Calculus of Functions of Complex Variables (Math 2072)

Chapter One: Complex numbers

- Define the complex numbers & their operations
- Geometric representation & polar form of complex numbers
- De-Moiver's formula
- Root extraction

- The Riemann and the extended complex plane

Complex Number

- Who uses them in real life?
- Here's a hint….

Cont'd

- The navigation system in the space shuttle depends on complex numbers

Definition of complex numbers

DEFINITION 1.1

Complex Number

A complex number is any number of the $z = a + ib$ where a and b are real numbers and $i = \sqrt{-1}$ is the imaginary units.

- $z = x + iy$, the real number x is called the real part and y is called the imaginary part:

$$\mathrm{Re}(z) = x, \quad \mathrm{Im}(z) = y$$

Equality of complex numbers

DEFINITION 1.2

Complex Number

Complex number $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, $z_1 = z_2$, if $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$

- $x + iy = 0$ iff $x = 0$ and $y = 0$.

Example: Let w=2+3i and r=a+bi are two complex numbers, then w=r iff a=2 and b=3

Arithmetic Operations

Suppose $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2)$$

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2{}^2 + y_2{}^2} + i\,\frac{y_1 x_2 - x_1 y_2}{x_2{}^2 + y_2{}^2}$$

Operation of Complex Numbers

❖ Addition of Complex Numbers

$$(x + yi) + (a + bi) = (x + a) + (y + b)i$$

Example:

Find each sum or difference

a. $(3 - 4i) + (-2 + 6i)$

Solution:

Add real parts — Add imaginary parts

$$(3 - 4i) + (-2 + 6i) = \underbrace{[3 + (-2)]}_{\text{Add real parts}} + \underbrace{[-4 + 6]}_{\text{Add imaginary parts}}i$$

$$= 1 + 2i$$

Practical Exercise

Perform each the following operations

a) $(8+3i)+(6-2i)$

b) $(8-6i)-(2i-7)$

c) $(5+7i)-(2+6i)$

d) $3+3i+8-2i-7$ Ans. $4+i$

e) $-1-8i-4-i$ Ans. $-5-9i$

Cont'd

Subtraction of Complex Numbers

$$(x + yi) - (a + bi) = (x - y) + (y - b)i$$

Examples

a) $-3 + 6i - \left(-5 - 3i\right) - 8i$ <u>Ans</u>. $2 + i$

b) $(5+3i)+(-1+2i)+(7-5i)$

c) $8+3i-(6-2i)$

The product of two complex numbers

$$(x+iy)(a+ib) = x(a+ib) + y(a+ib)$$

$$= xa + xbi + ayi + byi^2$$

$$= xa - by + by(-1) + (bx+ax)i$$

$$= xa - by + (bx+ax)i$$

Examples

a) (8-6i)(6-2i)

b) (2-i)(-3+2i)(5-4i)

c) 4i (−2 − 8i)

d) (−2 − 2i)(−4 − 3i)(7 + 8i)

e) (7 − 6i)(−8 + 3i)

Con’t

- **Remark**
- $(x+yi)(x-yi) = x^2 - y^2 i^2$

$$= x^2 - y^2(-1)$$

$$= x^2 + y^2$$

Hence

(x+iy)(x-iy)=x^2+y^2

<u>Example</u>

(2+3i)(2-3i)=2^2+3^2=4+9=13

Division of Complex Numbers

$z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$ then

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + ix_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$= \frac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$= \underbrace{\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}}_{Re(z)} + \underbrace{\frac{i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}}_{Im(z)}$$

Re(z) Im(z)

Example

Find x and y if $(2x - 3iy)(-2+i)^2 = 5(1-i)$

Solution:

$(2x - 3iy)(4+i^2-4i) = 5 - 5i$

$(2x - 3iy)(3 - 4i) = 5 - 5i$

$(6x - 12y - i(8x + 9y)) = 5 - 5i$

$6x - 12y = 5,\ 8x + 9y = 5$

$$\Rightarrow x = \frac{7}{10}, y = \frac{-1}{15}$$

Algebra properties of Complex Numbers –

Properties:

1) Closure: $z_1 + z_2$ is a complex number
2) Commutative: $z_1 + z_2 = z_2 + z_1$
3) Associative: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
4) Additive identity 0: $z + 0 = 0 + z = z$
5) Additive inverse $-z$: $z + (-z) = (-z) + z = 0$

Complex conjugate/Conjugaate/

Let z = x + yi , the complex conjugate of a complex number z is denoted by : $\bar{z} = \overline{x + yi} = x - yi$

$$z\bar{z} = (x + iy)(x - iy)$$

$$= x^2 + y^2 \qquad \text{(real number)}$$

The conjugate of a complex number changes the **sign of the imaginary part only!!!**

Obtained geometrically by **reflecting point *z* on the real axis**

Complex Conjugate

Suppose $z = x + iy, \overline{z} = x - iy$, and

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

$$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$$

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

cont'd

Find the complex conjugate of the following complex number

a) $z=7+3i$

b) $z=-5-2i$

c) $z=-3i$

d) $z=8$

Operations

Definition 1.5

(Division of Complex Numbers)

If $z_1 = a + bi$ and $z_2 = c + di$ then:

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di}$$

$$= \frac{a+bi}{c+di} \times \frac{c-di}{c-di}$$

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

Multiply with the **conjugate of denominator**

cont'd

- **Example: Simplify and write in standard form, *z*:**

$$a)\quad \frac{3+i}{1-i}$$

$$b)\quad \frac{2-4i}{3i}$$

$$c)\quad \frac{4-i^3}{i^{12}+5i^7}$$

$$d)\quad \frac{3}{1+3i}-\frac{1+i}{1-i}$$

$$a)\quad 1+2i$$

$$b)\quad -\frac{4}{3}-\frac{2}{3}i$$

$$c)\quad -\frac{1}{26}+\frac{21}{26}i$$

$$d)\quad \frac{3}{10}-\frac{19}{10}i$$

Perform the operation and write the result in standard form.

$$\frac{1+i}{i}-\frac{3}{4-i}=\frac{(1+i)}{i}\bullet\frac{i}{i}-\frac{3}{4-i}\bullet\frac{(4+i)}{(4+i)}$$

$$=\frac{i+i^2}{i^2}-\frac{12+3i}{4^2+1^2}=\frac{-1+i}{-1}-\frac{12+3i}{16+1}$$

$$=1-i-\frac{12}{17}-\frac{3}{17}i=1-\frac{12}{17}-i-\frac{3}{17}i$$

$$=\frac{17-12}{17}-\frac{17-3}{17}i$$

- Two important equations

$$z+\bar{z}=(x+iy)+(x-iy)=2x \qquad (1)$$

$$z\bar{z}=(x+iy)(x-iy)=x^2-i^2y^2=x^2+y^2 \qquad (2)$$

$$z-\bar{z}=(x+iy)-(x-iy)=2iy \qquad (3)$$

and

$$\mathrm{Re}(z)=\frac{z+\bar{z}}{2}, \mathrm{Im}(z)=\frac{z-\bar{z}}{2i}$$

The Properties of Conjugate Complex

$$i)\quad z = \overline{\overline{z}}$$

$$ii)\quad \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

$$iii)\quad \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$$

$$iv)\quad \overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$$

$$v)\quad \frac{1}{\overline{z}} = \overline{\left(\frac{1}{z}\right)}$$

$$vi)\quad \overline{z^n} = \left(\overline{z}\right)^n ; n$$

$$vii)\quad \frac{z + \overline{z}}{2} = \mathrm{Re}(z)$$

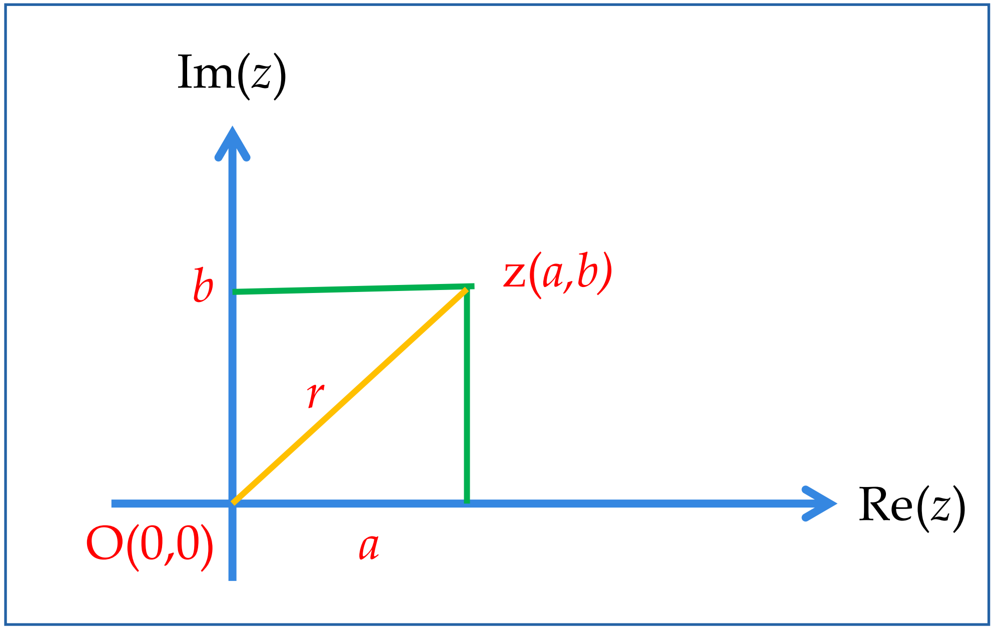
$$viii)\quad \frac{z - \overline{z}}{2} = \mathrm{Im}(z)$$

The Complex Plane Diagram

❖ **Definition : (Modulus of Complex Numbers)**

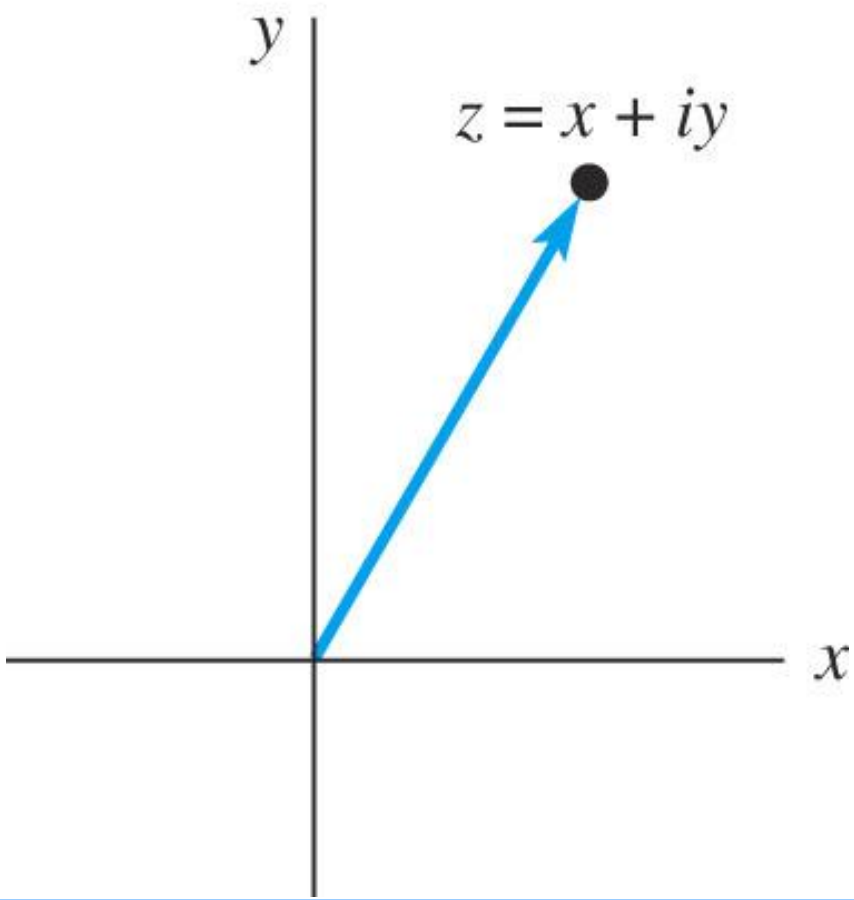
The modulus of z is defined by

$$r = |z| = \sqrt{a^2 + b^2}$$



Geometric Interpretation

- Fig 1.1 is called the complex plane and a complex number *z* is considered as a position vector.



