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Lecture Note

On Course Mathematical Methods of Physics II (Phys.2032)

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Content of the Lecture Note

Vector Analysis

Overview

Many of you will know a good deal already about Vector Algebra — how to add and subtract vectors, how to take scalar and vector products of vectors, and something of how to describe geometric and physical entities using vectors. This course will remind you about that good stuff, but goes on to introduce you to the subject of Vector Calculus which, like it says on the can, combines vector algebra with calculus.

To give you a feeling for the issues, suppose you were interested in the temperature T of water in a river. Temperature T is a *scalar*, and will certainly be a function of a position vector x = (x, y, z) and may also be a function of time t: T = T (x, t). It is a scalar field.

Suppose now that you kept y, z, t constant, and asked what is the change in temperature as you move a small amount in x? No doubt you'd be interested in calculating $\partial T/\partial x$. Similarly, if you kept the point fixed, and asked how does the temperature change of time, you would be interested in $\partial T/\partial t$.

But why restrict ourselves to movements up-down, left-right, etc.? Suppose you wanted to know what the change in temperature along an arbitrary direction. You would be interested in

$$\frac{\partial T}{\partial x},$$

but how would you calculate that? Is $\partial T/\partial x$ a vector or a scalar?

Now let's dive into the flow. At each point x in the stream, at each time t, there will be a stream velocity $v(x,t)$. The local stream velocity can be viewed directly using modern techniques such as laser Doppler anemometry, or traditional techniques such as throwing twigs in. The point now is that v is a function that has the same four input variables as temperature did, but its output result is a vector. We may be interested in places x where the stream suddenly accelerates, or vortices where the stream curls around dangerously. That is, we will be interested in finding the acceleration of the stream, the gradient of its velocity. We may be interested in the magnitude of the acceleration (a scalar). Equally, we may be interested in the acceleration as a vector, so that we can apply Newton's law and figure out the force.

1. Vector Analysis

1.1. Vectors

Many physical quantities, such a mass, time, temperature is fully specified by one number or magnitude. They are scalars. But other quantities require more than one number to describe them. They are vectors. You have already met vectors in their purer mathematical sense in your course on linear algebra (matrices and vectors), but often in the physical world, these numbers specify a magnitude and a direction — a total of two numbers in a 2D planar world, and three numbers in 3D.

Obvious examples are velocity, acceleration, electric field, and force. Below, probably all our examples will be of these "magnitude and direction" vectors, but we should not forget that many of the results extend to the wider realm of vectors.

There are three slightly different types of vectors:

- **Free vectors**: In many situations *only* the magnitude and direction of a vector is important, and we can *translate* them at will (with 3 degrees of freedom for a vector in 3-dimensions).
- **Sliding vectors:** In mechanics the line of action of a force is often important for deriving moments. The force vector can slide with 1 degree of freedom.
- **Bound or position vectors**: When describing lines, curves etc. in space, it is obviously important that the origin and head of the vector are not translated about arbitrarily. The origins of position vectors all coincide at an overall origin O.

One the advantages of using vectors is that it frees much of the analysis from the restriction of arbitrarily imposed coordinate frames. For example, if two free vectors are equal, we need only say that their magnitudes and directions are equal, and that can be done with a drawing that is independent of any coordinate system. However, coordinate systems are ultimately useful, so it useful to introduce the idea of vector components. Try to spot things in the notes that are independent of coordinate system.

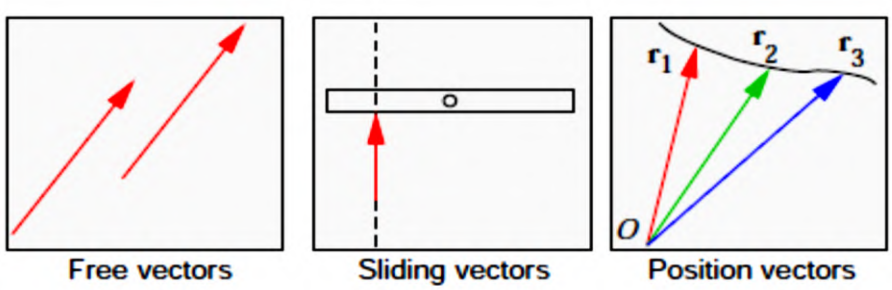


Figure 1.1:

1.1.1. Vector elements or components in a coordinate frame

A method of representing a vector is to list the values of its elements or components in a sufficient number of different (preferably mutually perpendicular) directions, depending on the dimension of the vector. These specified directions define ***a coordinate frame***. In this course we will mostly restrict our attention to the 3-dimensional Cartesian coordinate frame $O(x, y, z)$. When we come to examine vector fields later in the course you will use curvilinear coordinate frames, especially 3D spherical and cylindrical polar, and 2D plane polar, coordinate systems.

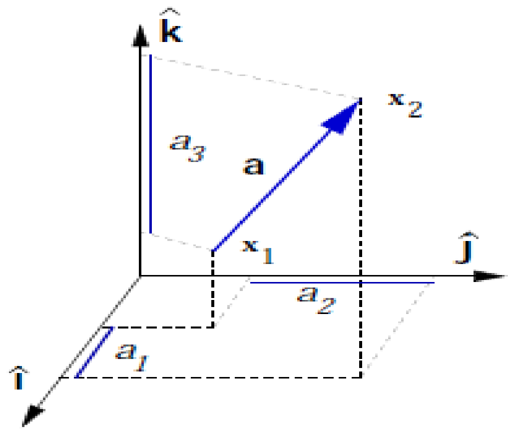


Figure 1.2: Vector components.

In a Cartesian coordinate frame, we write

$$a = [a_1, a_2, a_3] = [x_2 - x_1, y_2 - y_1, z_2 - z_1] \; or \; a = [a_x, a_y, a_z]$$

as sketched in Figure 1.2. Defining $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ as unit vectors in the x, y, z directions

$$\hat{\imath} = [1, 0, 0] \; \hat{\jmath} = [0, 1, 0] \; \hat{k} = [0, 0, 1]$$

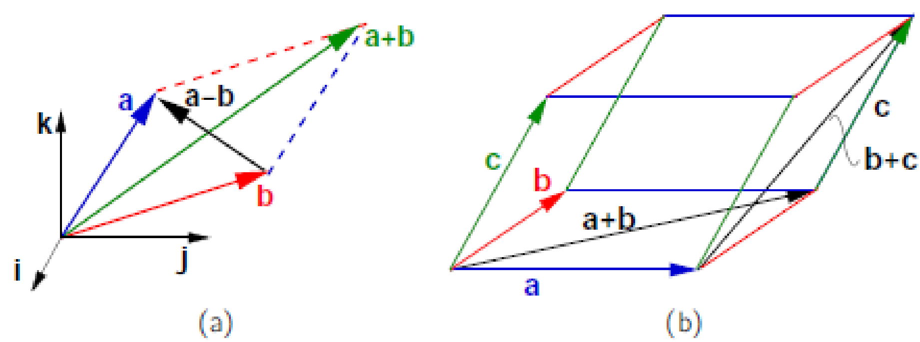


Fig.1.3. (a) Addition of two vectors is commutative, but subtraction isn't. Note that the coordinate frame is irrelevant. (b) Addition of three vectors is associative.

we could also write

$a = a_1\hat{\imath} + a_2\hat{\jmath} + a_2\hat{k}$.

Although we will be most often dealing with vectors in 3-space, you should not think that general vectors are limited to three components.

In these notes we will use bold font to represent vectors $\boldsymbol{a}, \boldsymbol{\omega}$, In your written work, underline the vector symbol $\underline{a, \omega}$ and be meticulous about doing so. We shall use the hat to denote a unit vector.

1.1.2. Vector equality

Two free vectors are said to be equal iff their lengths and directions are the same. If we use a coordinate frame, we might say that corresponding components of the two vectors must be equal. This definition of equality will also do for position vectors, but for sliding vectors we must add that the line of action must be identical too.

1.1.3. Vector magnitude and unit vectors

Provided we use an orthogonal coordinate system, the magnitude of a 3-vector is

$$a = |\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

To find the unit vector in the direction of a, simply divide by its magnitude

$$\hat{a} = \frac{a}{|a|}$$

1.1.4. Vector Addition and subtraction

Vectors are added/subtracted by adding/subtracting corresponding components, exactly as for matrices. Thus

$$a + b = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

Addition follows the parallelogram construction of Figure 1.3 (a). Subtraction $(\boldsymbol{a} - \boldsymbol{b})$ is defined as the addition $(\boldsymbol{a} + (-\boldsymbol{b}))$. It is useful to remember that the vector $a - b$ goes from b *to* a.

The following results follow immediately from the above definition of vector addition:

a) $a + b = b + a$ (commutativity) (Figure 1.3(a))
b) $(a + b) + c = a + (b + c) = a + b + c$ (associativity) (Figure 1.3(b))
c) $a + 0 = 0 + a = a$, where the zero vector is $0 = [0, 0, 0]$.
d) $a + (-a) = 0$

1.1.5. Multiplication of a vector by a scalar. (NOT the scalar product!)

Just as for matrices, multiplication of a vector a by a scalar **c** is defined as multiplication of each component by c, so that

$$ca = [ca_1, ca_2, ca_3].$$

It follows that:

$$|c\boldsymbol{a}| = \sqrt{(\boldsymbol{ca_1})^2 + (\boldsymbol{ca_2})^2 + (\boldsymbol{ca_3})^2} = |c||a|$$

The direction of the vector will reverse if c is negative, but otherwise is unaffected. (By the way, a vector where the sign is uncertain is called a director.)

♣ **Example**

Q. Coulomb's law states that the electrostatic force on charged particle Q due to another charged particle q_1 is

$$\mathbf{F} = K\frac{Qq_1}{r^2}\hat{\mathbf{e}}_r$$

where $\mathbf{r}$ is the vector from q_1 to Q and $\hat{\mathbf{r}}$ is the unit vector in that same direction. (Note that the rule "unlike charges attract, like charges repel" is built into this formula.) The force between two particles is not modified by the presence of other charged particles.

Hence write down an expression for the force on Q at $\mathbf{R}$ due to N charges q_i at $\mathbf{r}_i$.

A. The vector from q_i to Q is $\mathbf{R} - \mathbf{r}_i$. The unit vector in that direction is $(\mathbf{R} - \mathbf{r}_i)/|\mathbf{R} - \mathbf{r}_i|$, so the resultant force is

$$\mathbf{F}(\mathbf{R}) = \sum_{i=1}^{N} K\frac{Qq_i}{|\mathbf{R} - \mathbf{r}_i|^3}(\mathbf{R} - \mathbf{r}_i) \quad .$$

Note that $\mathbf{F}(\mathbf{R})$ is a vector *field.*

1.2. Scalar, dot, or inner product

This is a product of two vectors results in a scalar quantity and is defined as follows for 3-component vectors:

$$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3 .$$

Note that

$$a \cdot a = a_1^2 + a_2^2 + a_3^2 = |a|^2 = a^2$$

The following laws of multiplication follow immediately from the definition:

a) $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$ (commutativity)

b) $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$ (distributivity with respect to vector addition)

c) $(\lambda a) \cdot b = \lambda(a \cdot b) = a \cdot (\lambda b)$ scalar multiple of a scalar product of two vectors

1.2.1. Geometrical interpretation of scalar product

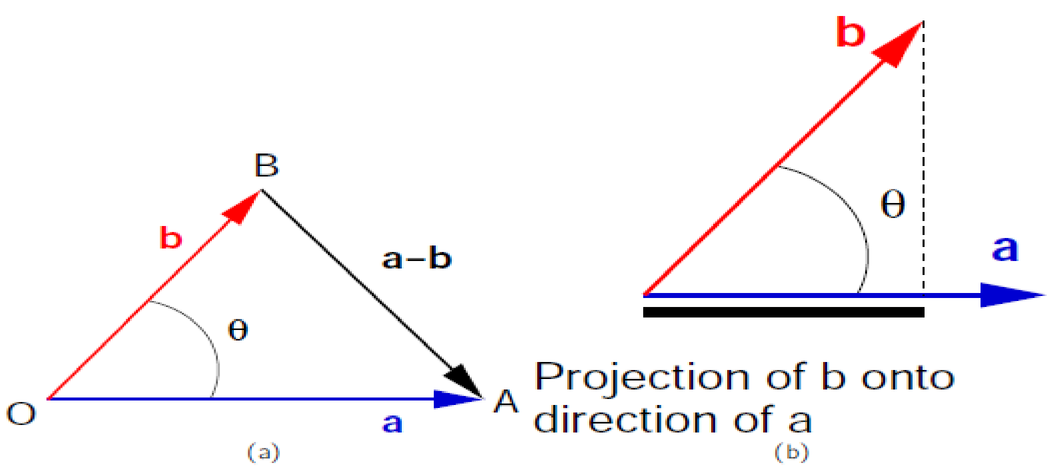


Fig 1.4. (a) Cosine rule. (b) Projection of b onto a.

Consider the square magnitude of the vector $a - b$. By the rules of the scalar product, this is

$$|\boldsymbol{a}-\boldsymbol{b}|^2 = (\boldsymbol{a}-\boldsymbol{b}) \cdot (\boldsymbol{a}-\boldsymbol{b})$$
$$= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{b} - 2(\boldsymbol{a} \cdot \boldsymbol{b})$$
$$= \boldsymbol{a}2 + \boldsymbol{b}2 - 2(\boldsymbol{a} \cdot \boldsymbol{b})$$

But, by the cosine rule for the triangle OAB (Figure 1.4a), the length AB^2 is given by

$$|\boldsymbol{a}-\boldsymbol{b}|^2 = \boldsymbol{a}2 + \boldsymbol{b}2 - 2\boldsymbol{ab}\cos\boldsymbol{\theta}$$

where θ is the angle between the two vectors. It follows that

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{ab}\cos\boldsymbol{\theta},$$

which is independent of the co-ordinate system used, and that $|a \cdot b| \leq ab$.
Conversely, the cosine of the angle between vectors a and b is given by

$$\cos\theta = a \cdot b/ab.$$

1.2.2. Projection of one vector onto the other

Another way of describing the scalar product is as the product of the magnitude of one vector and the component of the other in the direction of the first, since $\boldsymbol{b\cos\theta}$ is the component of b in the direction of a and vice versa (Figure 1.4b)?

Projection is particularly useful when the second vector is a unit vector — $\boldsymbol{a} \cdot \boldsymbol{\hat{\imath}}$ is the component of **a** in the direction of $\hat{\imath}$.

Notice that if we wanted the vector component of **b** in the direction of a we would write

$$(b \cdot \hat{a})\hat{a} = \frac{(\boldsymbol{b} \cdot \boldsymbol{a})\boldsymbol{a}}{a^2}.$$

In the particular case $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{0}$, the angle between the two vectors is a right angle and the vectors are said to be mutually orthogonal or perpendicular — neither vector has any component in the direction of the other.

An orthonormal coordinate system is characterized by $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$; and $\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0$.

1.2.3. A scalar product is an "inner product"

So far, we have been writing our vectors as row vectors $a = [a_1, a_2, a_3]$. This is convenient because it takes up less room than writing column vectors

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

In matrix algebra vectors are more usually defined as column vectors, as in

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

and a row vector is written as $\boldsymbol{a^T}$. Now for most of our work we can be quite relaxed about this minor difference, but here let us be fussy.

Why? Simply to point out at that the scalar product is also the inner product more commonly used in linear algebra. Defined as $\boldsymbol{a^T b}$ when vectors are column vectors as

$$a \cdot b = a^T b = [a_1, a_2, a_3]\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Here we treat a n-dimensional column vector as an $n \times 1$ matrix. (Remember that if you multiply two matrices $\boldsymbol{M_{m \times n} N_{n \times p}}$ then M must have the same columns as **N** has rows (here denoted by **n**) and the result has size (**rows × columns**) of $m \times p$. So for n dimensional column vectors a and **b,** $\boldsymbol{a^T}$ is a $\boldsymbol{1 \times n}$ matrix and b is $\boldsymbol{n \times 1}$ matrix, so the product $\boldsymbol{a^T} b$ is a 1×1 matrix, which is (at last!) a scalar.)

♣ **Examples**

Q1. A force **F** is applied to an object as it moves by a small amount $\delta\mathbf{r}$. What work is done on the object by the force?

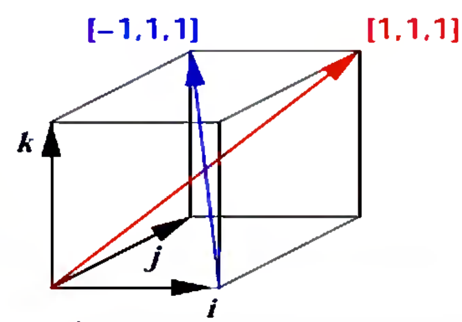
A1. The work done is equal to the component of force in the direction of the displacement multiplied by the displacement itself. This is just a scalar product:

$$\delta W = \mathbf{F} \cdot \delta\mathbf{r} \quad .$$

Q2. A cube has four diagonals, connecting opposite vertices. What is the angle between an adjacent pair?

A2. Well, you could plod through using Pythagoras' theorem to find the length of the diagonal from cube vertex to cube centre, and perhaps you should to check the following answer.

The directions of the diagonals are $[\pm 1, \pm 1, \pm 1]$. The ones shown in the figure are $[1, 1, 1]$ and $[-1, 1, 1]$. The angle is thus



$$\theta = \cos^{-1} \frac{[1,1,1] \cdot [-1,1,1]}{\sqrt{1^2+1^2+1^2}\sqrt{-1^2+1^2+1^2}} = \cos^{-1}\frac{1}{3}$$

Q3. A pinball moving in a plane with velocity **s** bounces (in a purely elastic impact) from a baffle whose endpoints are **p** and **q**. What is the velocity vector after the bounce?

A3. Best to refer everything to a coordinate frame with principal directions $\hat{\mathbf{u}}$ along and $\hat{\mathbf{v}}$ perpendicular to the baffle:

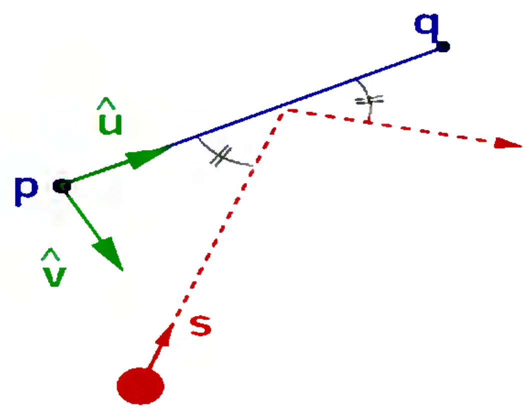
$$\hat{\mathbf{u}} = \frac{\mathbf{q}-\mathbf{p}}{|\mathbf{q}-\mathbf{p}|}$$
$$\hat{\mathbf{v}} = \mathbf{u}^{\perp} = [-u_y, \quad u_x]$$

Thus the velocity before impact is

$$\mathbf{s}_{\text{before}} = (\mathbf{s}.\hat{\mathbf{u}})\hat{\mathbf{u}} + (\mathbf{s}.\hat{\mathbf{v}})\hat{\mathbf{v}}$$

After the impact, the component of velocity in the direction of the baffle is unchanged and the component normal to the baffle is negated:

$$\mathbf{s}_{\text{after}} = (\mathbf{s}.\hat{\mathbf{u}})\hat{\mathbf{u}} - (\mathbf{s}.\hat{\mathbf{v}})\hat{\mathbf{v}}$$



1.3. Vector or cross product

The vector product of two vectors a and b is denoted by a × b and is defined as follows

$$a \times b = (a_2b_3 - a_3b_2)\hat{\imath} + (a_3b_1 - a_1b_3)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k}.$$

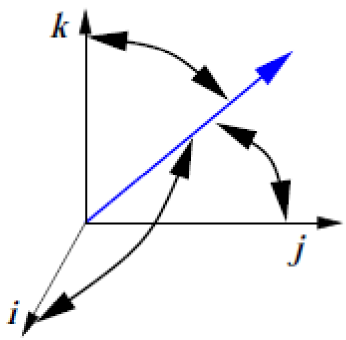


Fig. 1.5: The direction cosines are cosines of the angles shown.

It is MUCH more easily remembered in terms of the (pseudo-)determinant

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where the top row consists of the vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \widehat{\boldsymbol{k}}$ rather than scalars.
Since a determinant with two equal rows has value zero, it follows that $a \times a = 0$.
It is also easily verified that $(a \times b) \cdot a = (a \times b) \cdot b = 0$, so that $a \times b$ is orthogonal (perpendicular) to both a and b, as shown in Figure 1.6.
Note that $\hat{\imath} \times \hat{\jmath} = \hat{k}, \hat{\jmath} \times \hat{k} = \hat{\imath}, and\ \hat{k} \times \hat{\imath} = \hat{\jmath}$
The magnitude of the vector product can be obtained by showing that

$$|a \times b|^2 + (a \cdot b)^2 = a^2 b^2$$

from which it follows that

$$|a \times b| = ab\ sin\theta$$

which is again independent of the co-ordinate system used? This is left as an exercise.
Unlike the scalar product, the vector product does not satisfy commutativity but is in fact anti-commutative, in that $a \times b = -b \times a$. Moreover, the vector product does not satisfy the associative law of multiplication either since, as we shall see later
$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$.
Since the vector product is known to be orthogonal to both the vectors which form the product, it merely remains to specify its sense with respect to these vectors. Assuming that the co-ordinate vectors form a right-handed set in the order $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \widehat{\boldsymbol{k}}$ it can be seen that the sense of the vector product is also right-handed, i.e. the vector product has the same sense as the co-ordinate system used.

$$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \widehat{\boldsymbol{k}}$$

In practice, figure out the direction from a right-handed screw twisted from the first to second vector as shown in Figure 1.6(a).

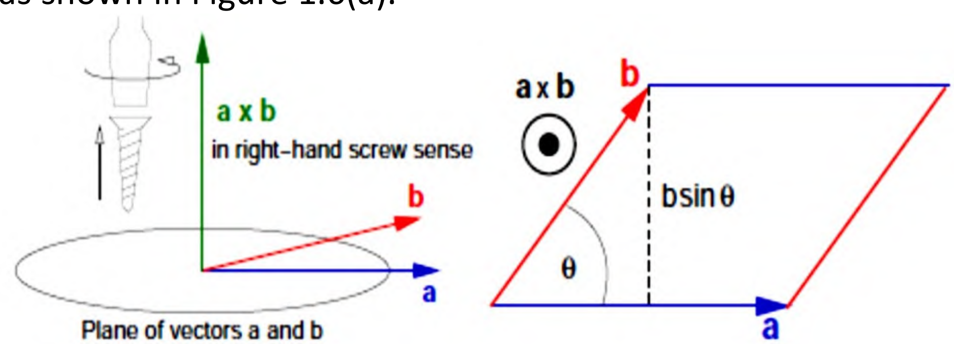


Figure 1.6: (a)The vector product is orthogonal to both a and b. Twist from first to second and move in the direction of a right-handed screw. (b) Area of parallelogram is ab sin θ.

1.3.1. Geometrical interpretation of vector product

The magnitude of the vector product $(\boldsymbol{a} \times \boldsymbol{b})$ is equal to the area of the parallelogram whose sides are parallel to, and have lengths equal to the magnitudes of, the vectors a and b (Figure 1.6b). Its direction is perpendicular to the parallelogram

♣ **Example**

Q. g is vector from A [1,2,3] to B [3,4,5].
$\hat{\ell}$ is the unit vector in the direction from O to A.
Find $\hat{m}$, a UNIT vector along $g \times \hat{\ell}$
Verify that $\hat{m}$ is perpendicular to $\hat{\ell}$.
Find $\hat{n}$, the third member of a right-handed coordinate set $\hat{\ell}, \hat{m}, \hat{n}$.

A.

$$g = [3,4,5] - [1,2,3] = [2,2,2]$$

$$\hat{\ell} = \frac{1}{\sqrt{14}}[1,2,3]$$

$$g \times \hat{\ell} = \frac{1}{\sqrt{14}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \frac{1}{\sqrt{14}}[2,-4,2]$$

Hence

$$\hat{m} = \cdot\frac{1}{\sqrt{24}}[2,-4,2]$$

and

$$\hat{n} = \hat{\ell} \times \hat{m}$$

2. Multiple Products. Geometry using Vectors

2.1. Triple and multiple products

Using mixtures of the pairwise scalar product and vector product, it is possible to derive "**triple products**" between three vectors, and indeed n-products between n vectors.
There is nothing about these that you cannot work out from the definitions of pairwise scalar and vector products already given, but some have interesting geometric interpretations, so it is worth looking at these.

2.1.1. Scalar triple product

This is the scalar product of a vector product and a third vector, i.e. $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$. Using the pseudo-determinant expression for the vector product, we see that the scalar triple product can be represented as the true determinant

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

You will recall that if you swap a pair of rows of a determinant, its sign changes; hence if you swap two pairs, its sign stays the same.

$$\underset{+}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}} \quad \text{1st SWAP} \quad \underset{-}{\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}} \quad \text{2nd SWAP} \quad \underset{+}{\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

This says that

(i) $a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$ (Called cyclic permutation.)

(ii) $a \cdot (b \times c) = -b \cdot (a \times c)$ and so on. (Called anti-cyclic permutation.)

(iii) The fact that $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$ allows the scalar triple product to be written as [a, b, c]. This notation is not very helpful, and we will try to avoid it below.

2.1.2. Geometrical interpretation of scalar triple product

The scalar triple product gives the volume of the parallelepiped whose sides are represented by the vectors a, b, and c. We saw earlier that the vector product (a × b) has magnitude equal to the area of the base, and direction perpendicular to the base. The component of c in this direction is equal to the height of the parallelepiped shown in Figure 2.1(a).

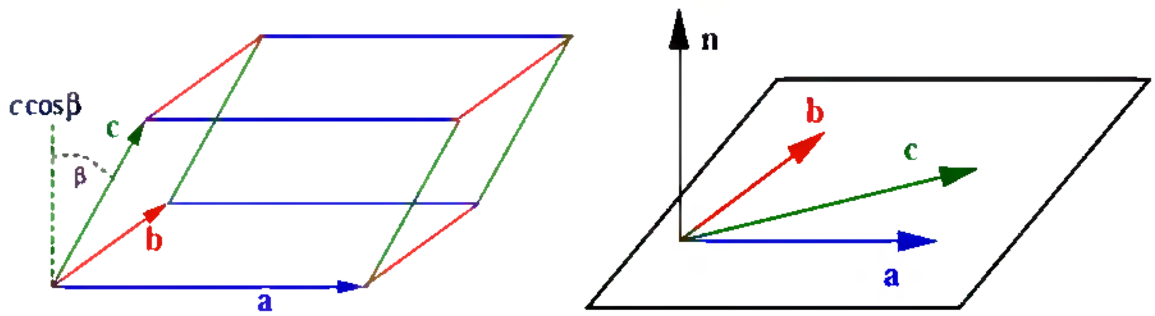


Figure 2.1: (a) Scalar triple product equals volume of parallelepiped. (b) Coplanarity yields zero scalar triple product.

2.1.3. Linearly dependent vectors

If the scalar triple product of three vectors is zero

$$a \cdot (b \times c) = 0$$

then the vectors are linearly dependent. That is, one can be expressed as a linear combination of the others. For example,

$$a = \lambda b + \mu c$$

where λ and μ are scalar coefficients.

You can see this immediately in two ways:

- The determinant would have one row that was a linear combination of the others. You'll remember that by doing row operations, you could reach a row of zeros, and so the determinant is zero.
- The parallelepiped would have zero volume if squashed flat. In this case all three vectors lie in a plane, and so any one is a linear combination of the other two. (Figure 2.1b.)

2.1.4. Vector triple product

This is defined as the vector product of a vector with a vector product, $a \times (b \times c)$. Now, the vector triple product $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ must be perpendicular to $(b \times c)$, which in turn is perpendicular to both b and c. Thus $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ can have no component perpendicular to b and c, and hence must be coplanar with them. It follows that the vector triple product must be expressible as a linear combination of b and c: $a \times (b \times c) = \lambda b + \mu c$.

The values of the coefficients can be obtained by multiplying out in component form. Only the first component need be evaluated, the others then being obtained by symmetry. That is

$$\big(a \times (b \times c)\big)_1 = a_2(b \times c)_3 - a_3(b \times c)_2$$
$$= a_2(b_1c_2 - b_2c_1) + a_3(b_1c_3 - b_3c_1)$$
$$= (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1$$
$$= (a_1c_1 + a_2c_2 + a_3c_3)b1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1$$
$$= (a \cdot c)b_1 - (a \cdot b)c_1$$

The equivalents must be true for the 2nd and 3rd components, so we arrive at the identity

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

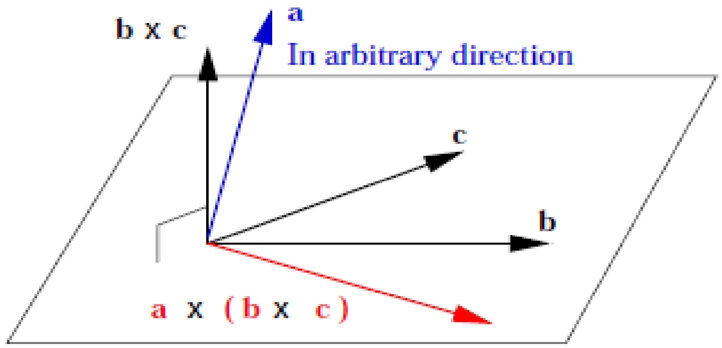


Figure 2.2: Vector triple product.

