Cauchy's Theorem

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Contents

3.1 Contour Integrals

DEFINITION 3.1 **Contour Integral**

Let f be defined at points of a smooth curve C given by $z = x(t) + iy(t), a \le t \le b$. The **contour integral** of f along C is

$$\int_C f(z)dz = \lim_{\|\Delta z_k\| \to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k \tag{1}$$

THEOREM 3.1 **Evaluation of a Contour Integral**

If f is continuous on a smooth curve C given by $z(t) = x(t) + iy(t),\ a \le t \le b$, then

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt \qquad (2)$$

Example 1

Evaluate $\int_C \bar{z}\,dz$

where C is given by $x = 3t, y = t^2, -1 \le t \le 4.$

Solution

$$z(t) = 3t + it^2,\ z'(t) = 3 + 2it$$

$$f(z(t)) = \overline{3t + it^2} = 3t - it^2$$

Thus, $\int_C \bar{z}\,dz = \int_{-1}^{4} (3t - it^2)(3 + 2it)\,dt$

$$= \int_{-1}^{4} (2t^3 + 9t)\,dt + i\int_{-1}^{4} 3t^2\,dt = 195 + 65i$$

Example 2

Evaluate $\oint_C \frac{1}{z} dz$

where C is the circle $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

Solution

$$z(t) = \cos t + i \sin t = e^{it}, \quad z'(t) = ie^{it}$$

$$f(z) = \frac{1}{z} = e^{-it}$$

Thus, $\oint_C \frac{1}{z} dz = \int_0^{2\pi} e^{-it} ie^{it} dt = 2\pi i$

$\int_{-i}^{i} \frac{dz}{z} = \operatorname{Ln} i - \operatorname{Ln}(-i) = \frac{i\pi}{2} - \left(-\frac{i\pi}{2}\right) = i\pi$. Here D is the complex plane without 0 and the negative real axis (where $\operatorname{Ln} z$ is not analytic). Obviously, D is a simply connected domain.

$$\int_{8+\pi i}^{8-3\pi i} e^{z/2}\,dz = 2e^{z/2}\Big|_{8+\pi i}^{8-3\pi i} = 2(e^{4-3\pi i/2} - e^{4+\pi i/2}) = 0$$

since e^z is periodic with period $2\pi i$.

$$\int_{0}^{1+i} z^2\,dz = \frac{1}{3}z^3\Big|_{0}^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3}i$$

Cont'd

Integral of a Nonanalytic Function. Dependence on Path

Integrate $f(z) = \text{Re}\, z = x$ from 0 to $1 + 2i$ (a) along C^* in Fig. 343, (b) along C consisting of C_1 and C_2.

Solution. (a) C^* can be represented by $z(t) = t + 2it$ $(0 \leqq t \leqq 1)$. Hence $\dot{z}(t) = 1 + 2i$ and $f[z(t)] = x(t) = t$ on C^*. We now calculate

$$\int_{C^*} \text{Re}\, z\, dz = \int_0^1 t(1 + 2i)\, dt = \frac{1}{2}(1 + 2i) = \frac{1}{2} + i.$$

(b) We now have

$$C_1: z(t) = t, \quad \dot{z}(t) = 1, \quad f(z(t)) = x(t) = t \quad (0 \leqq t \leqq 1)$$

$$C_2: z(t) = 1 + it, \quad \dot{z}(t) = i, \quad f(z(t)) = x(t) = 1 \quad (0 \leqq t \leqq 2).$$

Using (6) we calculate

$$\int_C \text{Re}\, z\, dz = \int_{C_1} \text{Re}\, z\, dz + \int_{C_2} \text{Re}\, z\, dz = \int_0^1 t\, dt + \int_0^2 1 \cdot i\, dt = \frac{1}{2} + 2i.$$

Note that this result differs from the result in (a).

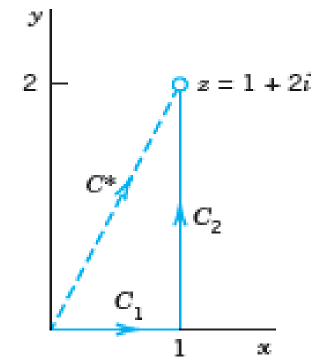


Fig. 343. Paths in Example 7

Cont'd

THEOREM 3.2 **Properties of Contour Integrals**

Suppose f and g are continuous in a domain D and C is a smooth curve lying entirely in D. Then:

(i) $\int_C k f(z)\,dz = k\int_C f(z)\,dz$, k a constant

(ii) $\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

(iii) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$, where C is the union of the smooth curve C_1 and C_2.

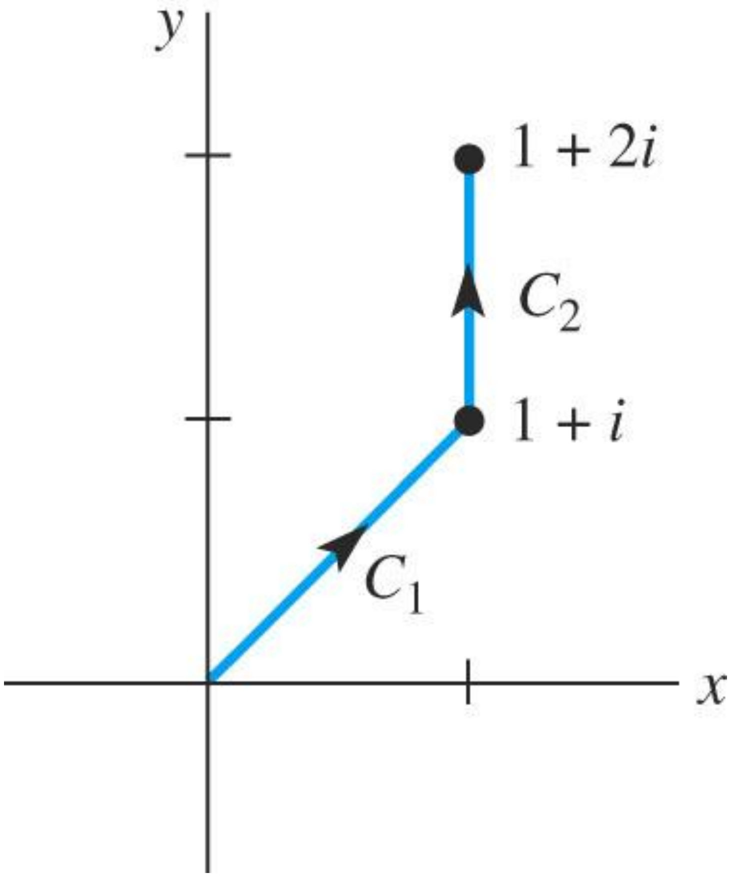
(iv) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$, where $-C$ denotes the curve having the opposite orientation of C.