DEFINITION 1.3

Modulus or Absolute Values

The modulus or absolute value of $z = x + iy$, denoted by $|z|$, is the real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

Example 3

If $z = 2 - 3i$, then

$$|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$$

As in Fig 1.2, the sum of the vectors z_1 and z_2 is the vector $z_1 + z_2$. Then we have

$$|z_1 + z_2| \le |z_1| + |z_2| \qquad (5)$$

The result in (5) is also known as the triangle inequality and extends to any finite sum:

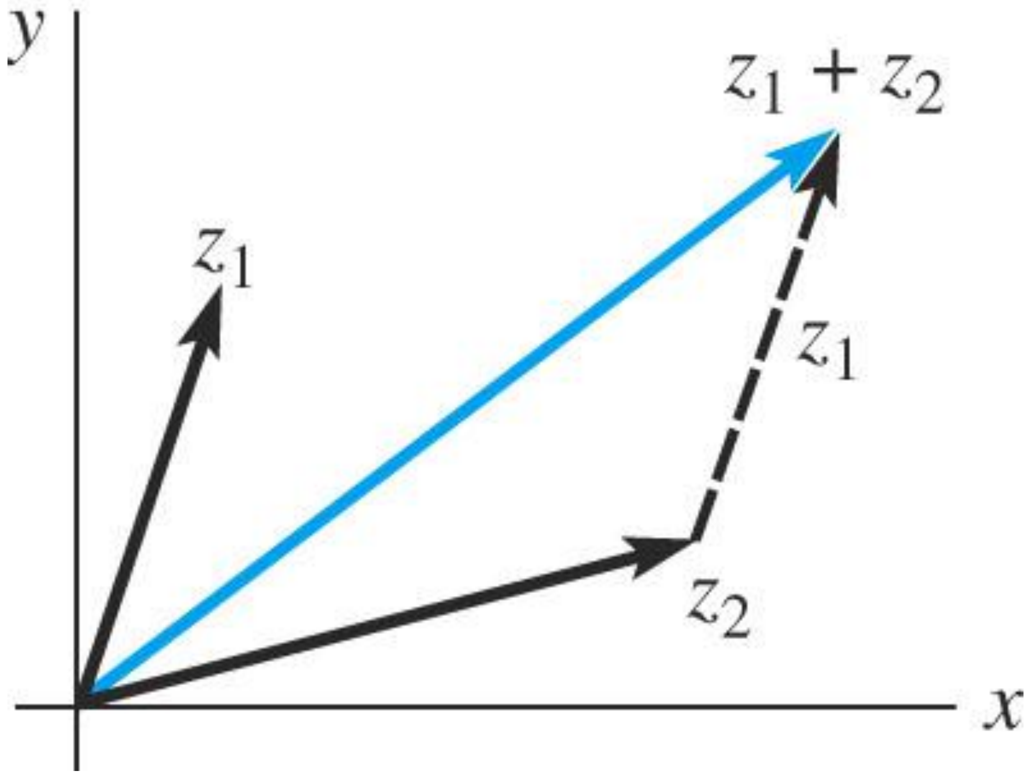
$$|z_1 + z_2 + \ldots + z_n| \le |z_1| + |z_2| + \ldots + |z_n| \qquad (6)$$

Using (5),

$$|z_1 + z_2 + (-z_2)| \le |z_1 + z_2| + |z_2|$$

$$|z_1 + z_2| \ge |z_1| - |z_2| \qquad (7)$$

Fig 1.2



Properties of modulus

$$\overline{|z_1 . z_2|} = |\overline{z_1}| . |\overline{z_2}| \qquad |z^n| = |z|^n$$

$$|\overline{z_1 \pm z_2}| = |\overline{z_1}| \pm |\overline{z_2}|$$

$$|\bar{z}| = |z|$$

$$|z . \bar{z}| = |z|^2 \qquad \text{Proof}: z = x + iy, z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$$

$$\left|\overline{\frac{z_1}{z_2}}\right| = \frac{|\overline{z_1}|}{|\overline{z_2}|}$$

$$|z_1 - z_2| \geq ||z_1| - |z_2|| \qquad |z_1 - z_2| \leq |z_1| + |z_2| \qquad \text{(Triangle inequality)}$$

$$|z_1 + z_2| \leq |z_1| + |z_2| \qquad |z_1 + z_2| \geq ||z_1| - |z_2||$$

$$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

The Properties of Modulus

$$i)\quad \left|\overline{z}\right| = |z|$$

$$ii)\quad \overline{zz} = \left|z^2\right|$$

$$iii)\quad |z_1 z_2| = |z_1||z_2|$$

$$iv)\quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},\ z_2 \neq 0$$

$$v)\quad \left|z^n\right| = |z|^n$$

$$vi)\quad |z_1 + z_2| \leq |z_1| + |z_2|$$

Conjugate of a Complex Number

$$x = \mathrm{Re}(z) = \frac{z + \bar{z}}{2}$$

$$y = \mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$$

$$|\bar{z}| = |x - iy| = \sqrt{x^2 + y^2} = |z|$$

$$\mathrm{Arg}(\bar{z}) = \mathrm{Arg}(x - iy) = -\tan^{-1}\frac{y}{x} = -\mathrm{Arg}(z)$$

1.2 Powers and Roots

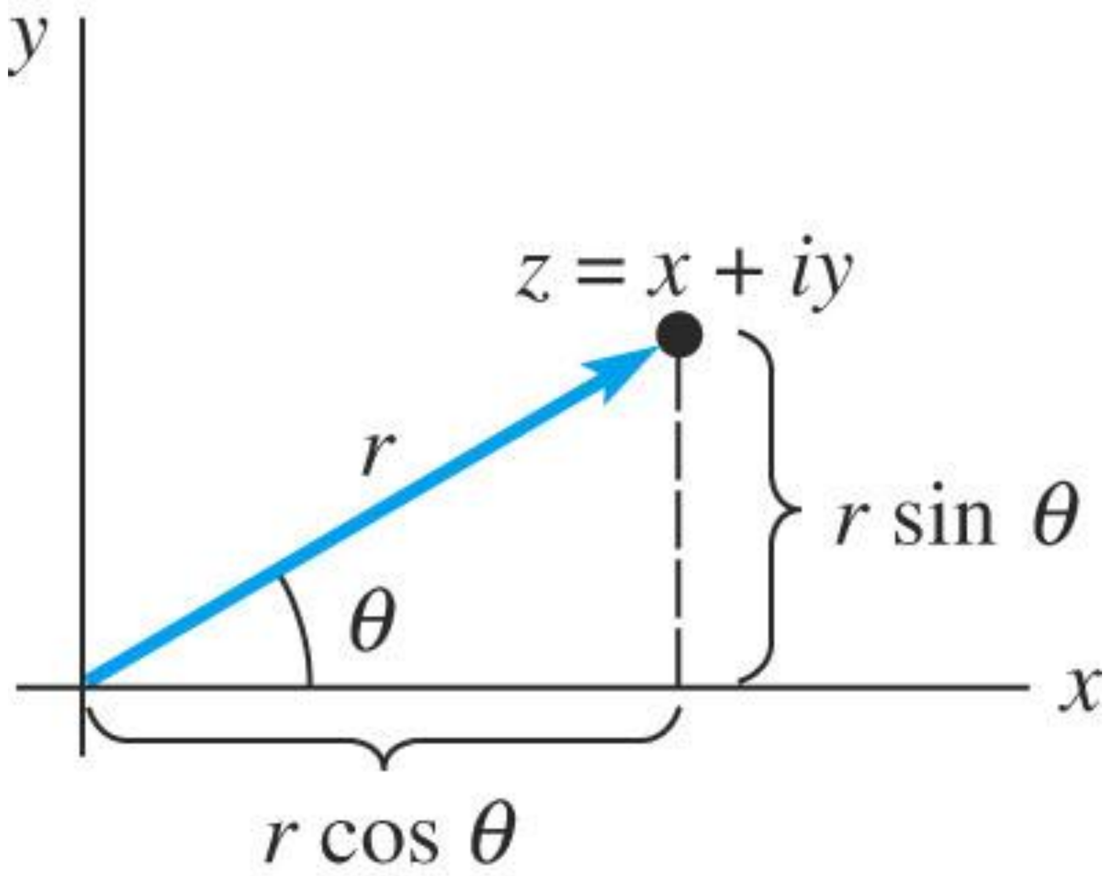
Polar Form

Referring to Fig 1.3, we have

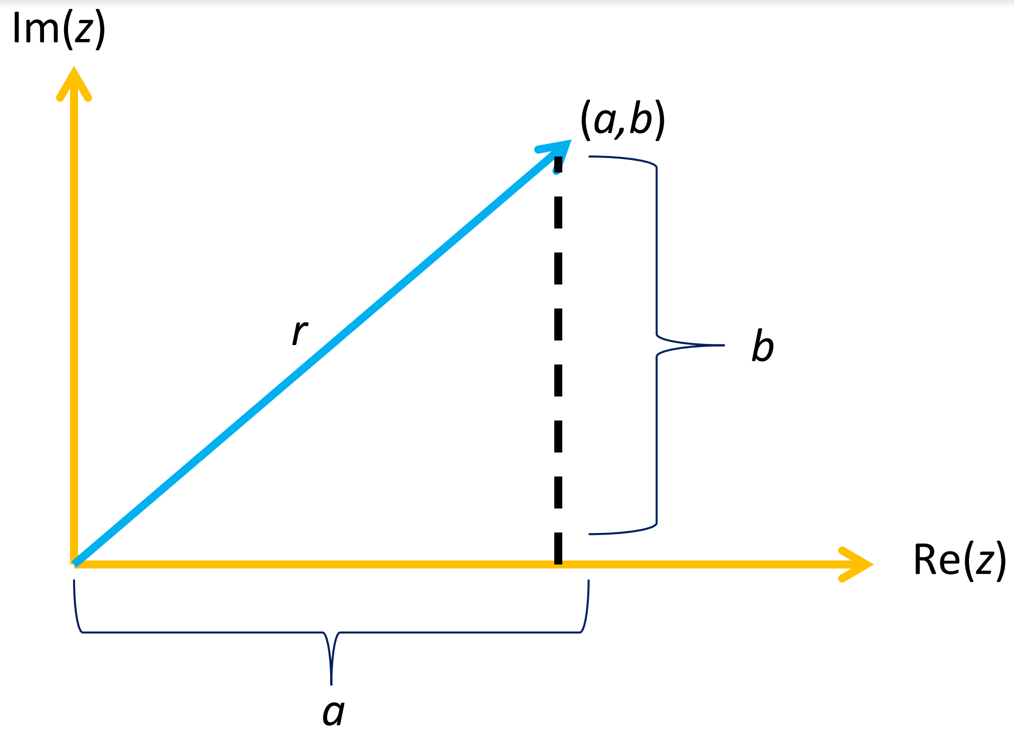
$$z = r(\cos\theta + i\sin\theta) \qquad (1)$$

where $r = |z|$ is the modulus of z and θ is the argument of z, $\theta = \arg(z)$. If θ is in the interval $-\pi < \theta \leq \pi$, it is called the principal argument, denoted by $\mathrm{Arg}(z)$.

