CHAPTER 2

Complex Integration

"The concept of definite integral of real functions does not directly extend to the case of complex functions, since real functions are usually integrated over intervals and complex functions are integrated over curves. Surprisingly complex integrations are not so complex to evaluate, oftenly simpler than the evaluation of real integrations. Some real integrals which are otherwise difficult to evaluate can be evaluated easily by complex integration, and moreover, some basic properties of analytic functions are established by complex integration only."

2.1 LINE INTEGRAL IN THE COMPLEX PLANE

The concept of definite integral $\int_a^b f(x)\,dx$, as studied in calculus of a real valued function f on a real variable x, was generalized to line integral as applied to vector field in Chapter 6(Vol I). Here we extend the concept once more and consider the line integral of a complex function. As in calculus of a real variable, here also we distinguish between definite integrals and indefinite integrals. Complex definite integrals are called the *line integrals* and are written as

$$\int_C f(z)dz.$$

The integrand $f(z)$ is integrated over a given curve C in the complex plane called the *path of integration* normally represented by a parametric representation

$$z(t) = x(t) + iy(t), \qquad a \leq t \leq b.$$

The sense of increasing t is called the positive sense on C. The curve C is assumed to be *smooth* curve, that is, it has continuous and non-zero derivative at each $t \in (a, b)$. In case the initial point and terminal point of a curve coincide, that is $z(a) = z(b)$, the curve is said to be closed one.

2. ***Sense reversal:*** $\int_{z_0}^{z} f(z)dz = -\int_{z}^{z_0} f(z)dz$

3. ***Partitioning of path:*** $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz,$

where the curve C consists of two smooth curves C_1 and C_2 joined end to end as shown in Fig. 2.2.

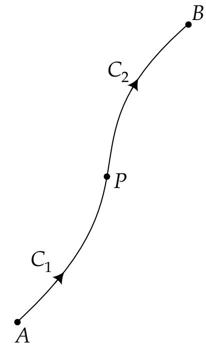


Fig. 2.2

4. ***ML-inequality:*** $\left|\int_C f(z)dz\right| \le ML,$

where M is a constant such that $|f(z)| \le M$ everywhere on C and L is the length of the curve.

Example 2.1: Evaluate $\int_C z^2 dz,$ where C is the straight line joining the origin O to the point $P(2, 1)$ in the complex plane.

Solution: The equation of the line OP is $x = 2y,\ 0 \le y \le 1.$

Thus, $dz = dx + idy = 2dy + idy = (2 + i)dy.$

Also, $z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy = 3y^2 + 4iy^2.$

Hence, $$\int_C z^2 dz = \int_0^1 (3 + 4i)y^2(2 + i)dy = (2 + 11i)\int_0^1 y^2 dy = \frac{1}{3}(2 + 11i).$$

Example 2.2: Evaluate $\oint_C (z - a)^n dz,$ where a is a given complex number, n is any integer and C is a circle of radius R centered at 'a' and oriented anticlockwise.

Solution: It is convenient here to use parametric equation of the circle in the form

$$C : z - a = \text{R}e^{i\theta},\ 0 \le \theta \le 2\pi,\ \text{ so }\ dz = i\text{R}e^{i\theta}\, d\theta.$$

Thus, $$\oint_C (z - a)^n dz = \int_0^{2\pi} R^n e^{ni\theta}\, iRe^{i\theta}\, d\theta = iR^{n+1}\int_0^{2\pi} e^{(n+1)i\theta}\, d\theta$$

$$= R^{n+1}\left|\frac{e^{(n+1)i\theta}}{n+1}\right|_0^{2\pi} = \frac{R^{n+1}}{n+1}\,[e^{2(n+1)i\pi} - 1] = 0, \text{ provided } n \ne -1.$$

For $n = -1$, we have $$\oint_C \frac{dz}{z - a} = \int_0^{2\pi} \frac{1}{\text{R}e^{i\theta}}\, i\text{R}e^{i\theta}\, d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$$

Example 2.3: Evaluate the integral $\int_0^{1+i} (x - y + ix^2)dz$

(a) along the straight line from $z = 0$ to $z = 1 + i$
(b) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + i$.

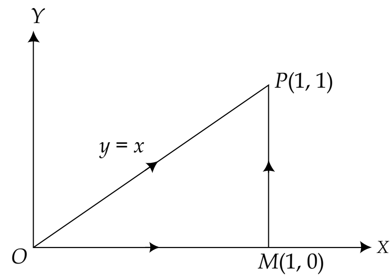


Fig. 2.3

Solution: (a) The equation of the straight line OP, refer to Fig. 2.3, is $y = x$. Thus along the line OP, $z = x + iy = x + ix = (1 + i)x$, which gives $dz = (1 + i)dx$, $0 \le x \le 1$, and hence

$$\int_0^{1+i} (x - y + ix^2)dz = \int_0^1 (x - x + ix^2)(1 + i)dx = i(1 + i)\int_0^1 x^2 dx = -\frac{1}{3}(1 - i).$$

(b) Along the path OM, we have $y = 0$ and thus $z = x + iy = x$ and hence $dz = dx$, $0 \le x \le 1$. Also, along the path MP, we have $x = 1$ and thus $z = x + iy = 1 + iy$, and hence $dz = idy$, $0 \le y \le 1$.