Example 3

Evaluate $\int_C (x^2 + iy^2)dz$
where C is the contour in Fig 3.1.

Solution

Fig 3.1



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We have

$$\int_C (x^2 + iy^2)dz = \int_{C_1} (x^2 + iy^2)dz + \int_{C_2} (x^2 + iy^2)dz$$

Since C_1 is defined by $y = x$, then $z(x) = x + ix$, $z'(x) = 1 + i$, $f(z(x)) = x^2 + ix^2$, and

$$\int_{C_1} (x^2 + iy^2)dz = \int_0^1 (x^2 + ix^2)(1+i)dx$$

$$= (1+i)^2 \int_0^1 x^2 dx = \frac{2}{3} i$$

Cont'd

The curve C_2 is defined by $x = 1, 1 \le y \le 2$. Then $z(y) = 1 + iy$, $z'(y) = i$, $f(z(y)) = 1 + iy^2$. Thus

$$\int_{C_2} (x^2 + iy^2)dz = \int_1^2 (1 + iy^2)idy$$

$$= -\int_1^2 y^2 dy + i\int_1^2 dy = -\frac{7}{3} + i$$

Finally, $\int_C (x^2 + iy^2)dz = \frac{2}{3}i + (-\frac{7}{3} + i) = -\frac{7}{3} + \frac{5}{3}i$

Cont'd

THEOREM 3.3 **A Bounding Theorem**

If f is continuous on a smooth curve C and if $|f(z)| \leq M$ for all z on C, then $\left|\int_C f(z)dz\right| \leq ML,$ where L is the length of C.

- This theorem is sometimes called the ML-inequality

Example 4

Find an upper bound for the absolute value of

$$\oint_C \frac{e^z}{z+1}\,dz$$

where C is the circle $|z| = 4$.

Solution

Since $|z+1| \geq |z| - 1 = 3$, then

$$\left|\frac{e^z}{z+1}\right| \leq \frac{|e^z|}{|z|-1} = \frac{|e^z|}{3} \qquad (3)$$

Cont'd

In addition, $|e^z| = e^x$, with $|z| = 4$, we have the maximum value of x is 4. Thus (3) becomes

$$\left|\frac{e^z}{z+1}\right| \leq \frac{e^4}{3}$$

Hence from Theorem 3.3,

$$\left|\oint_C \frac{e^z}{z+1}\,dz\right| \leq \frac{8\pi e^4}{3}$$

3.2 Cauchy-Goursat Theorem

Cauchy's Theorem

Suppose that a function f is analytic in a simply connected domain D and that f' is continuous in D. Then for every simple closed contour C in D,

$$\oint_C f(z)dz = 0$$

This proof is based on the result of Green's Theorem.

$$\int_C f(z)dz$$

$$= \int_C u(x, y)\,dx - v(x, y)\,dy + i\int_C v(x, y)\,dx + u(x, y)\,dy$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA \qquad (4)$$

❖ Now since f is analytic, the Cauchy-Riemann equations imply the integral in (4) is identical zero.

THEOREM 3.4 Cauchy-Goursat Theorem

Suppose a function f is a analytic in a simply connected domain D. Then for every simple closed C in D,

$$\oint_C f(z)\,dz = 0$$

Cont'd

- Since the interior of a simple closed contour is a simply connected domain, the Cauchy-Goursat Theorem can be stated as
  If f is analytic at all points within and on a simple closed contour C,

$$\oint_C f(z)\,dz = 0 \qquad (5)$$

Example 1

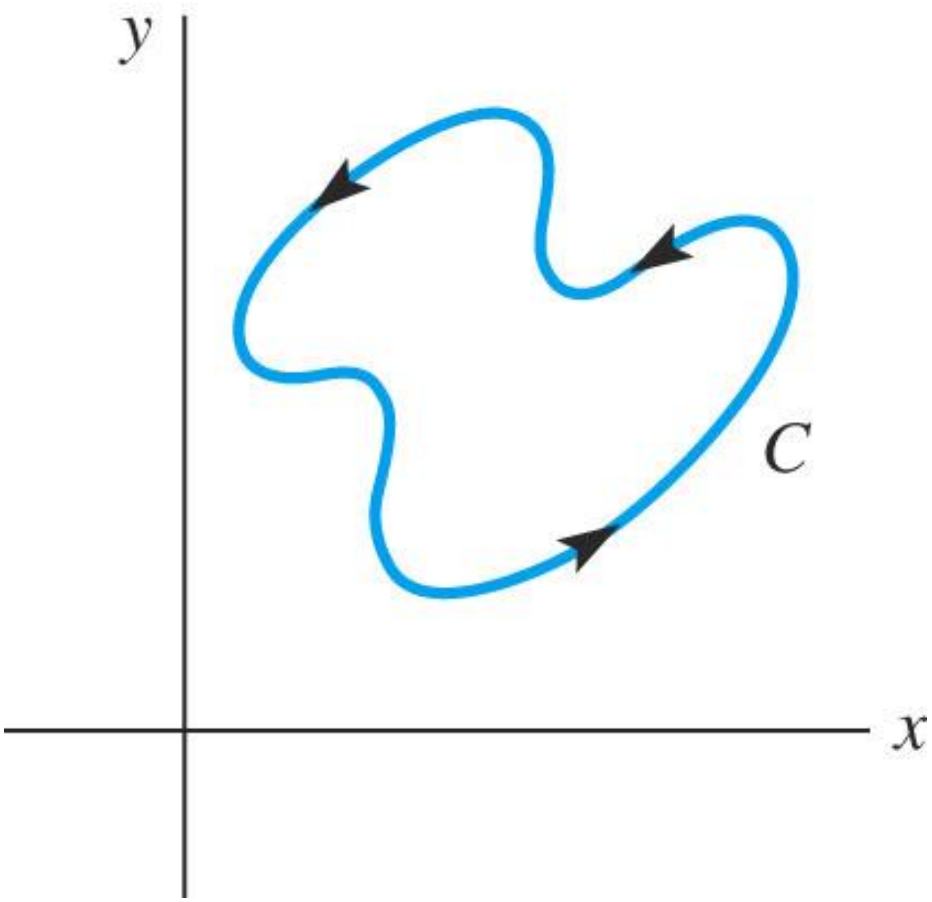
Evaluate $\oint_C e^z dz$

where C is shown in Fig 3.2.