Fig 1.3



THE POLAR FORM OF COMPLEX NUMBER

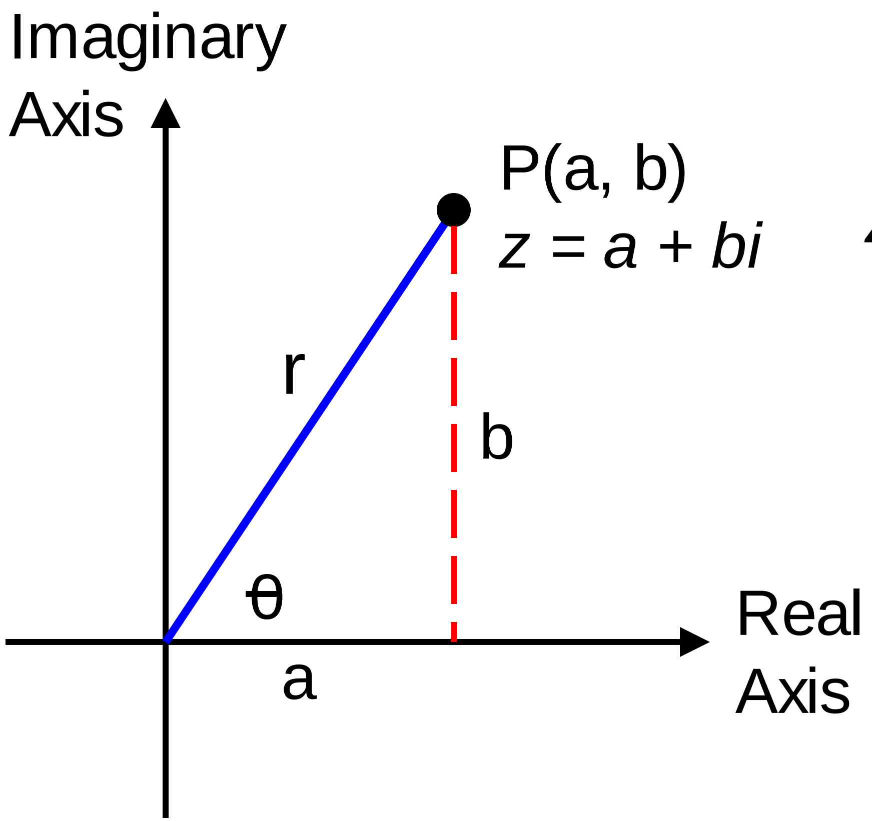


- Based on figure above: $a = r\cos\theta, \quad b = r\sin\theta,$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Recall how we *graph* complex numbers:

$$a = r\cos\theta \quad b = r\sin\theta$$



$$z = a + bi$$

$$= (r\cos\theta) + (r\sin\theta)i$$

$$= r(\cos\theta + i\sin\theta)$$

$$r = |z| = \sqrt{a^2 + b^2} \qquad \tan\theta = \frac{b}{a}$$

Trigonometric Form of a Complex Number

The **trigonometric form** of the complex number $z = a + bi$ is

$$z = r\left(\cos\theta + i\sin\theta\right)$$

The number r is the ***absolute value*** or ***modulus*** of z, and θ is an ***argument*** of z.

→ **Is the argument of any particular complex number *unique*?**

Practice *changing forms* of complex numbers

Switch forms of the given complex number, for $0 \le \theta < 2\pi$

(between trigonometric form and standard form)

$1-\sqrt{3}i$ *How about a graph???*

$$r = \left|1-\sqrt{3}i\right| = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2} = 2$$

Reference angle: $-\dfrac{\pi}{3}$ so… $\theta = 2\pi + \left(-\dfrac{\pi}{3}\right) = \dfrac{5\pi}{3}$

$$\Rightarrow \quad 1-\sqrt{3}i = 2\cos\frac{5\pi}{3} + 2i\sin\frac{5\pi}{3}$$

Example 1

Express $1-\sqrt{3}i$ in polar form.

Solution

See Fig 1.4 that the point lies in the fourth quarter.

$$r = |z| = \left|1-\sqrt{3}i\right| = \sqrt{1+3} = 2$$

$$\tan\theta = \frac{-\sqrt{3}}{1}, \theta = \arg(z) = \frac{5\pi}{3}$$

$$z = 2\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right]$$

Rewrite $-1+i$ in polar form with RADIANS.

$$(-1)^2 + 1^2 = r^2 \rightarrow r = \sqrt{2}$$

$$-1 = \sqrt{2}\cos\theta$$

$$1 = \sqrt{2}\sin\theta$$

$$\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$\theta = \frac{3\pi}{4}$$

Rewrite $1-\sqrt{3}i$ in polar form with RADIANS.

$$1^2 + \left(-\sqrt{3}\right)^2 = r^2 \rightarrow r = 2$$

$$1 = 2\cos\theta$$

$$-\sqrt{3} = 2\sin\theta$$

$$\text{Ans}: 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

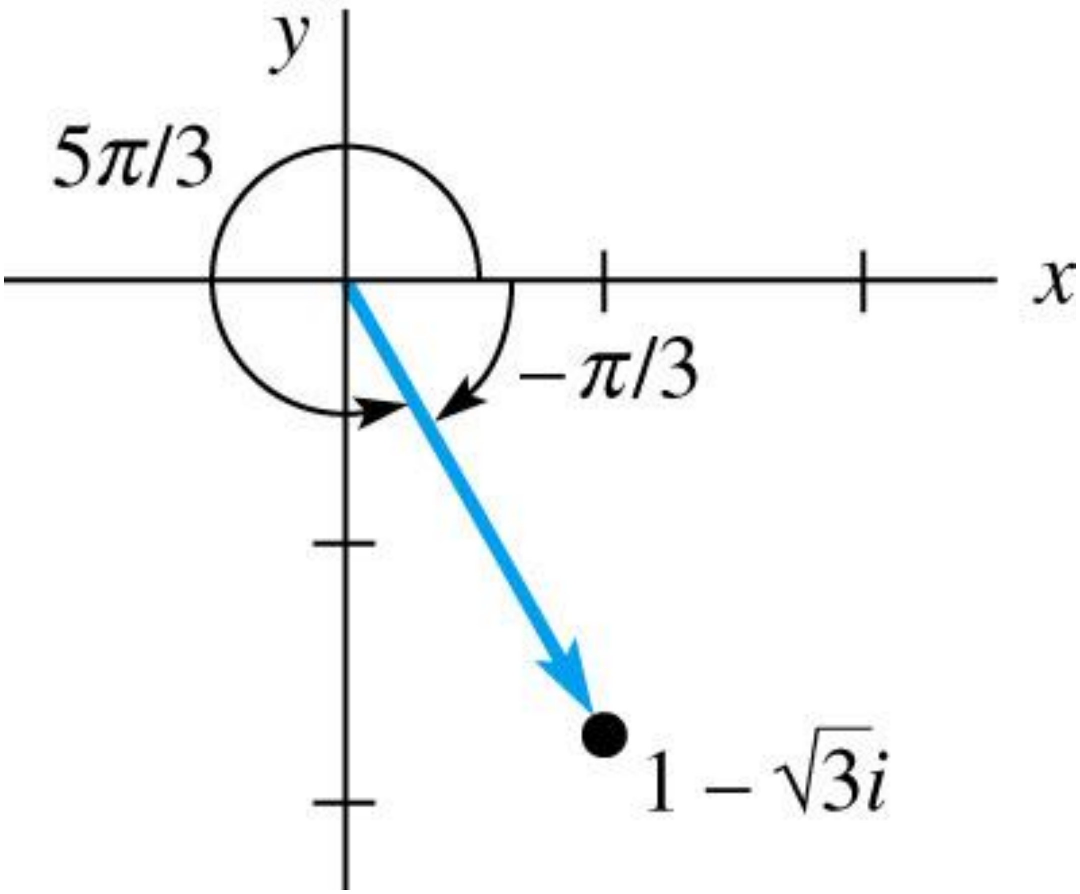
$$\theta = \frac{5\pi}{3}$$

Example 1 (2)

In addition, choose that $-\pi < \theta \leq \pi$, thus $\theta = -\pi/3$.

$$z = 2\left[\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})\right]$$

Fig 1.4



Multiplication and Division

- Suppose $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

  $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

  Then

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \qquad (2)$$

  for $z_2 \neq 0$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)] \qquad (3)$$

❖ From the addition formulas from trigonometry,

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \quad (4)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \quad (5)$$

Thus we can show

$$|z_1 z_2| = |z_1||z_2|, \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad (6)$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2, \quad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \quad (7)$$

Powers of z

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

$$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

$$z^n = r^n(\cos n\theta + i\sin n\theta) \qquad (8)$$

Demoivre's Formula

- When $r = 1$, then (8) becomes

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad (9)$$

Given $z = r_1 \cos(\alpha) + i\sin(\alpha)$ and

$w = r_2 \cos(\beta) + i\sin(\beta)$

$$\frac{z}{w} = \frac{r_1}{r_2}\cos(\alpha - \beta) + i\sin(\alpha - \beta) \qquad zw = r_1 r_2 \cos(\alpha + \beta) + i\sin(\alpha + \beta)$$

Compute zw and $\frac{z}{w}$, learning your answer in polar form in rad

$$z = 2(\cos 120° + i\sin 120°)$$

$$w = 3(\cos 100° + i\sin 100°)$$

$$zw = (2)(3)\cos(120° + 100°) + i\sin(120° + 100°)$$

$$zw = 6(\cos 220 + i\sin 220) = 6(\cos\frac{11\pi}{9} + i\sin\frac{11\pi}{9})$$

$$\frac{z}{w} = \frac{2}{3}(\cos(120 - 100) + i\sin(120 - 100))$$

$$\frac{z}{w} = \frac{2}{3}(\cos 20 + i\sin 20) = \frac{2}{3}(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9})$$

Compute zw and $\frac{z}{w}$, learning your answer in polar form in rad.

$$z = 4\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right),\ w = 3\left(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right)$$

$$zw = (4)(3)\left(\cos\left(\frac{3\pi}{8} + \frac{9\pi}{16}\right) + i\sin\left(\frac{3\pi}{8} + \frac{9\pi}{16}\right)\right)$$

$$zw = 12\left(\cos\frac{15\pi}{16} + i\sin\frac{15\pi}{16}\right)$$

$$\frac{z}{w} = \frac{4}{3}\left(\cos\left(\frac{3\pi}{8} - \frac{9\pi}{16}\right) + i\sin\left(\frac{3\pi}{8} - \frac{9\pi}{16}\right)\right)$$

$$\frac{z}{w} = \frac{4}{3}\left(\cos\left(\frac{-3\pi}{16}\right) + i\sin\left(\frac{-3\pi}{16}\right)\right)$$

