdot \mathbf{a}_k = \begin{cases} 0 & (j \neq k) \\ 1 & (j = k) \end{cases}$$

by the orthonormality relations. Hence, with this choice of  $\mathbf{P}$ ,

$$\mathbf{P}' \cdot \mathbf{T} \cdot \mathbf{P} = \mathbf{1}$$

where  $\mathbf{1}$  is the **identity matrix**. In the same way,

$$\mathbf{P}' \cdot \mathbf{V} \cdot \mathbf{P} = \begin{pmatrix} \mathbf{a}'_1 \\ \mathbf{a}'_2 \\ \vdots \\ \mathbf{a}'_n \end{pmatrix} \cdot \mathbf{V} \cdot (\mathbf{a}_1 | \mathbf{a}_2 | \cdots | \mathbf{a}_n).$$

The  $jk$ -th element of this matrix is given by

$$\begin{aligned} \mathbf{a}'_j \cdot \mathbf{V} \cdot \mathbf{a}_k &= \mathbf{a}'_j \cdot (\mathbf{V} \cdot \mathbf{a}_k) = \mathbf{a}'_j \cdot (\omega_k^2 \mathbf{T} \cdot \mathbf{a}_k) = \omega_k^2 (\mathbf{a}'_j \cdot \mathbf{T} \cdot \mathbf{a}_k) \\ &= \begin{cases} 0 & (j \neq k), \\ \omega_j^2 & (j = k). \end{cases} \end{aligned}$$

Hence, with this choice of  $\mathbf{P}$ ,

$$\mathbf{P}' \cdot \mathbf{V} \cdot \mathbf{P} = \omega^2$$

where  $\mathbf{X}$  is the diagonal matrix whose diagonal elements are the normal frequencies, that is,

$$\mathbf{\Omega} = \begin{pmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n \end{pmatrix}.$$

We have thus succeeded in reducing both  $\mathbf{V}$  and  $\mathbf{T}$  to diagonal form. Hence the coordinates  $\{\eta_j\}$  defined by (15.28) with  $\mathbf{P} = (\mathbf{a}_1 | \mathbf{a}_2 | \cdots | \mathbf{a}_n)$  are a set of **normal coordinates**. They are given explicitly by

$$\boldsymbol{\eta} = \mathbf{P}^{-1} \cdot \mathbf{q} = (\mathbf{P}' \cdot \mathbf{T}) \cdot \mathbf{q},$$

on using the formula  $\mathbf{P}' \cdot \mathbf{T} \cdot \mathbf{P} = \mathbf{1}$ . This can also be written in the semi-expanded form

$$\eta_j = (\mathbf{a}'_j \cdot \mathbf{T}) \cdot \mathbf{q} \quad (1 \leq j \leq n). \quad 3.27$$

From this last formula, we can see that, if the amplitude vectors  $\{\mathbf{a}_j\}$  are *not* normalised, then the coordinates  $\{\eta_j\}$  are simply multiplied by constants. They are therefore *still normal coordinates*. The corresponding  $V$ - and  $T$ -matrices are still diagonal, but  $\mathbf{T}$  is no longer reduced to the identity.

Our results are summarised as follows:

### Finding normal coordinates

Let  $a_1, a_2, \dots, a_n$  be the amplitude vectors of the normal modes when expressed in terms of the coordinates  $\{q_j\}$ . Then the coordinates  $\{\eta_j\}$  defined by

$$\eta_j = (\mathbf{a}'_j \cdot \mathbf{T}) \cdot \mathbf{q} \quad (1 \leq j \leq n)$$

are a set of **normal coordinates**, as are any constant multiples of them. (The amplitude vectors only need to be normalised if it is required to reduce the matrix  $\mathbf{T}$  to the identity.)

When expressed in terms of normal coordinates, the **small oscillation equations** become

$$\ddot{\boldsymbol{\eta}} + \boldsymbol{\Omega}^2 \cdot \boldsymbol{\eta} = \mathbf{0}.$$

In expanded form, this is

$$\ddot{\eta}_j + \omega_j^2 \eta_j = 0 \quad (1 \leq j \leq n),$$

a system of  $n$  *uncoupled* SHM equations. The solution  $\eta_1 = C_1 \cos(\omega_1 t - \gamma_1)$ ,  $\eta_2 = \eta_3 = \dots = \eta_n = 0$  is the first normal mode, the solution  $\eta_2 = C_2 \cos(\omega_2 t - \gamma_2)$ ,  $\eta_1 = \eta_3 = \dots = \eta_n = 0$  is the second normal mode, and so on.

*Note.* Using normal coordinates is *not* a practical way of solving normal mode problems. Indeed the problem has to be solved before the normal coordinates can be found! Normal coordinates are important because they simplify further developments of the general theory.

### 3.6 General problem of coupled oscillations

We now consider a general problem of a conservative system with  $n$  degrees of freedom and a corresponding set of generalized coordinates  $q_k$ , with  $k = 1, 2, \dots, n$ . We suppose that there exists a configuration where the system is at equilibrium, with the generalized coordinates having values  $q_{k0}$ . We expand the potential energy  $U$  of the system with a Taylor series around this configuration of equilibrium

$$U(q_1, q_2, \dots, q_n) = U_0 + \left. \frac{\partial U}{\partial q_k} \right|_0 (q_k - q_{k0}) + \frac{1}{2} \left. \frac{\partial^2 U}{\partial q_j \partial q_k} \right|_0 (q_j - q_{j0})(q_k - q_{k0}) + \dots \quad 3.28$$

where we neglected any terms of higher than second order, and summation over repeated indices is implied. We can arbitrarily set the first term on the right hand side  $U_0$  (the potential energy at equilibrium) to zero since the potential energy can only be defined up to a constant; therefore,  $U_0 = 0$ . Moreover, the existence of an equilibrium configuration implies that the first derivative of the potential energy relative to each generalized coordinate evaluated *at the corresponding positions of equilibrium* (i.e., at  $q_k = q_{k0}$ ) is also zero. That is,

$$\left. \frac{\partial U}{\partial q_k} \right|_0 = 0, \quad 3.29$$

and  $U$  is at a minimum when  $q_k = q_{k0}$ . Finally, if we further simplify the notation by setting  $q_{k0} = 0$ , we can approximate the potential energy by

$$U = \frac{1}{2} A_{jk} q_j q_k \quad 3.30$$

With

$$A_{jk} = \left. \frac{\partial^2 U}{\partial q_j \partial q_k} \right|_0. \quad 3.31$$

It is obvious from the form of equation (10.20) that  $A_{jk}$  is symmetric (i.e.,  $A_{jk} = A_{kj}$ ).

