Graph Theory

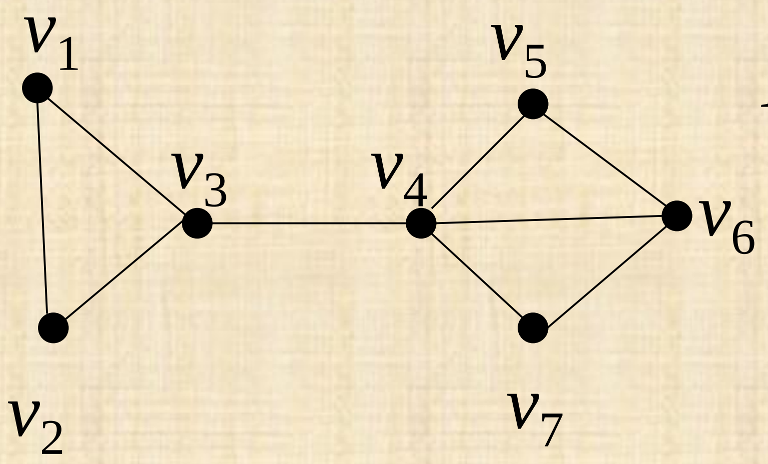
Outline

- Graph and Graph Models
- Graph Terminology and Special Types of Graphs
- Representing Graphs and Graph Isomorphism
- Connectivity
- Euler and Hamiltonian Paths
- Shortest-Path Problems
- Planar Graphs
- Graph Coloring

Introduction to Graphs

Def 1. A graph $\boldsymbol{G = (V, E)}$ consists of $\boldsymbol{V}$, a nonempty set of vertices (or nodes), and $\boldsymbol{E}$, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

eg.



$G=(V, E)$, where

$V=\{v_1,v_2,\ldots, v_7\}$

$E=\{\{v_1,v_2\}, \{v_1,v_3\}, \{v_2,v_3\}$
$\{v_3,v_4\}, \{v_4,v_5\}, \{v_4,v_6\}$
$\{v_4,v_7\}, \{v_5,v_6\}, \{v_6,v_7\}\}$

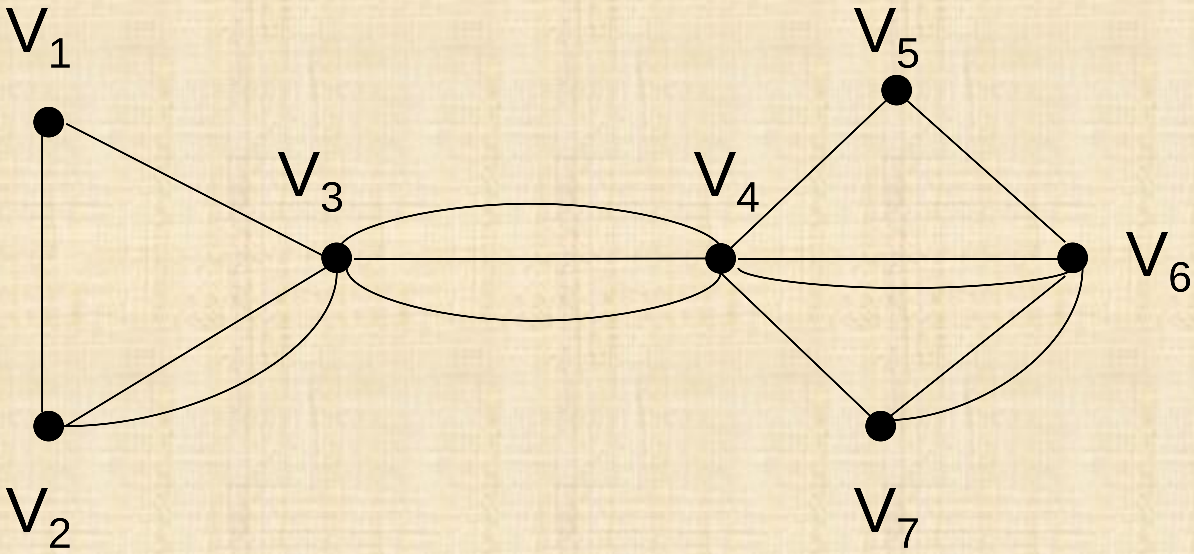
Def A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

Def Multigraph:

simple graph + multiple edges (multiedges)

(Between two points to allow multiple edges)

eg.



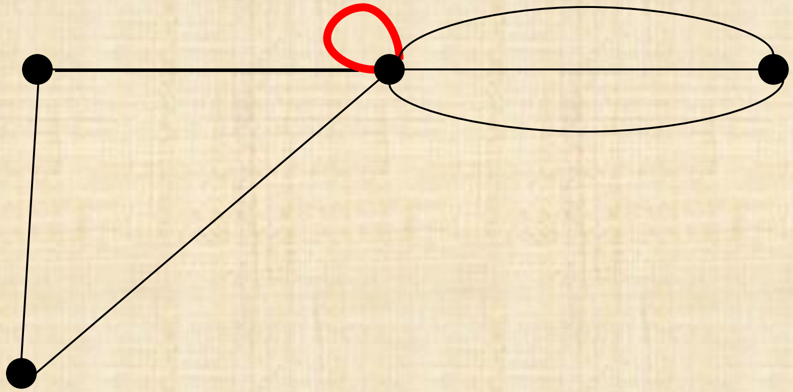
Def. Pseudograph:

simple graph + multiedge

+ loop

(a loop: )

eg.



Def 2. Directed graph (digraph):

simple graph with each edge directed

Note:  is allowed in a directed graph

Note:

The two edges (u,v),(u,v) are multiedges.

The two edges (u,v), (v,u) are not multiedges.

Def. Directed multigraph: digraph+multiedges

Table 1. Graph Terminology

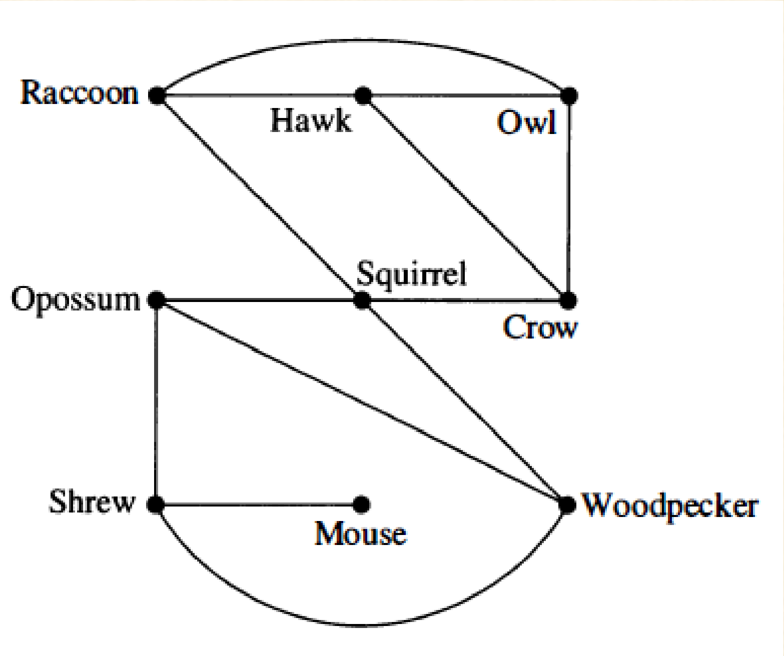
Type	Edges	Multiple Edges	Loops
(simple) graph	undirected edge: {u,v}	✗	✗
Multigraph		✓	✗
Pseudograph		✓	✓
Directed graph	directed edge: (u,v)	✗	✓
Directed multigraph		✓	✓

Graph Models

Example 1. (Niche Overlap graph)

We can use a simple graph to represent *interaction of different species of animals*. Each animal is represented by a *vertex*. An *undirected edge* connects two vertices if the two species represented by these vertices *compete*.

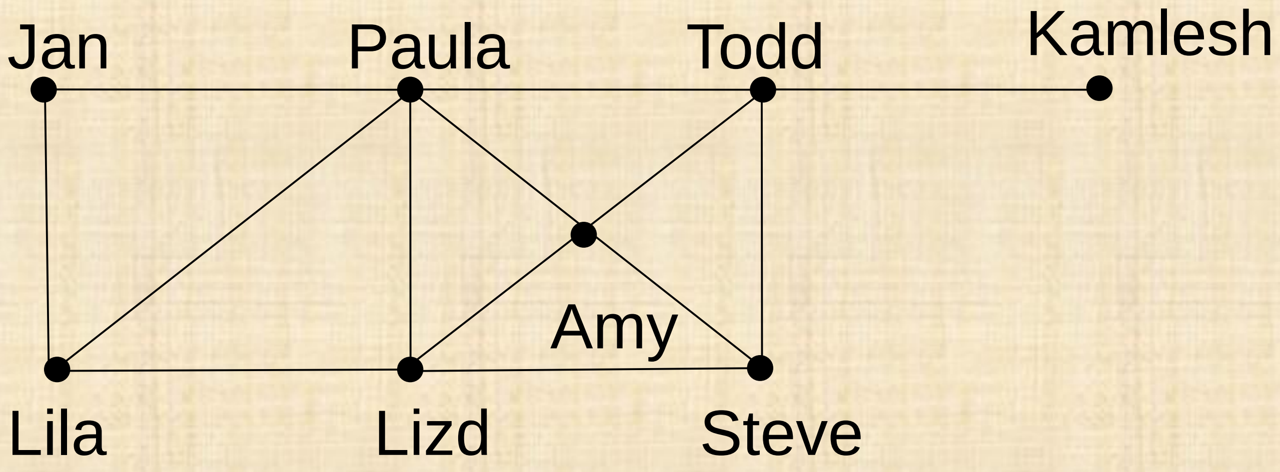
eg



Example 2. (Acquaintanceship graphs)

We can use a simple graph to represent whether two people *know* each other. Each *person* is represented by a *vertex*. An undirected *edge* is used to connect two people when these people *know* each other.

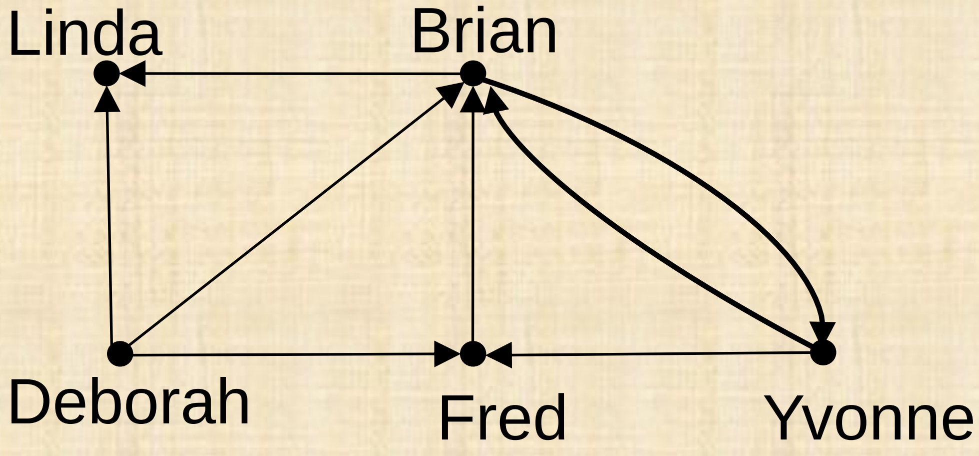
eg



Example 3. (Influence graphs)

In studies of *group behavior* it is observed that certain people can influence the thinking of others. Simple digraph $\Rightarrow$ Each *person* of the group is represented by a *vertex*. There is a *directed edge* from vertex a to vertex b when the person a *influences* the person b.

eg



Example 9. (Precedence graphs and concurrent processing)

Computer programs can be executed more rapidly by executing certain statements *concurrently*. It is important not to execute a statement that requires results of statements not yet executed.

Simple digraph $\Rightarrow$ Each statement is represented by a vertex, and there is an edge from a to b if the statement of b cannot be executed before the statement of a.

Eg.

S_1: a:=0

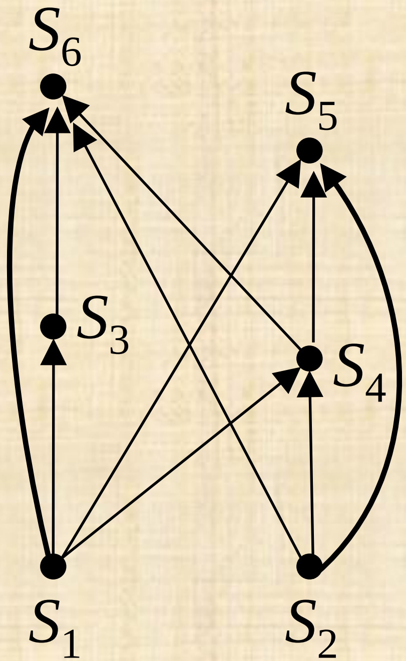
S_2: b:=1

S_3: c:=a+1

S_4: d:=b+a

S_5: e:=d+1

S_6: e:=c+d

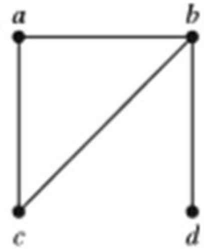


Exercises

1. Draw graph models, stating the type of graph (from Table 1) used, to represent airline routes where every day there are four flights from Boston to Newark, two flights from Newark to Boston, three flights from Newark to Miami, two flights from Miami to Newark, one flight from Newark to Detroit, two flights from Detroit to Newark, three flights from Newark to Washington, two flights from Washington to Newark, and one flight from Washington to Miami, with
   - **a)** an edge between vertices representing cities that have a flight between them (in either direction).
   - **b)** an edge between vertices representing cities for each flight that operates between them (in either direction).
   - **c)** an edge between vertices representing cities for each flight that operates between them (in either direction), plus a loop for a special sightseeing trip that takes off and lands in Miami.
   - **d)** an edge from a vertex representing a city where a flight starts to the vertex representing the city where it ends.
   - **e)** an edge for each flight from a vertex representing a city where the flight begins to the vertex representing the city where the flight ends.
2. What kind of graph (from Table 1) can be used to model a highway system between major cities where
   - **a)** there is an edge between the vertices representing cities if there is an interstate highway between them?
   - **b)** there is an edge between the vertices representing cities for each interstate highway between them?
   - **c)** there is an edge between the vertices representing cities for each interstate highway between them, and there is a loop at the vertex representing a city if there is an interstate highway that circles this city?

For Exercises 3–9, determine whether the graph shown has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops. Use your answers to determine the type of graph in Table 1 this graph is.

3.



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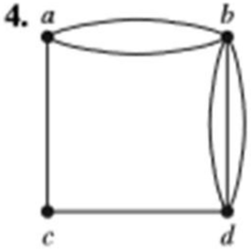

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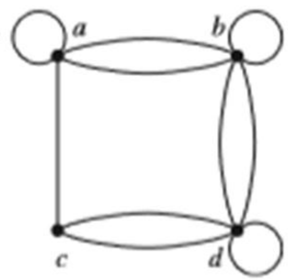

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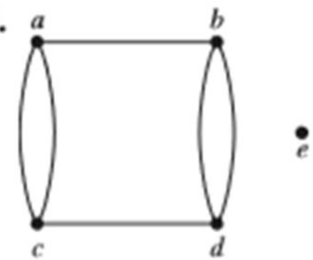

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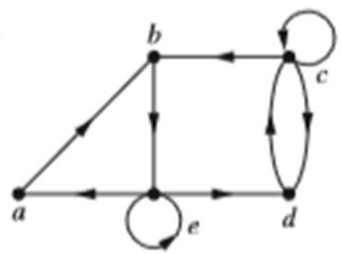
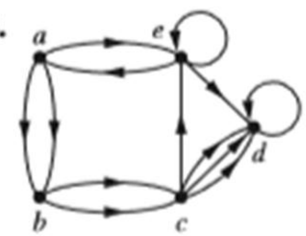
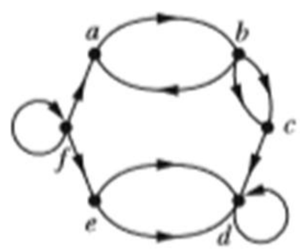


8.



9.



10. Construct an influence graph for the board members of a company if the President can influence the Director of Research and Development, the Director of Marketing, and the Director of Operations; the Director of Research and Development can influence the Director of Operations; the Director of Marketing can influence the Director of Operations; and no one can influence, or be influenced by, the Chief Financial Officer.

11. Construct the call graph for a set of seven telephone numbers 555-0011, 555-1221, 555-1333, 555-8888, 555-2222, 555-0091, and 555-1200 if there were three calls from 555-0011 to 555-8888 and two calls from 555-8888 to 555-0011, two calls from 555-2222 to 555-0091, two calls from 555-1221 to each of the other numbers, and one call from 555-1333 to each of 555-0011, 555-1221, and 555-1200.

12. Construct a precedence graph for the following program:

```
S1: x := 0
S2: x := x + 1
S3: y := 2
S4: z := y
S5: x := x + 2
S6: y := x + z
S7: z := 4
```

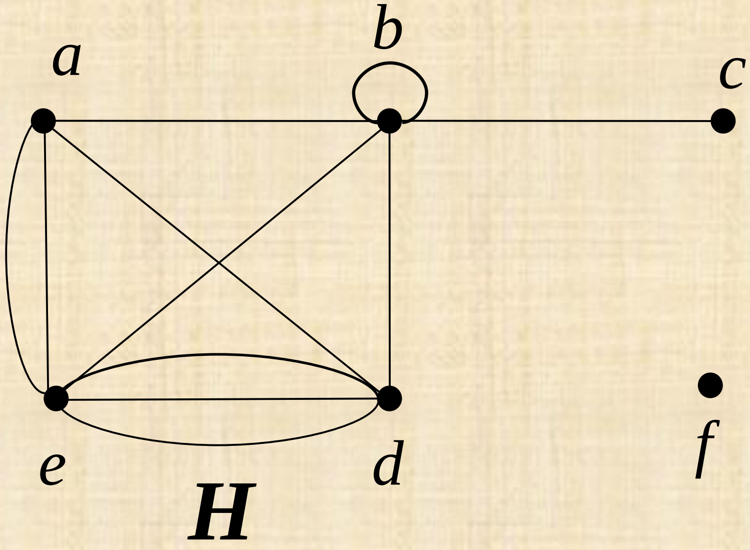
Graph Terminology

Def 1. Two vertices u and v in a undirected graph G are called adjacent (or neighbors) in G if $\{u, v\}$ is an edge of G.

Def 2. The degree of a vertex v, denoted by deg(v), in an undirected graph is the number of edges incident with it.

(*Note : A loop adds 2 to the degree*.)

Example 1. What are the degrees of the vertices in the graph ***H*** ?



Solution :

deg(a)=4, deg(b)=6, ***deg(c)=1***, deg(d)=5, deg(e)=6, and ***deg(f)=0***

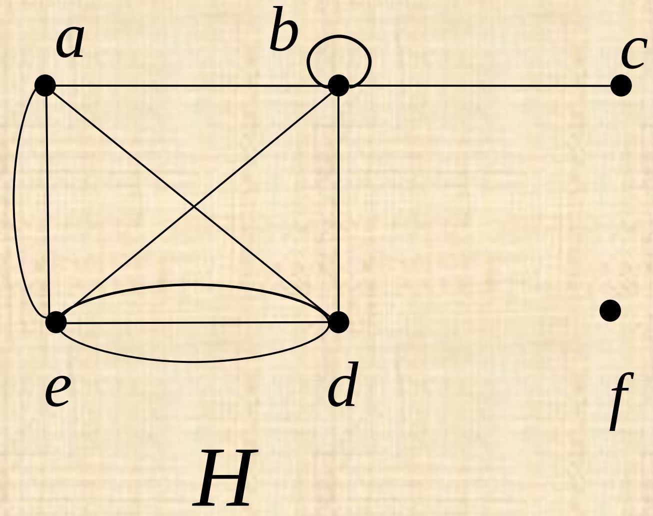
Def. A vertex of degree 0 is called ***isolated***.

Def. A vertex is ***pendant*** if and only if it has degree one.

Theorem 1. (*The Handshaking Theorem*)
Let $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ be an undirected graph with e edges (i.e., $|\boldsymbol{E}| = \boldsymbol{e}$). Then

☞ 1. Because of the analogy between an edge having two endpoints and a handshake involving two hands. 2. Each edge contributes two to the sum of the degrees of the vertices because an edge is incident with exactly two (possibly equal) vertices. This means that the sum of the degrees of the vertices is twice the number of edges.

eg.



The graph H has 11 edges, and

$$\sum_{v \in V} \deg(v) = 22$$

Example 3. How many edges are there in a graph with 10 vertices each of degree six?

Solution : $10 \cdot 6 = 2e \quad \Rightarrow \quad e = 30$

THEOREM 2: An undirected graph has an even number of vertices of odd degree.

Proof: Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph G = (V, E) with m edges.

Then

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$$

$$\Rightarrow \sum_{v \in V_2} \deg(v) \text{ is even.}$$

Thus, there are an even number of vertices of odd degree.