Therefore, the line integral

$$\int_0^{1+i} (x - y + ix^2)dz = \int_0^1 (x + ix^2)dx + \int_0^1 (1 - y + i)\,(idy).$$

$$= \left[\frac{x^2}{2} + \frac{ix^3}{3}\right]_0^1 + \left[(i-1)y - \frac{iy^2}{2}\right]_0^1 = \frac{1}{2} + \frac{i}{3} + (i - 1) - \frac{i}{2} = -\frac{1}{2} + \frac{5}{6}i.$$

Example 2.4: Evaluate $\oint_C \ln z\, dz$, where C is the unit circle $|z| = 1$ taken in counter clockwise sense.

Solution: Any point on the unit circle $|z| = 1$ in parametric form is $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, which gives $dz = ie^{i\theta}d\theta$. Thus the line integral becomes

$$\oint_C \ln z dz = \int_0^{2\pi} \ln e^{i\theta}.ie^{i\theta}d\theta = -\int_0^{2\pi} \theta e^{i\theta}d\theta = -\left[\theta\frac{e^{i\theta}}{i} - 1.\frac{e^{i\theta}}{i^2}\right]_0^{2\pi}$$

$$= -\left[\frac{2\pi e^{2\pi i}}{i} + e^{2\pi i} - 1\right] = -\frac{2\pi}{i} = 2\pi i.$$

Example 2.5: Evaluate $\oint_C |z|^2 dz$ around the square with vertices at (0, 0), (1, 0), (1, 1), (0, 1).

Solution: The contour of integration C is $OABCO$ as shown in Fig. 2.4.

We have, $|z|^2 = (x^2 + y^2)$, and also along

$$\begin{array}{lllll} OA: & y = 0, & 0 \le x \le 1, & dz = dx, & |z|^2 = x^2 \\ AB: & x = 1, & 0 \le y \le 1, & dz = idy, & |z|^2 = 1 + y^2 \\ BC: & y = 1, & x \text{ goes from 1 to 0}, & dz = dx, & |z|^2 = 1 + y^2 \\ CO: & x = 0, & y \text{ goes from 1 to 0}, & dz = idy, & |z|^2 = y^2 \end{array}$$

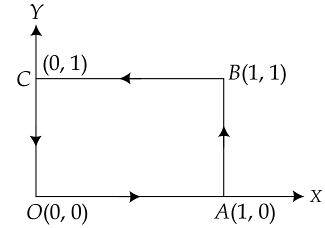


Fig. 2.4

Thus, $$\oint_C |z|^2 dz = \int_0^1 x^2 dx + i\int_0^1 (1 + y^2)dy + \int_1^0 (1 + x^2)dx + i\int_1^0 y^2 dy$$

$$= \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3} = -1 + i.$$

Example 2.6: Find an upper bound to the integral $I = \int_C \frac{e^z}{z^2} dz$, where C is the straight line from (0, 1) to (2, 0) in the complex plane.

Solution: The path C is the line segment AB as shown in Fig. 2.5. Consider

$$|f(z)| = \left|\frac{e^z}{z^2}\right| = \frac{|e^{x+iy}|}{|x+iy|^2} = \frac{|e^x||e^{iy}|}{x^2 + y^2} = \frac{e^x}{x^2 + y^2} \quad ...(2.3)$$

On C, e^x is maximum at $x = 2$, so maximum value of e^x is e^2.

Next the minimum value of $x^2 + y^2$ on C is the square of OP, the perpendicular distance from O to the line AB given by $x + 2y - 2 = 0$. This is $\left(2/\sqrt{5}\right)^2 = 4/5$.

Thus from (2.3) we have, $|f(z)| \le \frac{5e^2}{4}$. Also L, the length $|AB| = \sqrt{5}$.

Using the ML-inequality, we have

$$\left|\int_C \frac{e^z}{z^2} dz\right| \le \frac{5e^2}{4}(\sqrt{5}) = 20.65$$

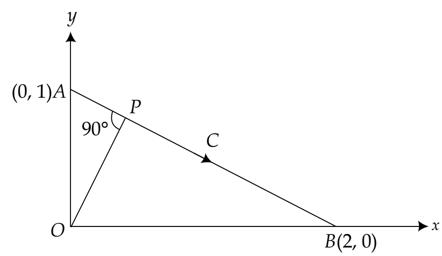


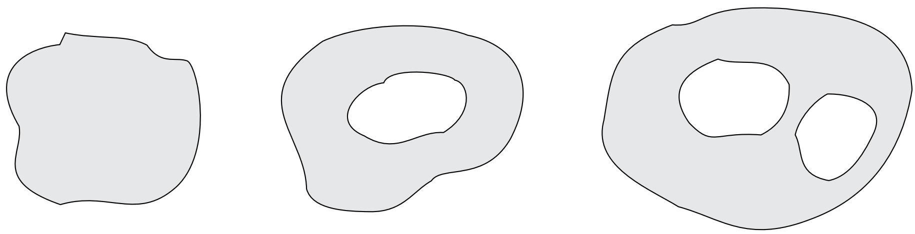
Fig. 2.5

EXERCISE 2.1

1. Evaluate $\int_C z^2 dz$, where C is the curve given by

(a) $z(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ 2 + i(t-1), & 1 \le t \le 2 \end{cases}$

Simply connected domain. A *simply connected domain D* in the complex plane is a domain such that every simple closed path in D encloses only points of D. A domain that is not simply connected is called *multiply connected*. For example, interior of an ellipse, or of a circle are examples of simply connected domains while interior of an annulus, for example $1 < |z| < 2$, is boubly connected domain. Figures 2.7(a), 2.7(b) and 2.7(c) represent respectively simply, doubly and triply connected domains.