If the potential energy is a quadratic function of the generalized coordinates, as is evident from equation (3.30), we can use already derived results for the kinetic energy of the system when the equations connecting the generalized coordinates and the Cartesian coordinates do not explicitly involve time. That is, if

$$x_{\alpha,i} = x_{\alpha,i}(q) \quad \text{or} \quad q_k = q_k(x_{\alpha,i}) \quad 3.32$$

then the kinetic energy is given by

$$T = \frac{1}{2} m_{jk} \dot{q}_j \dot{q}_k. \quad 3.33$$

With

$$m_{jk} = \sum_{\alpha} m_{\alpha} \frac{\partial x_{\alpha,j}}{\partial q_j} \frac{\partial x_{\alpha,k}}{\partial q_k}.$$

As was the case for  $A_{jk}$ ,  $m_{jk}$  is symmetric (i.e.,  $m_{jk} = m_{kj}$ ). Just as we did for the potential energy, we can expand the expression for the quantities  $m_{jk}$  about the position of equilibrium; we then get

$$m_{jk}(q_1, \dots, q_n) = m_{jk}(q_{l0}) + \left. \frac{\partial m_{jk}}{\partial q_l} \right|_0 q_l + \dots \quad 3.34$$

However, in order to be consistent in the accuracy kept for both the potential and kinetic energies, we only keep the first term on the right hand side of equation (3.34). This way, both expressions are valid to the second order (in velocities for the kinetic energy, and in displacement for the potential energy). We then write

$$\boxed{\begin{aligned} T &= \frac{1}{2} m_{jk} \dot{q}_j \dot{q}_k \\ U &= \frac{1}{2} A_{jk} q_j q_k \end{aligned}} \quad 3.35$$

with the understanding that  $m_{jk}$  consists only of the first term in the expansion on the right side of equation (3.34). We are now interested in solving for the equations of motion of the system, using the Lagrangian formalism. That is,

$$\frac{\partial \mathcal{L}}{\partial q_k} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) = 0, \quad 3.36$$

which, in this case simplifies to

$$\frac{\partial U}{\partial q_k} + \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_k} \right) = 0. \quad 3.37$$

Using equations (3.35), the equations of motion are reduced to the following

$$\boxed{A_{jk} q_j + m_{jk} \ddot{q}_j = 0} \quad 3.38$$

Equations (3.38) represent a set of coupled second-order differential equations with constant coefficients. Since we expect oscillatory motions, we propose a solution of the form

$$q_j(t) = a_j e^{i(\omega t - \delta)},$$

where the amplitudes  $a_j$  are real. Inserting this equation in equations (3.38), we find for the equations of motion

$$(A_{jk} - \omega^2 m_{jk})a_j = 0 \quad 3.40$$

Alternatively, the system of equations (3.40) can be written in a matrix form

$$(A - \omega^2 m) \cdot a = 0, \quad 3.41$$

where the matrices  $\mathbf{A}$  and  $\mathbf{m}$  are composed of the elements  $A_{jk}$  and  $m_{jk}$ , respectively (remember that  $\mathbf{A}$  and  $\mathbf{m}$  are symmetric). In order to get a non-trivial solution to this equation, the determinant of the quantity in parentheses must vanish

$$|\mathbf{A} - \omega^2 \mathbf{m}| = 0. \quad 3.42$$

This determinant is called the **characteristic** or **secular equation** and is an equation of degree  $n$  in  $\omega^2$ . The corresponding  $n$  roots  $\omega_r^2$  are the **characteristic frequencies** or **eigenfrequencies** of the system. The **eigenvector** associated with a given root  $\omega_r$  can be evaluated by inserting it back in equations (3.40) to determine the ratios  $a_1: a_2: \dots : a_n$ . If we represent by  $a_{jr}$  the  $j$ th component of the  $r$ th eigenvector, we can write the generalized coordinate  $q_j$  as a linear combination of the solutions for each root

$$q_j(t) = \sum_r a_{jr} e^{i(\omega_r t - \delta_r)}. \quad 3.43$$

It is, however, understood that the actual solution must be real (in a mathematical sense) and we must, therefore, take real part of equation (3.43). That is,

$$q_j(t) = \sum_r a_{jr} \cos(\omega_r t - \delta_r). \quad 3.44$$

### 3.9 Sympathetic vibrations and beats

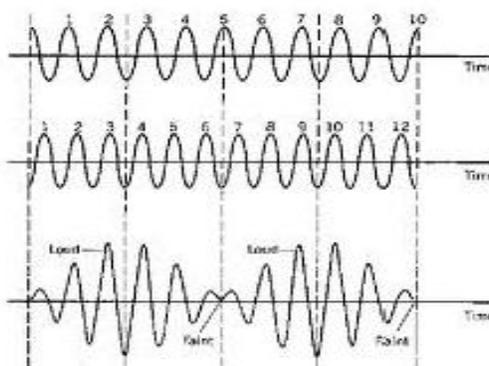
Sympathetic vibrations are when an object begins to oscillate or vibrate because of an external vibration which matches the resonance frequency of the object. In other words, when the resonance of one object causes another object to resonate, that is sympathetic resonance or sympathetic vibrations.