Definition 3:

$G = (V, E)$: directed graph,

$e = (u, v) \in E$: u is adjacent to v

v is adjacent from u

u : initial vertex of e

v : terminal (end) vertex of e

The initial vertex and terminal vertex of a loop are the same

Def 4.

$G = (V, E)$: directed graph, $v \in V$

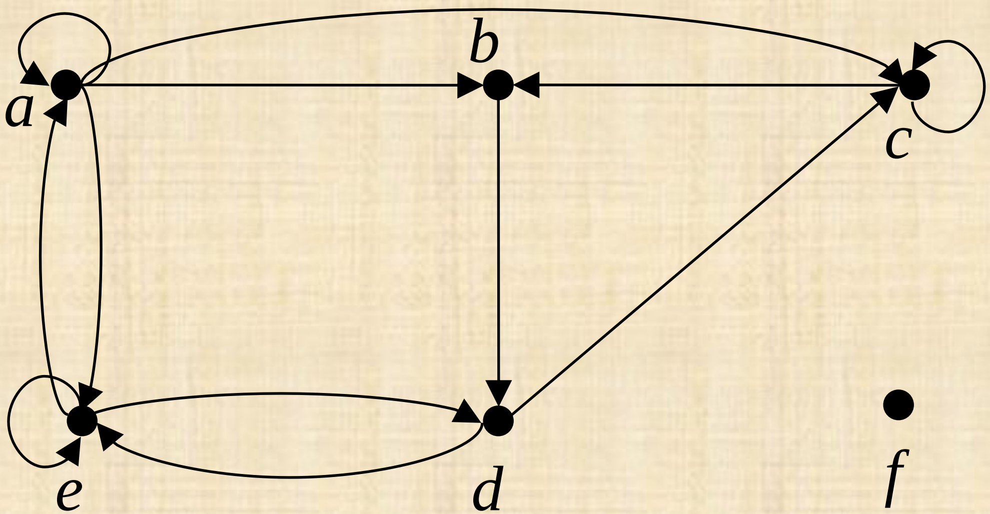
$\deg^-(v)$: # of edges with v as a terminal.

(in-degree)

$\deg^+(v)$: # of edges with v as a initial vertex

(out-degree)

Example 4.



$\deg^-(a)=2$, $\deg^+(a)=4$

$\deg^-(b)=2$, $\deg^+(b)=1$

$\deg^-(c)=3$, $\deg^+(c)=2$

$\deg^-(d)=2$, $\deg^+(d)=2$

$\deg^-(e)=3$, $\deg^+(e)=3$

$\deg^-(f)=0$, $\deg^+(f)=0$

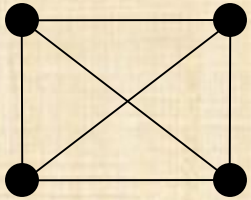
Thm 3. Let $G = (V, E)$ be a digraph. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$

Regular Graph

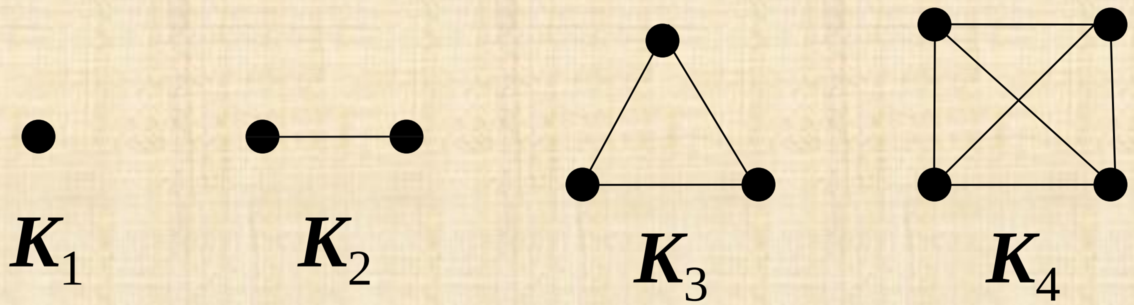
A simple graph $G=(V, E)$ is called regular if every vertex of this graph has the same degree. A regular graph is called n-regular if deg(v)=n , $\forall v \in V$.

eg. K_4 is 3-regular.



Some Special Simple Graphs

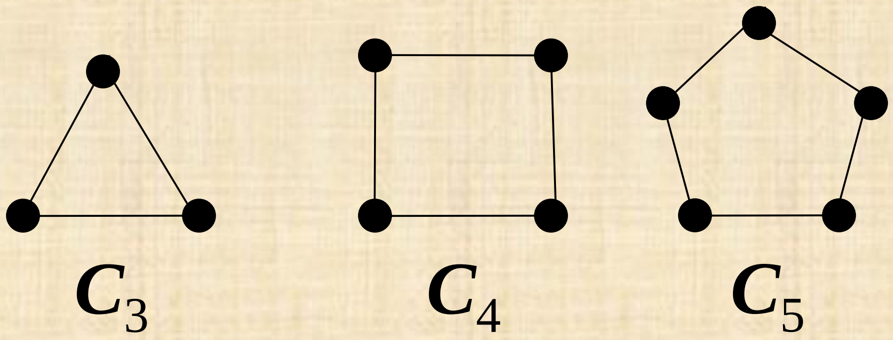
Example 5: The complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



Note. K_n is $(n–1)$-regular, $|V(K_n)|=n$,

$$|E(K_n)| = \binom{n}{2}$$

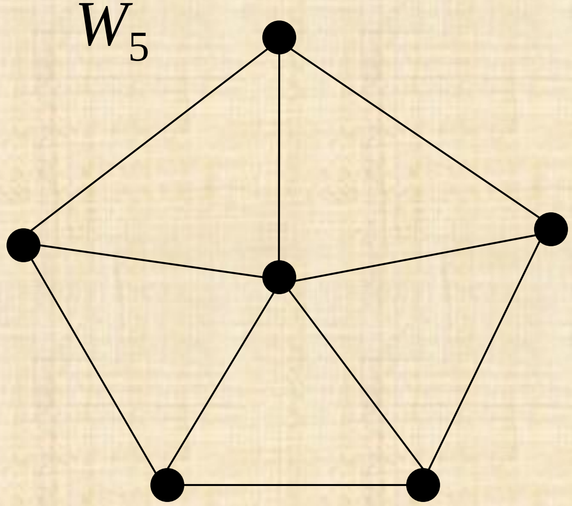
Example 6. The cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1,v_2\}$, $\{v_2,v_3\}, \ldots, \{v_{n-1},v_n\}, \{v_n,v_1\}$.



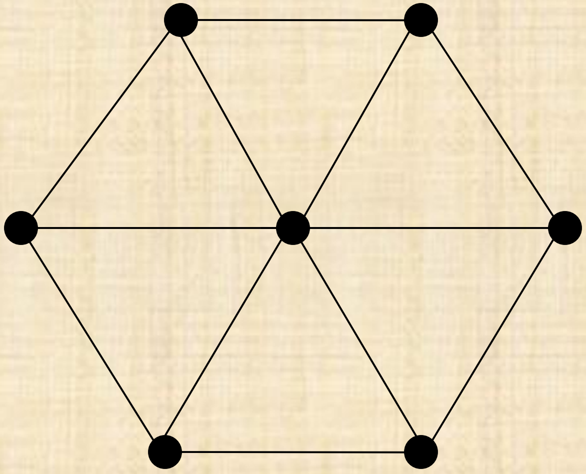
Note C_n is 2-regular, $|V(C_n)| = n$, $|E(C_n)| = n$

Example 7. We obtained the **wheel** W_n when we add an *additional vertex* to the cycle C_n for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.

W_5



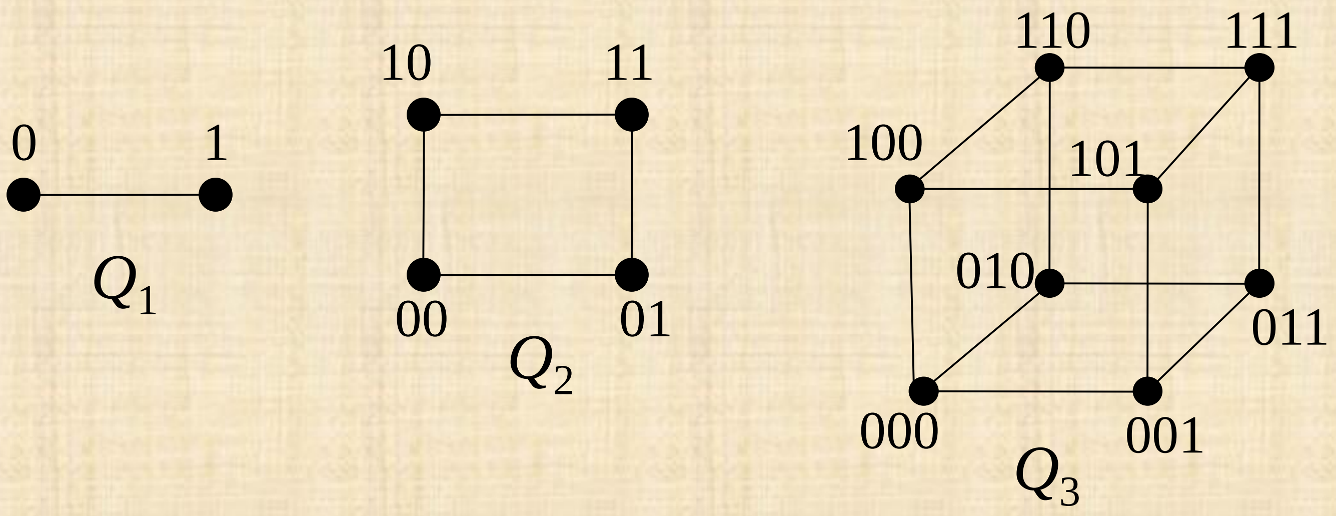
W_6



Note. $|V(W_n)| = n + 1$, $|E(W_n)| = 2n$,

W_n is not a regular graph if $n \neq 3$.

Example 8. The n-dimensional hypercube, or n-cube, denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

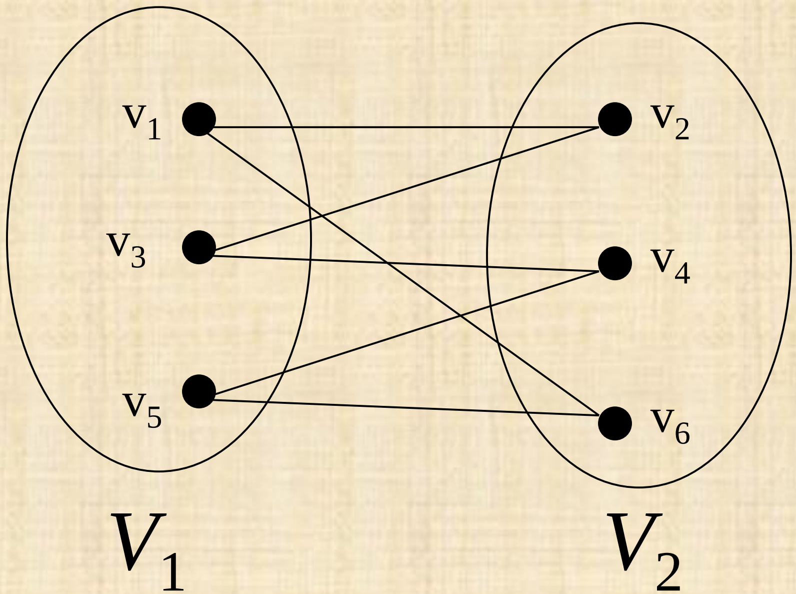


Note. Q_n is n-regular, $|V(Q_n)| = 2^n$, $|E(Q_n)| = (2^n n)/2 = 2^{n-1}n$

Some Special Simple Graphs

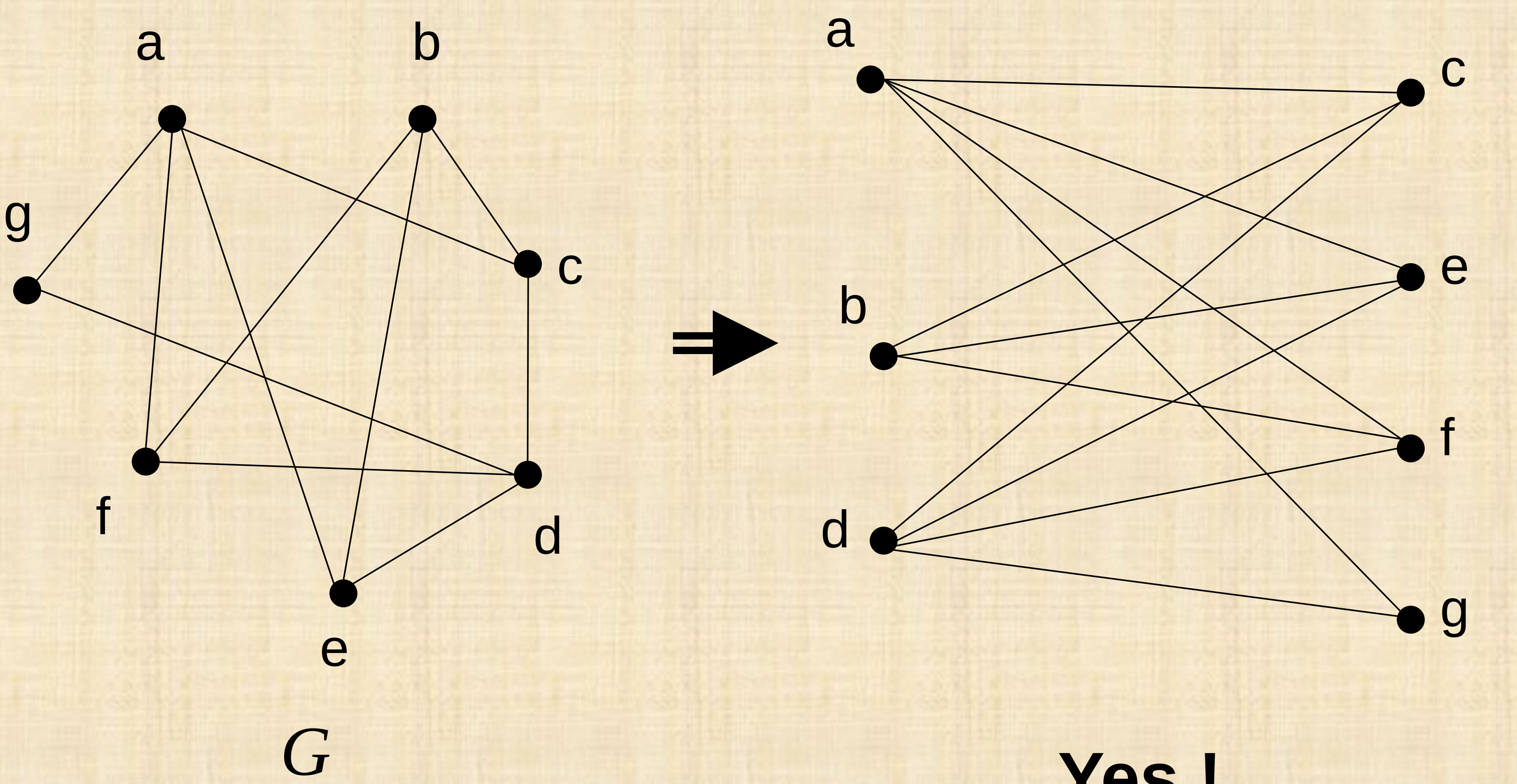
Def 5. A simple graph $G=(V, E)$ is called bipartite if V can be partitioned into V_1 and V_2, $V_1 \cap V_2 = \varnothing$, such that every edge in the graph connect a vertex in V_1 and a vertex in V_2.

Example 9.



$\therefore C_6$ is bipartite.

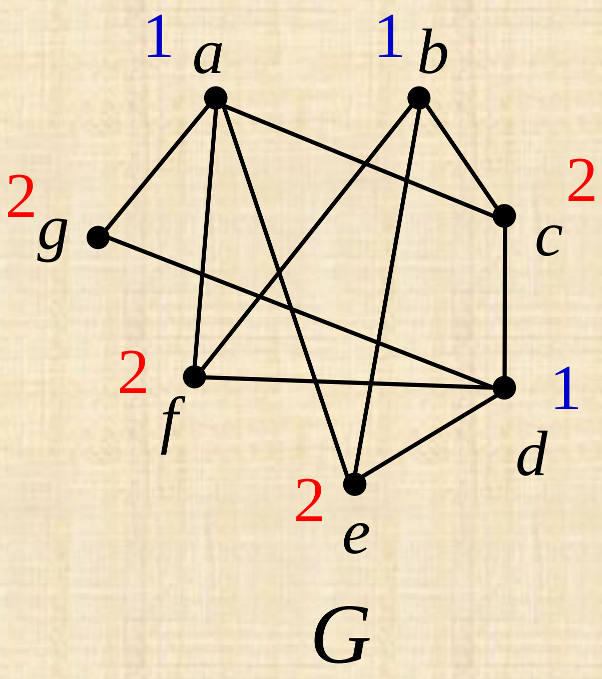
Example 10. Is the graph G bipartite ?



Yes !

Thm 4. A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

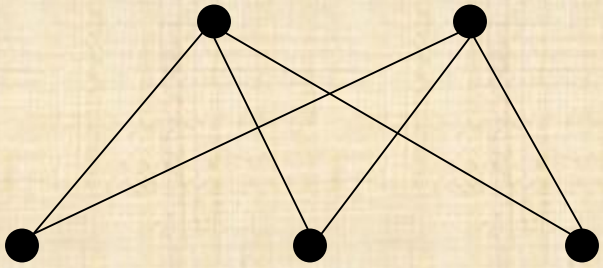
Example 12. Use Thm 4 to show that G is bipartite.



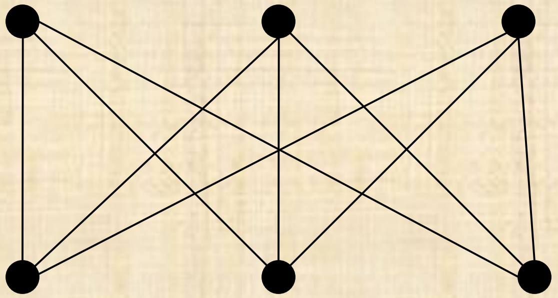
Definition: Complete Bipartite Graphs

A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into ***two subsets*** of m and n vertices, respectively with an ***edge between*** two vertices *if and only if* one vertex is in the first subset and the other vertex is in the second subset.

Example 11: Complete Bipartite graphs ($K_{m,n}$)



$K_{2,3}$



$K_{3,3}$

Note: $|V(K_{m,n})| = m+n$, $|E(K_{m,n})| = mn$,

$K_{m,n}$ is regular if and only if $m=n$.

EXERCISE

For which values of n are these graphs bipartite?

a) K_n

b) C_n

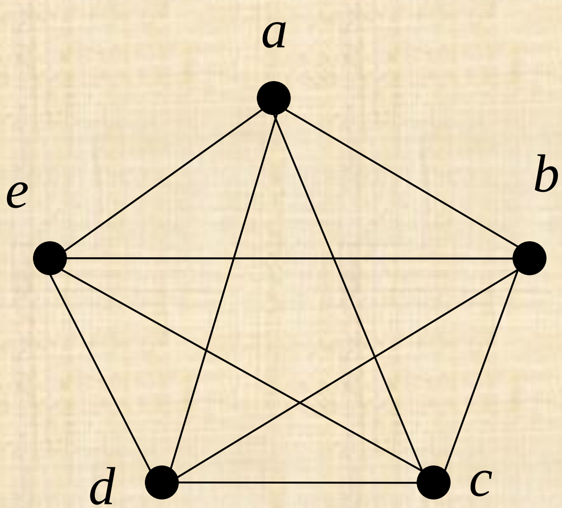
c) W_n

d) Q_n

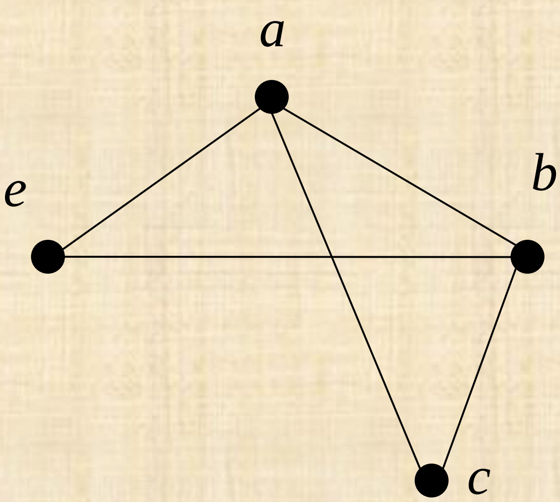
New Graphs from Old

Def 6. A subgraph of a graph $G=(V, E)$ is a graph $H=(W, F)$ where $W \subseteq V$ and $F \subseteq E$.