2.1.5. Projection using vector triple product

An example of the application of this formula is as follows. Suppose v is a vector and we want its projection into the xy -plane. The component of v in the z direction is $v \cdot \hat{k}$, so the projection we seek is $\boldsymbol{v} - (\boldsymbol{v} \cdot \hat{\boldsymbol{k}})\hat{\boldsymbol{k}}$. Writing $\hat{\boldsymbol{k}} \leftarrow$ **a**, **v** $\leftarrow$ **b**, $\hat{\boldsymbol{k}} \leftarrow$ **c,**

$$\begin{array}{rcl} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) & = & (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \\ \downarrow \quad \downarrow \quad \downarrow & & \\ \hat{\boldsymbol{k}} \times (\mathbf{v} \times \hat{\boldsymbol{k}}) & = & (\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}})\mathbf{v} - (\hat{\boldsymbol{k}}\cdot\mathbf{v})\hat{\boldsymbol{k}} \\ & = & \mathbf{v} - (\mathbf{v}\cdot\hat{\boldsymbol{k}})\hat{\boldsymbol{k}} \end{array}$$

So $\mathbf{v} - (\mathbf{v}\cdot\hat{\boldsymbol{k}})\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}} \times (\mathbf{v} \times \hat{\boldsymbol{k}})$.

(Hot stuff! But the expression $\mathbf{v} - (\mathbf{v}\cdot\hat{k})\hat{k}$ is much easier to understand, and cheaper to compute!)

2.1.6 Other repeated products

Many combinations of vector and scalar products are possible, but we consider only one more, namely the vector quadruple product $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$. By regarding $\mathbf{a} \times \mathbf{b}$ as a single vector, we see that this vector must be representable as a linear combination of $\mathbf{c}$ and $\mathbf{d}$. On the other hand, regarding $\mathbf{c} \times \mathbf{d}$ as a single vector, we see that it must also be a linear combination of $\mathbf{a}$ and $\mathbf{b}$. This provides a means of expressing one of the vectors, say $\mathbf{d}$, as linear combination of the other three, as follows:

$$\begin{array}{rcl} (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) & = & [(\mathbf{a} \times \mathbf{b})\cdot\mathbf{d}]\mathbf{c} - [(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}]\mathbf{d} \\ & = & [(\mathbf{c} \times \mathbf{d})\cdot\mathbf{a}]\mathbf{b} - [(\mathbf{c} \times \mathbf{d})\cdot\mathbf{b}]\mathbf{a} \end{array}$$

Hence

$$[(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}]\,\mathbf{d} = [(\mathbf{b} \times \mathbf{c})\cdot\mathbf{d}]\,\mathbf{a} + [(\mathbf{c} \times \mathbf{a})\cdot\mathbf{d}]\,\mathbf{b} + [(\mathbf{a} \times \mathbf{b})\cdot\mathbf{d}]\,\mathbf{c}$$

or

$$\mathbf{d} = \frac{[(\mathbf{b} \times \mathbf{c})\cdot\mathbf{d}]\,\mathbf{a} + [(\mathbf{c} \times \mathbf{a})\cdot\mathbf{d}]\,\mathbf{b} + [(\mathbf{a} \times \mathbf{b})\cdot\mathbf{d}]\,\mathbf{c}}{[(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}]} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \quad .$$

This is not something to remember off by heart, but it is worth remembering that the projection of a vector on any arbitrary basis set is unique.

♣ Example

Q1 Use the quadruple vector product to express the vector $\mathbf{d} = [3, 2, 1]$ in terms of the vectors $\mathbf{a} = [1, 2, 3]$, $\mathbf{b} = [2, 3, 1]$ and $\mathbf{c} = [3, 1, 2]$.

A1 Grinding away at the determinants, we find

$$[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}] = -18; \quad [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{d}] = 6; \quad [(\mathbf{c}\times\mathbf{a})\cdot\mathbf{d}] = -12; \quad [(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}] = -12$$

So, $\mathbf{d} = (-\mathbf{a} + 2\mathbf{b} + 2\mathbf{c})/3$.

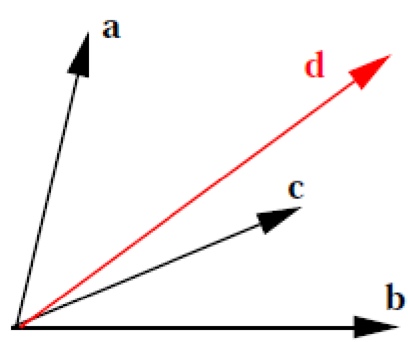


Figure 2.3: The projection of a (3-) vector onto a set of (3) basis vectors is unique. I.e. in $d = \alpha a + \beta b + \gamma c$, the set $\{\alpha, \beta, \gamma\}$ is unique.

2.2. Geometry using vectors: lines, planes

2.2.1. The equation of a line

The equation of the line passing through the point whose position vector is a and lying in the direction of vector b is

$$r = a + \lambda b$$

where $\boldsymbol{\lambda}$ is a scalar parameter. If you make b a unit vector, $r = a + \lambda \hat{b}$ then λ will represent metric length.

For a line defined by two points a_1 and a_2

$$r = a_1 + \lambda(a_2 - a_1)$$

or for the unit version

$$r = a_1 + \lambda \frac{(a_2 - a_1)}{|a_2 - a_1|}$$

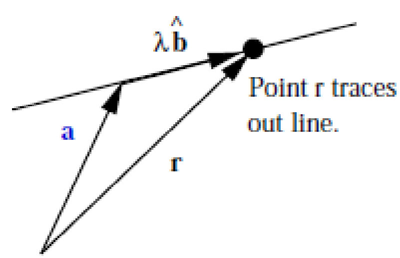


Figure 2.4: Equation of a line. With $\hat{b}$ a unit vector, λ is in the length units established by the definition of a.

2.2.2 The shortest distance from a point to a line

Referring to Figure 2.5(a) the vector $\mathbf{p}$ from $\mathbf{c}$ to any point on the line is $\mathbf{p} = \mathbf{a} + \lambda\hat{\mathbf{b}} - \mathbf{c} = (\mathbf{a}-\mathbf{c}) + \lambda\hat{\mathbf{b}}$ which has length squared $p^2 = (\mathbf{a}-\mathbf{c})^2 + \lambda^2 + 2\lambda(\mathbf{a}-\mathbf{c})\cdot\hat{\mathbf{b}}$. Rather than minimizing length, it is easier to minimize length-squared. The minumum is found when $d\,p^2/d\lambda = 0$, ie when

$$\lambda = -(\mathbf{a}-\mathbf{c})\cdot\hat{\mathbf{b}}\ .$$

So the minimum length vector is

$$\mathbf{p} = (\mathbf{a}-\mathbf{c}) - ((\mathbf{a}-\mathbf{c})\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}.$$

You might spot that is the component of $(\mathbf{a}-\mathbf{c})$ perpendicular to $\hat{\mathbf{b}}$ (as expected!). Furthermore, using the result of Section 2.1.5,

$$\mathbf{p} = \hat{\mathbf{b}} \times [(\mathbf{a}-\mathbf{c}) \times \hat{\mathbf{b}}]\ .$$

Because $\hat{\mathbf{b}}$ is a unit vector, and is orthogonal to $[(\mathbf{a}-\mathbf{c}) \times \hat{\mathbf{b}}]$, the modulus of the vector can be written rather more simply as just

$$p_{\min} = |(\mathbf{a}-\mathbf{c}) \times \hat{\mathbf{b}}|\ .$$

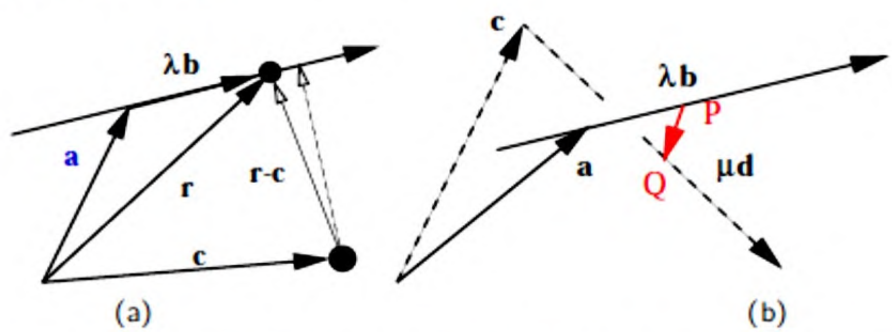


Figure 2.5: (a) Shortest distance point to line. (b) Shortest distance, line to line.

2.2.3 The shortest distance between two straight lines

If the shortest distance between a point and a line is along the perpendicular, then the shortest distance between the two straight lines $\mathbf{r} = \mathbf{a} + \lambda\hat{\mathbf{b}}$ and $\mathbf{r} = \mathbf{c} + \mu\hat{\mathbf{d}}$ must be found as the length of the vector which is mutually perpendicular to the lines.

The unit vector along the mutual perpendicular is

$$\hat{\mathbf{p}} = (\hat{\mathbf{b}} \times \hat{\mathbf{d}})/|\hat{\mathbf{b}} \times \hat{\mathbf{d}}|\ .$$

(Yes, don't forget that $\hat{\mathbf{b}} \times \hat{\mathbf{d}}$ is NOT a unit vector. $\hat{\mathbf{b}}$ and $\hat{\mathbf{d}}$ are not orthogonal, so there is a $\sin\theta$ lurking!)

The minimum length is therefore the component of $\mathbf{a}-\mathbf{c}$ in this direction

$$p_{\min} = \left|(\mathbf{a}-\mathbf{c})\cdot(\hat{\mathbf{b}} \times \hat{\mathbf{d}})/|\hat{\mathbf{b}} \times \hat{\mathbf{d}}|\right|\ .$$

♣ **Example**

Q Two long straight pipes are specified using Cartesian co-ordinates as follows: Pipe A has diameter 0.8 and its axis passes through points (2, 5, 3) and (7, 10, 8).
Pipe B has diameter 1.0 and its axis passes through the points (0, 6, 3) and (−12, 0, 9).
Determine whether the pipes need to be realigned to avoid intersection.

A Each pipe axis is defined using two points. The vector equation of the axis of pipe A is

$$\mathbf{r} = [2,5,3] + \lambda'[5,5,5] = [2,5,3] + \lambda[1,1,1]/\sqrt{3}$$

The equation of the axis of pipe B is

$$\mathbf{r} = [0,6,3] + \mu'[12,6,6] = [0,6,3] + \mu[-2,-1,1]/\sqrt{6}$$

The perpendicular to the two axes has direction

$$[1,1,1] \times [-2,-1,1] = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = [2,-3,1] = \mathbf{p}$$

The length of the mutual perpendicular is

$$(\mathbf{a}-\mathbf{c}) \cdot \frac{[2,-3,1]}{\sqrt{14}} = [2,-1,0] \cdot \frac{[2,-3,1]}{\sqrt{14}} = 1.87\ .$$

But the sum of the radii of the two pipes is $0.4 + 0.5 = 0.9$. Hence the pipes do not intersect.

2.2.4 The equation of a plane

There are a number of ways of specifying the equation of a plane.

1. If **b** and **c** are two non-parallel vectors (ie $\mathbf{b} \times \mathbf{c} \neq \mathbf{0}$), then the equation of the plane passing through the point **a** and parallel to the vectors **b** and **c** may be written in the form

   $$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

   where λ, μ are scalar parameters. Note that **b** and **c** are free vectors, so don't have to lie in the plane (Figure 2.6(a).)

2. Figure 2.6(b) shows the plane defined by three non-collinear points **a**, **b** and **c** in the plane (note that the vectors **b** and **c** are position vectors, not free vectors as in the previous case). The equation might be written as

   $$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}) + \mu(\mathbf{c}-\mathbf{a})$$

3. Figure 2.6(c) illustrates another description is in terms of the unit normal to the plane $\hat{\mathbf{n}}$ and a point **a** in the plane

   $$\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}}\ .$$

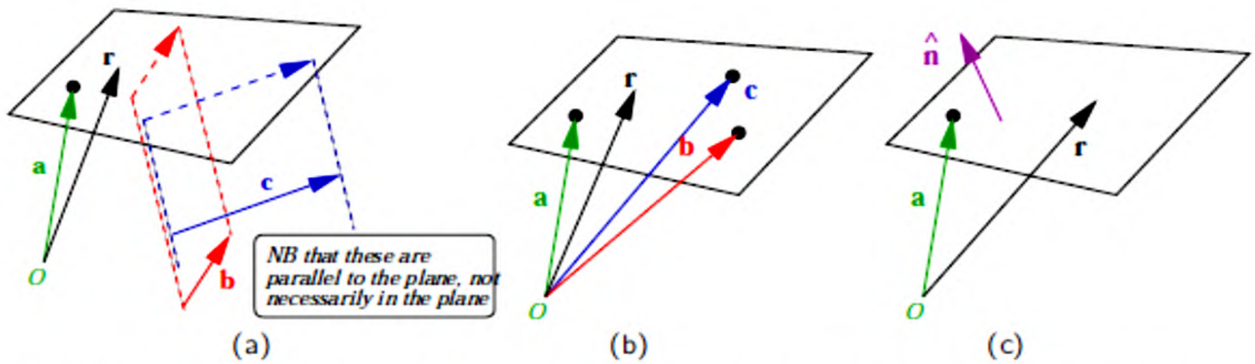


Figure 2.6: (a) Plane defined using point and two lines. (b) Plane defined using three points. (c) Plane defined using point and normal. Vector **r** is the position vector of a general point in the plane.

2.2.5 The shortest distance from a point to a plane

The shortest distance from a point **d** to the plane is along the perpendicular. Depending on how the plane is defined, this can be written as

$$D = |(\mathbf{d} - \mathbf{a}) \cdot \hat{\mathbf{n}}| \quad \text{or} \quad D = \frac{|(\mathbf{d} - \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{b} \times \mathbf{c}|} \ .$$

2.3 Solution of vector equations

It is sometimes required to obtain the most general vector which satisfies a given vector relationship. This is very much like obtaining the locus of a point. The best method of proceeding in such a case is as follows:

(i) Decide upon a system of three co-ordinate vectors using two non-parallel vectors appearing in the vector relationship. These might be **a**, **b** and their vector product $(\mathbf{a} \times \mathbf{b})$.

(ii) Express the unknown vector **x** as a linear combination of these vectors

$$\mathbf{x} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{a} \times \mathbf{b}$$

where λ, μ, ν are scalars to be found.

(iii) Substitute the above expression for **x** into the vector relationship to determine the constraints on λ, μ and ν for the relationship to be satisfied.

♣ Example

Q Solve the vector equation $\mathbf{x} = \mathbf{x} \times \mathbf{a} + \mathbf{b}$.

A Step (i): Set up basis vectors **a**, **b** and their vector product $\mathbf{a} \times \mathbf{b}$.
Step (ii): $\mathbf{x} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{a} \times \mathbf{b}$.

Step (iii): Bung this expression for **x** into the equation!

$$\begin{aligned}\lambda\mathbf{a}+\mu\mathbf{b}+\nu\mathbf{a}\times\mathbf{b} &= (\lambda\mathbf{a}+\mu\mathbf{b}+\nu\mathbf{a}\times\mathbf{b})\times\mathbf{a}+\mathbf{b}\\ &= \mathbf{0}+\mu(\mathbf{b}\times\mathbf{a})+\nu(\mathbf{a}\times\mathbf{b})\times\mathbf{a}+\mathbf{b}\\ &= -\nu(\mathbf{a}\cdot\mathbf{b})\mathbf{a}+(\nu a^2+1)\mathbf{b}-\mu(\mathbf{a}\times\mathbf{b})\end{aligned}$$

We have learned that any vector has a unique expression in terms of a basis set, so that the coefficients of **a**, **b** and **a** × **b** on either side of the equation much be equal.

$$\begin{aligned}\Rightarrow\lambda &= -\nu(\mathbf{a}\cdot\mathbf{b})\\ \mu &= \nu a^2+1\\ \nu &= -\mu\end{aligned}$$

so that

$$\mu=\frac{1}{1+a^2}\qquad \nu=-\frac{1}{1+a^2}\qquad \lambda=\frac{\mathbf{a}\cdot\mathbf{b}}{1+a^2}\ .$$

So finally the solution is the single point:

$$\mathbf{x}=\frac{1}{1+a^2}((\mathbf{a}\cdot\mathbf{b})\mathbf{a}+\mathbf{b}-(\mathbf{a}\times\mathbf{b}))$$

2.4 Rotation, angular velocity/acceleration and moments

A rotation can be represented by a vector whose direction is along the axis of rotation in the sense of a r-h screw, and whose magnitude is proportional to the size of the rotation (Fig. 2.7). The same idea can be extended to the derivatives, that is, angular velocity $\boldsymbol{\omega}$ and angular acceleration $\dot{\boldsymbol{\omega}}$.

Angular accelerations arise because of a moment (or torque) on a body. In mechanics, the moment of a force **F** about a point Q is defined to have magnitude $M = Fd$, where d is the perpendicular distance between Q and the line of action L of **F**.

The vector equation for moment is

$$\mathbf{M}=\mathbf{r}\times\mathbf{F}$$

where **r** is the vector from Q to *any* point on the line of action L of force **F**. The resulting angular acceleration vector is in the same direction as the moment vector.

The instantaneous velocity of any point P on a rigid body undergoing pure rotation can be defined by a vector product as follows. The angular velocity vector $\boldsymbol{\omega}$ has

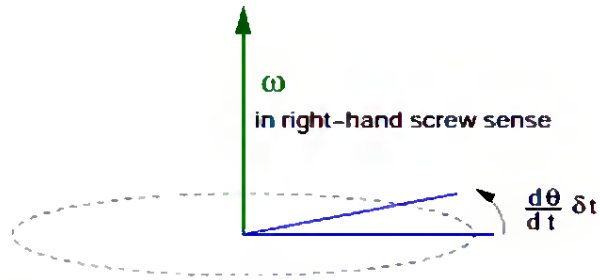


Figure 2.7: The angular velocity vector $\boldsymbol{\omega}$ is along the axis of rotation and has magnitude equal to the rate of rotation.

magnitude equal to the angular speed of rotation of the body and with direction the same as that of the r-h screw. If $\mathbf{r}$ is the vector OP, where the origin O can be taken to be any point on the axis of rotation, then the velocity $\mathbf{v}$ of P due to the rotation is given, in both magnitude and direction, by the vector product

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

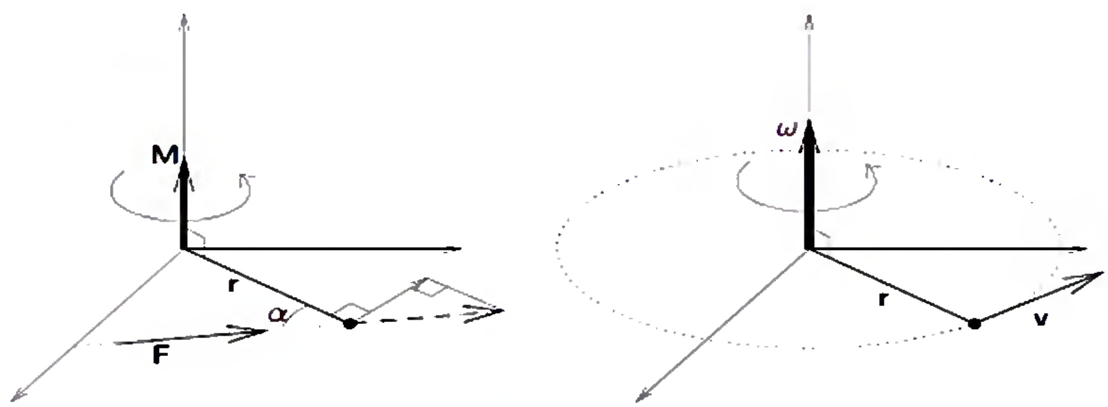


Figure 2.8:

Matrix Algebra

DEFINITION OF A MATRIX

A *matrix of order* $m \times n$, or m *by* n *matrix*, is a rectangular array of numbers having m rows and n columns. It can be written in the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \quad (1)$$

Each number a_{jk} in this matrix is called an *element*. The subscripts j and k indicate respectively the row and column of the matrix in which the element appears.

We shall often denote a matrix by a letter, such as A in (*1*), or by the symbol (a_{jk}) which shows a representative element.

A matrix having only one row is called a *row matrix* [or *row vector*] while a matrix having only one column is called a *column matrix* [or *column vector*]. If the number of rows m and columns n are equal the matrix is called a *square matrix* of order $n \times n$ or briefly n. A matrix is said to be a *real matrix* or *complex matrix* according as its elements are real or complex numbers.

SOME SPECIAL DEFINITIONS AND OPERATIONS INVOLVING MATRICES

1. **Equality of Matrices.** Two matrices $A = (a_{jk})$ and $B = (b_{jk})$ of the same order [i.e. equal numbers of rows and columns] are *equal* if and only if $a_{jk} = b_{jk}$.

2. **Addition of Matrices.** If $A = (a_{jk})$ and $B = (b_{jk})$ have the same order we define the *sum* of A and B as $A + B = (a_{jk} + b_{jk})$.

Example 1. If $A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -5 & 1 \\ 2 & 1 & 3 \end{pmatrix}$ then

$$A + B = \begin{pmatrix} 2+3 & 1-5 & 4+1 \\ -3+2 & 0+1 & 2+3 \end{pmatrix} = \begin{pmatrix} 5 & -4 & 5 \\ -1 & 1 & 5 \end{pmatrix}$$

Note that the commutative and associative laws for addition are satisfied by matrices, i.e. for any matrices A, B, C of the same order

$$A + B = B + A, \qquad A + (B + C) = (A + B) + C \quad (2)$$

3. **Subtraction of Matrices.** If $A = (a_{jk})$, $B = (b_{jk})$ have the same order, we define the *difference* of A and B as $A - B = (a_{jk} - b_{jk})$.

Example 2. If A and B are the matrices of Example 1, then

$$A - B = \begin{pmatrix} 2-3 & 1+5 & 4-1 \\ -3-2 & 0-1 & 2-3 \end{pmatrix} = \begin{pmatrix} -1 & 6 & 3 \\ -5 & -1 & -1 \end{pmatrix}$$

4. **Multiplication of a Matrix by a Number.** If $A = (a_{jk})$ and λ is any number [or *scalar*], we define the *product* of A by λ as $\lambda A = A\lambda = (\lambda a_{jk})$.

Example 3. If A is the matrix of Example 1 and $\lambda = 4$, then

$$\lambda A = 4\begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 4 & 16 \\ -12 & 0 & 8 \end{pmatrix}$$

5. **Multiplication of Matrices.** If $A = (a_{jk})$ is an $m \times n$ matrix while $B = (b_{jk})$ is an $n \times p$ matrix, then we define the *product* $A \cdot B$ or AB of A and B as the matrix $C = (c_{jk})$ where

$$c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk} \tag{3}$$

and where C is of order $m \times p$.

Note that matrix multiplication is defined if and only if the number of columns of A is the same as the number of rows of B. Such matrices are sometimes called *conformable*.

Example 4. Let $A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 4 & 2 \end{pmatrix}$. Then

$$AD = \begin{pmatrix} (2)(3)+(1)(2)+(4)(4) & (2)(5)+(1)(-1)+(4)(2) \\ (-3)(3)+(0)(2)+(2)(4) & (-3)(5)+(0)(-1)+(2)(2) \end{pmatrix} = \begin{pmatrix} 24 & 17 \\ -1 & -11 \end{pmatrix}$$

Note that in general $AB \neq BA$, i.e. the commutative law for multiplication of matrices is not satisfied in general. However, the associative and distributive laws are satisfied, i.e.

$$A(BC) = (AB)C, \quad A(B+C) = AB + AC, \quad (B+C)A = BA + CA \tag{4}$$

A matrix A can be multiplied by itself if and only if it is a square matrix. The product $A \cdot A$ can in such case be written A^2. Similarly we define powers of a square matrix, i.e. $A^3 = A \cdot A^2$, $A^4 = A \cdot A^3$, etc.

6. **Transpose of a Matrix.** If we interchange rows and columns of a matrix A, the resulting matrix is called the *transpose* of A and is denoted by A^T. In symbols, if $A = (a_{jk})$ then $A^T = (a_{kj})$.

Example 5. The transpose of $A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$ is

$$A^T = \begin{pmatrix} 2 & -3 \\ 1 & 0 \\ 4 & 2 \end{pmatrix}$$

We can prove that

$$(A+B)^T = A^T + B^T, \quad (AB)^T = B^T A^T, \quad (A^T)^T = A \tag{5}$$

7. **Symmetric and Skew-Symmetric Matrices.** A square matrix A is called *symmetric* if $A^T = A$ and *skew-symmetric* if $A^T = -A$.

Example 6. The matrix $E = \begin{pmatrix} 2 & -4 \\ -4 & 3 \end{pmatrix}$ is symmetric while $F = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ is skew-symmetric.

Any real square matrix [i.e. one having only real elements] can always be expressed as the sum of a real symmetric matrix and a real skew-symmetric matrix.

8. **Complex Conjugate of a Matrix.** If all elements a_{jk} of a matrix A are replaced by their complex conjugates $\bar{a}_{jk}$, the matrix obtained is called the *complex conjugate* of A and is denoted by $\bar{A}$.

9. **Hermitian and Skew-Hermitian Matrices.** A square matrix A which is the same as the complex conjugate of its transpose, i.e. if $A = \bar{A}^T$, is called *Hermitian*. If $A = -\bar{A}^T$, then A is called *skew-Hermitian*. If A is real these reduce to symmetric and skew-symmetric matrices respectively.

10. **Principal Diagonal and Trace of a Matrix.** If $A = (a_{jk})$ is a square matrix, then the diagonal which contains all elements a_{jk} for which $j = k$ is called the *principal* or *main diagonal* and the sum of all such elements is called the *trace* of A.

Example 7. The principal or main diagonal of the matrix

$$\begin{pmatrix} 5 & 2 & 0 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{pmatrix}$$

is indicated by the shading, and the trace of the matrix is $5 + 1 + 2 = 8$.

A matrix for which $a_{jk} = 0$ when $j \neq k$ is called a *diagonal matrix*.

11. **Unit Matrix.** A square matrix in which all elements of the principal diagonal are equal to 1 while all other elements are zero is called the *unit matrix* and is denoted by I. An important property of I is that

$$AI = IA = A, \quad I^n = I, \quad n = 1, 2, 3, \ldots \tag{6}$$

The unit matrix plays a role in matrix algebra similar to that played by the number one in ordinary algebra.

12. **Zero or Null Matrix.** A matrix whose elements are all equal to zero is called the *null* or *zero matrix* and is often denoted by O or simply 0. For any matrix A having the same order as 0 we have

$$A + 0 = 0 + A = A \tag{7}$$

Also if A and 0 are square matrices, then

$$A0 = 0A = 0 \tag{8}$$

The zero matrix plays a role in matrix algebra similar to that played by the number zero of ordinary algebra.

DETERMINANTS

If the matrix A in (*1*) is a square matrix, then we associate with A a number denoted by

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \tag{9}$$

called the *determinant* of A of *order n*, written $\det(A)$. In order to define the value of a determinant, we introduce the following concepts.

1. **Minor.** Given any element a_{jk} of Δ we associate a new determinant of order $(n-1)$ obtained by removing all elements of the jth row and kth column called the minor of a_{jk}.

Example 8. The minor corresponding to the element 5 in the 2nd row and 3rd column of the fourth order determinant

$$\begin{vmatrix} 2 & -1 & 1 & 3 \\ -3 & 2 & 5 & 0 \\ 1 & 0 & -2 & 2 \\ 4 & -2 & 3 & 1 \end{vmatrix} \quad \text{is} \quad \begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & -2 & 1 \end{vmatrix}$$

which is obtained by removing the elements shown shaded.

2. **Cofactor.** If we multiply the minor of a_{jk} by $(-1)^{j+k}$, the result is called the *cofactor* of a_{jk} and is denoted by A_{jk}.

Example 9. The cofactor corresponding to the element 5 in the determinant of Example 8 is $(-1)^{2+3}$ times its minor, or

$$-\begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & -2 & 1 \end{vmatrix}$$

The value of a determinant is then defined as the sum of the products of the elements in any row [or column] by their corresponding cofactors and is called the *Laplace expansion*. In symbols,

$$\det A = \sum_{k=1}^{n} a_{jk} A_{jk} \qquad (10)$$

Theorem of Determinants

Theorem 1:- The value of a determinant remains the same if rows and columns are interchanged. In symbols, det *(A)* = det (A^T).

Theorem 2: - If all elements of any row [or column] are zero except for one element, then the value of the determinant is equal to the product of that element by its cofactor. In particular, if all elements of a row [or column] are zero the determinant is zero.

Theorem 3: - An interchange of any two rows [or columns] changes the sign of the determinant.

Theorem *-4.* If all elements in any row [or column] are multiplied by a number, the determinant is also multiplied by this number.

Theorem *-5.* If any two rows [or columns] are the same or proportional, the determinant is zero.

Theorem *-6.* If we express the elements of each row [or column] as the sum of two terms, then the determinant can be expressed as the sum of two determinants having the same order.

Theorem *-7.* If we multiply the elements of any row [or column] by a given number and add to corresponding elements of any other row [or column], then the value of the determinant remains the same.

Theorem *-8.* If *A* and *B* are square matrices of the same order, then

$$\det(AB) = \det(A)\det(B) \qquad (11)$$

Theorem -9. The sum of the products of the elements of any row [or column] by the cofactors of another row [or column] is zero. In symbols,

$$\sum_{k=1}^{n} a_{qk} A_{pk} = 0 \quad \text{or} \quad \sum_{k=1}^{n} a_{kq} A_{kp} = 0 \quad \text{if } p \neq q \qquad (12)$$

If $p = q$, the sum is det(A) by *(10)*.

