Electrodynamics-II

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April/2020

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Outlines

- Maxwell's Equations
- Conservation Laws
- Potential and Fields
- Radiation
- Covariant Formulation of Electrodynamics

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Maxwell's Equations

❑ Electrodynamics before Maxwell's

(i) $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$ (Gauss's law),

(ii) $\nabla \cdot \mathbf{B} = 0$ (no name),

(iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (Faraday's law),

(iv) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ (Ampère's law).

(v) $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$ **(Charge conservation law)**
(→ Continuity equation)

❖ But, there is a fatal inconsistency

Ampere's law is bound to fail nonsteady currents

- Applying divergence to eqn. (iii), then

$$\nabla \cdot (\nabla \times \mathbf{E}) = \nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) \Longrightarrow \nabla \cdot B = 0$$

- Applying divergence to eqn. (iv)

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0(\nabla \cdot \mathbf{J}) \Longrightarrow \nabla \cdot (\nabla \times \mathbf{B}) = 0$$

$\nabla \cdot J = 0$ For steady current

Note: divergence of a curl vanishes: it's a vector identity.

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Cont'

- *For non-steady currents eqn.(iv)*

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0(\nabla \cdot \mathbf{J}) \Longrightarrow \nabla \cdot J = {}^{\partial\rho}/_{\partial t} \neq 0$$

- *Ampere's law cannot be right for non-steady currents!*

- *There's another way to see that Ampere's law is bound to fail for non-steady current.*
  - Consider the process of charging up a capacitor.

Ampere's law reads, $\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$

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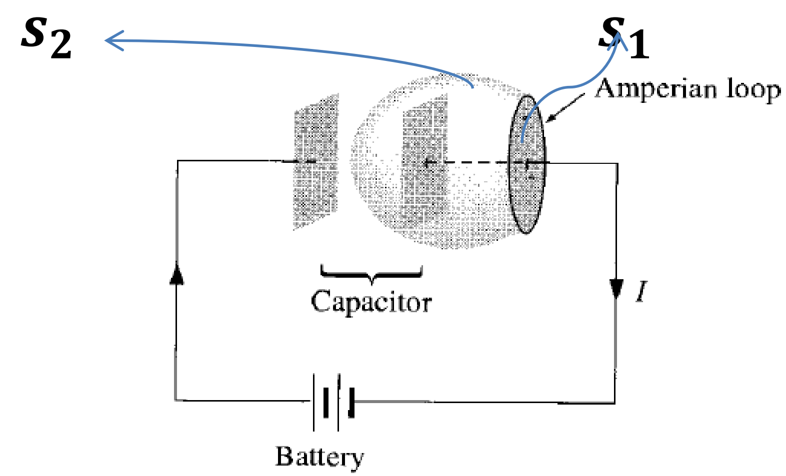
- We want to apply it to the Amperian loop or Amperian surface.
- *How do we determine* $\boldsymbol{I_{enc}}$?



- *For the surface* s_1, I_{enc} *= I*
- *For the surface* s_2, $I_{enc} = 0$ *(No current passes through it)*

- *The conflict arises only when charge is piling up somewhere (in this case, on the capacitor plates).*
- *For nonsteady currents, "the current enclosed by a loop" is an ill defined notion, since it depends entirely on what surface you use.*

How Maxwell fixed ampere's law

- The inconsistent problem arose on Ampere's law when;

$\nabla \cdot (\nabla X B) \neq \nabla . J$ for nonsteady currents

- Applying continuity equation and Gauss's law

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\nabla \cdot \left(\mathbf{J} + \varepsilon_0 \frac{\partial E}{\partial t}\right) = 0$$

$$\mathbf{J} \to \left(\mathbf{J} + \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot \left(\mathbf{J} + \varepsilon_0 \frac{\partial E}{\partial t}\right) = 0$$

$\Rightarrow$ The inconsistency in Ampere's law is now cured.

Cont'

❖ *Ampere's law can generally be expressed as*

$$\nabla \times \mathbf{B} = \varepsilon_0 \mathbf{J} \longrightarrow \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}}$$

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}} \Rightarrow$$ *A changing electric field induces a magnetic field*

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Maxwell's equation

(i) $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$ (Gauss's law),

(ii) $\nabla \cdot \mathbf{B} = 0$ (no name),

(iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (Faraday's law),

(iv) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (Ampère's law with Maxwell's correction).

……………………Eqn.3

Together with the force law $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$,

- *they summarize the entire theoretical content of classical electrodynamics.*

Magnetic Charge?

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} - \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} = \mu_0\mathbf{J}$$

The symmetry between E and B is spoiled by the charge term and the current.

If we had ρ_m (the density of magnetic charge) and J_m (the current of magnetic charge),

If we replace E $\longrightarrow$ B, B $\longrightarrow$ $-\mu_o\ \epsilon_o\ E$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = -\mu_0\mathbf{J}_m$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} - \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} = \mu_0\mathbf{J} \quad \rightarrow \nabla \cdot \mathbf{J}_m = -\frac{\partial \rho_m}{\partial t}$$

- *There could be a pleasing symmetry between E and B.*

Maxwell's equation in matter

- It would be nice to reformulate Maxwell's eqn. in the form (3) when you are working with material that are subjected to electric and magnetic polarization

- For inside polarized matter there will be accumulation of bound charges and current over which you exert no direct control.

$$\rho_b = \nabla.\mathbf{P}\text{..............................}(4)$$

Magnetic polarization M results in a bound current is

$$J_b = \nabla X\boldsymbol{M} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. (5)$$

Maxwell's Equations in Matter

$$\rho_b = \nabla . \mathrm{P}$$

- If P is time-varying, we expect there to be a current J_p associated with the resulting changes in ρ_b. In fact, the above expression suggests a good definition of J_n :

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t} \qquad \Longleftrightarrow \qquad \vec{\nabla} \cdot \vec{J}_p = -\vec{\nabla} \cdot \frac{\partial \vec{P}}{\partial t} = -\frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{P} = -\frac{\partial \rho_b}{\partial t}$$

- That is, the definition on the left naturally gives the continuity relation between J_p and ρ_b one would like.
- Time dependence of M yields time dependence of J_b, which produces time dependence of B and H

Cont'

- The charge and current densities have the following parts
- $$\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla} \cdot \vec{P} \qquad \vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p = \vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$$ 6

- Using Gauss's law,
- $\varepsilon_o \nabla . E = \rho_f - \nabla . P$ *and definition of* the displacement field

  D= $\varepsilon_o E$+P we obtain

$$\nabla . D = \rho_f \ldots \ldots \ldots \ldots \ldots \ldots \ldots 7$$

Ampere's Law with the displacement current term is

$$\nabla X B = \mu_o (J_f + \nabla \times M + \frac{\partial P}{\partial t}) + \mu_o \epsilon_o \frac{\partial E}{\partial t} \ldots\ldots\ldots 8$$