Example 14. A subgraph of K_5



K_5

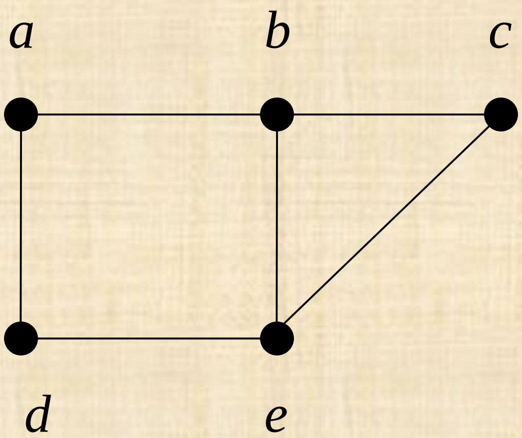


subgraph of K_5

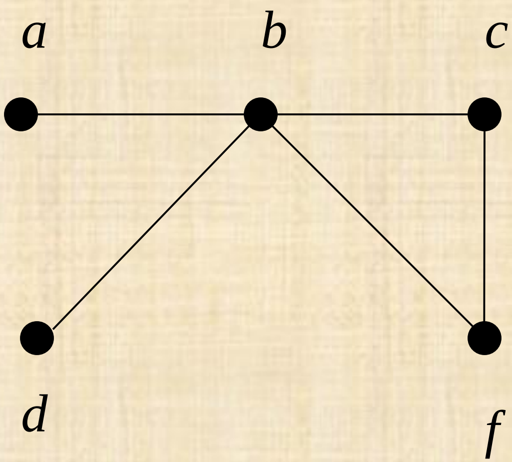
Def 7. The union of two simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ is the simple graph $G_1 \cup G_2=(V_1 \cup V_2, E_1 \cup E_2)$

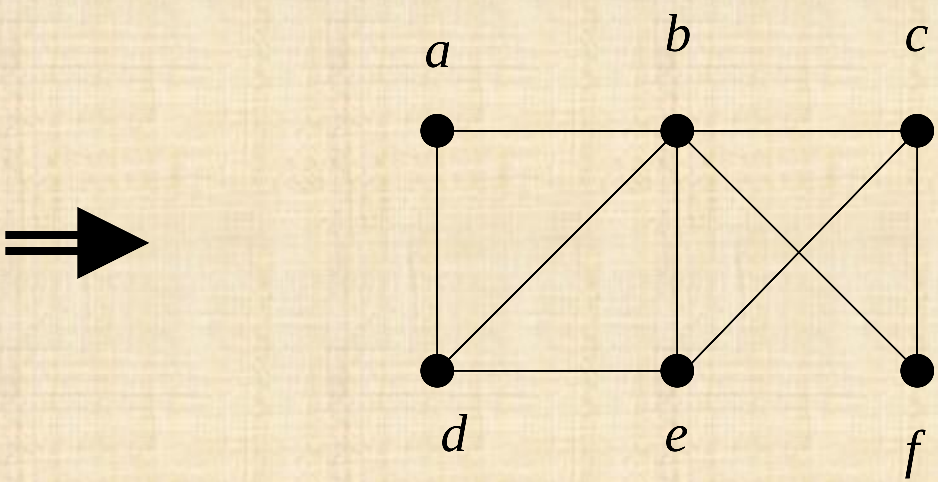
Example 15.

G_1



G_2



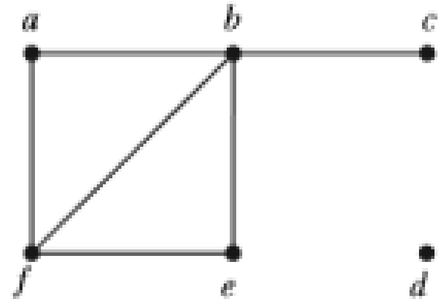


$G_1 \cup G_2$

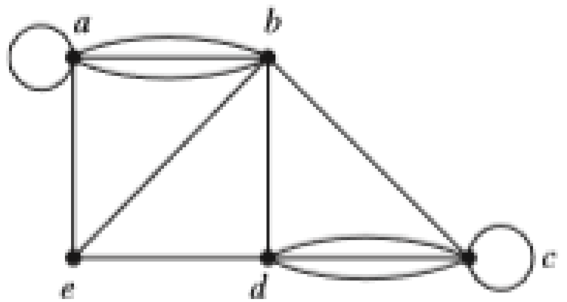
Exercises

In Exercises 1–3 find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.

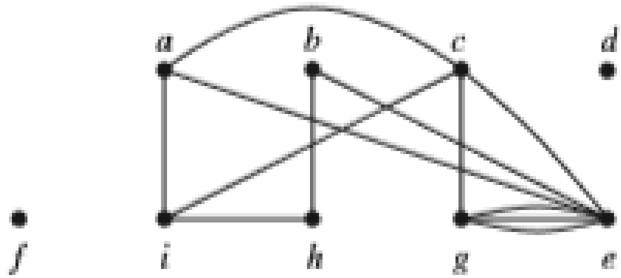
1.



2.



3.



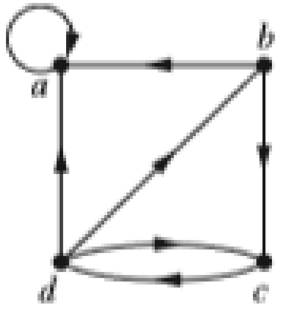
4. Find the sum of the degrees of the vertices of each graph in Exercises 1–3 and verify that it equals twice the number of edges in the graph.

5. Can a simple graph exist with 15 vertices each of degree five?

6. Show that the sum, over the set of people at a party, of the number of people a person has shaken hands with, is even. Assume that no one shakes his or her own hand.

In Exercises 7–9 determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.

7.



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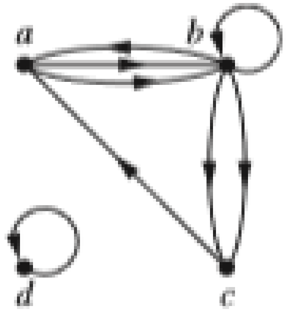

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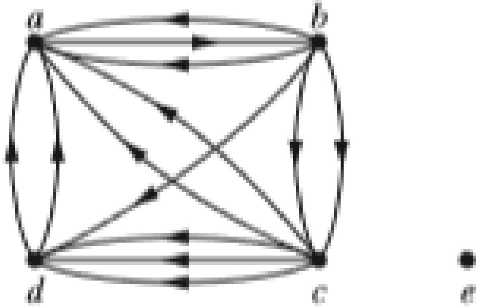
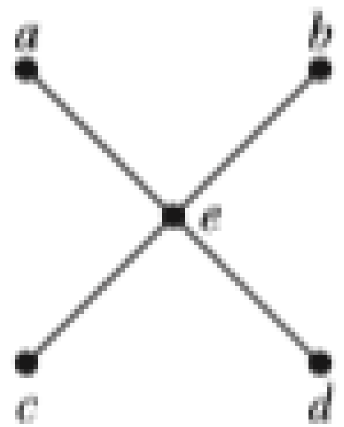


10. Draw these graphs.

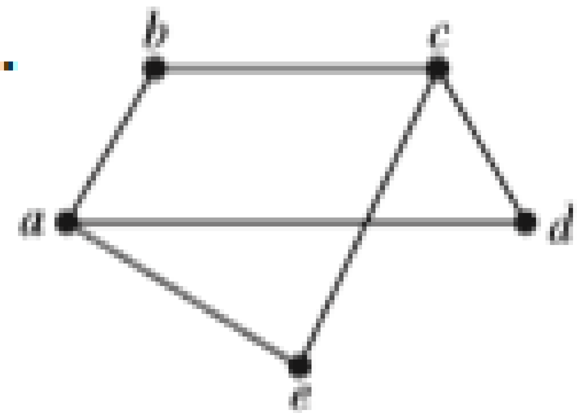
a) K_7 b) $K_{1,8}$ c) $K_{4,4}$
d) C_7 e) W_7 f) Q_4

In Exercises 11-15 determine whether the graph is bipartite. You may find it useful to apply Theorem 4 and answer the question by determining whether it is possible to assign either red or blue to each vertex so that no two adjacent vertices are assigned the same color.

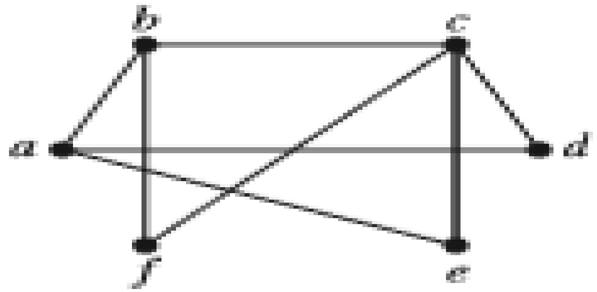
11.



12.



13.



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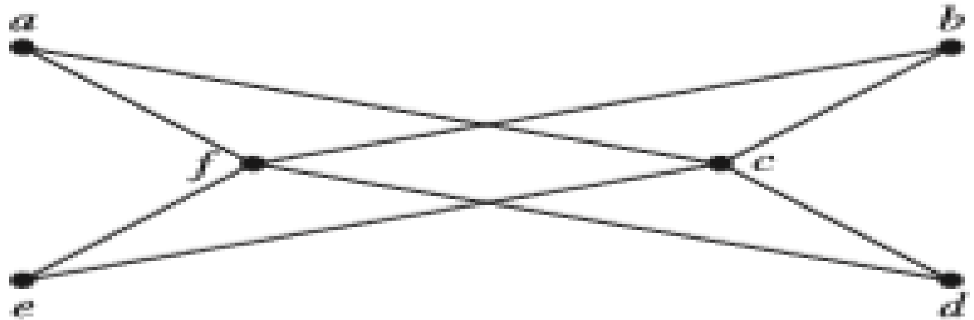
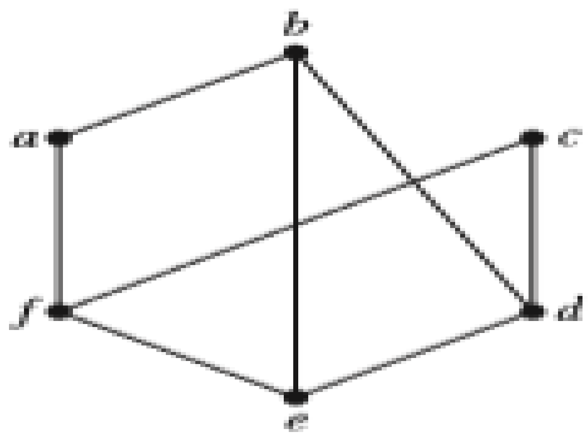
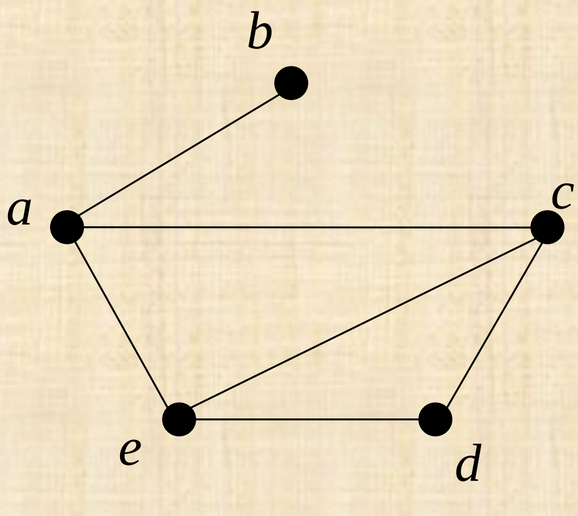


15.



Adjacency and Terminal vertices

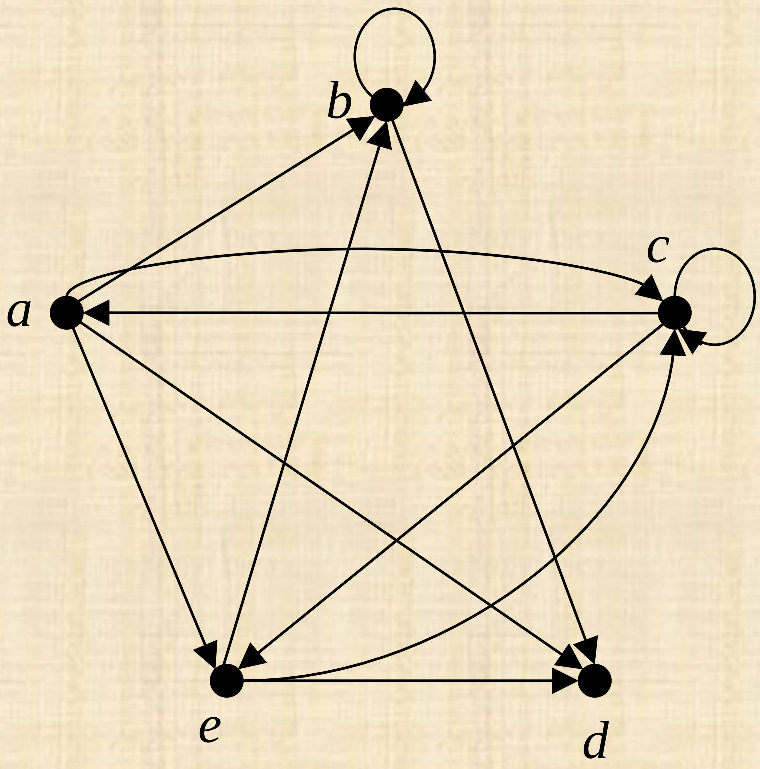
Example 1. Use adjacency lists to describe the simple graph given below.



Sol :

Vertex	Adjacent Vertices
a	*b,c,e*
b	*a*
c	*a,d,e*
d	*c,e*
e	*a,c,d*

Example 2. (digraph)



Initial vertex	Terminal vertices
a	*b,c,d,e*
b	*b,d*
c	*a,c,e*
d	
e	*b,c,d*

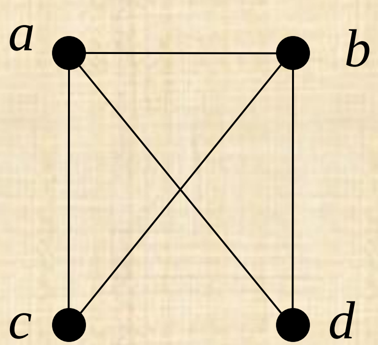
Adjacency Matrices

Def. $G=(V, E)$: simple graph, $V=\{v_1,v_2,\ldots,v_n\}$.

A matrix A is called the adjacency matrix of G

if $A=[a_{ij}]_{n\times n}$, where $a_{ij} = \begin{cases} 1, \text{ if } \{v_i,v_j\}\in E, \\ 0, \text{ otherwise.} \end{cases}$

Example 3.

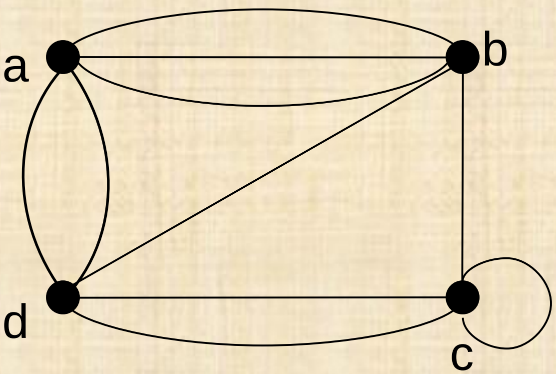


$$A_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ b \\ d \\ c \\ a \end{array} \begin{array}{c} \begin{array}{cccc} b & d & c & a \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Note:

1. There are $n!$ different adjacency matrices for a graph with n vertices.
2. The adjacency matrix of an undirected graph is symmetric.
3. $a_{ii} = 0$ (simple matrix has no loop)

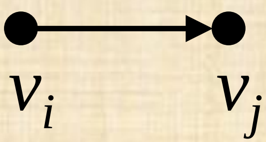
Example 5. (Pseudograph) (Matrix may not be 0,1 matrix.)



$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Def. If $A=[a_{ij}]$ is the adjacency matrix for the directed graph, then

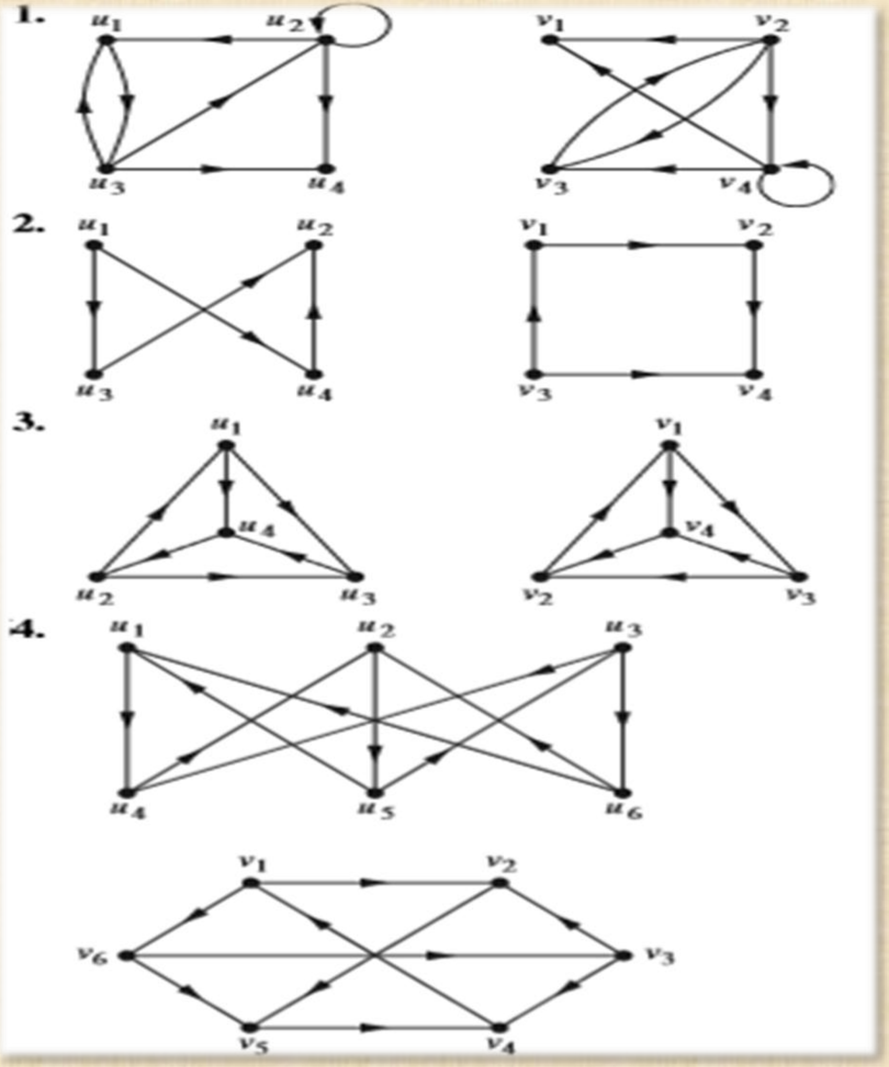
$$a_{ij} = \begin{cases} 1 & \text{, if } v_i \longrightarrow v_j \\ 0 & \text{, otherwise} \end{cases}$$



(So the matrix is not necessarily symmetrical)

EXERCISES

Identify whether the adjacency matrix of the following directed graphs is symmetrical or not.

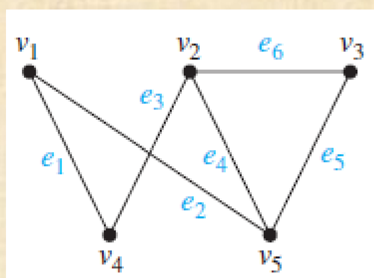


Incidence Matrices

Def. Let $G=(V, E)$: be an undirected graph. Suppose that $v_1,v_2,\ldots,v_n$ are the vertices and $e_1,e_2,\ldots,e_n$ are the edges of G . Then the incidence matrix with respect to this ordering of V and E is the n x m matrix $M=[m_{ij}]$, where

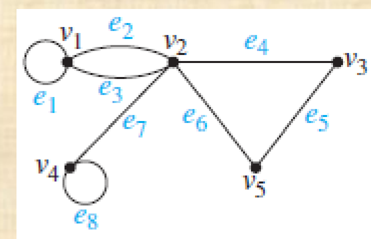
$$m_{i,j} = \begin{cases} 1 \text{ when edge } e_j \text{ is incident with } v_i, \\ 0 \text{ otherwise.} \end{cases}$$

Example 6.



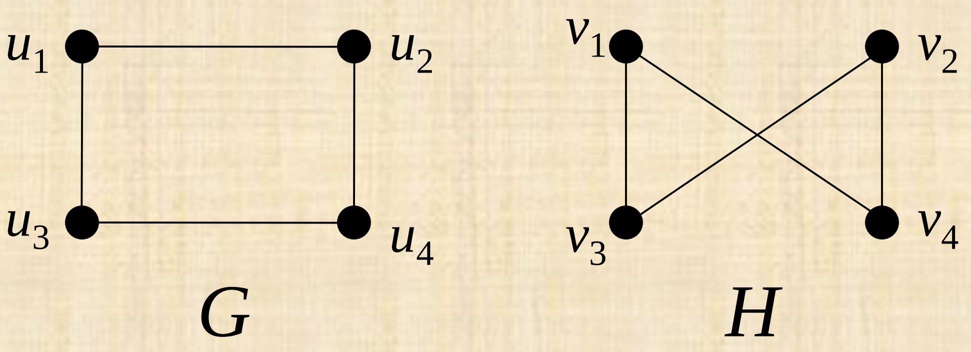
	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	0	1	1	0	1
v_3	0	0	0	0	1	1
v_4	1	0	1	0	0	0
v_5	0	1	0	1	1	0

Example 7.