Theorem -10. Let $v_1, v_2, \ldots, v_n$ represent row vectors [or column vectors] of a square matrix *A* of order *n*. Then $det\ (A) = 0$ if and only if there exist constants [scalars] $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all zero such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = O \quad (13)$$

where O is the null or zero row matrix. If condition *(13)* is satisfied we say that the vectors $v_1, v_2, \ldots, v_n$ are *linearly dependent*. Otherwise they are *linearly independent*. A matrix A such that $\det(A) = 0$ is called a *singular matrix*. If $\det(A) \neq 0$, then A is a *non-singular matrix*.

INVERSE OF A MATRIX

If for a given square matrix A there exists a matrix B such that $AB = I$, then B is called an ***inverse* of** A and is denoted by A^{-1}. The following theorem is fundamental.

Theorem 11. If A is a non-singular square matrix of order n [i.e. $det(A) \neq 0$], then there exists a unique inverse A^{-1}. such that $AA^{-1} = A^{-1}A = I$ and we can express A^{-1}, in the following form

$$A^{-1} = \frac{(A_{jk})^T}{\det(A)} \quad (14)$$

where (A_{jk}) is the matrix of cofactors A_{jk} and $(A_{jk})^T = (A_{kj})$ is its transpose.

The following express some properties of the inverse:

$$(AB)^{-1} = B^{-1}A^{-1}, \quad (A^{-1})^{-1} = A \quad (15)$$

ORTHOGONAL AND UNITARY MATRICES

A real matrix A is called an *orthogonal matrix* if its transpose is the same as its inverse, i.e. if $A^T = A^{-1}$ or $A^T A = I$.

A complex matrix A is called a *unitary matrix* if its complex conjugate transpose is the same as its inverse, i.e. if $\bar{A}^T = A^{-1}$ or $\bar{A}^T A = I$. It should be noted that a real unitary matrix is an orthogonal matrix.

ORTHOGONAL VECTORS

In Chapter 5 we found that the scalar or dot product of two vectors $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is $a_1b_1 + a_2b_2 + a_3b_3$ and that the vectors are perpendicular or orthogonal if $a_1b_1 + a_2b_2 + a_3b_3 = 0$. From the point of view of matrices we can consider these vectors as column vectors

$$A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

from which it follows that $\qquad A^T B = a_1 b_1 + a_2 b_2 + a_3 b_3$

This leads us to define the *scalar product of real column vectors* A and B as $A^T B$ and to define A and B to be *orthogonal* if $A^T B = 0$.

It is convenient to generalize this to cases where the vectors can have complex components and we adopt the following definition:

Definition 1. Two column vectors A and B are called *orthogonal* if $\bar{A}^T B = 0$, and $\bar{A}^T B$ is called the *scalar product* of A and B.

It should be noted also that if A is a unitary matrix then $\bar{A}^T A = 1$, which means that the scalar product of A with itself is 1 or equivalently A is a *unit vector*, i.e. having length 1. Thus a unitary column vector is a unit vector. Because of these remarks we have the following

Definition 2. A set of vectors $X_1, X_2, \ldots$ for which

$$\bar{X}_j^T X_k = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

is called a *unitary set or system of vectors* or, in the case where the vectors are real, an *orthonormal set* or an *orthogonal set of unit vectors*.

SYSTEMS OF LINEAR EQUATIONS

A set of equations having the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = r_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = r_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = r_n \end{array}\right\} \qquad (16)$$

is called a *system of m linear equations in the n unknowns* $x_1, x_2, \ldots, x_n$. If $r_1, r_2, \ldots, r_n$ are all zero the system is called *homogeneous*. If they are not all zero it is called *non-homogeneous*. Any set of numbers $x_1, x_2, \ldots, x_n$ which satisfies (*16*) is called a *solution* of the system.

In matrix form (*16*) can be written

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \qquad (17)$$

or more briefly

$$AX = R \qquad (18)$$

where A, X, R represent the corresponding matrices in (*17*).

SYSTEMS OF n EQUATIONS IN n UNKNOWNS. CRAMER'S RULE

If $m = n$ and if A is a non-singular matrix so that A^{-1} exists, we can solve (*17*) or (*18*) by writing

$$X = A^{-1}R \qquad (19)$$

and the system has a unique solution.

Alternatively we can express the unknowns $x_1, x_2, \ldots, x_n$ as

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad \ldots, \quad x_n = \frac{\Delta_n}{\Delta} \tag{20}$$

where $\Delta = \det(A)$, called the *determinant of the system*, is given by (*9*) and Δ_k, $k = 1, 2, \ldots, n$ is the determinant obtained from Δ by removing the kth column and replacing it by the column vector R. The rule expressed in (*20*) is called *Cramer's rule.*

The following four cases can arise.

Case 1, $\Delta \neq 0$, $R \neq 0$. In this case there will be a unique solution where not all x_k will be zero.

Case 2, $\Delta \neq 0$, $R = 0$. In this case the only solution will be $x_1 = 0, x_2 = 0, \ldots, x_n = 0$, i.e. $X = 0$. This is often called the *trivial solution.*

Case 3, $\Delta = 0$, $R = 0$. In this case there will be infinitely many solutions other than the trivial solution. This means that at least one of the equations can be obtained from the others, i.e. the equations are linearly dependent.

Case 4, $\Delta = 0$, $R \neq 0$. In this case infinitely many solutions will exist if and only if all of the determinants Δ_k in (*20*) are zero. Otherwise there will be no solution.

EIGENVALUES AND EIGENVECTORS

Let $A = (a_{jk})$ be an $n \times n$ matrix and X a column vector. The equation

$$AX = \lambda X \tag{21}$$

where λ is a number can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{22}$$

or

$$\left.\begin{array}{rcl} (a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n & = & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n & = & 0 \end{array}\right\} \tag{23}$$

The equation (*23*) will have non-trivial solutions if and only if

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix} = 0 \tag{24}$$

which is a polynomial equation of degree n in λ. The roots of this polynomial equation are called *eigenvalues* or *characteristic values* of the matrix A. Corresponding to each eigenvalue there will be a solution $X \neq 0$, i.e. a non-trivial solution, which is called an *eigenvector* or *characteristic vector* belonging to the eigenvalue. The equation (*24*) can also be written

$$\det(A - \lambda I) = 0 \tag{25}$$

and the equation in λ is often called the *characteristic equation.*

THEOREMS ON EIGENVALUES AND EIGENVECTORS

Theorem -12. The eigenvalues of a Hermitian matrix [or symmetric real matrix] are real. The eigenvalues of a skew-Hermitian matrix [or skew-symmetric real matrix] are zero or pure imaginary. The eigenvalues of a unitary [or real orthogonal matrix] all have absolute value equal to 1.

Theorem -13. The eigenvectors belonging to different eigenvalues of a Hermitian matrix [or symmetric real matrix] are orthogonal.

***Theorem -14* [Cayley-Hamilton].** A matrix satisfies its own characteristic equation

***Theorem -15* [Reduction of matrix to diagonal form].** If a non-singular matrix A has distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$ with corresponding eigenvectors written as columns in the matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \ldots \\ b_{21} & b_{22} & b_{23} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

then

$$B^{-1}AB = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots \\ 0 & \lambda_2 & 0 & \ldots \\ 0 & 0 & \lambda_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

i.e. $B^{-1}AB$, called the *transform* of A by B, is a diagonal matrix containing the eigenvalues of A in the main diagonal and zeros elsewhere. We say that A has been *transformed* or *reduced to diagonal form.*

***Theorem -16* [Reduction of quadratic form to canonical form].** Let A be a symmetric real matrix, for example,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad a_{12} = a_{21}, \ a_{13} = a_{31}, \ a_{23} = a_{32}$$

Then if $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, **we obtain the *quadratic form***

$$X^T A X = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}^2 x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$$

The cross-product terms of this quadratic form can be removed by letting $X = BU$ where U is the column vector with elements u_1, u_2, u_3 and B is an orthogonal matrix which diagonalizes A. The new quadratic form in u_1, u_2, u_3 with no cross-product terms is called the *canonical form.* A generalization can be made to Hermitian quadratic forms.

OPERATOR INTERPRETATION OF MATRICES

If A is an n x n matrix, we can think of it as an *operator* or *transformation* acting on a column vector X to produce AX which is another column vector. With this interpretation equation *(21)* asks for those vectors X which are transformed by A into constant multiples of themselves [or equivalently into vectors which have the same direction but possibly different magnitude].

If case A is an orthogonal matrix, the transformation is a *rotation* and explains why the absolute value of all the eigenvalues in such case are equal to one [***Theorem -12***], since an ordinary rotation of a vector would not change its magnitude.

The ideas of transformation are very convenient in giving interpretations to many properties of matrices.

Vector Calculus

2. Differentiating Vector Functions of a Single Variable

Your experience of differentiation and integration has extended as far as *scalar* functions of single and multiple variables — functions like $f\ (x)$ and $f\ (x, y\ , t)$. It should be no great surprise that we often wish to differentiate vector functions. For example, suppose you were driving along a wiggly road with position $r(t)$ at time t. Differentiating $r(t)$ w.r.t time should yield your velocity $v(t)$, and differentiating $v(t)$ should yield your acceleration. Let's see how to do this.

a. Differentiation of a vector

The derivative of a vector function a(p) of a single parameter p is

$$\mathbf{a}'(p) = \lim_{\delta p \to 0} \frac{\mathbf{a}(p + \delta p) - \mathbf{a}(p)}{\delta p} \ .$$

If we write **a** in terms of components relative to a FIXED coordinate system ($\hat{\imath}, \hat{\jmath}, \hat{k}$ constant)

$$\mathbf{a}(p) = a_1(p)\hat{\imath} + a_2(p)\hat{\jmath} + a_3(p)\hat{k}$$

then

$$\mathbf{a}'(p) = \frac{da_1}{dp}\hat{\imath} + \frac{da_2}{dp}\hat{\jmath} + \frac{da_3}{dp}\hat{k} \ .$$

That is, in order to differentiate a vector function, one simply differentiates each component separately. This means that all the familiar rules of differentiation apply, and they don't get altered by vector operations like scalar product and vector products.

Thus, for example:

$$\frac{d}{dp}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dp} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dp} \qquad \frac{d}{dp}(\mathbf{a} \cdot \mathbf{b}) = \frac{d\mathbf{a}}{dp} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dp} \ .$$

Note that da/dp has a different direction and a different magnitude from a. Likewise, as you might expect, the chain rule still applies. If $a\ =\ \boldsymbol{a}(u)$ and $u\ =\ u(t)$, say:

$$\frac{d}{dt}\boldsymbol{a} = \frac{d\boldsymbol{a}}{du}\frac{du}{dt}$$

♣ **Examples**

Q A 3D vector **a** of constant magnitude is varying over time. What can you say about the direction of $\dot{\mathbf{a}}$?

A Using intuition: if only the direction is changing, then the vector must be tracing out points on the surface of a sphere. We would guess that the derivative $\dot{\mathbf{a}}$ is orthogonal to **a**.

To prove this write

$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{a}) = \mathbf{a}\cdot\frac{d\mathbf{a}}{dt} + \frac{d\mathbf{a}}{dt}\cdot\mathbf{a} = 2\mathbf{a}\cdot\frac{d\mathbf{a}}{dt} \quad .$$

But $(\mathbf{a}\cdot\mathbf{a}) = a^2$ which we are told is constant. So

$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{a}) = 0 \qquad \Rightarrow 2\mathbf{a}\cdot\frac{d\mathbf{a}}{dt} = 0$$

and hence **a** and $d\mathbf{a}/dt$ must be perpendicular.

Q The position of a vehicle is $\mathbf{r}(u)$ where u is the amount of fuel consumed by some time t. Write down an expression for the acceleration.

A The velocity is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{du}\frac{du}{dt}$$

$$\mathbf{a} = \frac{d}{dt}\frac{d\mathbf{r}}{dt} = \frac{d^2\mathbf{r}}{du^2}\left(\frac{du}{dt}\right)^2 + \frac{d\mathbf{r}}{du}\frac{d^2u}{dt^2}$$

i. Geometrical interpretation of vector derivatives

Let $\boldsymbol{r}(\boldsymbol{p})$ be a position vector tracing a space curve as some parameter p varies. The vector $\boldsymbol{\delta r}$ is a secant to the curve, and $\boldsymbol{\delta r}/\boldsymbol{\delta p}$ lies in the same direction. (See Fig. 3.1.) In the limit as δp tends to zero $\boldsymbol{\delta r}/\boldsymbol{\delta p} = \boldsymbol{dr}/\boldsymbol{dp}$ becomes a tangent to the space curves. If the magnitude of this vector is 1 (i.e. a unit tangent), then
$|dr| = dp$ so, the parameter p is arc-length (metric distance). More generally, however, p will not be arc-length and we will have:

$$\frac{d\boldsymbol{r}}{dp} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dp}$$

So, the direction of the derivative is that of a tangent to the curve, and its magnitude is $|\boldsymbol{ds}/\boldsymbol{dp}|$, the rate of change of arc length w.r.t the parameter. Of course, if that parameter p is time, the magnitude $|\boldsymbol{dr}/\boldsymbol{dt}|$ is the speed.

♣ **Example**

Q Draw the curve

$$\mathbf{r} = a\cos\left(\frac{s}{\sqrt{a^2+h^2}}\right)\hat{\imath} \;+\; a\sin\left(\frac{s}{\sqrt{a^2+h^2}}\right)\hat{\jmath} \;+\; \frac{hs}{\sqrt{a^2+h^2}}\hat{\boldsymbol{k}}$$

where s is arc length and h, a are constants. Show that the tangent $d\mathbf{r}/ds$ to the curve has a constant elevation angle w.r.t the xy-plane, and determine its magnitude.

A

$$\frac{d\mathbf{r}}{ds} = -\frac{a}{\sqrt{a^2+h^2}}\sin()\,\hat{\imath} + \frac{a}{\sqrt{a^2+h^2}}\cos()\,\hat{\jmath} + \frac{h}{\sqrt{a^2+h^2}}\hat{\boldsymbol{k}}$$

The projection on the xy plane has magnitude $a/\sqrt{a^2+h^2}$ and in the z direction $h/\sqrt{a^2+h^2}$, so the elevation angle is a constant, $\tan^{-1}(h/a)$.
We are expecting $d\mathbf{r}/ds = 1$, and indeed

$$\sqrt{a^2\sin^2() + a^2\cos^2() + h^2}/\sqrt{a^2+h^2} = 1.$$

---

ii. Arc length is a special parameter!

It might seem that we can be completely relaxed about saying that any old parameter p is arc length, but this is not the case. Why not? The reason is that arc length is special is that, whatever the parameter p,

$$s = \int_{p_0}^{p} \left|\frac{d\boldsymbol{r}}{dp}\right| dp$$

Perhaps another way to grasp the significance of this is using Pythagoras' theorem on a short piece of curve: in the limit as dx etc. tend to zero,

$$ds^2 = dx^2 + dy^2 + dz^2 .$$

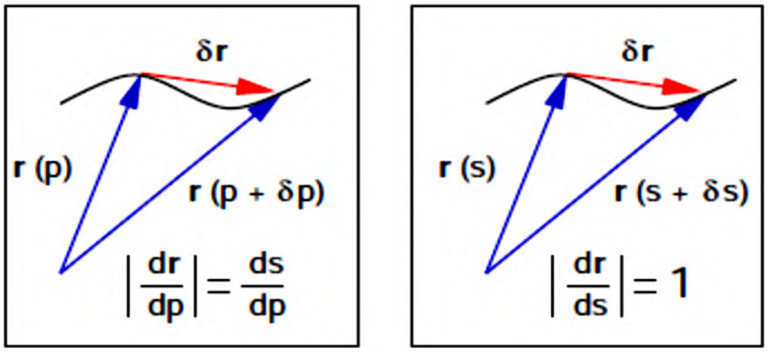


Figure 3.1: Left: $\delta\mathbf{r}$ is a secant to the curve but, in the limit as $\delta p \to 0$, becomes a tangent. Right: if the parameter is arc length s, then $|d\mathbf{r}| = ds$.

So if a curve is parameterized in terms of p

$$\frac{ds}{dp} = \sqrt{\frac{dx}{dp}^2 + \frac{dy}{dp}^2 + \frac{dz}{dp}^2} \ .$$

As an example, suppose in our earlier example we had parameterized our helix as

$$\mathbf{r} = a\cos p\hat{\imath} \quad + \quad a\sin p\hat{\jmath} \quad + \quad hp\hat{\mathbf{k}}$$

It would be easy just to *say* that p was arclength, but it would not be correct because

$$\begin{aligned}\frac{ds}{dp} &= \sqrt{\frac{dx}{dp}^2 + \frac{dy}{dp}^2 + \frac{dz}{dp}^2} \\ &= \sqrt{a^2\sin^2 p + a^2\cos^2 p + h^2} = \sqrt{a^2 + h^2}\end{aligned}$$

If p really was arclength, $ds/dp = 1$. So $p/\sqrt{a^2 + h^2}$ **is** arclength, not p.

Integration of a vector function

The integration of a vector function of a single scalar variable can be regarded simply as the reverse of differentiation. In other words

$$\int_{p_1}^{p_2} \frac{d\mathbf{a}(p)}{dp} dp$$

For example the integral of the acceleration vector of a point over an interval of time is equal to the change in the velocity vector during the same time interval. However, many other, more interesting and useful, types of integral are possible, especially when the vector is a function of more than one variable. This requires the introduction of the concepts of scalar and vector fields. See later!

Curves in 3 dimensions

In the examples above, parameter p has been either arc length s or time t. It doesn't have to be, but these are the main two of interest. Later we shall look at some important results when differentiating w.r.t. time, but now let use look more closely at 3D curves defined in terms of arc length, s.

Take a piece of wire, and bend it into some arbitrary non-planar curve. This is a *space curve*. We can specify a point on the wire by specifying $\mathbf{r}(s)$ as a function of distance or arc length s along the wire.

3.3.1 The Frénet-Serret relationships

We are now going to introduce a local orthogonal coordinate frame for each point s along the curve, ie one with its origin at $\mathbf{r}(s)$. To specify a coordinate frame we need three mutually perpendicular directions, and these should be *intrinsic* to the curve, not fixed in an external reference frame. The ideas were first suggested by two French mathematicians, F-J. Frénet and J. A. Serret.

1. **Tangent $\hat{\mathbf{t}}$**

   There is an obvious choice for the first direction at the point $\mathbf{r}(s)$, namely the **unit tangent $\hat{\mathbf{t}}$**. We already know that

   $$\hat{\mathbf{t}} = \frac{d\mathbf{r}(s)}{ds}$$

2. **Principal Normal $\hat{\mathbf{n}}$**

   Recall that earlier we proved that if $\mathbf{a}$ was a vector of constant magnitude that varies in direction over time then $d\mathbf{a}/dt$ was perpendicular to it. Because $\hat{\mathbf{t}}$ has constant magnitude but varies over s, $d\hat{\mathbf{t}}/ds$ must be perpendicular to $\hat{\mathbf{t}}$.

   Hence the principal normal $\hat{\mathbf{n}}$ is

   $$\frac{d\hat{\mathbf{t}}}{ds} = \kappa\hat{\mathbf{n}} : \text{ where } \kappa \geq 0 \ .$$

   κ is the **curvature**, and $\kappa = 0$ for a straight line. The plane containing $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ is called the **osculating plane**.

3. **The Binormal $\hat{\mathbf{b}}$**

   The local coordinate frame is completed by defining the binormal

   $$\hat{\mathbf{b}}(s) = \hat{\mathbf{t}}(s) \times \hat{\mathbf{n}}(s) \ .$$

Since $\hat{\mathbf{b}} \cdot \hat{\mathbf{t}} = 0$,

$$\frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{t}}}{ds} = \frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \kappa\hat{\mathbf{n}} = 0$$

from which

$$\frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} = 0.$$

But this means that $d\hat{\mathbf{b}}/ds$ is along the direction of $\hat{\mathbf{n}}$, or

$$\frac{d\hat{\mathbf{b}}}{ds} = -\tau(s)\hat{\mathbf{n}}(s)$$

where τ is the **torsion**, and the negative sign is a matter of convention.

Differentiating $\hat{\mathbf{n}} \cdot \hat{\mathbf{t}} = 0$ and $\hat{\mathbf{n}} \cdot \hat{\mathbf{b}} = 0$, we find

$$\frac{d\hat{\mathbf{n}}}{ds} = -\kappa(s)\hat{\mathbf{t}}(s) + \tau(s)\hat{\mathbf{b}}(s).$$

The Frénet-Serret relationships:

$$\begin{aligned} d\hat{\mathbf{t}}/ds &= \kappa\hat{\mathbf{n}} \\ d\hat{\mathbf{n}}/ds &= -\kappa(s)\hat{\mathbf{t}}(s) + \tau(s)\hat{\mathbf{b}}(s) \\ d\hat{\mathbf{b}}/ds &= -\tau(s)\hat{\mathbf{n}}(s) \end{aligned}$$

♣ Example

Q Derive $\kappa(s)$ and $\tau(s)$ for the helix

$$\mathbf{r}(s) = a\cos\left(\frac{s}{\beta}\right)\hat{\boldsymbol{\imath}} + a\sin\left(\frac{s}{\beta}\right)\hat{\boldsymbol{\jmath}} + h\left(\frac{s}{\beta}\right)\hat{\boldsymbol{k}}; \qquad \beta = \sqrt{a^2 + h^2}$$

and comment on their values.

A We found the unit tangent earlier as

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \left[-\frac{a}{\beta}\sin\left(\frac{s}{\beta}\right), \ \frac{a}{\beta}\cos\left(\frac{s}{\beta}\right), \ \frac{h}{\beta}\right].$$

Differentiation gives

$$\kappa\hat{\mathbf{n}} = \frac{d\hat{\mathbf{t}}}{ds} = \left[-\frac{a}{\beta^2}\cos\left(\frac{s}{\beta}\right), \ -\frac{a}{\beta^2}\sin\left(\frac{s}{\beta}\right), \ 0\right]$$

Curvature is always positive, so

$$\kappa = \frac{a}{\beta^2} \qquad \hat{\mathbf{n}} = \left[-\cos\left(\frac{s}{\beta}\right), \quad -\sin\left(\frac{s}{\beta}\right), \quad 0 \right] .$$

So the curvature is constant, and the normal is parallel to the xy-plane. Now use

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (-a/\beta)S & (a/\beta)C & (h/\beta) \\ -C & -S & 0 \end{vmatrix} = \left[\frac{h}{\beta}\sin\left(\frac{s}{\beta}\right), \quad -\frac{h}{\beta}\cos\left(\frac{s}{\beta}\right), \quad \frac{a}{\beta} \right]$$

and differentiate $\hat{\mathbf{b}}$ to find an expression for the torsion

$$\frac{d\hat{\mathbf{b}}}{ds} = \left[\frac{h}{\beta^2}\cos\left(\frac{s}{\beta}\right), \quad \frac{h}{\beta^2}\sin\left(\frac{s}{\beta}\right), \quad 0 \right] = \frac{-h}{\beta^2}\hat{\mathbf{n}}$$

so the torsion is

$$\tau = \frac{h}{\beta^2}$$

again a constant.

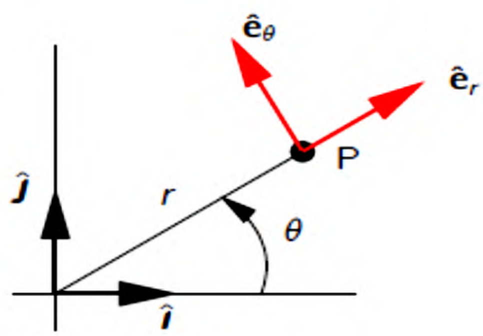
3.4 Radial and tangential components in plane polars

In plane polar coordinates, the radius vector of any point P is given by

$$\begin{aligned} \mathbf{r} &= r\cos\theta\hat{\imath} + r\sin\theta\hat{\jmath} \\ &= r\hat{\mathbf{e}}_r \end{aligned}$$

where we have introduced the unit radial vector

$$\hat{\mathbf{e}}_r = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} \quad .$$



The other "natural" (we'll see why in a later lecture) unit vector in plane polars is orthogonal to $\hat{\mathbf{e}}_r$ and is

$$\hat{\mathbf{e}}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$$

so that $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_\theta = 1$ and $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\theta = 0$.

Now suppose P is moving so that $\mathbf{r}$ is a function of time t. Its velocity is

$$\begin{aligned}\dot{\mathbf{r}} = \frac{d}{dt}(r\hat{\mathbf{e}}_r) &= \frac{dr}{dt}\hat{\mathbf{e}}_r + r\frac{d\hat{\mathbf{e}}_r}{dt} \\ &= \frac{dr}{dt}\hat{\mathbf{e}}_r + r\frac{d\theta}{dt}(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) \\ &= \frac{dr}{dt}\hat{\mathbf{e}}_r + r\frac{d\theta}{dt}\hat{\mathbf{e}}_\theta \\ &= \text{radial} + \text{tangential}\end{aligned}$$

The radial and tangential components of velocity of P are therefore dr/dt and $rd\theta/dt$, respectively.

Differentiating a second time gives the acceleration of P

$$\begin{aligned}\ddot{\mathbf{r}} &= \frac{d^2r}{dt^2}\hat{\mathbf{e}}_r + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\mathbf{e}}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\mathbf{e}}_\theta + r\frac{d^2\theta}{dt^2}\hat{\mathbf{e}}_\theta - r\frac{d\theta}{dt}\frac{d\theta}{dt}\hat{\mathbf{e}}_r \\ &= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{e}}_r + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\hat{\mathbf{e}}_\theta\end{aligned}$$

3.5. Rotating systems

Consider a body which is rotating with constant angular velocity $\boldsymbol{\omega}$ about some axis passing through the origin. Assume the origin is fixed, and that we are sitting in a fixed coordinate system $Oxyz$.

If ρ is a vector of constant magnitude and constant direction in the rotating system, then its representation r in the fixed system must be a function of t.

$$r(t) = R(t)\rho$$

At any instant as observed in the fixed system

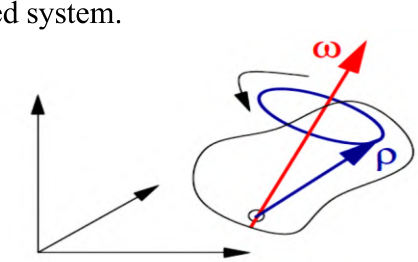
$$\frac{d\boldsymbol{r}}{dt} = \dot{R}\rho + R\dot{\rho}$$

but the second term is zero since we assumed ρ to be constant so we have

$$\frac{d\boldsymbol{r}}{dt} = \dot{R}R^T\boldsymbol{r}$$

Note that:

- dr/dt will have fixed magnitude;
- dr/dt will always be perpendicular to the axis of rotation;
- dr/dt will vary in direction within those constraints;
- $r(t)$ will move in a plane in the fixed system.



Now let's consider the term $\dot{R}R^{\top}$. First, note that $RR^{\top} = I$ (the identity), so differentiating both sides yields

$$\begin{aligned} \dot{R}R^{\top} + R\dot{R}^{\top} &= 0 \\ \dot{R}R^{\top} &= -R\dot{R}^{\top} \end{aligned}$$

Thus $\dot{R}R^{\top}$ is anti-symmetric:

$$\dot{R}R^{\top} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

Now you can verify for yourself that application of a matrix of this form to an arbitrary vector has precisely the same effect as the cross product operator, $\boldsymbol{\omega}\times$, where $\boldsymbol{\omega} = [xyz]^{\top}$. Loh-and-behold, we then we have

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$$

matching the equation at the end of lecture 2, $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, as we would hope/expect.