Cont'

- We use B=μ_o(H+M) as well as D= $\epsilon_o E$+P we obtain

$$\nabla \times H = J_f + \frac{\partial D}{\partial t} \quad \text{..............................9}$$

- Faraday's Law and $\nabla \cdot B$ = 0 are not affected since they do not depend on the free and bound currents.
- Thus, Maxwell's Equations in matter are (again, putting all the fields on the left sides and the sources on the right):

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho_f} \qquad \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

$$\boxed{\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0} \qquad \boxed{\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}_f}$$

Boundary Conditions for Maxwell's Equations

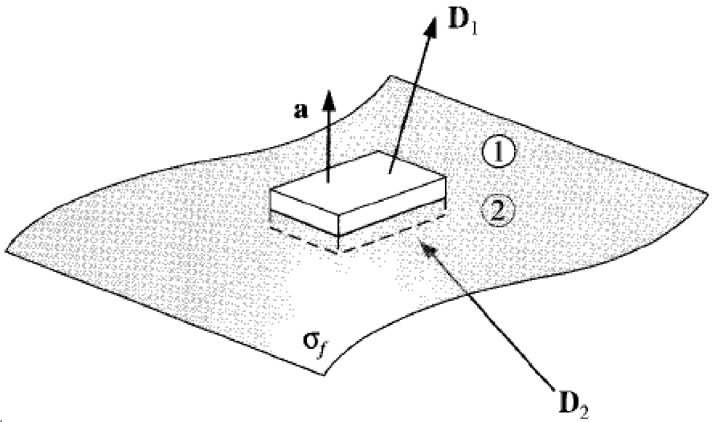
- In general the fields **E,B,D** & **H** will be discontinuous at a boundary between two different media or at a surface that carries charge density σ or current density k.
- *The integral form of Maxwell's equations can deduct the boundary conditions*

$$\left.\begin{aligned} &\text{(i)} \quad \oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}} \\ &\text{(ii)} \quad \oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = 0 \end{aligned}\right\} \text{ over any closed surface } \mathcal{S}.$$

$$\left.\begin{aligned} &\text{(iii)} \quad \oint_{\mathcal{P}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} \\ &\text{(iv)} \quad \oint_{\mathcal{P}} \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} + \frac{d}{dt}\int_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a} \end{aligned}\right\} \begin{gathered}\text{for any surface } \mathcal{S} \\ \text{bounded by the} \\ \text{closed loop } \mathcal{P}.\end{gathered}$$

Cont'

- Applying (i) to a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary we obtain
- $D_1 a - D_2 a$=$\sigma_f a$..............(12)

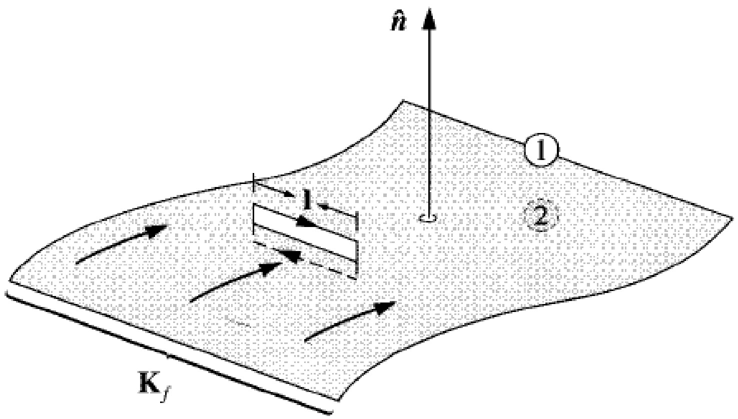


- Thus the component of D that is perpendicular to the interface is discontinuous in the amount

$$D_1^{\perp}\text{-}D_2^{\perp} = \sigma_f \text{ ....(13)}$$

cont'

- Identical reasoning applied to (ii) yields
    - $B_1^{\perp}$-$B_2^{\perp}$ =0 ……………………….(14)
- Turning to (iii) a very thin Amperian loop straddling the surface (fig.) yields
- $E_1 l - E_2 l = \frac{d}{dt}\int_s \; B \cdot da$



- But in the limit as width of the loop goes to zero the flux vanishes.

Cont'

- Therefore

$$E_1^{||}\text{-}E_2^{||}\text{=}0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

- That is the component of E parallel to the interface are continuous across the boundary.

- By the same token (iv) implies

$$H_1 I - H_2 I = I_{fenc}$$

Where I_{fenc} is free current passing through Amperian loop

Cont;

- $I_{fenc} = K_f \cdot (\bar{n} \times I) = (K_f \times \bar{n}) \cdot I$

And hence

$$H_1^{||} - H_2^{||} = (K_f \times \bar{n}) \cdot I \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 16$$

- So the parallel components of H are discontinuous by an amount proportional to the free surface current density.

Cont'

Equation 13-16 are the general boundary conditions for electrodynamics

$$\text{(i)}\quad \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, \qquad \text{(iii)}\quad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0,$$

$$\text{(ii)}\quad B_1^{\perp} - B_2^{\perp} = 0, \qquad \text{(iv)}\quad \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.$$

In the case of linear media,

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f \qquad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.$$

- If there is no free charge or free current at the interface

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0 \qquad B_1^{\perp} - B_2^{\perp} = 0 \qquad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0 \qquad \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = 0$$

Conservation Laws

- Under this we will study conservation of
  - Charge
  - Energy
  - Momentum
- Not only is there no creation or destruction of charge over the whole universe, there is also no creation or destruction of charge at a given point.
- Charge cannot jump from one place to another without a current owing to move that charge.