	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	1	1	1	0	0	0	0	0
v_2	0	1	1	1	0	1	1	0
v_3	0	0	0	1	1	0	0	0
v_4	0	0	0	0	0	0	1	1
v_5	0	0	0	0	1	1	0	0

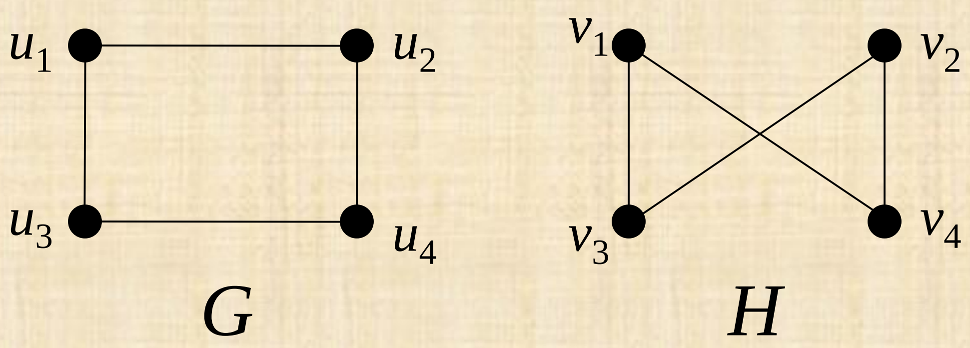
Isomorphism of Graphs



G is isomorphic to H

Def 1. The simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are isomorphic if there is an one-to-one and onto function f from V_1 to V_2 with the property that $a \sim b$ in G_1 iff $f(a) \sim f(b)$ in G_2, $\forall a,b \in V_1$. f is called an isomorphism.

Example 8. Show that G and H are isomorphic.



Sol. The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between $V(G)$ and $V(H)$.

※Isomorphism graphs there will be:

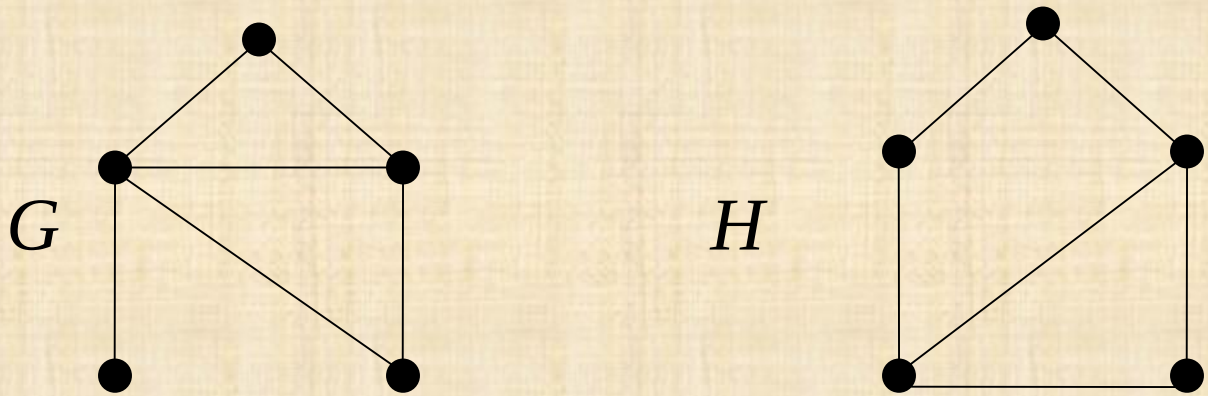
(1) The same number of points (vertices)

(2) The same number of edges

(3) The same number of degree

Given figures, judging whether they are isomorphic in general is not an easy task.

Example 9. Show that G and H are not isomorphic.

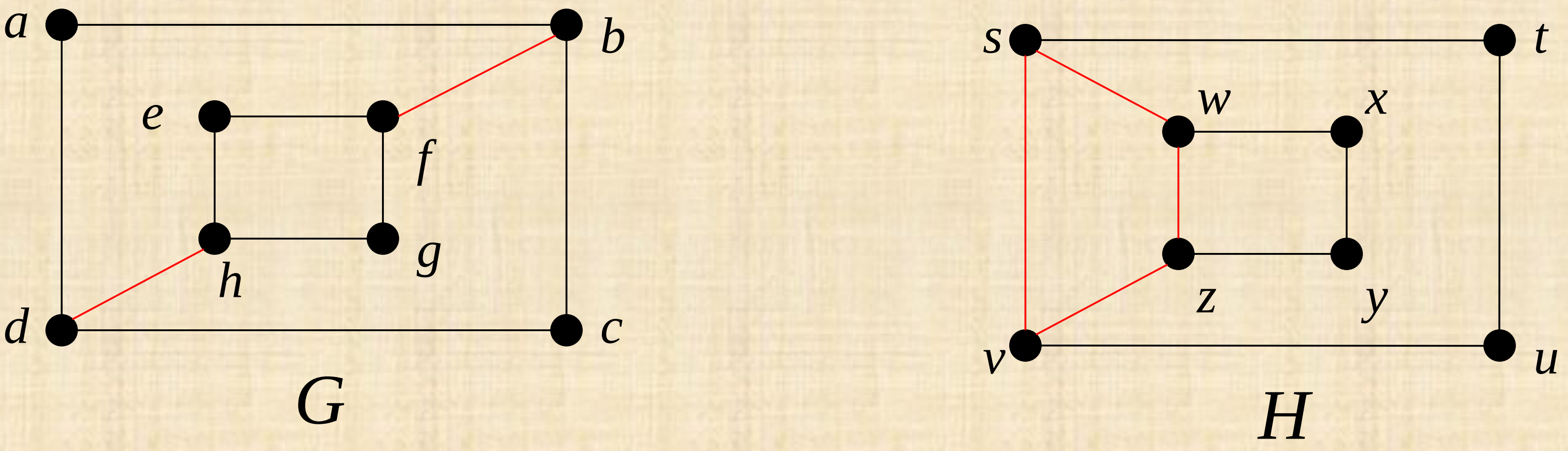


Sol :

G has a vertex of degree $= 1$, H don't

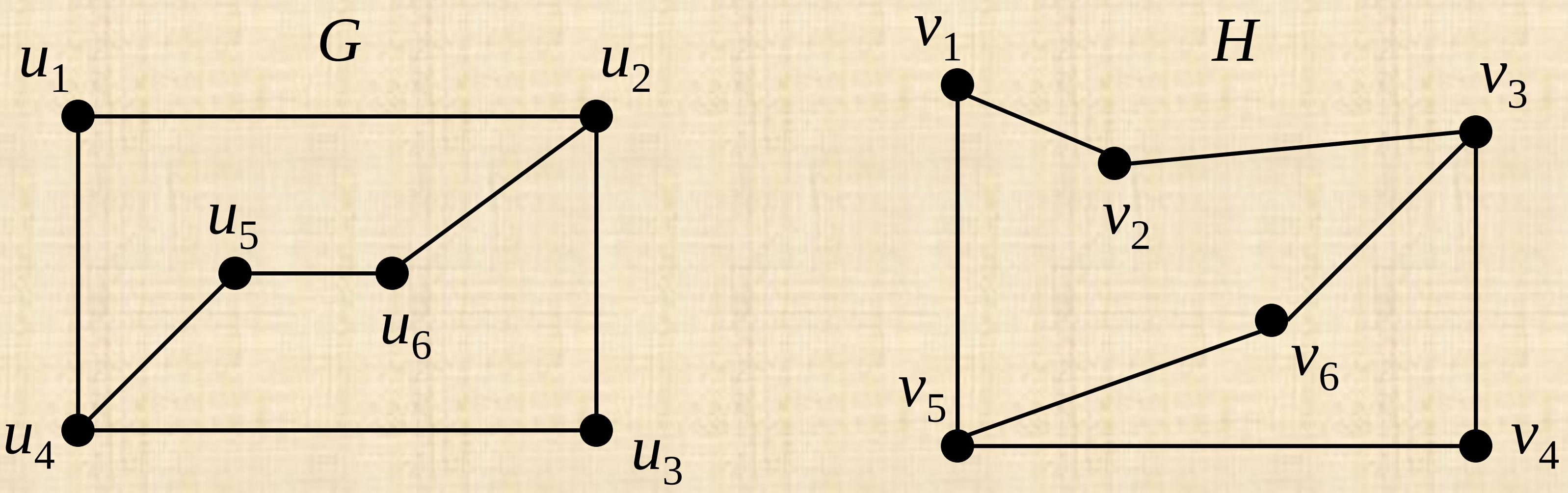
Example 10.

Determine whether G and H are isomorphic.



Sol : ∵ In G, deg(a)=2, which must correspond to either t, u, x, or y in H degree
Each of these four vertices in H is adjacent to another vertex of degree two in H,
which is not true for a in G
∴ G and H are not isomorphic.

Example 11. Determine whether the graphs G and H are isomorphic.



Sol:

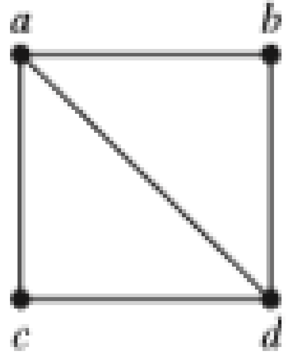
$f(u_1)=v_6$, $f(u_2)=v_3$, $f(u_3)=v_4$, $f(u_4)=v_5$, $f(u_5)=v_1$, $f(u_6)=v_2$

$\Rightarrow$ Yes

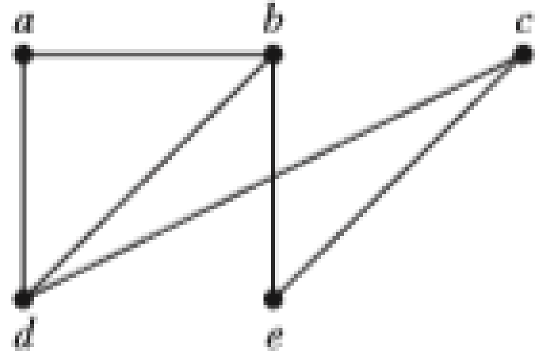
Exercises

In Exercises 1–4 use an adjacency list to represent the given graph.

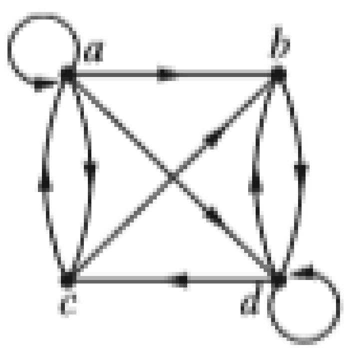
1.



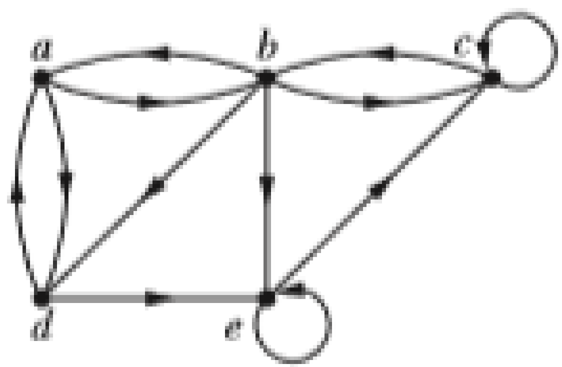
2.



3.



4.



5. Represent the graph in Exercise 1 with an adjacency matrix.

6. Represent the graph in Exercise 2 with an adjacency matrix.

7. Represent the graph in Exercise 3 with an adjacency matrix.

8. Represent the graph in Exercise 4 with an adjacency matrix.

9. Represent each of these graphs with an adjacency matrix.

a) K_4 **b)** $K_{1,4}$ **c)** $K_{2,3}$
d) C_4 **e)** W_4 **f)** Q_3

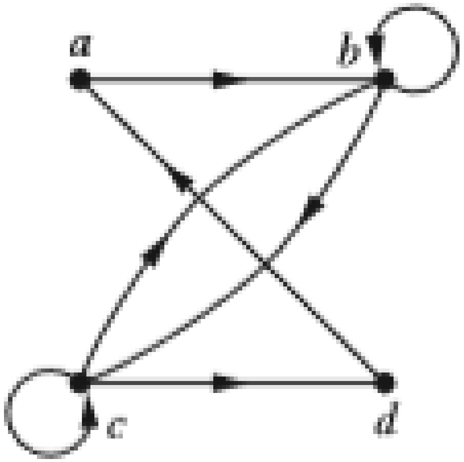
In Exercises 10-11 draw a graph with the given adjacency matrix.

10. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

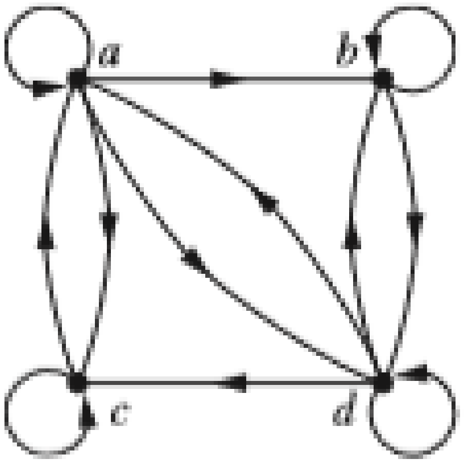
11. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

In Exercises 12-14 find the adjacency matrix of the given directed multigraph with respect to the vertices listed in alphabetic order.

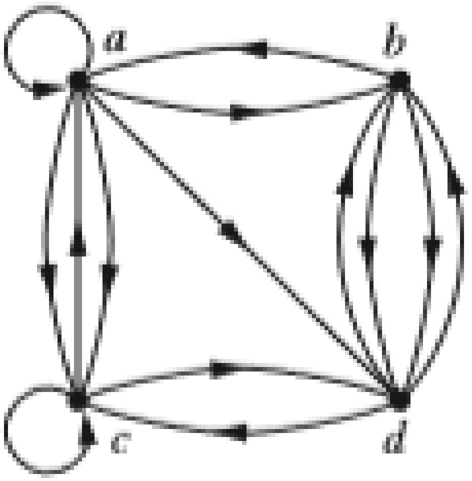
12.



14.

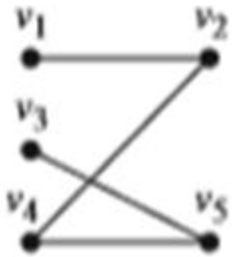


13.

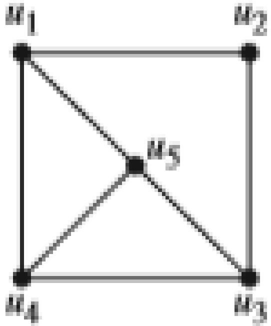


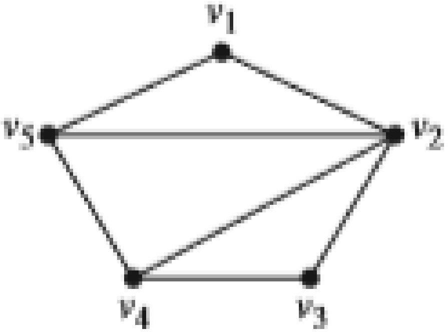
In Exercises 14-18 determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

14.

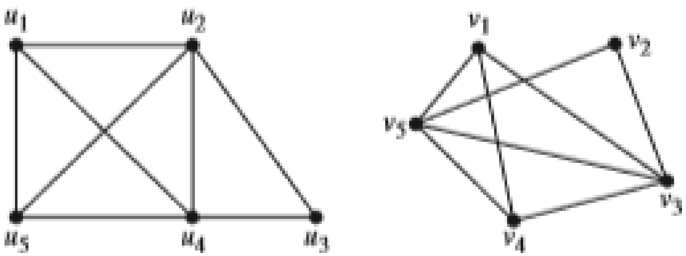


15.



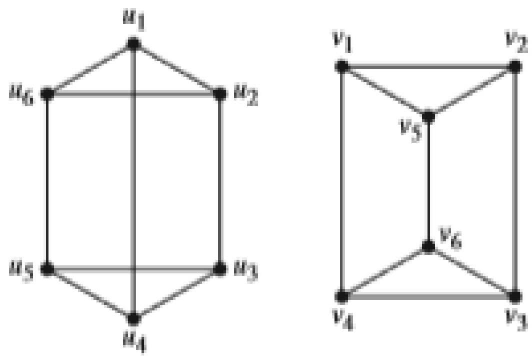


16.



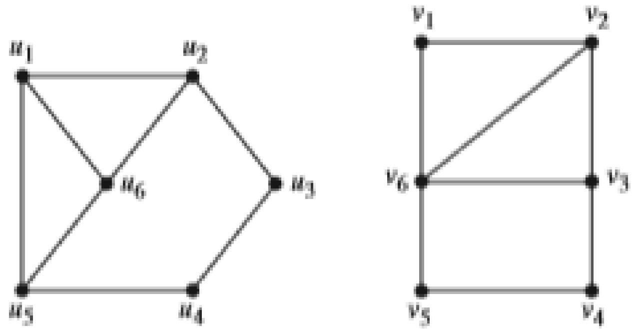
u_1
u_2
u_3
u_4
u_5
v_1
v_2
v_3
v_4
v_5

17.



u_1
u_2
u_3
u_4
u_5
u_6
v_1
v_2
v_3
v_4
v_5
v_6

18.



u_1
u_2
u_3
u_4
u_5
u_6
v_1
v_2
v_3
v_4
v_5
v_6

Connectivity

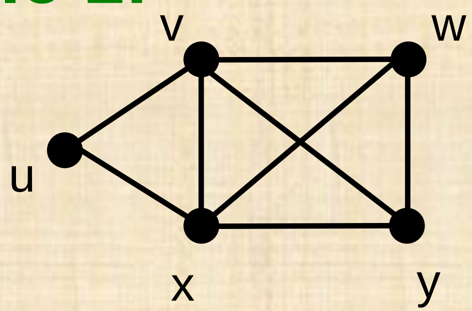
Def. 1 :

In an undirected graph, a path of length n from u to v is a sequence of $n + 1$ adjacent vertices going from vertex u to vertex v.
(e.g., P: $u=x_0, x_1, x_2, \ldots, x_n=v$.) (P has n edges.)

Def. 2:

path: Points and edges in unrepeatable
trail: Allows duplicate point (path not repeatable)
walk: Allows duplicate point and duplicate path

Example 1:



path: u, v, y
trail: u, v, w, y, v, x, y
walk: u, v, w, v, x, v, y

EXAMPLE 2 In the simple graph shown in Figure 1, a, d, c, f, e is a simple path of length 4, because $\{a, d\}$, $\{d, c\}$, $\{c, f\}$, and $\{f, e\}$ are all edges. However, d, e, c, a is not a path, because $\{e, c\}$ is not an edge. Note that b, c, f, e, b is a circuit of length 4 because $\{b, c\}$, $\{c, f\}$, $\{f, e\}$, and $\{e, b\}$ are edges, and this path begins and ends at b. The path a, b, e, d, a, b, which is of length 5, is not simple because it contains the edge $\{a, b\}$ twice. ◀

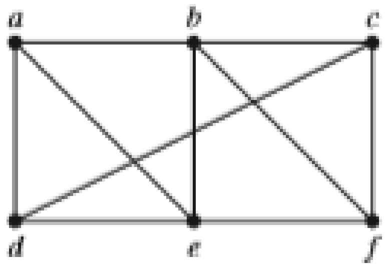


FIGURE 1 **A Simple Graph.**

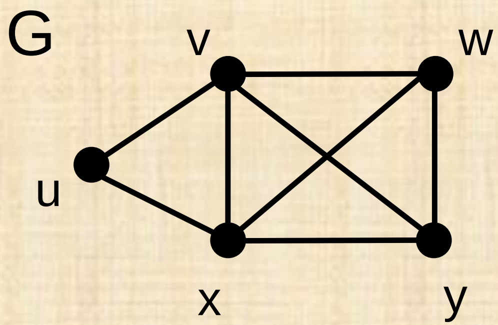
Def:

cycle: path with $u=v$

circuit: trail with $u=v$

closed walk: walk with $u=v$

Example



cycle: u, v, y, x, u

trail: u, v, w, y, v, x, u

walk: u, v, w, v, x, v, y, x, u

Paths in Directed Graphs

The same as in undirected graphs, but the path must go in the direction of the arrows.

Connectedness in Undirected Graphs

Def. 3:

An undirected graph is *connected* if there is a path between every pair of distinct vertices in the graph.

CONNECTED COMPONENTS: A connected component of a graph G is a connected sub-graph of G that is not a proper sub-graph of another connected sub-graph of G. That is, a connected component of a graph G is a maximal connected sub-graph of G. A graph G that is not connected has two or more connected components that are disjoint and have G as their union.

