Jimma University, College of Natural Sciences, Department of Physics

Mathematical Methods of Physics II (Phys 2032), Assignment III

1. Two-dimensional irrotational fluid flow is conveniently described by a complex potential

$$f(z) = u(x, y) + iv(x, y)$$

We label the real part, $u(x, y)$, the velocity potential, and the imaginary part, $v(x, y)$, the stream function. The fluid velocity $\mathbf{V}$ is given by $\mathbf{V} = \nabla u$. If $f(z)$ is analytic,

a. Show that $\frac{df}{dx} = V_x - iV_y$ $\quad$ ($V_x = (\nabla u)_x$, and $V_y = (\nabla u)_y$)

b. $\nabla . \mathbf{V} = 0$ (no sources or sinks)

c. $\nabla \times \mathbf{V} = 0$ (irrotational, nonturbulent flow)

2. Show that the real and imaginary parts of a complex function $f(z) = u(x, y) + iv(x, y)$ that is analytic in a domain D have continuous second order partial derivatives and are the solutions of Laplace's equations

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

A solution of Laplace's equation having continuous second order partial derivatives is called a **harmonic function**.

3. Determine the analytic function,

a. whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

b. whose imaginary part is $6xy - 5x + 3$

4. Evaluate the following integrals on a unit circle.

(a) $\oint \frac{\sin^2 z - z^2}{(z-a)^3} dz$ $\qquad$ (b) $\oint \frac{f(z)}{z(2z+1)^2} dz$

5. Calculate the Laurent series expansions for the following functions about the given point z_0 valid for the given region R.

$$f(z) = \frac{1}{(z+1)(z+3)}, z_0 = 1$$

$$(a) \quad R = \{z: 2 < |z-1| < 4\}$$

$$(b) \quad R = \{z: |z-1| > 4\}$$

6. Consider the differential equation

$$\frac{d^2y}{dx^2} + R(x)\frac{dy}{dx} + [Q(x) + \lambda P(x)]y = 0$$

Show that it can be put into the form of Sturm-Liouville equation

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [q(x) + \lambda p(x)]y = 0$$

with $r(x) = e^{\int R(x)dx}$, $q(x) = Q(x)e^{\int R(x)dx}$, and $p(x) = P(x)e^{\int R(x)dx}$

7. Show that the system

$$\frac{d^2y}{dx^2} + \lambda y = 0, \qquad y(0) = 0, \qquad \frac{dy}{dx}(\pi) = 0$$

is a Sturm-Liouville system. Find the eigenvalues and eigenfunctions of the system.

8. Write Legendre, Hermite, Bessel, and Laguerre differential equations and show that they can be converted to a Sturm-Liouville form.