DeMoivre's Theorem

compute $\left(\sqrt{3}+i\right)^6$

$$\left(r\,\text{cis}\,\theta\right)^n = r^n\,\text{cis}\left(n\theta\right)$$

$$\left(2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right)^6 = 2^6\left(\cos\pi + i\sin\pi\right)$$

$$= 2^6\left(-1+0i\right) = -64$$

$$\left(2-2i\right)^7 = \left(2\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)\right)^7$$

$$= \left(2\sqrt{2}\right)^7\left(\cos\frac{49\pi}{4}+i\sin\frac{49\pi}{4}\right)$$

$$= 1024\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = 1024+1024i$$

Find the cube roots of $-8-8i \Rightarrow x^3 = -8-8i$

$$x^3 = \sqrt{128}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \left(\sqrt{128}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)\right)^{1/3}$$

$$r^{1/n}\left(\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right)$$

$$\sqrt[6]{128} = 2\sqrt[6]{2}$$

$$= 2\sqrt[6]{2}\left(\cos\left(\frac{5\pi}{12} + \frac{2\pi k}{3}\right) + i\sin\left(\frac{5\pi}{12} + \frac{2\pi k}{3}\right)\right)$$

$$k = 0, \quad 2\sqrt[6]{2}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right)$$

$$k = 1, \quad 2\sqrt[6]{2}\left(\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right)$$

$$k = 2, \quad 2\sqrt[6]{2}\left(\cos\left(\frac{21\pi}{12}\right) + i\sin\left(\frac{21\pi}{12}\right)\right)$$

DE MOIVRE'S THEOREM

Theorem 3

If $z = r(\cos\theta + i\sin\theta)$ is a complex number in polar form to any power of n, then

$$z^n = r^n(\cos\theta + i\sin\theta)^n$$

De Moivre's Theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Therefore :

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Roots

- A number w is an nth root of a nonzero number z if $w^n = z$. If we let $w = \rho(\cos\phi + i\sin\phi)$ and $z = r(\cos\theta + i\sin\theta)$, then

$$\rho^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$

$$\rho^n = r, \rho = r^{1/n}$$

$$\cos n\phi = \cos\theta,\ \sin n\phi = \sin\theta$$

$$\phi = \frac{\theta + 2k\pi}{n}, k = 0,1,2,\ldots,n-1$$

The root corresponds to k=0 called the principal nth root.

FINDING ROOTS

Theorem 4

If $z^n = r(\cos\theta + i\sin\theta)$ then, the n root of z is:

(θ in degrees)

$$z = r^{\frac{1}{n}}\left(\cos\frac{\theta + 360°k}{n} + i\sin\frac{\theta + 360°k}{n}\right)$$

OR

(θ in radians)

$$z = r^{\frac{1}{n}}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$$

Where $k = 0, 1, 2, .. n-1$

Using DeMoivre to Find Roots

- Again, starting with

  a + b***i*** = $z = r\cdot\left(\cos\theta + i\cdot\sin\theta\right)$

also

$$z^n = r^n\left(\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right)$$

works when *n* is a fraction

  - Thus we can take a root of a complex number

$$z^{1/n} = r^{1/n}\cdot\left(\cos\left(\frac{\theta + 360\cdot k}{n}\right) + i\cdot\sin\left(\frac{\theta + 360\cdot k}{n}\right)\right)$$

Using DeMoivre to Find Roots

- Note that there will be *n* such roots

$$z^{1/n} = r^{1/n} \cdot \left(\cos\left(\frac{\theta + 360 \cdot k}{n} \right) + i \cdot \sin\left(\frac{\theta + 360 \cdot k}{n} \right) \right)$$

  - One each for k = 0, k = 1, … k = n – 1

- Find the two square roots of $z = -1 + i\sqrt{3}$
  - Represent as $z = r(\cos\theta + i\sin\theta)$
  - What is r?
  - What is θ?

Graphical Interpretation of Roots

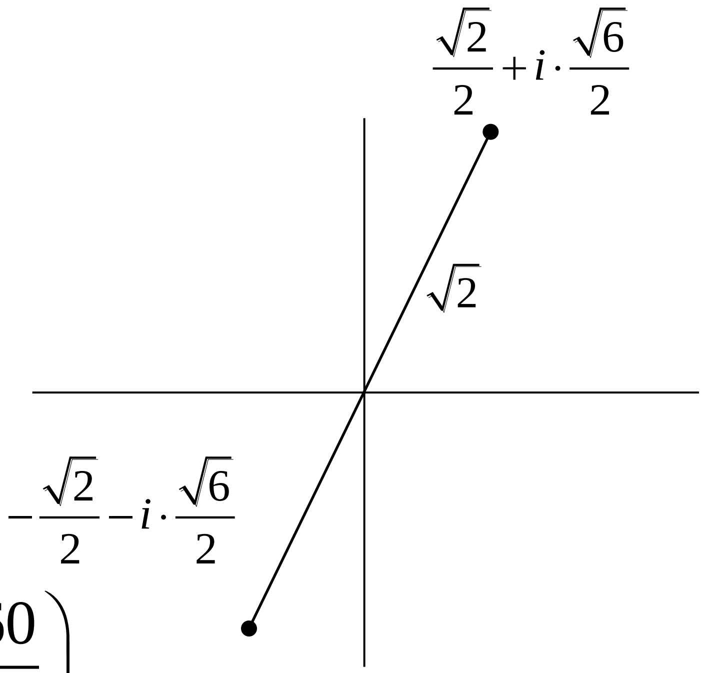
- Solutions $z=\sqrt{-1+i\sqrt{3}}$ are:

$$\sqrt{-1+i\cdot\sqrt{3}}$$

$$=\left(2\left(\cos 120+i\cdot\sin 120\right)\right)^{1/2}$$

$$=\sqrt{2}\cdot\left(\cos\frac{120}{2}+i\cdot\sin\frac{120}{2}\right)$$

$$\textit{and}\ \sqrt{2}\cdot\left(\cos\frac{120+360}{2}+i\cdot\sin\frac{120+360}{2}\right)$$



Roots will be equally spaced around a circle with radius $r^{1/2}$

FINDING COMPLEX ROOTS

Find the two square roots of $4i$. Write the roots in rectangular form.

Write $4i$ in trigonometric form: $4i = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$$r = 4,\ \theta = \frac{\pi}{2}$$

The square roots have absolute value $\sqrt{4} = 2$ and argument

$$\alpha = \frac{\frac{\pi}{2}}{2} + \frac{2\pi k}{2} = \frac{\pi}{4} + \pi k.$$

Since there are two square roots, let $k = 0$ and 1.

$$\text{If } k = 0, \text{ then } \alpha = \frac{\pi}{4} + \pi \cdot 0 = \frac{\pi}{4}.$$

$$\text{If } k = 1, \text{ then } \alpha = \frac{\pi}{4} + \pi \cdot 1 = \frac{5\pi}{4}.$$

Using these values for α, the square roots are

$$2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \text{ and } 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

FINDING COMPLEX ROOTS (continued)

$$2\,\text{cis}\,\frac{\pi}{4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2}$$

$$2\,\text{cis}\,\frac{\pi}{4} = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= -\sqrt{2} - i\sqrt{2}$$

FINDING COMPLEX ROOTS (continued)

Since there are four roots, let k = 0, 1, 2, and 3.

$$\text{If } k = 0, \text{ then } \alpha = 30° + 90° \cdot 0 = 30°.$$

$$\text{If } k = 1, \text{ then } \alpha = 30° + 90° \cdot 1 = 120°.$$

$$\text{If } k = 2, \text{ then } \alpha = 30° + 90° \cdot 2 = 210°.$$

$$\text{If } k = 3, \text{ then } \alpha = 30° + 90° \cdot 3 = 300°.$$

Using these values for α, the fourth roots are

$$2(\cos 30 + i \sin 30), \ 2(\cos 120 + i \sin 120),$$

$$2(\cos 210 + i \sin 210), \ 2(\cos 300 + i \sin 300),$$

Converting from Rectangular form to Trig form

1. Find r. $r = \sqrt{a^2 + b^2}$
2. Find θ. $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
3. Fill in the blanks in $z = r(\cos\theta + i\sin\theta)$

Convert z = 4 + 3i to trig form.

1. Find r

$$r = \sqrt{4^2 + 3^2} = \sqrt{16 + 9}$$
$$r = \sqrt{25} = 5$$

2. Find θ

$$\theta = \tan^{-1}\frac{3}{4} \approx 36.9$$

3. Fill in the blanks

$$z = 5(\cos 36.9 + i\sin 36.9)$$
$$Polar\ \ form\ (5, 36.9)$$

Chapter 2: Analytic Functions

2.1. Elementary Functions

- 2.1.1 Exponential and Logarthimic Functions
- 2.1.2. Trignometric and Hyperbolic Functions
- 2.1.3 Inverse Trigonometric and Hyperbolic Functions

2.2. Open and closed sets ,connected sets & regions in complex plane

2.3. Definitions of limit and continuity

2.4. Limit theorem

2.5. Definition of derivative &its properties

2.6. Analytic function &their algebraic properties

2.7. Conformal mappings

2.8. The Cauchy Riemann equations and Harmonic functions

2.1 Exponential and Logarithmic Functions

❖ Exponential Functions

Recall that the function $f(x) = e^x$ has the property

$$f'(x) = f(x) \text{ and } \quad f(x_1 + x_2) = f(x_1)f(x_2) \tag{1}$$

and the Euler's formula is

$$e^{iy} = \cos y + i \sin y, \quad y: \text{ a real number} \tag{2}$$

Thus

$$e^{x+iy} = e^x(\cos y + i \sin y)$$

DEFINITION 2.9 **Exponential Functions**

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y) \qquad (3)$$

Example 1: **Evaluate e^{1+3i}.**

Solution

$$e^{1+4i} = e^1(\cos 4 + i\sin 4)$$

Cont'd

$$\frac{de^z}{dz} = e^z$$

$$e^{z_1}e^{z_2} = e^{z_1+z_2}, \frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$$

Periodicity

$$e^{z+i2\pi} = e^z e^{i2\pi}$$

$$= e^z(\cos 2\pi + i\sin 2\pi) = e^z$$

- Polar From of a Complex number

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Logarithm Function

- Given a complex number $z = x + iy$, $z \neq 0$, we define

$$w = \ln z \quad \text{if} \quad z = e^{w} \qquad (5)$$

Let $w = u + iv$, then

$$x + iy = e^{u+iv} = e^{u}(\cos v + i\sin v) = e^{u}\cos v + ie^{u}\sin v$$

We have

$$x = e^{u}\cos v, \;\; y = e^{u}\sin v$$

and also

$$e^{2u} = x^{2} + y^{2} = r^{2} = |z|^{2}, \; u = \log_{e}|z|$$

$$\tan v = \frac{y}{x}, \; v = \theta + 2n\pi, \quad \theta = \arg z, \; n = 0, \; \pm 1, \; \pm 2, \ldots$$

DEFINITION 2.10 **Logarithm of a Complex Number**

For $z \neq 0$, and $\theta = \arg z$,

$$\ln z = \log_e |z| + i(\theta + 2n\pi), \quad n = 0, \pm 1, \pm 2, \cdots \qquad (6)$$

Example 2

Find the values of (a) ln (−2) (b) ln i, (c) ln (−1 – i).