3.5.1 Rotation: Part 2

Now suppose $\boldsymbol{\rho}$ is the position vector of a point P which **moves** in the rotating frame. There will be two contributions to motion with respect to the fixed frame, one due to its motion within the rotating frame, and one due to the rotation itself. So, returning to the equations we derived earlier:

$$\mathbf{r}(t) = R(t)\boldsymbol{\rho}(t)$$

and the instantaenous differential with respect to time:

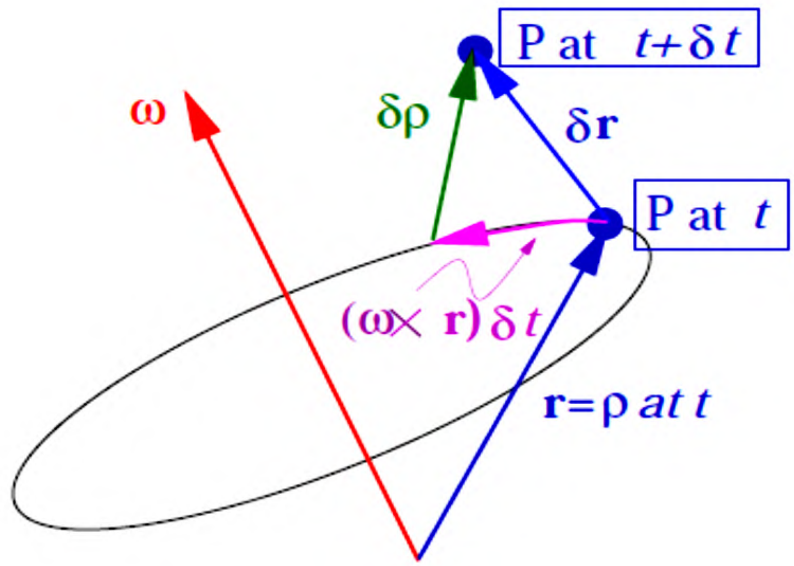
$$\frac{d\mathbf{r}}{dt} = \dot{R}\boldsymbol{\rho} + R\dot{\boldsymbol{\rho}} = \dot{R}R^{\top}\mathbf{r} + R\dot{\boldsymbol{\rho}}$$

Now $\boldsymbol{\rho}$ is not constant, so its differential is not zero; hence rewriting this last equations we have that

The **instantaneous velocity** of P in the fixed frame is

$$\frac{d\mathbf{r}}{dt} = R\dot{\boldsymbol{\rho}} + \boldsymbol{\omega} \times \mathbf{r}$$

The second term of course, is the contribution from the rotating frame which we saw previously. The first is the linear velocity measured in the rotating frame $\dot{\boldsymbol{\rho}}$, referred to the fixed frame (via the rotation matrix R which aligns the two frames)



3.5.2 Rotation 3: Instantaneous acceleration

Our previous result is a general one relating the time derivatives of any vector in rotating and non-rotating frames. Let us now consider the second differential:

$$\ddot{\mathbf{r}} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\mathsf{R}}\dot{\boldsymbol{\rho}} + \mathsf{R}\ddot{\boldsymbol{\rho}}$$

We shall assume that the angular acceleration is zero, which kills off the first term, and so now, substituting for $\dot{\mathbf{r}}$ we have

$$\begin{aligned}
\ddot{\mathbf{r}} &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r} + \mathsf{R}\dot{\boldsymbol{\rho}}) + \dot{\mathsf{R}}\dot{\boldsymbol{\rho}} + \mathsf{R}\ddot{\boldsymbol{\rho}} \\
&= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times \mathsf{R}\dot{\boldsymbol{\rho}} + \dot{\mathsf{R}}\dot{\boldsymbol{\rho}} + \mathsf{R}\ddot{\boldsymbol{\rho}} \\
&= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times \mathsf{R}\dot{\boldsymbol{\rho}} + \dot{\mathsf{R}}(\mathsf{R}^{\mathsf{T}}\mathsf{R})\dot{\boldsymbol{\rho}} + \mathsf{R}\ddot{\boldsymbol{\rho}} \\
&= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times (\mathsf{R}\dot{\boldsymbol{\rho}}) + \mathsf{R}\ddot{\boldsymbol{\rho}}
\end{aligned}$$

The **instantaneous acceleration** is therefore

$$\ddot{\mathbf{r}} = \mathsf{R}\ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega} \times (\mathsf{R}\dot{\boldsymbol{\rho}}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

- The first term is the acceleration of the point P in the rotating frame measured in the rotating frame, but referred to the fixed frame by the rotation R
- The last term is the centripetal acceleration to due to the rotation. (Yes! Its magnitude is $\omega^2 r$ and its direction is that of $-\mathbf{r}$. Check it out.)

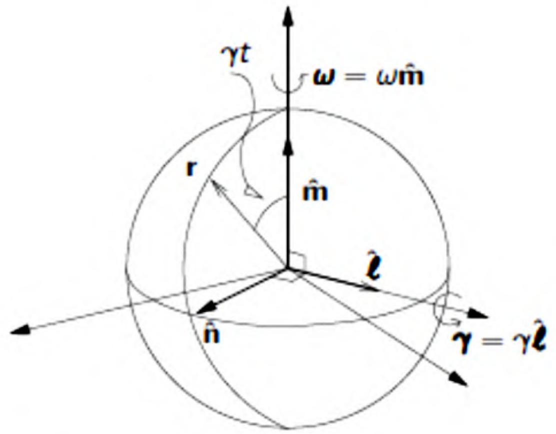


Figure 3.2: Coriolis example.

- The middle term is an extra term which arises because of the velocity of P in the rotating frame. It is known as the **Coriolis acceleration**, named after the French engineer who first identified it.

Because of the rotation of the earth, the Coriolis acceleration is of great importance in meteorology and accounts for the occurrence of high pressure anti-cyclones and low pressure cyclones in the northern hemisphere, in which the Coriolis acceleration is produced by a pressure gradient. It is also a very important component of the acceleration (hence the force exerted) by a rapidly moving robot arm, whose links whirl rapidly about rotary joints.

♣ Example

Q Find the instantaneous acceleration of a projectile fired along a line of longitude (with angular velocity of $\boldsymbol{\gamma}$ constant relative to the sphere) if the sphere is rotating with angular velocity $\boldsymbol{\omega}$.

A Consider a coordinate frame defined by mutually orthogonal unit vectors, $\hat{\boldsymbol{\ell}}$, $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$, as shown in Fig. 3.2. We shall assume, without loss of generality, that the fixed and rotating frames are instantaneously aligned at the moment shown in the diagram, so that $\mathsf{R} = \mathsf{I}$, the identity, and hence $\mathbf{r} = \boldsymbol{\rho}$.

In the rotating frame

$$\dot{\boldsymbol{\rho}} = \boldsymbol{\gamma} \times \boldsymbol{\rho} \qquad \text{and} \qquad \ddot{\boldsymbol{\rho}} = \boldsymbol{\gamma} \times \dot{\boldsymbol{\rho}} = \boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \boldsymbol{\rho})$$

So the in the fixed reference frame, because these two frames are instantaneously aligned

$$\ddot{\mathbf{r}} = \boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \boldsymbol{\rho}) + 2\boldsymbol{\omega} \times (\boldsymbol{\gamma} \times \boldsymbol{\rho}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) .$$

The first term is the centripetal acceleration due to the projectile moving around the sphere — which it does because of the gravitational force. The last term is the centripetal acceleration resulting from the rotation of the sphere. The middle term is the Coriolis acceleration.

Using Fig. 3.2, at some instant t

$$\mathbf{r}(t) = \boldsymbol{\rho}(t) = r\cos(\gamma t)\hat{\mathbf{m}} \quad + \quad r\sin(\gamma t)\hat{\mathbf{n}}$$

and

$$\boldsymbol{\gamma} = \gamma\hat{\boldsymbol{\ell}}$$

Then

$$\boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \boldsymbol{\rho}) = (\boldsymbol{\gamma} \cdot \boldsymbol{\rho})\boldsymbol{\gamma} - \gamma^2\boldsymbol{\rho} = -\gamma^2\boldsymbol{\rho} = -\gamma^2\mathbf{r},$$

Check the direction — the negative sign means it points *towards* the centre of the sphere, which is as expected.

Likewise the last term can be obtained as

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\omega^2 r\sin(\gamma t)\hat{\mathbf{n}}$$

Note that it is perpendicular to the axis of rotation $\hat{\mathbf{m}}$, and because of the minus sign, directed towards the axis)

The Coriolis term is derived as:

$$\begin{aligned} 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}} &= 2\boldsymbol{\omega} \times (\boldsymbol{\gamma} \times \boldsymbol{\rho}) \\ &= 2\begin{bmatrix} 0 \\ \omega \\ 0 \end{bmatrix} \times \left(\begin{bmatrix} \gamma \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ r\cos\gamma t \\ r\sin\gamma t \end{bmatrix} \right) \\ &= 2\omega\gamma r\cos\gamma t\hat{\boldsymbol{\ell}} \end{aligned}$$

Instead of a projectile, now consider a rocket on rails which stretch north from the equator. As the rocket travels north it experiences the Coriolis force (exerted by the rails):

2	γ	ω	$R\cos\gamma t$	$\hat{\boldsymbol{\ell}}$
+ve	-ve	+ve	+ve	

Hence the coriolis force is in the direction opposed to $\hat{\boldsymbol{\ell}}$ (i.e. in the opposite direction to the earth's rotation). In the absence of the rails (or atmosphere) the rocket's tangetial speed (relative to the surface of the earth) is *greater* than the speed of the surface of the earth underneath it (since the radius of successive lines of latitude decreases) so it would (to an observer on the earth) appear to deflect to the east. The rails provide a coriolis force keeping it on the same meridian.

3. Line, Surface and Volume Integrals. Curvilinear coordinates.

In this lecture we introduce line, *surface and volume integrals, and consider how these are defined in non-Cartesian, curvilinear coordinates*

a. Scalar and vector fields

When a scalar function $\boldsymbol{u}(r)$ is determined or defined at each position r in some region, we say that $\boldsymbol{u}$ is a scalar field in that region.
Similarly, if a vector function v(r) is defined at each point, then v is a vector field in that region. As you will see, in field theory our aim is to derive statements about the bulk properties of scalar and vector fields, rather than to deal with individual scalars or vectors. Familiar examples of each are shown in figure 4.1.
In part 1 we worked out the force $\boldsymbol{F}(\boldsymbol{r})$ on a charge $\boldsymbol{Q}$ arising from a number of charges $\boldsymbol{q_i}$. The electric field is $\boldsymbol{F}/\boldsymbol{Q}$, so

$$E(r) = \sum_{i=0}^{N} K\frac{q_i}{|r - r_i|^3}(r - r_i) . \left(k = \frac{1}{4\pi \in_r \in_0} \right)$$

For example; you could work out the velocity field, in plane polar, at any point on

(a) (b)

Figure 4.1: Examples of (a) a scalar field (pressure); (b) avector field (wind velocity)

a wheel spinning about its axis

$$\mathbf{v}(\mathbf{r}) = \boldsymbol{\omega} \times \mathbf{r}$$

or the fluid flow field around a wing.

If the fields are independent of time, they are said to be *steady*. Of course, most vector fields of practical interest in engineering science are not steady, and some are unpredictable.

Let us first consider how to perform a variety of types of integration in vector and scalar fields.

b. Line integrals through fields

Line integrals are concerned with measuring the integrated interaction with a field as you move through it on some defined path. E.g., given a map showing the pollution density field in Oxford, you may wish to work out how much pollution you breathe in when cycling from college to the Department via different routes.

First recall the definition of an integral for a scalar function $f\ (x)$ of a single scalar variable x. One assumes a set of n samples $fi\ =\ f(xi\ )$ spaced by δxi . One forms the limit of the sum of the products $f\ (xi\ )\delta xi$ as the number of samples tends to infinity

$$\int f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f_i\ \delta x_i, \qquad \delta x_i \to 0$$

For a smooth function, it is irrelevant how the function is subdivided.

i. Vector line integrals

In a vector line integral, the path L along which the integral is to be evaluated is split into a large number of *vector* segments δr_i . Each line segment is then

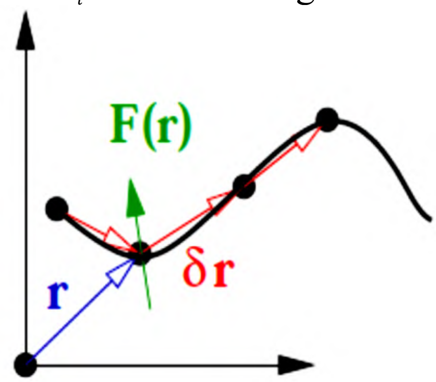


Figure 4.2: Line integral. In the diagram $F(r)$ is a vector field, but it could be replaced with scalar field $U(r)$.

multiplied by the quantity associated with that point in space, the products are then summed and the limit taken as the lengths of the segments tend to zero.

There are three types of integral we have to think about, depending on the nature of the product:

1. Integrand $U(\mathbf{r})$ is a scalar field, hence the integral is a vector.

$$\mathbf{I} = \int_L U(\mathbf{r})d\mathbf{r} \qquad \left(= \lim_{\delta\mathbf{r}_i\to 0} \sum_i U_i \boldsymbol{\delta}\mathbf{r}_i.\right)$$

2. Integrand $\mathbf{a}(\mathbf{r})$ is a vector field dotted with $d\mathbf{r}$ hence the integral is a scalar:

$$I = \int_L \mathbf{a}(\mathbf{r}) \cdot d\mathbf{r} \qquad \left(= \lim_{\delta\mathbf{r}_i\to 0} \sum_i \mathbf{a}_i \cdot \boldsymbol{\delta}\mathbf{r}_i.\right)$$

3. Integrand $\mathbf{a}(\mathbf{r})$ is a vector field crossed with $d\mathbf{r}$ hence vector result.

$$\mathbf{I} = \int_L \mathbf{a}(\mathbf{r}) \times d\mathbf{r} \qquad \left(= \lim_{\delta\mathbf{r}_i\to 0} \sum_i \mathbf{a}_i \times \boldsymbol{\delta}\mathbf{r}_i.\right)$$

Note immediately that unlike an integral in a single scalar variable, there are many paths L from start point $\mathbf{r}_A$ to end point $\mathbf{r}_B$, and the integral will in general depend on the path taken.

Physical examples of line integrals

- The total work done by a force F as it moves a point from A to B along a given path C is given by a line integral of type 2 above. If the force acts at point r and the instantaneous displacement along curve C is dr then the infinitesimal work done is $dW = F.dr$, and so the total work done traversing the path is

$$W_c = \int_C F.dr$$

- Ampere's law relating magnetic field B to linked current can be written as

$$\oint B.dr = \mu_0 I$$

  where I is the current enclosed by (closed) path C.
- The force on an element of wire carrying current **I**, placed in a magnetic field of strength B, is $d\boldsymbol{F} = Idr \times \boldsymbol{B}$. So, if a loop this wire C is placed in the field then the total force will be an integral of type 3 above:

$$F = I\oint_c dr \times B$$

Note that the expressions above are beautifully compact in vector notation, and are all independent of coordinate system. Of course, when evaluating them we need to choose a coordinate system: often this is the standard Cartesian coordinate system (as in the worked examples below),

♣ Examples

Q1 An example in the xy-plane. A force $\mathbf{F} = x^2y\hat{\imath} + xy^2\hat{\jmath}$ acts on a body as it moves between $(0,0)$ and $(1,1)$.

Determine the work done when the path is

1. along the line $y = x$.
2. along the curve $y = x^n$, $n > 0$.
3. along the x axis to the point $(1,0)$ and then along the line $x = 1$.

A1 This is an example of the "type 2" line integral. In planar Cartesians, $d\mathbf{r} = \hat{\imath}dx + \hat{\jmath}dy$. Then the work done is

$$\int_L \mathbf{F}\cdot d\mathbf{r} = \int_L (x^2y\,dx + xy^2dy)\ .$$

1. For the path $y = x$ we find that $dy = dx$. So it is easiest to convert all y references to x.

$$\int_{(0,0)}^{(1,1)} (x^2ydx+xy^2dy) = \int_{x=0}^{x=1} (x^2xdx+xx^2dx) = \int_{x=0}^{x=1} 2x^3dx = \left[x^4/2\right|_{x=0}^{x=1} = 1/2\ .$$

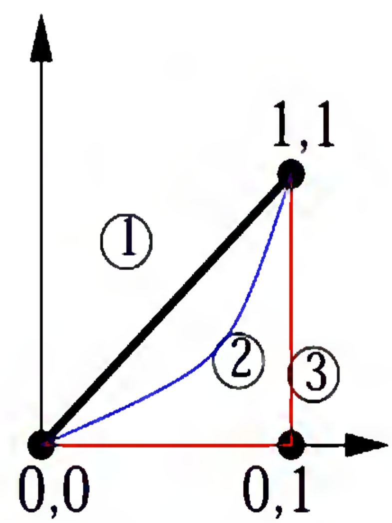


Figure 4.3: Line integral taken along three difference paths

2. For the path $y = x^n$ we find that $dy = nx^{n-1}dx$, so again it is easiest to convert all y references to x.

$$\begin{aligned}\int_{(0,0)}^{(1,1)} (x^2y\,dx + xy^2dy) &= \int_{x=0}^{x=1} (x^{n+2}dx + nx^{n-1}.x.x^{2n}dx)\\ &= \int_{x=0}^{x=1} (x^{n+2}dx + nx^{3n}dx)\\ &= \frac{1}{n+3} + \frac{n}{3n+1}\end{aligned}$$

3. This path is not smooth, so break it into two. Along the first section, $y = 0$ and $dy = 0$, and on the second $x = 1$ and $dx = 0$, so

$$\int_A^B (x^2y\,dx + xy^2dy) = \int_{x=0}^{x=1} (x^2 0dx) + \int_{y=0}^{y=1} 1y^2dy = 0 + \left[y^3/3\right|_{y=0}^{y=1} = 1/3\ .$$

So in general the integral depends on the path taken. Notice that answer (1) is the same as answer (2) when $n = 1$, and that answer (3) is the limiting value of answer (2) as $n \to \infty$.

Q2 Repeat part (2) using the Force $\mathbf{F} = xy^2\hat{\imath} + x^2y\hat{\jmath}$.

A2 For the path $y = x^n$ we find that $dy = nx^{n-1}dx$, so

$$\begin{aligned}\int_{(0,0)}^{(1,1)} (y^2x\,dx + yx^2dy) &= \int_{x=0}^{x=1} (x^{2n+1}dx + nx^{n-1}.x^2.x^n dx)\\ &= \int_{x=0}^{x=1} (x^{2n+1}dx + nx^{2n+1}dx)\\ &= \frac{1}{2n+2} + \frac{n}{2n+2}\\ &= \frac{1}{2} \quad \text{independent of } n\end{aligned}$$

4.3 Line integrals in Conservative fields

In the second example, the line integral has the same value for the whole range of paths. In fact it is wholly independent of path. This is easy to see if we write $g(x, y) = x^2y^2/2$. Then using the definition of the perfect differential

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy$$

we find that

$$\begin{aligned} \int_A^B (y^2 x dx + y x^2 dy) &= \int_A^B dg \\ &= g_B - g_A \end{aligned}$$

which depends solely on the value of g at the start and end points, and not at all on the path used to get from A to B. Such a vector field is called **conservative**.

One sort of line integral performs the integration around a complete loop and is denoted with a ring. If **E** is a conservative field, determine the value of

$$\oint \mathbf{E} \cdot d\mathbf{r} \ .$$

In electrostatics, if **E** is the electric field then the potential function is

$$\phi = -\int \mathbf{E} \cdot d\mathbf{r} \ .$$

Do you think **E** is conservative?

4.3.1 A note on line integrals defined in terms of arc length

Line integrals are often defined in terms of scalar arc length. They don't appear to involve vectors (but actually they are another form of type 2 defined earlier).

The integrals usually appear as follows

$$I = \int_L F(x, y, z) ds$$

and most often the path L is along a curve defined parametrically as $x = x(p)$, $y = y(p)$, $z = z(p)$ where p is some parameter. Convert the function to $F(p)$, writing

$$I = \int_{p_{\text{start}}}^{p_{\text{end}}} F(p) \frac{ds}{dp}\, dp$$

where

$$\frac{ds}{dp} = \left[\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2 + \left(\frac{dz}{dp}\right)^2 \right]^{1/2} .$$

Note that the parameter p could be arc-length s itself, in which case $ds/dp = 1$ of course! Another possibility is that the parameter p is x — that is we are told $y = y(x)$ and $z = z(x)$. Then

$$I = \int_{x_{\text{start}}}^{x_{\text{end}}} F(x) \left[1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2 \right]^{1/2} dx .$$

4.4 Surface integrals

These can be defined by analogy with line integrals.

The surface S over which the integral is to be evaluated is now divided into infinitesimal vector elements of area $d\mathbf{S}$, the direction of the vector $d\mathbf{S}$ representing the direction of the surface normal and its magnitude representing the area of the element.

Again there are three possibilities:

- $\int_S U d\mathbf{S}$ — scalar field U; vector integral.
- $\int_S \mathbf{a} \cdot d\mathbf{S}$ — vector field $\mathbf{a}$; scalar integral.
- $\int_S \mathbf{a} \times d\mathbf{S}$ — vector field $\mathbf{a}$; vector integral.

(in addition, of course, to the purely scalar form, $\int_S U dS$).

Physical example of surface integral

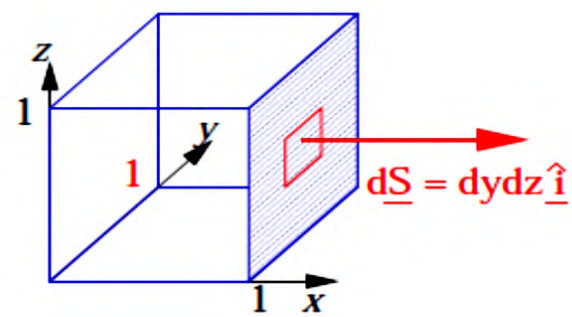
- Physical examples of surface integrals with vectors often involve the idea of *flux* of a vector field through a surface, $\int_S \mathbf{a}.d\mathbf{S}$ For example the mass of fluid crossing a surface S in time dt is $dM = \rho\mathbf{v}.d\mathbf{S}dt$ where $\rho(\mathbf{r})$ is the fluid density and $\mathbf{v}(\mathbf{r})$ is the fluid velocity. The total mass flux can be expressed as a surface integral:

$$\Phi_M = \int_S \rho(r)\mathbf{v}(\mathbf{r}).d\mathbf{S}$$

Again, though this expression is coordinate free, we evaluate an example below using Cartesians. Note, however, that in some problems, symmetry may lead us to a different more natural coordinate system.

♣ Example

Evaluate $\int \mathbf{F}\cdot d\mathbf{S}$ over the $x = 1$ side of the cube shown in the figure when $\mathbf{F} = y\hat{\boldsymbol{\imath}} + z\hat{\boldsymbol{\jmath}} + x\hat{\mathbf{k}}$.
$d\mathbf{S}$ is perpendicular to the surface. Its $\pm$ direction actually depends on the nature of the problem. More often than not, the surface will enclose a volume, and the surface direction is taken as everywhere emanating from the interior.
Hence for the $x = 1$ face of the cube



$$d\mathbf{S} = dydz\hat{\boldsymbol{\imath}}$$

and

$$\int \mathbf{F}\cdot d\mathbf{S} = \int\int y\,dy\,dz = \frac{1}{2}y^2\Big|_0^1 z\Big|_0^1 = \frac{1}{2} \quad .$$

4.5 Volume integrals

The definition of the volume integral is again taken as the limit of a sum of products as the size of the volume element tends to zero. One obvious difference though is that the element of volume is a scalar (how could you define a direction with an infinitesimal volume element?). The possibilities are:

- $\int_V U(\mathbf{r})dV$ — scalar field; scalar integral.
- $\int_V \mathbf{a}dV$ — vector field; vector integral.

You have covered these (more or less) in your first year course, so not much more to say here. The next section considers these again in the context of a change of coordinates.

4.6. Changing variables: curvilinear coordinates

Up to now we have been concerned with Cartesian coordinates x, y, z with coordinate axes $\hat{\imath}, \hat{\jmath}, \hat{k}$. When performing a line integral in Cartesian coordinates, you write $\boldsymbol{r} = \boldsymbol{x\hat{\imath}} + \boldsymbol{y\hat{\jmath}} + \boldsymbol{z\hat{k}}$ and $dr = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$ and can be sure that length scales are properly handled because – as we saw in the above Lecture

$$|dr| = ds^2 = \sqrt{dx^2 + dy^2 + dz^2}$$

The reason for using the basis $\hat{\imath}, \hat{\jmath}, \hat{k}$ rather than any other orthonormal basis set is that $\hat{\imath}$ represents a direction in which x is increasing while the other two coordinates remain constant (and likewise for $\hat{\jmath}$ and $\hat{k}$ with y and z respectively), simplifying the representation and resulting mathematics. Often the symmetry of the problem strongly hints at using another coordinate
system:

- likely to be plane, cylindrical, or spherical polars,
- but can be something more exotic

The general name for any different "u, v, w" coordinate system is a curvilinear coordinate system. We will see that the idea hinted at above – of defining a basis set by considering directions in which only one coordinate is (instantaneously) increasing – provides the appropriate generalization.

We begin by discussing common special cases: cylindrical polar and spherical polar, and conclude with a more general formulation.

4.6.1. Cylindrical polar coordinates

As shown in figure 4.4 a point in space P having cartesian coordinates x, y, z can be expressed in terms of cylindrical polar coordinates, r, ϕ, z as follows:

$$\boldsymbol{r} = \boldsymbol{x\hat{\imath}} + \boldsymbol{y\hat{\jmath}} + \boldsymbol{z\hat{k}}$$

$$= r\cos\phi\hat{\imath} + r\sin\varphi\hat{\jmath} + z\hat{k}$$

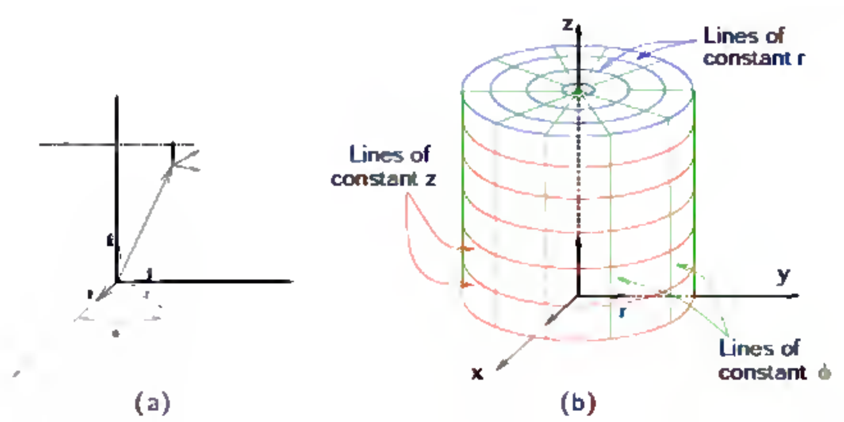


Figure 4.4: Cylindrical polar: (a) coordinate definition; (b) "iso" lines in r , ϕ and z.

Note that, by definition, $\frac{\partial \mathbf{r}}{\partial r}$ represents a direction in which (instantaneously) r is changing while the other two coordinates stay constant. That is, it is tangent to lines of constant ϕ and z. Likewise for $\frac{\partial \mathbf{r}}{\partial \phi}$ and $\frac{\partial \mathbf{r}}{\partial z}$, Thus the vectors:

$$
\begin{aligned}
\mathbf{e}_r &= \frac{\partial \mathbf{r}}{\partial r} = \cos\phi \hat{\imath} + \sin\phi \hat{\jmath} \\
\mathbf{e}_\phi &= \frac{\partial \mathbf{r}}{\partial \phi} = -r\sin\phi \hat{\imath} + r\cos\phi \hat{\jmath} \\
\mathbf{e}_z &= \frac{\partial \mathbf{r}}{\partial z} = \hat{\mathbf{k}}
\end{aligned}
$$

> Aside on notation: some texts use the notation $\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \ldots$ to represent the unit vectors that form the local basis set. Though I prefer the notation used here, where the basis vectors are written as $\hat{\mathbf{e}}$ with appropriate subscripts (as used in Riley *et al*), you should be aware of, and comfortable with, either possibility.

form a basis set in which we may describe infinitessimal vector displacements in the position of P, $d\mathbf{r}$. It is more usual, however, first to normalise the vectors to obtain their corresponding unit vectors, $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\phi$, $\hat{\mathbf{e}}_z$. Following the usual rules of calculus we may write:

$$
\begin{aligned}
d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial r}dr + \frac{\partial \mathbf{r}}{\partial \phi}d\phi + \frac{\partial \mathbf{r}}{\partial z}dz \\
&= dr\mathbf{e}_r + d\phi\mathbf{e}_\phi + dz\mathbf{e}_z \\
&= dr\hat{\mathbf{e}}_r + rd\phi\hat{\mathbf{e}}_\phi + dz\hat{\mathbf{e}}_z
\end{aligned}
$$

Now here is the important thing to note. In cartesian coordinates, a small change in (eg) x while keeping y and z constant would result in a displacement of

$$ds = |d\mathbf{r}| = \sqrt{d\mathbf{r}.d\mathbf{r}} = \sqrt{dx^2 + 0 + 0} = dx$$

But in cylindrical polars, a small change in ϕ of $d\phi$ while keeping r and z constant results in a displacement of

$$ds = |d\mathbf{r}| = \sqrt{r^2(d\phi)^2} = rd\phi$$

Thus the size of the (infinitessimal) displacement is dependent on the value of r. Factors such as this r are known as **scale factors** or **metric coefficients**, and we must be careful to take them into account when, eg, performing line, surface or volume integrals, as you will below. For cylindrical polars the metric coefficients are clearly 1, r and 1.

Example: line integral in cylindrical coordinates

Q Evaluate $\oint_C \mathbf{a} \cdot d\mathbf{l}$, where $\mathbf{a} = x^3\hat{\boldsymbol{\jmath}} - y^3\hat{\boldsymbol{\imath}} + x^2y\hat{\boldsymbol{k}}$ and C is the circle of radius r in the $z = 0$ plane, centred on the origin.

A Consider figure 4.5. In this case our cylindrical coordinates effectively reduce to plane polars since the path of integration is a circle in the $z = 0$ plane, but let's persist with the full set of coordinates anyway; the $\hat{\boldsymbol{k}}$ component of $\mathbf{a}$ will play no role (it is normal to the path of integration and therefore cancels as seen below).