(a) (b) (c)

Fig. 2.7

Now we are in a position to state the Cauchy's integral theorem.

Theorem 2.1: (Cauchy's Integral Theorem) *If f(z) is analytic and f′(z) is continuous in a simply connected domain D, then for every piecewise smooth closed curve C in D the contour integral*

$$\oint_C f(z)dz = 0 \qquad \ldots(2.4)$$

Proof. Writing $f(z) = u + iv$ and $dz = dx + idy$, we have

$$\oint_C f(z)dz = \oint_C (u + iv)(dx + idy)$$

$$= \int_C (udx - vdy) + i\int_C (vdx + udy) \qquad \ldots(2.5)$$

Since $f'(z)$ is continuous, therefore, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in D, and hence in the region enclosed by C. Thus Green's theorem, refer to Section 6.4 (Vol. 1), is applicable to the right side of (2.5) and hence it becomes

$$\oint_C f(z)dz = -\iint_E \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy + i\iint_E \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy, \qquad \ldots(2.6)$$

where E is the region bounded by the closed curve C, refer to Fig. 2.7(a).

Since $f(z)$ is analytic, u and v satisfy the Cauchy-Riemann equations (1.28), and thus the integrands of the two double integrals on the right side of (2.6) are identically zero and hence we obtain (2.4).

We must note that *analyticity of f(z) is only sufficient but not a necessary condition for (2.4) to be true.* We can check very easily that $\oint_C \frac{dz}{z^2} = 0$, where C is the unit circle, refer Example 2.2 for $a = 0$ and $n = -2$, but this result does not follow from Cauchy's theorem since $f(z) = 1/z^2$ is not analytic in $|z| < 1$, zero being the point of singularity.

On the other hand, *simple connectedness of the domain is essential one*. For example, $\oint_C \frac{dz}{z} = 2\pi i$, where C is the unit circle lying in the annulus $1/2 < |z| < 3/2$, refer to Example 2.2. Here, $f(z) = 1/z$ is analytic in the given domain but this domain is not simply connected so Cauchy theorem is not applicable.

Example 2.7: Evaluate the following integrals by applying Cauchy's integral theorem, in case applicable

(a) $\oint_C \cos z \, dz$ (b) $\oint_C \sec z \, dz$ (c) $\oint_C \frac{dz}{z^2 - 5z + 6} dz$ (d) $\oint_C \bar{z} \, dz$,

where C is the unit circle $|z| = 1$.

Solution: (a) The integrand $f(z) = \cos z$ is analytic for all z and also $f'(z) = \sin z$ is continuous everywhere, and hence on and inside C also. Thus by Cauchy's theorem

$$\oint_C \cos z \, dz = 0.$$

(b) The integrand $f(z) = \sec z = \frac{1}{\cos z}$ is not analytic at the points $z = \pm\pi/2, \pm 3\pi/2, \ldots$ but all these points lie outside the unit circle $|z| = 1$. Hence $f(z)$ is analytic and $f'(z)$ is continuous in and on C, and thus

$$\oint_C \sec z \, dz = 0.$$

(c) The integrand $f(z) = \frac{1}{z^2 - 5z + 6} = \frac{1}{(z-2)(z-3)}$ is analytic everywhere except at $z = 2, 3$, the points which lie outside the unit circle $|z| = 1$, and hence by Cauchy's theorem

$$\oint_C \frac{1}{z^2 - 5z + 6} dz = 0.$$

(d) The integrand $f(z) = \bar{z}$ is not analytic and hence the Cauchy's theorem is not applicable. In fact, about C: $|z| = 1$, we have

$$\oint_C \bar{z} \, dz = \int_0^{2\pi} e^{-i\theta} \, ie^{i\theta} \, d\theta = i \int_0^{2\pi} d\theta = 2\pi i.$$

2.2.1 Independence of Path

In the preceding section, we have noted that a line integral of a function $f(z)$ depends not merely on the end points of the path but also the path itself, refer to Example 2.3. We say that an integral of $f(z)$ is *independent of path in a domain D*, if for every z_1, z_2 in D the value of $\int_{z_1}^{z_2} f(z)dz$ depends only on the end points z_1 and z_2 and not on the choice of the path C joining z_1 to z_2. An important consequence of Cauchy's theorem is to look for the situations when the line integral is independent of path in a domain D. We have the following result:

Theorem 2.2: **(Independence of path)** *If $f(z)$ is analytic in a simply connected domain D, then $\int_C f(z)dz$ is independent of the path for every piecewise smooth curve C lying entirely within D.*

Proof. Let $P(z_1)$ and $Q(z_2)$ be any two points in D and let C_1 and C_2 be two arbitrary paths in D from P to Q intersecting each other only at the end points P and Q, as shown in Fig. 2.8a. Consider the curve $C_2^{\bullet}$ same as C_2 but with reverse orientation as shown in Fig. 2.8b. We observe that $C_1 \cup C_2^{\bullet}$ is a piecewise smooth simple closed curve in D, and so according to Cauchy's integral theorem

$$\int_{C_1 \cup C_2^{\bullet}} f(z)dz = 0, \text{ which gives } \int_{C_1} f(z)dz = -\int_{C_2^{\bullet}} f(z)dz, \text{ or } \int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

The minus sign disappears in case we integrate in the reverse direction.