Solution

$$(a)\quad \theta = \arg(-2) = \pi,\ \log_e|-2| = 0.6932$$

$$\ln(-2) = 0.6932 + i(\pi + 2n\pi)$$

$$(b)\quad \theta = \arg(i) = \frac{\pi}{2},\ \log_e 1 = 0$$

$$\ln(i) = i(\frac{\pi}{2} + 2n\pi)$$

$$(c)\quad \theta = \arg(-1-i) = \frac{5\pi}{4},\ \log_e|-1-i| = \log_e\sqrt{2} = 0.3466$$

$$\ln(-1-i) = 0.3466 + i(\frac{5\pi}{4} + 2n\pi)$$

Example 3

Find all values of z such that $e^z = \sqrt{3} + i.$

Solution

$$z = \ln(\sqrt{3} + i), |\sqrt{3} + i| = 2, \arg(\sqrt{3} + i) = \frac{\pi}{6}$$

$$z = \ln(\sqrt{3} + i) = \log_e 2 + i(\frac{\pi}{6} + 2n\pi)$$

$$= 0.6931 + i(\frac{\pi}{6} + 2n\pi)$$

Principal Value

- $$\operatorname{Ln} z = \log_e |z| + i\operatorname{Arg} z \qquad (7)$$

Since $\operatorname{Arg} z \in (-\pi, \pi]$ is unique, there is only one value of $\operatorname{Ln} z$ for which $z \neq 0$.

Example 4

- The principal values of example 2 are as follows.

(*a*) $\operatorname{Arg}(-2) = \pi$

$\operatorname{Ln}(-2) = 0.6932 + i\pi$

(*b*) $\operatorname{Arg}(i) = \frac{\pi}{2}, \ \operatorname{Ln}(i) = i\frac{\pi}{2}$

(*c*) $\operatorname{Arg}(-1-i) = \frac{5\pi}{4}$ is not the principal value.

Let $n = -1$, then $\operatorname{Ln}(-1-i) = 0.3466 - \frac{3\pi}{4}i$

Example 4 (2)

- Each function in the collection of ln z is called a branch. The function Ln z is called the principal branch or the principal logarithm function.

- Some familiar properties of logarithmic function hold in complex case:

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2 \qquad (8)$$

Example 5

Suppose $z_1 = 1$ and $z_2 = -1$. If we take $\ln z_1 = 2\pi i$, $\ln z_2 = \pi i$, we get

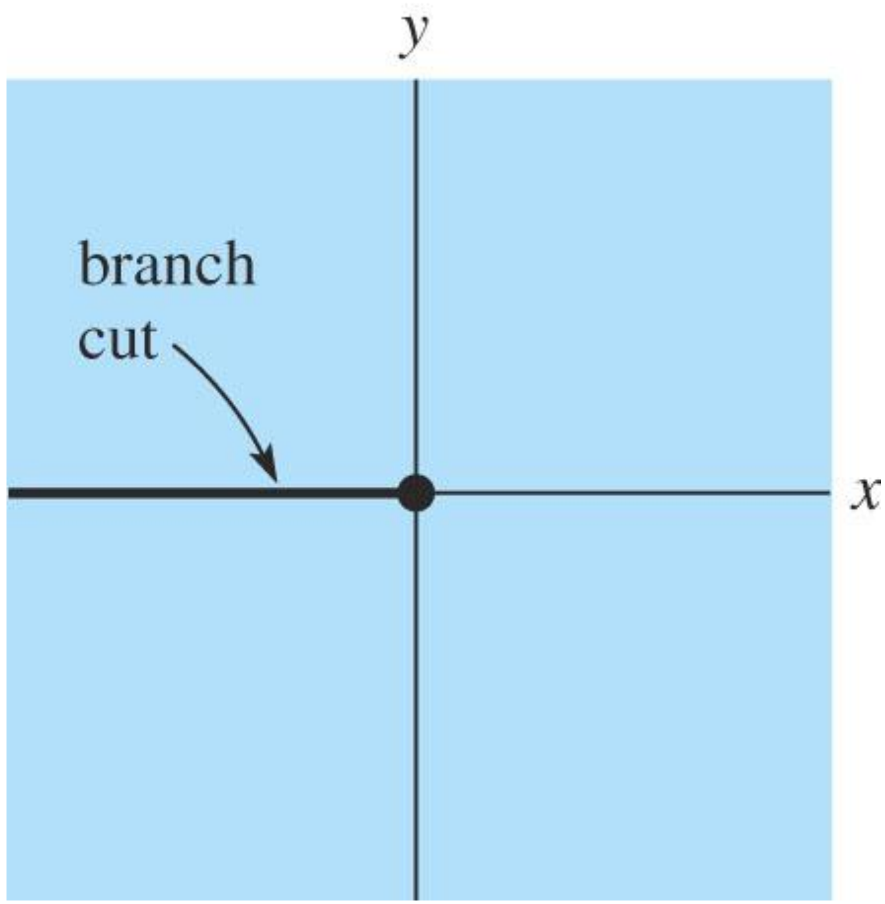
$$\ln(z_1 z_2) = \ln(-1) = \ln z_1 + \ln z_2 = 3\pi i$$

$$\ln\left(\frac{z_1}{z_2}\right) = \ln(-1) = \ln z_1 - \ln z_2 = \pi i$$

Analyticity

- The function Ln z is not analytic at $z = 0$, since Ln 0 is not defined. Moreover, Ln z is discontinuous at all points of the negative real axis. Since Ln z is the principal branch of ln z, the nonpositive real axis is referred to as a branch cut. See Fig 2.19.

Fig 2.91



- It is left as exercises to show

$$\frac{d}{dz}\text{Ln}\ z = \frac{1}{z} \tag{9}$$

for all z in D *(the complex plane except those on the non-positive real axis)*.

Complex Powers

- In real variables, we have $x^{\alpha} = e^{\alpha \ln x}$.
  If α is a complex number, $z = x + iy$, we have

$$z^{\alpha} = e^{\ln z^{\alpha}} = e^{\alpha \ln z}, \quad z \neq 0 \qquad (10)$$

Example 6

Find the value of i^{2i}.

Solution

$$\text{With } z = i, \arg z = \pi / 2, \alpha = 2i, \text{from (9)},$$

$$i^{2i} = e^{2i[\log_e 1 + i(\pi/2 + 2n\pi)]} = e^{-(1+4n)\pi}$$

$$\text{where } n = 0, \pm 1, \pm 2, \ldots$$

Trigonometric Functions

From Euler's Formula, we have

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{1}$$

DEFINITION 2.11 **Trigonometric Sine and Cosine**

For any complex number $z = x + iy$,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad (2)$$

- Four additional trigonometric functions:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{1}{\tan z}, \qquad (3)$$

$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

Analyticity

- Since e^{iz} and e^{-iz} are entire functions, then sin z and cos z are entire functions.
- sin $z = 0$ only for the real numbers $z = n\pi$ &
- cos $z = 0$ only for the real numbers $z = (2n+1)\pi/2$. Thus tan z and sec z are analytic except $z = (2n+1)\pi/2$, and cot z and
  csc z are analytic except $z = n\pi$.

Derivatives

$$\frac{d}{dz}\sin z = \frac{d}{dz}\frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2} = \cos z$$

Similarly we have

$$\frac{d}{dz}\sin z = \cos z \qquad \frac{d}{dz}\cos z = -\sin z$$
$$\frac{d}{dz}\tan z = \sec^2 z \qquad \frac{d}{dz}\cot z = -\csc^2 z \qquad (4)$$
$$\frac{d}{dz}\sec z = \sec z \tan z \qquad \frac{d}{dz}\csc z = -\csc z \cot z$$

Identities

$$\sin(-z) = -\sin z \qquad \cos(-z) = \cos z$$

$$\cos^2 z + \sin^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin 2z = 2\sin z \cos z \qquad \cos 2z = \cos^2 z - \sin^2 z$$

Zeros

- If y is real, we have

$$\sinh y = \frac{e^{y} - e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^{y} + e^{-y}}{2} \tag{5}$$

From Definition 11.17 and Euler's formula

$$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \sin x\left(\frac{e^{y} + e^{-y}}{2}\right) + i\cos x\left(\frac{e^{y} - e^{-y}}{2}\right)$$

Thus we have

$$\sin z = \sin x \cosh y + i \cos x \sinh y \qquad (6)$$

and

$$\cos z = \cos x \cosh y - i \sin x \sinh y \qquad (7)$$

From (6) and (7) and $\cosh^2 y = 1 + \sinh^2 y$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \qquad (8)$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \qquad (9)$$

Example 1

- From (6) we have

$$\sin(2+i) = \sin 2 \cosh 1 + i \cos 2 \sinh 1$$
$$= 1.4301 - 0.4891i$$

Example 2

Solve $\cos z = 10.$

Solution

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 10$$

$$e^{2iz} - 20e^{iz} + 1 = 0, e^{iz} = 10 \pm 3\sqrt{11}$$

$$iz = \log_e(10 \pm 3\sqrt{11}) + 2n\pi i$$

Since $\log_e(10 - 3\sqrt{11}) = -\log_e(10 + 3\sqrt{11})$

we have

$$z = 2n\pi \pm i\log_e(10 + 3\sqrt{11})$$

DEFINITION 2.12

Hyperbolic Sine and Cosine

For any complex number $z = x + iy$,

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \cosh z = \frac{e^z + e^{-z}}{2} \tag{10}$$

- Additional functions are defined as

$$\tanh z = \frac{\sinh z}{\cosh z} \qquad \coth z = \frac{1}{\tanh z} \tag{11}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} \qquad \operatorname{csch} z = \frac{1}{\sinh z}$$

❖ Similarly we have

$$\frac{d}{dz}\sinh z = \cosh z \quad \text{and} \quad \frac{d}{dz}\cosh z = \sinh z \qquad (12)$$

$$\sin z = -i\sinh(iz)\,, \quad \cos z = \cosh(iz) \qquad (13)$$

$$\sinh z = -i\sin(iz)\,, \quad \cosh z = \cos(iz) \qquad (14)$$

Zeros

- $$\sinh z = -i\sin iz = -i\sin(-y+ix)$$
$$= -i[\sin(-y)\cosh x + i\cos(-y)\sinh x]$$

Since $\sin(-y) = -\sin y$, $\cos(-y) = \cos y$, then

$$\sinh z = \sinh x\cos y + i\cosh x\sin y \tag{15}$$

$$\cosh z = \cosh x\cos y + i\sinh x\sin y \tag{16}$$

It also follows from (14) that the zeros of sinh z and cosh z are respectively,

$$z = n\pi i \text{ and } z = (2n+1)\pi i/2,\ n = 0, \pm 1, \pm 2, \ldots.$$

Periodicity

- From (6),

$$\begin{aligned}
&\sin(z + 2\pi i) \\
&= \sin(x + iy + 2\pi) \\
&= \sin(x + 2\pi)\cosh y + i\cos(x + 2\pi)\sinh y \\
&= \sin x \cosh y + i\cos x \sinh y = \sin z
\end{aligned}$$

The period is then 2π.

2.8 Inverse Trigonometric and Hyperbolic Functions

Inverse Sine

We define

$$z = \sin w \quad \text{if} \quad w = \sin^{-1} z \qquad (1)$$

From (1),

$$\frac{e^{iw} - e^{-iw}}{2i} = z, \quad e^{2iw} - 2ize^{iw} - 1 = 0$$

$$e^{iw} = iz + (1 - z^2)^{1/2} \qquad (2)$$

- Solving (2) for w then gives

$$\sin^{-1} z = -i \ln[iz + (1 - z^2)^{1/2}] \qquad (3)$$

Similarly we can get

$$\cos^{-1} z = -i \ln[z + i(1 - z^2)^{1/2}] \qquad (4)$$

$$\tan^{-1} z = \frac{i}{2} \ln \frac{i + z}{i - z} \qquad (5)$$

Example 1

Find all values of $\sin^{-1}\sqrt{5}$.

