CHAPTER TWO

ELEMENTARY PROBABILITY

Why Learn Probability?

- Nothing in life is certain. In everything we do, we gauge the chances of successful outcomes, from business to medicine to the weather
- A probability provides a quantitative description of the chances or likelihoods associated with various outcomes
- We gauge the chances of successful outcomes in business, medicine, weather,

and other everyday situations such as the lottery

- It provides a bridge between descriptive and inferential statistics

Definitions of some probability terms

- **Experiment**: Any process of observation or measurement or any process which generates well defined outcome.
- **Probability Experiment**: It is an experiment that can be repeated any number of times under similar conditions and it is possible to enumerate the total number of outcomes with out predicting an individual out come. It is also called random experiment

Example: If a fair die is rolled once it is possible to list all the possible outcomes *i.e.1*, 2, 3, 4, 5, 6 but it is not possible to predict which outcome will occur.

- **Outcome** :The result of a single trial of a random experiment
- **Sample Space**: Set of all possible outcomes of a probability experiment
- **Event**: It is a subset of sample space. It is a statement about one or more outcomes of a random experiment .They are denoted by capital letters.

Example: Consider the above experiment.

Let A be the event of odd numbers,

B be the event of even numbers, and

C be the event of number 8, then

A = {1, 3, 5}, B = {2, 4, 6}, and C = { }.

Remark: If S (sample space) has n members then there are exactly 2^n subsets or events.

- **Equally Likely Events**: Events which have the same chance of occurring.
- **Complement of an Event:** the complement of an event A means non-occurrence

  of A and is denoted by $A^{'}, or\ A^{c}, or\ \overline{A}$

  contains those points of the sample space which don't belong to A.

- **Elementary Event**: an event having only a single element or sample point.
- **Mutually Exclusive Events**: Two events which cannot happen at the same time.
- **Independent Events**: Two events are independent if the occurrence of one does not affect the probability of the other occurring.
- **Dependent Events**: Two events are dependent if the first event affects the outcome or occurrence of the second event in a way the probability is changed.

Sample Space Examples

Experiment	Sample Space
• Toss a Coin, Note Face	{Head, Tail}
• Toss 2 Coins, Note Faces	{HH, HT, TH, TT}
• Select 1 Card, Note Kind	{2♥, 2♠, ..., A♦} (52)
• Select 1 Card, Note Color	{Red, Black}
• Play a Football Game	{Win, Lose, Tie}
• Inspect a Part, Note Quality	{Defective, Good}
• Observe Gender	{Male, Female}

The classical approach

This approach is used when:

- All outcomes are equally likely.
- Total number of outcome is finite, say N.

$$P(A) = \frac{N_A}{N} = \frac{No. of\ outcomes\ favourable\ to\ A}{Total\ number\ of\ \text{outcomes}} = \frac{n(A)}{n(S)}$$

Relative frequency method

- This method used for an experiment where it is not possible to apply the classical approach (usually because outcomes not equally likely or the experiment is not repeatable under uniform conditions).
- The probability of an event E is the relative frequency of occurrence of E or the proportion of times E occurs in a large number of trials of the experiment.

Example

Household appliances classified by color and style are given in the following table:

Color				
Style	**Blue (C1)**	**White (C2)**	**Green (C3)**	**Total**
S1	1400	450	900	2750
S2	1300	350	800	2450
S3	900	700	750	2350
S4	1000	250	1250	2450
Total	4600	1750	3650	10000

a) Calculate the probability of an appliance picked at random will be blue.

b) What is the probability that an appliance picked at random will be Blue, given that it is style S1?

c) What is the probability that an appliance picked at random will be Green or it is style S4?

d) Compute the joint occurrence of color C1 and style S1.