EXAMPLE 6: The graph G_1 in Figure 2 is connected, because for every pair of distinct vertices there is a path between them (the reader should verify this). However, the graph G_2 in Figure 2 is not connected. For instance, there is no path in G_2 between vertices a and d.

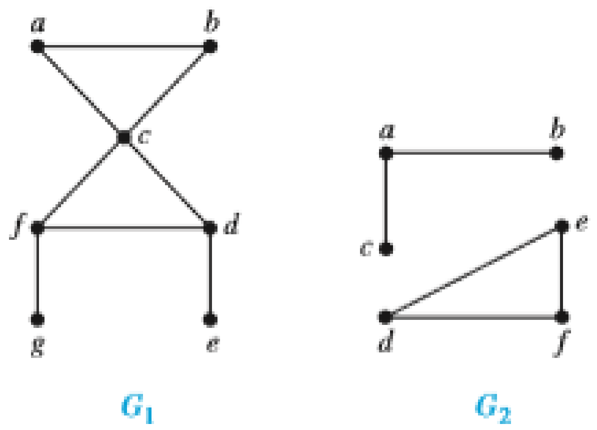


FIGURE 2 The Graphs G_1 and G_2.

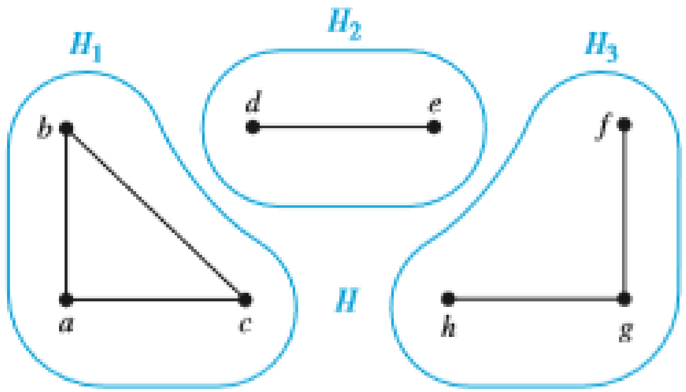
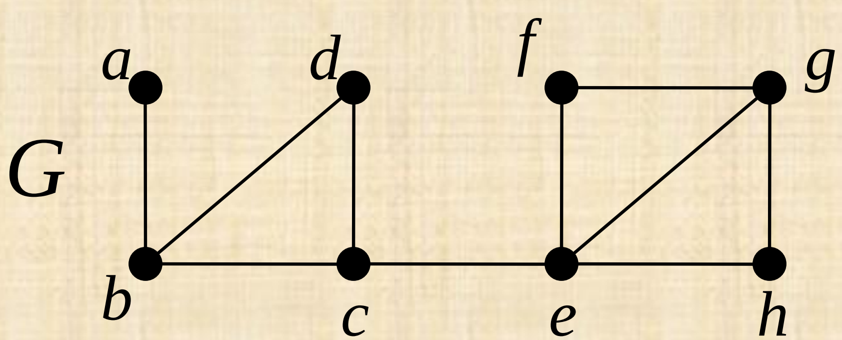


FIGURE 3 The Graph H and Its Connected Components H_1, H_2, and H_3.

Def:

A *cut vertex* separates one connected component into several components if it is removed.
A *cut edge* separates one connected component into two components if it is removed.

Example 8. Find the cut vertices and cut edges in the graph G.



Sol:

cut vertices: b, c, e

cut edges:

$\{a, b\}, \{c, e\}$

Connectedness in Directed Graphs

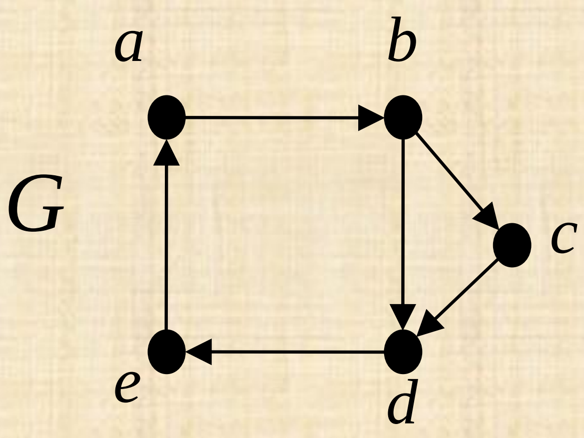
Def. 4: (a) A directed graph is *strongly connected* if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

For a directed graph to be strongly connected there must be a sequence of directed edges from any vertex in the graph to any other vertex. A directed graph can fail to be strongly connected but still be in “one piece.”

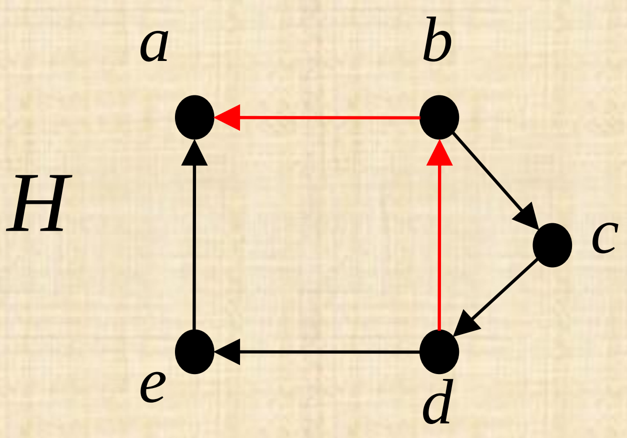
(b) A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graph.

That is, a directed graph is weakly connected if and only if there is always a path between two vertices when the directions of the edges are disregarded. Clearly, any strongly connected directed graph is also weakly connected.

Example 9 Are the directed graphs G and H strongly connected or weakly connected?



strongly connected

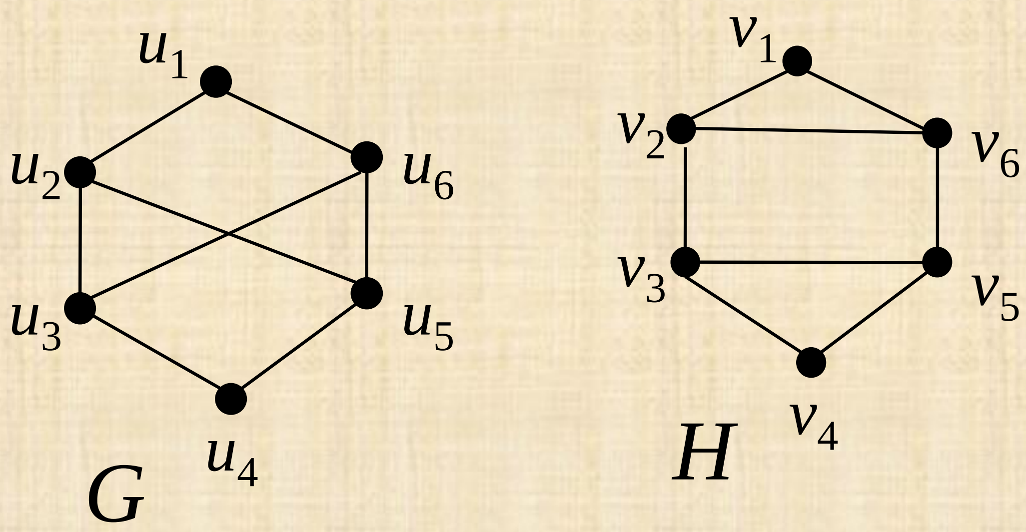


weakly connected

Paths and Isomorphism

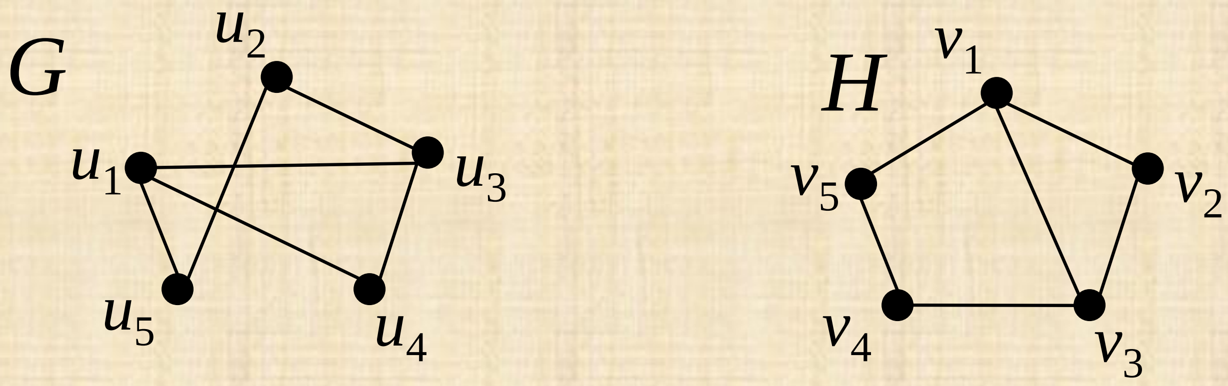
Note that connectedness, and the existence of a circuit or simple circuit of length *k* are graph invariants with respect to isomorphism.

Example 12. Determine whether the graphs G and H are isomorphic.



Sol: No, Because G has no simple circuit of length three, but H does

Example 13. Determine whether the graphs G and H are isomorphic.



Sol.

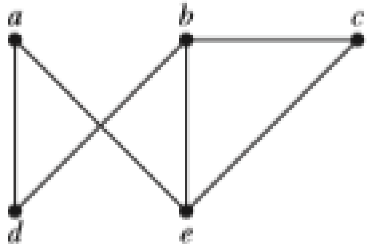
Both G and H have 5 vertices, 6 edges, two vertices of deg 3, three vertices of deg 2, a 3-cycle, a 4-cycle, and a 5-cycle. $\Rightarrow$ G and H may be isomorphic.

The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$ and $f(u_5) = v_5$ is a one-to-one correspondence between $V(G)$ and $V(H)$. $\Rightarrow$ G and H are isomorphic.

Exercises

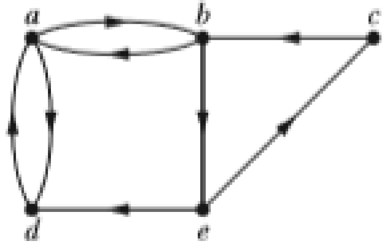
1. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

   a) a, e, b, c, b
   b) a, e, a, d, b, c, a
   c) e, b, a, d, b, e
   d) c, b, d, a, e, c



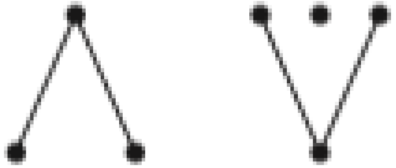
2. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

   a) a, b, e, c, b
   b) a, d, a, d, a
   c) a, d, b, e, a
   d) a, b, e, c, b, d, a

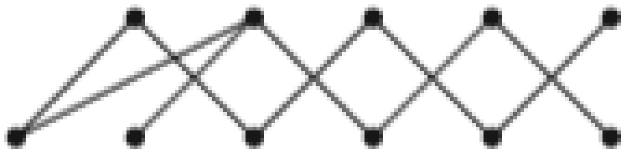


In Exercises 3–5 determine whether the given graph is connected.

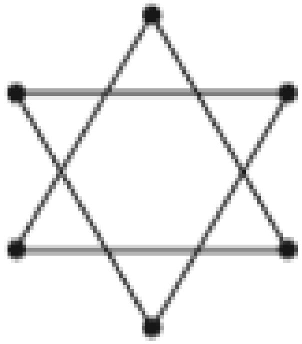
3.



4.

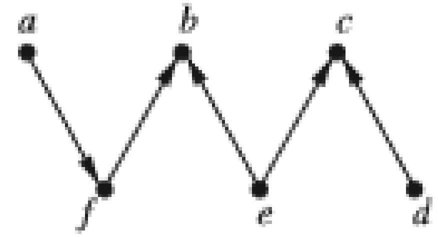


5.

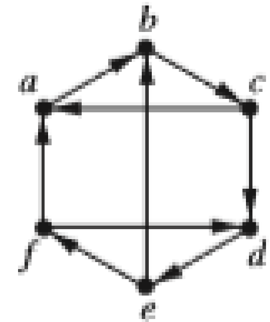


6. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

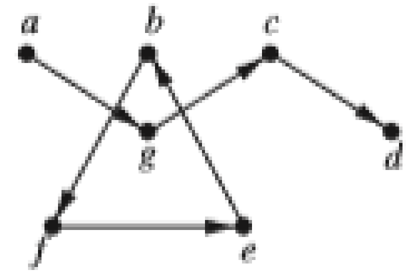
a)



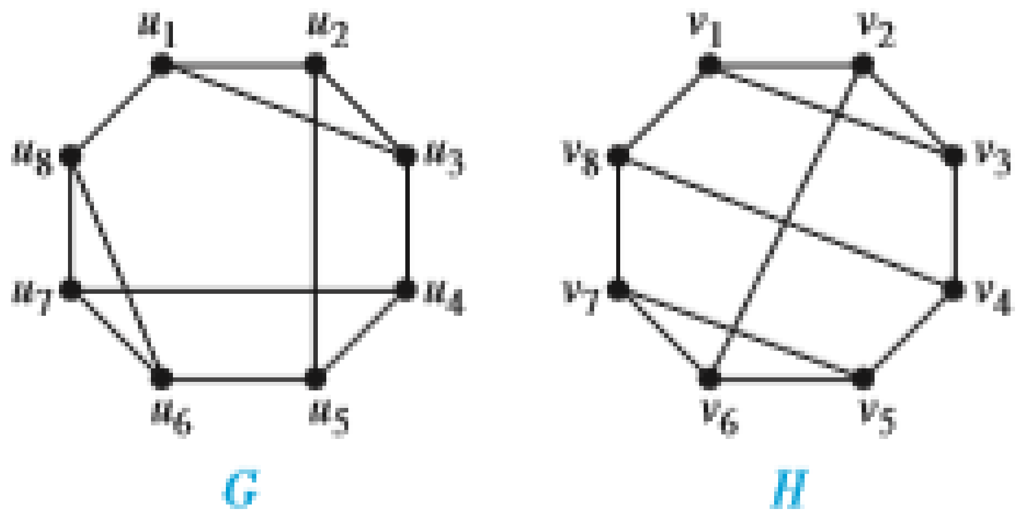
b)



c)



7. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.

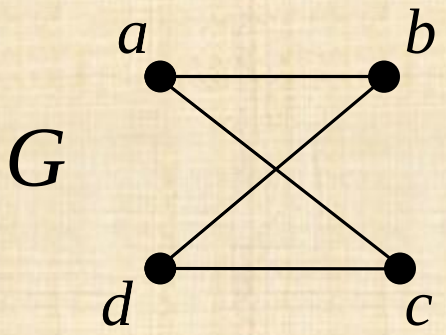


Counting Paths between Vertices

Theorem 2:

Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \ldots, v_n$. The number of walks of length r from v_i to v_j is equal to $(A^r)_{i,j}$.

Example 14. How many walks of length 4 are there from a to d in the graph G?



Sol.

The adjacency matrix of G (ordering as a, b, c, d) is

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \Rightarrow \quad A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix} \Rightarrow 8$$

a-b-a-b-d, a-b-a-c-d, a-c-a-b-d, a-c-a-c-d,
a-b-d-b-d, a-b-d-c-d, a-c-d-b-d, a-c-d-c-d

Def 1:

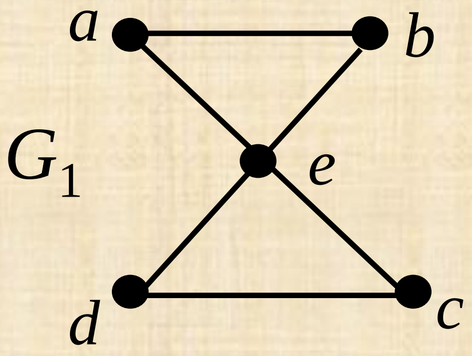
(a) An *Euler circuit* in a graph *G* is a simple circuit containing every edge of *G*.

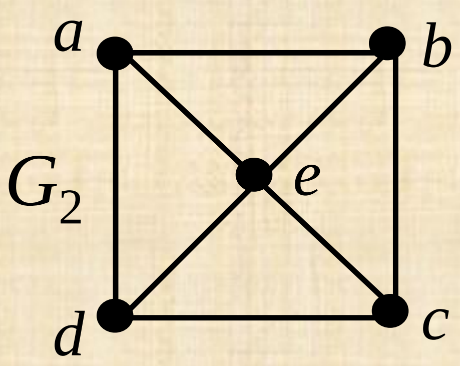
(b) An *Euler path* in *G* is a simple path containing every edge of *G*.

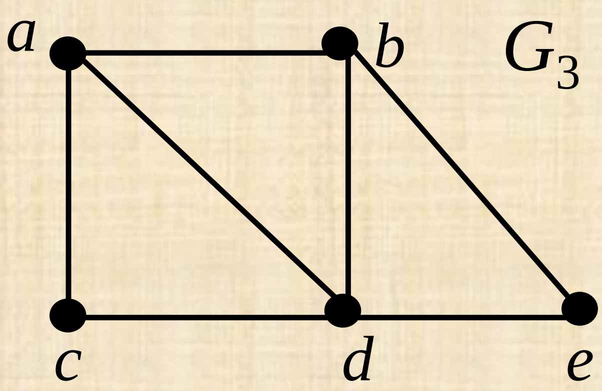
Thm. 1: A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Thm. 2: A connected multigraph has an Euler path (but not an Euler circuit) if and only if it has exactly 2 vertices of odd degree.

Example 1. Which of the following graphs have an Euler circuit or an Euler path?







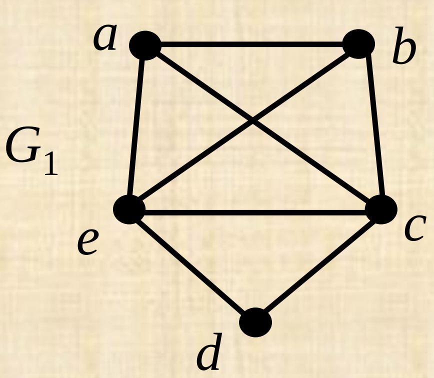
Euler circuit

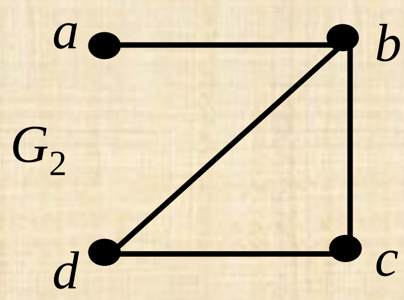
none

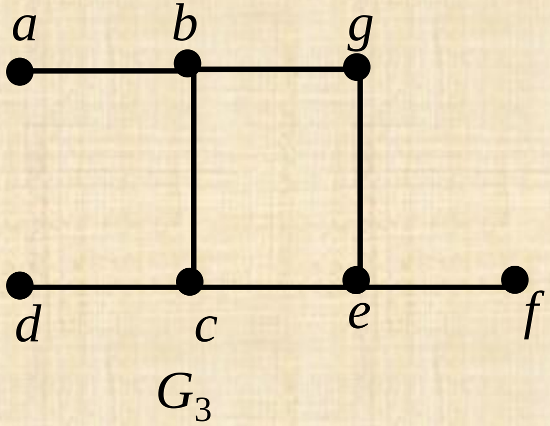
Euler path

Def. 2: A simple path in a graph G that passes through every vertex exactly once is called a Hamilton path, and a simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit.

Example 1. Which of the following graphs have a Hamilton circuit or a Hamilton path?