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Poynting's Theorem: Conservation of Energy

- Work necessary to assemble charge distribution (againist coloumbs repulsion of like charge)
- $$W_e = \frac{\epsilon_0}{2} \int E^2 \, d\tau, \quad \text{.......................17}$$
- Likewise, the work required to get currents going (againist the back emf)

$$W_m = \frac{1}{2\mu_0} \int B^2 \, d\tau, \quad \text{...........................18}$$

- Therefore the total energy stored in EMF is
- $$U_{em} = \frac{1}{2} \int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau \quad \text{............19}$$

cont;

- We will prove this by considering the work done to move charges as currents
- Given a single particle of charge q acted on by the electromagnetic field, the work done on it as it moves by dl is
- $$dW = \vec{F} \cdot d\vec{\ell} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) dt = q\,\vec{E} \cdot \vec{v}\, dt \quad \ldots\ldots\ldots\ldots..20$$
- Now, q= $\rho d\tau$ and $\rho v = J$ so the rate at which work is done on all the charges in volume V is
- $$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J})\, d\tau. \quad \ldots\ldots\ldots\ldots..21$$

cont'

- Let's manipulate the integrand using Ampere's Law:

$$\vec{E}\cdot\vec{J}=\frac{1}{\mu_o}\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)-\epsilon_o\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

- From product rule $\nabla\cdot(E\times B)=B\cdot(\nabla\times E)-E\cdot(\nabla\times B)$
- Invoking Faraday's law

$$E\cdot(\nabla\times B)=-B\cdot\frac{\partial B}{\partial t}-\nabla\cdot(E\times B)$$

Meanwhile $-B\cdot\frac{\partial B}{\partial t}=\frac{1\partial}{2\partial t}(B^2)$

So E$\cdot J=\frac{1\partial}{2\partial t}\left(\epsilon_o E^2+\frac{1}{\mu_o}B^2\right)-\frac{1}{\mu_o}\nabla\cdot(E\times B)$……………..22

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Cont'

- Putting this into equation (21) and applying divergence theorem to the 2nd term

$$\frac{dW}{dt} = -\frac{d}{dt}\int \frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_o E^2 + \frac{1}{\mu_o}B^2\right)d\tau - \frac{1}{\mu_o}\oint (E \times B)\cdot da \ldots .23$$

Where S is the surface bounding V.

- This poynting's theorem; it is the "work-energy theorem" of electrodynamics.

Cont'

- The first integral on the right is the total energy stored in the fields. The 2^{nd} term evidently represent the rate at which energy is carried out of V, across its boundary surface by the EMF.
- Poynting's theorem state that, the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the fields, less the energy that flowed out through the surface.

Cont'

- The energy per unit time, per unit area, transported by fields is called the poynting vector:

$$S = \frac{1}{\mu_o}(E \times B) \ldots\ldots\ldots\ldots 24$$

where S is the energy flux density

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \frac{1}{\mu_o}\oint_S \; S \cdot da \ldots\ldots\ldots 25$$

Another useful form is given by putting the eld energy density term on the left side:

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Conservation of Momentum

- According to Newton's second law

$$F = \frac{dP_{mech}}{dt}$$

$$\frac{d\mathbf{p}_{\mathrm{mech}}}{dt} = -\epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau + \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 26$$

Where P_{mech} is the total momentum of particle contained in the volume V and T is the Maxwell Stress Tensor

$$T_{ij} \equiv \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right) \quad \ldots\ldots\ldots 27$$

Cont'

- The indices I and j refer to the coordinates x,y and z, so the stress tensor has a total of nine components (T_{xx}, T_{yy}, T_{zz} and so on).
- The kronecker delta δ_{ij} *is* 1 *if are the same and zero if not*
- Therefore

$$T_{xx} = \frac{1}{2}\epsilon_0(E_x^2 - E_y^2 - E_z^2) + \frac{1}{2\mu_0}(B_x^2 - B_y^2 - B_z^2),$$

$$T_{xy} = \epsilon_0(E_x E_y) + \frac{1}{\mu_0}(B_x B_y),$$

and so on

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Cont'

- Eqn. (26) states that the rate of change of the mechanical momentum in a volume V is equal to the integral over the surface of the volume of the stress tensor's flux through that surface minus the rate of change of the volume integral of the Poynting vector.
- Let p_{em} *be* the density of momentum in fields and p_{mech} *density of mechanical momentum*
- $p_{em}=\epsilon_o\mu_o S$
- $\frac{\partial}{\partial t}(p_{em}+p_{mech})=\nabla\cdot\overleftrightarrow{T}$

Potential and Fields

❑ Potential formulation

- If you are asking your self how the sources ($\rho\ and\ J$) generate electric field and magnetic fields; so this topic answer for your question.
- RECALL MAXWELL’S EQUATION…………..
- Recall how we arrived at Maxwell's Equations. We first developed Faraday's Law by incorporating both empirical information and the requirement of the Lorentz Force being consistent with Galilean relativity.

Cont'

- $\vec{B} = \vec{\nabla} \times \vec{A}$ ………………….1
- However, Faraday's Law implies that $\nabla \times E \neq 0$ when B has time dependence.
- Therefore, we cannot assume $E = \nabla V$ However, using $B = \nabla \times A$, we see that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{A}\right) \quad \Longrightarrow \quad \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

- Thus, the Helmholtz Theorem implies
- $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V \quad \Longrightarrow \quad \boxed{\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}}$ ………….2

Cont'

- Eqn. (1) & (2) fulfill the homogenous Maxwell equation (ii) & (iii) what about Gauss (i) & Ampere's-Maxwell (iv) law. Putting (2) in (i) ; $\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho$ 3
- Putting (1) & (2) in (iv) yields

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0\epsilon_0 \nabla \left(\frac{\partial V}{\partial t}\right) - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} \text{.....................4}$$

- Or using the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, and some rearranging terms a bit

$$\left(\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J} \text{..........5}$$

Cont’

Eqn. (3) & (5) contain all the information in Maxwell’s equations.

Coulombs and Lorentz gauges

❖ Coulombs gauges

- As in magnetostatics we pick ;

$$\nabla \cdot A = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots 6$$

- This eqn. (3) yields ;

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho. \quad \ldots\ldots\ldots\ldots\ldots 7$$

- That is, the charge density sets the potential in the same way as in electrostatics, so changes in charge density propagate into the potential instantaneously.
- Of course, you know from special relativity that this is not possible. We will see that there are corrections from $\partial A/_{\partial t}$ that prevent E from responding instantaneously to such changes.

Cont'

- The differential equation for **A** becomes

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \mu_0 \epsilon_0 \nabla \left(\frac{\partial V}{\partial t} \right) \text{..................} 8$$

- *Advantage* is that the *scalar* potential is particularly simple to calculate;
- *Disadvantage* is that **A** is particularly *difficult* to calculate.