Axiomatic Approach:
Let E be a random experiment and S be a sample space associated with E. With each event A a real number called the probability of A if it satisfies the following properties called axioms of probability or postulates of probability.

$$1)\ P(S)=1,\ S\ \text{ is the sure event.}$$

$$2)\ P(\emptyset)=0$$

$$3)\ P(A\cup B)=P(A)+P(B)\ \text{where } A \text{ and } B \text{ are mutually exclusive events}$$

$$4)\ P(A\cap B)=P(A)*P(B)\ \text{where } A \text{ and } B \text{ are independent events}$$

$$5)\ P(A')=1-P(A)$$

$$6)\ 0\le P(A)\le 1$$

Conditional probability and Independency

- **Conditional Events:** If the occurrence of one event has an effect on the next occurrence of the other event then the two events are conditional or dependent events.
- The conditional probability of an event A given that B has already occurred, denoted by

$$p(A/B) = \frac{p(A \cap B)}{p(B)}, \qquad p(B) \neq 0$$

Remark:

$$p(A^{'}/B) = 1 - p(A/B)$$

$$p(B^{'}/A) = 1 - p(B/A)$$

Probability of Independent Events

Two events A and B are independent if and only if

$$p\left(A\cap B\right)=p\left(A\right).p\left(B\right)$$

Here

$$p(A/B)=p(A), \qquad P(B/A)=p(B)$$

EXAMPLES

1. If two coins are tossed once, what is the probability of getting

   a) Both heads, b) At least one head?

2. A fair die is tossed once. What is the probability of getting

   a) Number 4? b) An even number?

   b) An odd number? c) Number 8?

3. Compare the chances of throwing 4 with one dice, 8 with two dice and 12 with three dice.

4. A box of 80 candles consists of 30 defective and 50 non defective candles. If 10 of this candles are selected at random, what is the probability that

   a) All will be defective, b) 6 will be non defective, c) All will be non defective

5. A committee of 4 people is to be appointed from 3 officers of the production department, 4 officers of the purchase department, two officers of the sales department, and 1 chartered accountant. Find the probability of forming the committee in the following manner:

a) There must be one from each category.
b) The committee should have at least one from the purchase department.
c) The chartered accountant must be in the committee.

6. Suppose we have two red and three white balls in a bag

a) Draw a ball with replacement

Let A = the event that the first draw is red and

B = the event that the second draw is red, then

P(A) = p(B) = 2/5.

b) Draw a ball with out replacement

Let A= the event that the first draw is red and

B= the event that the second draw is red, then

P(A) = 2/5 and p(B) = 1/4 .

7. For a student enrolling at freshman at certain university the probability is 0.25 that he/she will get scholarship and 0.75 that he/she will graduate. If the probability is 0.2 that he/she will get scholarship and will also graduate. ***What is the probability that a student who get a scholarship given that he/she is already graduated***?

8. If the probability that a research project will be well planned is 0.60 and the probability that it will be well planned and well executed is 0.54, ***what is the probability that it will be well executed given that it is well planned?***

9. The following table contains the raw values and corresponding probability matrix for the results of a national survey of 200 executives. They were asked to identify the geographic location of their company's industry type:

Industry Type	**Geographic Location**				**Total**
	North (D)	South (E)	East (F)	West (G)	
Finance (A)	24	10	8	14	56
Manufacturing (B)	30	6	22	12	70
Communications (C)	28	18	12	16	74
Total	82	34	42	42	200

a) If a respondent is randomly selected from these data, what is the probability that this executive is from the east?
b) What is the probability that a respondent is from the communications industry or from the North?
c) What is the probability that a randomly selected respondent is from South or the West?

10. A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0? (We assume that 0 bits and 1 bits are equally likely.)

11. What is the conditional probability that a family with two children has two boys, given they have at least one boy? Assume that each of the possibilities BB, BG, GB, and GG is equally likely, where B represents a boy and G represents a girl.(Note that BG represents a family with an older boy and a younger girl while GB represents a family with an older girl and a younger boy.)

12. Suppose E is the event that a randomly generated bit string of length four begins with a 1 and F is the event that this bit string contains an even number of 1s. Are E and F independent, if the 16 bit strings of length four are equally likely?