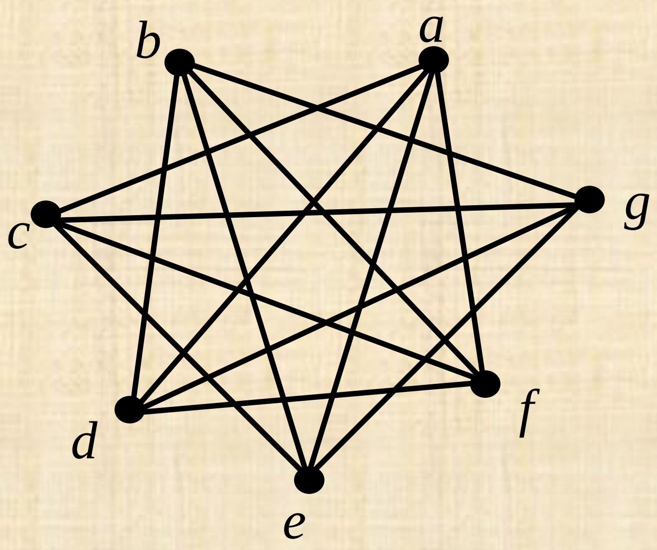


Hamilton circuit: G_1 Hamilton path: G_1, G_2

Thm. 3 (Dirac's Thm.):

If G is a simple graph with $n \geq 3$ vertices such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

Example



each vertex has deg $\geq n/2 = 3.5$

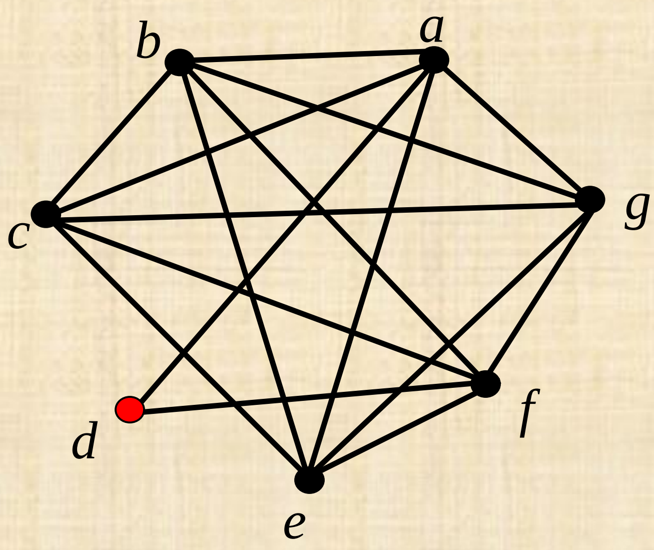
$\Rightarrow$Hamilton circuit exists

Such as: a, c, e, g, b, d, f, a

Thm. 4 (Ore's Thm.):

If G is a simple graph with $n \geq 3$ vertices such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v, then G has a Hamilton circuit.

Example



each nonadjacent vertex pair has deg sum $\geq n = 7$

$\Rightarrow$Hamilton circuit exists

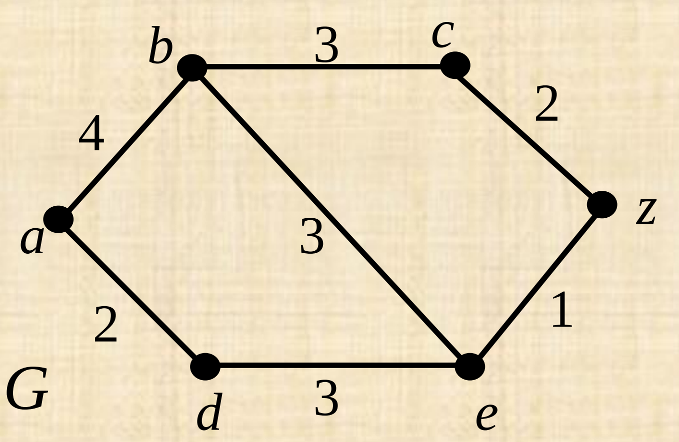
Such as: a, d, f, e, c, b, g, a

Shortest-Path Problems

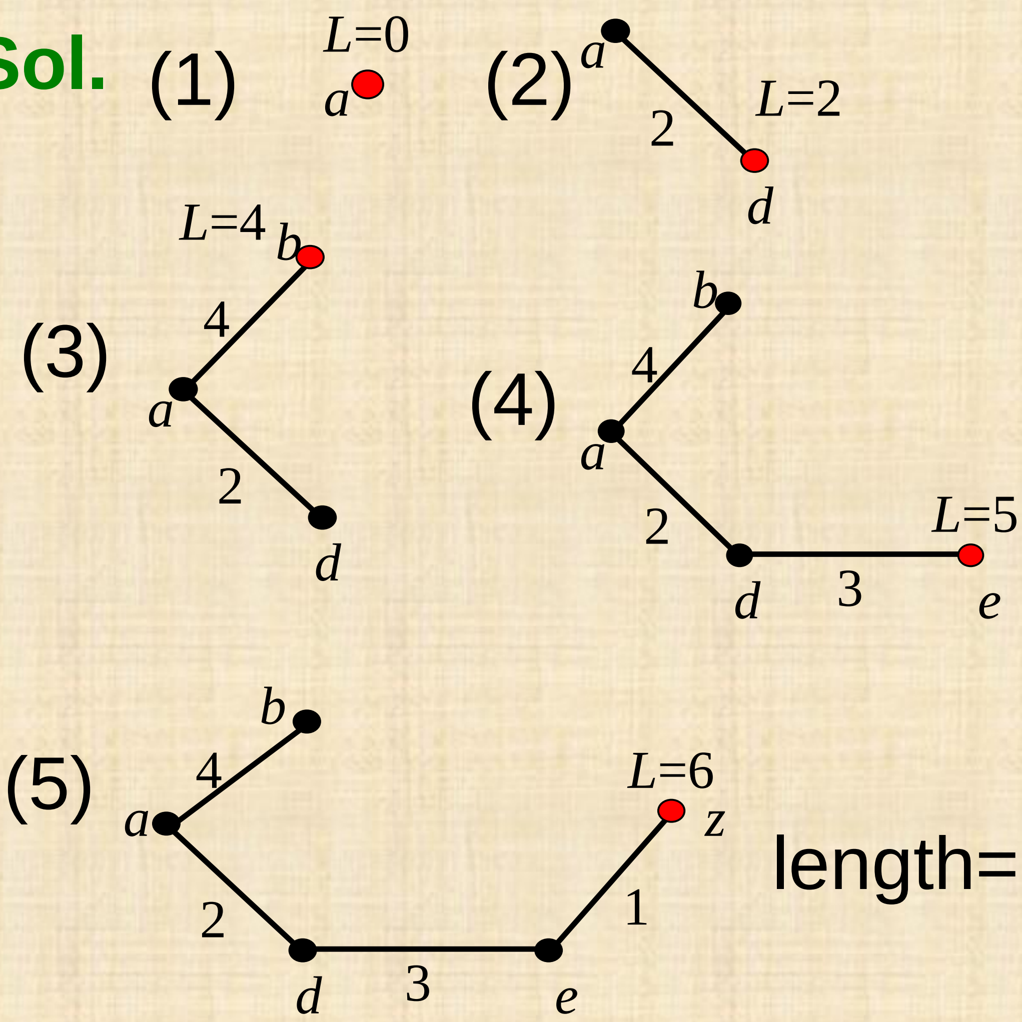
Def:

1. Graphs that have a number assigned to each edge are called *weighted graphs*.
2. The length of a path in a weighted graph is the sum of the weights of the edges of this path.
3. Short-Path is the path of least sum of the weights between two vertices in a weighted graph.

Example 1. What is the length of a shortest path between a and z in the weighted graph G?

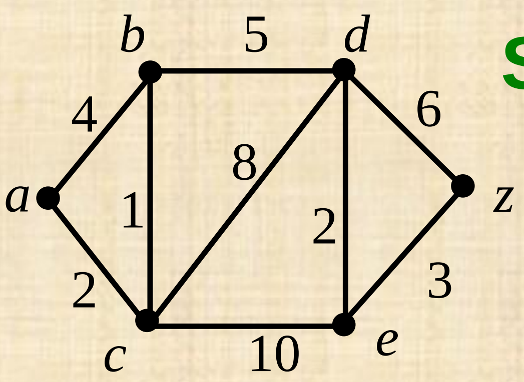


Sol.

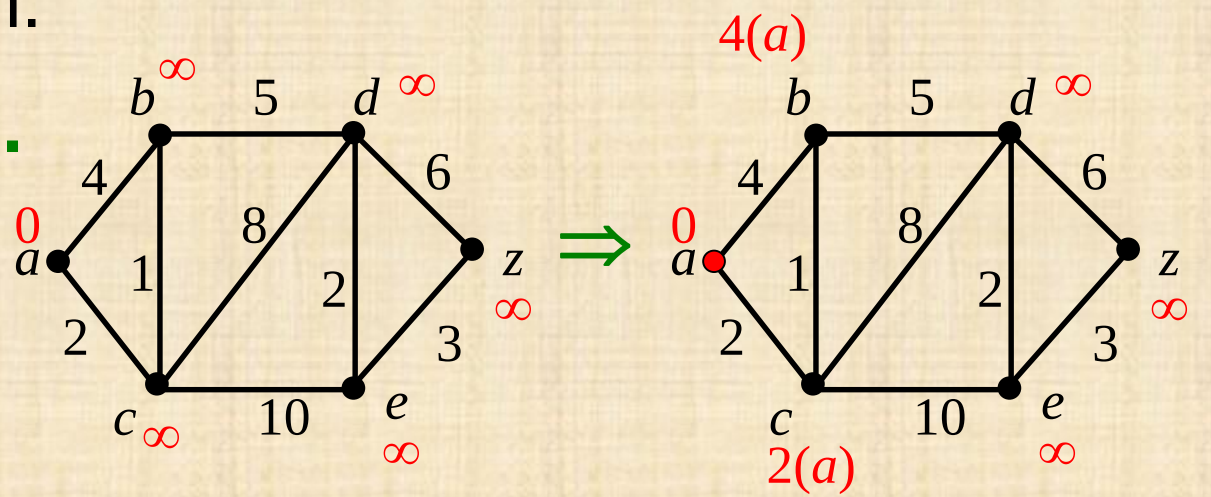


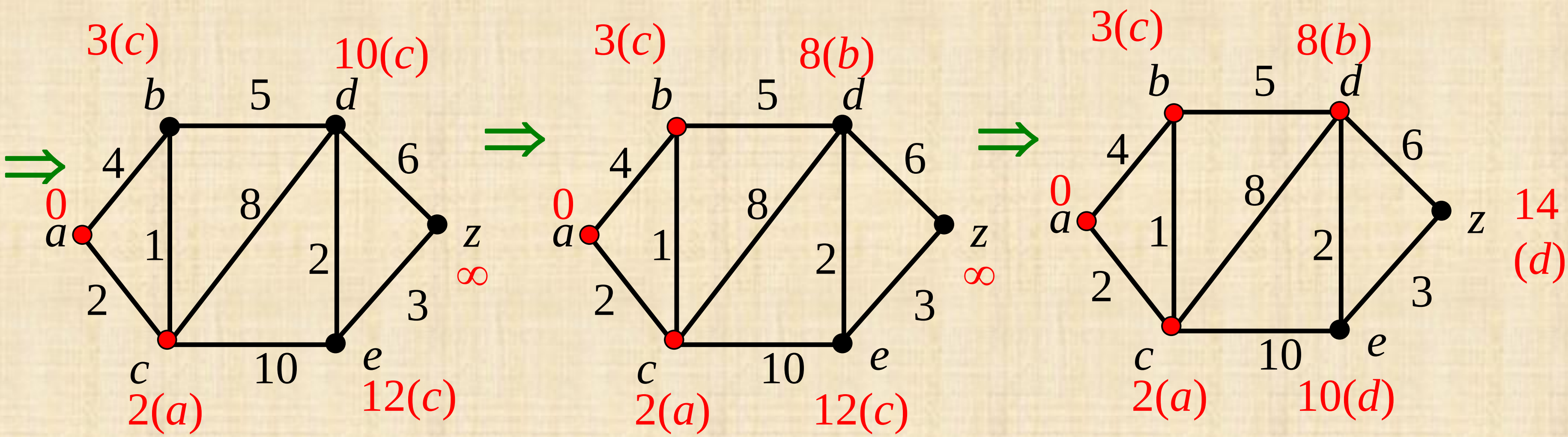
length=6

Example 2. Use Dijkstra's algorithm to find the length of a shortest path between a and z in the weighted graph.

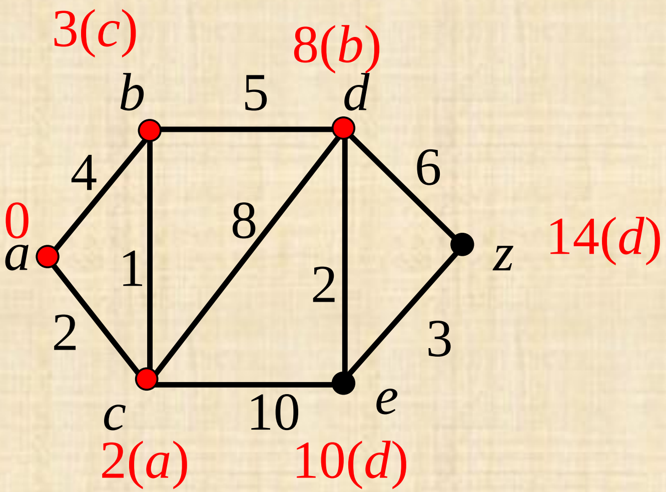


Sol.

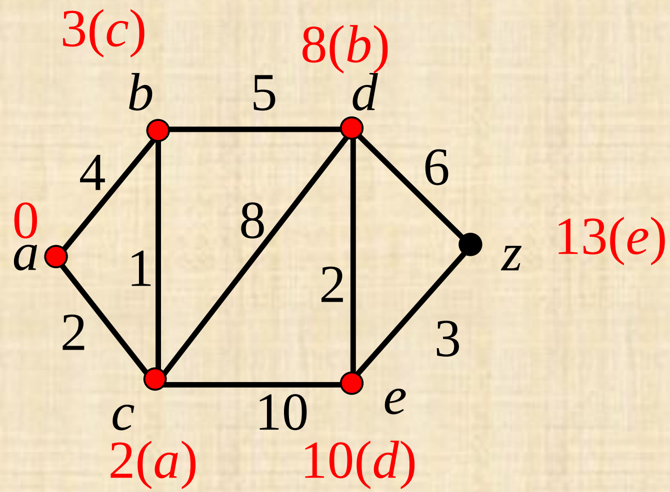




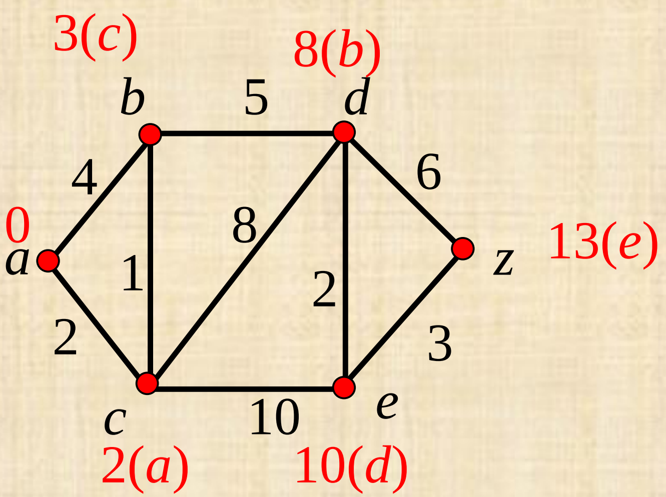
Contd



⇒



⇒



⇒ path: a, c, b, d, e, z

length: 13

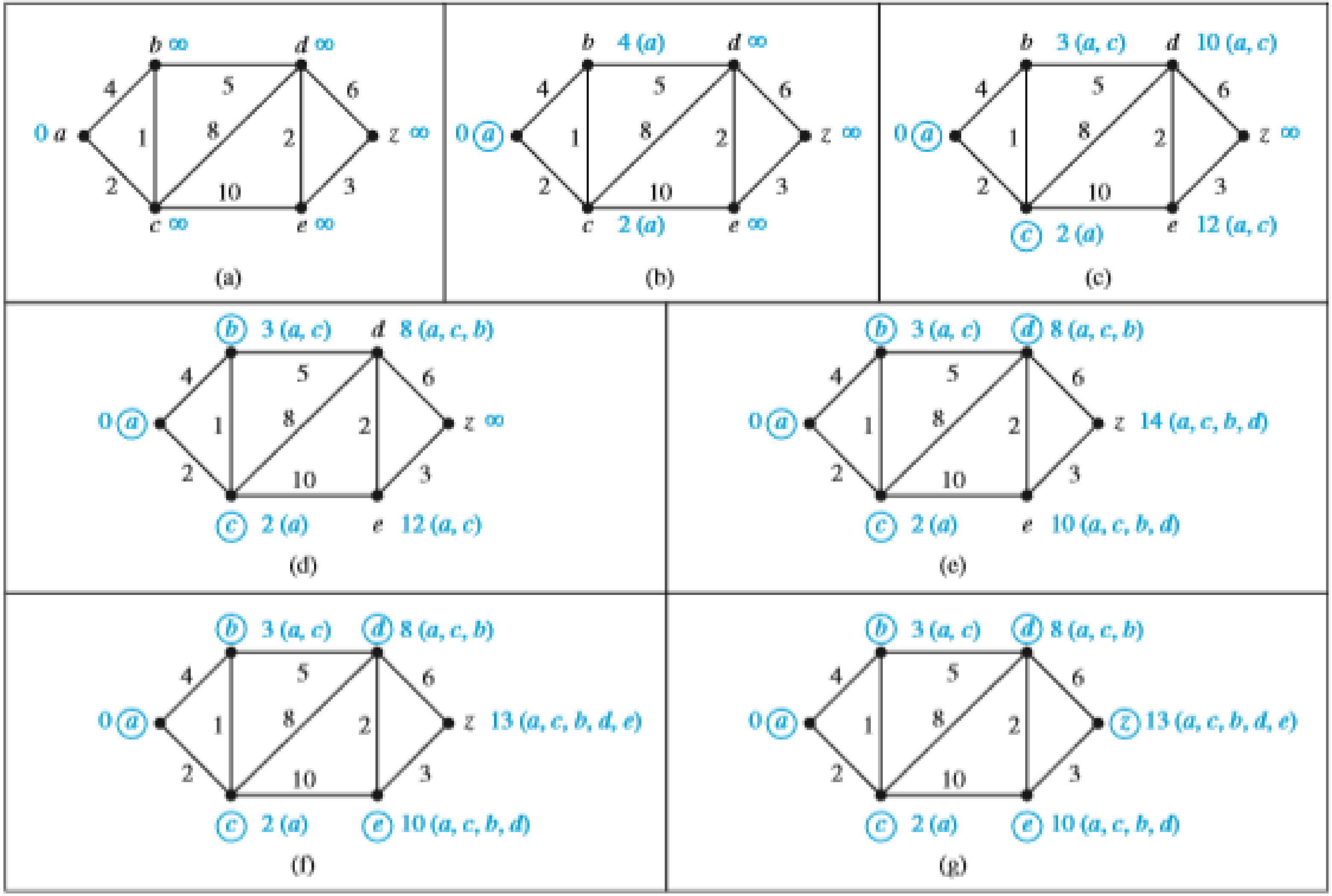
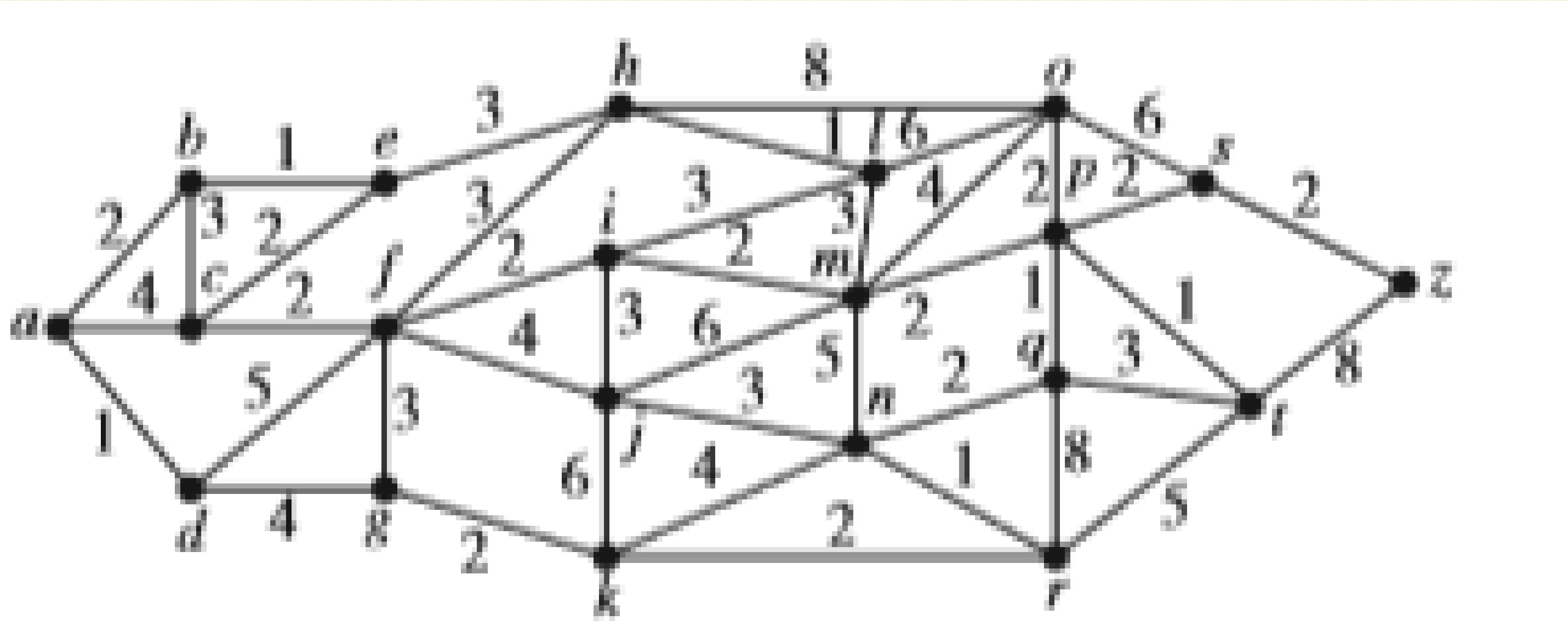


FIGURE **Using Dijkstra's Algorithm to Find a Shortest Path from *a* to *z*.**

excersice

1. Use Dijkstra's algorithm (on page 712 of your text book) to find the length of a shortest path between a and z in the weighted graph.