This proves the theorem.

In case the two paths have finitely many points in common as shown in Fig. 2.9, then the independence of path can be proved by applying the argument to each loop separately.

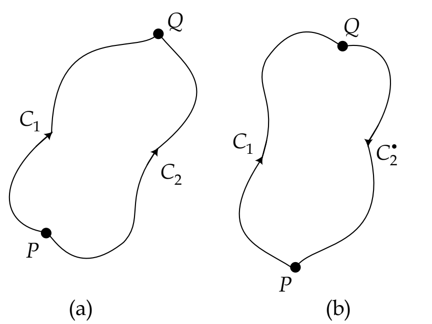


Fig. 2.8

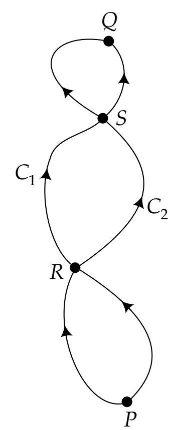


Fig. 2.9

2.2.2 Deformation of Path

It is useful to consider path independence in terms of process of *path deformation*. We can visualize deforming C_1 continuously into C_2, refer to Fig. 2.10, keeping the end points P and Q fixed. If f is

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz \qquad \ldots(2.7)$$

This result can be extended to multiply connected regions also as shown in Fig. 2.12. The result is as follow:

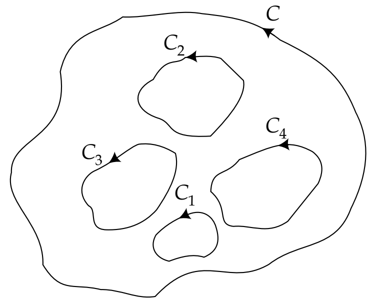


Fig. 2.12

Theorem 2.4: *If f(z) is analytic on and between the region included in the closed curves C, C_1, C_2, C_3 etc., then*

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \ldots . \qquad \ldots(2.8)$$

Example 2.8: Verify that the line integral $I = \int_C z^2 dz$ is the same in each of the following cases

(a) C is the straight line OP joining the points $O(0, 0)$ and $P(1, 2)$.
(b) C is the straight line from $O(0, 0)$ to $A(1, 0)$ and then from $A(1, 0)$ to $P(1, 2)$.
(c) C is the parabolic path $y = 2x^2$.

Solution: The three paths are shown in the Fig. 2.13.

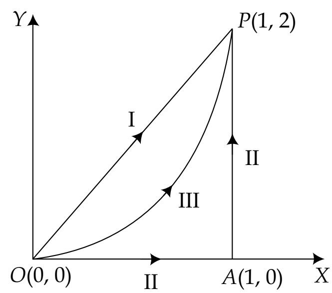


Fig. 2.13

(a) Equation of the line OP is $y = 2x, 0 \le x \le 1$.

Thus, $z^2 = (x + iy)^2 = (1 + 2i)^2 x^2$ and $dz = dx + idy = (1 + 2i)dx$

Therefore, $I = \int_C z^2 dz = \int_0^1 (1 + 2i)^3 x^2\, dx = \frac{1}{3}(1 + 2i)^3 = -\frac{1}{3}(11 + 2i)$

(b) Along OA; $y = 0, 0 \le x \le 1, z = x$. Thus we have $z^2 = (x + iy)^2 = x^2$ and $dz = dx$.

Along AP; $x = 1, 0 \le y \le 2$, thus we have $z = 1 + iy$, $z^2 = (1 + iy)^2$ and $dz = idy$.

Therefore, $$I = \int_0^1 x^2 dx + i\int_0^2 (1 + iy)^2 dy = \frac{1}{3} + \frac{(1 + 2i)^3}{3} - \frac{1}{3} = -\frac{1}{3}(11 + 2i).$$

(c) Along the curve $y = 2x^2$, $0 \le x \le 1$, we have, $z = x + iy = x + 2ix^2$, thus $z^2 = (1 + 2ix)^2 x^2$ and $dz = dx + 4ix\, dx = (1 + 4ix)dx$

Therefore, $$I = \int_C z^2 dz = \int_0^1 (1 + 2ix)^2 x^2 (1 + 4ix)dx = \int_0^1 (x^2 - 4x^4 + 4ix^3)(1 + 4ix)dx$$

$$= \int_0^1 [(x^2 - 20x^4) + i(8x^3 - 16x^5)]dx = -\frac{1}{3}(11 + 2i)$$

Thus along all the three paths the value of the line integral is the same. In fact the integrand z^2 is analytic in the entire complex plane, the value of the line integral I depends only on the end points.