13. A box contains four black and six white balls. What is the probability of getting two black balls in drawing one after the other under the following conditions?

a) The first ball drawn is not replaced

b) The first ball drawn is replaced

Chapter Three

Solving Linear Recurrence Relations

Introduction

Links

A wide variety of recurrence relations occur in models. Some of these recurrence relations can be solved using iteration or some other ad hoc technique. However, one important class of recurrence relations can be explicitly solved in a systematic way. These are recurrence relations that express the terms of a sequence as linear combinations of previous terms.

DEFINITION 1 A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

The recurrence relation in the definition is **linear** because the right-hand side is a sum of previous terms of the sequence each multiplied by a function of n. The recurrence relation is **homogeneous** because no terms occur that are not multiples of the a_js. The coefficients of the terms of the sequence are all **constants**, rather than functions that depend on n. The **degree** is k because a_n is expressed in terms of the previous k terms of the sequence.

A consequence of the second principle of mathematical induction is that a sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the k initial conditions

$$a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}.$$

EXAMPLE 1 The recurrence relation $P_n = (1.11)P_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two. The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five. ◄

Example 2 presents some examples of recurrence relations that are not linear homogeneous recurrence relations with constant coefficients.

EXAMPLE 2 The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear. The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous. The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients. ◄

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

THEOREM 1 Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r_1^n + \alpha_2r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

THEOREM 2 Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r_0^n + \alpha_2nr_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

THEOREM 3 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

THEOREM 4 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Linear Homogeneous Recurrence Relations

- Always has a solution of the form $\boldsymbol{a_n = r^n}$
- Plug into the recurrence and solve for r

Example 4:

- $a_n = 5a_{n-1} - 6a_{n-2}$
- Characteristic equation for $a_n = 5a_{n-1} - 6a_{n-2}$ is

  $r^2 - 5r + 6 = 0$
- Roots: $r = 2$, $r = 3$. (Case: distinct, real roots)
- Solutions: $a_n = 2^n$ and $a_n = 3^n$

- Any linear combination
  - ✓ $a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 3^n$ satisfies: $a_n = 5a_{n-1} - 6a_{n-2}$
  - ✓ $a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 3^n$ is called the **general solution**
- *Initial conditions ($a_0 = 3$ and $a_1 = 8$)* are used to solve for the constants α_1 and α_2 in

  $a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 3^n$
- Hence $a_n = 2^n + 2(3^n)$

Linear Homogeneous Recurrence Relations

1. Plug in $\boldsymbol{a_n = r^n}$ to get characteristic equation

2. Solve for roots of characteristic equation.

3. Set up general solution.

4. Use initial conditions to set up linear equations to solve for constants in general solution:

   ✓ Degree d recurrence relation -> degree d characteristic equation -> d constants (unknown coefficients) in general solution

   ✓ d initial conditions -> d equations.

Linear Homogeneous Recurrence Relations: degree 3

Example 5:1. Plug in $g_{\boldsymbol{n}} = \boldsymbol{r}^{\boldsymbol{n}}$ to get characteristic equation

$g_n = 4g_{n-1} - g_{n-2} - 6g_{n-3}$

$g_0 = 5$

$g_1 = 0$

$g_2 = 18$

Cont'd…

2. Solve for roots of characteristic equation.
3. Set up general solution.
4. Use initial conditions to set up linear equations to solve for constants in general solution: $g_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot (-1)^n$

   $g_0 = 5$

   $g_1 = 0$

   $g_2 = 18$

5. Solve linear equations for coefficients and plug back in to general solution to get the specific solution for this sequence.

Class Work

1. What is the solution of the recurrence relation $a_n = a_{n-1} + a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?
2. What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$?
3. Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 2,\ a_1 = 5,$ and $a_2 = 15.$
4. Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with the initial conditions $a_0 = 1,\ a_1 = -2,$ and $a_2 = -1.$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

- The recurrence relation $a_n = 3a_{n-1} + 2n$ is an example of a linear nonhomogeneous recurrence relation with constant coefficients, that is,
- A recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$, where $c_1, c_2, \ldots, c_k$ are real numbers and F(n) is a function not identically zero depending only on n.
- The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ is called the *associated* homogeneous recurrence relation.
- It plays an important role in the solution of the nonhomogeneous recurrence relation.