Solution

The function e^z is entire and C is a simple closed contour. Thus the integral is zero.

Fig 3.2



Example 5

Evaluate $\oint_C \frac{dz}{z^z}$

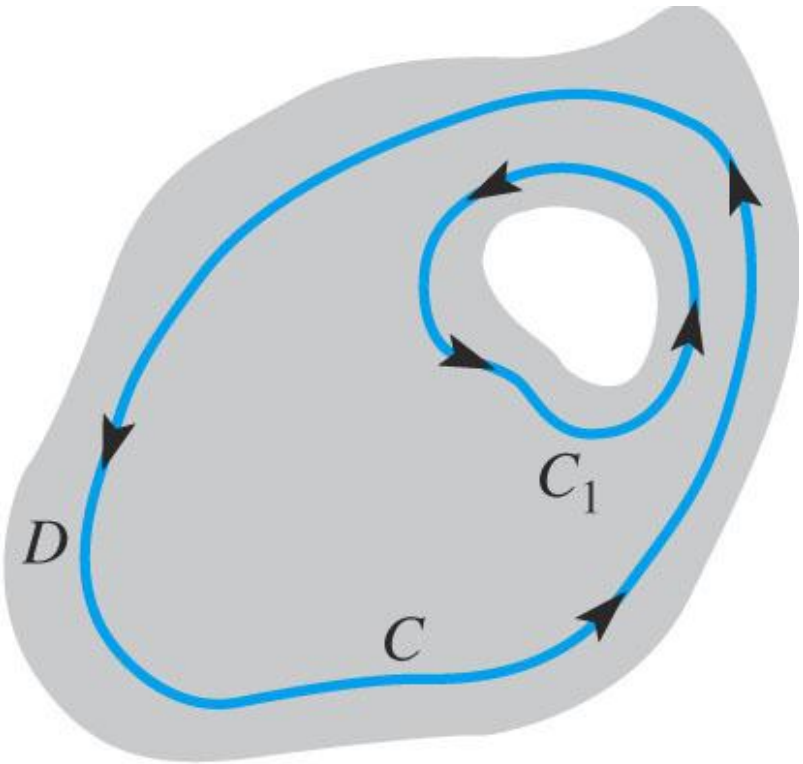
where C is the ellipse $(x - 2)^2 + (y - 5)^2/4 = 1$.

Solution

We find that $1/z^2$ is analytic except at $z = 0$ and $z = 0$ is not a point interior to or on C. Thus the integral is zero.

Cauchy-Goursat Theorem for Multiply Connected Domains

- Fig 3.3(a) shows that C_1 surrounds the "hole" in the domain and is interior to C.



(a)

Cont'd

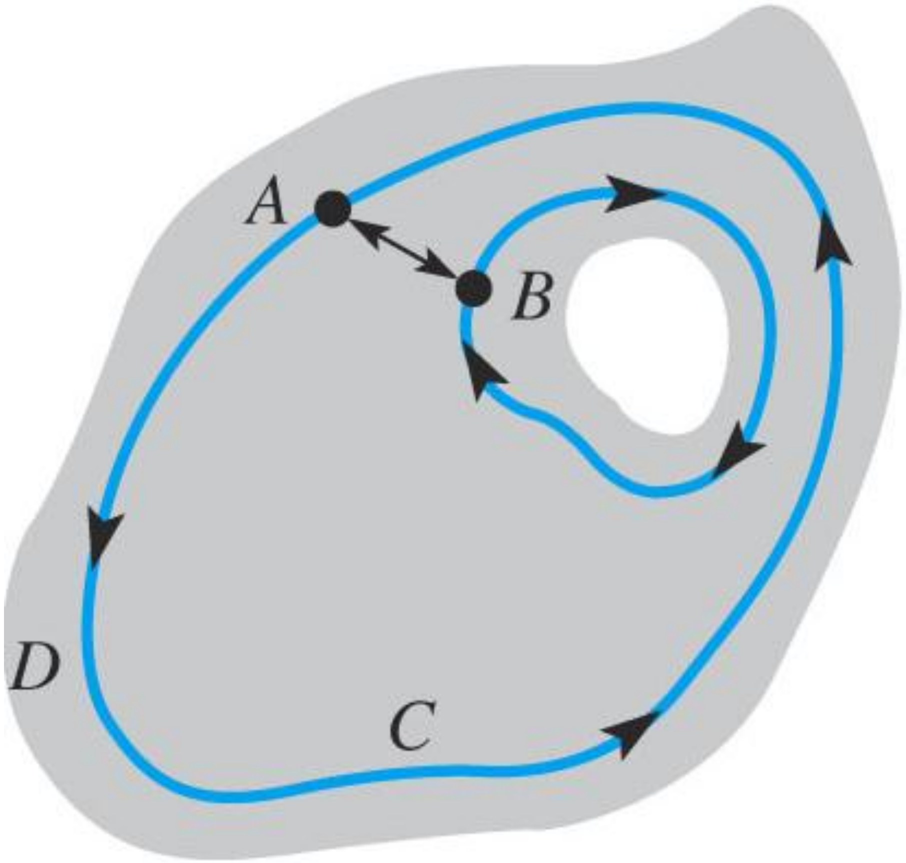
- Suppose also that f is analytic on each contour and at each point interior to C but exterior to C_1. When we introduce the cut AB shown in Fig 3.3(b), the region bounded by the curves is simply connected. Thus from (5)

$$\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz = 0$$

and

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz \qquad (6)$$

Fig 3.3 (b)



(b)