Example 2.9: Evaluate $\oint_\Gamma \frac{1}{z-a}dz$ over any closed path enclosing the given point 'a'.

Solution: Figure 2.14 shows a typical such path but it cannot be parameterized, since we do not know the contour Γ specifically. Let C be a circle of radius r with centre 'a'. Since the function $f(z)$ is analytic on and between Γ and C, thus by principle of deformation of path

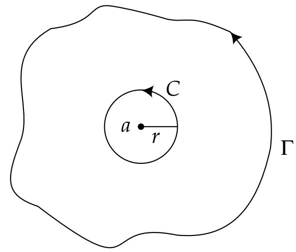


Fig. 2.14

$$\int_\Gamma \frac{1}{z-a}\,dz = \int_C \frac{1}{z-a}\,dz.$$

The contour C can be parametrized as $z = a + re^{i\theta}, 0 \le \theta \le 2\pi$, and thus

$$\int_\Gamma \frac{1}{z-a}\,dz = \int_C \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{e^{-i\theta}}{r}.\,ir\,e^{i\theta}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$$

In general, we note that for any closed anticlockwise contour Γ about a point 'a', we have

$$\oint_\Gamma (z-a)^n\,dz = \begin{cases} 2\pi i & n = -1 \\ 0, & n \ne -1 \end{cases} \qquad \ldots(2.9)$$

This result follows from the principle of deformation and Example 2.2.

Example 2.10: Evaluate $I = \oint_C \frac{dz}{z^2(z-2)(z-4)}$, where C is the rectangle joining the points (– 1, – 1), (3, – 1), (3, 1) and (– 1, 1) in the complex plane.

Solution: The curve C is the rectangle $ABCD$ as shown in Fig. 2.15. Expanding the integrand in the partial fractions, we obtain

$$I = \frac{3}{32}\oint_C \frac{dz}{z} + \frac{1}{8}\oint_C \frac{dz}{z^2} - \frac{1}{8}\oint_C \frac{dz}{z-2} + \frac{1}{32}\oint_C \frac{dz}{z-4} \qquad \ldots(2.10)$$

$$= \frac{3}{32}(2\pi i) + \frac{1}{8}(0) - \frac{1}{8}(2\pi i) + \frac{1}{32}(0) = -\frac{\pi i}{16}$$

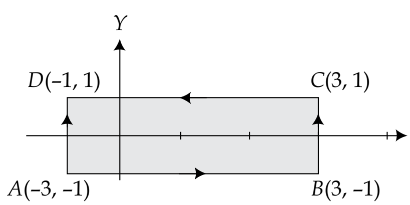


Fig. 2.15

The first three integrals on the right side of (2.10) are evaluated by using (2.9), and the last integral is zero by Cauchy's integral theorem.

Example 2.11: Evaluate the integral $\oint_C \frac{dz}{z(z+2)}$, where C is any rectangle containing the points $z = 0$ and $z = -2$ inside it.

Solution: The integrand $f(z)$ is analytic everywhere except at the points $z = 0$ and $z = -2$, both the points lying inside the rectangle C. Draw circles C_1 and C_2 respectively enclosing the points $z = 0$ and $z = -2$ as shown in Fig. 2.16. The function $f(z)$ is analytic on and between the curves C, C_1 and C_2 and hence by the extension of Cauchy's theorem we have

$$\oint_C \frac{dz}{z(z+2)} = \oint_{C_1} \frac{dz}{z(z+2)} + \oint_{C_2} \frac{dz}{z(z+2)} = \frac{1}{2}\left[\oint_{C_1} \frac{dz}{z} - \oint_{C_1} \frac{dz}{z+2} + \oint_{C_2} \frac{dz}{z} - \oint_{C_2} \frac{dz}{z+2}\right] \qquad \ldots(2.11)$$

By Cauchy's theorem, $\oint_{C_1} \frac{dz}{z+2} = 0, \int_{C_2} \frac{dz}{z} = 0.$

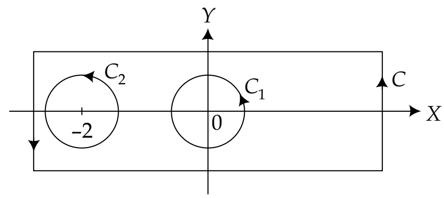


Fig. 2.16

Also, $\int_{C_1} \frac{dz}{z} = 2\pi i$ and $\int_{C_2} \frac{dz}{z+2} = 2\pi i,$

refer to Example 2.9. Therefore, from (2.11), we obtain

$$\oint_C \frac{dz}{z(z+2)} = \frac{1}{2}(2\pi i - 2\pi i) = 0$$

2.2.3 Fundamental Theorem of the Complex Integral Calculus

The *fundamental theorem of the complex integral calculus* is a result analogous to the fundamental theorem of integral calculus. The theorem stated below is useful for evaluating the integrals for which an antiderivative can be found simply by inspection.