Solution

From (3),

$$\sin^{-1}\sqrt{5} = -i\ln[\sqrt{5}i + (1-(\sqrt{5})^2)^{1/2}]$$

$$(1-(\sqrt{5})^2)^{1/2} = (-4)^{1/2} = \pm 2i$$

$$\sin^{-1}\sqrt{5} = -i\ln[(\sqrt{5}\pm 2)i]$$

$$= -i[\log_e(\sqrt{5}\pm 2) + (\frac{\pi}{2} + 2n\pi)i],$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Example 1 (2)

Noting that

$$\log_e(\sqrt{5}-2)=\log_e\frac{1}{\sqrt{5}+2}=-\log_e(\sqrt{5}+2).$$

Thus for $n=0,\pm 1,\pm 2,\ldots$

$$\sin^{-1}\sqrt{5}=\frac{\pi}{2}+2n\pi\pm i\log_e(\sqrt{5}+2) \qquad (6)$$

Derivatives

- If we define $w = \sin^{-1} z$, $z = \sin w$, then

$$\frac{d}{dz} z = \frac{d}{dz} \sin w \quad \text{gives} \quad \frac{dw}{dz} = \frac{1}{\cos w}$$

Using $\cos^2 w + \sin^2 w = 1$, $\cos w = (1 - \sin^2 w)^{1/2}$

$= (1 - z^2)^{1/2}$, thus

$$\frac{d}{dz} \sin^{-1} z = \frac{1}{(1 - z^2)^{1/2}} \qquad (7)$$

$$\frac{d}{dz} \cos^{-1} z = \frac{-1}{(1 - z^2)^{1/2}} \qquad (8)$$

$$\frac{d}{dz} \tan^{-1} z = \frac{1}{1 + z^2} \qquad (9)$$

Example 2

Find the derivative of $w = \sin^{-1} z$ at $z = \sqrt{5}$.

Solution

$$(1-(\sqrt{5})^2)^{1/2} = (-4)^{1/2} = 2i$$

$$\left.\frac{dw}{dz}\right|_{z=\sqrt{5}} = \frac{1}{(1-(\sqrt{5})^2)^{1/2}} = \frac{1}{2i} = -\frac{1}{2}i$$

Inverse Hyperbolic Functions

- Similarly we have

$$\sinh^{-1} z = \ln[z + (z^2 + 1)^{1/2}] \quad (10)$$

$$\cosh^{-1} z = \ln[z + (z^2 - 1)^{1/2}] \quad (11)$$

$$\tanh^{-1} z = \frac{1}{2} \ln \frac{1+z}{1-z} \quad (12)$$

$$\frac{d}{dz} \sinh^{-1} z = \frac{1}{(z^2 + 1)^{1/2}} \quad (13)$$

$$\frac{d}{dz}\cosh^{-1} z = \frac{1}{(z^2+1)^{1/2}} \tag{14}$$

$$\frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2} \tag{15}$$

Example 3

Find all values of $\cosh^{-1}(-1)$.

Solution

From (11),

$$\cosh^{-1}(-1) = \ln(-1) = \log_e 1 + (\pi + 2n\pi)i$$

$$= (\pi + 2n\pi)i = (2n+1)\pi i$$

$$n = 0, \pm 1, \pm 2, \ldots$$

2.3 Sets in the Complex Plane

Terminology

$$z = x + iy,\ z_0 = x_0 + iy_0$$

$$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

If z satisfies $|z - z_0| = \rho$, this point lies on a circle of radius ρ centered at the point z_0.

Example 1

(a) $|z| = 1$ is the equation of a unit circle centered at the origin.

(b) $|z - 1 - 2i| = 5$ is the equation of a circle of radius 5 centered at $1 + 2i$.

- If z satisfies $|z - z_0| < \rho$, this point lies within (not on) a circle of radius ρ centered at the point z_0. The set is called a **neighborhood** of z_0, or an **open disk**.
- A point z_0 is an **interior point** of a set S if there exists some neighborhood of z_0 that lies entirely within S.
- If every point of S is an interior point then S is an **open set**. See Fig 2.7.

Fig 2.7

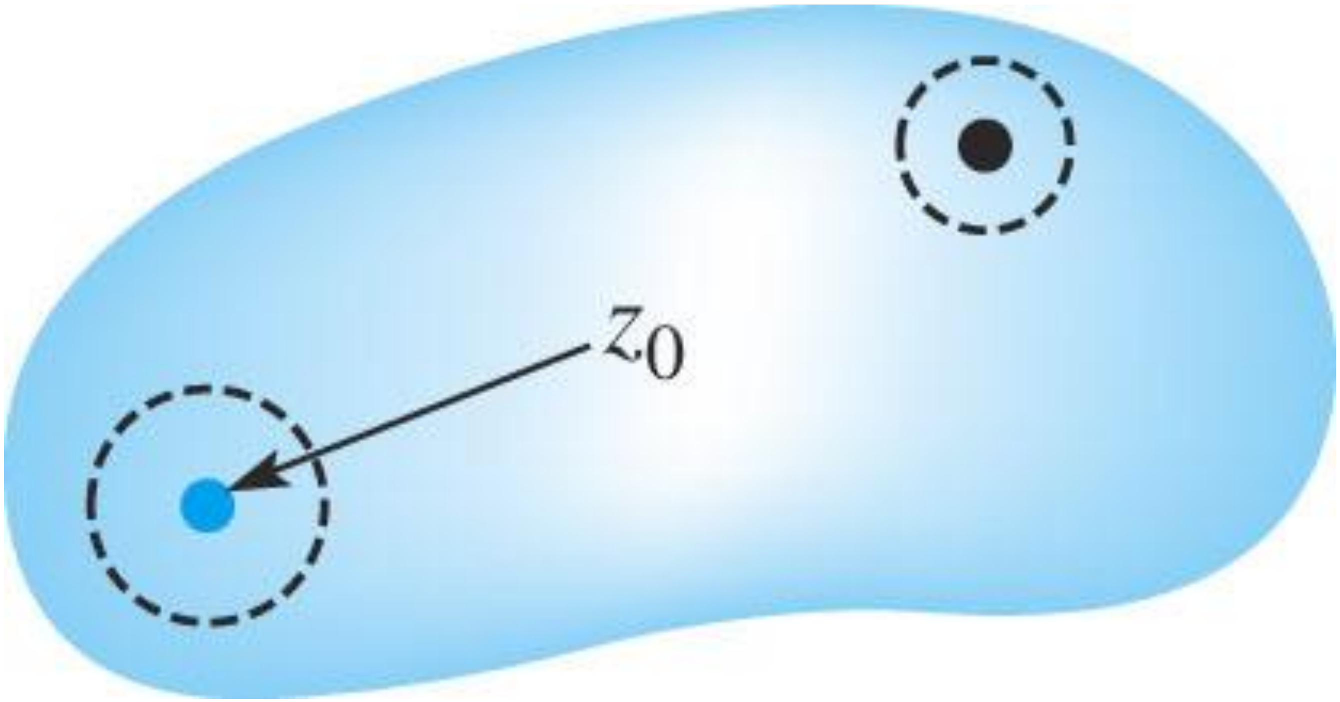


Fig 2.8

- The graph of $|z - (1.1 + 2i)| < 0.05$ is shown in Fig 2.8. It is an open set.

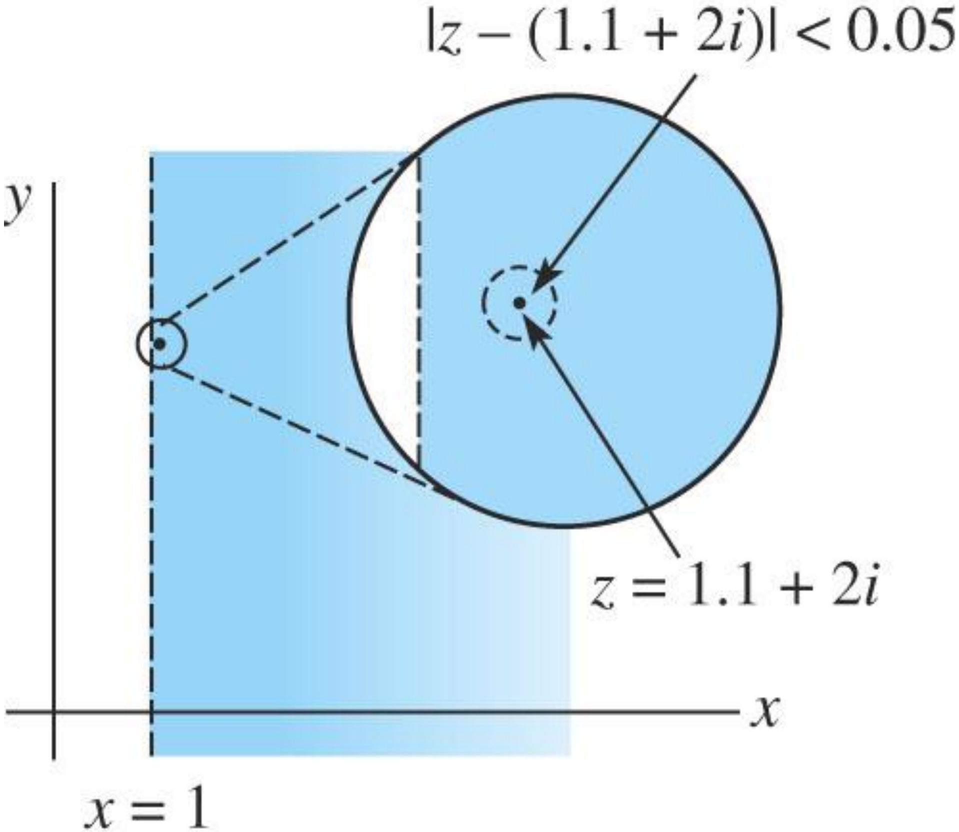
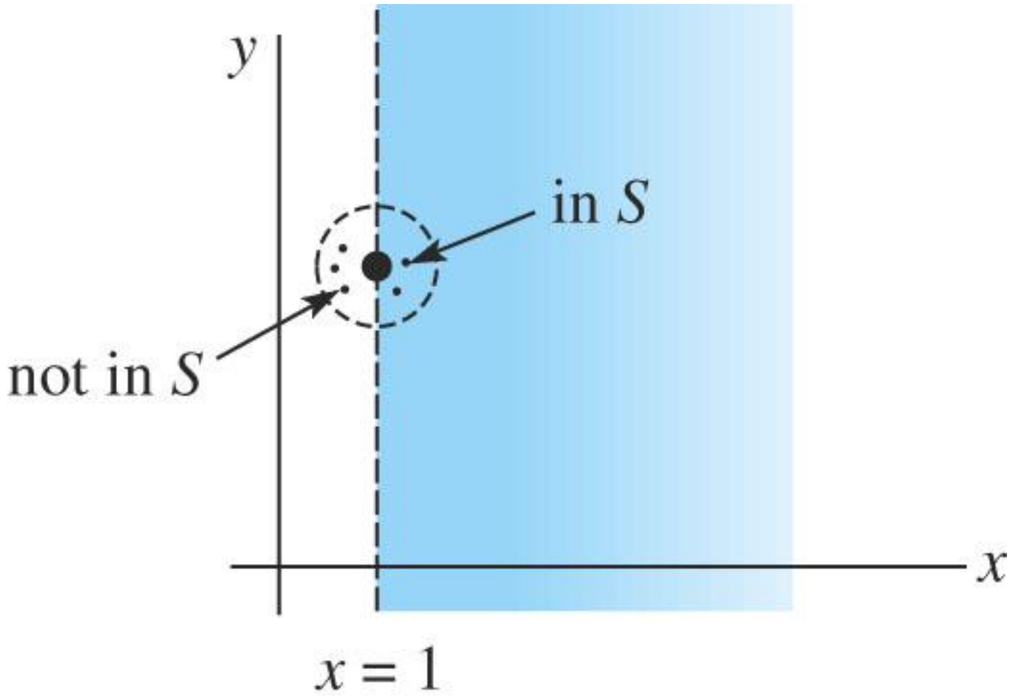