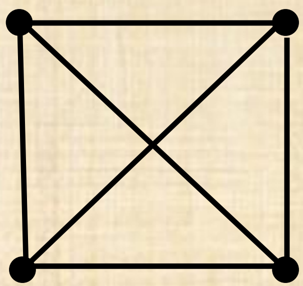


Planar Graphs

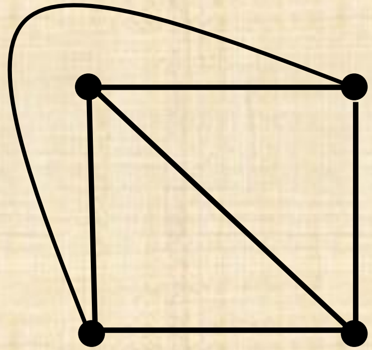
Def 1.

A graph is called *planar* if it can be drawn in the plane without any edge crossing. Such a drawing is called a *planar representation* of the graph.

Example 1: Is K_4 planar?



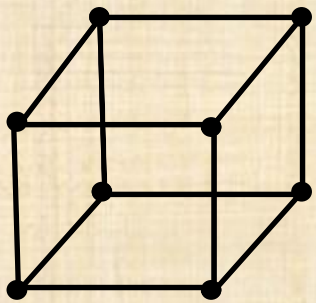
K_4



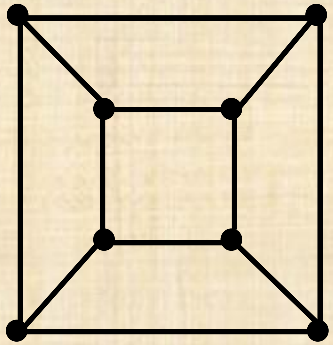
K_4 drawn with no crossings

$\therefore K_4$ is planar

Example 2: Is Q_3 planar?



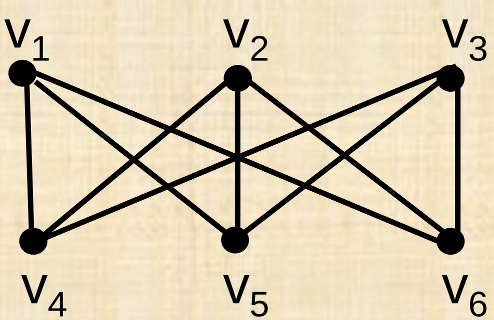
Q_3



Q_3 drawn with no crossings

$\therefore$ Q_3 is planar

Example 3: Show that $K_{3,3}$ is nonplanar.



Sol. In any planar representation of $K_{3,3}$, the vertices v_1 and v_2 must be connected to both v_4 and v_5. These four edges form a closed curve that splits the plane into two regions, R_1 and R_2, as shown in Figure 7(a).

The vertex v_3 is in either R_1 or R_2. When v_3 is in R_2, the inside of the closed curve, the edges between v_3 and v_4 and between v_3 and v_5 separate R_2 into two sub-regions, R_{21} and R_{22}, as shown in Figure 7(b).

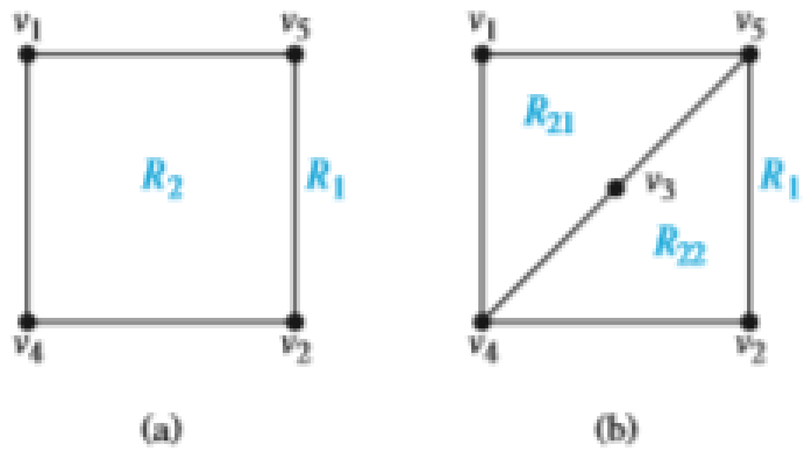


FIGURE 7 **Showing that $K_{3,3}$ Is Nonplanar.**

Next, note that there is no way to place the final vertex v_6 without forcing a crossing. For if v_6 is in R1, then the edge between v_6 and v_3 cannot be drawn without a crossing.

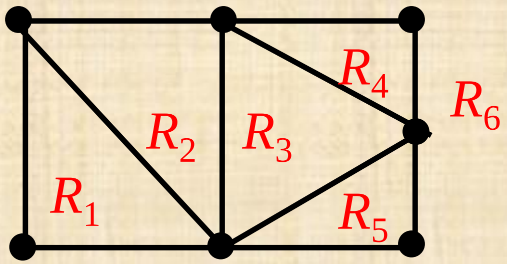
If v_6 is in R_{21}, then the edge between v_2 and v_6 cannot be drawn without a crossing. If v_6 is in R_{22}, then the edge between v_1 and v_6 cannot be drawn without a crossing.

A similar argument can be used when v_3 is in R_1. The completion of this argument is left for the reader. It follows that $K_{3,3}$ is not planar.

Euler's Formula

A planar representation of a graph splits the plane into regions, including an unbounded region.

Example : How many regions are there in the following graph?



Sol. 6

Thm 1 (Euler's Formula)

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Example 4: Suppose that a connected planar graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Sol.

$v = 20,\ 2e = 3\times20 = 60,\ e = 30$

$r = e-v+2 = 30-20+2 = 12$

Corollary 1

If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Example 5: Show that K_5 is nonplanar.

Sol.

$v = 5,\ e = 10,$ but $3v - 6 = 9.$

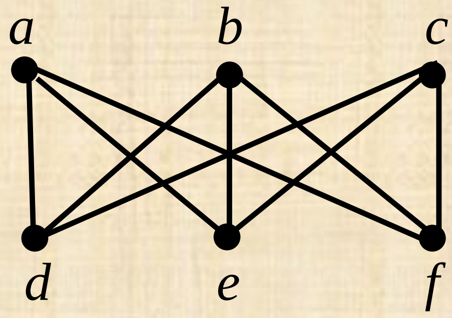
Corollary 3

If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and <u>no circuits of length three</u>, then $e \leq 2v - 4$.

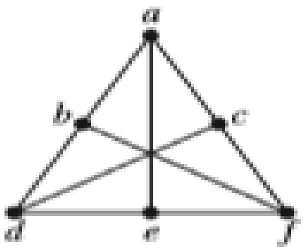
Example 6: Show that $K_{3,3}$ is nonplanar by Cor. 3.

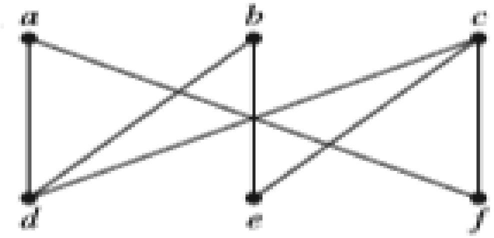
Sol.

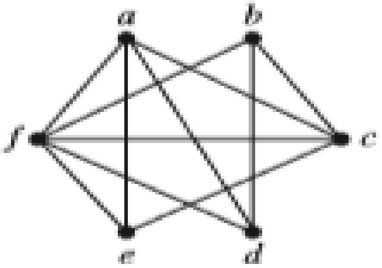
Because $K_{3,3}$ has no circuits of length three, and $v = 6$, $e = 9$, but $e = 9 > 2v - 4$.

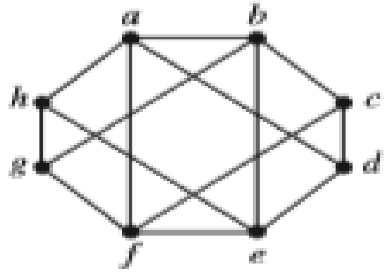


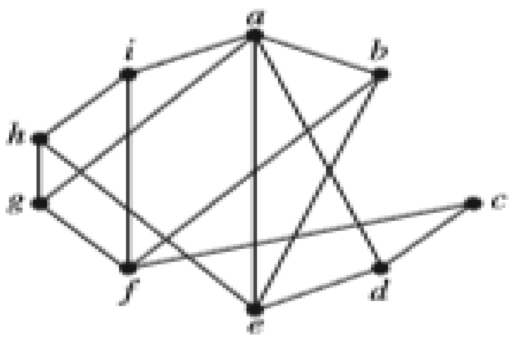
Exercise: Determine whether the given graph is planar.





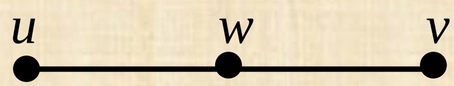






Kuratowski's Theorem

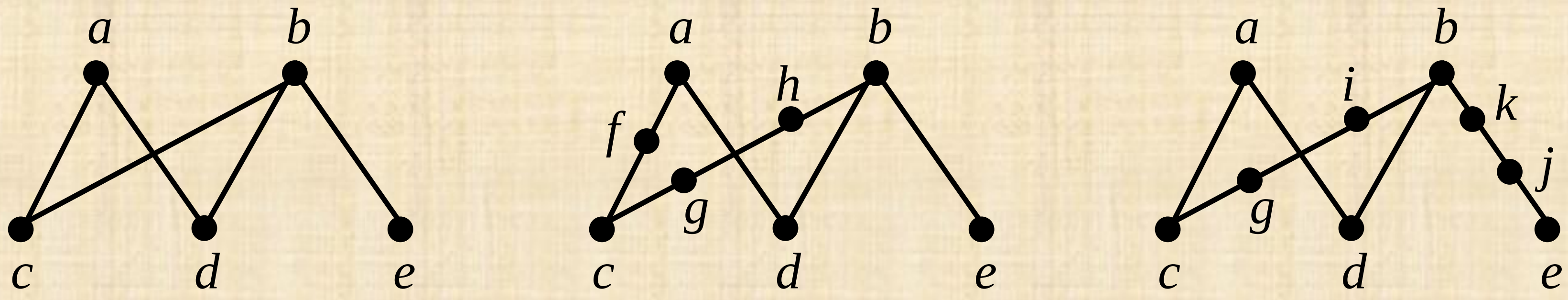
If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{v, w\}$.



Such an operation is called an elementary subdivision.

Two graphs $G_1 = (V_1, E_1)$, $G_2=(V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisons.

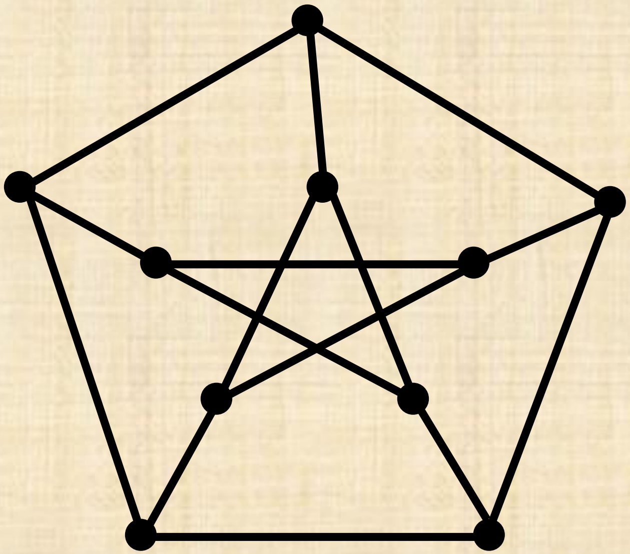
Example 7: Show that the graphs G_1, G_2, and G_3 are all homeomorphic.



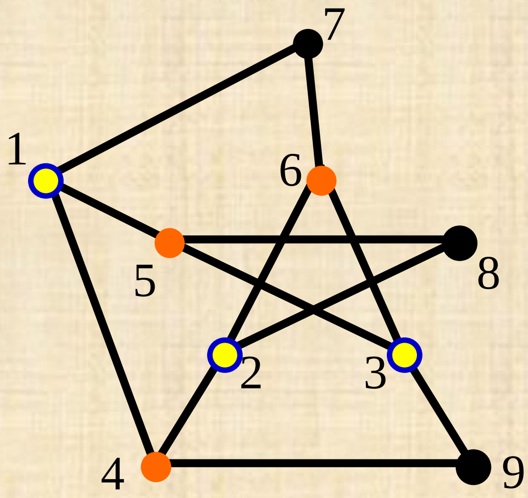
Sol: all three can be obtained from G_1

Thm 2. (Kuratowski Theorem)
A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

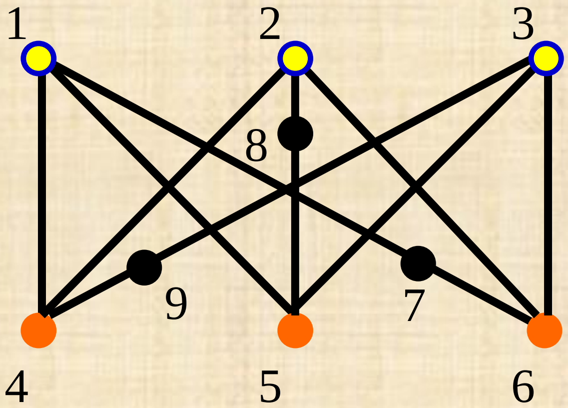
Example 9: Show that the Petersen graph is not planar.



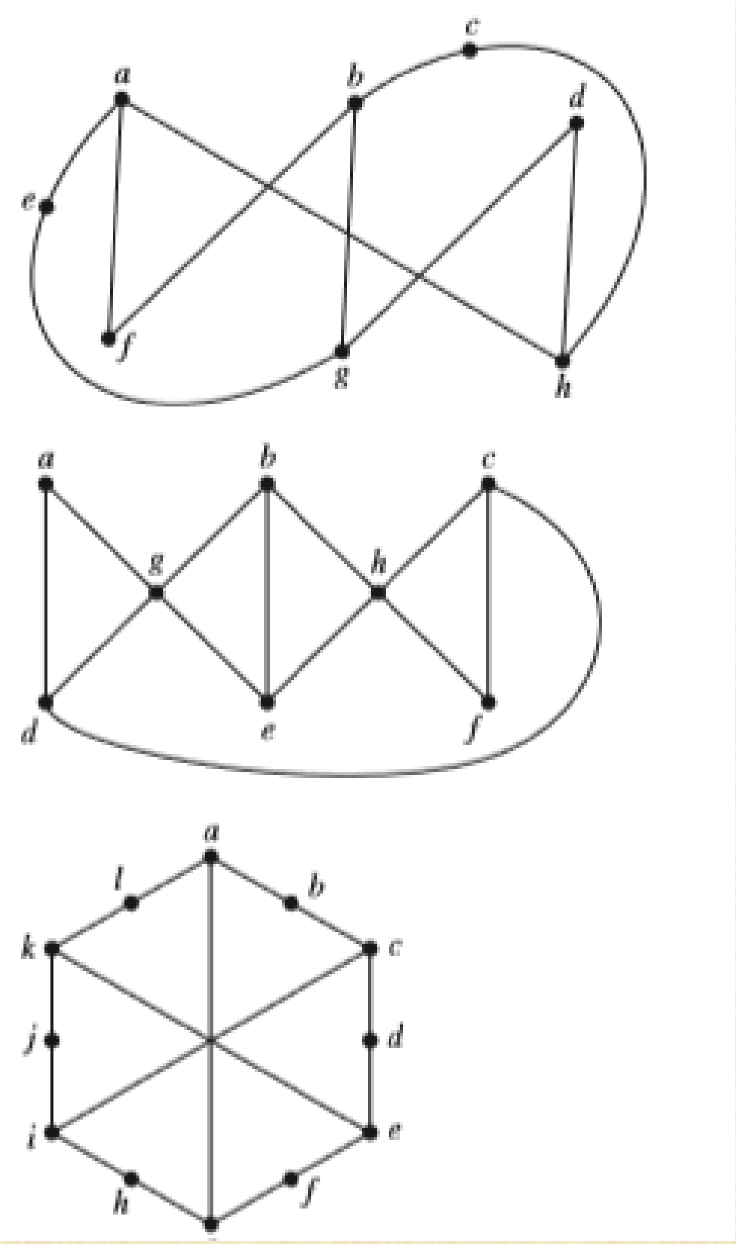
Sol:



It is homeomorphic to $K_{3,3}$.



Exercise: Determine whether the given graph is homeomorphic to $K_{3,3}$.

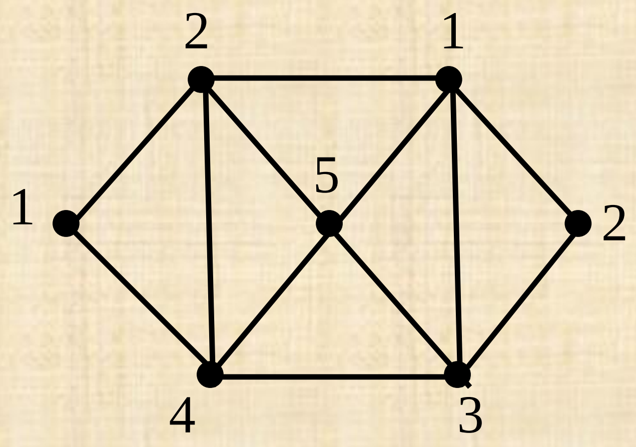


Graph Coloring

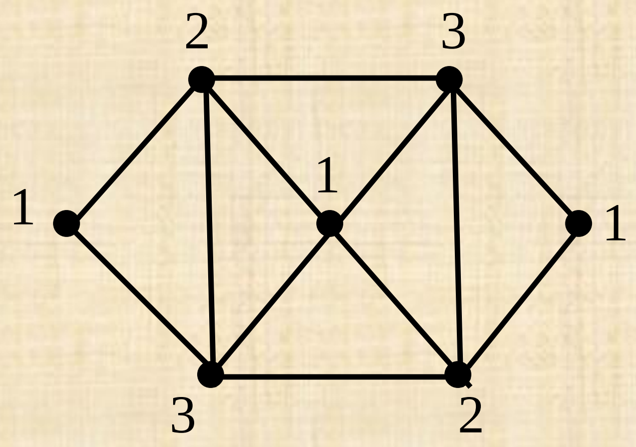
Def. 1:

A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Example:



5-coloring

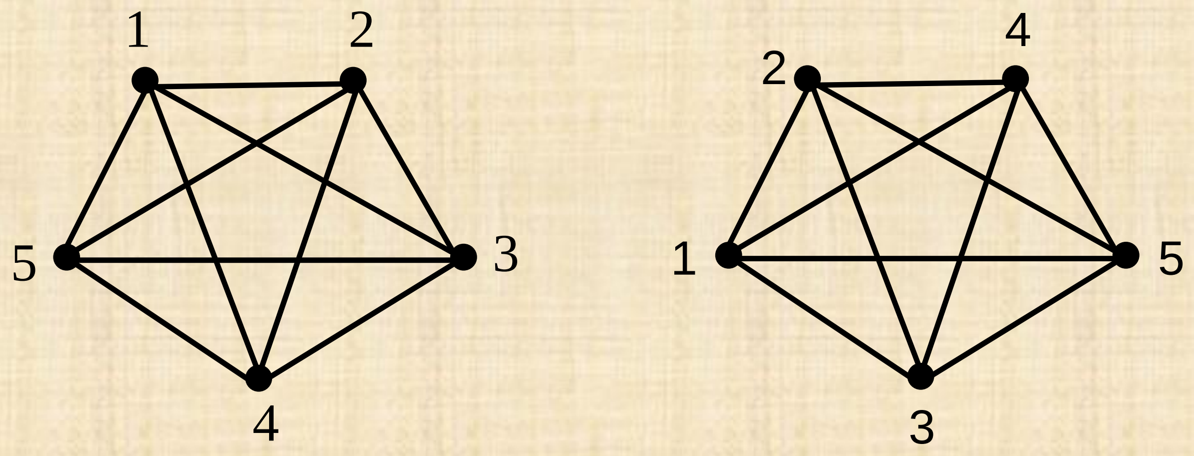


3-coloring

Less the number of colors, the better

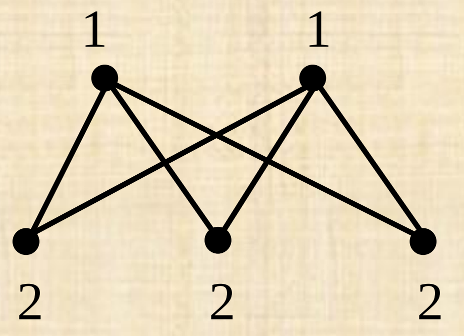
Def. 2: The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph. (denoted by $\chi(G)$)

Example 2: $\chi(K_5)=5$



Note: $\chi(K_n)=n$

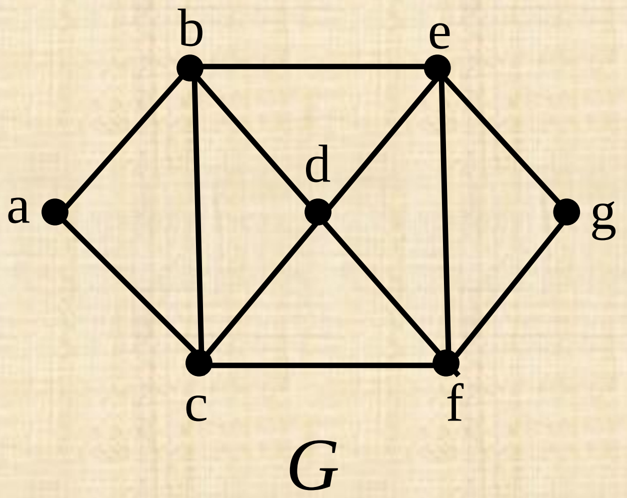
Example: $\chi(K_{2,3}) = 2$.