The Lorenz Gauge

- In Lorentz gauge we pick

$$\nabla \cdot A = -\epsilon_o \mu_o \frac{\partial V}{\partial t} \text{..........................9}$$

This is designed to eliminate the middle term in Eqn. 5.with this

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \quad \text{..... ....10}$$

- Meanwhile , the differential eqn. for V, (3) becomes

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0} \rho. \quad \text{...................11}$$

The virtual of Lorentz gauge is that it treats V and **A o**n equal footing: the same differential operator

$$\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \equiv \square^2 \quad \textbf{...................13}$$

Cont'

$$\left.\begin{aligned} &\text{(i)} \quad \Box^2 V = -\frac{1}{\epsilon_0}\rho \\ &\text{(ii)} \quad \Box^2 \mathbf{A} = -\mu_0 \mathbf{J}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots 14$$

where $\Box^2$ called the **d' Alembertian**

- *V and A have the same differential operator of l'Alembertian (4-dim. operator)*
- *Under the Lorentz gauge, the whole of electrodynamics reduces to the problem of solving the inhomogeneous wave equation for specified sources.*

Continuous charge distributions

- Retarded Potentials
- In static case Eqs. 14 reduce to poisson's eqs.
- $$\nabla^2 V = -\frac{1}{\epsilon_0}\rho, \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad \text{..........15}$$

- With the familiar solution
- $$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau', \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau' \quad \text{............16}$$

Cont'

- In non-static case, therefore it's not the status of the sources right now that matters, but rather its condition at some earlier time t_r(called retarded time) when the message left.
- Since this message must travel a distance r, the delay is $\frac{r}{c}$

$$t_r \equiv t - \frac{r}{c} \;\dots\dots\dots..17$$

- The natural generalization of eqn. (16) for non-static sources is therefore

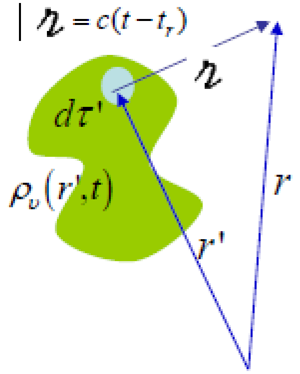
Cont'

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau', \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau' \quad \ldots(18)$$

Here $\rho(r', t_r)$ is the charge density that prevailed at point r' *at the retarded time*. Because the integrand are evaluated at the retarded time, these called 'Retarded potentials'.

Jefimenko's Equations

- *Time-varying charges and currents generate retarded scalar potential, retarded vector potential.*



- Potentials at a distance *r* from the source at time t depend on the values of ρ and J at an earlier time **(t - *r*/u)**

- **Retarded in time**($t_r = t - \frac{\mathscr{r}}{c} = t - \frac{\mathscr{r}}{c}$ **in vacuum)**

Cont'

$$V(r,t)=\frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho_v(r',t_r)}{\mathscr{r}}d\tau' \qquad \xrightarrow[(t_r=t-\mathscr{r}/c)]{(\mathscr{r}=|r-r'|)} \qquad \left(\mathbf{E}=-\nabla V-\frac{\partial \mathbf{A}}{\partial t}\right)$$

$$\mathbf{A}(r,t)=\frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbf{J}(r',t_r)}{\mathscr{r}}d\tau' \qquad \left(\mathbf{B}=\nabla\times\mathbf{A}\right)$$

$$\nabla V=\frac{1}{4\pi\epsilon_0}\int\left[(\nabla\rho)\frac{1}{\mathscr{r}}+\rho\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

$$\nabla V=\frac{1}{4\pi\epsilon_0}\int\left[-\frac{\dot\rho}{c}\frac{\hat{\mathscr{r}}}{\mathscr{r}}-\rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau'$$

- $$\frac{\partial \mathbf{A}}{\partial t}=\frac{\mu_0}{4\pi}\int\frac{\dot{\mathbf{J}}}{\mathscr{r}}d\tau' \Rightarrow \mathbf{E}(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}}+\frac{\dot\rho(\mathbf{r}',t_r)}{c\mathscr{r}}\hat{\mathscr{r}}-\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c^2\mathscr{r}}\right]d\tau'$$
- **time-dependent generalization of Coulomb's law**

Cont'

- Similarly

$$\mathbf{B}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\left[\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}^2}+\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c\mathscr{r}}\right]\times\hat{\mathscr{r}}\,d\tau'$$

❖ time-dependent generalization of the Biot Savart law

Taking the divergence

$$\nabla V=\frac{1}{4\pi\epsilon_0}\int\left[-\frac{\dot{\rho}}{c}\frac{\hat{\mathscr{r}}}{\mathscr{r}}-\rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau'$$

$$\nabla^2 V=\frac{1}{4\pi\epsilon_0}\int\left\{-\frac{1}{c}\left[\frac{\hat{\mathscr{r}}}{\mathscr{r}}\cdot(\nabla\dot{\rho})+\dot{\rho}\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right)\right]-\left[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot(\nabla\rho)+\rho\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]\right\}d\tau'$$

$$\nabla^2 V=\frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{c^2}\frac{\ddot{\rho}}{\mathscr{r}}-4\pi\rho\delta^3(\mathscr{r})\right]d\tau'=\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}-\frac{1}{\epsilon_0}\rho(\mathbf{r},t)$$

The retarded potential also satisfies the inhomogeneous wave equation.

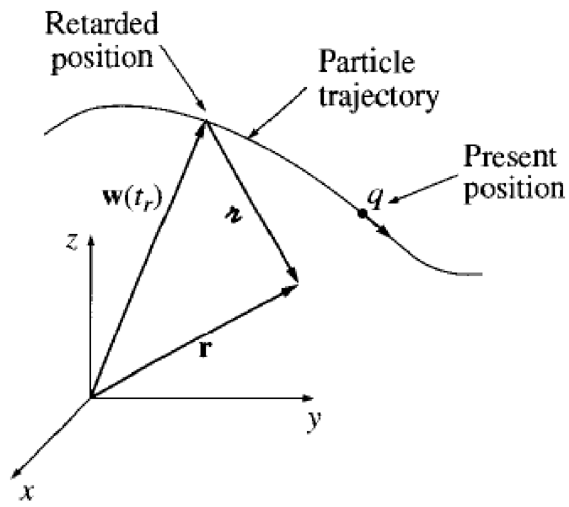
Lienard-Wiechert Potentials

❖**Lienard-Wiechert potentials** for a moving point charge

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})}$$

$$\mathbf{J} = \rho\mathbf{v} \longrightarrow \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\rho(\mathbf{r}', t_r)\mathbf{v}(t_r)}{\mathscr{r}} d\tau' = \frac{\mu_0}{4\pi} \frac{\mathbf{v}}{\mathscr{r}} \int \rho(\mathbf{r}', t_r)\, d\tau'$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t)$$



The Fields of a Moving Point Charge

- $$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A} \qquad V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})}, \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t)$$

- ❑ Let’s begin with the gradient of V:

$$\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2} \left[\mathbf{v} + (c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a}) \nabla t_r \right]$$

To complete the calculation, we need to know ∇t_r.

$$\nabla V = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3} \left[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}) \mathbf{v} - (c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a}) \boldsymbol{\mathscr{r}} \right]$$

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3} \left[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})(-\mathbf{v} + \mathscr{r}\mathbf{a}/c) + \frac{\mathscr{r}}{c}(c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a}) \mathbf{v} \right]$$