Example 9: Each of the recurrence relations

a) $a_n = a_{n-1} + 2n$,

b) $a_n = a_{n-1} + a_{n-2} + n + 1$,

c) $a_n = 3a_{n-1} + n3^n$ and

d) $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$ is a linear nonhomogeneous recurrence relation with constant coefficients.

- The associated linear homogeneous recurrence relations are

a) $a_n = a_{n-1}$,

b) $a_n = a_{n-1} + a_{n-2}$,

c) $a_n = 3a_{n-1}$, and

d) $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, respectively.

- The key fact about linear nonhomogeneous recurrence relations with constant coefficients is that *every solution is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation*, as Theorem 5 shows.

THEOREM 5

If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

EXAMPLE 10: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2\mathrm{n}$. What is the solution with $a_1 = 3$?

EXAMPLE 11: Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

THEOREM 6 Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

EXAMPLE 1: What form does a *particular solution* of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when

a) $\mathrm{F}(n) = 3^n$

b) $F(n) = n3^n$

c) $F(n) = n^2 2^n$

d) $F(n) = (n^2 + 1)3^n$?

EXAMPLE 2: Let S_n be the sum of the first n positive integers. *Find a recurrence relation for* S_n *and obtain its solution.*

EXAMPLE 3: a) Find all solutions of the recurrence relation

$$a_n = -5a_{n-1} - 6a_{n-2} + 42(4^n).$$

b) Find the solution of this recurrence relation with

$$a_0 = 56 \text{ and } a_1 = 278$$

EXAMPLE 4: Find all solutions of the recurrence relation

$$a_n = 2a_{n-1} + 3 \cdot 2^n.$$

Exercises

1. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

   a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$
   b) $a_n = 2na_{n-1} + a_{n-2}$
   c) $a_n = a_{n-1} + a_{n-4}$
   d) $a_n = a_{n-1} + 2$
   e) $a_n = a_{n-1}^2 + a_{n-2}$
   f) $a_n = a_{n-2}$
   g) $a_n = a_{n-1} + n$

2. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

   a) $a_n = 3a_{n-2}$
   b) $a_n = 3$
   c) $a_n = a_{n-1}^2$
   d) $a_n = a_{n-1} + 2a_{n-3}$
   e) $a_n = a_{n-1}/n$
   f) $a_n = a_{n-1} + a_{n-2} + n + 3$
   g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$

3. Solve these recurrence relations together with the initial conditions given.

a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$
b) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$
c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$
d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$
e) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$
f) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$
g) $a_n = a_{n-2}/4$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

4. Solve these recurrence relations together with the initial conditions given.

a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$
b) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$
c) $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 10$
d) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 1$
e) $a_n = a_{n-2}$ for $n \geq 2$, $a_0 = 5$, $a_1 = -1$
f) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = -3$
g) $a_{n+2} = -4a_{n+1} + 5a_n$ for $n \geq 0$, $a_0 = 2$, $a_1 = 8$

5. Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

a) Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.

b) Use Theorem 5 to find all solutions of this recurrence relation.

c) Find the solution with $a_0 = 1$.

6. Consider the nonhomogeneous linear recurrence relation $a_n = 2a_{n-1} + 2^n$.

a) Show that $a_n = n2^n$ is a solution of this recurrence relation.

b) Use Theorem 5 to find all solutions of this recurrence relation.

c) Find the solution with $a_0 = 2$.

7. a) Determine values of the constants A and B such that $a_n = An + B$ is a solution of recurrence relation $a_n = 2a_{n-1} + n + 5$.
   b) Use Theorem 5 to find all solutions of this recurrence relation.
   c) Find the solution of this recurrence relation with $a_0 = 4$.