Theorem 2.5 (Fundamental theorem of complex integral calculus): *If $f(z)$ is analytic in a simply connected domain D and z_0 be any fixed point in D, then*

$$F(z) = \int_{z_0}^{z} f(z^\bullet)dz^\bullet$$

is analytic in D given by $F'(z) = f(z)$, and

$$\int_{z_0}^{z} f(z^\bullet)dz^\bullet = F(z) - F(z_0).$$

A function $F(z)$ satisfying $F'(z) = f(z)$ is called an 'indefinite integral' or 'primitive' of f.

(a) $\oint_C \frac{e^z}{z^2 - 5iz - 6} dz$; C: $|z| = 1$ (b) $\oint_C \left(z + \frac{3}{z^2}\right) dz$; C: $|z| = 1$

(c) $\oint_C \frac{\cosh^2 2z}{(z + 3i)(z^2 + 16)} dz$; C: $|z| = 2$

6. Evaluate the following integrals using the extension of the Cauchy's integral theorem to multiply connected domains

(a) $\oint_C \frac{2z - 3}{z^2 - 3z - 18} dz$; C: $|z| = 8$ (b) $\oint_C \frac{2z^3 + z^2 + 4}{z^4 + 4z^2} dz$; C: $|z - 2| = 4$

(c) $\oint_C \frac{dz}{(z - 1)(z - 2)(z - 3)}$; C: $|z| = 4$

7. By evaluating $\oint_C e^z dz$, C: $|z| = 1$, show that $\int_0^{2\pi} e^{\cos\theta} \cos(\theta + \sin\theta) d\theta = 0$ and $\int_0^{2\pi} e^{\cos\theta} \sin(\theta + \sin\theta) d\theta = 0$.

8. Prove that $\int_C (z^2 + 2)^2 dz = 8\pi a(12\pi^4 a^4 + 20\pi^2 a^2 + 15)/15$, where C is the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ joining the points (0, 0) and $(2\pi a, 0)$.

9. Show that the integral $\int_C e^{-2z} dz$, where C is the path joining the points $z = 1 + 2\pi i$ and $z = 3 + 4\pi i$ is independent of the path of integration. Evaluate it by taking a suitable path.

10. Use the fundamental theorem to evaluate the following integrals:

(a) $\int_i^0 \cos 3z \, dz$ (b) $\int_0^{3i} z e^{z^2} dz$ (c) $\int_0^{1+2i} z \sin(z^2) dz$

(d) $\int_0^{1+\pi i} (z^2 + \cosh 2z) dz$ (e) $\int_0^1 \frac{\tan^{-1} z}{1 + z^2} dz$ (f) $\int_{-i}^{i} z \cosh^2 z \, dz$

2.3 CAUCHY'S INTEGRAL FORMULA. DERIVATIVES OF AN ANALYTIC FUNCTION

Cauchy's integral formula is an important consequence of Cauchy's integral theorem. This gives a representation of an analytic function $f(z)$ at any interior point z_0 of a simply connected domain D as a contour integral evaluated along the boundary of a simple closed curve C which lies inside D and encloses the point z_0. The result is of fundamental importance and is stated as follows.

Theorem 2.6 (Cauchy's integral formula): *Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0*

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz, \qquad \text{...(2.12)}$$

the integration being taken counter-clockwise.

Proof. Let C_1 be a circle with centre z_0 and radius r lying entirely within C. The function $\frac{f(z)}{z - z_0}$ is analytic on and within the closed curves C and C_1 as shown in Fig. 2.17, thus by the extension of Cauchy's integral theorem,

$$\oint_C \frac{f(z)}{z - z_0}\,dz = \oint_{C_1} \frac{f(z)}{z - z_0}\,dz = \oint_{C_1} \frac{[f(z_0) + f(z) - f(z_0)]}{z - z_0}\,dz$$

$$= f(z_0)\oint_{C_1} \frac{dz}{z - z_0} + \oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0}\,dz \qquad \text{...(2.13)}$$

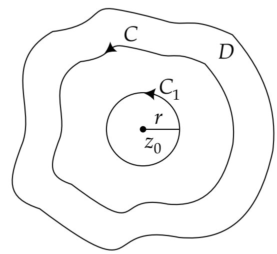


Fig. 2.17

Consider the first integral on the right side of (2.13). Put $z - z_0 = r e^{i\theta}$, we have $dz = ir e^{i\theta}\, d\theta$, and hence

$$\oint_{C_1} \frac{dz}{z - z_0} = \int_0^{2\pi} i\,d\theta = 2\pi i$$

Next, if I denotes the second integral on the right hand side of (2.13), then

$$|I| = \left|\oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0}\,dz\right| \le \oint_{C_1} \left|\frac{f(z) - f(z_0)}{z - z_0}\right| |dz| = \oint_{C_1} \frac{|f(z) - f(z_0)|}{|z - z_0|}\,|dz| \qquad \text{...(2.14)}$$

Since $f(z)$ is continuous in D, (for it is analytic in D), thus for a given $\in > 0$, there exists a number $\delta > 0$ such that $|f(z) - f(z_0)| < \in$, wherever $|z - z_0| < \delta$.