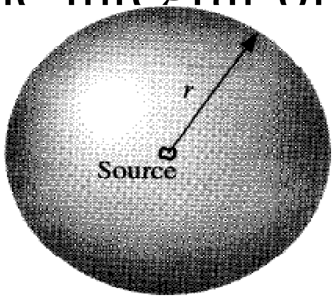
$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} [(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] \qquad \mathbf{u} \equiv c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t)$$

Radiation

- How accelerating charges and changing currents produce electromagnetic waves, how they radiate?
- Assume a radiation source is localized near the origin. Total power passing out through a spherical shell is the integral of the Poynting vector

$$P(r) = \oint \mathbf{S} \cdot d\mathbf{a} = \frac{1}{\mu_0} \oint (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}$$



- The total power *radiated from the source* is the limit of this quantity as *r* goes to infinity:

$$P_{\text{rad}} \equiv \lim_{r \to \infty} P(r)$$

Cont'

- *Since the area of the sphere is $4\pi r^2$, so for radiation to occur (for Prad not to be zero), the Poynting vector must decrease (at large r) no faster than $1/r^2$.*

- But, according to Coulomb's law and Bio-Savart law, S ~ $1/r^4$ for static configurations.

- **Static sources do not radiate!**

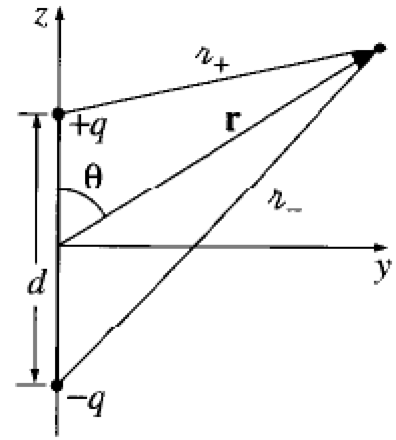
- **Jefimenko's Equations** indicate that *time dependent* fields include terms that go like 1/ *r;(ρ & J)* it is *these* terms that are responsible for electromagnetic radiation.

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}+\frac{\dot{\rho}(\mathbf{r}',t_r)}{c\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}}-\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c^2\mathscr{r}}\right]d\tau'$$

$$\mathbf{B}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\left[\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}^2}+\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c\mathscr{r}}\right]\times\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

Electric Dipole Radiation

❑ Suppose the charge back and forth through the wire, from one end to the other, at an angular frequency ω:



❖ Dipole charge: $q(t) = q_0 \cos(\omega t)$

❖ Current: $\mathbf{I}(t) = \dfrac{dq}{dt}\,\hat{\mathbf{z}} = -q_0\omega \sin(\omega t)\,\hat{\mathbf{z}}$

❖ Electric dipole: $\mathbf{p}(t) = p_0 \cos(\omega t)\,\hat{\mathbf{z}},$

❑ The retarded scalar and vector potentials at P are

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0\cos[\omega(t-\mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0\cos[\omega(t-\mathscr{r}_-/c)]}{\mathscr{r}_-}\right\} \qquad \mathbf{A}(s,t) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\int_{-\infty}^{\infty}\frac{I(t_r)}{\mathscr{r}}\,dz$$

Retarded scalar potential $V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0\cos[\omega(t - \mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0\cos[\omega(t - \mathscr{r}_-/c)]}{\mathscr{r}_-}\right\}$

- Approximation 1 : *d* « *r* ⟹ To make a perfect dipole, assume *d* to be extremely small
- $$\mathscr{r}_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2}. \implies \frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$$\cos[\omega(t - \mathscr{r}_\pm/c)] \cong \cos\left[\omega(t - r/c) \pm \frac{\omega d}{2c}\cos\theta\right]$$

$$= \cos[\omega(t - r/c)]\cos\left(\frac{\omega d}{2c}\cos\theta\right) \mp \sin[\omega(t - r/c)]\sin\left(\frac{\omega d}{2c}\cos\theta\right)$$

Approximation 2 : *d* « λ = *2πc/ω* ⟹ Assume *d* to be extremely smaller than wavelength

$$\cos[\omega(t - \mathscr{r}_\pm/c)] \cong \cos[\omega(t - r/c)] \mp \frac{\omega d}{2c}\cos\theta\sin[\omega(t - r/c)]$$

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0\cos[\omega(t - \imath_+/c)]}{\imath_+} - \frac{q_0\cos[\omega(t - \imath_-/c)]}{\imath_-}\right\}$$

- In the static limit ($\omega \rightarrow 0$)

$$V = \frac{p_0\cos\theta}{4\pi\epsilon_0 r^2}$$

- Approximation 2 : $d >> \lambda = 2\pi c/\omega$ $\Rightarrow$ Assume ***r*** to be larger than wavelength (far-field radiation)

$$\Rightarrow V(r, \theta, t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t - r/c)]$$

Retarded vector potential: $\mathbf{A}(s,t) = \frac{\mu_0}{4\pi}\hat{\mathbf{z}}\int_{-\infty}^{\infty}\frac{I(t_r)}{\mathscr{r}}\,dz$

$$\mathbf{I}(t) = \frac{dq}{dt}\hat{\mathbf{z}} = -q_0\omega\sin(\omega t)\hat{\mathbf{z}} \longrightarrow \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2}\frac{-q_0\omega\sin[\omega(t-\mathscr{r}/c)]\hat{\mathbf{z}}}{\mathscr{r}}\,dz$$

- $d \ll \lambda \ll r \longrightarrow \mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-r/c)]\hat{\mathbf{z}}$

❑ Retarded potentials:

$$V(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t-r/c)]$$

$$\begin{aligned}
\nabla V &= \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}} \\
&= -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\cos\theta\left(-\frac{1}{r^2}\sin[\omega(t-r/c)] - \frac{\omega}{rc}\cos[\omega(t-r/c)]\right)\hat{\mathbf{r}} - \frac{\sin\theta}{r^2}\sin[\omega(t-r/c)]\hat{\boldsymbol{\theta}}\right\} \\
&\cong \frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\mathbf{r}}
\end{aligned}$$

Cont'

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t - r/c)]\,\hat{\mathbf{z}}$$

$$\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi r}\cos[\omega(t - r/c)](\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}} = -\frac{\mu_0 p_0 \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos[\omega(t - r/c)] + \frac{\sin\theta}{r}\sin[\omega(t - r/c)]\right\}\hat{\boldsymbol{\phi}}$$

- $$\nabla V = \frac{p_0 \omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\mathbf{r}}.$$

$$\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi r}\cos[\omega(t - r/c)](\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\nabla \times \mathbf{A} = -\frac{\mu_0 p_0 \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos[\omega(t - r/c)] + \frac{\sin\theta}{r}\sin[\omega(t - r/c)]\right\}\hat{\boldsymbol{\phi}}$$