8. What is the general form of the particular solution guaranteed to exist by Theorem 5 of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n)$ if
   a) $F(n) = n^2$?
   b) $F(n) = 2^n$?
   c) $F(n) = n2^n$?
   d) $F(n) = (-2)^n$?
   e) $F(n) = n^2 2^n$?
   f) $F(n) = n^3(-2)^n$?
   g) $F(n) = 3$?

GENERATING FUNCTION

DEFINITION 1 The *generating function for the sequence* $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

EXAMPLE 1 The generating functions for the sequences $\{a_k\}$ with $a_k = 3, a_k = k+1$, and $a_k = 2^k$ are $\sum_{k=0}^{\infty} 3x^k$, $\sum_{k=0}^{\infty}(k+1)x^k$, and $\sum_{k=0}^{\infty} 2^k x^k$, respectively. ◀

We can define generating functions for finite sequences of real numbers by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0$, $a_{n+2} = 0$, and so on. The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n because no terms of the form $a_j x^j$ with $j > n$ occur, that is,

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

EXAMPLE 2 What is the generating function for the sequence 1, 1, 1, 1, 1, 1?

Solution: The generating function of 1, 1, 1, 1, 1, 1 is

$$1 + x + x^2 + x^3 + x^4 + x^5 = (x^6 - 1)/(x - 1)$$

when $x \neq 1$. Consequently, $G(x) = (x^6 - 1)/(x - 1)$ is the generating function of the sequence 1, 1, 1, 1, 1, 1. [Because the powers of x are only place holders for the terms of the sequence in a generating function, we do not need to worry that $G(1)$ is undefined.] ◀

EXAMPLE 3 Let m be a positive integer. Let $a_k = C(m, k)$, for $k = 0, 1, 2, \ldots, m$. What is the generating function for the sequence $a_0, a_1, \ldots, a_m$?

Solution: The generating function for this sequence is

$$G(x) = C(m, 0) + C(m, 1)x + C(m, 2)x^2 + \cdots + C(m, m)x^m.$$

The binomial theorem shows that $G(x) = (1 + x)^m$. ◀

Using Generating Functions to Solve Recurrence Relations

We can find the solution to a recurrence relation and its initial conditions by finding an explicit formula for the associated generating function. This is illustrated in Examples 16 and 17.

EXAMPLE 4 Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \ldots$ and initial condition $a_0 = 2$.

Solution: Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_k x^k$. First note that

$$xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k.$$

Using the recurrence relation, we see that

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k \\ &= 2, \end{aligned}$$

because $a_0 = 2$ and $a_k = 3a_{k-1}$. Thus,

$$G(x) - 3xG(x) = (1 - 3x)G(x) = 2.$$

Solving for $G(x)$ shows that $G(x) = 2/(1-3x)$. Using the identity $1/(1-ax) = \sum_{k=0}^{\infty} a^k x^k$, from Table 1, we have

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k.$$

Consequently, $a_k = 2 \cdot 3^k$. ◀

EXAMPLE 6 $a_n = 8a_{n-1} + 10^{n-1}$

and the initial condition $a_1 = 9$. Use generating functions to find an explicit formula for a_n.

Solution: We multiply both sides of the recurrence relation by x^n to obtain

$$a_n x^n = 8a_{n-1}x^n + 10^{n-1}x^n.$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $a_0, a_1, a_2, \ldots$. We sum both sides of the last equation starting with $n = 1$, to find that

$$\begin{aligned}
G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n) \\
&= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n \\
&= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\
&= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n \\
&= 8xG(x) + x/(1 - 10x),
\end{aligned}$$

$$G(x) - 1 = 8xG(x) + x/(1 - 10x).$$

Solving for $G(x)$ shows that

$$G(x) = \frac{1 - 9x}{(1 - 8x)(1 - 10x)}.$$

Expanding the right-hand side of this equation into partial fractions (as is done in the integration of rational functions studied in calculus) gives

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right).$$

Using Example 5 twice (once with $a = 8$ and once with $a = 10$) gives

$$\begin{aligned} G(x) &= \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n. \end{aligned}$$