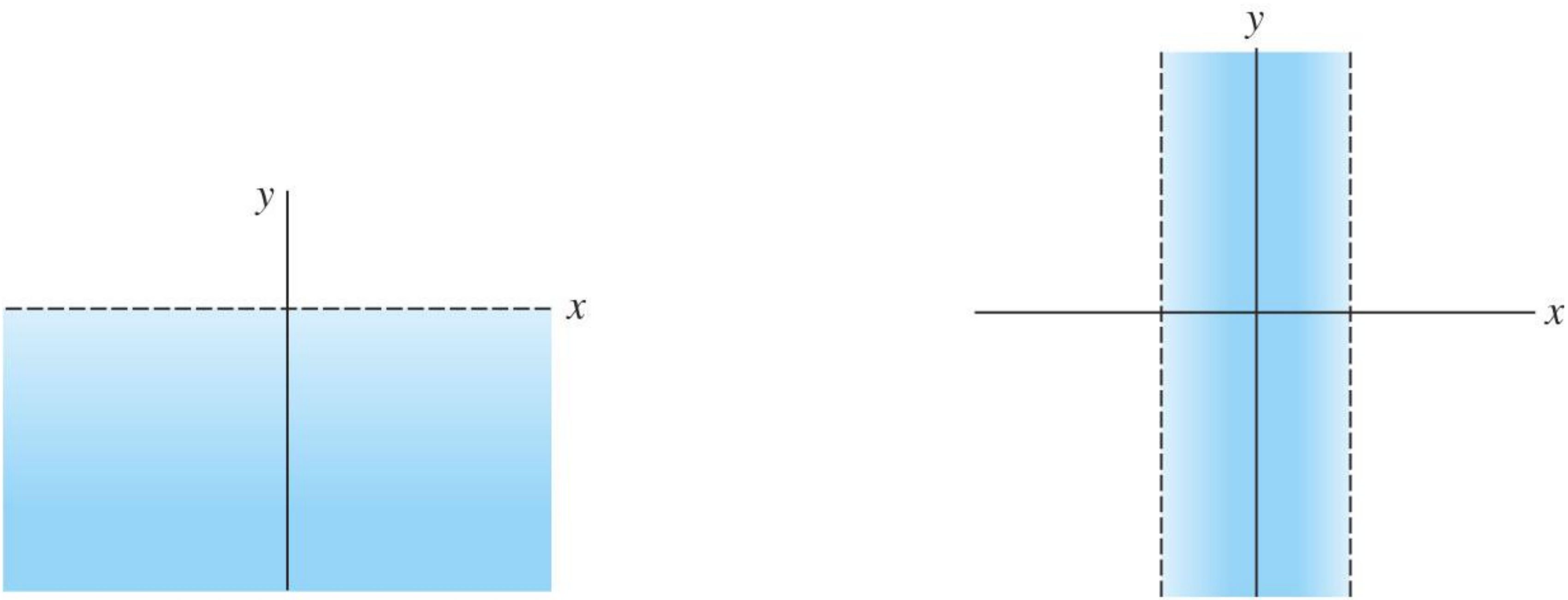
Fig 2.9

- The graph of Re(z) ≥ 1 is shown in Fig 2.9. It is not an open set.



Example 2

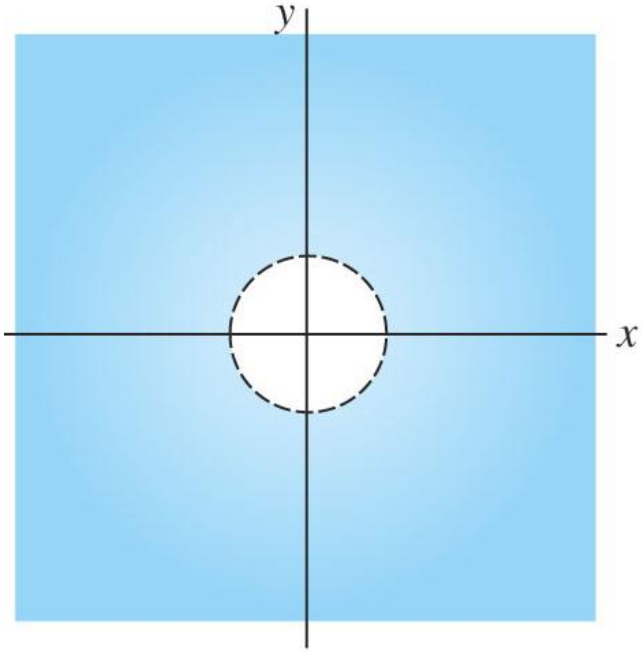
- Fig 2.10 illustrates some additional open sets.



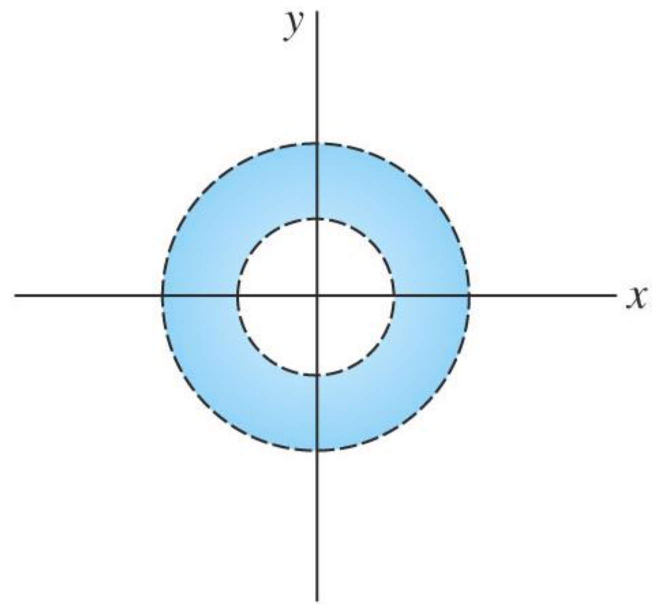
$\mathrm{Im}(z) < 0$
lower half-plane
(a)

$-1 < \mathrm{Re}(z) < 1$
infinite strip
(b)

Example 2 (2)



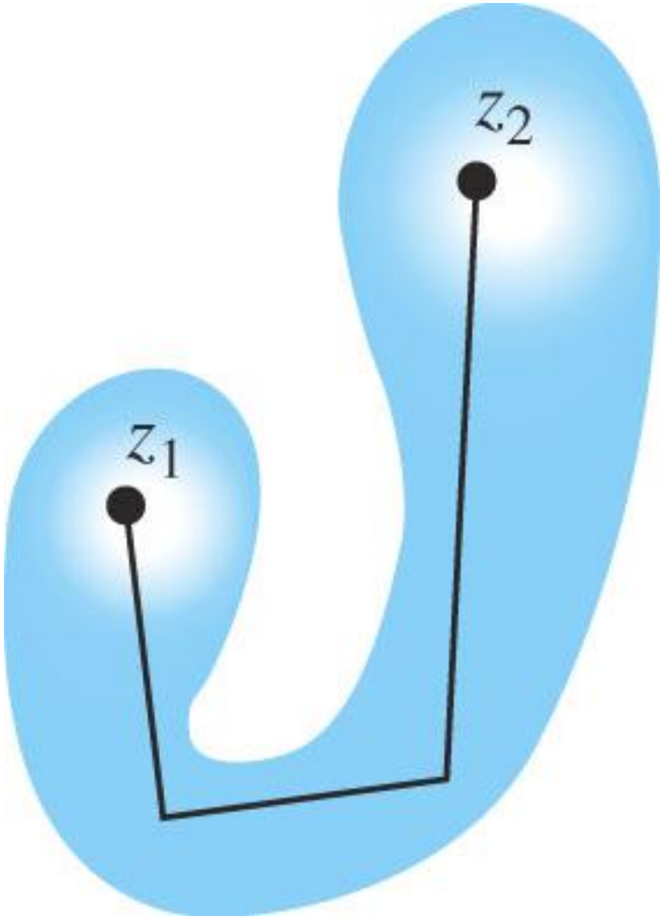
$|z| > 1$
exterior of unit circle
(c)



$1 < |z| < 2$
circular ring
(d)

- If every neighborhood of z_0 contains at least one point that is in a set S and at least one point that is not in S, z_0 is said to be a **boundary point** of S. The **boundary** of S is the set of all boundary points.
- If any pair of points z_1 and z_2 in an open set S can be connected by a polygonal line that lies entirely in S is said to be **connected**. See Fig 2.11. An open connected set is called a **domain**.

Fig2.11



- A **region** is a domain in the complex plane with all, some or none of its boundary points. Since an open connected set does not contain any boundary points, it is a region. A region containing all its boundary points is said to be **closed**.

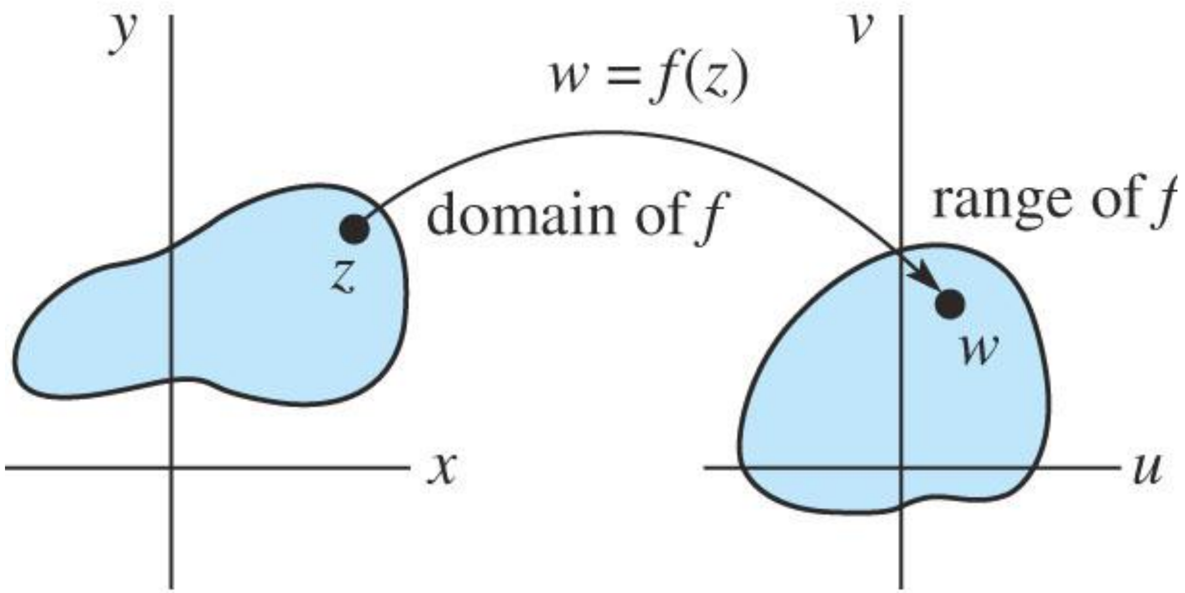
2.4 Functions of a Complex Variable

❖ Complex Functions

$$w = f(z) = u(x, y) + iv(x, y) \tag{1}$$

where u and v are real-valued functions.
Also, $w = f(z)$ can be interpreted as a mapping or transformation from the z-plane to the w-plane. See Fig 2.12.