Example 6

Evaluate $\oint_C \frac{dz}{z-i}$

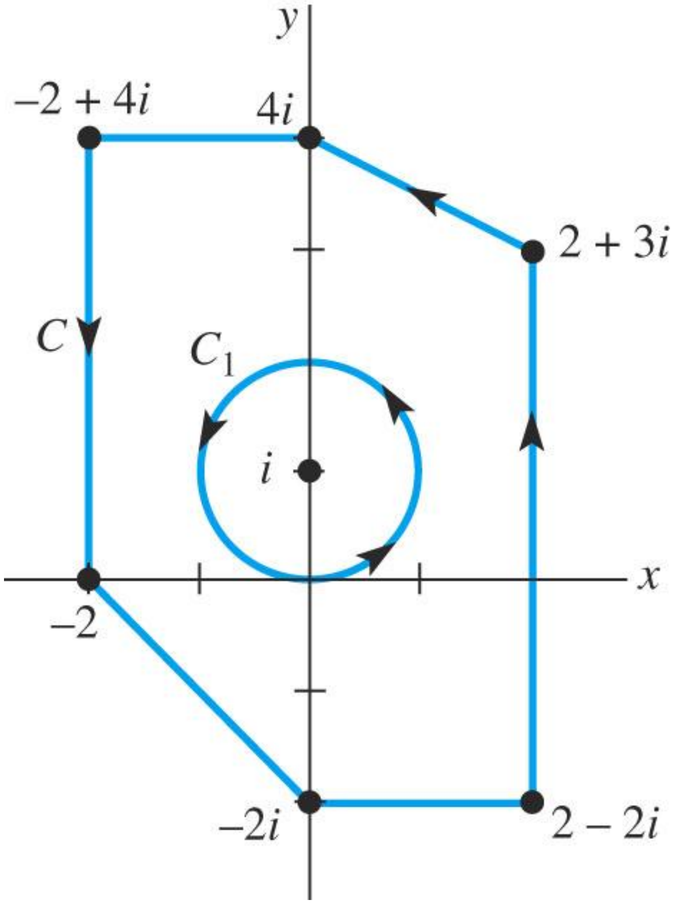
where C is the outer contour in Fig 4.

Solution

From (6), we choose the simpler circular contour C_1: $|z - i| = 1$ in the figure. Thus $x = \cos t$, $y = 1 + \sin t$, $0 \le t \le 2\pi$, or $z = i + e^{it}$, $0 \le t \le 2\pi$. Then

$$\oint_C \frac{dz}{z-i} dz = \oint_{C_1} \frac{dz}{z-i} dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = i\int_0^{2\pi} dt = 2\pi i$$

Fig. 4



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- The result in Example 6 can be generalized. We can show that if z_0 is any constant complex number interior to any simple closed contour C, then

$$\oint_C \frac{dz}{(z - z_0)^n} = \begin{cases} 2\pi i\,, & n = 1 \\ 0\,, & n \text{ an integer} \neq 1 \end{cases} \tag{7}$$

Example 7

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}\,dz$

where C is the circle $|z - 2| = 2$.

Solution

$$\frac{5z+7}{z^2+2z-3} = \frac{3}{z-1} + \frac{2}{z+3}$$

and so

$$\oint_C \frac{5z+7}{z^2+2z-3}\,dz = 3\oint_C \frac{dz}{z-1} + 2\oint_C \frac{dz}{z+3} \qquad (8)$$

Cont'd

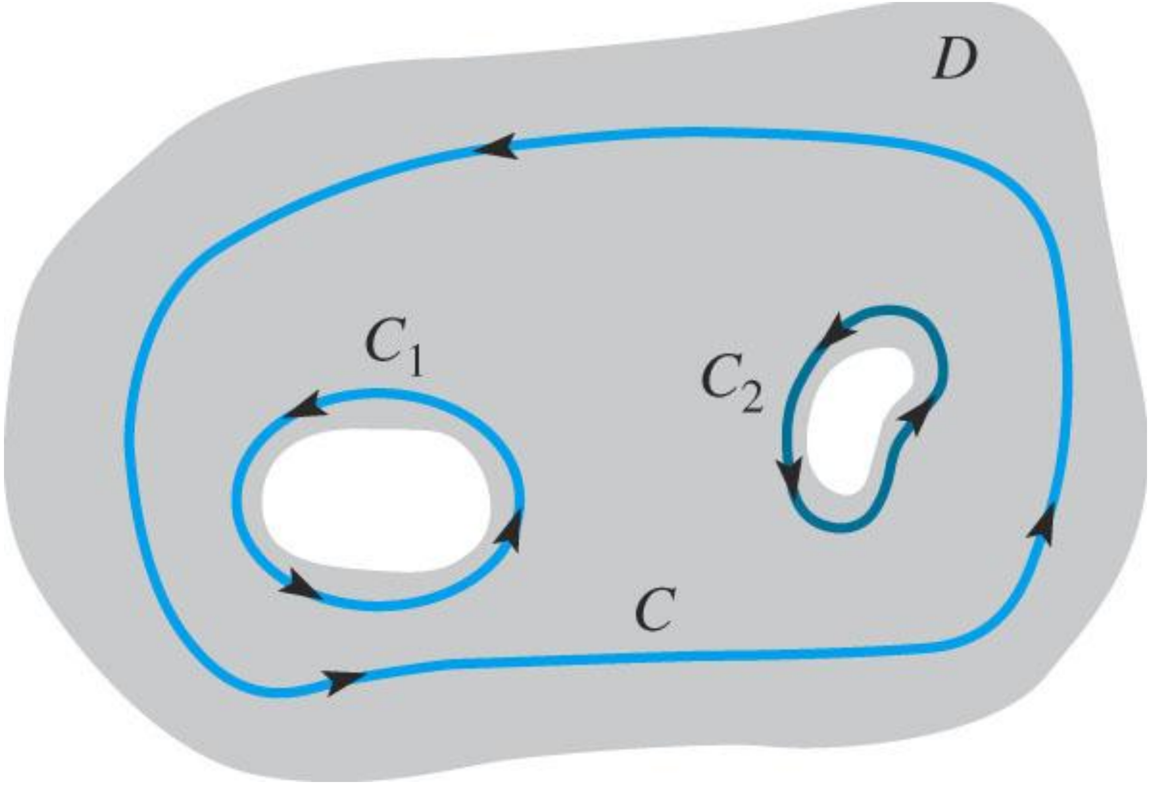
Since $z = 1$ is interior to C and $z = -3$ is exterior to C, we have

$$\oint_C \frac{5z+7}{z^2+2z-3}\,dz = 3(2\pi i) + 2(0) = 6\pi i$$

Fig 5

- See Fig 5. We can show that

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz$$



THEOREM 3.5

Cauchy-Goursat Theorem for Multiply Connected Domain

Suppose $C, C_1, \ldots, C_n$ are simple closed curves with a positive orientation such that $C_1, C_2, \ldots, C_n$ are interior to C but the regions interior to each C_k, $k = 1, 2, \ldots, n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, \ldots, n$, then

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz \qquad (7)$$

Example 8

Evaluate $\oint_C \frac{dz}{z^2+1}$

where C is the circle $|z| = 3$.