Or, a Sympathetic vibrations is an induced resonant vibration. All objects have some natural frequency at which they vibrate. External vibrations at this frequency will be more readily absorbed, and the object will begin to vibrate “in sympathy” with the inducing energy. An object that absorbs

sufficient vibrational energy will shake apart. The term “sympathetic vibration” has generally given way to the more accurate and general term “resonance”.

## Beats

When two waves with different frequencies interfere, the resulting wave will oscillate between high and low amplitudes with a given frequency. This oscillation is referred to as **Beats**. In the diagram below, two sound waves are added together. One wave has a frequency of 10 Hz, while the other has a frequency of 12 Hz. Their sum (the resulting wave when they interfere) is shown in the bottom graph. Notice that the resulting wave rises and falls in a sinusoidal way; at some points, the interference is constructive; at other times, it is destructive. The net result is that the sound wave we hear (the sum of the two waves) will oscillate between loud and faint. The frequency of the Beat is equal to the difference in the frequency of the two initial waves. In this case, the sound wave we hear (the combination of the initial waves) will have a Beat Frequency of 2 Hz [12 Hz – 10 Hz].



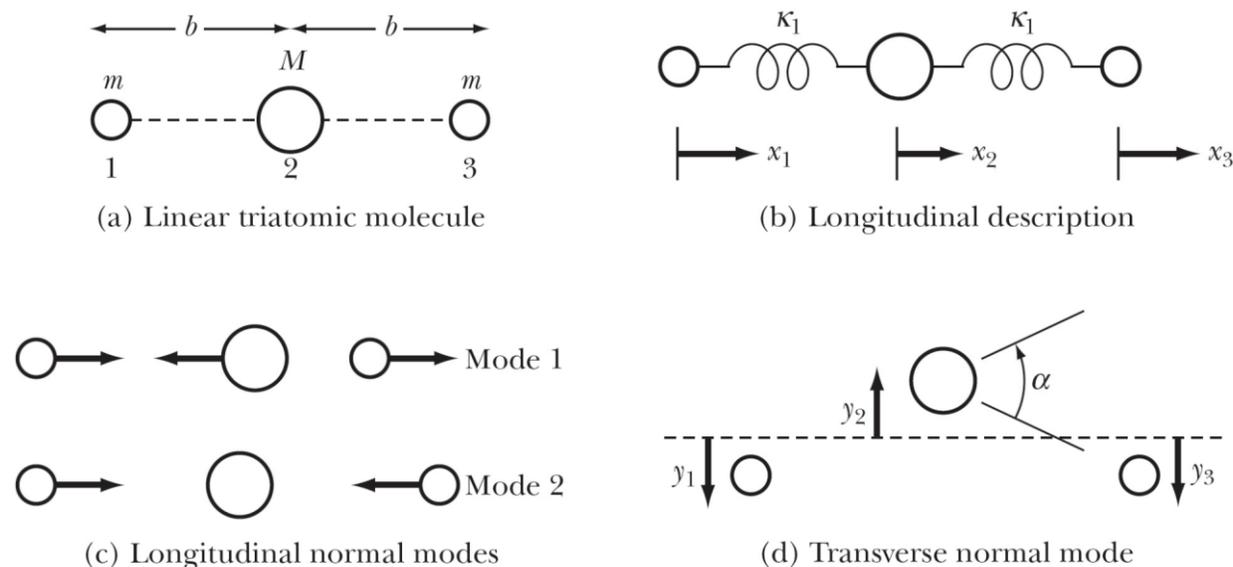
Beats are used to tune musical instruments. When a piano tuner is tuning a piano, he or she will use a tuning fork (that puts out a pure sound frequency) and listen for beats between it and the vibrating string. When no beats occur, the piano is correctly tuned. The same is true of when a symphonic band tunes their instruments.

### 3.10 Molecular vibrations

Our theory of coupled oscillations has many important applications in molecular physics. Each atom in a molecule has 3 degrees of freedom, and if we are looking at a molecule with  $n$  atoms, we have a total of  $3n$  degrees of freedom. Three different types of motion can be carried freedom), and vibration ( $3n - 6$  degrees of freedom).

Consider a linear molecule (the equilibrium positions of all atoms are located along a straight line) with  $n$  atoms. The number of degrees of freedom associated with Vibrational motion is  $3n - 5$  since there are only 2 rotational degrees of freedom. The vibrations in a linear molecule can be longitudinal vibrations (there are  $n - 1$  degrees of freedom associated with this type of vibrations) and transverse vibrations (there are  $(3n - 5) - (n - 1) = (2n - 4)$  degrees of freedom associated with this type of vibration). If the vibrations are planar vibrations (the motion of all atoms is carried out in a single plane) we can specify any transverse vibration in terms of vibrations in two mutually perpendicular planes. The characteristic frequencies in each of these planes will be the same (symmetry) and the number of characteristic frequencies will thus be equal to  $n - 2$ .

To illustrate molecular vibrations let us consider the dynamics of a triatomic molecule (see the Figure ).



Figure; Vibrational motion of a linear triatomic molecule.

In order to determine the vibrational modes of this system we look at the longitudinal and transversal modes separately. Since we are not interested in pure translational motion we can

require that the center of mass of the system is at rest. This means that we do not have 3 independent position coordinates, but only 2. For example, we can eliminate the position of the heavy atom:

$$x_2 = -\frac{m}{M}(x_1 + x_3)$$

In order to determine the normal modes, we will follow the following procedure.