On the circle of interest

$$\mathbf{a} = r^3(-\sin^3\phi\hat{\boldsymbol{\imath}} + \cos^3\phi\hat{\boldsymbol{\jmath}} + \cos^2\phi\sin\phi\hat{\boldsymbol{k}})$$

and (since dz = dr = 0 on the path)

$$\begin{aligned} d\mathbf{r} &= r\ d\phi\ \hat{\mathbf{e}}_\phi \\ &= r d\phi(-\sin\phi\hat{\boldsymbol{\imath}} + \cos\phi\hat{\boldsymbol{\jmath}}) \end{aligned}$$

so that

$$\oint_C \mathbf{a} \cdot d\mathbf{r} = \int_0^{2\pi} r^4(\sin^4\phi + \cos^4\phi)d\phi = \frac{3\pi}{2}r^4$$

since

$$\int_0^{2\pi} \sin^4\phi d\phi = \int_0^{2\pi} \cos^4\phi d\phi = \frac{3\pi}{4}$$

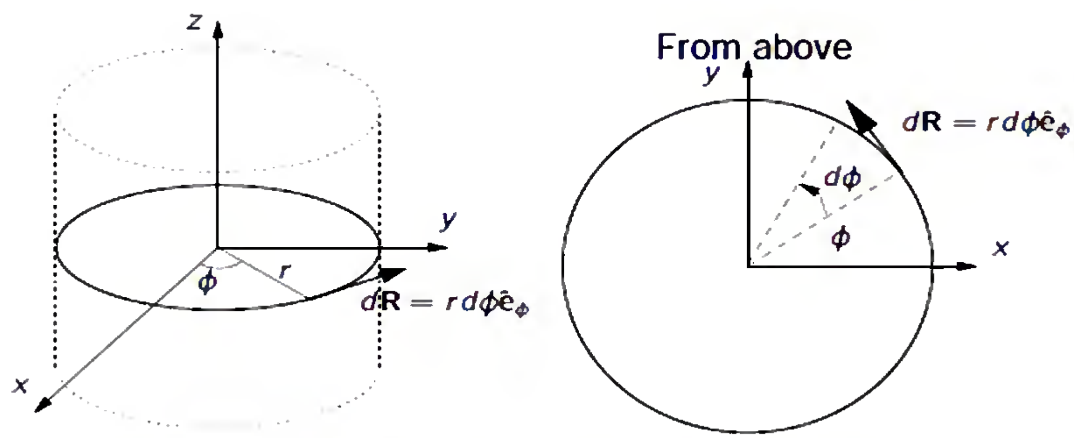


Figure 4.5: Line integral example in cylindrical coordinates

Volume integrals in cylindrical polars

In Cartesian coordinates a volume element is given by (see figure 4.6a):

$$dV = dx\,dy\,dz$$

Recall that the volume of a parallelopiped is given by the scalar triple product of the vectors which define it (see section 2.1.2). Thus the formula above can be derived (even though it is "obvious") as:

$$dV = dx\hat{\imath}.(dy\hat{\jmath} \times dz\hat{\mathbf{k}}) = dx\,dy\,dz$$

since the basis set is orthonormal.
In cylindrical polars a volume element is given by (see figure 4.6b):

$$dV = dr\hat{\mathbf{e}}_r.(r\,d\phi\hat{\mathbf{e}}_\phi \times dz\hat{\mathbf{e}}_z) = r\,d\phi\,dr\,dz$$

Note also that this volume, because it is a scalar triple product, can be written as a determinant:

$$dV = \begin{vmatrix} \hat{\mathbf{e}}_r dr \\ \hat{\mathbf{e}}_\phi r d\phi \\ \hat{\mathbf{e}}_z dz \end{vmatrix} = \begin{vmatrix} \mathbf{e}_r dr \\ \mathbf{e}_\phi d\phi \\ \mathbf{e}_z dz \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} dr\,d\phi\,dz$$

where the equality on the right-hand side follows from the definitions of $\hat{\mathbf{e}}_r = \frac{\partial \mathbf{r}}{\partial r} = \frac{\partial x}{\partial r}\hat{\imath} + \frac{\partial y}{\partial r}\hat{\jmath} + \frac{\partial z}{\partial r}\hat{\mathbf{k}}$, etc. This is the explanation for the "magical" appearance of the determinant in change-of-variables integration that you encountered in your first year maths!

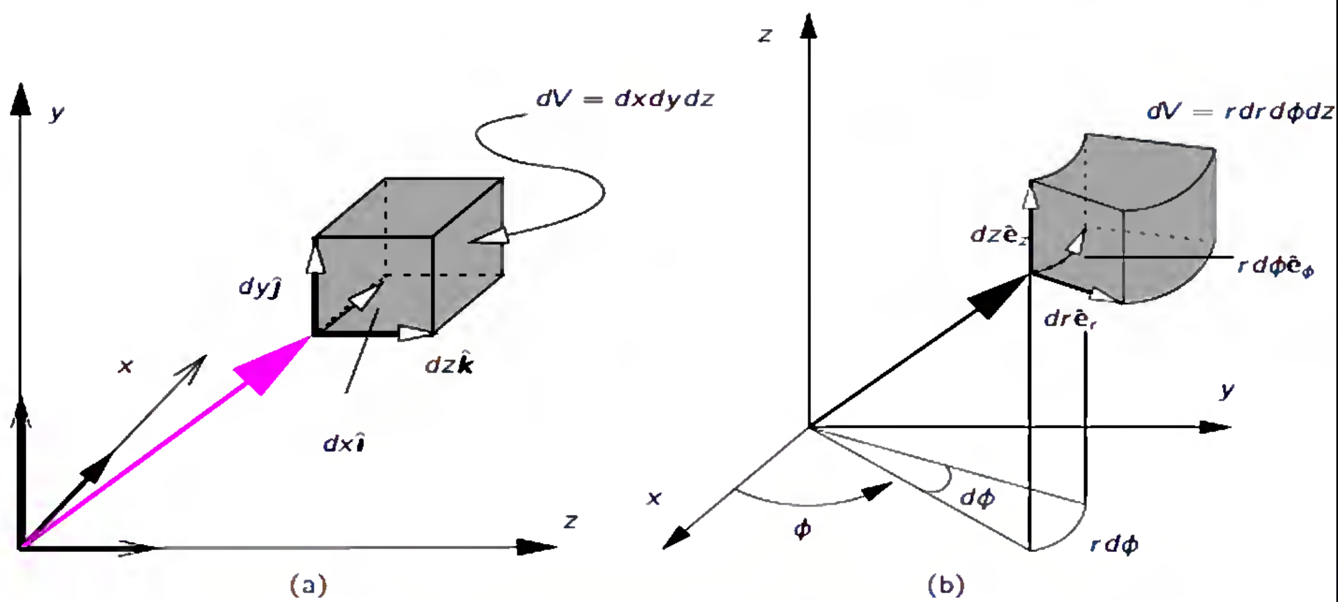


Figure 4.6: Volume elements dV in (a) Cartesian coordinates; (b) Cylindrical polar coordinates

Surface integrals in cylindrical polars

Recall from section 4.4 that for a surface element with normal along $\hat{\boldsymbol{\imath}}$ we have:

$$d\mathbf{S} = dydz\hat{\boldsymbol{\imath}}$$

More explicitly this comes from finding normal to the plane that is tangent to the surface of constant x and from finding the area of an infinitessimal area element on the plane. In this case the plane is spanned by the vectors $\hat{\jmath}$ and $\hat{\boldsymbol{k}}$ and the area of the element given by (see section 1.3):

$$dS = \left| dy\hat{\boldsymbol{\jmath}} \times dz\hat{\boldsymbol{k}} \right|$$

Thus

$$d\mathbf{S} = dy\hat{\boldsymbol{\jmath}} \times dz\hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}}dS = dydz\hat{\boldsymbol{\imath}}$$

In cylindrical polars, surface area elements (see figure 4.7) are given by:

$$d\mathbf{S} = dr\hat{\mathbf{e}}_r \times rd\phi\hat{\mathbf{e}}_\phi = rdrd\phi\hat{\mathbf{e}}_z \qquad \text{(for surfaces of constant } z\text{)}$$
$$d\mathbf{S} = rd\phi\hat{\mathbf{e}}_\phi \times dz\hat{\mathbf{e}}_z = rd\phi dz\hat{\mathbf{e}}_r \qquad \text{(for surfaces of constant } r\text{)}$$

Similarly we can find $d\mathbf{S}$ for surfaces of constant ϕ, though since these aren't as common this is left as a (relatively easy) exercise.

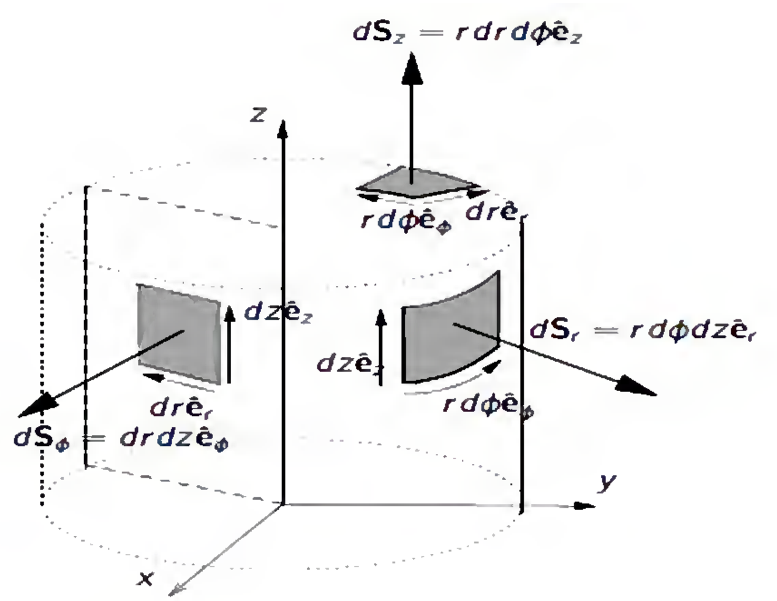


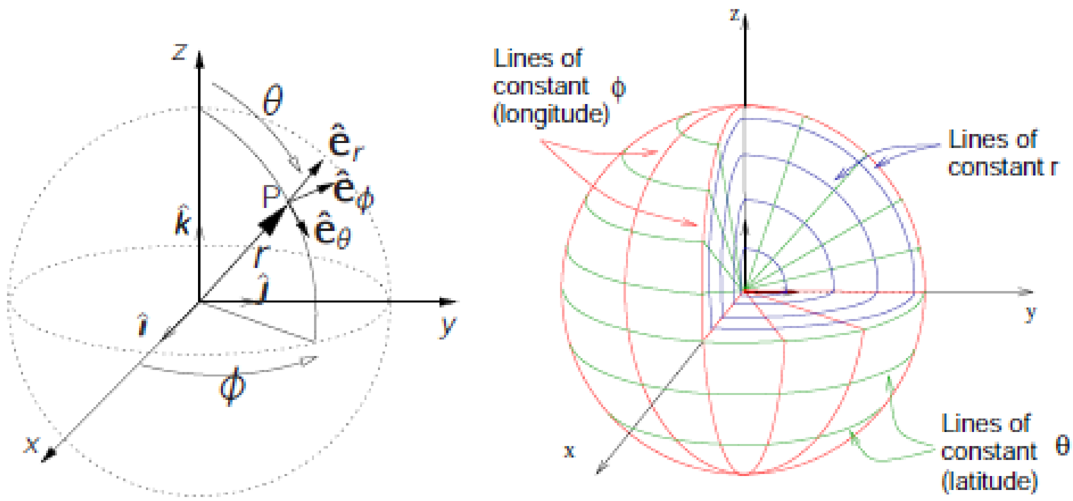
Figure 4.7: Surface elements in cylindrical polar coordinates

4.6.2 Spherical polars

Much of the development for spherical polars is similar to that for cylindrical polars. As shown in figure 4.6.2 a point in space P having cartesian coordinates x, y, z can be expressed in terms of spherical polar coordinates, r, θ, ϕ as follows:

$$\begin{aligned}\mathbf{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{\boldsymbol{k}} \\ &= r\sin\theta\cos\phi\hat{\imath} + r\sin\theta\sin\phi\hat{\jmath} + r\cos\theta\hat{\boldsymbol{k}}\end{aligned}$$

The basis set in spherical polars is obtained in an analogous fashion: we find unit



vectors which are in the direction of increase of each coordinate:

$$\begin{aligned}\mathbf{e}_r = \frac{\partial \mathbf{r}}{\partial r} &= \sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{\boldsymbol{k}} &= \hat{\mathbf{e}}_r \\ \mathbf{e}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} &= r\cos\theta\cos\phi\hat{\imath} + r\cos\theta\sin\phi\hat{\jmath} - r\sin\theta\hat{\boldsymbol{k}} &= r\hat{\mathbf{e}}_\theta \\ \mathbf{e}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} &= -r\sin\theta\sin\phi\hat{\imath} + r\sin\theta\cos\phi\hat{\jmath} &= r\sin\theta\hat{\mathbf{e}}_\phi\end{aligned}$$

As with cylindrical polars, it is easily verified that the vectors $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$ form an orthonormal basis.
A small displacement $d\mathbf{r}$ is given by:

$$\begin{aligned}d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial r}dr + \frac{\partial \mathbf{r}}{\partial \theta}d\theta + \frac{\partial \mathbf{r}}{\partial \phi}d\phi \\ &= dr\mathbf{e}_r + d\theta\mathbf{e}_\theta + d\phi\mathbf{e}_\phi \\ &= dr\hat{\mathbf{e}}_r + r\,d\theta\hat{\mathbf{e}}_\theta + r\sin\theta\, d\phi\hat{\mathbf{e}}_\phi\end{aligned}$$

Thus the metric coefficients are $1, r, r\sin\theta$.

Volume integrals in spherical polars

In spherical polars a volume element is given by (see figure 4.8):

$$dV = dr\hat{\mathbf{e}}_r.(r\,d\theta\hat{\mathbf{e}}_\theta \times r\sin\theta d\phi\hat{\mathbf{e}}_\phi) = r^2 \sin\theta dr\,d\theta d\phi$$

Note again that this volume could be written as a determinant, but this is left as an exercise.

Surface integrals in spherical polars

The most (the only?) useful surface elements in spherical polars are those tangent to surfaces of constant r (see figure 4.9). The surface direction (unnormalised) is given by $\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\phi = \hat{\mathbf{e}}_r$ and the area of an infinitessimal surface element is given by $|r\,d\theta\hat{\mathbf{e}}_\theta \times r\sin\theta d\phi\hat{\mathbf{e}}_\phi| = r^2 \sin\theta d\theta d\phi$.

Thus a surface element $d\mathbf{S}$ in spherical polars is given by

$$d\mathbf{S} = r\,d\theta\hat{\mathbf{e}}_\theta \times r\sin\theta d\phi\hat{\mathbf{e}}_\phi = r^2 \sin\theta\hat{\mathbf{e}}_r$$

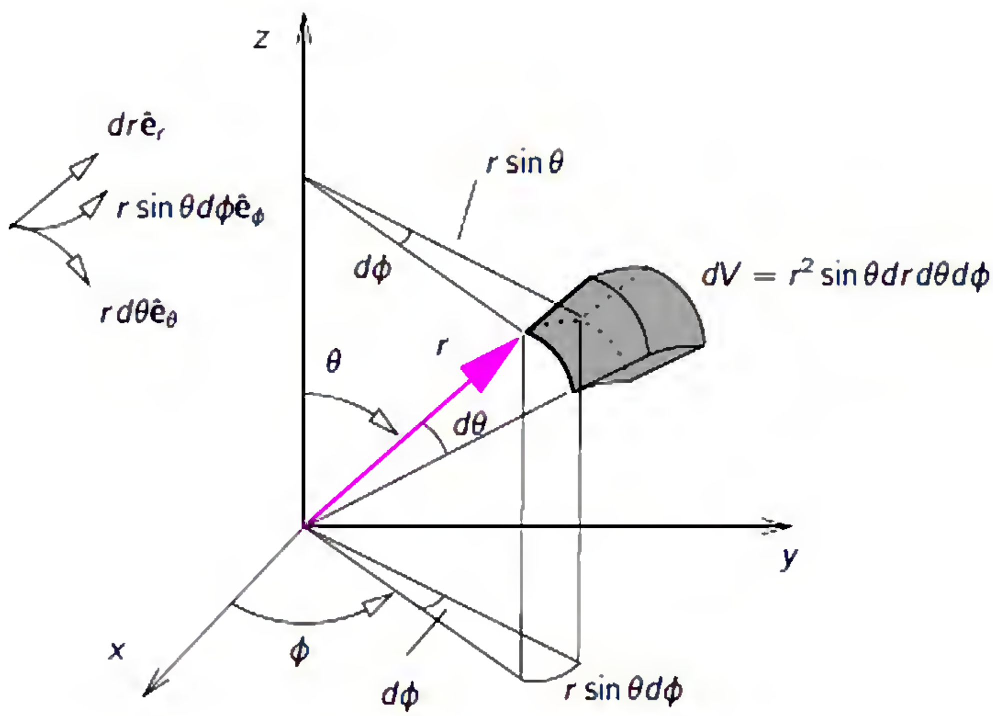


Figure 4.8: Volume element dV in spherical polar coordinates

♣ **Example: surface integral in spherical polars**

Q Evaluate $\int_S \mathbf{a} \cdot d\mathbf{S}$, where $\mathbf{a} = z^3\hat{\boldsymbol{k}}$ and S is the sphere of radius A centred on the origin.

A On the surface of the sphere:

$$\mathbf{a} = A^3 cos^3\theta \hat{\boldsymbol{k}} \qquad d\mathbf{S} = A^2 \sin\theta \, d\theta \, d\phi \hat{\mathbf{e}}_r$$

Hence

$$\begin{aligned}
\int_S \mathbf{a} \cdot d\mathbf{S} &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} A^3 cos^3\theta \; A^2 \sin\theta \; [\hat{\mathbf{e}}_r \cdot \hat{\boldsymbol{k}}] \; d\theta d\phi \\
&= A^5 \int_0^{2\pi} d\phi \; \int_0^{\pi} cos^3\theta \sin\theta [\cos\theta] \; d\theta \\
&= 2\pi A^5 \frac{1}{5} \left[-\cos^5\theta\right]_0^{\pi} \\
&= \frac{4\pi A^5}{5}
\end{aligned}$$

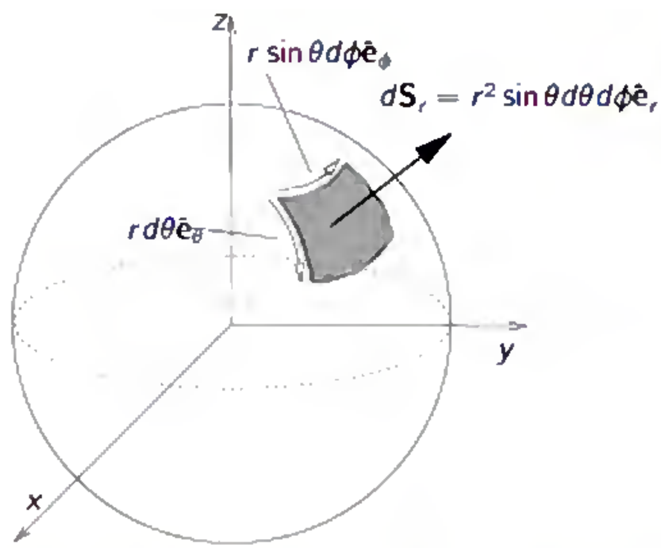


Figure 4.9: Surface element $d\mathbf{S}$ in spherical polar coordinates

4.6.3 General curvilinear coordinates

Cylindrical and spherical polar coordinates are two (useful) examples of general curvilinear coordinates. In general a point P with Cartesian coordinates x, y, z can be expressed in terms of the curvilinear coordinates u, v, w where

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w)$$

Thus

$$\mathbf{r} = x(u, v, w)\hat{\imath} + y(u, v, w)\hat{\jmath} + z(u, v, w)\hat{\boldsymbol{k}}$$

and

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\hat{\imath} + \frac{\partial y}{\partial u}\hat{\jmath} + \frac{\partial z}{\partial u}\hat{\boldsymbol{k}}$$

and similarly for partials with respect to v and w, so

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv + \frac{\partial \mathbf{r}}{\partial w}dw$$

We now define the local coordinate system as before by considering the directions in which each coordinate "unilaterally" (and instantaneously) increases:

$$\begin{aligned}
\mathbf{e}_u &= \frac{\partial \mathbf{r}}{\partial u} = \left|\frac{\partial \mathbf{r}}{\partial u}\right| \hat{\mathbf{e}}_u = h_u \hat{\mathbf{e}}_u \\
\mathbf{e}_v &= \frac{\partial \mathbf{r}}{\partial v} = \left|\frac{\partial \mathbf{r}}{\partial v}\right| \hat{\mathbf{e}}_v = h_v \hat{\mathbf{e}}_v \\
\mathbf{e}_w &= \frac{\partial \mathbf{r}}{\partial w} = \left|\frac{\partial \mathbf{r}}{\partial w}\right| \hat{\mathbf{e}}_w = h_w \hat{\mathbf{e}}_w
\end{aligned}$$

The metric coefficients are therefore $h_u = |\frac{\partial \mathbf{r}}{\partial u}|$, $h_v = |\frac{\partial \mathbf{r}}{\partial v}|$ and $h_w = |\frac{\partial \mathbf{r}}{\partial w}|$.

A volume element is in general given by

$$dV = h_u du \hat{\mathbf{e}}_u.(h_v dv \hat{\mathbf{e}}_v \times h_w dw \hat{\mathbf{e}}_w)$$

and simplifies *if the coordinate system is orthonormal* (since $\hat{\mathbf{e}}_u.(\hat{\mathbf{e}}_v \times \hat{\mathbf{e}}_w) = 1$) to

$$dV = h_u h_v h_w du dv dw$$

A surface element (normal to constant w, say) is in general

$$d\mathbf{S} = h_u du \hat{\mathbf{e}}_u \times h_v dv \hat{\mathbf{e}}_v$$

and simplifies *if the coordinate system is orthogonal* to

$$d\mathbf{S} = h_u h_v du dv \hat{\mathbf{e}}_w$$

Summary

To summarise

General curvilinear coordinates

$$
\begin{aligned}
x &= x(u,v,w), \qquad y = y(u,v,w), \qquad z = z(u,v,w) \\
\mathbf{r} &= x(u,v,w)\hat{\imath} + y(u,v,w)\hat{\jmath} + z(u,v,w)\hat{\boldsymbol{k}} \\
h_u &= \left|\frac{\partial \mathbf{r}}{\partial u}\right|, \qquad h_v = \left|\frac{\partial \mathbf{r}}{\partial v}\right|, \qquad h_w = \left|\frac{\partial \mathbf{r}}{\partial w}\right| \\
\hat{\mathbf{u}} &= \hat{\mathbf{e}}_u = \frac{1}{h_u}\frac{\partial \mathbf{r}}{\partial u}, \qquad \hat{\mathbf{v}} = \hat{\mathbf{e}}_v = \frac{1}{h_v}\frac{\partial \mathbf{r}}{\partial v}, \qquad \hat{\mathbf{w}} = \hat{\mathbf{e}}_w = \frac{1}{h_w}\frac{\partial \mathbf{r}}{\partial w} \\
d\mathbf{r} &= h_u du\hat{\mathbf{u}} + h_v dv\hat{\mathbf{v}} + h_w dw\hat{\mathbf{w}} \\
dV &= h_u h_v h_w du dv dw \; \hat{\mathbf{u}}.(\hat{\mathbf{v}} \times \hat{\mathbf{w}}) \\
d\mathbf{S} &= h_u h_v du dv \; \hat{\mathbf{u}} \times \hat{\mathbf{v}} \quad \text{(for surface element tangent to constant } w\text{)}
\end{aligned}
$$

Plane polar coordinates

$$
\begin{aligned}
x &= r\cos\theta, \qquad y = r\sin\theta \\
\mathbf{r} &= r\cos\theta\hat{\imath} + r\sin\theta\hat{\jmath} \\
h_r &= 1, \qquad h_\theta = r \\
\hat{\mathbf{e}}_r &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}, \qquad \hat{\mathbf{e}}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} \\
d\mathbf{r} &= dr\hat{\mathbf{e}}_r + rd\theta\hat{\mathbf{e}}_\theta \\
d\mathbf{S} &= rdrd\theta\hat{\boldsymbol{k}}
\end{aligned}
$$

Cylindrical polar coordinates

$$
\begin{aligned}
x &= r\cos\phi, \qquad y = r\sin\phi, \qquad z = z \\
\mathbf{r} &= r\cos\phi\hat{\imath} + r\sin\phi\hat{\jmath} + z\hat{\boldsymbol{k}} \\
h_r &= 1, \qquad h_\phi = r, \qquad h_z = 1 \\
\hat{\mathbf{e}}_r &= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath}, \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}, \qquad \hat{\mathbf{e}}_z = \hat{\boldsymbol{k}} \\
d\mathbf{r} &= dr\hat{\mathbf{e}}_r + rd\phi\hat{\mathbf{e}}_\phi + dz\hat{\mathbf{e}}_z \\
d\mathbf{S} &= rdrd\phi\hat{\boldsymbol{k}} \quad \text{(on the flat ends)} \\
d\mathbf{S} &= rd\phi dz\hat{\mathbf{e}}_r \quad \text{(on the curved sides)} \\
dV &= rdrd\phi dz
\end{aligned}
$$

Spherical polar coordinates

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$
$$\mathbf{r} = r\sin\theta\cos\phi\hat{\boldsymbol{\imath}} + r\sin\theta\sin\phi\hat{\boldsymbol{\jmath}} + r\cos\theta\hat{\boldsymbol{k}}$$
$$h_r = 1, \qquad h_\theta = r, \qquad h_\phi = r\sin\theta$$
$$\hat{\mathbf{e}}_r = \sin\theta\cos\phi\hat{\boldsymbol{\imath}} + \sin\theta\sin\phi\hat{\boldsymbol{\jmath}} + \cos\theta\hat{\boldsymbol{k}}$$
$$\hat{\mathbf{e}}_\theta = \cos\theta\cos\phi\hat{\boldsymbol{\imath}} + \cos\theta\sin\phi\hat{\boldsymbol{\jmath}} + \sin\theta\hat{\boldsymbol{k}}$$
$$\hat{\mathbf{e}}_\phi = -\sin\phi\hat{\boldsymbol{\imath}} + \cos\phi\hat{\boldsymbol{\jmath}}$$
$$d\mathbf{r} = dr\hat{\mathbf{e}}_r + rd\theta\hat{\mathbf{e}}_\theta + r\sin\theta d\phi\hat{\mathbf{e}}_\phi$$
$$d\mathbf{S} = r^2\sin\theta dr\, d\theta d\phi\hat{\mathbf{e}}_r \quad \text{(on a spherical surface)}$$
$$dV = r^2\sin\theta dr\, d\theta d\phi$$

Vector Operators: Grad, Div. and Curl

We introduce three field operators which reveal interesting collective field properties, the gradient of a scalar field,

- the ***divergence*** of a vector field, and
- the ***curl*** of a vector field.

There are two points to get over about each:

- The mechanics of taking the grad, div or curl, for which you will need to brush up your multivariate calculus.
- The underlying physical meaning — that is, why they are worth bothering about.

The gradient of a scalar field

Recall the discussion of temperature distribution throughout a room in the overview, where we wondered how a scalar would vary as we moved off in an arbitrary direction. Here we find out how. If $U(x, y, z)$ is a scalar field, ie a scalar function of position $r = [x, y, z]$ in 3 dimensions, then its gradient at any point is defined in Cartesian coordinates by

$$gradU = \frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}$$

It is usual to define the vector operator which is called "del" or "nabla"

$$\boldsymbol{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Then

$$grad\ U = \boldsymbol{\nabla}\mathbf{U}$$

Note immediately that $\boldsymbol{\nabla U}$ is a vector field!

Without thinking too carefully about it, we can see that the gradient of a scalar field tends to point in the direction of greatest change of the field. Later we will be more precise.

♣ Worked examples of gradient evaluation

1. $U = x^2$

$$\Rightarrow \nabla U = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) x^2 = 2x\hat{\imath} \ .$$

2. $U = r^2$

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2 \\ \Rightarrow \nabla U &= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right)(x^2 + y^2 + z^2) \\ &= 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{\mathbf{k}} \quad = \quad 2\,\mathbf{r} \ . \end{aligned}$$

3. $U = \mathbf{c} \cdot \mathbf{r}$, where $\mathbf{c}$ is constant.

$$\Rightarrow \nabla U = \left(\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\right)(c_1 x + c_2 y + c_3 z) = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{\mathbf{k}} = \mathbf{c} \ .$$

4. $U = f(r)$, where $r = \sqrt{(x^2 + y^2 + z^2)}$

U is a function of r alone so df/dr exists. As $U = f(x, y, z)$ also,

$$\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} \qquad \frac{\partial f}{\partial y} = \frac{df}{dr}\frac{\partial r}{\partial y} \qquad \frac{\partial f}{\partial z} = \frac{df}{dr}\frac{\partial r}{\partial z} \ .$$

$$\Rightarrow \nabla U = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \quad = \frac{df}{dr}\left(\frac{\partial r}{\partial x}\hat{\imath} + \frac{\partial r}{\partial y}\hat{\jmath} + \frac{\partial r}{\partial z}\hat{\mathbf{k}}\right)$$

But $r = \sqrt{x^2 + y^2 + z^2}$, so $\partial r/\partial x = x/r$ and similarly for y, z.

$$\Rightarrow \nabla U = \frac{df}{dr}\left(\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}}{r}\right) = \frac{df}{dr}\left(\frac{\mathbf{r}}{r}\right) \ .$$

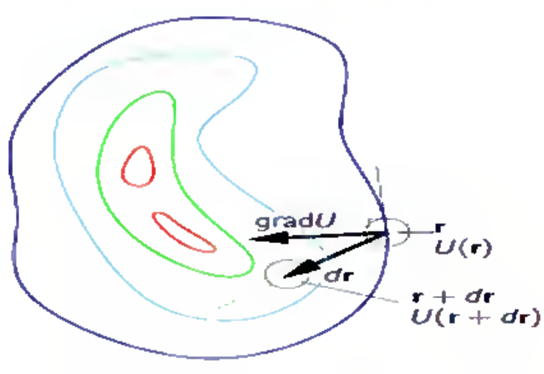


Figure 5.1: The directional derivative

The significance of grad

If our current position is $\mathbf{r}$ in some scalar field U (Fig. 5.1), and we move an infinitesimal distance $d\mathbf{r}$, we know that the change in U is

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \ .$$

But we know that $d\mathbf{r} = (\hat{\imath}dx + \hat{\jmath}dy + \hat{\mathbf{k}}dz)$ and $\nabla U = (\hat{\imath}\partial U/\partial x + \hat{\jmath}\partial U/\partial y + \hat{\mathbf{k}}\partial U/\partial z)$, so that the change in U is also given by the scalar product

$$dU = \nabla U \cdot d\mathbf{r} \ .$$

Now divide both sides by ds

$$\frac{dU}{ds} = \nabla U \cdot \frac{d\mathbf{r}}{ds} \ .$$

But remember that $|d\mathbf{r}| = ds$, so $d\mathbf{r}/ds$ is a unit vector in the direction of $d\mathbf{r}$.