Solution

$$\frac{1}{z^2+1} = \frac{1/2i}{z-i} - \frac{1/2i}{z+i}$$

$$\oint_C \frac{dz}{z^2+1} = \frac{1}{2i}\oint_C \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz$$

Cont'd

We now surround the points $z = i$ and $z = -i$ by circular contours C_1 and C_2. See Fig 6, we have

$$\oint_C \frac{dz}{z^2+1}$$

$$= \frac{1}{2i}\oint_{C_1}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]dz + \oint_{C_2}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]dz \qquad (8)$$

$$= \frac{1}{2i}\oint_{C_1}\frac{dz}{z-i} - \frac{1}{2i}\oint_{C_1}\frac{dz}{z+i} + \frac{1}{2i}\oint_{C_2}\frac{dz}{z-i} - \frac{1}{2i}\int_{C_2}\frac{dz}{z+i}$$

Since $\int_{C_1}\frac{dz}{z-i}i = 2\pi i, \quad \int_{C_2}\frac{dz}{z+i}i = 2\pi i$

thus (8) becomes zero.

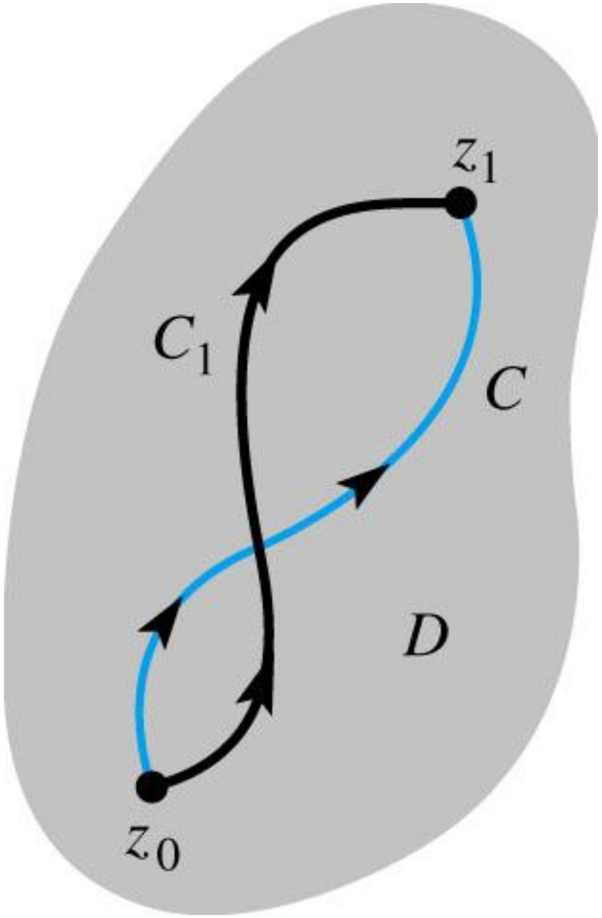
3.3 Independence of Path

DEFINITION 3.2 **Independence of the Path**

Let z_0 and z_1 be points in a domain D. A contour integral $\oint_C f(z)\,dz$ is said to be **independent of the path** if its value is the same for all contours C in D with an initial point z_0 and a terminal point z_1.

- See Fig 7

Fig 7



- Note that C and C_1 form a closed contour. If f is analytic in D then

$$\int_C f(z)\,dz + \int_{-C_1} f(z)\,dz = 0 \qquad (8)$$

Thus

$$\int_C f(z)\,dz = \int_{-C_1} f(z)\,dz \qquad (9)$$

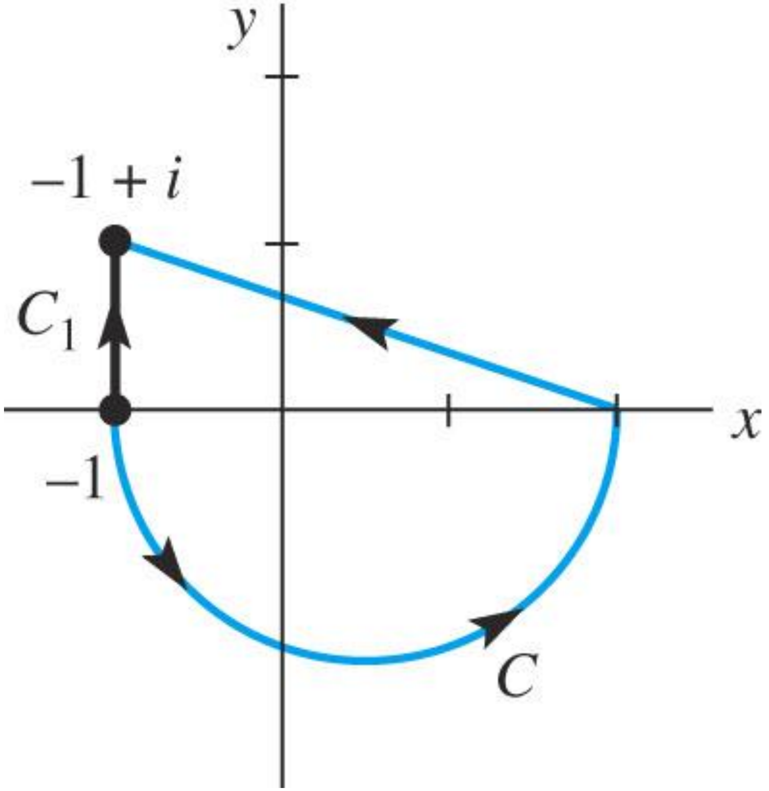
THEOREM 3.6 **Analyticity Implies Path Independence**

If f is an analytic function in a simply connected domain D, then $\int_C f(z)\,dz$ is independent of the path C.

Example 9

Evaluate $\int_C 2z\,dz$

where C is shown in Fig 8.



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Solution

Since $f(z) = 2z$ is entire, we choose the path C_1 to replace C (see Fig 7). C_1 is a straight line segment $x = -1$, $0 \le y \le 1$. Thus $z = -1 + iy$, $dz = idy$.

$$\int_C 2z\,dz = \int_{C_1} 2z\,dz$$

$$= -2\int_0^1 y\,dy - 2i\int_0^1 dy = -1 - 2i$$

Cont'd

DEFINITION 3.3

Antiderivative

Suppose f is continuous in a domain D. If there exists a function F such that $F'(z) = f(z)$ for each z in D, then F is called an **antiderivative** of f.