Far-field radiation

$d \ll \lambda \ll r$

$$\boxed{\begin{aligned}\mathbf{E} &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}}.\\ \mathbf{B} &= \nabla \times \mathbf{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}}.\end{aligned}}$$

Cont'

- E and B are in phase, mutually perpendicular, and transverse; the ratio of their amplitudes is E_o/B_o=c.
- *These are actually spherical waves, not plane waves, and their amplitude decreases like 1/r.*
- The energy radiated by an oscillating electric dipole is determined by the Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{\mathbf{r}}$$

Cont'

- $$\langle \mathbf{S} \rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \frac{\sin^2\theta}{r^2} \hat{\mathbf{r}}$$ Intensity obtained by averaging
- $$\langle P \rangle = \int \langle \mathbf{S} \rangle \cdot d\mathbf{a} = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$ total power radiated

Magnetic Dipole Radiation

- Magnetic dipole moment of an oscillating loop current:
- $I(t) = I_0 \cos(\omega t)$ where $m_0 \equiv \pi b^2 I_0$

$$\mathbf{m}(t) = \pi b^2 I(t)\,\hat{\mathbf{z}} = m_0 \cos(\omega t)\,\hat{\mathbf{z}}$$

➢ The loop is uncharged, so the scalar potential is zero.

➢ The retarded vector potential is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{I_0 \cos[\omega(t - \imath/c)]}{\imath}\, d\mathbf{l}'$$

➢ For a point ***r*** directly above the *x* axis, **A** must aim in the *y* direction, since the *x* components from symmetrically placed points on either side of the *x* axis will cancel.

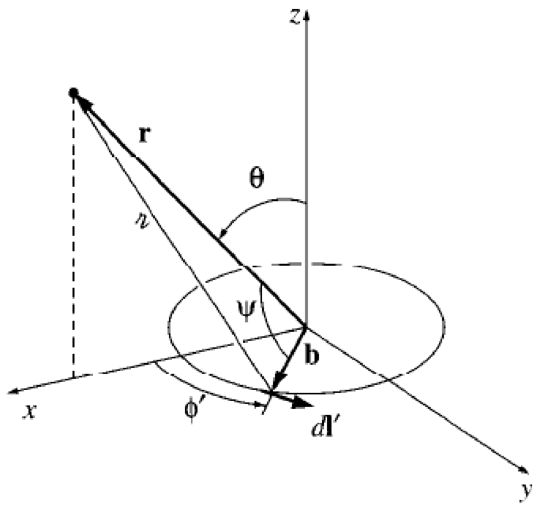
Cont’

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 I_0 b}{4\pi}\,\hat{\mathbf{y}} \int_0^{2\pi} \frac{\cos[\omega(t - \mathscr{r}/c)]}{\mathscr{r}} \cos\phi'\,d\phi'$$

($cos\varphi$ serves to pick out the y- component of dI').

$$\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\cos\psi}$$

$$\mathbf{r} = r\sin\theta\,\hat{\mathbf{x}} + r\cos\theta\,\hat{\mathbf{z}}, \quad \mathbf{b} = b\cos\phi'\,\hat{\mathbf{x}} + b\sin\phi'\,\hat{\mathbf{y}}$$



$$rb\cos\psi = \mathbf{r}\cdot\mathbf{b} = rb\sin\theta\cos\phi'$$

$$\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\sin\theta\cos\phi'}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 I_0 b}{4\pi}\,\hat{\mathbf{y}} \int_0^{2\pi} \frac{\cos[\omega(t - \mathscr{r}/c)]}{\mathscr{r}} \cos\phi'\,d\phi'$$

$$\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\sin\theta\cos\phi'}$$

Cont'

- **Approximation 1 : *b* « *r*** For a "perfect" dipole, the loop must be extremely small

$$\mathscr{r} \cong r\left(1 - \frac{b}{r}\sin\theta\cos\phi'\right) \longrightarrow \frac{1}{\mathscr{r}} \cong \frac{1}{r}\left(1 + \frac{b}{r}\sin\theta\cos\phi'\right)$$

$$\cos[\omega(t - \mathscr{r}/c)] \cong \cos\left[\omega(t - r/c) + \frac{\omega b}{c}\sin\theta\cos\phi'\right]$$

$$= \cos[\omega(t - r/c)]\cos\left(\frac{\omega b}{c}\sin\theta\cos\phi'\right) - \sin[\omega(t - r/c)]\sin\left(\frac{\omega b}{c}\sin\theta\cos\phi'\right)$$

- **Approximation 2 : *b* « λ= *2πc*/ω** → Assume ***b*** to be extremely smaller than wavelength

$$\cos[\omega(t - \mathscr{r}/c)] \cong \cos[\omega(t - r/c)] - \frac{\omega b}{c}\sin\theta\cos\phi'\sin[\omega(t - r/c)]$$

Cont’

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0 I_0 b}{4\pi r}\,\hat{\mathbf{y}} \int_0^{2\pi} \left\{\cos[\omega(t - r/c)] + b\sin\theta\cos\phi' \left(\frac{1}{r}\cos[\omega(t - r/c)] - \frac{\omega}{c}\sin[\omega(t - r/c)]\right)\right\} \cos\phi'\, d\phi'$$

In general **A** points in the ϕ-direction.

$$\mathbf{A}(r, \theta, t) = \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}\cos[\omega(t - r/c)] - \frac{\omega}{c}\sin[\omega(t - r/c)]\right\}\hat{\boldsymbol{\phi}}$$

❖ In the static limit ($\omega = 0$),

$$\mathbf{A}(r, \theta) = \frac{\mu_0}{4\pi}\frac{m_0 \sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

Approximation 3 : $r \gg \lambda = 2\pi c/\omega$ → Assume r to be larger than wavelength (far-field radiation)

$$\mathbf{A}(r, \theta, t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin[\omega(t - r/c)]\hat{\boldsymbol{\phi}}$$

Cont'

Far-field radiation

$d \ll \lambda \ll r$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}}$$

- These fields are in phase, mutually perpendicular, and transverse to the direction of propagation (r) and the ratio of their amplitudes is $E_0/B_0 = c$, all of which is as expected for electromagnetic waves.
- **Energy flux:** $$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{\mathbf{r}}$$

$$\langle \mathbf{S} \rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}}$$

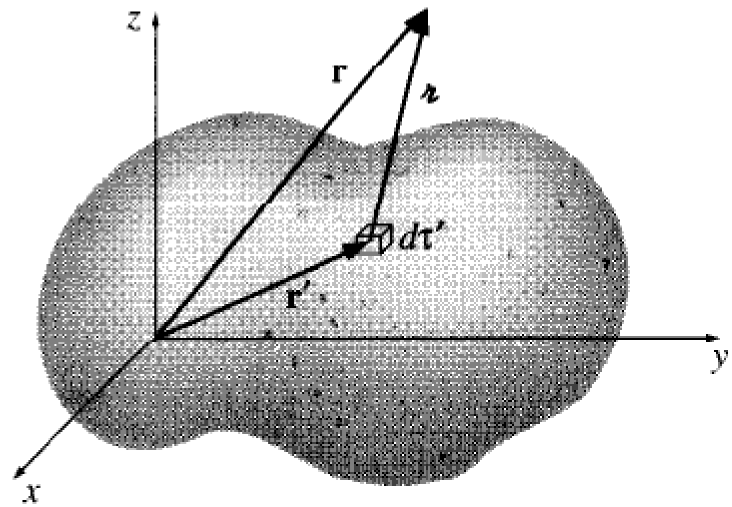
Cont'