This result can be paraphrased as:

- grad$U$ has the property that the rate of change of U wrt distance in a particular direction ($\hat{\mathbf{d}}$) is the projection of grad$U$ onto that direction (or the component of grad$U$ in that direction).

The quantity dU/ds is called a **directional derivative**, but note that in general it has a different value for each direction, and so has no meaning until you specify the direction.

Another nice property emerges if we think of a surface of constant U – that is the locus (x, y, z) for

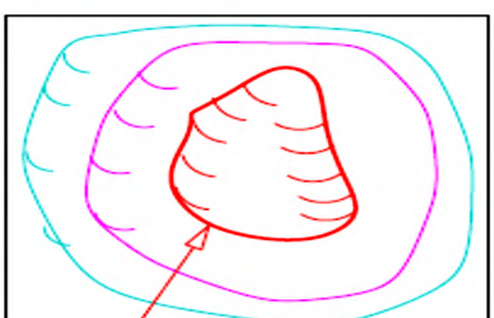
$$U(x, y, z) = \text{constant} \ .$$

If we move a tiny amount within that iso-U surface, there is no change in U, so $dU/ds = 0$. So for any $d\mathbf{r}/ds$ in the surface

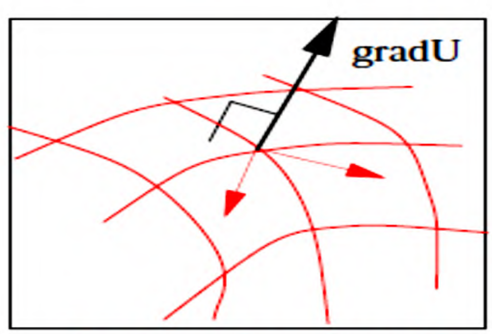
$$\nabla U \cdot \frac{d\mathbf{r}}{ds} = 0 \ .$$

But $d\mathbf{r}/ds$ is a tangent to the surface, so this result shows that

- grad$U$ is everywhere NORMAL to a surface of constant U.



Surface of constant U
These are called Level Surfaces



Surface of constant U

The divergence of a vector field

The divergence computes a scalar quantity from a vector field by differentiation. If $a(x, y, z)$ is a vector function of position in 3 dimensions, that is $\boldsymbol{a} = \boldsymbol{a_1}\hat{\boldsymbol{\imath}} + \boldsymbol{a_2}\hat{\boldsymbol{\jmath}} + \boldsymbol{a_3}\hat{\boldsymbol{k}}$ then its divergence at any point is defined in Cartesian co-ordinates by

$$diva = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

We can write this in a simplified notation using a scalar product with the $\boldsymbol{\nabla}$ vector differential operator:

$$diva = \left(\hat{\boldsymbol{\imath}}\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}\right) \cdot \boldsymbol{a} = \boldsymbol{\nabla} \cdot \boldsymbol{a}$$

Notice that the divergence of a vector field is a scalar field.

♣ **Examples of divergence evaluation**

	a	diva
1)	$x\hat{\boldsymbol{\imath}}$	1
2)	$\mathbf{r}(= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}})$	3
3)	$\mathbf{r}/r^3$	0
4)	$r\mathbf{c}$, for $\mathbf{c}$ constant	$(\mathbf{r}\cdot\mathbf{c})/r$

We work through example 3).

The x component of $\mathbf{r}/r^3$ is $x.(x^2+y^2+z^2)^{-3/2}$, and we need to find $\partial/\partial x$ of it.

$$\begin{aligned}\frac{\partial}{\partial x}x.(x^2+y^2+z^2)^{-3/2} &= 1.(x^2+y^2+z^2)^{-3/2} + x\frac{-3}{2}(x^2+y^2+z^2)^{-5/2}.2x \\ &= r^{-3}\left(1-3x^2r^{-2}\right) .\end{aligned}$$

The terms in y and z are similar, so that

$$\begin{aligned}\text{div}(\mathbf{r}/r^3) &= r^{-3}\left(3-3(x^2+y^2+z^2)r^{-2}\right) = r^{-3}(3-3) \\ &= 0\end{aligned}$$

The significance of div

Consider a typical vector field, water flow, and denote it by $a(r)$. This vector has magnitude equal to the mass of water crossing a unit area perpendicular to the direction of a per unit time.

Now take an infinitesimal volume element dV and figure out the balance of the flow of a in and out of dV .

To be specific, consider the volume element $d\boldsymbol{V} = dxdydz$ in Cartesian coordinates, and think first about the face of area $dxdz$ perpendicular to the y axis and facing outwards in the negative y direction. (That is, the one with surface area $dS = -dxdz\hat{\jmath}$.)

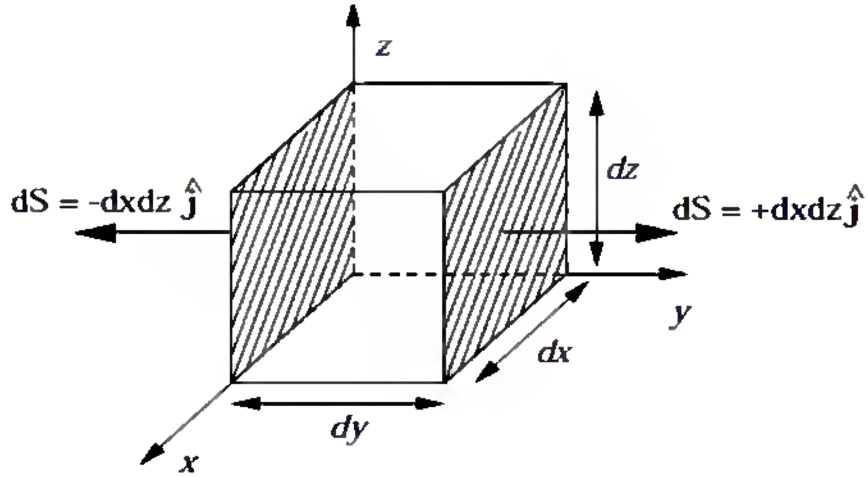


Figure 5.2: Elemental volume for calculating divergence.

The component of the vector a normal to this face is $\boldsymbol{a}\cdot\hat{\boldsymbol{\jmath}} = \boldsymbol{a}_y$, and is pointing inwards, and so its contribution to the OUTWARD flux from this surface is

$$\boldsymbol{a} \cdot dS = -a_y(y)dzdx$$

where $a_y(y)$ means that a_y is a function of y . (By the way***, flux here denotes mass per unit time***.)

A similar contribution, but of opposite sign, will arise from the opposite face, but we must remember that we have moved along y by an amount dy , so that this OUTWARD amount is

$$a_y(y+dy)dzdx = (a_y + \frac{\partial a_y}{\partial y}dy)dxdz$$

The total outward amount from these two faces is

$$\frac{\partial a_y}{\partial y}dxdydz = \frac{\partial a_y}{\partial y}dv$$

Summing the other faces gives a total outward flux of

$$\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right)dV = \boldsymbol{\nabla}\cdot\boldsymbol{a}\,dV$$

So we see that

The divergence of a vector field represents the flux generation per unit volume at each point of the field. (Divergence because it is an efflux not an influx.)

Interestingly we also saw that the total efflux from the infinitesimal volume was equal to the flux integrated over the surface of the volume.

The Laplacian: div ($gradU$) of a scalar field

Recall that $gradU$ of *any* scalar field U is a vector field. Recall also that we can compute the divergence of any vector field. So, we can certainly compute $div(gradU)$, even if we don't know what it means yet.
Here is where the ∇ operator starts to be really handy.

$$\begin{aligned}
\nabla\cdot(\nabla U) &= \left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)U\right)\\
&= \left(\left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\right)U\\
&= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)U\\
&= \left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)
\end{aligned}$$

The last expression is used next in solving Laplace's Equation in partial differential equations. For this reason, the operator ∇^2 is called the ***"Laplacian"***

$$\nabla^2 U = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)U$$

Laplace's equation itself is

$$\nabla^2 U = 0$$

♣ **Examples of $\nabla^2 U$ evaluation**

	U	$\nabla^2 U$
1)	$r^2(= x^2+y^2+z^2)$	6
2)	xy^2z^3	$2xz^3+6xy^2z$
3)	$1/r$	0

Let's prove example (3) (which is particularly significant – can you guess why?).

$$1/r = (x^2+y^2+z^2)^{-1/2}$$

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} &= \frac{\partial}{\partial x} - x.(x^2+y^2+z^2)^{-3/2}\\
&= -(x^2+y^2+z^2)^{-3/2}+3x.x.(x^2+y^2+z^2)^{-5/2}\\
&= (1/r^3)(-1+3x^2/r^2)
\end{aligned}$$

Adding up similar terms for y and z

$$\nabla^2\frac{1}{r} = \frac{1}{r^3}\left(-3+3\frac{(x^2+y^2+x^2)}{r^2}\right) = 0$$

The curl of a vector field

So far we have seen the operator ∇ applied to a scalar field ∇U; and dotted with a vector field $\nabla \cdot \mathbf{a}$.

We are now overwhelmed by an irrestible temptation to

- cross it with a vector field $\nabla \times \mathbf{a}$

This gives the **curl of a vector field**

$$\nabla \times \mathbf{a} \equiv \text{curl}(\mathbf{a})$$

We can follow the pseudo-determinant recipe for vector products, so that

$$\nabla \times \mathbf{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \quad \text{(remember it this way)}$$

$$= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{\boldsymbol{k}}$$

♣ **Examples of curl evaluation**

	$\mathbf{a}$	$\nabla \times \mathbf{a}$
1)	$-y\hat{\imath} + x\hat{\jmath}$	$2\hat{\boldsymbol{k}}$
2)	$x^2y^2\hat{\boldsymbol{k}}$	$2x^2y\hat{\imath} - 2xy^2\hat{\jmath}$

The Significance of Curl

Perhaps the first example gives a clue. The field $\mathbf{a} = -y\hat{\imath} + x\hat{\jmath}$ is sketched in Figure 5.3(a). (It is the field you would calculate as the velocity field of an object rotating with $\boldsymbol{\omega} = [0, 0, 1]$.) This field has a curl of $2\hat{\boldsymbol{k}}$, which is in the r-h screw sense out of the page. You can also see that a field like this must give a finite value to the line integral around the complete loop $\oint_C \mathbf{a} \cdot d\mathbf{r}$.

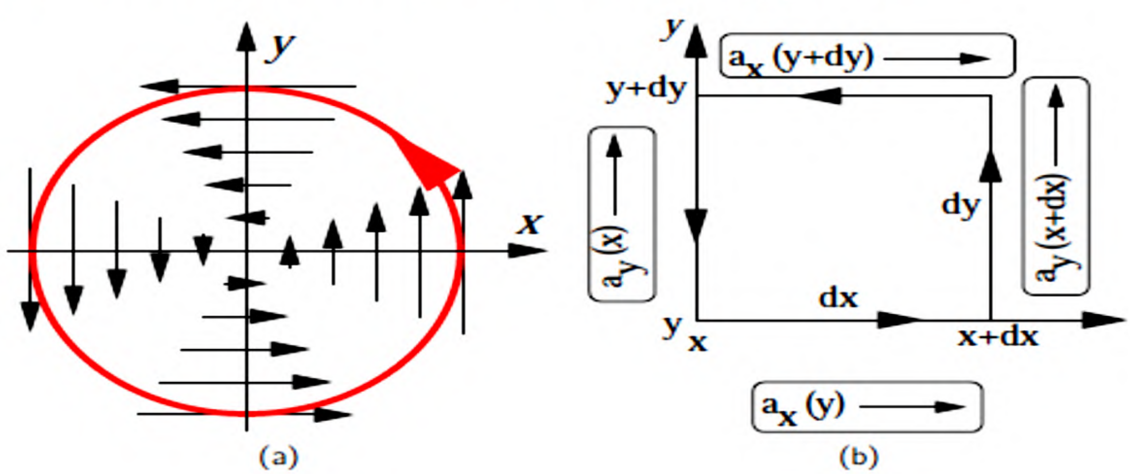


Figure 5.3: (a) A rough sketch of the vector field $-y\hat{\imath} + x\hat{\jmath}$. (b) An element in which to calculate curl.

In fact, curl is closely related to the line integral around a loop.

The circulation of a vector a round any closed curve C is defined to be $\oint_C \mathbf{a} \cdot d\mathbf{r}$ *and the curl of the vector field a represents the* ***vorticity****, or* ***circulation per unit area****, of the field*.

Our proof uses the small rectangular element dx by dy shown in Figure 5.3(b). Consider the circulation round the perimeter of a rectangular element.

The fields in the x direction at the bottom and top are

$$a_x(y) \qquad \text{and} \qquad a_x(y+dy) = a_x(y) + \frac{\partial a_x}{\partial y} dy,$$

where $a_x(y)$ denotes a_x is a function of y, and the fields in the y direction at the left and right are

$$a_y(x) \qquad \text{and} \qquad a_y(x+dx) = a_y(x) + \frac{\partial a_y}{\partial x} dx$$

Starting at the bottom and working round in the anticlockwise sense, the four contributions to the circulation dC are therefore as follows, where the minus signs take account of the path being opposed to the field:

$$\begin{aligned} dC &= +[a_x(y)\ dx] + [a_y(x+dx)\ dy] - [a_x(y+dy)\ dx] - [a_y(x)\ dy] \\ &= +[a_x(y)\ dx] + \left[\left(a_y(x) + \frac{\partial a_y}{\partial x} dx\right) dy\right] - \left[\left(a_x(y) + \frac{\partial a_x}{\partial y} dy\right) dx\right] - [a_y(x)\ dy] \\ &= \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) dx\, dy \\ &= (\boldsymbol{\nabla} \times \mathbf{a}) \cdot d\mathbf{S} \end{aligned}$$

where $d\mathbf{S} = dx\,dy\,\hat{\boldsymbol{k}}$.

NB: Again, this is not a completely rigorous proof as we have not shown that the result is independent of the co-ordinate system used.so proof it

Some definitions involving div, curl and grad

- A vector field with zero divergence is said to be ***solenoidal.***
- A vector field with zero curl is said to ***be irrotational***.
- A scalar field with zero gradient is said to be***, constant***.

Vector Calculus Expressions and Identities

In this Topics we look at more complicated identities involving vector operators. The main thing to appreciate it that the operators behave both as vectors and as differential operators, so that the usual rules of taking the derivative of, say, a product must be observed. why we need vector calculus?

First, since grad, div and curl describe key aspects of vectors fields, they arise often in practice, and so the identities can save you a lot of time and hacking of partial derivatives, as we will see when we consider Maxwell's equation as an example later.

Secondly, they help to identify other practically important vector operators.

1. ***Identity 1: curl grad U = 0***

$$\begin{aligned}\nabla \times \nabla U &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial U/\partial x & \partial U/\partial y & \partial U/\partial z \end{vmatrix} \\ &= \hat{\imath}\left(\frac{\partial^2 U}{\partial y \partial z} - \frac{\partial^2 U}{\partial z \partial y}\right) + \hat{\jmath}() + \hat{k}() \\ &= \mathbf{0}\ ,\end{aligned}$$

as $\partial^2/\partial y \partial z = \partial^2/\partial z \partial y$.

Note that the output is a null *vector*.

2. **Identity 2: div curl a = 0**

$$\begin{aligned}\nabla \cdot \nabla \times \mathbf{a} &= \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix} \\ &= \frac{\partial^2 a_z}{\partial x \partial y} - \frac{\partial^2 a_y}{\partial x \partial z} - \frac{\partial^2 a_z}{\partial y \partial x} + \frac{\partial^2 a_x}{\partial y \partial z} + \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial z \partial y} \\ &= 0\end{aligned}$$

3. ***Identity 3: div and curl of Ua***

Suppose that $U(r)$ is a scalar field and that a(r) is a vector field and we are interested in the product Ua. This is a vector field, so we can compute its divergence and curl. For example, the density $\rho(r)$ of a fluid is a scalar field, and the instantaneous velocity of the fluid $v(r)$ is a vector field, and we are probably interested in mass flow rates for which we will be interested in $\rho(r)\boldsymbol{v}(\boldsymbol{r})$.

The divergence (a scalar) of the product Ua is given by:

$$\begin{aligned} \nabla \cdot (U\mathbf{a}) &= U(\nabla \cdot \mathbf{a}) + (\nabla U) \cdot \mathbf{a} \\ &= U\text{div}\mathbf{a} + (\text{grad}U) \cdot \mathbf{a} \end{aligned}$$

In a similar way, we can take the curl of the vector field $U\boldsymbol{a}$, and the result should be a vector field:

$$\nabla \times (Ua) = U\nabla \times \boldsymbol{a} + (\nabla U) \times \boldsymbol{a}\,.$$

4. **Identity 4: div of $\boldsymbol{a} \times \boldsymbol{b}$**

Life quickly gets trickier when vector or scalar products are involved: For example, it is not *that* obvious that

$$\text{div}(\mathbf{a} \times \mathbf{b}) = \text{curl}\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \text{curl}\mathbf{b}$$

To show this, use the determinant:

$$\begin{vmatrix} \partial/\partial x_i & \partial/\partial x_j & \partial/\partial x_k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \frac{\partial}{\partial x}[a_y b_z - a_z b_y] + \frac{\partial}{\partial y}[a_z b_x - a_x b_z] + \frac{\partial}{\partial z}[a_x b_y - a_y b_x]$$

$$= \ \ldots \text{ bash out the products } \ldots$$

$$= \text{curl}\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot (\text{curl } \mathbf{b})$$

5. **Identity 5: $\boldsymbol{curl(a \times b)}$**

$$\text{curl}(\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_y b_z - a_z b_y & a_z b_x - a_x b_z & a_x b_y - a_y b_x \end{vmatrix}$$

so the $\hat{\imath}$ component is

$$\frac{\partial}{\partial y}(a_x b_y - a_y b_x) - \frac{\partial}{\partial z}(a_z b_x - a_x b_z)$$

which can be written as the sum of four terms:

$$a_x\left(\frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z}\right) - b_x\left(\frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right) + \left(b_y\frac{\partial}{\partial y} + b_z\frac{\partial}{\partial z}\right)a_x - \left(a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}\right)b_x$$

Adding $a_x(\partial b_x/\partial x)$ to the first of these, and subtracting it from the last, and doing the same with $b_x(\partial a_x/\partial x)$ to the other two terms, we find that (you should of course check this):

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\nabla \cdot \mathbf{b})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} + [\mathbf{b} \cdot \nabla]\mathbf{a} - [\mathbf{a} \cdot \nabla]\mathbf{b}$$

where $[\mathbf{a} \cdot \nabla]$ can be regarded as new, and very useful, scalar differential operator.

6. Definition of the operator $[\boldsymbol{a}\ \cdot\ \boldsymbol{\nabla}]$

This is a *scalar operator*, but it can obviously can be applied to a scalar field, resulting in a scalar field, or to a vector field resulting in a vector field:

$$[\boldsymbol{a}\ \cdot\ \boldsymbol{\nabla}] \equiv \left[a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}\right]$$

7. Identity 6: $curl(curl\boldsymbol{a})$ for you to derive

The following important identity is stated, and left as an exercise:

$$curl(curla)\ =\ graddiv\boldsymbol{a}\ -\ \nabla^2\boldsymbol{a}$$

where

$$\nabla^2\boldsymbol{a}\ =\ \nabla^2\boldsymbol{a_x}\hat{\imath} +\ \nabla^2\boldsymbol{a_y}\hat{\jmath} +\ \nabla^2\boldsymbol{a_z}\hat{k}$$

♣ Example of Identity 6: electromagnetic waves

Q: James Clerk Maxwell established a set of four vector equations which are fundamental to working out how electromagnetic waves propagate. The entire telecommunications industry is built on these.

$$\begin{aligned} \text{div}\mathbf{D} &= \rho \\ \text{div}\mathbf{B} &= 0 \\ \text{curl}\mathbf{E} &= -\frac{\partial}{\partial t}\mathbf{B} \\ \text{curl}\mathbf{H} &= \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D} \end{aligned}$$

In addition, we can assume the following, which should all be familiar to you: $\mathbf{B} = \mu_r\mu_0\mathbf{H}$, $\mathbf{J} = \sigma\mathbf{E}$, $\mathbf{D} = \epsilon_r\epsilon_0\mathbf{E}$, where all the scalars are constants.

Now show that in a material with zero free charge density, $\rho = 0$, and with zero conductivity, $\sigma = 0$, the electric field $\mathbf{E}$ must be a solution of the wave equation

$$\nabla^2\mathbf{E} = \mu_r\mu_0\epsilon_r\epsilon_0(\partial^2\mathbf{E}/\partial t^2)\ .$$

A: First, a bit of respect. Imagine you are the first to do this — this is a tingle moment.

$$\begin{aligned} \text{div}\mathbf{D} &= \text{div}(\epsilon_r\epsilon_0\mathbf{E}) = \epsilon_r\epsilon_0\text{div}\mathbf{E} = \rho = 0 \quad \Rightarrow \text{div}\mathbf{E} = 0. \quad (a) \\ \text{div}\mathbf{B} &= \text{div}(\mu_r\mu_0\mathbf{H}) = \mu_r\mu_0\text{div}\mathbf{H} = 0 \quad \Rightarrow \text{div}\mathbf{B} = 0 \quad (b) \\ \text{curl}\mathbf{E} &= -\partial\mathbf{B}/\partial t = -\mu_r\mu_0(\partial\mathbf{H}/\partial t) \quad (c) \\ \text{curl}\mathbf{H} &= \mathbf{J} + \partial\mathbf{D}/\partial t = \mathbf{0} + \epsilon_r\epsilon_0(\partial\mathbf{E}/\partial t) \quad (d) \end{aligned}$$

But we know (or rather you worked out in Identity 6) that curlcurl = graddiv− ∇^2, and using (c)

$$\text{curlcurl}\mathbf{E} = \text{graddiv}\mathbf{E} - \nabla^2\mathbf{E} = \text{curl}\left(-\mu_r\mu_0(\partial\mathbf{H}/\partial t)\right)$$

so interchanging the order of partial differentation, and using (a) $\text{div}\mathbf{E} = 0$:

$$\begin{aligned} -\nabla^2\mathbf{E} &= -\mu_r\mu_0\frac{\partial}{\partial t}(\text{curl}\mathbf{H}) \\ &= -\mu_r\mu_0\frac{\partial}{\partial t}\left(\epsilon_r\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) \\ \Rightarrow \nabla^2\mathbf{E} &= \mu_r\mu_0\epsilon_r\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} \end{aligned}$$

This equation is actually three equations, one for each component:

$$\nabla^2 E_x = \mu_r\mu_0\epsilon_r\epsilon_0\frac{\partial^2 E_x}{\partial t^2}$$

and so on for E_y and E_z.

8. *Grad, div, curl and ∇^2 in curvilinear co-ordinate systems*

It is possible to obtain general expressions for grad, div and curl in any orthogonal curvilinear co-ordinate system by making use of the h factors which were introduced in Lecture 4.

We recall that the unit vector in the direction of increasing u, with v and w being kept constant, is

$$\hat{\mathbf{u}} = \frac{1}{h_u}\frac{\partial \mathbf{r}}{\partial u}$$

where $\mathbf{r}$ is the position vector, and

$$h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right|$$

is the metric coefficient. Similar expressions apply for the other co-ordinate directions. Then

$$d\mathbf{r} = h_u \hat{\mathbf{u}} du + h_v \hat{\mathbf{v}} dv + h_w \hat{\mathbf{w}} dw \ .$$

9. ***Grad in curvilinear coordinates***

Noting that $U = U(\mathbf{r})$ and $U = U(u, v, w)$, and using the properties of the gradient of a scalar field obtained previously

$$\nabla U \cdot d\mathbf{r} = dU = \frac{\partial U}{\partial u}du + \frac{\partial U}{\partial v}dv + \frac{\partial U}{\partial w}dw$$

It follows that

$$\nabla U \cdot (h_u \hat{\mathbf{u}} du + h_v \hat{\mathbf{v}} dv + h_w \hat{\mathbf{w}} dw) = \frac{\partial U}{\partial u}du + \frac{\partial U}{\partial v}dv + \frac{\partial U}{\partial w}dw$$

The only way this can be satisfied for independent du, dv, dw is when

$$\boxed{\nabla U = \frac{1}{h_u}\frac{\partial U}{\partial u}\hat{\mathbf{u}} + \frac{1}{h_v}\frac{\partial U}{\partial v}\hat{\mathbf{v}} + \frac{1}{h_w}\frac{\partial U}{\partial w}\hat{\mathbf{w}}}$$

10. ***Divergence in curvilinear coordinates***

Expressions can be obtained for the divergence of a vector field in orthogonal curvilinear co-ordinates by making use of the flux property.

We consider an element of volume $d\boldsymbol{V}$. If the curvilinear coordinates are orthogonal then the little volume is a cuboid (to first order in small quantities) and

$$dV = hu\, hv\, hw\ \ du\, dv\, dw$$

However, it is not quite a cuboid: the area of two opposite faces will differ as the scale parameters are functions of u, v and w in general.

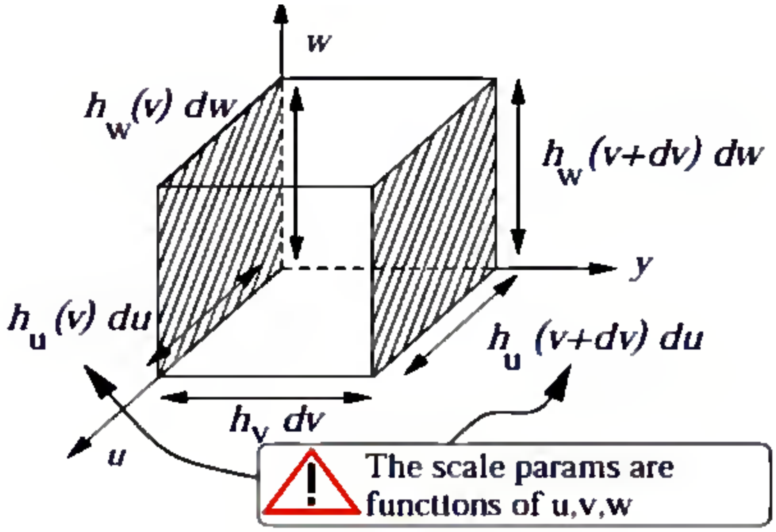


Figure 6.1: Elemental volume for calculating divergence in orthogonal curvilinear coordinates

So, the net efflux from the two faces in the ˆv direction shown in Figure 6.1 is

$$
\begin{aligned}
&= \left[a_v + \frac{\partial a_v}{\partial v}dv\right]\left[h_u + \frac{\partial h_u}{\partial v}dv\right]\left[h_w + \frac{\partial h_w}{\partial v}dv\right] du dw - a_v h_u h_w du dw \\
&= \frac{\partial(a_v h_u h_w)}{\partial v} du dv dw
\end{aligned}
$$

which is easily shown by multiplying the first line out and dropping second order terms (i.e. $(dv)^2$).