Cont'd

THEOREM 3.7 **Fundamentals Theorem for Contour Integrals**

Suppose f is continuous in a domain D and F is an antiderivative of f in D. Then for any contour C in D with initial point z_0 and terminal point z_1,

$$\int_C f(z)\,dz = F(z_1) - F(z_0) \tag{10}$$

Cont'd

Proof

With $F'(z) = f(z)$ for each z in D, we have

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b F'(z(t))z'(t)dt$$

$$= \int_a^b \frac{d}{dt} F(z(t))dt \qquad \leftarrow \text{Chain Rule}$$

$$= F(z(t))\Big|_a^b$$

$$= F(z(b)) - F(z(a)) = F(z_1) - F(z_0)$$

Example 10

In Example 9, the contour is from -1 to $-1 + i$. The function $f(z) = 2z$ is entire and $F(z) = z^2$ such that $F'(z) = 2z = f(z)$. Thus

$$\int_{-1}^{-1+i} 2z\,dz = z^2 \Big|_{-1}^{-1+i} = -1 - 2i$$

Example 11

Evaluate $\int_C \cos z dz$

where C is any contour fro $z = 0$ to $z = 2 + i$.

Solution

$$\int_C \cos z dz = \int_0^{2+i} \cos z dz = \sin z \Big|_0^{2+i}$$

$$= \sin(2+i) = 1.4031 - 0.4891i$$

Some Conclusions from Theorem 3.7

- If C is closed then $z_0 = z_2$, then

$$\oint_C f(z)\,dz = 0 \qquad (11)$$

- In other words:
  If a continuous function f has an antiderivative F in D, then $\int_C f(z)\,dz$ is independent of the path.
- Sufficient condition for the existence of an antiderivative:
  If f is continuous and $\int_C f(z)\,dz$ is independent of the path in a domain D, then f has an antiderivative everywhere in D.

Cont'd

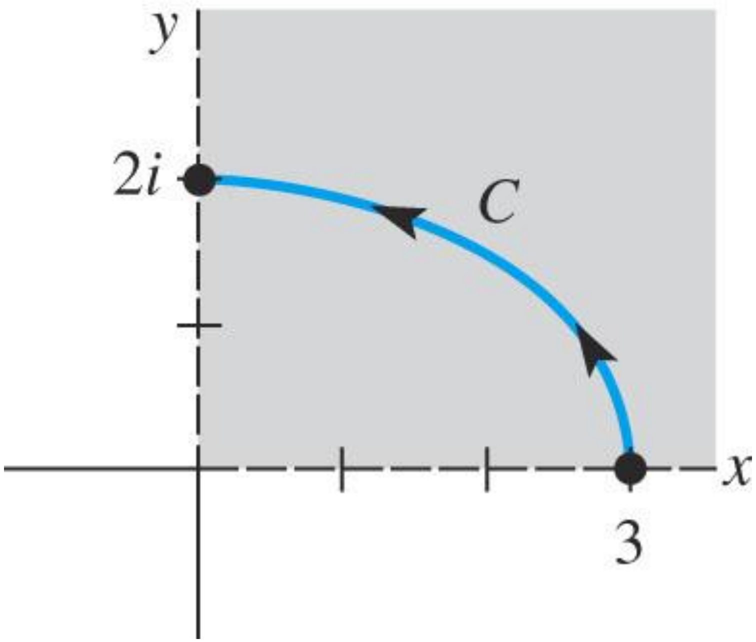
THEOREM 3.8 **Existence of a Antiderivative**

If f is analytic in a simply connected domain D, then f has an antiderivative in D; that is, there existence a function F such that $F'(z) = f(z)$ for all z in D.

Example 12

Evaluate $\int_C \frac{dz}{z}$

where C is shown in Fig 8.



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Solution

Suppose that D is the simply connected domain defined by $x > 0$, $y > 0$. In this case Ln z is an antiderivative of $1/z$. Hence

$$\int_3^{2i} \frac{dz}{z} = \text{Ln } z \Big|_3^{2i} = \text{Ln } 2i - \text{Ln } 3$$

$$\text{Ln } 2i = \log_e 2 + \frac{\pi}{2} i, \ \text{Ln } 3 = \log_e 3$$

$$\int_3^{2i} \frac{dz}{z} = \log_e \frac{2}{3} + \frac{\pi}{2} i$$

3.4 Cauchy Integral Formulas

THEOREM 3.9 **Cauchy's Integral Formula**

Let f be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz \qquad (11)$$

Cont'd

Proof

Let C_1 be a circle centered at z_0 with radius small enough that it is interior to C. Then we have

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_{C_1} \frac{f(z)}{z-z_0}\,dz \qquad (12)$$

For the right side of (12)

$$\oint_{C_1} \frac{f(z)}{z-z_0}\,dz = \oint_{C_1} \frac{f(z_0)-f(z_0)+f(z)}{z-z_0}\,dz \qquad (13)$$

$$= f(z_0)\oint_{C_1} \frac{dz}{z-z_0} + \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

Cont'd

From (4) of Sec. 3.2, we know

$$\oint_C \frac{dz}{z - z_0} = 2\pi i$$

Thus (13) becomes

$$\oint_{C_1} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0) + \oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz \qquad (14)$$

However from the ML-inequality and the fact that the length of C_1 is small enough, the second term of the right side in (4) is zero. We complete the proof.

$$\left| \oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \le \frac{\delta}{\delta/2} 2\pi \left(\frac{\delta}{2} \right) = 2\pi\varepsilon$$

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- A more practical restatement of Theorem 3.9 is :

  If f is analytic at all points within and on a simple closed contour C, and z_0 *is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz \qquad (15)$$

Example 13

Evaluate $\oint_C \frac{z^2 - 4z + 4}{z + i} \, dz$

where C is the circle $|z| = 2$.