- **Total power radiated:** $\langle P \rangle = \dfrac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$
- One important difference between electric and magnetic dipole radiation is that for configurations with comparable dimensions, the power radiated electrically is enormously greater.

$$\frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \left(\frac{m_0}{p_0 c}\right)^2 \xrightarrow[\text{Setting } d = \pi b]{m_0 = \pi b^2 I_0, \text{ and } p_0 = q_0 d \quad I_0 = q_0\omega} \frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \left(\frac{\omega b}{c}\right)^2 \xrightarrow[b \ll \lambda = 2\pi c/\omega]{\text{Approximation 2}} \ll 1$$

Radiation from an Arbitrary Source

- Consider a configuration of charge and current that is entirely arbitrary
- The retarded scalar potential is



$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t - \imath/c)}{\imath}\, d\tau'$$

$$\imath = \sqrt{r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}'}.$$

- Approximation 1 : $r' \ll r$ (far field)

$$\imath \cong r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right) \longrightarrow \frac{1}{\imath} \cong \frac{1}{r}\left(1 + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)$$

$$\longrightarrow \rho(\mathbf{r}', t - \imath/c) \cong \rho\left(\mathbf{r}', t - \frac{r}{c} + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)$$

Cont'

- *Expanding* ρ *as a Taylor series in t about the retarded time at the origin* $t_0 \equiv t - \frac{r}{c}$,

$$\rho(\mathbf{r}', t - r/c) \cong \rho(\mathbf{r}', t_0) + \dot{\rho}(\mathbf{r}', t_0)\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right) + \frac{1}{2}\ddot{\rho}\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)^2, \quad \frac{1}{3!}\dddot{\rho}\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)^3, \ldots$$

- *Approximation 2 :* $r' \ll \lambda = 2\pi c/\omega \longrightarrow r' \ll \frac{c}{|\ddot{\rho}/\dot{\rho}|}, \frac{c}{|\dddot{\rho}/\dot{\rho}|^{1/2}}, \frac{c}{|\ddot{\ddot{\rho}}/\dot{\rho}|^{1/3}}, \ldots$

- $r' << \lambda \ll r \longrightarrow V(\mathbf{r}, t) \cong \frac{1}{4\pi\epsilon_0 r}\left[\int \rho(\mathbf{r}', t_0)\, d\tau' + \frac{\hat{\mathbf{r}}}{r}\cdot\int \mathbf{r}'\rho(\mathbf{r}', t_0)\, d\tau' + \frac{\hat{\mathbf{r}}}{c}\cdot\frac{d}{dt}\int \mathbf{r}'\rho(\mathbf{r}', t_0)\, d\tau'\right]$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}(t_0)}{r^2} + \frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right]$$

➢ In the static case, the first two terms are the monopole and dipole contributions

Cont'

- Now, consider the vector potential:

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t - \mathscr{r}/c)}{\mathscr{r}} \, d\tau'$$

- To first order in r' it suffices to replace by r in the integrand:

$\mathscr{r} \cong r \longrightarrow t_0 \cong t - \frac{r}{c}$ (*Ignore the effect of magnetic* dipole moment)

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r} \int \mathbf{J}(\mathbf{r}', t_0) \, d\tau' \cong \frac{\mu_0}{4\pi} \frac{\dot{\mathbf{p}}(t_0)}{r}$$

- Approximation 3 : $r \gg \lambda = 2\pi c/\omega$ (discard 1/r2 terms in E and B)

$$V(\mathbf{r}, t) \cong \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{p}(t_0)}{r^2} + \frac{\hat{\mathbf{r}} \cdot \dot{\mathbf{p}}(t_0)}{rc} \right]$$

Cont'

- $$\nabla V \cong \nabla\left[\frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right] \cong \frac{1}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)}{rc}\right]\nabla t_0 = -\frac{1}{4\pi\epsilon_0 c^2}\frac{[\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)]}{r}\hat{\mathbf{r}}$$

$$\nabla\times\mathbf{A} \cong \frac{\mu_0}{4\pi r}[\nabla\times\dot{\mathbf{p}}(t_0)] = \frac{\mu_0}{4\pi r}[(\nabla t_0)\times\ddot{\mathbf{p}}(t_0)] = -\frac{\mu_0}{4\pi rc}[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t_0)]$$

$$\frac{\partial\mathbf{A}}{\partial t} \cong \frac{\mu_0}{4\pi}\frac{\ddot{\mathbf{p}}(t_0)}{r}$$

$$\mathbf{E}(\mathbf{r},t) \cong \frac{\mu_0}{4\pi r}[(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}})\hat{\mathbf{r}} - \ddot{\mathbf{p}}] = \frac{\mu_0}{4\pi r}[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\ddot{\mathbf{p}})]$$

$$\mathbf{B}(\mathbf{r},t) \cong -\frac{\mu_0}{4\pi rc}[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}]$$

Cont'

- In particular, if we use spherical polar coordinates, with the z axis in the direction of $\ddot{\mathbf{p}}(t_0)$,

$$\mathbf{E}(r, \theta, t) \cong \frac{\mu_0 \ddot{p}(t_0)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\theta}}$$

$$\mathbf{B}(r, \theta, t) \cong \frac{\mu_0 \ddot{p}(t_0)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\phi}}$$

- Notice that E and B are mutually perpendicular, transverse to the direction of propagation (r) and in the ratio $E/B = c$, as always for radiation fields.
- Poynting vector: $\mathbf{S} \cong \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{\mu_0}{16\pi^2 c}[\ddot{p}(t_0)]^2\left(\frac{\sin^2\theta}{r^2}\right)\hat{\mathbf{r}}$
- Total radiated power $P \cong \int \mathbf{S} \cdot d\mathbf{a} = \frac{\mu_0 \ddot{p}^2}{6\pi c}$

Cont'

- If the electric dipole moment should happen to vanish (or, at any rate, if its second time derivative is zero), then there is no electric dipole radiation, and one must look to the next term: the one of *second* order in r'.
- As it happens, this term can be separated into two parts, one of which is related to the *magnetic* dipole moment of the source, the other to its electric *quadrupole* moment (The former is a generalization of the magnetic dipole radiation).
- If the magnetic dipole and electric quadrupole contributions vanish, the $(r')^3$ term must be considered.