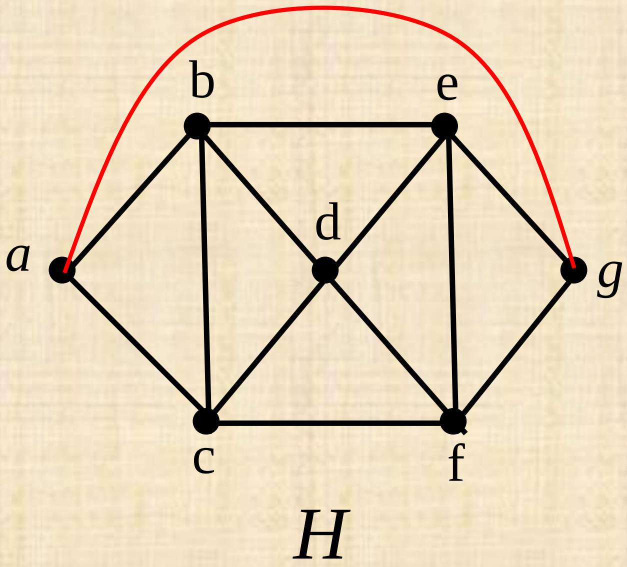


Note: $\chi(K_{m,n}) = 2$

Note: If G is a bipartite graph, $\chi(G) = 2$.

Example 1: What are the chromatic numbers of the graphs *G* and *H*?





Solution: The chromatic number of G is at least three, because the vertices a,b, and c must be assigned different colors. To see if G can be colored with three colors, assign red to a, blue to b, and green to c. Then, d can (and must) be colored red because it is adjacent to b and c.

Furthermore, e can (and must) be colored green because it is adjacent only to vertices colored red and blue, and f can (and must) be colored blue because it is adjacent only to vertices colored red and green. Finally, g can (and must) be colored red because it is adjacent only to vertices colored blue and green. This produces a coloring of G using exactly three colors. Figure 4 displays such a coloring.

The graph H is made up of the graph G with an edge connecting a and g. Any attempt to color H using three colors must follow the same reasoning as that used to color G, except at the last stage, when all vertices other than g have been colored. Then, because g is adjacent (in H) to vertices colored red, blue, and green, a fourth color, say brown, needs to be used. Hence, H has a chromatic number equal to 4. A coloring of H is shown in Figure 4.

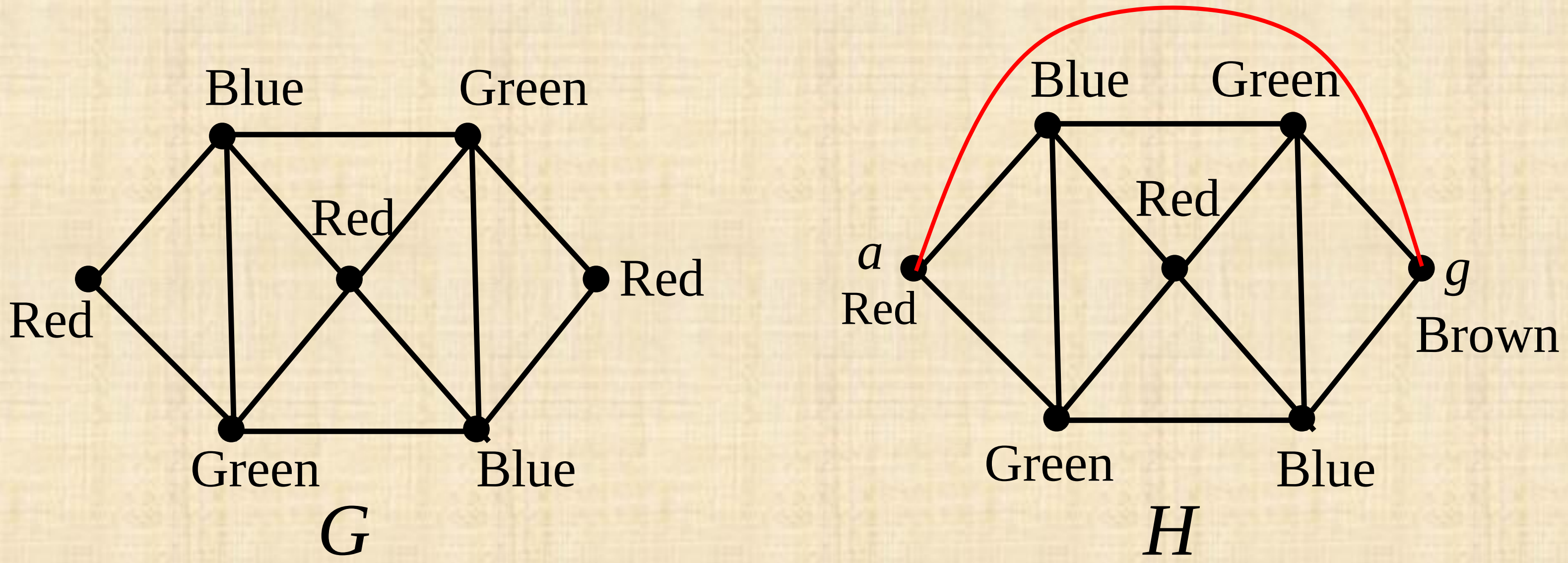
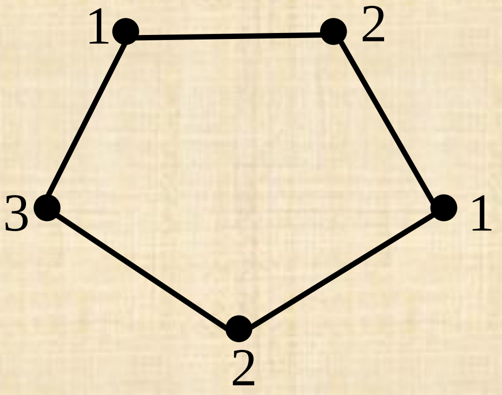


Figure 4: Colorings of the graphs G and H.

Example 4: $\chi(C_n) = \begin{cases} 2 \text{ if } n \text{ is even,} \\ 3 \text{ if } n \text{ is odd.} \end{cases}$

C_n is bipartite when n is even.



Thm 1. (The Four Color Theorem)
The chromatic number of a planar graph is no greater than four.

Corollary

Any graph with chromatic number > 4 is non-planar.

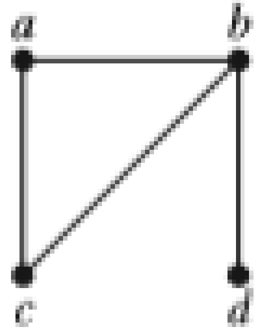
Example 5: K_7 is not planar because its chromatic number is 7 which is greater than 4.

excercise

2. Schedule the final exams for Math 115, Math 116, Math 185, Math 195, CS 101, CS 102, CS 273, and CS 473, using the fewest number of different time slots, if there are no students taking both Math115 and CS473, both Math 116 and CS 473, both Math 195 and CS 101, both Math195 and CS102, both Math115 and Math116, both Math 115 and Math 185, and both Math 185 and Math 195, but there are students in every other pair of courses.(***Hint***: See Example 5 on page 732.)

In Exercises 5–11 find the chromatic number of the given graph.

5.



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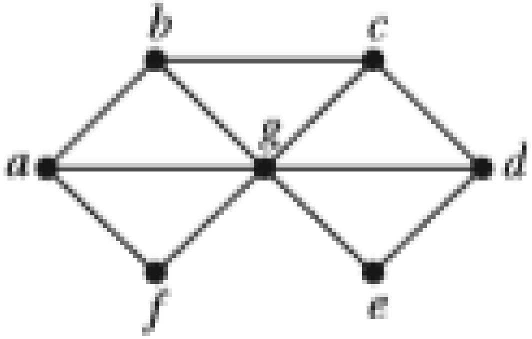

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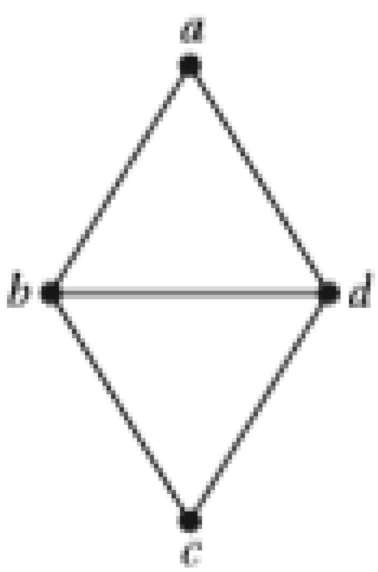

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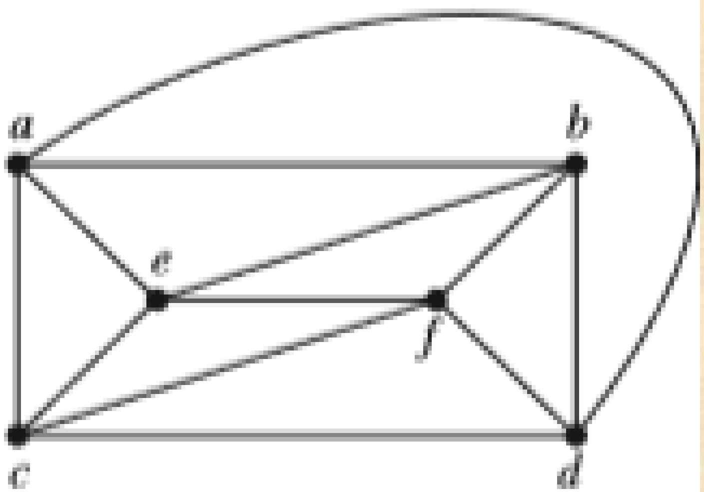

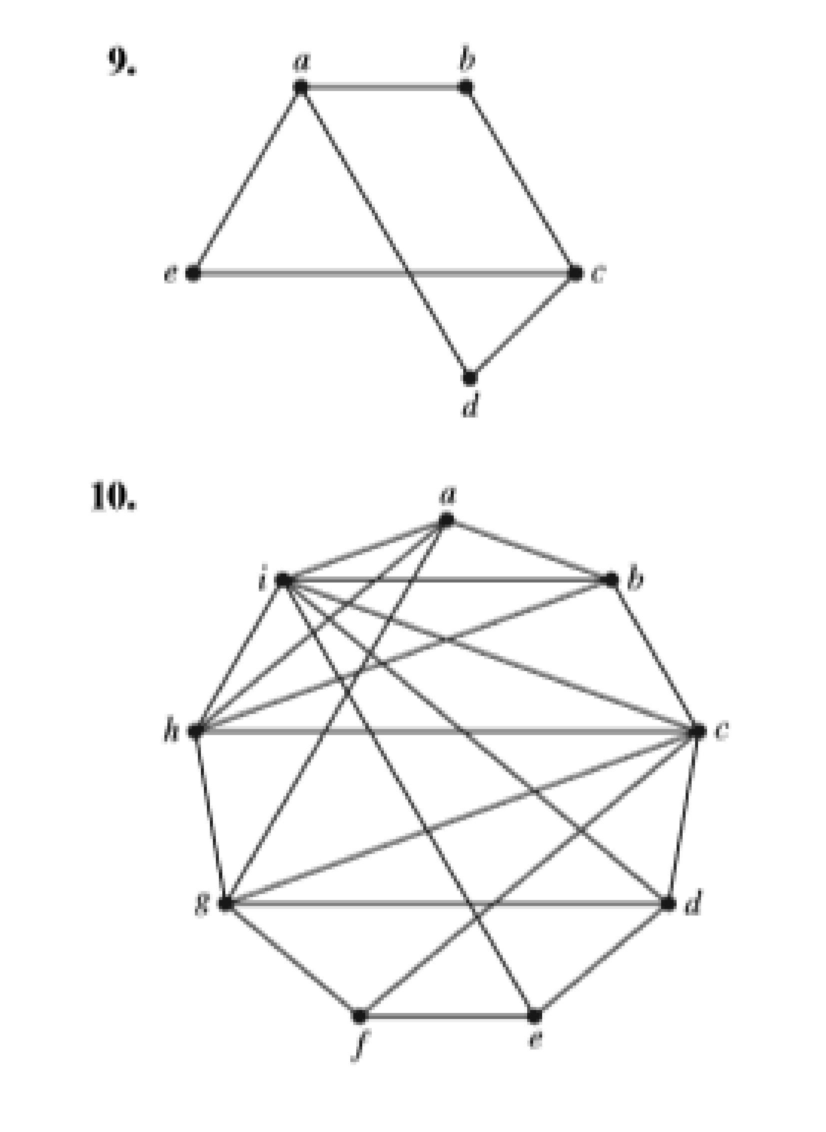
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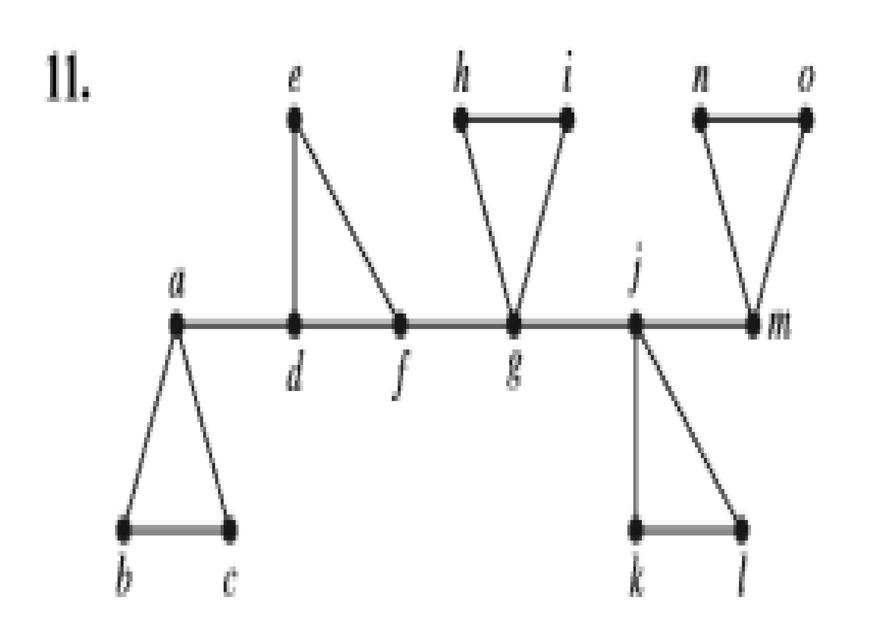
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DEFINITION 1 A *tree* is a connected undirected graph with no simple circuits.

Because a tree cannot have a simple circuit, a tree cannot contain multiple edges or loops. Therefore any tree must be a simple graph.

EXAMPLE 1 Which of the graphs shown in Figure 2 are trees?

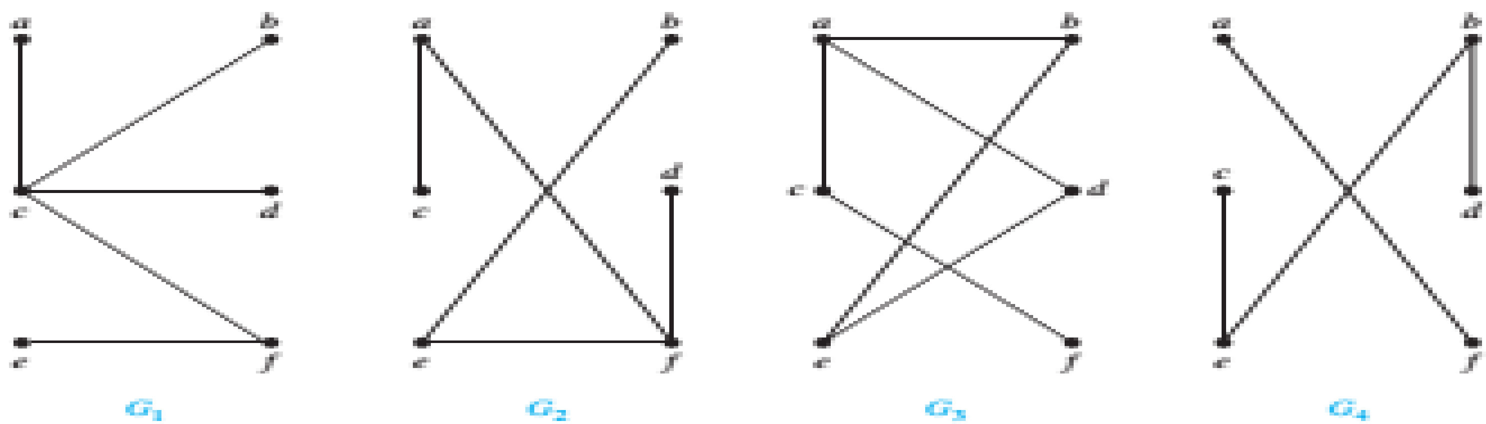

FIGURE 2 Examples of Trees and Graphs That Are Not Trees.

Solution: G_1 and G_2 are trees, because both are connected graphs with no simple circuits. G_3 is not a tree because e, b, a, d, e is a simple circuit in this graph. Finally, G_4 is not a tree because it is not connected. ◀

Any connected graph that contains no simple circuits is a tree. What about graphs containing no simple circuits that are not necessarily connected? These graphs are called **forests** and have the property that each of their connected components is a tree. Figure 3 displays a forest.

Trees are often defined as undirected graphs with the property that there is a unique simple path between every pair of vertices. Theorem 1 shows that this alternative definition is equivalent to our definition.

THEOREM 1 An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

DEFINITION 2 A *rooted tree* is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

Rooted trees can also be defined recursively. Refer to Section 5.3 to see how this can be done. We can change an unrooted tree into a rooted tree by choosing any vertex as the root. Note that different choices of the root produce different rooted trees. For instance, Figure 4 displays the rooted trees formed by designating a to be the root and c to be the root, respectively, in the tree T. We usually draw a rooted tree with its root at the top of the graph. The arrows indicating the directions of the edges in a rooted tree can be omitted, because the choice of root determines the directions of the edges.

The terminology for trees has botanical and genealogical origins. Suppose that T is a rooted tree. If v is a vertex in T other than the root, the **parent** of v is the unique vertex u such that there is a directed edge from u to v (the reader should show that such a vertex is unique). When u is the parent of v, v is called a **child** of u. Vertices with the same parent are called **siblings**. The **ancestors** of a vertex other than the root are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root (that is, its parent, its parent's parent, and so on, until the root is reached). The **descendants** of a vertex v are those vertices that have v as an ancestor. A vertex of a rooted tree is called a **leaf** if it has no children. Vertices that have children are called **internal vertices**. The root is an internal vertex unless it is the only vertex in the graph, in which case it is a leaf.

If a is a vertex in a tree, the **subtree** with a as its root is the subgraph of the tree consisting of a and its descendants and all edges incident to these descendants.

DEFINITION 3 A rooted tree is called an *m-ary tree* if every internal vertex has no more than m children. The tree is called a *full m-ary tree* if every internal vertex has exactly m children. An m-ary tree with $m = 2$ is called a *binary tree*.

Links

Properties of Trees

THEOREM 2 A tree with n vertices has $n - 1$ edges.

THEOREM 3 A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices.

THEOREM 4 A full m-ary tree with

(*i*) n vertices has $i = (n - 1)/m$ internal vertices and $l = [(m - 1)n + 1]/m$ leaves,
(*ii*) i internal vertices has $n = mi + 1$ vertices and $l = (m - 1)i + 1$ leaves,
(*iii*) l leaves has $n = (ml - 1)/(m - 1)$ vertices and $i = (l - 1)/(m - 1)$ internal vertices.

EXAMPLE A chain letter starts with a person sending a letter out to 10 others. Each person is asked to send the letter out to 10 others, and each letter contains a list of the previous six people in the chain. Unless there are fewer than six names in the list, each person sends one dollar to the first person in this list, removes the name of this person from the list, moves up each of the other five names one position, and inserts his or her name at the end of this list. If no person breaks the chain and no one receives more than one letter, how much money will a person in the chain ultimately receive?