Consequently, we have shown that

$$a_n = \frac{1}{2}(8^n + 10^n).$$

TABLE 1 **Useful Generating Functions.**

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k

$$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$$	1 if $r \mid k$; 0 otherwise
$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$$	$k+1$
$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1, k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$$	$C(n+k-1,k) = C(n+k-1,n-1)$
$$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1, k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1, k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$	$1/k!$
$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$	$(-1)^{k+1}/k$

CLASS WORK

Use generating functions to solve the recurrence relation $a_k = a_{k-1} + 2a_{k-2} + 2^k$ with initial conditions $a_0 = 4$ and $a_1 = 12$.

Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$ (by changing the name of the variable from k to $k+1$), and $x^2 G(x) = \sum_{k=0}^{\infty} a_k x^{k+2} = \sum_{k=2}^{\infty} a_{k-2} x^k$. Thus

$$\begin{aligned} G(x) - xG(x) - 2x^2 G(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} a_{k-1} x^k - \sum_{k=2}^{\infty} 2a_{k-2} x^k = a_0 + a_1 x - a_0 x + \sum_{k=2}^{\infty} 2^k \cdot x^k \\ &= 4 + 8x + \frac{1}{1-2x} - 1 - 2x = \frac{4 - 12x^2}{1-2x}, \end{aligned}$$

because of the given recurrence relation, the initial conditions, Table 1, and algebra. Since the left-hand side of this equation factors as $G(x)(1-2x)(1+x)$, we have $G(x) = (4-12x^2)/((1+x)(1-2x)^2)$. At this point we must use partial fractions to break up the denominator. Setting

$$\frac{4-12x^2}{(1+x)(1-2x)^2} = \frac{A}{1+x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2},$$

multiplying through by the common denominator, and equating coefficients, we find that $A = -8/9$, $B = 38/9$, and $C = 2/3$. Thus

$$G(x) = \frac{-8/9}{1+x} + \frac{38/9}{1-2x} + \frac{2/3}{(1-2x)^2} = \sum_{k=0}^{\infty} \left(-\frac{8}{9}(-1)^k + \frac{38}{9} \cdot 2^k + \frac{2}{3}(k+1)2^k \right) x^k$$

(from Table 1). Therefore $a_k = (-8/9)(-1)^k + (38/9)2^k + (2/3)(k+1)2^k$.

Exercises

1. Find the generating function for the finite sequence 2, 2, 2, 2, 2, 2.
2. Find the generating function for the finite sequence 1, 4, 16, 64, 256.

3. Find a closed form for the generating function for each of these sequences. (Assume a general form for the terms of the sequence, using the most obvious choice of such a sequence.)

a) $-1, -1, -1, -1, -1, -1, -1, 0, 0, 0, 0, 0, 0, \ldots$

b) $1, 3, 9, 27, 81, 243, 729, \ldots$

c) $0, 0, 3, -3, 3, -3, 3, -3, \ldots$

d) $1, 2, 1, 1, 1, 1, 1, 1, 1, \ldots$

e) $\binom{7}{0}, 2\binom{7}{1}, 2^2\binom{7}{2}, \ldots, 2^7\binom{7}{7}, 0, 0, 0, 0, \ldots$

f) $-3, 3, -3, 3, -3, 3, \ldots$

g) $0, 1, -2, 4, -8, 16, -32, 64, \ldots$

h) $1, 0, 1, 0, 1, 0, 1, 0, \ldots$

4. Use generating functions to solve the recurrence relation $a_k = 7a_{k-1}$ with the initial condition $a_0 = 5$.

5. Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$.

6. Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 4^{k-1}$ with the initial condition $a_0 = 1$.

7. Use generating functions to solve the recurrence relation $a_k = 5a_{k-1} - 6a_{k-2}$ with initial conditions $a_0 = 6$ and $a_1 = 30$.

8. Use generating functions to solve the recurrence relation $a_k = 4a_{k-1} - 4a_{k-2} + k^2$ with initial conditions $a_0 = 2$ and $a_1 = 5$.