Magnetism as a Relativistic Phenomenon

- In the reference frame where q is at rest, system $\bar{S}$,

$$\lambda_{\text{tot}} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda u v}{c^2\sqrt{1-u^2/c^2}}$$

- The line charge sets up an *electric* field

$$E = \frac{\lambda_{\text{tot}}}{2\pi\epsilon_0 s}$$

so there is an electrical force on q in $\bar{S}$,

$$\bar{F} = qE = -\frac{\lambda v}{\pi\epsilon_0 c^2 s}\frac{qu}{\sqrt{1-u^2/c^2}}$$

- The force can be transformed into in S

$$F = \frac{1}{\gamma}\bar{F} = \sqrt{1-u^2/c^2}\,\bar{F} = -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{s}$$

- But, in the wire frame (S) the total charge is neutral !

Cont'

- Electrostatics and relativity imply the existence of another force in view point of S frame ⇨ **magnetic force**.
- by using $c^2 = (\epsilon_0\mu_0)^{-1}$ and $I = 2\lambda v$

$$F = -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{s} = -qu\left(\frac{\mu_0 I}{2\pi s}\right)$$

$$B = \left(\frac{\mu_0 I}{2\pi s}\right)$$

- One observer's electric field is another's magnetic field!
- Therefore, the relativistic force F is the Lorentz force in system S, not Minkowski!

Field transformation

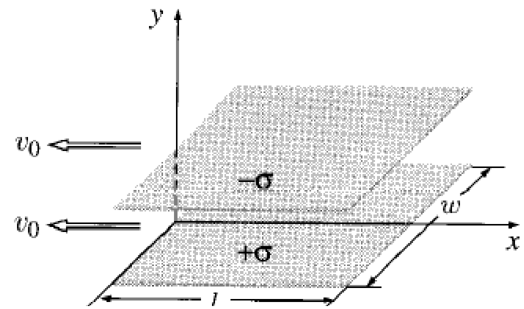
- Let's find the general transformation rules for electromagnetic fields:
- Given the fields in a frame (s), what are the fields in another frame (s')?
- consider the *simplest possible* electric field in a large parallel-plate capacitor in s_o frame.

$$\mathbf{E}_0 = \frac{\sigma_0}{\epsilon_0}\hat{\mathbf{y}}$$

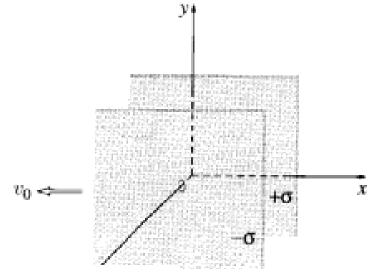
- In the system ***S***, moving to the right at speed *v0*,
- the plates are moving to the left with the different surface charge ?:

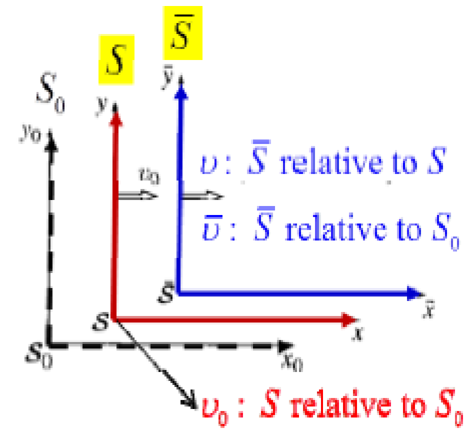
Field transformation

$$\left.\begin{aligned}\bar{E}_y &= \gamma(E_y - vB_z),\\ \bar{B}_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right).\end{aligned}\right\}$$



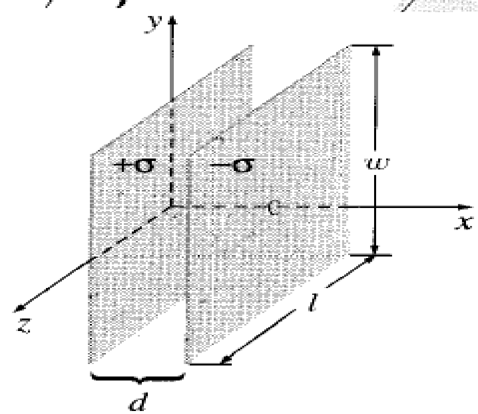
$$\left.\begin{aligned}\bar{E}_z &= \gamma(E_z + vB_y),\\ \bar{B}_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right).\end{aligned}\right\}$$





$$\bar{E}_x = E_x$$

$$\bar{B}_x = B_x$$



Cont':

- **Two special cases:**

- (1) *If B* = 0 *in S frame*, (*E* ≠0);

$$\bar{\mathbf{B}} = \gamma \frac{v}{c^2}(E_z \,\hat{\mathbf{y}} - E_y \,\hat{\mathbf{z}})$$

$$\text{or, since } \mathbf{E}^{\perp} = \gamma_0 \mathbf{E}_0{}^{\perp} \longrightarrow \bar{\mathbf{B}} = \frac{v}{c^2}(\bar{E}_z \,\hat{\mathbf{y}} - \bar{E}_y \,\hat{\mathbf{z}})$$

$$\text{or, since } \mathbf{v} = v\,\hat{\mathbf{x}}, \longrightarrow \bar{\mathbf{B}} = -\frac{1}{c^2}(\mathbf{v} \times \bar{\mathbf{E}})$$

- (2) *If E* =0 *in S frame*, (*B* ≠0);

$$\bar{\mathbf{E}} = -\gamma\, v(B_z \,\hat{\mathbf{y}} - B_y \,\hat{\mathbf{z}}) = -v(\bar{B}_z \,\hat{\mathbf{y}} - \bar{B}_y \,\hat{\mathbf{z}}) \longrightarrow \bar{\mathbf{E}} = \mathbf{v} \times \bar{\mathbf{B}}$$

- If either E or B is zero (at a particular point) in *one* system, then in any other system the fields (at that point) are very simply related.

Electromagnetic field tensor

- The components of **E** and **B** are stirred together when you go from one inertial system to another.
- What sort of an object is this, which has six components and transforms according to the above relations? It's an *antisymmetric, second-rank tensor.*
- $\bar{a}^{\mu} = \Lambda^{\mu}_{\nu} a^{\nu}$ and a second rank tensor is $\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda} \Lambda^{\nu}_{\sigma} t^{\lambda\sigma}$
- Where $\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\Rightarrow$ Lorentz transformation matrix


THE END
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