Fig 2.12



(a) z-plane (b) w-plane

Example 1

Find the image of the line Re(z) = 1 under $f(z) = z^2$.

Solution

$$f(z) = z^2 = (x + iy)^2$$

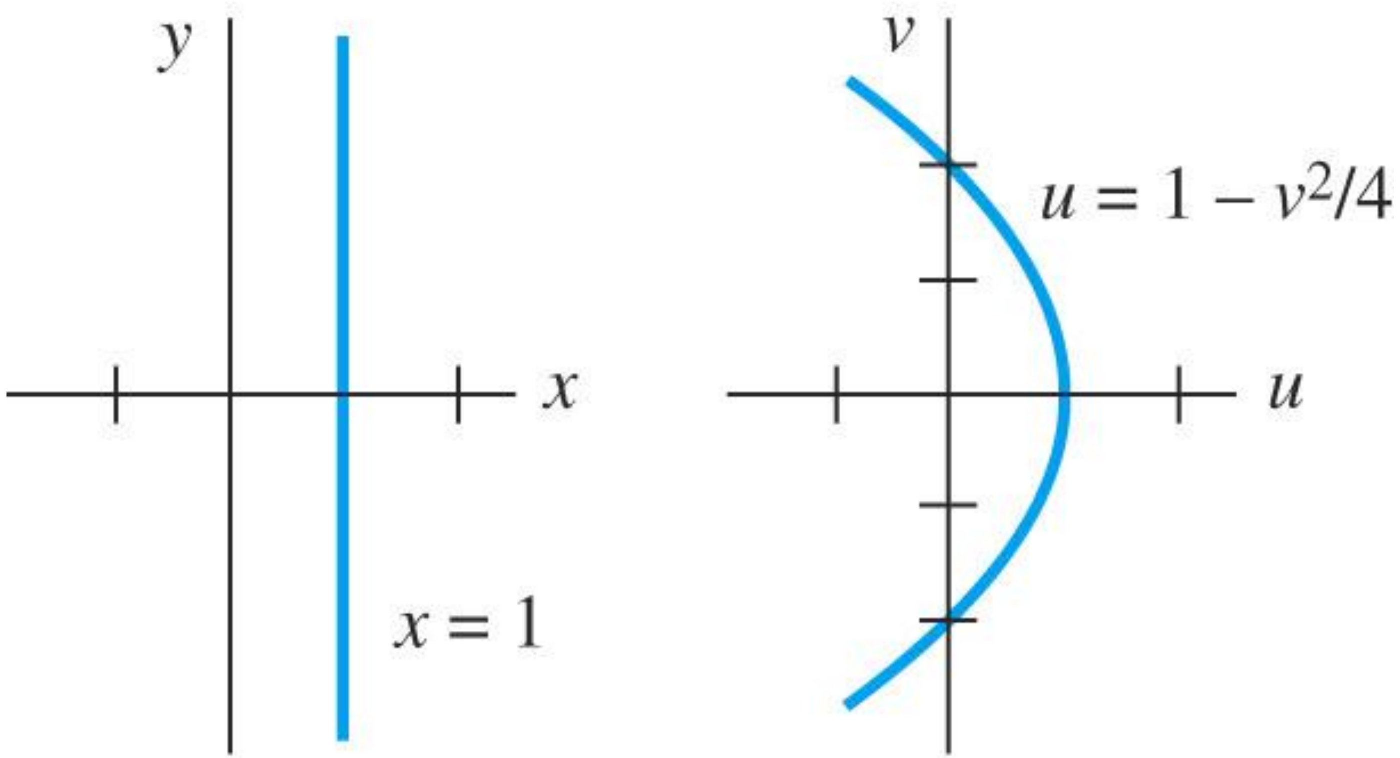
$$u(x, y) = x^2 - y^2, v(x, y) = 2xy$$

Now Re(z) = x = 1, $u = 1 - y^2$, $v = 2y$.

$$y = v/2, \text{ then } u = 1 - v^2/4$$

See Fig 2.13.

Fig 2.13



(a) *z*-plane (b) *w*-plane

DEFINITION 2.4

Limit of a Function

Suppose the function f is defined in some neighborhood of z_0, except possibly at z_0 itself. Then f is said to possess a **limit** at z_0, written

$$\lim_{z \to z_0} f(z) = L$$

if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

THEOREM 2.1 **Limit of Sum, Product, Quotient**

Suppose $\lim_{z\to z_0} f(z) = L_1$ and $\lim_{z\to z_0} g(z) = L_2$. Then

(i) $$\lim_{z\to z_0}[f(z)+g(z)] = L_1 + L_2$$

(ii) $$\lim_{z\to z_0} f(z)g(z) = L_1 L_2$$

(iii) $$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2}, \quad L_2 \neq 0$$

DEFINITION 2.5 **Continuous Function**

A function f is continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

- A function f defined by

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0\,, \quad a_n \neq 0 \qquad (2)$$

where n is a nonnegative integer and a_i, $i = 0, 1, 2, \ldots,$ n, are complex constants, is called a polynomial of degree n.

DEFINITION 2.6

Derivative

Suppose the complex function f is defined in a neighborhood of a point z_0. The **derivative** of f at z_0 is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \qquad (3)$$

provided this limit exists.

- If the limit in (3) exists, f is said to be differentiable at z_0. Also,
  if f is differentiable at z_0, then f is continuous at z_0.

Rules of differentiation

- **Constant Rules:**

$$\frac{d}{dz}c = 0\,,\quad \frac{d}{dz}cf(z) = cf'(z) \qquad (4)$$

- **Sum Rules:**

$$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z) \qquad (5)$$

- **Product Rule:**

$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + g(z)f'(z) \qquad (6)$$

- **Quotient Rule:**

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right]=\frac{g(z)f'(z)-f(z)g'(z)}{[g(z)]^2} \qquad (7)$$

- **Chain Rule:**

$$\frac{d}{dz}f(g(z))=f'(g(z))g'(z) \qquad (8)$$

- Usual rule

$$\frac{d}{dz}z^n=nz^{n-1},\ \ n \text{ an integer} \qquad (9)$$

Example 3

Differentiate $(a)\, f(z) = 3z^4 - 5z^3 + 2z$, $(b)\, f(z) = \dfrac{z^2}{4z+1}$.

Solution

$$(a)\, f'(z) = 12z^3 - 15z^2 + 2$$

$$(b)\, f'(z) = \frac{(4z+1)2z - z^2 4}{(4z+1)^2} = \frac{4z^2 + 2z}{(4z+1)^2}$$

Example 4

Show that $f(z) = x + 4iy$ is nowhere differentiable.

Solution

With $\Delta z = \Delta x + i\Delta y$, we have

$$f(z+\Delta z) - f(z)$$

$$= (x+\Delta x) + 4i(y+\Delta y) - x - 4iy$$

And so

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x + 4i\Delta y}{\Delta x + i\Delta y} \qquad (10)$$

Example 4 (2)

Now if we let $\Delta z \to 0$ along a line parallel to the x-axis then $\Delta y=0$ and the value of (10) is 1. On the other hand, if we let $\Delta z \to 0$ along a line parallel to the y-axis then $\Delta x=0$ and the value of (10) is 4. Therefore $f(z)$ is not differentiable at any point z.

DEFINITION 2.7

Analyticity at a Point

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

- A function is analytic at every point z is said to be an entire function. Polynomial functions are entire functions.

17.5 Cauchy-Riemann Equations

THEOREM 2.2 Cauchy-Riemann Equations

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exists and satisfy the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

THEOREM 2.2 Proof

- **Proof**

  Since $f'(z)$ exists, we know that

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \qquad (2)$$

By writing $f(z) = u(x, y) + iv(x, y)$, and $\Delta z = \Delta x + i\Delta y$, form (2)

$$f'(z) \qquad (3)$$

$$= \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}$$

THEOREM 2.2 Proof (2)

Since the limit exists, Δz can approach zero from any direction. In particular, if $\Delta z \to 0$ horizontally, then $\Delta z = \Delta x$ and (3) becomes

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \quad (4)$$

By the definition, the limits in (4) are the first partial derivatives of *u* and *v* w.r.t. *x*. Thus

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad (5)$$

THEOREM 2.2 Proof (3)

Now if $\Delta z \to 0$ vertically, then $\Delta z = i\Delta y$ and (3) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{iv(x, y + \Delta y) - iv(x, y)}{i\Delta y} \qquad (6)$$

which is the same as

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad (7)$$

Then we complete the proof.

Example 1

- The polynomial $f(z) = z^2 + z$ is analytic for all z and $f(z) = x^2 - y^2 + x + i(2xy + y)$. Thus $u = x^2 - y^2 + x$, $v = 2xy + y$. We can see that

$$\frac{\partial u}{\partial x} = 2x + 1 = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

Example 2

Show that $f(z) = (2x^2 + y) + i(y^2 - x)$ is not analytic at any point.

Solution

$$\frac{\partial u}{\partial x} = 4x \quad \text{and} \quad \frac{\partial v}{\partial y} = 2y$$

$$\frac{\partial u}{\partial y} = 1 \quad \text{and} \quad \frac{\partial v}{\partial x} = -1$$

We see that $\partial u/\partial y = -\partial v/\partial x$ but $\partial u/\partial x = \partial v/\partial y$ is satisfied only on the line $y = 2x$. However, for any z on this line, there is no neighborhood or open disk about z in which f is differentiable. We conclude that f is nowhere analytic.

THEOREM 2.3

Criterion for Analyticity

Suppose the real-valued function $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the Cauchy-Riemann equations at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

Example 3

For the equation $f(z) = \dfrac{x}{x^2 + y^2} - i\dfrac{y}{x^2 + y^2}$, we have

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}$$

That is, the Cauchy-Riemann equations are satisfied except at the point $x^2 + y^2 = 0$, that is $z = 0$. We conclude that f is analytic in any domain not containing the point $z = 0$.

- From (5) and (7), we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad (8)$$

This is a formula to compute $f'(z)$ if $f(z)$ is differentiable at the point z.

DEFINITION 2.8

Harmonic Functions

A real-valued function $\phi(x, y)$ that has continuous second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D.

THEOREM 2.4

A Source of Harmonic Functions

Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. Then the functions $u(x, y)$ and $v(x, y)$ are harmonic functions.

THEOREM 2.4

Proof **we assume u and v have continuous second order derivative**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \text{ then}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

$$\text{Thus} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Similarly we have

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Conjugate Harmonic Functions

- If *u* and *v* are harmonic in *D*, and u(x,y)+iv(x,y) is an analytic function in D, then *u* and *v* are called the conjugate harmonic function of each other.

Example 4

(a) Verify $u(x, y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane.

(b) Find the conjugate harmonic function of u.

Solution

$$(a)\ \frac{\partial u}{\partial x} = 3x^2 - 3y^2, \frac{\partial^2 u}{\partial x^2} = 6x, \frac{\partial u}{\partial y} = -6xy - 5, \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

Example 4 (2)

(*b*) $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = 6\mathrm{xy} + 5$

Integrating the first one, $v(x, y) = 3x^2 y - y^3 + h(x)$

and $\dfrac{\partial v}{\partial x} = 6xy + h'(x),\ h'(x) = 5,\ h(x) = 5x + C$

Thus $v(x,y) = 3x^2 y - y^3 + 5x + C$

Thank You !