1. **Choose generalized coordinates.** The proper generalized coordinates in this problem are the displacements  $x_1$  and  $x_2$ . The kinetic and the potential energy of the system can be easily expressed in terms of these displacements. The kinetic energy of the system is thus just equal to the kinetic energy of the three atoms, and thus equal to

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 = \\ &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\left(\frac{m}{M}\right)^2(\dot{x}_1^2 + 2\dot{x}_1\dot{x}_3 + \dot{x}_3^2) + \frac{1}{2}m\dot{x}_3^2 \\ &= \frac{1}{2}\left(m + \frac{m^2}{M}\right)\dot{x}_1^2 + \left(\frac{m^2}{M}\right)\dot{x}_1\dot{x}_3 + \frac{1}{2}\left(m + \frac{m^2}{M}\right)\dot{x}_3^2 \end{aligned}$$

The potential energy of the system is the sum of the potential energy associated with the compression of the springs. The total potential energy is thus equal to

$$\begin{aligned} U &= \frac{1}{2}\kappa_1(x_2 - x_1)^2 + \frac{1}{2}\kappa_1(x_3 - x_2)^2 = \\ &= \frac{1}{2}\kappa_1\left\{[(x_1^2 + x_3^2) + 2x_2^2 - 2x_2(x_1 + x_3)]\right\} = \\ &= \frac{1}{2}\kappa_1\left\{(x_1^2 + x_3^2) + 2\frac{m^2}{M^2}(x_1 + x_3)^2 + 2\frac{m}{M}(x_1 + x_3)^2\right\} = \\ &= \frac{1}{2}\kappa_1\left\{\left(1 + 2\frac{m}{M} + 2\frac{m^2}{M^2}\right)(x_1^2 + x_3^2) + 2\left(2\frac{m}{M} + 2\frac{m^2}{M^2}\right)x_1x_3\right\} \end{aligned}$$

2. **Determine the  $A$  and  $m$  tensors.** In order to calculate these tensors we use the expressions for  $T$  and  $U$  obtained in step 1. Since the kinetic energy obtained in step 1 does not contain products of the generalized velocity of mass 1 and the generalized velocity of mass 2, the mass tensor will be a diagonal tensor. We can see this by looking at the definition of the mass tensor elements:

$$T = \frac{1}{2} \sum_{j,k} m_{j,k} \dot{q}_j \dot{q}_k$$

The mass tensor is thus equal to

$$F_{xi} = \begin{pmatrix} \left( \frac{mM + mM^2}{M} \right) & \left( \frac{mM^2}{M} \right) \\ \left( \frac{mM^2}{M} \right) & \left( \frac{mM + mM^2}{M} \right) \end{pmatrix}$$

The A tensor is equal to

$$A = \begin{Bmatrix} \left. \frac{\partial^2 U}{\partial x_1 \partial x_1} \right|_0 & \left. \frac{\partial^2 U}{\partial x_1 \partial x_3} \right|_0 \\ \left. \frac{\partial^2 U}{\partial x_3 \partial x_1} \right|_0 & \left. \frac{\partial^2 U}{\partial x_3 \partial x_3} \right|_0 \end{Bmatrix} = \begin{Bmatrix} \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) & 2\kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \\ 2\kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) & \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) \end{Bmatrix}$$

3. **Determine the eigen frequency and the eigen vectors.** The eigen frequencies can be determined by requiring that the determinant of the coefficients of the equations of motions vanishes:

$$\begin{aligned} | \{A\} - \{m\}\omega^2 | &= \begin{vmatrix} \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) & 2\kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \\ 2\kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) & \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) \end{vmatrix} - \begin{vmatrix} \left( m + \frac{m^2}{M} \right) & \left( \frac{m^2}{M} \right) \\ \left( \frac{m^2}{M} \right) & \left( m + \frac{m^2}{M} \right) \end{vmatrix} \omega^2 = \\ &= \begin{vmatrix} \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) - \left( m + \frac{m^2}{M} \right) \omega^2 & 2\kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) - \left( \frac{m^2}{M} \right) \omega^2 \\ 2\kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) - \left( \frac{m^2}{M} \right) \omega^2 & \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) - \left( m + \frac{m^2}{M} \right) \omega^2 \end{vmatrix} = 0 \end{aligned}$$

This requires that

$$\left( \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) - \left( m + \frac{m^2}{M} \right) \omega^2 \right)^2 - \left( 2\kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) - \left( \frac{m^2}{M} \right) \omega^2 \right)^2 = 0$$

Or

$$\left( \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) - \left( m + \frac{m^2}{M} \right) \omega^2 \right) = \pm \left( 2\kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) - \left( \frac{m^2}{M} \right) \omega^2 \right)$$

Consider the two signs. First the positive sign:

$$\left( \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) - \left( m + \frac{m^2}{M} \right) \omega^2 \right) = \left( 2 \kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) - \left( \frac{m^2}{M} \right) \omega^2 \right)$$

This is equivalent to

$$m \omega^2 = \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) - 2 \kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) = \kappa_1$$

Or

$$\omega = \sqrt{\frac{\kappa_1}{m}}$$

Now consider the negative sign:

$$\left( \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) - \left( m + \frac{m^2}{M} \right) \omega^2 \right) = - \left( 2 \kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) - \left( \frac{m^2}{M} \right) \omega^2 \right)$$

This is equivalent to

$$m \left( m + 2 \frac{m}{M} \right) \omega^2 = \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) + 2 \kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) = \kappa_1 \left( 1 + 4 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) = \kappa_1 \left( 1 + 2 \frac{m}{M} \right)^2$$

Or

$$\omega = \sqrt{\frac{\kappa_1 \left( 1 + 2 \frac{m}{M} \right)^2}{m \left( 1 + 2 \frac{m}{M} \right)}} = \sqrt{\frac{\kappa_1 \left( 1 + 2 \frac{m}{M} \right)}{m}} = \sqrt{\frac{\kappa_1 \left( 1 + 2 \frac{m}{M} \right)}{m}} = \sqrt{\kappa_1 \left( \frac{M + 2m}{mM} \right)}$$

Consider the first eigen frequency, and assume the corresponding eigen vector is  $(a_{11}, a_{21})$ .