Choosing the radius r of the circle C_1 such that $r < \delta$ and hence from (2.14), we have

$$|I| \le \oint_{C_1} \frac{|f(z) - f(z_0)|}{|z - z_0|}\,|dz| < \oint_{C_1} \frac{\in}{r}\,|dz| = \frac{\in}{r} 2\pi r = 2\pi\in$$

Since $\in > 0$ can be chosen arbitrary small, thus $|I|$ can be made arbitrary small tending to zero, and thus Eq. (2.13) becomes

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0), \text{ or } f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz,$$

which is (2.12).

Example 2.13: Evaluate the integral $\oint_C \frac{z^2 + 1}{z^2 - 1} dz$, C: $|z - 1| = 1$

Solution: Writing the integrand as $\frac{z^2 + 1}{z^2 - 1} = \frac{(z^2 + 1)/(z + 1)}{z - 1}$

We observe that $f(z) = (z^2 + 1)/(z + 1)$ is analytic on and inside C, and here $z_0 = 1$, as shown in Fig. 2.18. Hence by Cauchy's integral formula

$$\oint_C \frac{z^2 + 1}{z^2 - 1} dz = 2\pi i f(1) = 2\pi i.$$

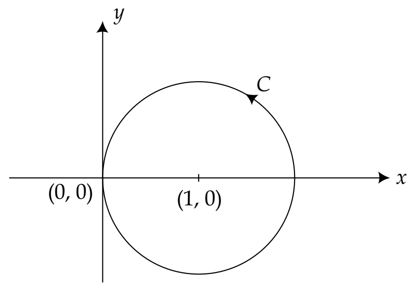


Fig. 2.18

Example 2.14: Evaluate the integral $\oint_C \frac{z^2 + 1}{z(2z - 1)} dz$, C: $|z| = 1$

Solution: Let $I = \oint_C \frac{z^2 + 1}{z(2z - 1)} dz$.

The integrand $(z^2 + 1)/z(2z - 1)$ is not analytic at the point $z = 0$ and $z = 1/2$ both of which lie inside C. Writing it as

$$\frac{z^2 + 1}{z(2z - 1)} = (z^2 + 1)\left[\frac{1}{(z - 1/2)} - \frac{1}{z}\right]$$

Therefore, $I = \oint_C \frac{z^2 + 1}{z - 1/2} dz - \oint_C \frac{z^2 + 1}{z} dz = 2\pi i\, [z^2 + 1]_{z = 1/2} - 2\pi i\, [z^2 + 1]_{z=0} = \frac{5\pi i}{2} - 2\pi i = \frac{\pi i}{2}$,

using the Cauchy's integral formula.

Example 2.15: Evaluate the integral $\oint_C \frac{dz}{(z - z_0)(z - z_1)}$, where the points z_0 and z_1 lie inside the simple closed curve C and integration is taken in counter-clockwise sense.

Solution: Let C_0 and C_1 be two small simple closed non-intersecting curves surrounding z_0 and z_1 respectively and lying entirely within C. Then by the extension of the Cauchy's integral theorem, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz, n = 1, 2, ..., \quad ...(2.16)$$

where C is any simple closed path in D taken in counter-clockwise sense.

Proof. The Cauchy's integral formula is

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz.$$

Differentiating it under the integral sign w.r.t. z_0 we obtain

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz \quad ...(2.17)$$

Similarly, $$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^3} dz$$ and, in general,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

This completes the proof.

Example 2.17: Evaluate the integral $\oint_C \frac{e^z}{z^3} dz$, C: $|z| = 1$ taken in counter-clockwise sense.

Solution: Let $I = \oint_C \frac{e^z}{z^3} dz$. Here $f(z) = e^z$ is analytic in the region bounded by the simple closed curve $|z| = 1$. The singular point $z = 0$ of $1/z^3$ lies inside $|z| = 1$. Hence, applying the generalized Cauchy's integral formula

$$I = \oint_C \frac{e^z}{z^3} dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2}(e^z)\Big|_{z=0} = \pi i.$$

Example 2.18: Evaluate $\oint_C \frac{(z + 1)}{z(z - 2)(z - 4)^3} dz$, C: $|z - 3| = 2$ in the counter- clockwise sense.

Solution: Let $I = \oint_C \frac{(z + 1)}{z(z - 2)(z - 4)^3} dz.$

The integrand has singularities at $z = 0, 2$, and 4, out of these $z = 2$ and 4 lie inside C.

Consider two non-intersecting closed contour C_1 and C_2, as shown in Fig. 2.19, lying completely within C, respectively about the point $z = 2$ and $z = 4$. Applying the principle of deformation the integral I becomes

$$I = \oint_C \frac{z+1}{z(z-2)(z-4)^3} dz$$

$$= \oint_{C_1} \left[\frac{z+1}{z(z-4)^3}\right] \frac{dz}{z-2} + \oint_{C_2} \left[\frac{z+1}{z(z-2)}\right] \frac{dz}{(z-4)^3} = I_1 + I_2, \text{ say.}$$

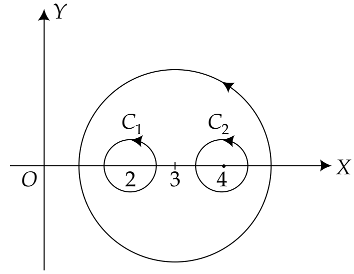


Fig. 2.19

Now, $I_1 = \oint_{C_1} \left[\frac{z+1}{z(z-4)^3}\right] \frac{dz}{z-2} = 2\pi i \left[\frac{z+1}{z(z-4)^3}\right]_{z=2} = -\frac{3\pi i}{8}$, using Cauchy's integral formula.