Solution

First $f = z^2 - 4z + 4$ is analytic and $z_0 = -i$ is within C. Thus

$$\oint_C \frac{z^2 - 4z + 4}{z + i} \, dz = 2\pi i f(-i) = 2\pi i(3 + 4i) = 2\pi(-4 + 3i)$$

Example 14

Evaluate $\oint_C \frac{z}{z^2+9}\,dz$

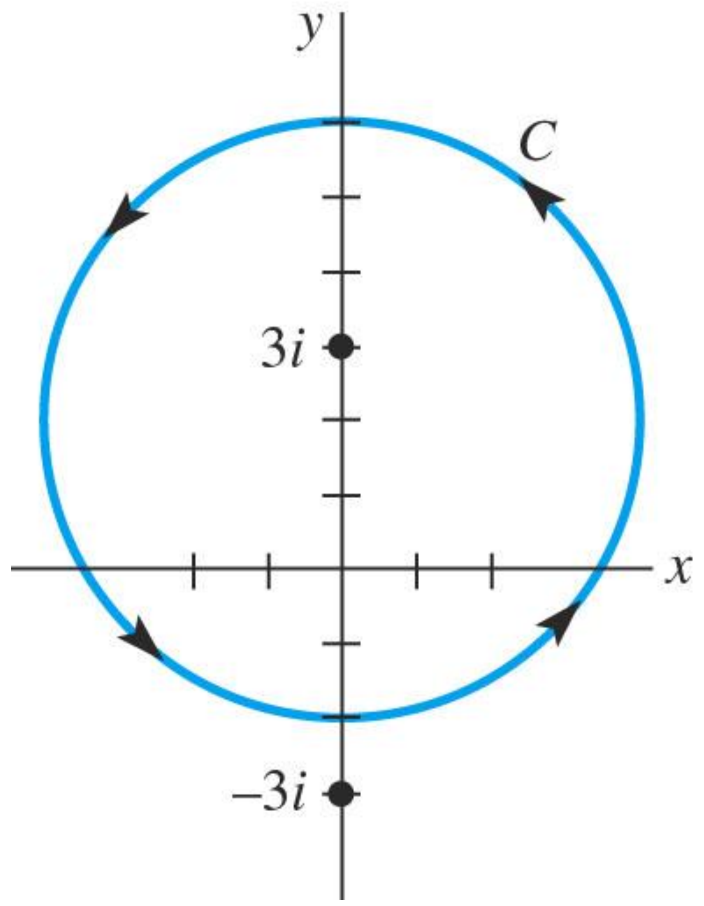
where C is the circle $|z - 2i| = 4$.

Solution

See Fig 8. Only $z = 3i$ is within C, and

$$\frac{z}{z^2+9} = \frac{\frac{z}{z+3i}}{z-3i}$$

Fig 8.



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Let $f(z)=\dfrac{z}{z+3i}$, then

$$\oint_C \frac{z}{z^2+9}\,dz=\oint_C \frac{\dfrac{z}{z+3i}}{z-3i}\,dz=2\pi i f(3i)=2\pi i\frac{3i}{6i}=\pi i$$

Example 15

The complex function $f(z) = k/(\bar{z} - \bar{z}_1)$ where $k = a + ib$ and z_1 are complex numbers, gives rise to a flow in the domain $z \neq z_1$. If C is a simple closed contour containing $z = z_1$ in its interior, then we have

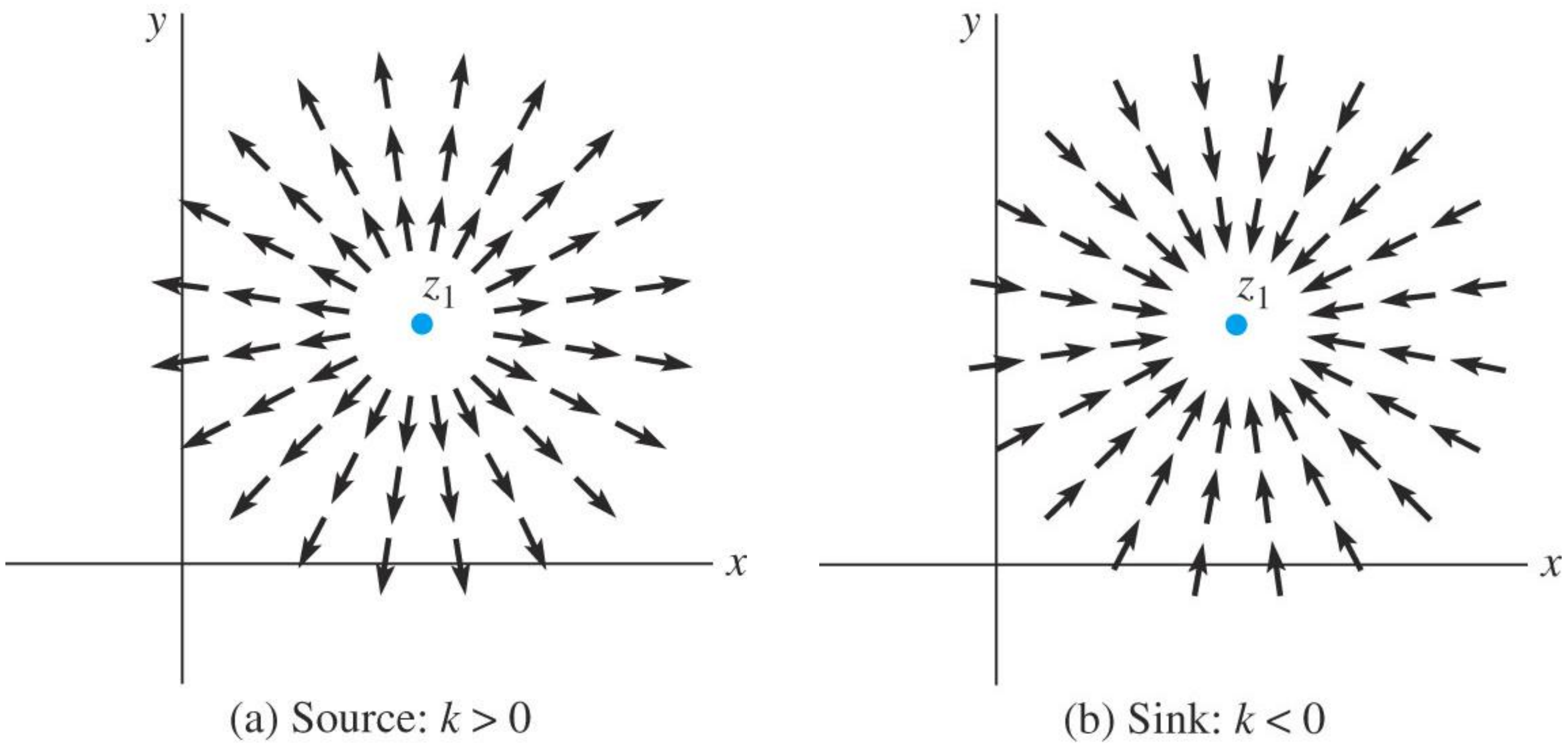
$$\oint_C \overline{f(z)}\, dz = \oint_C \frac{a - ib}{z - z_1}\, dz = 2\pi i(a - ib)$$

Cont'd

The circulation around C is $2\pi b$ and the net flux across C is $2\pi a$. If z_1 were in the exterior of C both of them would be zero. Note that when k is real, the circulation around C is zero but the net flux across C is $2\pi k$. The complex number z_1 is called a source when $k > 0$ and is a sink when $k < 0$.