The equations of motion for this frequency are

$$\left\{ \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) - \left( m + \frac{m^2}{M} \right) \omega^2 \right\} a_{11} + \left\{ 2 \kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) - \left( \frac{m^2}{M} \right) \omega^2 \right\} a_{31} = 0$$

$$\left\{ 2 \kappa_1 \frac{m}{M} \left( 1 + \frac{m}{M} \right) - \left( \frac{m^2}{M} \right) \omega^2 \right\} a_{11} + \left\{ \kappa_1 \left( 1 + 2 \frac{m}{M} \left( 1 + \frac{m}{M} \right) \right) - \left( m + \frac{m^2}{M} \right) \omega^2 \right\} a_{31} = 0$$

Substituting the expression of the first eigen frequency in these equations we obtain for each equation the following expression:

$$\left\{ \kappa_1 \left( \frac{m}{M} \left( 1 + 2 \frac{m}{M} \right) \right) \right\} a_{11} + \left\{ \kappa_1 \frac{m}{M} \left( 1 + 2 \frac{m}{M} \right) \right\} a_{31} = \kappa_1 \left( \frac{m}{M} \left( 1 + 2 \frac{m}{M} \right) \right) (a_{11} + a_{31}) = 0$$

This equations tells us that  $a_{11} = -a_{31}$ . Since the eigen vectors are orthogonal, we expect that the eigen vector for the second eigen frequency is given by  $a_{12} = a_{32}$ .

4. **Determine the scale factors required to match the initial conditions.** In this example, we do not need to match initial conditions (such as the initial displacement or the initial velocity and we thus do not need to determine scale factor).
5. **Determine the normal coordinates.** The normal coordinates are those coordinates that oscillate with a single frequency. In the current example we thus observe the following normal coordinates:

$$\eta_1 = a_{11} x_1 + a_{31} x_3 = a_{11} (x_1 - x_3)$$

$$\eta_2 = a_{12} x_1 + a_{32} x_3 = a_{22} (x_1 + x_3)$$

- *Note:* the constants in these equations need to be adjusted to match the initial conditions. The system will carry out a motion with normal frequency 1 when  $\eta_2 = 0$ . This requires that  $x_1 = -x_3$  and the motion is asymmetric. The system will carry out a motion with normal frequency 2 when  $\eta_1 = 0$ . This requires that  $x_1 = x_3$  and the motion is symmetric. Note: the normal frequency 1 is equal to the frequency of a mass  $m$  on a spring whose other end remains fixed. This mode requires the central atom to remain fixed, and this can be

achieved when the motion is asymmetric since the forces exerted by the two springs on the central mass cancel.

The transverse vibration of the molecule can be specified in terms of a single parameter  $\alpha$ . For this mode of vibration we will get a single "uncoupled" differential equation with a single corresponding characteristic frequency.

## Chapter Four

### Wave propagation

#### 4.1 Introduction

Why are we able to see? Answer: Because there is light.

And...what is light? Answer: Light is a wave.

So...what is a wave?

Answer: A wave is a disturbance that carries energy from place to place. A wave does NOT carry matter with it! It just moves the matter as it goes through it.

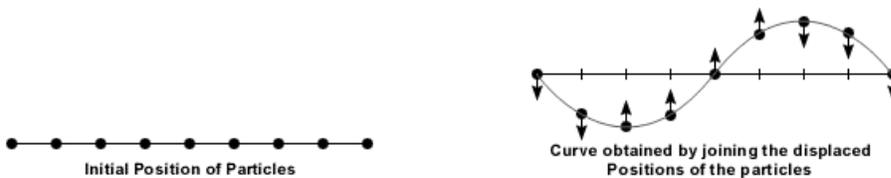
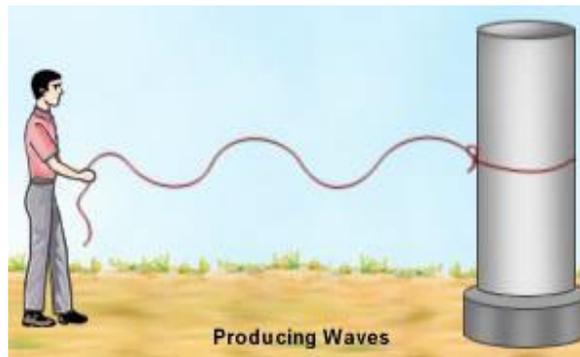
Some waves do not need matter (called a “medium”) to be able to move (for example, through space).

These are called electromagnetic waves (or EM waves).

Some waves **MUST** have a medium in order to move. These are called mechanical waves.

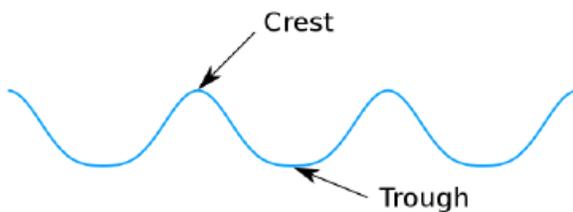
## Wave types

1. *Transverse waves*: Waves in which the medium moves at right angles to the direction of the wave

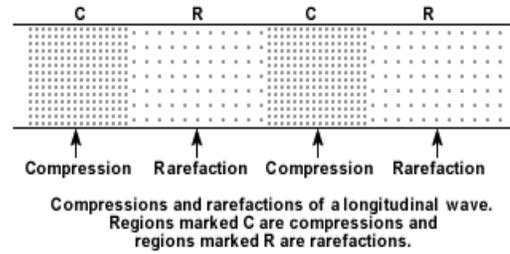
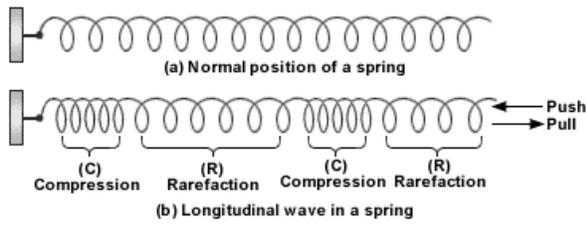
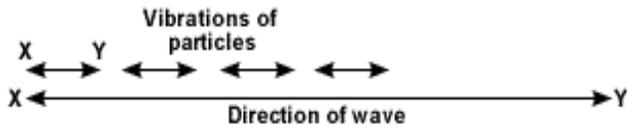