Similarly, $I_2 = \oint_{C_2} \left[\frac{z+1}{z(z-2)}\right] \frac{dz}{(z-4)^3} = \frac{2\pi i}{2!} \frac{d^2}{dz^2} \left[\frac{z+1}{z(z-2)}\right]_{z=4} = \frac{23\pi i}{64}$

Therefore, $I = \frac{-3\pi i}{8} + \frac{23\pi i}{64} = -\frac{\pi i}{64}$.

Example 2.19: If $F(a) = \oint_C \frac{4z^2 + z + 5}{z-a} dz$, where $C: (x/2)^2 + (y/3)^2 = 1$, taken in counter-clockwise sense, then find $F(3.5)$, $F(i)$, $F'(-1)$ and $F''(-i)$.

Solution: We have, $F(3.5) = \oint_C \frac{4z^2 + z + 5}{z - 3.5} dz$

The integrand $\frac{4z^2 + z + 5}{z - 3.5}$ is analytic everywhere except at the point $z = 3.5$ which lies outside the ellipse $(x/2)^2 + (y/3)^2 = 1$, as shown in Fig. 2.20. Therefore, it is analytic everywhere within C and hence by Cauchy's integral theorem $F(3.5) = 0$.

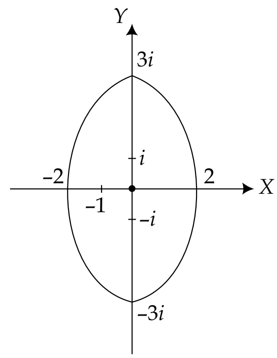


Fig. 2.20

Next the numerator $f(z) = 4z^2 + z + 5$ of the integrand is analytic everywhere in C and $a = i, -1$ and $-i$ all lie within C. Therefore by Cauchy's integral theorem, $f(a) = \frac{1}{2\pi i} \oint_C \frac{4z^2 + z + 5}{z-a} dz$, which gives

$$\oint_C \frac{4z^2 + z + 5}{z-a} dz = 2\pi i f(a) = 2\pi i[4a^2 + a + 5]$$

Hence $F(a) = 2\pi i[4a^2 + a + 5]$, which implies

$$F'(a) = 2\pi i[8a + 1] \text{ and } F''(a) = 16\pi i.$$

Thus, $F(i) = 2\pi(i + 1)$, $F'(-1) = -14\pi i$ and $F''(-i) = 16\pi i$.

6. If the function $f(z)$ is analytic inside and on a simple closed curve C containing the point $z = a$ inside it, then show that

$$f^{(n)}(a) = \frac{n!}{2\pi}\int_0^{2\pi} e^{-in\theta} f(a + e^{i\theta})\,d\theta, \quad n = 0, 1, 2, ...$$

2.4 OBJECTIVE TYPE QUESTIONS

EXERCISE 2.4

Choose the correct answer or fill up the blanks in the following problems:

1. In case the path of integral C is a closed curve then the integration $\oint_c f(z)\,dz$ is called the ....

2. If $|f(z)| \le M$ every where on a curve C and l is the length of C, then $\left|\int_C f(z)\,dz\right| \le$ ....

3. If C is a circle of radius r with center at a and oriented anticlockwise, then $\oint_c \frac{dz}{z-a} =$

   (a) 2π (b) $2\pi i$ (c) πi (d) none of these.

4. $\oint_{|z|=1} ln\, z dz$, when the curve $|z| = 1$ is oriented anticlockwise, is equal to

   (a) $2\pi i$ (b) 2π (c) $-2\pi i$ (d) $-\pi i$

5. The domain $1 < |z| < 2$ is

   (a) simply connected (b) doubly connected
   (c) triply connected (d) none of these.

6. $\int_{|z|=1} \sin z dz =$

   (a) $2\pi i$ (b) 2π (c) 0 (d) none of these.

7. The contour integral $\oint_{|z|=1} \bar{z}\,dz$ can not be evaluated using Cauchy's theorem, since..............

8. If $f(z)$ is analytic in a simply connected domain D, then $\int_c f(z)\,dz$ is dependent on the path for every piecewise smooth curve C lying entirely within D.

3. (a) πi (b) i/π (c) $\pi/16$

(d) $(11e^{-1} - 4)\pi i$ (e) $-\pi(8 + i)$ (f) $\dfrac{2\pi i}{(m-1)!}\sin\left[(m-1)\dfrac{\pi}{2}\right]$

4. $-512\pi\,(1 - 2i)\cos(256)$

Exercise 2.4

1. Contour integral
2. Ml
3. b
4. a
5. b
6. c
7. $\bar{z}$ is analytic nowhere.
8. b
9. sin z is analytic everywhere
10. b
11. c
12. d
13. b
14. c
15. derivatives of all orders
16. c
17. b
18. c
19. indefinite integral
20. c