By definition div is the net efflux per unit volume, so summing up the other faces:

$$
\begin{aligned}
\text{div}\mathbf{a}\ dV &= \left(\frac{\partial(a_u\ h_v\ h_w)}{\partial u} + \frac{\partial(a_v\ h_u\ h_w)}{\partial v} + \frac{\partial(a_w\ h_u\ h_v)}{\partial w}\right) du dv dw \\
\Rightarrow \text{div}\mathbf{a}\ h_u h_v h_w\ du dv dw &= \left(\frac{\partial(a_u\ h_v\ h_w)}{\partial u} + \frac{\partial(a_v\ h_u\ h_w)}{\partial v} + \frac{\partial(a_w\ h_u\ h_v)}{\partial w}\right) du dv dw
\end{aligned}
$$

So, finally,

$$
\boxed{\text{div}\mathbf{a} = \frac{1}{h_u h_v h_w}\left(\frac{\partial(a_u\ h_v\ h_w)}{\partial u} + \frac{\partial(a_v\ h_u\ h_w)}{\partial v} + \frac{\partial(a_w\ h_u\ h_v)}{\partial w}\right)}
$$

11. *Curl in curvilinear coordinates*

Recall from Lecture 5 that we computed the z component of curl as the circulation per unit area from

$$dC = \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\ dx\ dy$$

By analogy with our derivation of divergence, you will realize that for an orthogonal curvilinear coordinate system we can write the area as $h_u h_v du dw$. But the opposite sides are no longer quite of the same length. The lower of the pair in Figure 6.2 is length $h_u(v)du$, but the upper is of length $h_u(v+dv)du$

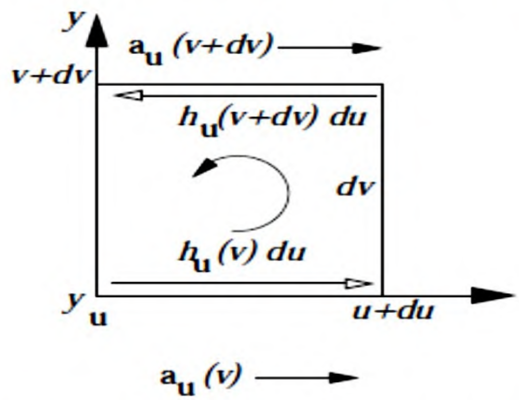


Figure 6.2: Elemental loop for calculating curl in orthogonal curvilinear coordinates

Summing this pair gives a contribution to the circulation

$$a_u(v)h_u(v)du - a_u(v+dv)h_u(v+dv)du = -\frac{\partial(h_u a_u)}{\partial v}dv du$$

and together with the other pair:

$$dC = \left(-\frac{\partial(h_u a_u)}{\partial v} + \frac{\partial(h_v a_v)}{\partial u}\right)\ du dv$$

So the circulation per unit area is

$$\frac{dC}{h_u h_v du dv} = \frac{1}{h_u h_v}\left(\frac{\partial(h_v a_v)}{\partial u} - \frac{\partial(h_u a_u)}{\partial v}\right)$$

and hence curl is

$$\begin{aligned}\text{curl}\mathbf{a}(u,v,w) \;=\; & \frac{1}{h_v h_w}\left(\frac{\partial(h_w a_w)}{\partial v} - \frac{\partial(h_v a_v)}{\partial w}\right)\hat{\mathbf{u}} \;+ \\ & \frac{1}{h_w h_u}\left(\frac{\partial(h_u a_u)}{\partial w} - \frac{\partial(h_w a_w)}{\partial u}\right)\hat{\mathbf{v}} \;+ \\ & \frac{1}{h_u h_v}\left(\frac{\partial(h_v a_v)}{\partial u} - \frac{\partial(h_u a_u)}{\partial v}\right)\hat{\mathbf{w}}\end{aligned}$$

You should check that this can be written as

Curl in curvilinear coords:

$$\text{curl}\mathbf{a}(u,v,w) = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{\mathbf{u}} & h_v\hat{\mathbf{v}} & h_w\hat{\mathbf{w}} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u a_u & h_v a_v & h_w a_w \end{vmatrix}$$

12. *The Laplacian in curvilinear coordinates*

Substitution of the components of grad$U$ into the expression for diva immediately (!*?) gives the following expression for the Laplacian in general orthogonal co-ordinates:

$$\nabla^2 U = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial U}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_w h_u}{h_v}\frac{\partial U}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial U}{\partial w}\right)\right].$$

13. Grad Div, Curl, ∇^2 in cylindrical polars

Here $(u, v, w) \to (r, \phi, z)$. The position vector is $\mathbf{r} = r\cos\phi\hat{\boldsymbol{\imath}} + r\sin\phi\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$, and $h_r = |\partial\mathbf{r}/\partial r|$, etc.

$$\begin{aligned}
\Rightarrow h_r &= \sqrt{(\cos^2\phi + sin^2\phi)} = 1, \\
h_\phi &= \sqrt{(r^2\sin^2\phi + r^2\cos^2\phi)} = r, \\
h_z &= 1
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \text{grad}U &= \frac{\partial U}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\mathbf{e}}_\phi + \frac{\partial U}{\partial z}\hat{\boldsymbol{k}} \\
\text{div}\mathbf{a} &= \frac{1}{r}\left(\frac{\partial(r a_r)}{\partial r} + \frac{\partial a_\phi}{\partial \phi}\right) + \frac{\partial a_z}{\partial z} \\
\text{curl}\mathbf{a} &= \left(\frac{1}{r}\frac{\partial a_z}{\partial \phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\hat{\mathbf{e}}_\phi + \frac{1}{r}\left(\frac{\partial(r a_\phi)}{\partial r} - \frac{\partial a_r}{\partial \phi}\right)\hat{\boldsymbol{k}} \\
\nabla^2 U &= \text{Tutorial Exercise}
\end{aligned}$$

14. Grad Div, Curl, ∇^2 in spherical polars

Here $(u, v, w) \to (r, \theta, \phi)$. The position vector is $\mathbf{r} = r\sin\theta\cos\phi\hat{\boldsymbol{\imath}} + r\sin\theta\sin\phi\hat{\boldsymbol{\jmath}} + r\cos\theta\hat{\boldsymbol{k}}$.

$$\begin{aligned}
\Rightarrow h_r &= \sqrt{(\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta)} = 1 \\
h_\theta &= \sqrt{(r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta)} = r \\
h_\phi &= \sqrt{(r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)} = r\sin\theta
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \text{grad}U &= \frac{\partial U}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\mathbf{e}}_\phi \\
\text{div}\mathbf{a} &= \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(a_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\
\text{curl}\mathbf{a} &= \frac{\hat{\mathbf{e}}_r}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(a_\phi\sin\theta) - \frac{\partial}{\partial\phi}(a_\theta)\right) + \frac{\hat{\mathbf{e}}_\theta}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(a_r) - \frac{\partial}{\partial r}(a_\phi r\sin\theta)\right) + \\
&\quad \frac{\hat{\mathbf{e}}_\phi}{r}\left(\frac{\partial}{\partial r}(a_\theta r) - \frac{\partial}{\partial\theta}(a_r)\right) \\
\nabla^2 U &= \text{Tutorial Exercise}
\end{aligned}$$

♣ **Examples**

Q1 Find curla in (i) Cartesians and (ii) Spherical polars when $\mathbf{a} = x(x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}})$.

A1 (i) In Cartesians

$$\text{curl}\mathbf{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & xz \end{vmatrix} = -z\hat{\jmath} + y\hat{\mathbf{k}} \ .$$

(ii) In spherical polars, $x = r\sin\theta\cos\phi$ and $\mathbf{r} = (x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}})$. So

$$\begin{aligned} \mathbf{a} &= r^2\sin\theta\cos\phi\hat{\mathbf{e}}_r \\ \Rightarrow a_r &= r^2\sin\theta\cos\phi; \qquad a_\theta = 0; \qquad a_\phi = 0 \ . \end{aligned}$$

Hence as

$$\text{curl}\mathbf{a} = \frac{\hat{\mathbf{e}}_r}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(a_\phi\sin\theta) - \frac{\partial}{\partial\phi}(a_\theta)\right) + \frac{\hat{\mathbf{e}}_\theta}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(a_r) - \frac{\partial}{\partial r}(a_\phi r\sin\theta)\right) + \frac{\hat{\mathbf{e}}_\phi}{r}\left(\frac{\partial}{\partial r}(a_\theta r) - \frac{\partial}{\partial\theta}(a_r)\right)$$

$$\begin{aligned} \text{curl}\mathbf{a} &= \frac{\hat{\mathbf{e}}_\theta}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(r^2\sin\theta\cos\phi)\right) + \frac{\hat{\mathbf{e}}_\phi}{r}\left(-\frac{\partial}{\partial\theta}(r^2\sin\theta\cos\phi)\right) \\ &= \frac{\hat{\mathbf{e}}_\theta}{r\sin\theta}(-r^2\sin\theta\sin\phi) + \frac{\hat{\mathbf{e}}_\phi}{r}(-r^2\cos\theta\cos\phi)) \\ &= \hat{\mathbf{e}}_\theta(-r\sin\phi) + \hat{\mathbf{e}}_\phi(-r\cos\theta\cos\phi) \end{aligned}$$

Checking: these two results should be the same, but to check we need expressions for $\hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$ in terms of $\hat{\imath}$ etc.
Remember that we can work out the unit vectors $\hat{\mathbf{e}}_r$ and so on in terms of $\hat{\imath}$ etc using

$$\hat{\mathbf{e}}_r = \frac{1}{h_1}\frac{\partial\mathbf{r}}{dr}; \qquad \hat{\mathbf{e}}_\theta = \frac{1}{h_2}\frac{\partial\mathbf{r}}{d\theta}; \qquad \hat{\mathbf{e}}_\phi = \frac{1}{h_3}\frac{\partial\mathbf{r}}{d\phi} \quad \text{where } \mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}} \ .$$

Grinding through we find

$$\begin{bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{\mathbf{k}} \end{bmatrix} = [R] \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{\mathbf{k}} \end{bmatrix}$$

Don't be shocked to see a rotation matrix $[R]$: we are after all rotating one right-handed orthogonal coord system into another.
So the result in spherical polars is

$$\begin{aligned} \text{curl}\mathbf{a} &= (\cos\theta\cos\phi\hat{\imath} + \cos\theta\sin\phi\hat{\jmath} - \sin\theta\hat{\mathbf{k}})(-r\sin\phi) + (-\sin\phi\hat{\imath} + \cos\phi\hat{\jmath})(-r\cos\theta\cos \\ &= -r\cos\theta\hat{\jmath} + r\sin\theta\sin\phi\hat{\mathbf{k}} \\ &= -z\hat{\jmath} + y\hat{\mathbf{k}} \end{aligned}$$

which is exactly the result in Cartesians.

Applications of Vector Calculus

In Lecture under identity we saw one classic example of the application of vector calculus to Maxwell's equation.

In this lecture we explore a few more examples from fluid mechanics and heat transfer. As with Maxwell's equations, the examples show how vector calculus provides a powerful way of representing underlying physics.

The power come from the fact that div, grad and curl have a significance or meaning which is more immediate than a collection of partial derivatives. Vector calculus will, with practice, become a convenient shorthand for you.

- Electricity – Ampere's Law
- Fluid Mechanics - The Continuity Equation
- Thermo: The Heat Conduction Equation
- Mechanics/Electrostatics - Conservative fields
- The Inverse Square Law of force
- Gravitational field due to distributed mass
- Gravitational field inside body
- Pressure forces in non-uniform flows

a. Electricity – Ampere's Law

If the frequency is low, the displacement current in Maxwell's equation $\text{curl}\mathbf{H} = \mathbf{J} + \partial\mathbf{D}/\partial t$ is negligible, and we find

$$\text{curl}\mathbf{H} = \mathbf{J}$$

Hence

$$\int_S \text{curl}\mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

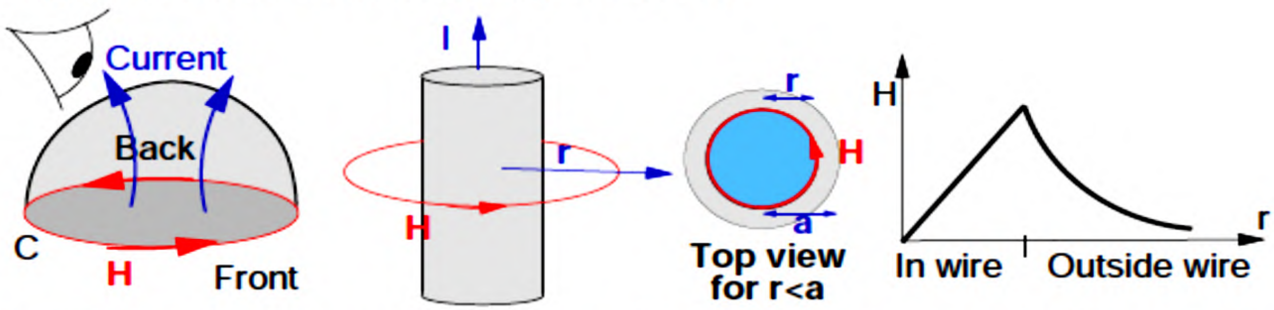
or

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

where $\int_S \mathbf{J} \cdot d\mathbf{S}$ is total current through the surface.

Now consider the $\mathbf{H}$ around a straight wire carrying current I. Symmetry tells us the $\mathbf{H}$ is in the $\hat{\mathbf{e}}_\theta$ direction, in a rhs screw sense with respect to the current. (You might check this against Biot-Savart's law.)

Suppose we asked what is the magnitude of $\mathbf{H}$?



Inside the wire, the bounding contour only encloses a fraction $(\pi r^2)/(\pi a^2)$ of the current, and so

$$\begin{aligned} H2\pi r &= \int \mathbf{J} \cdot d\mathbf{S} = I(r^2/A^2) \\ \Rightarrow H &= Ir/2\pi A^2 \end{aligned}$$

whereas outside we enclose all the current, and so

$$\begin{aligned} H2\pi r &= \int \mathbf{J} \cdot d\mathbf{S} = I \\ \Rightarrow H &= I/2\pi r \end{aligned}$$

A plot is shown in the Figure.

b. Fluid Mechanics - The Continuity Equation

The **Continuity Equation** expresses the condition of conservation of mass in a fluid flow. The continuity principle applied to *any* volume (called a *control volume*) may be expressed in words as follows:

> "The net rate of mass flow of fluid out of the control volume must equal the rate of decrease of the mass of fluid within the control volume"

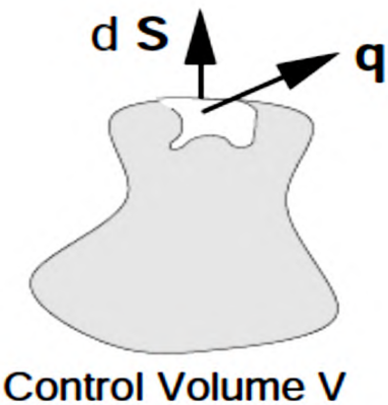


Figure 8.1:

To express the above as a mathematical equation, we denote the velocity of the fluid at each point of the flow by $\mathbf{q}(\mathbf{r})$ (a vector field) and the density by $\rho(\mathbf{r})$ (a scalar field). The element of rate-of-volume-loss through surface $d\mathbf{S}$ is $d\dot{V} = \mathbf{q}\cdot d\mathbf{S}$, so the rate of mass loss is

$$d\dot{M} = \rho\mathbf{q}\cdot d\mathbf{S},$$

so that the total rate of mass loss from the volume is

$$-\frac{\partial}{\partial t}\int_V \rho(\mathbf{r})dV = \int_S \rho\mathbf{q}\cdot d\mathbf{S}.$$

Assuming that the volume of interest is fixed, this is the same as

$$-\int_V \frac{\partial\rho}{\partial t}dV = \int_S \rho\mathbf{q}\cdot d\mathbf{S}\ .$$

Now we use Gauss' Theorem to transform the RHS into a volume integral

$$-\int_V \frac{\partial\rho}{\partial t}dV = \int_V \text{div }(\rho\mathbf{q})dV\ .$$

The two volume integrals can be equal for any control volume V only if the two integrands are equal at each point of the flow. This leads to the mathematical formulation of

The Continuity Equation:

$$\text{div }(\rho\mathbf{q}) = -\frac{\partial\rho}{\partial t}$$

Notice that if the density doesn't vary with time, div $(\rho\mathbf{q}) = 0$, and if the density doesn't vary with position then

The Continuity Equation for uniform, time-invariant density:

$$\text{div }(\mathbf{q}) = 0 \quad .$$

In this last case, we can say that the flow **q** is solenoidal.

c. Thermodynamics - The Heat Conduction Equation

Flow of heat is very similar to flow of fluid, and heat flow satisfies a similar continuity equation. The flow is characterized by the heat current density q(r) (heat flow per unit area and time), sometimes misleadingly called heat flux.
Assuming that there is no mass flow across the boundary of the control volume and no source of heat inside it, the rate of flow of heat out of the control volume by conduction must equal the rate of decrease of internal energy (constant volume) or enthalpy (constant pressure) within it. This leads to the equation

$$\text{div }\ \mathbf{q} = -\rho c\frac{\partial T}{\partial t},$$

where ρ is the density of the conducting medium, c its specific heat (both are assumed constant) and T is the temperature.

In order to solve for the temperature field another equation is required, linking q to the temperature gradient. This is

$$\mathbf{q} = -\kappa\text{grad }\ T,$$

where κ is the thermal conductivity of the medium. Combining the two equations gives the *heat conduction equation*:

$$-\text{div }\mathbf{q} = \kappa\text{div }\ \text{grad }\ T = \kappa\nabla^2 T = \rho c\frac{\partial T}{\partial t}$$

where it has been assumed that κ is a constant. In steady flow the temperature field satisfies Laplace's Equation $\nabla^2 T = 0$.

d. Mechanics - Conservative fields of force

A conservative field of force is one for which the work done

$$\int_A^B \mathbf{F} \cdot d\mathbf{r},$$

moving from A to B is indep. of path taken. As we saw in Lecture 4, conservative fields must satisfy the condition

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0,$$

Stokes' tells us that this is

$$\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0,$$

where S is *any* surface bounded by C.
But if true for *any* C containing A and B, it must be that

$$\text{curl } \mathbf{F} = \mathbf{0}$$

Conservative fields are irrotational
All radial fields are irrotational

One way (actually the only way) of satisfying this condition is for

$$\mathbf{F} = \boldsymbol{\nabla} U$$

The scalar field $U(\mathbf{r})$ is the Potential Function

e. The Inverse Square Law of force

Radial forces are found in electrostatics and gravitation — so they are certainly irrotational and conservative.

But in nature these radial forces are also inverse square laws. One reason why this may be so is that it turns out to be the only central force field which is **solenoidal**, i.e. has zero divergence.

If $\mathbf{F} = f(r)\mathbf{r}$,

$$\text{div } \mathbf{F} = 3f(r) + rf'(r).$$

For div $\mathbf{F} = 0$ we conclude

$$r\frac{df}{dr} + 3f = 0$$

or

$$\frac{df}{f} + 3\frac{dr}{r} = 0.$$

Integrating with respect to r gives $fr^3 = const = A$, so that

$$\mathbf{F} = \frac{A\mathbf{r}}{r^3}, \quad |\mathbf{F}| = \frac{A}{r^2}.$$

The condition of zero divergence of the inverse square force field applies everywhere except at $\mathbf{r} = \mathbf{0}$, where the divergence is infinite.

To show this, calculate the outward normal flux out of a sphere of radius R centered on the origin when $\mathbf{F} = F\hat{\mathbf{r}}$. This is

$$\int_S \mathbf{F} \cdot d\mathbf{S} = F \int_{\text{Sphere}} \hat{\mathbf{r}} \cdot d\mathbf{S} = F \int_{\text{Sphere}} d = F4\pi R^2 = 4\pi A = \text{Constant}.$$

Gauss tells us that this flux must be equal to

$$\int_V \text{div } \mathbf{F} dV = \int_0^R \text{div } \mathbf{F} 4\pi r^2 dr$$

where we have done the volume integral as a summation over thin shells of surface area $4\pi r^2$ and thickness dr.

But for all finite r, div$\mathbf{F} = 0$, so div$\mathbf{F}$ must be infinite at the origin.

The flux integral is thus

- zero — for any volume which does not contain the origin
- $4\pi A$ for any volume which does contain it.

f. Gravitational field due to distributed mass: Poisson's Equation

If one tried the same approach as §8.4 for the gravitational field, $A = Gm$, where m is the mass at the origin and G the universal gravitational constant, one would run into the problem that there is no such thing as point mass.

We can make progress though by considering distributed mass.

The mass contained in each small volume element dV is ρdV and this will make a contribution $-4\pi\rho G dV$ to the flux integral from the control volume. Mass outside the control volume makes no contribution, so that we obtain the equation

$$\int_S \mathbf{F} \cdot d\mathbf{S} = -4\pi G \int_V \rho dV.$$

Transforming the left hand integral by Gauss' Theorem gives

$$\int_V \text{div } \mathbf{F} dV = -4\pi G \int_V \rho dV$$

which, since it is true for any V, implies that

$$-\text{div } \mathbf{F} = 4\pi\rho G.$$

Since the gravitational field is also conservative (i.e. irrotational) it must have an associated potential function U, so that $\mathbf{F} = \text{grad } U$. It follows that the gravitational potential U satisfies

Poisson's Equation

$$\nabla^2 U = 4\pi\rho G \ .$$

Using the integral form of Poisson's equation, it is possible to calculate the gravitational field inside a spherical body whose density is a function of radius only. We have

$$4\pi R^2 F = 4\pi G \int_0^R 4\pi r^2 \rho dr,$$

where $F = |\mathbf{F}|$, or

$$|F| = \frac{G}{R^2} \int_0^R 4\pi r^2 \rho dr = \frac{MG}{R^2},$$

where M is the total mass inside radius R. For the case of uniform density, this is equal to $M = \frac{4}{3}\pi\rho R^3$ and $|F| = \frac{4}{3}\pi\rho G R$.

g. Pressure forces in non-uniform flows

When a body is immersed in a flow it experiences a net pressure force

$$\mathbf{F}_p = -\int_S p d\mathbf{S},$$

where S is the surface of the body. If the pressure p is non-uniform, this integral is not zero. The integral can be transformed using Gauss' Theorem to give the alternative expression

$$\mathbf{F}_p = -\int_V \text{grad } p \, dV,$$

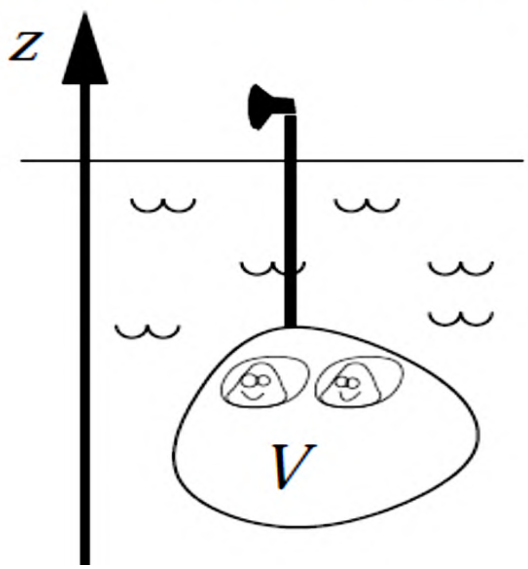
where V is the volume of the body. In the simple hydrostatic case $p + \rho g z = constant$, so that

$$\text{grad } p = -\rho g k$$

and the net pressure force is simply

$$\mathbf{F}_p = g\hat{\mathbf{k}} \int_V \rho dV$$

which, in agreement with Archimedes' principle, is equal to the weight of fluid displaced.



Chapter Three

Complex Variables

FUNCTIONS

If to each of a set of complex numbers which a variable z may assume there corresponds one or more values of a variable w, then w is called a *function of the complex variable* z, written $w = f(z)$. The fundamental operations with complex numbers have already been considered in Chapter 1.

A function is *single-valued* if for each value of z there corresponds only one value of w; otherwise it is *multiple-valued* or *many-valued*. In general we can write $w = f(z) = u(x, y) + iv(x, y)$, where u and v are real functions of x and y.

Example 1. $w = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy = u + iv$ so that $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$. These are called the *real* and *imaginary parts* of $w = z^2$ respectively.

Unless otherwise specified we shall assume that $f(z)$ is single-valued. A function which is multiple-valued can be considered as a collection of single-valued functions.

LIMITS AND CONTINUITY

Definitions of limits and continuity for functions of a complex variable are analogous to those for a real variable. Thus $f(z)$ is said to have the *limit* l as z approaches z_0 if, given any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

Similarly, $f(z)$ is said to be *continuous* at z_0 if, given any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$. Alternatively, $f(z)$ is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

DERIVATIVES

If $f(z)$ is single-valued in some region of the z plane the *derivative* of $f(z)$, denoted by $f'(z)$, is defined as

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad (1)$$

provided the limit exists independent of the manner in which $\Delta z \to 0$. If the limit (1) exists for $z = z_0$, then $f(z)$ is called *analytic* at z_0. If the limit exists for all z in a region $\mathcal{R}$, then $f(z)$ is called *analytic in* $\mathcal{R}$. In order to be analytic, $f(z)$ must be single-valued and continuous. The converse, however, is not necessarily true.

We define elementary functions of a complex variable by a natural extension of the corresponding functions of a real variable. Where series expansions for real functions $f(x)$ exist, we can use as definition the series with x replaced by z.

Example 2. We define $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$, $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$, $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$. From these we can show that $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$, as well as numerous other relations.

Example 3. We define a^b as $e^{b \ln a}$ even when a and b are complex numbers. Since $e^{2k\pi i} = 1$, it follows that $e^{i\phi} = e^{i(\phi + 2k\pi)}$ and we define $\ln z = \ln(\rho e^{i\phi}) = \ln \rho + i(\phi + 2k\pi)$. Thus $\ln z$ is a many-valued function. The various single-valued functions of which this many-valued function is composed are called its *branches*.

Rules for differentiating functions of a complex variable are much the same as for those of real variables. Thus $\frac{d}{dz}(z^n) = nz^{n-1}$, $\frac{d}{dz}(\sin z) = \cos z$, etc.

CAUCHY-RIEMANN EQUATIONS

A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region $\mathcal{R}$ is that u and v satisfy the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

(see Problem 13.7). If the partial derivatives in (2) are continuous in $\mathcal{R}$, the equations are sufficient conditions that $f(z)$ be analytic in $\mathcal{R}$.

If the second derivatives of u and v with respect to x and y exist and are continuous, we find by differentiating (2) that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{3}$$

Thus the real and imaginary parts satisfy Laplace's equation in two dimensions. Functions satisfying Laplace's equation are called *harmonic functions*.

INTEGRALS

If $f(z)$ is defined, single-valued and continuous in a region $\mathcal{R}$, we define the *integral* of $f(z)$ along some path C in $\mathcal{R}$ from point z_1 to point z_2, where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, as

$$\int_C f(z)\,dz = \int_{(x_1,y_1)}^{(x_2,y_2)} (u+iv)(dx + i\,dy) = \int_{(x_1,y_1)}^{(x_2,y_2)} u\,dx - v\,dy + i\int_{(x_1,y_1)}^{(x_2,y_2)} v\,dx + u\,dy$$

with this definition the integral of a function of a complex variable can be made to depend on line integrals for real functions already considered in Chapter 6. An alternative definition based on the limit of a sum, as for functions of a real variable, can also be formulated and turns out to be equivalent to the one above.

The rules for complex integration are similar to those for real integrals. An important result is

$$\left|\int_C f(z)\,dz\right| \leqq \int_C |f(z)|\,|dz| \leqq M\int_C ds = ML \tag{4}$$

where M is an upper bound of $|f(z)|$ on C, i.e. $|f(z)| \leqq M$, and L is the length of the path C.

CAUCHY'S THEOREM

Let C be a simple closed curve. If $f(z)$ is analytic within the region bounded by C as well as on C, then we have *Cauchy's theorem* that

$$\int_C f(z)\,dz = \oint_C f(z)\,dz = 0 \tag{5}$$

where the second integral emphasizes the fact that C is a simple closed curve.

Expressed in another way, *(5)* is equivalent to the statement that $\int_{z_1}^{z_2} \boldsymbol{f(z)dz}$ has a value *independent of the path* joining z_1 and z_2. Such integrals can be evaluated as $F(zz) - F(Z_1)$ where $F'(z) = f(z)$.

Example 4. Since $f(z) = 2x$ is analytic everywhere, we have for any simple closed curve C

$$\oint 2z\, dz = 0$$

Also $\int_{2i}^{1+i} 2z\, dz = z^2|_{2i}^{1+i} = (1+i)^2 - (2i)^2 = 2i + 4$

CAUCHY'S INTEGRAL FORMULAS

If $f(z)$ is analytic within and on a simple closed curve C and a is any point interior to C, then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz \tag{6}$$

where C is traversed in the positive (counterclockwise) sense.

Also, the nth derivative of $f(z)$ at $z = a$ is given by

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \tag{7}$$

These are called *Cauchy's integral formulas*. They are quite remarkable because they show that if the function $f(z)$ is known *on* the closed curve C then it is also known *within* C, and the various derivatives at points within C can be calculated. Thus if a function of a complex variable has a first derivative, it has all higher derivatives as well. This of course is not necessarily true for functions of real variables.

TAYLOR'S SERIES

Let $f(z)$ be analytic inside and on a circle having its center at $z = a$. Then for all points z in the circle we have the *Taylor series* representation of $f(z)$ given by

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots \tag{8}$$

SINGULAR POINTS

A singular point of a function $f(z)$ is a value of z at which $f(z)$ fails to be analytic. If $f(z)$ is analytic everywhere in some region except at an interior point $z = a$, we call $z = a$ an *isolated singularity* of $f(z)$.

Example 5. If $f(z) = \frac{1}{(z-3)^2}$, then $z = 3$ is an isolated singularity of $f(z)$.

POLES

If $f(z) = \frac{\phi(z)}{(z-a)^n}$, $\phi(a) \neq 0$, where $\phi(z)$ is analytic everywhere in a region including $z = a$, and if n is a positive integer, then $f(z)$ has an isolated singularity at $z = a$ which is called a *pole of order n*. If $n = 1$, the pole is often called a *simple pole*; if $n = 2$ it is called a *double pole*, etc.

Example 6. $f(z) = \frac{z}{(z-3)^2(z+1)}$ **has two singularities: a pole of order 2 or double pole at $z = 3$, and a pole of order 1 or simple pole at $z = -1$.**

Example 7. $f(z) = \frac{3z-1}{z^2+4} = \frac{3z-1}{(z+2i)(z-2i)}$ **has two simple poles at $z = \pm 2i$.**

A function can have other types of singularities besides poles. For example, $f(z) = \sqrt{z}$ has a *branch point* at $z = 0$. The function $f(z) = \frac{sinZ}{z}$ has a singularity at $z = 0$. However, due to the fact that $\lim_{z\to 0} \frac{sinZ}{z}$ is finite, we call such a singularity a ***removable singularity***.

LAURENT'S SERIES

If $f(z)$ has a pole of order n at $z = a$ but is analytic at every other point inside and on a circle *C* with center at $\boldsymbol{a}$, then $(z - a)^n f(z)$ is analytic at all points inside and on *C* and has a Taylor series about $z = a$ so that

$$\boldsymbol{f(z) = \frac{a_{-n}}{(z-a)^n} + \frac{a_{-n+1}}{(z-a)^{n-1}} + \cdots + \frac{a_{-n}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots} \qquad (9)$$

This is called a ***Laurent series* for *f(z)*.** The part $a_0 + a_1(z - a) + a_2(z - a)2 + \bullet\bullet\bullet$ is called the *analytic part,* while the remainder consisting of inverse powers of *z* — *a* is called the "*principal part*".