Parts of transverse waves:

- i. Crest: the highest point of the wave
- ii. Trough: the lowest point of the wave

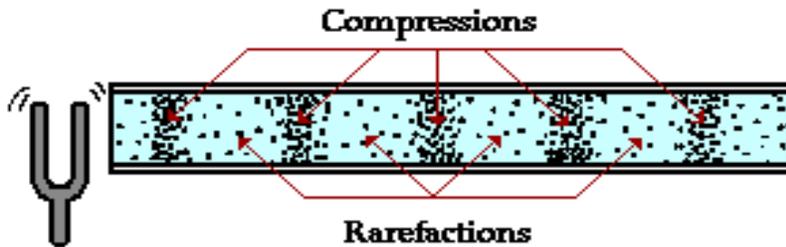


2. Compressional (or longitudinal) waves: Waves in which the medium moves back and forth in the same direction as the wave



Parts of longitudinal waves:

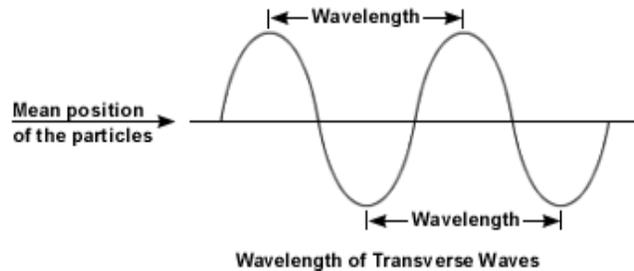
- i. Compression: where the particles are close together
- ii. Rarefaction: where the particles are spread apart



## Wave Properties

Wave properties depend on what (type of energy) is making the waves.

1. Wavelength: The distance between one point on a wave and the exact same place on the next wave.



2. Frequency: How many waves go past a point in one second; unit of measurement is hertz (Hz).

The higher the frequency, the more energy in the wave.

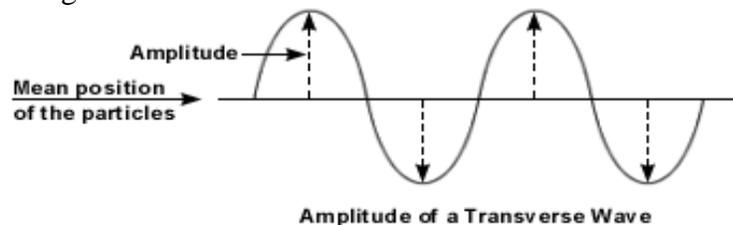
10 waves going past in 1 second = 10 Hz

1,000 waves go past in 1 second = 1,000 Hz

1 million waves going past = 1 million Hz

3. Amplitude: How far the medium moves from rest position (where it is when not moving).

Remember that for transverse waves, the highest point is the crest, and the lowest point is the trough.



4. Wave speed: Depends on the medium in which the wave is traveling. It varies in solids, liquids and gases.

A mathematical way to calculate speed:

wave speed = wavelength x frequen

OR

$$v = f \times \lambda$$

## 4.2 Wave equation

1. The basic description of a right/left moving pulse is a function of the form:

$$y(x, t) = f(x - vt) \quad \text{right mover} \qquad y(x, t) = f(x + vt) \quad \text{left mover} \qquad (1)$$

where  $y(x, t)$  is the displacement from equilibrium.

2. For a sinusoidal wave with amplitude  $A$

$$y(x, t) = A \cos(kx - \omega t) \qquad (2)$$

where the wave number  $k$  and angular frequency  $\omega$  are related to the wavelength and frequency/period

$$k = \frac{2\pi}{\lambda} \quad \text{or} \quad \omega = \frac{2\pi}{T} = 2\pi f \qquad (3)$$

The speed of the wave is

$$v = \lambda f \quad \text{or} \quad \omega = vk \qquad (4)$$

3. The string is characterized by:

$$F_T \equiv \text{Then tension in the rope a constant} \qquad (5)$$

$$\mu \equiv \text{The mass per unit length.} \qquad (6)$$

4. Analyzing Newton Laws for a small bit of string shows that the displacement obeys the wave equation

$$\underbrace{F_T \frac{\partial^2 y}{\partial x^2}}_{\text{net force}} = \underbrace{\mu \frac{\partial^2 y}{\partial t^2}}_{\text{mass} \times \text{accel}} \qquad (7)$$

5. The wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad (8)$$

where for the particular example of the string

$$v = \sqrt{\frac{F_T}{\mu}} \quad (9)$$

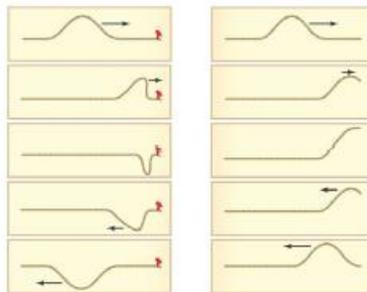
6. The most general solution to the wave equation is anything of the form