See Fig 9.

Fig 9.



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THEOREM 3.8 **Cauchy's Integral Formula For Derivative**

Let f be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If z_0 is any point within C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (16)$$

Cont'd

Partial Proof

Prove only for $n = 1$. From the definition of the derivative and (11): $f(z) = k/(\bar{z} - \bar{z}_1)$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \left[\oint_C \frac{f(z)}{z - (z_0 + \Delta z)} dz - \oint_C \frac{f(z)}{z - z_0} dz \right]$$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)} dz$$

Cont'd

From the ML-inequality and

$$\left| \oint_C \frac{f(z)}{(z-z_0)^2}\, dz - \oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\, dz \right|$$

$$= \left| \oint_C \frac{-\Delta z\, f(z)}{(z-z_0)^2(z-z_0-\Delta z)}\, dz \right| \le \frac{2ML\,|\Delta z|}{\delta^3} \to 0 \text{ as } \Delta z \to 0$$

Thus

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\, dz$$

Example 16

Evaluate $\oint_C \frac{z+1}{z^4+4z^3}\,dz$

where C is the circle $|z| = 1$.

Solution

This integrand is not analytic at $z = 0, -4$ but only $z = 0$ lies within C. Since

$$\frac{z+1}{z^4+4z^3} = \frac{\frac{z+1}{z+4}}{z^3}$$

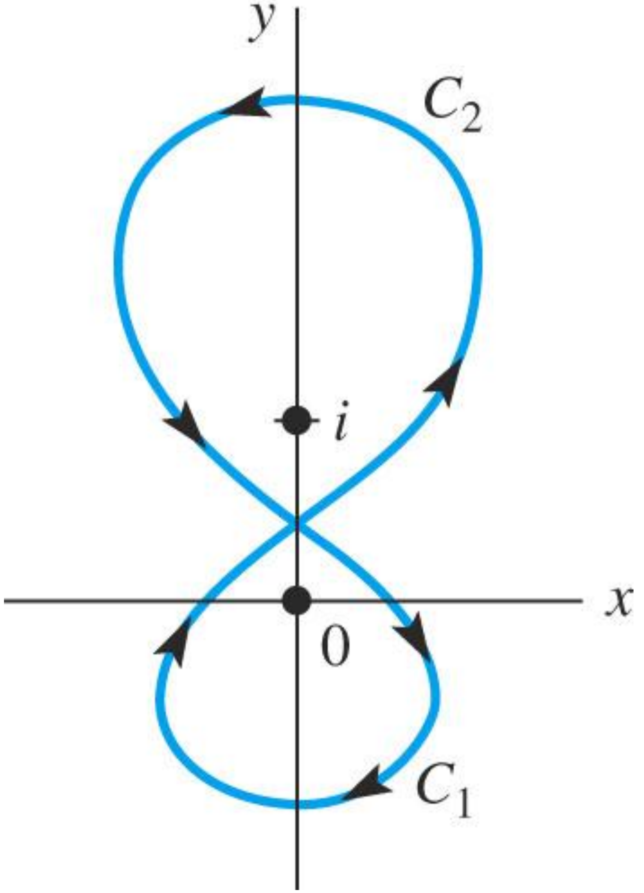
We get $z_0 = 0$, $n = 2$, $f(z) = (z + 1)/(z + 4)$, $f''(z) = -6/(z + 4)^3$. By (6):

$$\oint_C \frac{z+1}{z^4+4z^3}\,dz = \frac{2\pi i}{2!} f''(0) = -\frac{3\pi}{32} i$$

Example 17

Evaluate $\oint_C \frac{z^3+3}{z(z-i)^2}\,dz$

where C is shown in Fig 9.



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Solution

Though C is not simple, we can think of it is as the union of two simple closed contours C_1 and C_2 in Fig 10.

$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}\,dz + \oint_{C_2} \frac{z+3}{z(z-i)^2}\,dz$$

$$= -\oint_{C_1} \frac{\dfrac{z^3+3}{z(z-i)^2}}{z}\,dz + \oint_{C_2} \frac{\dfrac{z^3+3}{z}}{z(z-i)^2}\,dz$$

$$= -I_1 + I_2$$

Cont'd

For I_1 : $z_0 = 0$, $f(z) = (z^3 + 3)/(z - i)^2$:

$$I_1 = \oint_{C_1} \frac{\frac{z^3+3}{z(z-i)^2}}{z} dz = 2\pi i f(0) = -6\pi i$$

For I_2 : $z_0 = i$, $n = 1$, $f(z) = (z^3 + 3)/z$, $f'(z) = (2z^3 - 3)/z^2$:

$$I_2 = \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2} dz = \frac{2\pi i}{1!} f'(i) = -2\pi i(3+2i) = 2\pi(-2+3i)$$

We get

$$\oint_C \frac{z^3+3}{z(z-i)^2} dz = -I_1 + I_2 = 6\pi i + 2\pi(-2+3i) = 4\pi(-1+3i)$$

Liouville's Theorem

- If we take the contour C to be the circle $|z - z_0| = r$, from (16) and ML-inequality that

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz\right|$$

$$\leq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = \frac{n!M}{r^n} \qquad (17)$$

where $|f(z)| \leq M$ for all points on C. The result in (17) is called Cauchy's inequality.

THEOREM 3.11

Liouville's Theorem

The only bounded entire functions are constants.

Proof

For $n = 1$, (17) gives $|f\,'(z_0)| \le M/r$. By taking r arbitrarily large, we can make $|f\,'(z_0)|$ as small as we wish. That is, $|f\,'(z_0)| = 0$, f is a constant function.

Thank You !