A function which is analytic in a region bounded by two concentric circles having center at $z = a$ can always be expanded into such a Laurent series.

It is possible to define various types of singularities of a function *f(z)* from its Laurent series. For example, when the principal part of a Laurent series has a finite number of terms and $a_{-n} \neq 0$ while $a_{-n-1}, a_{-n-z}, \ldots$ are all zero, then *z* = *a* is a pole of order *n.* If the principal part has infinitely many terms, *z* = *a* is called an *essential singularity* or sometimes a *pole of infinite order.*

Example 8. The function $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots$ has an essential singularity at $z = 0$.

RESIDUES

The coefficients in (9) can be obtained in the customary manner by writing the coefficients for the Taylor series corresponding to $(z-a)^n f(z)$. In further developments, the coefficient a_{-1}, called the *residue* of $f(z)$ at the pole $z = a$, is of considerable importance. It can be found from the formula

$$a_{-1} = \lim_{z\to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \{(z-a)^n f(z)\} \qquad (10)$$

where n is the order of the pole. For simple poles the calculation of the residue is of particular simplicity since it reduces to

$$a_{-1} = \lim_{z\to a} (z-a) f(z) \qquad (11)$$

RESIDUE THEOREM

If $f(z)$ is analytic in a region $\mathcal{R}$ except for a pole of order n at $z = a$ and if C is any simple closed curve in $\mathcal{R}$ containing $z = a$, then $f(z)$ has the form (9). Integrating (9), using the fact that

$$\oint_C \frac{dz}{(z-a)^n} = \begin{cases} 0 & \text{if } n \neq 1 \\ 2\pi i & \text{if } n = 1 \end{cases} \qquad (12)$$

(see Problem 13.13), it follows that

$$\oint_C f(z)\,dz = 2\pi i a_{-1} \qquad (13)$$

i.e. the integral of $f(z)$ around a closed path enclosing a single pole of $f(z)$ is $2\pi i$ times the residue at the pole.

More generally, we have the following important

Theorem 13. If $f(z)$ is analytic within and on the boundary C of a region R except at a finite number of poles $a, b, c, \ldots$ within R having residues $a_{-1}, b_{-1}, c_{-1}, \ldots$ respectively, then

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots) \qquad (14)$$

i.e. the integral of $f(z)$ is $2\pi i$ times the sum of the residues of $f(z)$ at the poles enclosed by C.

Cauchy's theorem and integral formulas are special cases of this result which we call the *residue theorem.*

4. Partial Differential Equations (PDEs)

4.1. SOME DEFINITIONS INVOLVING PARTIAL DIFFERENTIAL EQUATIONS

A *partial differential equation* is an equation containing an unknown function of two or more variables and its partial derivatives with respect to these variables. The *order* of a partial differential equation is that of the highest ordered derivative present.

Example 1. $\frac{\partial^2 u}{\partial x\,\partial y} = 2x - y$ is a partial differential equation of order two, or a second order partial differential equation.

A *solution* of a partial differential equation is any function which satisfies the equation identically.
The *general solution* is a solution which contains a number of arbitrary independent functions equal to the order of the equation.
A *particular solution* is one which can be obtained from the general solution by particular choice of the arbitrary functions.

Example 2. As seen by substitution, $u = x^2y - \frac{1}{2}xy^2 + F(x) + G(y)$ is a *solution* of the partial differential equation of Example 1. Because it contains two arbitrary independent functions $F(x)$ and $G(y)$, it is the *general solution*. If in particular $F(x) = 2\sin x$, $G(y) = 3y^4 - 5$, we obtain the *particular solution* $u = x^2y - \frac{1}{2}xy^2 + 2\sin x + 3y^4 - 5$.

A *singular solution* is one which cannot be obtained from the general solution by particular choice of the arbitrary functions.
A *boundary-value problem* involving a partial differential equation seeks all solutions of a partial differential equation which satisfy conditions called *boundary conditions.* Theorems relating to the existence and uniqueness of such solutions are called *existence and uniqueness theorems.*

4.2. LINEAR PARTIAL DIFFERENTIAL EQUATIONS

The general *linear partial differential equation* of order two in two independent variables has the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \qquad (1)$$

where *A, B, ..., G* may depend on *x* and *y* but not on *u.* A second order equation with independent variables *x* and *y* which does not have the form *(1)* is called *nonlinear.*
If $G = 0$ the equation is called *homogeneous,* while if $G \neq 0$ it is called *non-homogeneous.* Generalizations to higher order equations are easily made. Because of the nature of the solutions of *(1)* the equation is often classified as *elliptic, hyperbolic* or *parabolic* according as $B^2 - 4AC$ is less than, greater than or equal to zero respectively.

4.3. SOME IMPORTANT PARTIAL DIFFERENTIAL EQUATIONS

1. Heat Conduction Equation

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u$$

Here $u(x, y, z, t)$ is the temperature in a solid at position (x, y, z) at time t. The constant κ, called the *diffusivity*, is equal to $K/\sigma\tau$ where the *thermal conductivity* K, the *specific heat* σ and the density (mass per unit volume) τ are assumed constant.

In case u does not depend on y and z, the equation reduces to $\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$ called the *one-dimensional heat conduction equation.*

2. Vibrating String Equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

This equation is applicable to the small transverse vibrations of a taut, flexible string, such as a violin string, initially located on the x axis and set into motion [see Fig. 12-1]. The function $y(x, t)$ is the displacement of any point x of the string at time t. The constant $a^2 = T/\mu$, where T is the (constant) tension in the string and μ is the (constant) mass per unit length of the string. It is assumed that no external forces act on the string but that it vibrates only due to its elasticity.

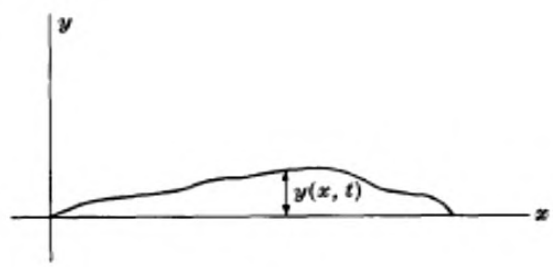


Fig. 12-1

The equation can easily be generalized to higher dimensions as for example the vibrations of a membrane or drum head in two dimensions. In two dimensions, for example, the equation is

$$\frac{\partial^2 z}{\partial t^2} = a^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

3. Laplace's Equation

$$\nabla^2 v = 0$$

This equation occurs in many fields. In the theory of heat conduction, for example, v is the *steady-state temperature,* i.e. the temperature after a long time has elapsed, and is equivalent to putting $\partial u/\partial t = 0$ in the heat conduction equation above. In the theory of gravitation or electricity v represents the *gravitational* or *electric potential* respectively. For this reason the equation is often called the *potential equation.*

4. Longitudinal Vibrations of a Beam

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

This equation describes the motion of a beam [Fig. 12-2] which can vibrate longitudinally [i.e. in the x direction]. The variable $u(x, t)$ is the longitudinal displacement from the equilibrium position of the cross section at x. The constant $c^2 = gE/\tau$ where g is the acceleration due to gravity, E is the modulus of elasticity [stress divided by strain] and depends on the properties of the beam, τ is the density [mass per unit volume].

Note that this equation is the same as that for a vibrating string.

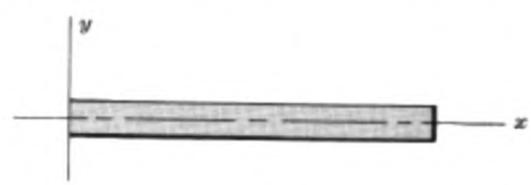


Fig. 12-2

5. Transverse Vibrations of a Beam

$$\frac{\partial^2 y}{\partial t^2} + b^2 \frac{\partial^4 y}{\partial x^4} = 0$$

This equation describes the motion of a beam [initially located on the x axis, see Fig. 12-3] which is vibrating transversely (i.e. perpendicular to the x direction). In this case $y(x, t)$ is the transverse displacement or deflection at any time t of any point x. The constant $b^2 = EIg/\mu$ where E is the modulus of elasticity, I is the moment of inertia of any cross section about the x axis, g is the acceleration due to gravity and μ is the mass per unit length. In case an external transverse force $F(x, t)$ is applied, the right hand side of the equation is replaced by $b^2 F(x, t)/EI$.

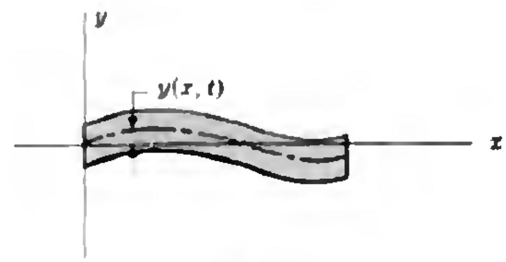


Fig. 12-3

METHODS OF SOLVING BOUNDARY-VALUE PROBLEMS

There are many methods by which boundary-value problems involving linear partial differential equations can be solved. The following are among the most important.

1. **General Solutions.**

In this method we first find the general solution and then that particular solution which satisfies the boundary conditions? The following theorems are of fundamental importance.

Theorem 1: [Superposition principle]. If $u_1, u_2, \ldots,$ «„ are solutions of a linear homogeneous partial differential equation, then $c_1u_1 + c_2u_2 + \bullet\bullet\bullet + c_nu_n$ where $c_1, c_2, \ldots, c_n$ are constants is also a solution.

Theorem 2. The general solution of a linear non-homogeneous partial differential equation is obtained by adding a particular solution of the non-homogeneous equation to the general solution of the homogeneous equation. We can sometimes find general solutions by using the methods of ordinary differential equations.

If *A, B, ..., F* in *(1)* are constants, then the general solution of the homogeneous equation can be found by assuming that "$u = e^{ax+by}$" where a and b are constants to be determined.

2. **Separation of Variables.**

In this method it is assumed that a solution can be expressed as a product of unknown functions each of which depends on only one of the independent variables.

The success of the method hinges on being able to write the resulting equation so that one side depends only on one variable while the other side depends on the remaining variables so that each side must be a constant. By repetition of this the unknown functions can then be determined. Superposition of these solutions can then be used to find the actual solution.

The method often makes use of ***Fourier series***, ***Fourier integrals, Bessel series*** *and* ***Legendre series***.

3. **Laplace Transform Methods.**

In this method the Laplace transform of the partial differential equation and associated boundary conditions are first obtained with respect to one of the independent variables. We then solve the resulting equation for the Laplace transform of the required solution which is then found by taking the inverse Laplace transform.

LEGENDRE'S DIFFERENTIAL EQUATION

Legendre functions arise as solutions of the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{1}$$

which is called *Legendre's differential equation*. The general solution of (1) in the case where $n = 0, 1, 2, 3, \ldots$ is given by

$$y = c_1 P_n(x) + c_2 Q_n(x)$$

where $P_n(x)$ are polynomials called *Legendre polynomials* and $Q_n(x)$ are called *Legendre functions of the second kind* which are unbounded at $x = \pm 1$.

LEGENDRE POLYNOMIALS

The Legendre polynomials are defined by

$$P_n(x) = \frac{(2n-1)(2n-3)\cdots 1}{n!}\left\{x^n - \frac{n(n-1)}{2(n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \cdots\right\} \tag{2}$$

Note that $P_n(x)$ is a polynomial of degree n. The first few Legendre polynomials are as follows:

1. $P_0(x) = 1$
2. $P_1(x) = x$
3. $P_2(x) = \frac{1}{2}(3x^2 - 1)$
4. $P_3(x) = \frac{1}{2}(5x^3 - 3x)$
5. $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$
6. $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

In all cases $P_n(1) = 1$, $P_n(-1) = (-1)^n$.

The Legendre polynomials can also be expressed by *Rodrigue's formula* given by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{3}$$

GENERATING FUNCTION FOR LEGENDRE POLYNOMIALS

The function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{4}$$

is called the *generating function* for Legendre polynomials and is useful in obtaining their properties.

RECURRENCE FORMULAS

1. $P_{n+1}(x) = \dfrac{2n+1}{n+1}xP_n(x) - \dfrac{n}{n+1}P_{n-1}(x)$
2. $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$

LEGENDRE FUNCTIONS OF THE SECOND KIND

If $|x| < 1$, the Legendre functions of the second kind are given by the following according as n is even or odd respectively:

$$Q_n(x) = \frac{(-1)^{n/2}2^n[(n/2)!]^2}{n!}\left\{x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots\right\} \tag{5}$$

$$Q_n(x) = \frac{(-1)^{(n+1)/2}2^{n-1}[(n-1)/2]!^2}{1\cdot 3\cdot 5\cdots n}\left\{1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots\right\} \tag{6}$$

For $n > 1$, these coefficients are taken so that the recurrence formulas for $P_n(x)$ above apply also to $Q_n(x)$.

ORTHOGONALITY OF LEGENDRE POLYNOMIALS

The following results are fundamental:

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0 \qquad \text{if } m \neq n \tag{7}$$

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1} \tag{8}$$

The first shows that any two different Legendre polynomials are orthogonal in the interval $-1 < x < 1$.

SERIES OF LEGENDRE POLYNOMIALS

If $f(x)$ satisfies the Dirichlet conditions [Page 183], then at every point of continuity of $f(x)$ in the interval $-1 < x < 1$ there will exist a Legendre series expansion having the form

$$f(x) = A_0P_0(x) + A_1P_1(x) + A_2P_2(x) + \cdots = \sum_{k=0}^{\infty} A_kP_k(x) \tag{9}$$

where

$$A_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)\,P_k(x)\,dx \tag{10}$$

At any point of discontinuity the series on the right in (*9*) converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ which can be used to replace the left side of (*9*).

ASSOCIATED LEGENDRE FUNCTIONS

The differential equation

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0 \tag{11}$$

is called *Legendre's associated differential equation*. If $m = 0$ this reduces to Legendre's equation (*1*). Solutions to (*11*) are called *associated Legendre functions*. We consider the case where m and n are non-negative integers. In this case the general solution of (*11*) is given by

$$y = c_1P_n^m(x) + c_2Q_n^m(x) \tag{12}$$

where $P_n^m(x)$ and $Q_n^m(x)$ are called *associated Legendre functions of the first and second kinds* respectively. They are given in terms of the ordinary Legendre functions by

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \tag{13}$$

$$Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}Q_n(x) \tag{14}$$

Note that if $m > n$, $P_n^m(x) = 0$.

As in the case of Legendre polynomials, the Legendre functions $P_n^m(x)$ are orthogonal in $-1 < x < 1$, i.e.

$$\int_{-1}^{1} P_n^m(x)\, P_k^m(x)\, dx \quad = \quad 0 \qquad n \neq k \tag{15}$$

We also have

$$\int_{-1}^{1} [P_n^m(x)]^2\, dx \quad = \quad \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} \tag{16}$$

Using these, we can expand a function $f(x)$ in a series of the form

$$f(x) \quad = \quad \sum_{k=0}^{\infty} A_k P_k^m(x) \tag{17}$$

OTHER SPECIAL FUNCTIONS

The following special functions are some of the important ones arising in science and engineering.

1. Hermite polynomials. These polynomials, denoted by $H_n(x)$, are solutions of *Hermite's differential equation*

$$y'' - 2xy' + 2ny \; = \; 0 \tag{18}$$

The polynomials are given by a corresponding *Rodrigue's formula*

$$H_n(x) \quad = \quad (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \tag{19}$$

Their generating function is given by

$$e^{2tx-t^2} \quad = \quad \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \tag{20}$$

and they satisfy the *recursion formulas*

$$H_{n+1}(x) \; = \; 2xH_n(x) - 2nH_{n-1}(x) \tag{21}$$

$$H_n'(x) \; = \; 2nH_{n-1}(x) \tag{22}$$

The important results

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\, H_n(x)\, dx \; = \; 0 \qquad m \neq n \tag{23}$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)\, dx \; = \; 2^n n!\sqrt{\pi} \tag{24}$$

enable us to expand a function into a *Hermite series* of the form

$$f(x) \quad = \quad \sum_{k=0}^{\infty} A_k H_k(x) \tag{25}$$

where

$$A_k \quad = \quad \frac{1}{2^k k!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x)\, H_k(x)\, dx \tag{26}$$

2. Laguerre polynomials. These polynomials, denoted by $L_n(x)$, are solutions of *Laguerre's differential equation*

$$xy'' + (1-x)y' + ny \; = \; 0 \tag{27}$$

The polynomials are given by the *Rodrigue's formula*

$$L_n(x) \quad = \quad e^x \frac{d^n}{dx^n}(x^n e^{-x}) \tag{28}$$

Their generating function is given by

$$\frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n \tag{29}$$

and they satisfy the *recursion formulas*

$$L_{n+1}(x) = (2n+1-x)L_n(x) - n^2 L_{n-1}(x) \tag{30}$$

$$nL_{n-1}(x) = nL'_{n-1}(x) - L'_n(x) \tag{31}$$

The important results

$$\int_0^{\infty} e^{-x} L_m(x)\, L_n(x)\, dx = 0 \qquad m \neq n \tag{32}$$

$$\int_0^{\infty} e^{-x} L_n^2(x)\, dx = (n!)^2 \tag{33}$$

enable us to expand a function into a *Laguerre series* of the form

$$f(x) = \sum_{k=0}^{\infty} A_k L_k(x) \tag{34}$$

where

$$A_k = \frac{1}{(k!)^2} \int_0^{\infty} e^{-x} f(x)\, L_k(x)\, dx \tag{35}$$

STURM-LIOUVILLE SYSTEMS

A boundary-value problem having the form

$$\left.\begin{array}{l} \dfrac{d}{dx}\left[p(x)\dfrac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0 \qquad a \leqq x \leqq b \\ a_1 y(a) + a_2 y'(a) = 0, \qquad b_1 y(b) + b_2 y'(b) = 0 \end{array}\right\} \tag{36}$$

where a_1, a_2, b_1, b_2 are given constants; $p(x), q(x), r(x)$ are given functions which we shall assume to be differentiable and λ is an unspecified parameter independent of x, is called a *Sturm-Liouville boundary-value problem* or *Sturm-Liouville system.*

A non-trivial solution of this system, i.e. one which is not identically zero, exists in general only for a particular set of values of the parameter λ. These values are called the *characteristic values*, or more often *eigenvalues*, of the system. The corresponding solutions are called *characteristic functions* or *eigenfunctions* of the system. In general to each eigenvalue there is one eigenfunction, although exceptions can occur.

If $p(x), q(x)$ are real, then the eigenvalues are real. Also the eigenfunctions form an orthogonal set with respect to the *density function* $r(x)$ which is generally taken as non-negative, i.e. $r(x) \geqq 0$. It follows that by suitable normalization the set of functions can be made an orthonormal set with respect to $r(x)$ in $a \leqq x \leqq b$.

BESSEL'S DIFFERENTIAL EQUATION

Bessel functions arise as solutions of the differential equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \qquad n \geqq 0 \tag{1}$$

which is called *Bessel's differential equation*. The general solution of (*1*) is given by

$$y = c_1 J_n(x) + c_2 Y_n(x) \tag{2}$$

The solution $J_n(x)$, which has a finite limit as x approaches zero, is called a *Bessel function of the first kind and order n*. The solution $Y_n(x)$ which has no finite limit [i.e. is unbounded] as x approaches zero, is called a *Bessel function of the second kind and order n* or *Neumann function*.

If the independent variable x in (*1*) is changed to λx where λ is a constant, the resulting equation is

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0 \tag{3}$$

with general solution

$$y = c_1 J_n(\lambda x) + c_2 Y_n(\lambda x) \tag{4}$$

BESSEL FUNCTIONS OF THE FIRST KIND

We define the Bessel function of the first kind of order n as

$$J_n(x) = \frac{x^n}{2^n\,\Gamma(n+1)}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right\} \tag{5}$$

or

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r!\,\Gamma(n+r+1)} \tag{6}$$

where $\Gamma(n+1)$ is the *gamma function* [Chapter 9]. If n is a positive integer, $\Gamma(n+1) = n!$, $\Gamma(1) = 1$. For $n = 0$, (*6*) becomes

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots \tag{7}$$

The series (*6*) converges for all x. Graphs of $J_0(x)$ and $J_1(x)$ are shown in Fig. 10-1.

If n is half an odd integer, $J_n(x)$ can be expressed in terms of sines and cosines. See Problems 10.4 and 10.7.

A function $J_{-n}(x)$, $n > 0$, can be defined by replacing n by $-n$ in (*5*) or (*6*). If n is an integer then we can show that [see Problem 10.3]

$$J_{-n}(x) = (-1)^n J_n(x) \tag{8}$$

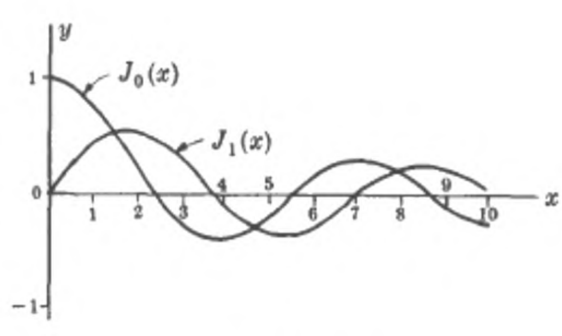


Fig. 10-1

If n is not an integer, $J_n(x)$ and $J_{-n}(x)$ are linearly independent, and for this case the general solution of (*1*) is

$$y = AJ_n(x) + BJ_{-n}(x) \qquad n \neq 0, 1, 2, 3, \ldots \tag{9}$$

BESSEL FUNCTIONS OF THE SECOND KIND

We shall define the Bessel function of the second kind of order n as

$$Y_n(x) = \begin{cases} \dfrac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi} & n \neq 0, 1, 2, 3, \ldots \\ \lim\limits_{p \to n} \dfrac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi} & n = 0, 1, 2, 3, \ldots \end{cases} \tag{10}$$

For the case where $n = 0, 1, 2, 3, \ldots$ we obtain the following series expansion for $Y_n(x)$.

$$Y_n(x) = \frac{2}{\pi}\{\ln(x/2) + \gamma\}J_n(x) - \frac{1}{\pi}\sum_{k=0}^{n-1}(n-k-1)!\,(x/2)^{2k-n} - \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\{\Phi(k) + \Phi(n+k)\}\frac{(x/2)^{2k+n}}{k!\,(n+k)!} \tag{11}$$

where $\gamma = .5772156\ldots$ is *Euler's constant* and

$$\Phi(p) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p}, \qquad \Phi(0) = 0 \tag{12}$$

GENERATING FUNCTION FOR $J_n(x)$

The function

$$e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n \tag{13}$$

is called the *generating function* for Bessel functions of the first kind of integral order. It is very useful in obtaining properties of these functions for integer values of n which can then often be proved for all values of n.

RECURRENCE FORMULAS

The following results are valid for all values of n.

1. $J_{n+1}(x) = \dfrac{2n}{x}J_n(x) - J_{n-1}(x)$
2. $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$
3. $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$
4. $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$
5. $\dfrac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$
6. $\dfrac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

If n is an integer these can be proved by using the generating function. Note that results 3 and 4 are equivalent respectively to 5 and 6.

The functions $Y_n(x)$ satisfy exactly the same results as those above, where $Y_n(x)$ replaces $J_n(x)$.

FUNCTIONS RELATED TO BESSEL FUNCTIONS

1. **Hankel Functions of First and Second Kinds** are defined respectively by

$$H_n^{(1)}(x) = J_n(x) + iY_n(x), \quad H_n^{(2)}(x) = J_n(x) - iY_n(x)$$

2. **Modified Bessel Functions.** The *modified Bessel function of the first kind of order n* is defined as

$$I_n(x) = i^{-n}J_n(ix) = e^{-n\pi i/2}J_n(ix) \tag{14}$$

If n is an integer,

$$I_{-n} = I_n(x) \tag{15}$$

but if n is not an integer, $I_n(x)$ and $I_{-n}(x)$ are linearly independent.

The *modified Bessel function of the second kind of order n* is defined as

$$K_n(x) = \begin{cases} \dfrac{\pi}{2}\left[\dfrac{I_{-n}(x) - I_n(x)}{\sin n\pi}\right] & n \neq 0, 1, 2, 3, \ldots \\ \lim\limits_{p \to n} \dfrac{\pi}{2}\left[\dfrac{I_{-p}(x) - I_p(x)}{\sin p\pi}\right] & n = 0, 1, 2, 3, \ldots \end{cases} \tag{16}$$

These functions satisfy the differential equation

$$x^2y'' + xy' - (x^2 + n^2)y = 0 \tag{17}$$

and the general solution of this equation is

$$y = c_1I_n(x) + c_2K_n(x) \tag{18}$$

or if $n \neq 0, 1, 2, 3, \ldots$

$$y = AI_n(x) + BI_{-n}(x) \tag{19}$$

3. **Ber, Bei, Ker, Kei Functions.** The functions $\text{Ber}_n(x)$ and $\text{Bei}_n(x)$ are the real and imaginary parts of $J_n(i^{3/2}x)$ where $i^{3/2} = e^{3\pi i/4} = (\sqrt{2}/2)(1-i)$, i.e.

$$J_n(i^{3/2}x) = \text{Ber}_n(x) + i\,\text{Bei}_n(x) \tag{20}$$

The functions $\text{Ker}_n(x)$ and $\text{Kei}_n(x)$ are the real and imaginary parts of $e^{-n\pi i/2}K_n(i^{1/2}x)$ where $i^{1/2} = e^{\pi i/4} = (\sqrt{2}/2)(1+i)$, i.e.

$$e^{-n\pi i/2}K_n(i^{1/2}x) = \text{Ker}_n(x) + i\,\text{Kei}_n(x) \tag{21}$$

The functions are useful in connection with the equation

$$x^2y'' + xy' - (ix^2 + n^2)y = 0 \tag{22}$$

which arises in electrical engineering and other fields. The general solution of this equation is

$$y = c_1J_n(i^{3/2}x) + c_2K_n(i^{1/2}x) \tag{23}$$

EQUATIONS TRANSFORMED INTO BESSEL'S EQUATION

The equation

$$x^2y'' + (2k+1)xy' + (\alpha^2x^{2r} + \beta^2)y = 0 \tag{24}$$

where k, α, r, β are constants, has the general solution

$$y = x^{-k}\left[c_1J_{\kappa/r}(\alpha x^r/r) + c_2Y_{\kappa/r}(\alpha x^r/r)\right] \tag{25}$$

where $\kappa = \sqrt{k^2 - \beta^2}$. If $\alpha = 0$ the equation is solvable as an Euler or Cauchy equation

ASYMPTOTIC FORMULAS FOR BESSEL FUNCTIONS

For large values of x we have the following asymptotic formulas

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right), \qquad Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) \tag{26}$$

ZEROS OF BESSEL FUNCTIONS

We can show that if n is any real number, $J_n(x) = 0$ has an infinite number of roots which are all real. The difference between successive roots approaches π as the roots increase in value. This can be seen from (*26*). We can also show that the roots of $J_n(x) = 0$ lie between those of $J_{n-1}(x) = 0$ and $J_{n+1}(x) = 0$. Similar remarks can be made for $Y_n(x)$.

ORTHOGONALITY OF BESSEL FUNCTIONS

If λ and μ are two different constants, we can show [see Problem 10.21] that

$$\int_0^1 x J_n(\lambda x) J_n(\mu x)\, dx = \frac{\mu J_n(\lambda) J_n'(\mu) - \lambda J_n(\mu) J_n'(\lambda)}{\lambda^2 - \mu^2} \tag{27}$$

while [see Problem 10.22]

$$\int_0^1 x J_n^2(\lambda x)\, dx = \frac{1}{2}\left[J_n'^2(\lambda) + \left(1 - \frac{n^2}{\lambda^2}\right) J_n^2(\lambda)\right] \tag{28}$$

From (*27*) we can see that if λ and μ are any two different roots of the equation

$$R J_n(x) + S x J_n'(x) = 0 \tag{29}$$

where R and S are constants, then

$$\int_0^1 x J_n(\lambda x) J_n(\mu x)\, dx = 0 \tag{30}$$

which states that the functions $\sqrt{x}\, J_n(\lambda x)$ and $\sqrt{x}\, J_n(\mu x)$ are *orthogonal* in $(0, 1)$. Note that as special cases of (*29*) we see that λ and μ can be any two different roots of $J_n(x) = 0$ or $J_n'(x) = 0$. We can also say that the functions $J_n(\lambda x)$, $J_n(\mu x)$ are orthogonal with respect to the density function x.

SERIES OF BESSEL FUNCTIONS

As in the case of Fourier series, we can show that if $f(x)$ satisfies the Dirichlet conditions [page 183] then at every point of continuity of $f(x)$ in the interval $0 < x < 1$ there will exist a Bessel series expansion having the form

$$f(x) = A_1 J_n(\lambda_1 x) + A_2 J_n(\lambda_2 x) + \cdots = \sum_{p=1}^{\infty} A_p J_n(\lambda_p x) \tag{31}$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of (*29*) with $R/S \geqq 0$, $S \neq 0$ and

$$A_p = \frac{2\lambda_p^2}{(\lambda_p^2 - n^2 + R^2/S^2) J_n^2(\lambda_p)} \int_0^1 x J_n(\lambda_p x) f(x)\, dx \tag{32}$$

At any point of discontinuity the series on the right in (*31*) converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ which can be used in place of the left side of (*31*).

In case $S = 0$ so that $\lambda_1, \lambda_2, \ldots$ are the roots of $J_n(x) = 0$,

$$A_p = \frac{2}{J_{n+1}^2(\lambda_p)} \int_0^1 x J_n(\lambda_p x) f(x)\, dx \tag{33}$$

If $R = 0$ and $n = 0$, then the series (31) starts out with the constant term

$$A_1 = 2 \int_0^1 x f(x)\, dx \tag{34}$$

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