$$y(x, t) = f(x - vt) + g(x + vt) \quad (10)$$

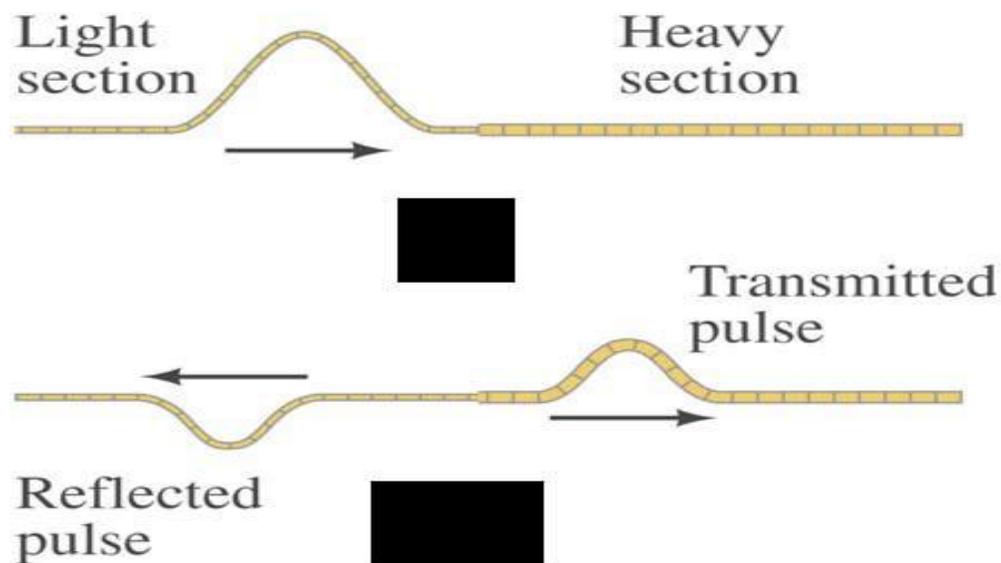
7. The wave equation is linear so if  $y_1$  is a solution to the wave equation and  $y_2$  is a solution to the wave equation so  $y_1 + y_2$ . This leads to the notion of interference between waves. If two waves tend to cancel there is destructive interference. If two waves tend to add there is constructive interference.

### 4.3 Reflection and Transmission

A wave reaching the end of its medium, but where the medium is still free to move, will be reflected (b), and its reflection will be upright.

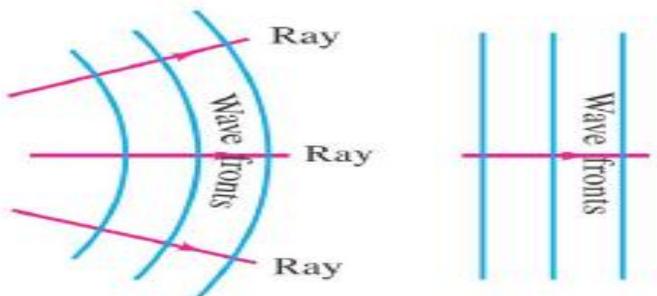


A wave hitting an obstacle will be reflected (a), and its reflection will be inverted.



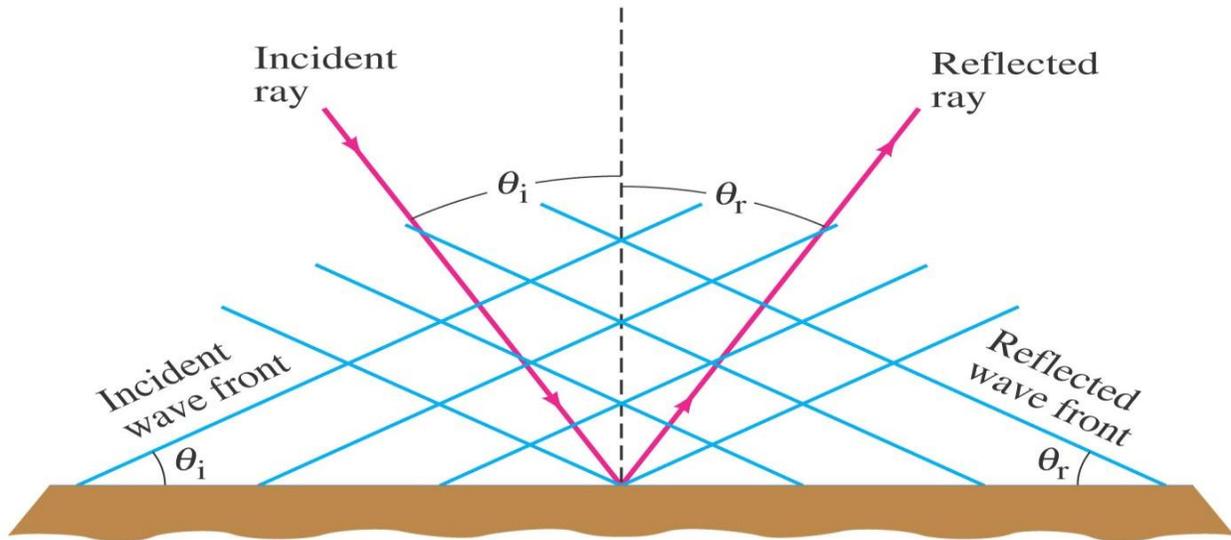
A wave encountering a denser medium will be partly reflected and partly transmitted; if the wave speed is less in the denser medium, the wavelength will be shorter.

Two- or three-dimensional waves can be represented by wave fronts, which are curves of surfaces where all the waves have the same phase.



Lines perpendicular to the wave fronts are called rays; they point in the direction of propagation of the wave.

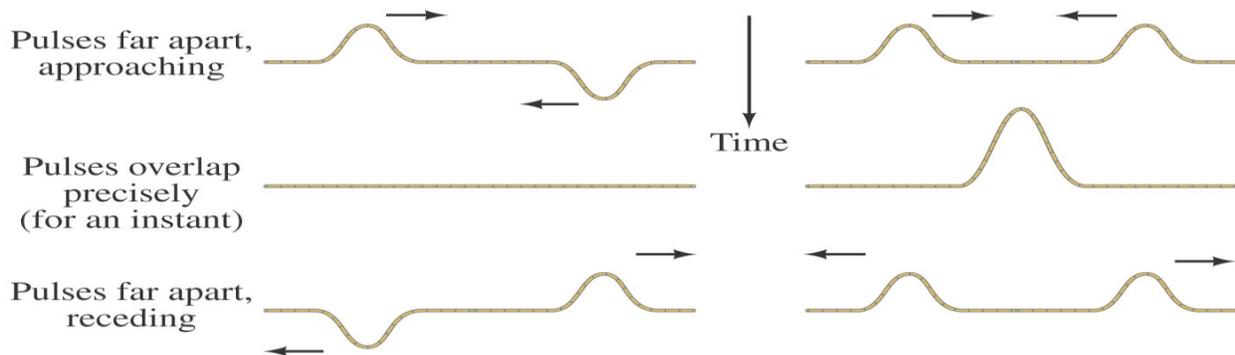
- **The law of reflection:** the angle of incidence equals the angle of reflection.



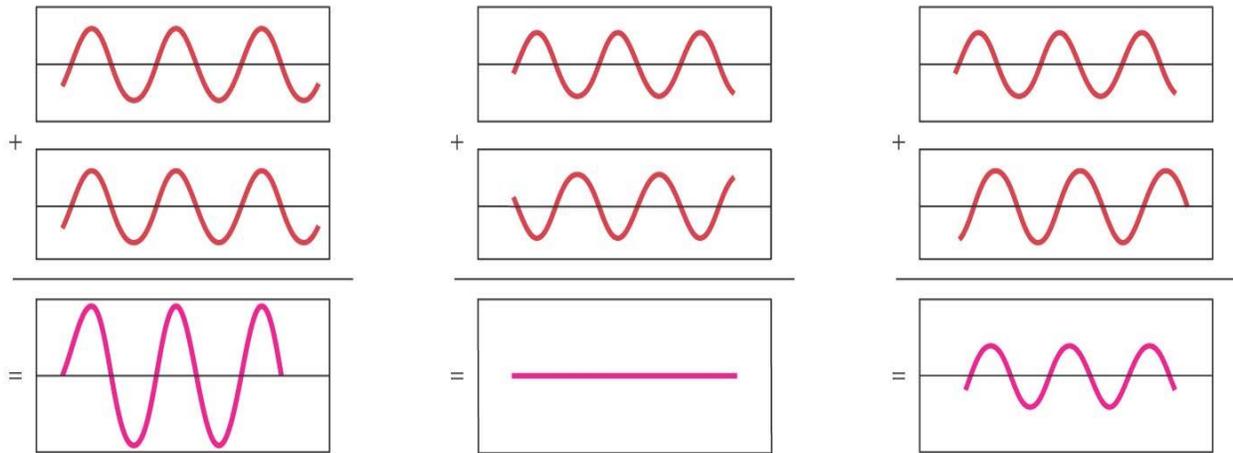
#### 4.4 Interference

The superposition principle says that when two waves pass through the same point, the displacement is the arithmetic sum of the individual displacements.

In the figure below, (a) exhibits destructive interference and (b) exhibits constructive interference.



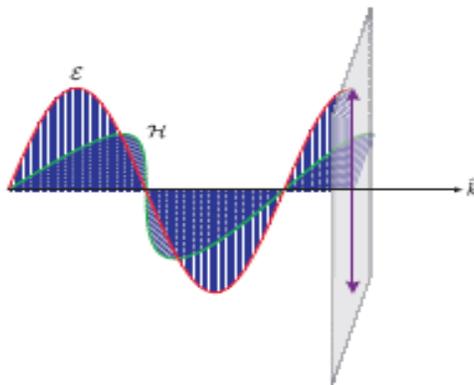
These graphs show the sum of two waves. In (a) they add constructively; in (b) they add destructively; and in (c) they add partially destructively.



## 4.5 Polarization

The polarization of a wave becomes very important when we consider radio communication systems, and radio wave propagation. The performance of communication systems can be strongly affected by the polarization of a wave, if it is not "matched" to the intended polarization. Along similar lines, propagation of a wave introduces potential changes to its polarization which will in turn affect communication system performance. Hence, it is important to understand how waves are polarized and the different polarization classifications.

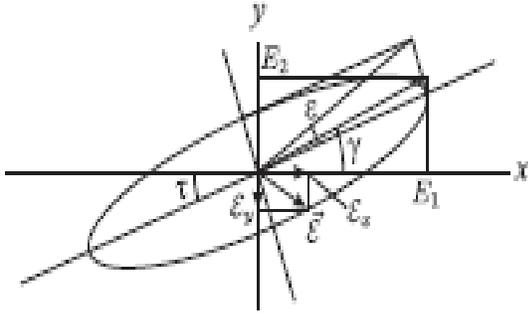
The polarization of a wave is defined as the figure that the instantaneous electric field traces out with time at a fixed observation point that is normal to the direction of propagation. Perhaps the most common example of polarization you have heard of is linear polarization. A linearly polarized plane wave is illustrated graphically below. We might define this as vertically polarized but in general linear polarization could refer to linear polarization vectors pointed in any direction (e.g. a horizontally polarized wave is also a form of linear polarization).



In general, the figure traced out by the electric field is not a line, but in fact an ellipse, of which a line is a degenerate case. Consider a wave travelling in the  $+z$ -direction. Its polarization will then be traced out in the  $xy$ -plane, and appears as shown in the figure below.

Some important definitions are called for here:

- $\tau$  is the major axis angle, i.e. how far the major axis of the ellipse is tilted away from the  $x$ -axis;
- $E_1$  and  $E_2$  are the two components of the electric field, which will be defined momentarily;



Axial ratio is a measure of how close the polarization is to circular; if  $AR = 1$ , the polarization traced out is a circle. More specifically, a wave can be defined as being right hand circularly polarized (RHCP) or left hand circularly polarized (LHCP) if  $AR = 1$ . The "handedness" of the polarization is observed by viewing the rotation of the wave vector as it travels. If your thumb points in the direction of propagation, your fingers should curl in the direction of polarization, hence the use of "hand" in the polarization description. A sign can be arbitrarily added to the axial ratio to be more specific as to the handedness of the wave it is referring to. A positive  $AR$  indicates a right-hand polarized wave while a negative  $AR$  indicates a left-hand polarized wave (note, not necessarily circularly-polarized). Note that the sign is simply used to indicate the handedness of the wave; it is not possible to generate a negative  $AR$  from the formulae shown above.